ORTHOGONAL EMBEDDING THEORY FOR CONTRACTIVE TIME-VARYING SYSTEMS

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Abstract – This paper discusses a constructive solution of the problem of the realization of a given (strictly) contractive time-varying system as the partial transfer operator of a lossless system. The construction is done in a state space context and gives rise to a time-varying Ricatti-type equation. It is the generalization to the time-varying case of the time-invariant Darlington synthesis.

I. INTRODUCTION

The orthogonal embedding problem is, given a causal bounded transfer operator T, to extend this system by adding more inputs and outputs to it such that the resulting system Σ,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

is lossless: $\Sigma^*\Sigma = I$, $\Sigma\Sigma^* = I$, and has T as its partial transfer when the extra inputs are forced to zero: $T = \Sigma_{11}$. See figure 1. This problem is also known as the Darlington problem in classical network theory [1, 2]. Since unitarity of Σ implies $T^*T + T_c^*T_c = I$, (where $T_c = \Sigma_{21}$), it will be possible to find solutions to the embedding problem only if T is contractive: $I - T^*T \geq 0$.

We will solve the lossless embedding problem for strictly contractive time-varying systems in a state space context, under the assumption that the number of states of T is finite at any moment. Such an approach is discussed in [2] for continuous-time time-invariant systems, and hinges on what is called the Bounded Real Lemma. This lemma states that T is contractive if and only if certain conditions on the state space realization matrices are fulfilled. If this is the case, the conditions imply the existence of a realization for T_c that has the same A and C matrices as the realization of T, and which can be determined by solving a Ricatti equation. The Bounded Real Lemma can without much effort be translated to the discrete time context, as the idea behind it is based on the conservation of energy. Using this law, a first version of this appears in [3], a resulting Ricatti equation is stated in [4].

While it is clear that, also in the time-varying context, contractivity of T is a necessary condition, it will be shown in the sequel that strict contractivity of T is sufficient to construct a solution to the embedding problem. The extension to the boundary case invokes some mathematical complications (and is omitted here), but results in almost the same algorithm. The solution to the embedding problem is shown to depend at each time instant k on the positivity of some quantity M_k, which can be computed recursively from the given state space realization as

$$\begin{aligned} M_{k+1} &= A_k^*M_kA_k + B_k^*B_k \quad + \\ &+ [A_k^*M_kC_k + B_k^*D_k]\,(I - D_k^*D_k - C_k^*M_kC_k)^{-1}\,[D_k^*B_k + C_k^*M_kA_k]\,. \end{aligned}$$

This recursion is not unlike the recursive solution of the Ricatti equation that occurs in the time-invariant embedding problem, but now with time-varying coefficients. In this respect, note that in a Ricatti equation as it occurs in *e.g.*, optimal control problems, the term that is inverted is typically positive automatically, while here we have $(I - D_k^*D_k - C_k^*M_kC_k)$, which can potentially become negative and cause M_{k+1} to become negative (or indefinite) too. The main contribution of the paper is to show that this recursion does not break down (*i.e.*, all M_k are uniformly positive), under the condition that T is strictly contractive and the given realization for T is uniformly controllable.

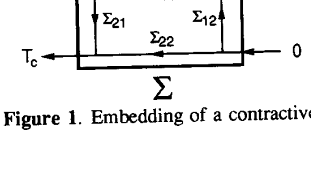

Figure 1. Embedding of a contractive TV system T.

II. NOTATION AND PRELIMINARIES

The notation in this paper is mostly as in [5], see also [6, 7]. We are interested in bounded causal ("upper") operators that map ℓ_2-sequences u to ℓ_2-sequences y via $y = uT$. With $u = [\cdots\ u_{-1}\ \boxed{u_0}\ u_1\ u_2\ \cdots]$, and y likewise, we will identify T with its (doubly-infinite) matrix representation

$$T = \begin{bmatrix} \ddots & \ddots & & \ddots & & \cdot^{\cdot^{\cdot}} \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & T_{-1,2} & \\ & & \boxed{T_{00}} & T_{01} & T_{02} & \ddots \\ & & & T_{11} & T_{12} & \\ & \mathbf{0} & & & T_{22} & \ddots \\ & & & & & \ddots \end{bmatrix}.$$

