Local Generalized Nash Equilibria with Nonconvex Coupling Constraints

Scarabaggio, Paolo; Carli, Raffaele; Grammatico, Sergio; Dotoli, Mariagrazia

DOI
10.1109/TAC.2024.3462553

Publication date
2025

Document Version
Final published version

Published in
IEEE Transactions on Automatic Control

Citation (APA)
Scarabaggio, P., Carli, R., Grammatico, S., & Dotoli, M. (2025). Local Generalized Nash Equilibria with Nonconvex Coupling Constraints. *IEEE Transactions on Automatic Control*, *70*(3), 1427-1439. https://doi.org/10.1109/TAC.2024.3462553

Local Generalized Nash Equilibria With Nonconvex Coupling Constraints

Paolo Scarabaggio, *Member, IEEE*, Raffaele Carli, *Senior Member, IEEE*, Sergio Grammatico, *Senior Member, IEEE*, and Mariagrazia Dotoli, *Fellow, IEEE*

***Abstract*—In this article, we address a class of Nash games with nonconvex coupling constraints for which we define a novel notion of local equilibrium, here named local generalized Nash equilibrium (LGNE). Our first technical contribution is to show the stability in the game theoretic sense of these equilibria on a specific local subset of the original feasible set. Remarkably, we show that the proposed notion of local equilibrium can be equivalently formulated as the solution of a quasi-variational inequality with equal Lagrange multipliers. Next, under the additional proximal smoothness assumption of the coupled feasible set, we define conditions for the existence and local uniqueness of an LGNE. To compute such an equilibrium, we propose two discrete-time dynamics, or fixed-point iterations implemented in a centralized fashion. Our third technical contribution is to prove convergence under (strongly) monotone assumptions on the pseudogradient mapping of the game and proximal smoothness of the coupled feasible set. Finally, we apply our theoretical results to a noncooperative version of the optimal power flow control problem.**

***Index Terms*—Generalized Nash equilibrium (GNE), multiagent systems, nonconvex generalized games, variational inequalities (VIs).**

I. Introduction

IN NONCOOPERATIVE games, a number of self-interested agents with their own individual dynamics and constraints aim at optimizing their objective functions, possibly in competition with each other, e.g., due to the scarcity of shared resources. In this context, one of the most important concepts is the *Nash equilibrium* (NE) [1], which has been adopted as a solution concept in many applications, from electricity markets to mobile-edge computing [2], [3]. Whenever the feasible set of an agent depends on the strategies of the other agents, the concept of equilibrium is extended to the so-called *generalized Nash equilibrium* (GNE) [4], which is recently getting strong attention from researchers of different fields, since the presence of coupled feasible sets is widespread in real-world applications [5], [6], [7], [8], [9], [10].

From a control-theoretic perspective, the objective is to develop a mechanism, namely, a discrete-time dynamical system, for updating the strategies of the agents towards an equilibrium. Solving this problem is difficult, since the objective function and the constraints of each agent are interdependent. While the NE of a game with compact local feasible sets and strongly monotone pseudogradient is unique, uniqueness is instead not guaranteed in games with shared constraints. Thus, most methods to reach an equilibrium typically rely on the *variational inequality* (VI) theory [11], which has the advantage of possibly selecting a particular solution named *normalized solution* or *variational generalized Nash equilibrium* (vGNE); the latter is usually referred to as "economically meaningful" [11], "more socially stable" or "fair" [12], [13].

Most of the related literature focuses on *jointly convex* (JC) games that have locally convex objective functions and convex feasible set [14]. These assumptions allow ensuring the existence of equilibria and global convergence to solution algorithms [15], [16]. Furthermore, thanks to convexity, the global convergence of several classes of multiagent dynamics (centralized, decentralized, and distributed) to a GNE can be guaranteed [14], [17], [18], [19], [20].

Nevertheless, in many applications, convexity does not hold, and thus, some alternative approaches have been proposed in the literature. Among the works considering nonconvex games, let us mention the equilibrium notions of weak NE [21], local NE (LNE) [22], generalized equilibrium [23], and critical NE [15]. All these concepts focus on the *Nash equilibrium problem* (NEP) and do not apply to games with coupling constraints. For instance, in [16] and [24], the authors develop an optimization-based theory for games with nonconvex objective functions and nonconvex side constraints. Specifically, the authors define the concept of quasi-Nash equilibrium (QNE), defined as the solution of the VI obtained by aggregating the first-order optimality

Received 7 November 2023; revised 22 July 2024; accepted 29 August 2024. Date of publication 17 September 2024; date of current version 28 February 2025. The work of Paolo Scarabaggio was supported by the NRRP - Mission 4 Component 2 Investment 1.3 - Call for tender No. 341 of March 15, 2022 of MUR - project "MICS (Circular and Sustainable Made-in-Italy)" (project code: PE00000004). The work of Raffaele Carli and Mariagrazia Dotoli was supported in part by the NRRP - Mission 4 Component 2 Investment 1.3 - Call for tender No. 341 of March 15, 2022 of MUR - project "NEST (Network 4 Energy Sustainable Transition)" (project no. PE00000021) and in part by NRRP - Mission 4 Component 2 Investment 1.1 - Call for "Projects of significant national interest - Progetti di rilevante interesse nazionale (PRIN)" - Decree no. 104 of February 2, 2022 of MUR (project: CORRECT, code: 202248PWRY). The work of Sergio Grammatico was supported in part by ERC under project COSMOS (802348). Recommended by Associate Editor G. Hu. *(Corresponding author: Paolo Scarabaggio.)*

Paolo Scarabaggio, Raffaele Carli, and Mariagrazia Dotoli are with the Department of Electrical and Information Engineering, Polytechnic of Bari, 70126 Bari, Italy (e-mail: paolo.scarabaggio@poliba.it; raffaele.carli@poliba.it; mariagrazia.dotoli@poliba.it).

Sergio Grammatico is with the Delft Center for Systems and Control, TU Delft, 2628 CD Delft, The Netherlands (e-mail: s.grammatico@tudelft.nl).

Digital Object Identifier 10.1109/TAC.2024.3462553

conditions of the individual agents. This approach has been applied to nonconvex power allocation games in cognitive radio networks [24], [25].