(The square identifies the 00-th entry of the matrix.) If T is viewed as the transfer operator of a non-stationary causal linear system with input u and corresponding output y then the i-th row of T corresponds to the impulse response of the system when it is excited at time instant i. A time-varying state space realization of T has traditionally the form

$$\begin{cases} x_{i+1} &= x_iA_i + u_iB_i \\ y_i &= x_iC_i + u_iD_i \end{cases}$$

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for all time instances i. Let A_i have dimensions $N_i \times N_{i+1}$. The above indexed time varying description can be collected into one expression in which is operated on sequences rather than entries, by defining

$$\begin{array}{rcl} x & = & [\cdots \; x_{-1} \; \boxed{x_0} \; x_1 \; x_2 \; \cdots] \in \ell_2^N \\ xZ^{-1} & = & [\cdots \; x_0 \; \boxed{x_1} \; x_2 \; x_3 \; \cdots] \in \ell_2^{N^{(-1)}} . \end{array} \tag{1}$$

In these expressions, x is a generalization of an ℓ_2 sequence in which each of the entries x_i is an element of a (row) vector space $\mathbf{C}^{N_i}$, with varying dimensions $N_i \in \mathbf{N}$, and such that the total energy $\|x\|_2^2 = \sum_{-\infty}^{\infty} \|x_i\|_2^2$ is bounded. We say that the above x is an element of $\ell_2(\mathbf{C}^N)$, or ℓ_2^N for brevity. We adopt the shorthand "$\bullet n$" for the index sequence N with all N_i equal to the same integer n. The k-th index shift $N^{(k)}$ is defined by $(N^{(k)})_i = N_{i-k}$. The shift operator $Z : \ell_2^N \to \ell_2^{N^{(+1)}}$ is defined by $(xZ)_i = x_{i-1}$; in equation (1), xZ^{-1} is the 'next state' sequence. The resulting state space description is

$$\left[\begin{array}{rcl} xZ^{-1} & = & xA + uB \\ y & = & xC + uD \end{array}\right] \qquad \mathbf{T} = \left[\begin{array}{cc} A & C \\ B & D \end{array}\right]$$

in which $A = \mathrm{diag}(A_i)$ is a "diagonal" mapping of ℓ_2^N sequences to $\ell_2^{N^{(-1)}}$ sequences. An equivalent description is obtained via a state space transformation $x \to x'R$ by an invertible operator R which results in a realization

$$\mathbf{T}' = \left[\begin{array}{cc} R & \\ & I \end{array}\right] \left[\begin{array}{cc} A & C \\ B & D \end{array}\right] \left[\begin{array}{cc} R^{-(-1)} & \\ & I \end{array}\right] .$$

Following [6], we denote by $\mathcal{X}(\ell_2^N, \ell_2^P)$ the class of bounded operators $(\ell_2^N \to \ell_2^P)$. Standard subsets of $\mathcal{X}$ are the space of upper (causal), lower and diagonal operators:

$$\begin{array}{rcl} \mathcal{U} & = & \{A \in \mathcal{X} : A_{ij} = 0, \; i > j\} \\ \mathcal{L} & = & \{A \in \mathcal{X} : A_{ij} = 0, \; i < j\} \\ \mathcal{D} & = & \mathcal{U} \cap \mathcal{L} . \end{array}$$

E.g., a causal system transfer operator T with n_I input ports and n_O output ports is an operator in $\mathcal{U}(\ell_2^{\bullet n_I}, \ell_2^{\bullet n_O})$. Let $F \in \mathcal{X}$. The k-th diagonal shift on F is $F^{(k)} = Z^{*k} F Z^k$: it shifts F down over k positions along the direction of the main diagonal. We define the j-th diagonal $F_{[j]} \in \mathcal{D}$ of F by $(F_{[j]})_i = F_{i-j,i}$. Hence $F_{[0]}$ is the main diagonal of the operator F, and for positive j, $F_{[j]}$ is the j-th subdiagonal above $F_{[0]}$. With this notation, F can formally be written in terms of its diagonals as $F = \sum_{-\infty}^{\infty} Z^j F_{[j]}$, although this expression need not converge at all. A class of operators that do allow this representation is the set of Hilbert-Schmidt operators [6]:

$$\mathcal{X}_2 = \{F \in \mathcal{X} : \|F\|_{HS}^2 = \textstyle\sum_{i,j} \|F_{ij}\|_2^2 < \infty\} .$$

Standard subspaces in $\mathcal{X}_2$ are $\mathcal{U}_2 = \mathcal{U} \cap \mathcal{X}_2$, $\mathcal{L}_2 = \mathcal{L} \cap \mathcal{X}_2$, $\mathcal{D}_2 = \mathcal{L}_2 \cap \mathcal{U}_2$, and standard projectors onto these spaces are denoted by $\mathbf{P}$. In particular, we define $\mathbf{P}_0 = \mathbf{P}_{\mathcal{D}_2}$, which maps to the main diagonal. We typically take inputs and outputs in subspaces of $\mathcal{X}_2$, because, *e.g.*, an output Y in $\mathcal{U}_2$ has rows y_i in ℓ_2 that start at time i (for the i-th row) and thus plays the role of an output in the "future", with respect to each time instant i. This collection of outputs $\{y_i\}$ can also be thought of as an instance of a generalized output sequence which is isomorphic to Y and which has entries in $\mathcal{D}$: for $Y \in \mathcal{U}_2$ the diagonal expansion of Y is $\tilde{Y}$, defined by

$$\tilde{Y} = \left[Y_{[0]} \;\; Y_{[1]}^{(-1)} \;\; Y_{[2]}^{(-2)} \;\; \cdots \right] .$$

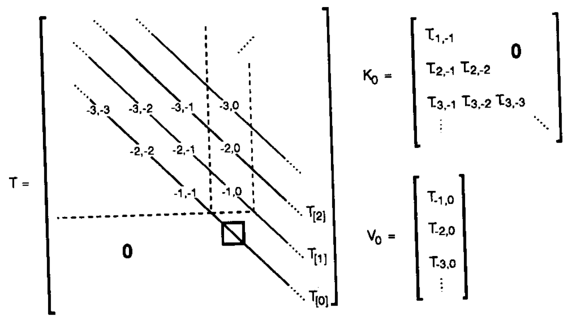

Figure 2. K_i and V_i matrices are submatrices of T.

This definition keeps entries of Y that are on the same row in Y also on the same row in $\tilde{Y}$, and is useful because it synchronizes the "future" to start collectively at the first diagonal entry of the sequence $\tilde{Y}$. Analogously, for $U \in \mathcal{L}_2 Z^{-1}$, the diagonal expansion of U is also designated by $\tilde{U}$, now defined by

$$\begin{array}{rcl} U & = & Z^{-1} U_{[-1]} + Z^{-2} U_{[-2]} + \cdots \\ \tilde{U} & = & \left[U_{[-1]}^{(+1)} \;\; U_{[-2]}^{(+2)} \;\; \cdots \right] \end{array}$$

III. DIAGONAL EXPANSIONS

In the sequel, we will need the notion of a "top left" part of an operator T in $\mathcal{U}$ in the sense of that part of T that maps inputs in the "past", $\mathcal{L}_2 Z^{-1}$, to outputs in the past, $\mathcal{L}_2 Z^{-1}$, which will be shown to correspond to the top left part of the matrix representation of T. To this end, define the operators K_T and V_T in the following way.

$$\begin{array}{llll} K_T : & \mathcal{L}_2 Z^{-1} \to \mathcal{L}_2 Z^{-1} , & & U K_T = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(UT) \\ V_T : & \mathcal{L}_2 Z^{-1} \to \mathcal{D}_2 , & & U V_T = \mathbf{P}_0(UT) . \end{array}$$