Ratliff et al. [26] present a framework to characterize local Nash equilibria in continuous games with nonconvex feasible sets. The approach relies on necessary and sufficient first- and second-order conditions to ensure optimality of the local equilibrium point, thanks to the local convexity of the solution space around the equilibrium point. Notably, Ratliff et al. [26] does not propose any algorithm, nor demonstrate the convergence to these equilibria.

Differently from the state of the art, in this article, we introduce a novel local equilibrium concept that we call *local generalized Nash equilibrium* (LGNE). We define this equilibrium over the linearized feasible directions set as a local subset, and we show that it satisfies the first-order optimality conditions of the optimization problems of the agents. By characterizing such equilibria, we leverage on the theory of quasi-variational inequalities (QVIs) to define locally variational, and thus, locally fair, equilibrium points. Then, by introducing an additional assumption on the proximal smoothness of the coupled feasible set, we demonstrate the existence of these equilibria and their local uniqueness in a well-defined subset of the original nonconvex feasible set. To compute a variational LGNE, we design two discrete-time autonomous dynamics, or fixed-point iterations, that we implement in a centralized fashion. We prove convergence of our proposed dynamics to an equilibrium under certain technical conditions, namely (strongly) monotone pseudogradient mapping of the game and proximally smoothness of the coupled feasible set. Finally, we apply our theoretical results to power systems control and particularly to the *optimal power flow* (OPF) problem. Several game-theoretic methods have been proposed for solving this problem [12], yet most disregard physical constraints due to their nonconvexity. Instead, we propose a noncooperative version of the OPF including the actual power flow equations, namely, a set of nonlinear nonconvex algebraic equations. The resulting noncooperative game with shared nonconvex constraints falls exactly in our proposed framework, thus allowing us to analyze the proposed concept in one of the most important problems in power systems control.

The rest of this article is organized as follows. In Section II, we define the problem setup, and we report the basic definitions and assumptions used in the sequel. In Section III, we introduce the novel theoretical concept of LGNE together with some examples. Existence and uniqueness are discusses in Section IV, while two algorithms to search for an equilibrium are presented in Section V. In Section VI we show the illustrative application of our framework and some related numerical examples. Finally, Section VII concludes this article. Appendix A reports some useful notions on cones theory. All proofs are given in Appendix B.

Basic notation: $\mathbb{R}^n$, $\mathbb{R}^n_{>0}$, and $\mathbb{R}^n_{\geq 0}$ denote the set of real, positive real, and nonnegative real n-dimensional vectors, respectively. $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{B}$ denotes the closed unit ball centered at zero. $A^\top$ denotes the transpose of A. $\|A\|$ is the square norm of A. $\mathbf{0}_n$ and $\mathbf{1}_n$ indicate the column vectors with n entries all equal to 0 and to 1, i.e., $\mathbf{0}_n := (0,\ldots,0)^\top \in \mathbb{R}^n$ and $\mathbf{1}_n := (1,\ldots,1)^\top \in \mathbb{R}^n$, respectively. Moreover, $\mathbf{x} := \mathrm{col}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is equal to $\mathbf{x} := (\mathbf{x}_1^\top,\ldots,\mathbf{x}_n^\top)^\top$. We define the mapping $\mathrm{proj}_{\mathcal{X}}(\cdot) : \mathbb{R}^n \to \mathcal{X}$ as the projection into the generic closed nonempty set $\mathcal{X} \subseteq \mathbb{R}^n$, i.e., $\mathrm{proj}_{\mathcal{X}}(\mathbf{y}) = \mathrm{argmin}_{\mathbf{x}\in\mathcal{X}}\|\mathbf{x}-\mathbf{y}\|$. Moreover, we define the mapping $\mathrm{dist}(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|$ as the distance operator between two points and $\mathrm{dist}(\mathbf{y},\mathcal{X}) := \min_{\mathbf{x}\in\mathcal{X}}\|\mathbf{x}-\mathbf{y}\|$ as the distance operator between a point and a set. For a generic closed nonempty set $\mathcal{X} \subseteq \mathbb{R}^n$, we define the topological closure $\mathrm{cl}(\mathcal{X})$ and the boundary $\mathrm{bd}(\mathcal{X})$. The mapping $F(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with a constant $\ell \in \mathbb{R}_{>0}$ if $\|F(\mathbf{x}) - F(\mathbf{y})\| \leq \ell\|\mathbf{x}-\mathbf{y}\|\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$. F is strongly monotone with a constant $\mu \in \mathbb{R}_{>0}$ if $(F(\mathbf{x}) - F(\mathbf{y}))^\top(\mathbf{x} - \mathbf{y})) \geq \mu\|\mathbf{x}-\mathbf{y}\|^2\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$ while F is pseudo monotone if $F(\mathbf{y})^\top(\mathbf{x}-\mathbf{y})) \geq 0 \Rightarrow F(\mathbf{x})^\top(\mathbf{x}-\mathbf{y})) \geq 0,\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$. Finally, we define $\bar{F}_{\mathcal{X}} := \sup_{\mathbf{x}\in\mathcal{X}}\|F(\mathbf{x})\|$.

II. Generalized Nash Equilibrium Problems

We consider a game composed of a set of N agents, indexed by $i \in \mathcal{N} := \{1,\ldots,N\} \subseteq \mathbb{N}$ each with decision variables $\mathbf{x}_i \in \mathbb{R}^n$. Moreover, we define vectors $\mathbf{x}_{-i} := \mathrm{col}(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\ldots,\mathbf{x}_N) \in \mathbb{R}^{(N-1)n}$ and $\mathbf{x} := \mathrm{col}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N) \in \mathbb{R}^{Nn}$, collecting the strategies of all agents different from i and the strategies of all agents, respectively. Each agent $i \in \mathcal{N}$ tries to minimize its cost function $f_i(\mathbf{x}_i,\mathbf{x}_{-i}) : \mathbb{R}^n \times \mathbb{R}^{(N-1)n} \to \mathbb{R}$ by choosing a strategy in its local feasible set $\mathbf{x}_i \in \Omega_i \subseteq \mathbb{R}^n$, hence, $\mathbf{x} \in \mathbf{\Omega} = \prod_{i=1}^N \mathbf{\Omega}_i$.