We can define operators $\tilde{K}_T$ and $\tilde{V}_T$ that act on diagonal expansions $\tilde{U}$ and $\tilde{Y}$ of U and Y. Unlike K_T and V_T, these operators have a matrix representation, which is obtained by reverting to diagonal expansions: if $Y = UK_T \in \mathcal{L}_2 Z^{-1}$ and $D = UV_T \in \mathcal{D}_2$, with $U \in \mathcal{L}_2 Z^{-1}$, then the matrix representations of the operators $\tilde{K}_T$ and $\tilde{V}_T$ such that $\tilde{Y} = \tilde{U}\tilde{K}_T$ and $D = \tilde{U}\tilde{V}_T$ is given by

$$\tilde{K}_T = \left[\begin{array}{ccccc} T_{[0]}^{(+1)} & 0 & \cdots & & \\ T_{[1]}^{(+1)} & T_{[0]}^{(+2)} & 0 & \cdots & \\ T_{[2]}^{(+1)} & T_{[1]}^{(+2)} & T_{[0]}^{(+3)} & 0 & \cdots \\ \vdots & & & \ddots & \ddots \end{array}\right] \qquad \tilde{V}_T = \left[\begin{array}{c} T_{[1]} \\ T_{[2]} \\ T_{[3]} \\ \vdots \end{array}\right] .$$

It is clear from the above that $\tilde{K}_T$ satisfies the relation

$$\tilde{K}_T^{(-1)} = \left[\begin{array}{cc} T_{[0]} & 0\,0\cdots \\ \tilde{V}_T & \tilde{K}_T \end{array}\right] . \tag{2}$$

There is a useful connection of T with $\tilde{K}_T$ and $\tilde{V}_T$, obtained by selecting the i-th entry of each diagonal in $\tilde{K}_T$ and using these to construct (infinite size) submatrices K_i $(-\infty < i < \infty)$ of $\tilde{K}_T$. The K_i are double-mirrored "top-left" submatrices of T (see figure 2), and can be viewed as a sequence of *time-varying* matrices that would be Toeplitz in the time-invariant case. In the same way, V_i is the vector representation of the operator V_T, obtained by selecting the i-th entry of the diagonal representation of $\tilde{V}_T$. (This technique was used in [5] to construct time-varying Hankel operators.)

IV. CONTRACTIVITY

A hermitian operator A in $\mathcal{X}$ is said to be strictly positive definite if there exists an $\varepsilon > 0$ such that, for all sequences u in ℓ_2, $uAu^* \geq \varepsilon uu^*$. Notation: $A \gg 0$. It is a known result that an operator A is strictly positive definite if and only if $A = A^*$ and A^{-1} exists in $\mathcal{X}$.

Definition 1. *Let T be a system transfer operator in $\mathcal{U}$. T is strictly contractive if $I - TT^* \gg 0$.*

Because of the identity $I + T^*(I - TT^*)^{-1}T = (I - T^*T)^{-1}$ it is clear that $I - TT^* \gg 0$ implies that $I - T^*T \gg 0$ also. It is straightforward to show that if T is strictly contractive, then K_T is strictly contractive on its domain: $I - K_TK_T^* \gg 0$, $I - K_T^*K_T \gg 0$. We also have that $\tilde{K}_T$ is strictly contractive: $I - \tilde{K}_T\tilde{K}_T^* \gg 0$, $I - \tilde{K}_T^*\tilde{K}_T \gg 0$, and all K_i are strictly contractive. Letting $i \to \infty$ it follows that T is strictly contractive. Hence contractivity of T, K_T and $\tilde{K}_T$ are equivalent.