In addition, let us consider a finite number of constraints indexed by $m \in \mathcal{M} := \{1,\ldots,M\} \subseteq \mathbb{N}$, each denoted as $g_m(\mathbf{x}) \leq 0$, defining the coupled feasible set as follows:

$$\mathcal{X} = \mathbf{\Omega} \cap \left\{\mathbf{x} \in \mathbb{R}^{Nn} \mid g(\mathbf{x}) \leq \mathbf{0}_M\right\} \tag{1}$$

where $g(\mathbf{x}) := ((g_m(\mathbf{x}))_{m\in\mathcal{M}})$. By defining the set-valued mapping $\mathcal{X}_i(\mathbf{x}_{-i}) := \{\mathbf{y}_i \in \mathbb{R}^n \mid (\mathbf{y}_i,\mathbf{x}_{-i}) \in \mathcal{X}\}$, one can define the N interdependent optimization problems as follows:

$$\forall i \in \mathcal{N} : \quad \begin{cases} \min\limits_{\mathbf{x}_i} & f_i(\mathbf{x}_i,\mathbf{x}_{-i}) \\ \text{s.t.} & \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}). \end{cases} \tag{2}$$

To make the notation easier to follow, in the rest of this article, we ignore the presence of local constraints. However, they can be included directly in the coupled feasible set or approximated via barrier functions in the objective function.

The latter problem is a GNEP whose solution is the GNE, formally defined as follows.

Definition 1 (GNE): A GNE is a collective strategy $\mathbf{x}^* \in \mathcal{X}$ such that for each $i \in \mathcal{N}$ it holds

$$f_i(\mathbf{x}_i^*,\mathbf{x}_{-i}^*) \leq \inf\left\{f_i(\mathbf{x}_i,\mathbf{x}_{-i}^*) \,\middle|\, \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}^*)\right\}. \tag{3}$$

■

In other words, a GNE is a collective strategy profile satisfying the property that no single agent in the game can improve its objective function by unilaterally changing its strategy with another feasible one.

Properties such as stability, uniqueness, and optimality of the GNE have been studied under different assumptions. The convexity of the coupling feasible set $\mathcal{X}$ in (1) is one of the most employed assumptions [27]. Due to the nature of several applications, this set may be nonconvex. Here, we aim to analyze the particular case of nonconvex feasible sets. Let us first introduce some preliminary assumptions used in the rest of this article.

Assumption 1: For each $i \in \mathcal{N}$ and for every $\mathbf{x}_{-i}$, the function $f_i(\cdot, \mathbf{x}_{-i})$ in (2) is convex and continuously differentiable. ■

Assumption 2: For each $m \in \mathcal{M}$ and for every $\mathbf{x}_{-i}$, the function $g_m(\cdot, \mathbf{x}_{-i})$ in (1) is continuously differentiable (possibly nonconvex). The coupled feasible set $\mathcal{X}$ in (1) is nonempty, compact, and satisfies the Mangasarian–Fromovitz constraint qualification (MFCQ), i.e., for each $\mathbf{x} \in \mathcal{X}$, the gradients of the equality constraints are linearly independent and there exists $a \in \mathbb{R}^n$ such that $\nabla g_i(\mathbf{x})^\top a < 0$ for all active inequality constraints and $\nabla g_j(\mathbf{x})^\top a = 0$ for all equality constraints [28]. ■

III. Local Generalized Nash Equilibria: Definition and Characterization

Let us search for weaker equilibrium conditions, following the approach commonly used in nonconvex optimization, which consists of looking for a stationary (possibly locally optimal) solution. In particular, let us propose a novel concept, namely *LGNE problem* (LGNEP) and let us search for its possible solution, i.e., the LGNE. Our approach relies on the definition of the *linearized feasible directions set* at a point $\mathbf{x}$ for the (nonconvex) set $\mathcal{X}$ in (1), denoted as $\mathcal{F}(\mathcal{X}, \mathbf{x})$. We note that $\mathcal{F}(\mathcal{X}, \mathbf{x})$ is convex even if $\mathcal{X}$ is not, while due to Assumption 2, it equals the so-called tangent cone, i.e., $\mathcal{F}(\mathcal{X}, \mathbf{x}) = T(\mathcal{X}, \mathbf{x})$ (see Appendix A for technical details). For the sake of keeping the notation light, let us define $\tilde{\mathcal{X}}(\mathbf{x}) := \mathbf{x} + \mathcal{F}(\mathcal{X}, \mathbf{x})$.

Definition 2 (LGNE): An LGNE is a collective strategy $\mathbf{x}^* \in \mathcal{X}$ such that for each $i \in \mathcal{N}$

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf\left\{ f_i(\mathbf{y}, \mathbf{x}_{-i}^*) \,|\, \mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*)\right\} \tag{4}$$

where the set-valued mapping $\tilde{\mathcal{X}}_i(\mathbf{x}_{-i}) := \{\mathbf{y}_i \in \mathbb{R}^n \mid (\mathbf{y}_i, \mathbf{x}_{-i}) \in \tilde{\mathcal{X}}(\mathbf{x})\}$ is the linearized feasible directions set of the ith agent and thus $\tilde{\mathcal{X}}(\mathbf{x}) = \prod_{i=1}^N \tilde{\mathcal{X}}_i(\mathbf{x}_{-i})$. ■

In other words, an LGNE is a stable collective strategy profile with the property that no single agent can benefit by unilaterally changing its strategy with another feasible one contained in the linearized feasible directions set.

Remark 1: Whenever the feasible set $\mathcal{X}$ in (1) is convex, the LGNEP is equivalent to the GNEP, since the set of LGNEs is equal to that of GNEs. Indeed, the linearized feasible directions set of a convex set includes the convex set itself [29]. ■

Note that, in general, we may have multiple LGNEs for the game and the sets of GNEs might be a subset of LGNEs.