Theorem 2. *Let $T \in \mathcal{U}$ be a system transfer operator. If T is strictly contractive, then*

$$I - T_{[0]}^*T_{[0]} - \tilde{V}_T^*(I - \tilde{K}_T\tilde{K}_T^*)^{-1}\tilde{V}_T \gg 0 .$$

PROOF Since T is strictly contractive, $\tilde{K}_T$ and $\tilde{K}_T^{(-1)}$ are also strictly contractive. Using equation (2), we have that

$$I - \tilde{K}_T^{(-1)*}\tilde{K}_T^{(-1)} = \begin{bmatrix} I - T_{[0]}^*T_{[0]} - \tilde{V}_T^*\tilde{V}_T & -\tilde{V}_T^*\tilde{K}_T \\ -\tilde{K}_T^*\tilde{V}_T & I - \tilde{K}_T^*\tilde{K}_T \end{bmatrix} \qquad (3)$$

From an application of Schur's inversion formula (see *e.g.*, [8]), it is seen that this expression is positive definite iff

$$\begin{cases} (1) & I - \tilde{K}_T^*\tilde{K}_T \gg 0 \\ (2) & I - T_{[0]}^*T_{[0]} - \tilde{V}_T^*\tilde{V}_T - \tilde{V}_T^*\tilde{K}_T(I - \tilde{K}_T^*\tilde{K}_T)^{-1}\tilde{K}_T^*\tilde{V}_T \gg 0 . \end{cases}$$

The first condition is satisfied because T is strictly contractive. The second condition is equal to the result. □

V. CONTRACTIVITY OF A REALIZATION

Let $T \in \mathcal{U}$ have a state space realization $\{A, B, C, D\}$, with $A \in \mathcal{D}(\ell_2^N, \ell_2^{N(-1)})$. We denote by $\mathcal{C}$ the controllability operator:

$$\mathcal{C} = \begin{bmatrix} B^{(+1)} \\ B^{(+2)}A^{(+1)} \\ B^{(+3)}A^{(+2)}A^{(+1)} \\ \vdots \end{bmatrix}$$

and we shall say that a realization is uniformly controllable if $\mathcal{C}^*\mathcal{C} \gg 0$. ($\mathcal{C}$ is an extension of the usual controllability operator to the present context. It is such that $\mathcal{C}^*$ is the diagonal expansion of $[BZ(I - AZ)^{-1}]^*$ in $\mathcal{L}_2Z^{-1}$; its derivation and many related issues are discussed in [5].) It can be derived that $\tilde{V}_T = C \cdot \mathcal{C}$, and by using the above structure of $\mathcal{C}$ we also have

$$\tilde{V}_T^{(-1)} = \begin{bmatrix} B \\ CA \end{bmatrix} \cdot \mathcal{C}^{(-1)} . \qquad (4)$$

Let the hermitian operator M in $\mathcal{D}(\ell_2^N, \ell_2^N)$ be defined by

$$M = \mathcal{C}^*(I - \tilde{K}_T\tilde{K}_T^*)^{-1}\mathcal{C} . \qquad (5)$$

M is well-defined if T is strictly contractive. It will play an important role in the embedding theory to follow in the next section. In that respect, the following observation is important. The contractivity condition implies that $M \geq 0$. If in addition the state space realization is uniformly controllable, $\mathcal{C}^*\mathcal{C} \gg 0$, then also $M \gg 0$ and invertible.

Theorem 3. *Let $T \in \mathcal{U}$ be a system transfer operator with state space realization $\{A, B, C, D\}$. If T is strictly contractive, then the above defined M satisfies the relations $I - D^*D - C^*MC \gg 0$, and*

$$\begin{aligned} M^{(-1)} &= A^*MA + B^*B + \\ &+ [A^*MC + B^*D]\,(I - D^*D - C^*MC)^{-1}\,[D^*B + C^*MA] . \end{aligned}$$

If in addition the state space realization is uniformly controllable, then $M \gg 0$.

PROOF The proof is straightforward but tedious, and hence omitted. It uses the definition of M, equations (2),(4), $D = T_{[0]}$, Theorem 2, and is based on an application of Schur's inversion formula to equation (3). □

VI. ORTHOGONAL EMBEDDING

We will construct a solution to the embedding problem as stated in the Introduction under the following conditions.