A first approach to characterize an LGNE $\mathbf{x}^* \in \mathcal{X}$ is based on employing normal cone $N(\mathcal{X}, \mathbf{x}^*) = T^\circ(\mathcal{X}, \mathbf{x}^*) = \mathcal{F}^\circ(\mathcal{X}, \mathbf{x}^*)$; see Appendix A. For each agent $i \in \mathcal{N}$, the following optimality condition must be verified: $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$. This version of the optimality conditions is however not useful enough, due to the difficulty of constructing the normal cone in the nonconvex case. Thus, let us characterize the LGNE by deriving the *Karush–Kuhn–Tucker* (KKT) conditions for each agent $i \in \mathcal{N}$. For a constrained nonlinear program with a differentiable objective function, the KKT conditions are necessary conditions to be satisfied by a locally optimal solution under an appropriate constraint qualification. More formally, the KKT conditions for each agent are

$$\mathrm{KKT}_i : \begin{cases} 0 \in \nabla_{\mathbf{x}_i} f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) + \nabla_{\mathbf{x}_i} g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)\boldsymbol{\lambda}_i \\ \mathbf{0}_M \leq \boldsymbol{\lambda}_i \quad \perp \quad g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \mathbf{0}_M \end{cases} \tag{5}$$

where we assume the existence of dual variables $\boldsymbol{\lambda}_i = \mathrm{col}(\lambda_i^1, \ldots, \lambda_i^m, \ldots, \lambda_i^M) \in \mathbb{R}^M_{\geq 0}$ satisfying the KKT optimality conditions for each individual optimization problem in (2). Note that, if a constraint g_m is not active at $\mathbf{x}$, then the corresponding Lagrangian multiplier λ_i^m is necessarily zero for all agents $i \in \mathcal{N}$. On the other hand, multipliers corresponding to the same active constraint can have different values among agents.

Before exploiting the aforementioned optimality conditions, let us make the following remark.

Theorem 1: Let Assumptions 1 and 2 hold. Then, the following statements are equivalent:

1) $\mathbf{x}^* \in \mathcal{X}$ is an LGNE;
2) for each $i \in \mathcal{N}$, $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$;
3) for each $i \in \mathcal{N}$, there exists a vector $\boldsymbol{\lambda}_i = \mathrm{col}(\lambda_i^1, \ldots, \lambda_i^m, \ldots, \lambda_i^M) \in \mathbb{R}^M_{\geq 0}$ satisfying the KKT conditions in (5). ■

The KKT conditions are, in general, only necessary conditions for optimality. Nevertheless, these are both necessary and sufficient in our setting, due to Assumptions 1 and 2. Specifically, we can ensure that at each point $\mathbf{x} \in \mathcal{X}$, the LGNEP is convex even though the original problem is nonconvex. Consequently, the optimization problems of all agents $i \in \mathcal{N}$ become convex, complying with Slater's condition (via MFCQ) [30], [31].

In the related literature [32], JC games are usually solved by finding a solution to the associated variational inequality problem (VIP), since the resulting vGNE not only exists but is also unique whenever the cost functions are strongly monotone [33]. By relaxing the convexity condition on the coupling feasible set $\mathcal{X}$ in (1), we can no longer rely on the existence and uniqueness of a vGNE. However, we can still obtain a first-order necessary condition for a local minimizer. To this aim, by introducing the pseudogradient mapping

$$F(\mathbf{x}) = \begin{bmatrix} \nabla_{\mathbf{x}_1} f_1(\mathbf{x}_1, \mathbf{x}_{-1}) \\ \vdots \\ \nabla_{\mathbf{x}_N} f_N(\mathbf{x}_N, \mathbf{x}_{-N}) \end{bmatrix} \tag{6}$$

we consider the following QVI associated with the LGNEP in (2).

Definition 3 (QVI [34]): Given the set $\tilde{\mathcal{X}}(\mathbf{x})$ and the mapping F in (6), the *quasi-variational inequality problem QVIP*$(\tilde{\mathcal{X}}, F)$ consists in finding a vector $\mathbf{x}^* \in \tilde{\mathcal{X}}(\mathbf{x}^*)$, the so-called *quasi-variational equilibrium* (QVE), such that

$$\inf_{\mathbf{y} \in \tilde{\mathcal{X}}(\mathbf{x}^*)} (\mathbf{y} - \mathbf{x}^*)^\top F(\mathbf{x}^*) \geq 0. \tag{7}$$

■

We can now define the *variational LGNE* (vLGNE).

Definition 4 (Variational LGNE): A vLGNE is an LGNE in (4) that satisfies the QVIP in (7). ■

Similarly to the relation between GNEP and VIP, not all solutions of the LGNEP are a solution of the QVIP; viceversa, a solution of the QVIP is always a solution of the original LGNEP.

Furthermore, if we consider the KKT conditions of the corresponding QVIP$(\tilde{\mathcal{X}}(\mathbf{x}), F(\mathbf{x}))$, we have that

$$\begin{cases} 0 \in F(\mathbf{x}) + \nabla_{\mathbf{x}} g(\mathbf{x})\boldsymbol{\lambda} \\ \boldsymbol{\lambda} \geq 0 \quad \perp \quad g(\mathbf{x}) \leq 0 \end{cases} \tag{8}$$

where the solutions of the LGNEP that are preserved passing to the QVIP are exactly those for which all agents have the same multipliers for the respective constraints. Thus, we have the following results.

Theorem 2: Let Assumptions 1 and 2 hold.

1) Let $\mathbf{x}^*$ be a solution of the LGNEP in (2), where the KKT conditions in (5) for all agents hold with the same Lagrangian multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i \; \forall i \in \mathcal{N}$. Then, $\mathbf{x}^*$ is a solution of the QVI in (7), and thus, it is a vLGNE.
2) Viceversa, let $\mathbf{x}^*$ be a solution of the QVI in (7), and thus, be a vLGNE. Then, $\mathbf{x}^*$ is a solution of the LGNEP in (2) at which the KKT conditions in (5) hold with the same Lagrangian multipliers, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i \; \forall i \in \mathcal{N}$. ■

In other words, at a vLGNE, we have that in a local subset of $\mathcal{X}$ in (1) each agent cannot unilaterally minimize their own function while keeping the strategies of the other agents fixed. Moreover, when at this point, the common optimal multiplier for all the agents associated with the individual constraints are the same, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i \; \forall i \in \mathcal{N}$, then the point is a QVE, and thus, a locally fair equilibrium point [12], [13].

We can also characterize a vLGNE by employing the normal cone. By Theorem 1 we have that in a LGNE $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*) \; \forall i \in \mathcal{N}$. Thus, by leveraging on Theorem 2, we can prove the following corollary.

Corollary 1: Let Assumptions 1 and 2 hold. $\mathbf{x}^*$ is a vLGNE if and only if

$$-F(\mathbf{x}^*) \in N(\mathcal{X}, \mathbf{x}^*) \tag{9}$$

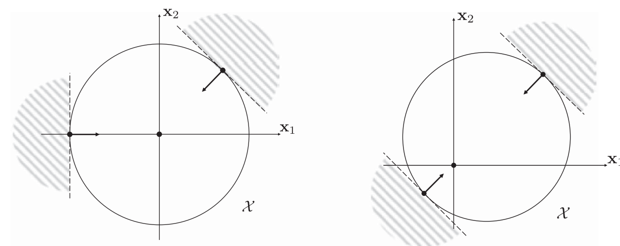
with F as in (6) and $\mathcal{X}$ as in (1). ■

To illustrate the proposed concept of vLGNE, let us present the following examples.

Example 1: Let us consider two agents, with strategies $\mathbf{x}_1 \in \mathbb{R}$ and $\mathbf{x}_2 \in \mathbb{R}$ with the same cost function. Both agents must respect the global constraint $\mathbf{x}_1^2 + \mathbf{x}_2^2 \geq 1$, represented in Fig. 1(a) by the set $\mathcal{X}$ of points outside the unit circle, which is clearly nonconvex. Moreover, we include further constraints $-2 \leq \mathbf{x}_i \leq 2$ ($\forall i = 1, 2$) to ensure the compactness of the set. Thus, we can formally define the game as

$$\forall i \in \{1,2\} : \begin{cases} \min\limits_{-2 \leq \mathbf{x}_i \leq 2} & f_i(\mathbf{x}_i) = \mathbf{x}_i^2 \\ \text{s.t.} & \mathbf{x}_1^2 + \mathbf{x}_2^2 \geq 1. \end{cases} \tag{10}$$

Hence, the cost functions of the agents are decoupled and strictly convex, while the coupling constraint is nonconvex. It is easy to note that the game has an infinite number of LGNEs.



(a) (b)

Fig. 1. Illustration of LGNE points in (a) Example 1 and (b) Example 2.

Moreover, by studying the KKT conditions, we get

$$\forall i \in \{1,2\} : \begin{cases} 2\mathbf{x}_i - \lambda_i 2\mathbf{x}_i = 0 \\ \lambda_i \geq 0 \perp \mathbf{x}_1^2 + \mathbf{x}_2^2 \geq 1. \end{cases} \tag{11}$$

From the aforementioned equation, we have for all points on the unitary circle $\lambda_1 = \lambda_2 = 1$; hence, all these points are also vLGNE. In this example, the analysis of local equilibrium points is still useful, since we can ensure the uniqueness of each vLGNE in their respective linearized feasible directions set (indicated in light gray in Fig. 1). ■

Example 2: Let us now consider a modified version of the game described in Example 1. In particular, we modify the coupling constraint as shown in Fig. 1(b), thus formulating the following game:

$$\forall i \in \{1,2\}\colon \begin{cases} \min\limits_{-2 \leq \mathbf{x}_i \leq 2} & f_i(\mathbf{x}_i) = \mathbf{x}_i^2 \\ \text{s.t.} & (\mathbf{x}_1 - 1/4)^2 + (\mathbf{x}_2 - 1/4)^2 \geq 1. \end{cases} \tag{12}$$

Here, the KKT conditions are

$$\forall i \in \{1,2\}\colon \begin{cases} 2\mathbf{x}_i - \lambda_i(2\mathbf{x}_i - 1/2) = 0 \\ \lambda_i \geq 0 \perp 1 - (\mathbf{x}_1 - 1/4)^2 - (\mathbf{x}_2 - 1/4)^2 \leq 0. \end{cases} \tag{13}$$

From (13), we have that not all points on the circle are LGNE; moreover, only two vLGNE exist that are $\mathbf{x}_1 = \mathbf{x}_2 = 1/4 + \sqrt{2}/2$ and $\mathbf{x}_1 = \mathbf{x}_2 = 1/4 - \sqrt{2}/2$. However, even if in these two vLGNEs the agents are fairly penalized (i.e., same value for the Lagrange multiplier), the corresponding characteristics are different. The vLGNE in quadrant I $(+;+)$ is not stable with respect to the coupling set, since the agents' strategies can jump into other quadrants improving the respective objectives. However, this equilibrium is stable within its linearized feasible directions set. Conversely, the vLGNE in quadrant III $(-;-)$ is not only a vLGNE but also a vGNE. The difference between these two points can be seen also analyzing the respective multipliers. Indeed, we have $\lambda_1 = \lambda_2 = \sqrt{2}/4 + 1$ in the first case (quadrant I) while $\lambda_1 = \lambda_2 = 1 - \sqrt{2}/4$ in the second one (quadrant III). ■

IV. Existence and Uniqueness

Since the projection onto a nonconvex set is not a nonexpansive operator, classical existence and convergence proofs based on projected gradient approaches do not directly apply to our setting.

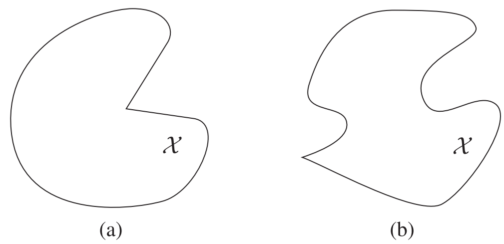


Fig. 2. Example of (a) nonproximally smooth set and of (b) proximally smooth set.

Let us thus focus on a particular class of nonconvex sets, namely *proximally smooth sets* firstly proposed by *Federer* [35]; this concept is present in the literature with different but equivalent definitions, such as $\mathcal{O}(2)$-*convexity* [36], *weak convexity* [37], and *proximal regularity* [38], [39], [40]

Definition 5: A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be proximally smooth if there exists $r > 0$ such that the distance function $\mathrm{dist}(\cdot, \mathcal{X})$ is continuously differentiable on the r-enlargement $U(\mathcal{X}, r) := \mathcal{X} + r\mathbb{B}$. ■

Proximally smooth sets include several classes of nonconvex sets and also convex sets as special case (for any $r > 0$) [38], [40].

Let us recall the following properties of proximally smooth sets.