Theorem 4. *Let T be a bounded causal LTV operator with n_1 inputs and n_0 outputs: $T \in \mathcal{U}(\ell_2^{n_1}, \ell_2^{n_0})$, and let $\mathbf{T}$ be a state space realization of T. Suppose $A \in \mathcal{D}(\ell_2^N, \ell_2^{N(-1)})$. A solution to the embedding problem can be constructed if T is strictly contractive and the given realization $\mathbf{T}$ is uniformly controllable. This construction will yield a lossless realization $\mathbf{\Sigma}$ for the embedding system Σ with the following properties.*

(1) Σ is in $\mathcal{U}(\ell_2^n, \ell_2^n)$, with $n = n_1 + n_0$, i.e., the embedding adds n_0 more inputs and n_1 more outputs to those of T. This n cannot be smaller unless the number of added outputs is allowed to vary in time.

(2) $\mathbf{\Sigma} = \{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ has $A_\Sigma \in \mathcal{D}(\ell_2^m, \ell_2^m)$, where $m = \max_i(N_i)$. This m cannot be smaller unless the number of added outputs is allowed to vary in time.

To introduce the strategy to solve the embedding problem, consider the following simplified problem. Let $T \in \mathcal{U}(\ell_2^1, \ell_2^1)$ be a single-input, single-output system, with state space realization $\mathbf{T} \in \mathcal{D}(\ell_2^n, \ell_2^n)$ of constant dimensions. Then the objective is to find a lossless embedding system Σ, having two inputs and two outputs, and such that its state space realization $\mathbf{\Sigma}$,

$$\mathbf{\Sigma} = \begin{bmatrix} R & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A & C & C_2 \\ B & D & D_{12} \\ B_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} R^{(-1)} & & \\ & I & \\ & & I \end{bmatrix}$$

is unitary. $\mathbf{\Sigma}$ contains the given realization $\mathbf{T}$, suitably state space transformed by R (hence Σ_{11} is equal to the given T), and is extended by matrices B_2, C_2, D_{21}, D_{12}, D_{22} corresponding to the second input and output. The embedding problem is to find the state transformation R, and the embedding matrices. The problem can be split into two parts:

1. Determine R, B_2, D_{21} to make the columns of $\mathbf{\Sigma}_2$ orthogonal, with

$$\mathbf{\Sigma}_2 = \begin{bmatrix} R & & \\ & I & \\ & & I \end{bmatrix} \left[\begin{array}{cc} A & C \\ B & D \\ \hline B_2 & D_{21} \end{array}\right] \begin{bmatrix} R^{-(-1)} & \\ & I \end{bmatrix}$$

That is, $(\mathbf{\Sigma}_2)^*\mathbf{\Sigma}_2 = I$.

2. Add one orthogonal column $\mathbf{\Sigma}_3$ to $\mathbf{\Sigma}_2$ to make $\mathbf{\Sigma} = [\mathbf{\Sigma}_2 \;\; \mathbf{\Sigma}_3]$ unitary. The resulting realization $\mathbf{\Sigma}$ is a diagonal whose entries are square finite-dimensional matrices, hence this can always be done.

The key step in the above construction is step 1. With proper care on the dimensions of the embedding, it is always possible to find the solution to step 2. In this respect, note that the condition [Σ is lossless] implies that the number of inputs of Σ is equal to its number of outputs, and that the condition [$\mathbf{\Sigma}$ is unitary] implies that $\mathbf{\Sigma}$ is a diagonal of square matrices and hence that the system order of the embedding is constant. This determines the minimal system order of the embedding. The system order can be made smaller (in particular: equal to the system order of T at any moment) only if the number of added outputs is allowed to vary in time; we omit a discussion of this.

We will now present the construction referred to in theorem 4 for the general case.

Step 1. As before, find a state transformation R and matrices B_2 and D_{21} such that the columns of $\mathbf{\Sigma}_2$,

$$\mathbf{\Sigma}_2 = \begin{bmatrix} R & & \\ & I & \\ & & I \end{bmatrix} \left[\begin{array}{cc} A & C \\ B & D \\ \hline B_2 & D_{21} \end{array}\right] \begin{bmatrix} R^{-(-1)} & \\ & I \end{bmatrix}$$

are unitary, *i.e.*, $(\mathbf{\Sigma}_2)^*\mathbf{\Sigma}_2 = I$.