Lemma 1 (Clarke et al. [40]): Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty closed set. If $\mathcal{X}$ in (1) is r-proximally smooth, then the following properties hold for any $r' \in (0, r)$:

1) $\mathrm{proj}_{\mathcal{X}}(\mathbf{x}) \neq \varnothing$ for all $\mathbf{x} \in U(\mathcal{X}, r')$;
2) $\mathrm{proj}_{\mathcal{X}}(\mathbf{x})$ is a singleton for all $\mathbf{x} \in U(\mathcal{X}, r')$;
3) $\mathrm{proj}_{\mathcal{X}}(\cdot)$ is p-Lipschitz continuous on $U(\mathcal{X}, r')$, where $p = r/(r-r')$;
4) the proximal normal cone $N_{\mathrm{px}}(\mathcal{X}, \cdot)$ is closed as a set-valued mapping;
5) the proximal normal cone and normal cone are equivalent (see Appendix A). ■

Informally speaking, a set is proximally smooth when the local nonconvexities are smooth [41], [42]. For instance, the set in Fig. 2(a) is nonproximally smooth since it is nonsmooth in its nonconvex area. Conversely, in Fig. 2(b), despite the presence of a "sharp point," the set is proximally smooth since it is locally convex around that point.

When a single constraint defines the feasible set, the only requirement for the set to be proximally smooth is the continuous differentiability of the constraint. This property is unfortunately not preserved under intersection without additional conditions [41], [43]. For sets defined by a finite number of inequality and equality constraints, most of the required conditions are fulfilled whenever the associated functions are differentiable and Lipschitz continuous [41]. Proximally smooth sets encompass a wide range of sets, including p-convex sets [44], submanifolds (possibly with boundary), sets that are the images under a diffeomorphism of convex sets, and various other nonconvex sets [38], [45].

Let us ensure the existence of a vLGNE employing the weaker properties of the projection operator recalled in Lemma 1 and let us prove that a vLGNE is equivalent to a fixed point of the forward–backward mapping.

Lemma 2: Let Assumptions 1 and 2 hold and let the set $\mathcal{X}$ in (1) be r-proximally smooth. For any $\gamma \in (0, \frac{r'}{1+F_{U(\mathcal{X},r)}})$ with $r' \in (0, r)$, the following statements are equivalent:

1) $\mathbf{x}^* \in \mathcal{X}$ is a vLGNE;
2) $\mathbf{x}^* = \mathrm{proj}_{\mathcal{X}}(\mathbf{x}^* - \gamma F(\mathbf{x}^*))$, i.e., $\mathbf{x}^* \in \mathrm{fix}(\mathrm{proj}_{\mathcal{X}}(\mathrm{Id} - \gamma F(\cdot)))$. ■

Thanks to Lemma 2, we show the existence of a vLGNE when no assumptions are made on the pseudogradient mapping F in (6).

Proposition 1 (Existence): Let Assumptions 1 and 2 hold and let the set $\mathcal{X}$ in (1) be r-proximally smooth and simply connected. Then, the LGNEP in (2) has at least one vLGNE. ■

An alternative way to ensure the existence of the QVI in (7), and thus, of a vLGNE, is presented in [34] by defining a convex set $\mathcal{T}$ such that for every $\mathbf{x} \in \mathcal{T}$, $\tilde{\mathcal{X}}(\mathbf{x})$ is a nonempty, closed, convex subset of $\mathcal{T}$. Note that in [34], the smoothness property is refereed as *continuity*.

Regarding uniqueness, we cannot ensure that a vLGNE is globally unique, nevertheless, with an additional assumption on the mapping F in in (6), local uniqueness holds in the linearized feasible directions set.

Remark 2 (Local Uniqueness): Under Assumptions 1 and 2, if the mapping F in (6) is strictly monotone, then the strict inequality holds in (7), and thus, any vLGNE $\mathbf{x}^* \in \mathcal{X}$ is unique in the set $\tilde{\mathcal{X}}(\mathbf{x}^*)$. This statement can be derived from [46, Proposition 12.11] as $\tilde{\mathcal{X}}(\mathbf{x}^*)$ is a convex set. ■

V. Equilibrium Computation

Differently from convex sets, that are necessarily connected, requiring a nonconvex set to be simply connected is a rather strong assumption. To relax it, we strengthen assumptions on the pseudogradient mapping. In this section, we propose two alternative algorithms for seeking a vLGNE when the pseudogradient mapping is strongly monotone and merely monotone, respectively.

A. Existence and Convergence to a vLGNE Under Strongly Monotone Pseudogradient Mappings

A popular algorithm for solving VIs with strongly monotone mappings is the projected pseudogradient method that can be formally redefined as a discrete-time system

$$\mathbf{x}^{k+1} = \mathrm{proj}_{\mathcal{X}}(\mathbf{x}^k - \gamma F(\mathbf{x}^k)). \quad (14)$$

This algorithm generates, given a starting point $\mathbf{x}^0 \in \mathcal{X}$ and a step size $\gamma > 0$, a sequence that approaches the solution set. In particular, at each iteration k, a gradient step of length γ is projected onto the feasible set $\mathcal{X}$.

By recalling the definition of quasi-asymptotically stable (QAS) equilibrium point, we can prove the following result.

Definition 6 (QAS equilibrium point): A point $\mathbf{x}^* \in \mathcal{X}$ is a QAS equilibrium point if and only if there exists a basin of attraction $B(\mathbf{x}^*)$ such that for all $\mathbf{x}^0 \in B(\mathbf{x}^*)$, it holds $\lim_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\| = 0$. ■

Lemma 3: Let Assumptions 1 and 2 hold and let $\gamma \in (0, \frac{r'}{1+\bar{F}_{U(\mathcal{X},r)}})$ with $r' \in (0, r)$. If $\mathbf{x}^*$ is a QAS equilibrium point for (14), then $\mathbf{x}^*$ is a fixed point and it is a vLGNE. ∎

Therefore, to prove the convergence to a vLGNE, we can analyze the autonomous evolution of the discrete-time system in (14) and search for possible QAS equilibrium points. By introducing a technical assumption, let us show the following convergence result.

Assumption 3: The set $\mathcal{X}$ in (1) is r-proximally smooth. The mapping F in (6) is strongly monotone with constant $\mu > 0$ and Lipschitz continuous with constant $\ell > 0$. ∎

Theorem 3: Let Assumptions 1–3 hold and let $\gamma \in (0, \min\{\frac{\sqrt{\ell^2+p^2(\mu^2-\ell^2)}+\mu p}{\ell^2 p}, \frac{r'}{1+\bar{F}_{U(\mathcal{X},r)}}\})$, such that $\mu \geq \ell$, $r' \in (0, r)$ and $p = {r}/{(r-r')}$. Then

1) the LGNEP in (2) has at least one vLGNE;
2) the sequence generated by (14), given a starting point $\mathbf{x}^0 \in \mathcal{X}$, converges to a vLGNE. ∎

B. Existence and Convergence to a vLGNE Under Monotone Pseudogradient Mappings

Let us now consider a weaker requirement for the pseudogradient mapping. Inspired by the classical Korpelevich's method [47], let us propose a novel algorithm for computing a vLGNE, defined as follows:

$$\begin{cases} \mathbf{y}^k = \text{proj}_{\mathcal{X}}(\mathbf{x}^k - \gamma F(\mathbf{x}^k)) \\ \mathbf{x}^{k+1} = \text{proj}_{\tilde{\mathcal{X}}(\mathbf{y}^k)}(\mathbf{x}^k - \gamma F(\mathbf{y}^k)). \end{cases} \quad (15)$$

Our method generates from a starting point $\mathbf{x}^0 \in \mathcal{X}$ a sequence approaching the solution set. In particular, at each iteration k, the algorithm requires two consecutive steps. Starting from $\mathbf{x}^k \in \mathcal{X}$, a temporary point $\mathbf{y}^k$ is computed by a gradient step of length γ projected onto the feasible set $\mathcal{X}$. Next, the point for the subsequent iteration $\mathbf{x}^{k+1}$ is computed by taking a gradient step of length γ, with the gradient of the mapping calculated in $\mathbf{y}^k$, and projecting it onto the linearized feasible direction set computed at the same point $\tilde{\mathcal{X}}(\mathbf{y}^k)$.

Introducing a technical assumption, we demonstrate the convergence to a vLGNE.

Assumption 4: The set $\mathcal{X}$ in (1) is r-proximally smooth. The mapping F in (6) is monotone and Lipschitz continuous with constant $\ell > 0$. ∎

Lemma 4: Let Assumptions 1, 2, and 4 hold and let $\gamma \in (0, \frac{r}{1+\bar{F}_{U(\mathcal{X},r)}})$ with $r' \in (0, r)$. If the sequence generated by (15) reaches $\mathbf{y}^k = \mathbf{x}^k$, then $\mathbf{x}^k$ is a QAS equilibrium point for (15). ∎

Lemma 5: Let Assumptions 1, 2, and 4 hold and let $\gamma \in (0, \frac{r}{1+\bar{F}_{U(\mathcal{X},r)}})$ with $r' \in (0, r)$. If $\mathbf{x}^*$ is a QAS equilibrium point for (15), then $\mathbf{x}^*$ is a fixed point and it is a vLGNE. ∎

Theorem 4: Let Assumptions 1, 2, and 4 hold and let $\gamma \in (0, \min\{\frac{1}{\ell}, \frac{r}{4(1+\bar{F}_{U(\mathcal{X},r)})}\})$ with $r' \in (0, r)$. Then

1) the LGNEP in (2) has at least one vLGNE;
2) the sequence generated by (15), given a starting point $\mathbf{x}^0 \in \mathcal{X}$, converges to a vLGNE. ∎

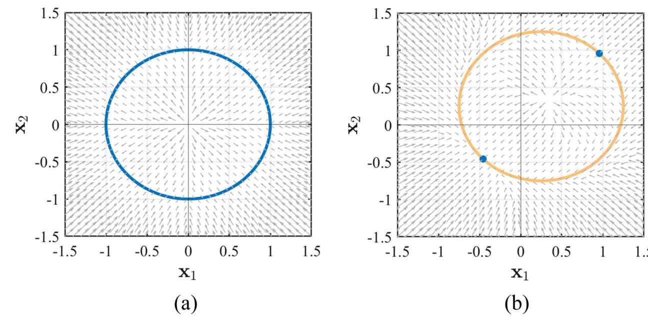


Fig. 3. Pseudophase plane plots for (a) Example 1 and (b) Example 2, where we indicate the vLGNE in blue.

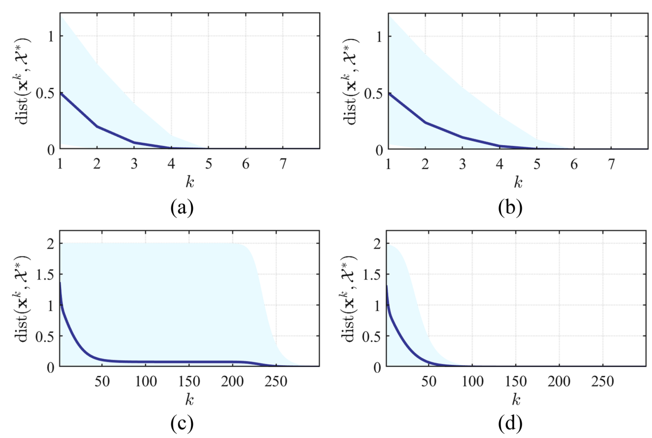


Fig. 4. Distance to a vLGNE with respect to the iteration (line corresponds to the mean, while the shaded area represents the boundaries): (a) Algorithm (14) with Example 1, (b) Algorithm (15) with Example 1, (c) Algorithm (14) with Example 2, and (d) Algorithm (15) with Example 2.

C. Discussion on the Convergence Properties

Algorithms (14) and (15) may yield different local solutions, i.e., vLGNEs, depending on the initial condition. Let us illustrate this fact by analyzing the convergence of the proposed algorithms for Examples 1 and 2. We first note that these two sets are r-proximally smooth with $r < 1/2$, and thus, both algorithms can be employed, for instance, with a step size $\gamma = 0.1$.