Lemma 5. *A solution to step 1. is obtained by putting $M = R^*R$ and solving for M in*

$$\begin{aligned} M^{(-1)} &= A^*MA + B^*B \quad + \\ &\quad + [A^*MC + B^*D]\,(I - D^*D - C^*MC)^{-1}\,[D^*B + C^*MA]\;. \end{aligned}$$

The solution M exists under the condition [T is strictly contractive] and is strictly positive definite if [$\mathbf{T}$ is uniformly controllable]. Because of Theorem 3, it is given in closed form by equation (5). $B_2 \in \mathcal{D}(\ell_2^{\bullet n_0}, \ell_2^{N^{(-1)}})$ and $D_{21} \in \mathcal{D}(\ell_2^{\bullet n_0}, \ell_2^{\bullet n_0})$ are determined as

$$\left[\begin{array}{lcl} D_{21} &=& (I - D^*D - C^*MC)^{\frac{1}{2}} \\ B_2 &=& -(I - D^*D - C^*MC)^{-\frac{1}{2}}\,[D^*B + C^*MA] \end{array}\right.$$

PROOF To solve step 1, compute $(\mathbf{\Sigma}_2)^*\mathbf{\Sigma}_2$, and put $M = R^*R$. From the orthogonality conditions the equations mentioned in the theorem follow directly. At this point, recall Theorem 3, and observe that the solution to the last equation is precisely given by

$$M = C^*(I - \tilde{K}_T\tilde{K}_T^*)^{-1}C\;.$$

Since the realization is uniformly controllable, Theorem 3 asserts that this $M \gg 0$, so that it can be factored as $M = R^*R$ with R invertible. It also follows that $D_{21}^*D_{21} \gg 0$, so that D_{21} cannot have less than $\bullet n_0$ rows, and hence no less than n_0 inputs must be added to T to yield a lossless embedding. □

Step 2. Define $\mathbf{\Sigma}_2'$ to consist of $\mathbf{\Sigma}_2$ extended by zero rows to $\mathbf{\Sigma}_2' \in \mathcal{D}(\ell_2^{\bullet(m+n_1+n_0)}, \ell_2^{N^{(-1)}})$:

$$\begin{aligned} \mathbf{\Sigma}_2' &= \left[\begin{array}{c} 0_{(\bullet m - N)\times N^{(-1)}} \\ \hline \mathbf{\Sigma}_2 \end{array}\right] \\ &= \begin{bmatrix} I & & & \\ & R & & \\ & & I & \\ & & & I \end{bmatrix} \left[\begin{array}{cc} \multicolumn{2}{c}{0_{(\bullet m - N)\times N^{(-1)}}} \\ \hline A & C \\ B & D \\ \hline B_2 & D_{21} \end{array}\right] \begin{bmatrix} R^{-(-1)} & \\ & I \end{bmatrix} \end{aligned}$$

Find matrices $\mathbf{\Sigma}_1 \in \mathcal{D}(\ell_2^{\bullet(m+n_1+n_0)}, \ell_2^{\bullet m - N^{(-1)}})$ and $\mathbf{\Sigma}_3 \in \mathcal{D}(\ell_2^{\bullet(m+n_1+n_0)}, \ell_2^{\bullet n_1})$ in the orthogonal complement of $\mathbf{\Sigma}_2'$ such that

$$\mathbf{\Sigma} = [\mathbf{\Sigma}_1 \;\; \mathbf{\Sigma}_2' \;\; \mathbf{\Sigma}_3]$$

is a diagonal of square unitary matrices of constant size $(m + n_1 + n_0)$. Put into this form, step 2. is always possible and reduces to a standard exercise in linear algebra.

ACKNOWLEDGEMENT

This research was supported in part by the commission of the EC under the ESPRIT BRA program 3280 (NANA).

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