In Fig. 3, we show two phase plane plots for Examples 1 and 2. In particular, Fig. 3(a) refers to Example 1, while Fig. 3(b) refers to Example 2. These plots are constructed by exploring the trajectories of the agents' dynamics from different initial conditions. In the first example [see Fig. 3(a)], as all the points of the circumference are vLGNE, the two algorithms converge to a different point based on the initial values. Conversely, the second example has only two vLGNE. Nevertheless, both algorithms converge to the lower vLGNE with almost all initial conditions [see Fig. 3(b)].

Moreover, in Fig. 4, we show the difference between the two algorithms in terms of iterations required to converge to a vLGNE. In order to perform this analysis, we select 10^5 initial conditions equally outdistanced and we analyze the agents' dynamics by showing the distance between $\mathbf{x}^k$ and the set $\mathcal{X}^*$

that collects all the vLGNEs of the games. In particular, we show the mean value of $\mathrm{dist}(\mathbf{x}^k, \mathcal{X}^*)$ in blue together with the upper and lower boundaries in light blue. As expected, the distance with respect to a vLGNE decreases to zero; in some circumstances Algorithm (14) may require several iterations to converge.

Remark 3: Algorithm (14) has a simple update rule, making it easier to implement, but it requires stricter conditions on the pseudogradient mapping. On the other hand, Algorithm (15) is suitable for a broader range of problems. Nevertheless, it involves a projection onto the linearized feasible directions set at each update, which has a higher computational cost. ■

VI. Application: Noncooperative OPF in DC Microgrids

DC microgrids have gained influence in modern electrical systems due to their high efficiency and natural interface to many types of renewable energy resources (RESs) and energy storage systems (ESSs).

One of the critical issues in DC microgrids—inherited from the operation of power systems—is the well-known OPF problem. The standard method to solve the OPF problem involves formulating an optimization problem to determine an optimal operating point regarding power losses or energy production costs while satisfying system constraints, considering variables such as power generation, voltage levels, and maximum line flows. The OPF relies on the power flow model, which in DC microgrids corresponds to a set of nonlinear nonconvex algebraic equations that cannot be solved analytically [48]. Various approaches exist to solve this problem, including mathematical programming techniques [49] and heuristic algorithms [50], possibly considering the nonconvexity of power flow constraints [51], [52].

When the OPF is addressed by a centralized approach, a central unit must have access to all system parameters in order to reach the optimal working condition for the entire grid. However, several independent parties are progressively involved in controlling and optimizing power grids, strongly affecting their dynamics. The cooperation of such independent entities, which is necessary for a centralized approach, is clearly challenging: on the one hand, they act selfishly; on the other hand, since they are physically interconnected through power lines, they must cooperate to ensure safe and secure grid operations. In this context, several game-theoretic methods have been proposed for distributed generation and storage control in power grids. Nevertheless, most of these works focus on the economic dispatch only [12], [53], [54], [55], yet disregard physical constraints also due to their nonconvexity.

Therefore, here we apply the proposed novel noncooperative approach for managing DC microgrids, including the full power flow equations leading to a noncooperative game with shared nonconvex constraints.

A. OPF as a Noncooperative Game

Let us consider a DC microgrid model composed of several interconnected buses and connected to the ac distribution grid through the so-called slack bus. We suppose that the grid is controlled over a control horizon denoted as $\mathcal{H} := \{1, ..., h, ..., H\}$, with H discrete time slots with equal length Δh, where h is a generic time slot.

The microgrid is described by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes with cardinality N, while $\mathcal{E} \in \mathcal{N} \times \mathcal{N}$ is the set of pairs of distinct nodes called edges with cardinality E. The nodes $i \in \mathcal{N}$ of the graph represent the buses and the edges $(i, r) \in \mathcal{E}$ represent the lines between these buses. Each bus $i \in \mathcal{N}$ is connected with several noncontrollable loads and RESs whose aggregated per-slot power profile over the control horizon $\mathcal{H}$ is $\mathbf{P}_i := (P_i(1), ..., P_i(h), ..., P_i(H))^\top \in \mathbb{R}^H$.

We assume that the variables related to each bus are controlled by an active user $i \in \mathcal{N}$ that can modify the energy scheduling profile of the bus aiming at decreasing its total cost, while providing flexibility to the overall microgrid. In particular, each user schedules over the control horizon both the voltage magnitude of the bus $\mathbf{V}_i := (V_i(1), ..., V_i(h), ..., V_i(H))^\top \in \mathbb{R}^H_{\geq 0}$ and the power injected in the bus $\mathbf{e}_i := (e_i(1), ..., e_i(h), ..., e_i(H))^\top \in \mathbb{R}^H_{\geq 0}$. Hence, let us define for each active agent $i \in \mathcal{N}$ its strategy by $\mathbf{x}_i = \mathrm{col}(\mathbf{V}_i, \mathbf{e}_i) \in \mathbb{R}^{2H}_{\geq 0}$.

We further assume that user $i = 1$, i.e., the slack bus, can only buy energy from the main grid with a linear function $W_1(\mathbf{e}_1) = \Delta\mathrm{h}\, \eta_1 \mathbf{1}_H^\top \mathbf{e}_1$, where η_1 is the energy price coefficient. Conversely, we assume that users $i \in \mathcal{N} \setminus \{1\}$ are equipped with dispatchable energy generation devices and are subject to variable production costs in accordance with a linear cost function $W_i(\mathbf{e}_i) = \Delta\mathrm{h}\, \eta_i \mathbf{1}_H^\top \mathbf{e}_i$, where η_i is the generation cost of the user i. As commonly done in power distribution grids, we include a constraint for the slack bus whose voltage magnitude must be fixed

$$\mathbf{V}_1 = V_{\text{ref}} \mathbf{1}_H \tag{16}$$

where V_{ref} is the reference voltage magnitude. Moreover, we impose a minimum and a maximum voltage magnitude for the remaining buses

$$V_i^{\min} \mathbf{1}_H \leq \mathbf{V}_i \leq V_i^{\max} \mathbf{1}_H \quad \forall i \in \mathcal{N} \setminus \{1\}. \tag{17}$$

Due to technological limitations, the generation profile is bounded by a minimum and a maximum per-slot energy generation capacity

$$e_i^{\min} \mathbf{1}_H \leq \mathbf{e}_i \leq e_i^{\max} \mathbf{1}_H \quad \forall i \in \mathcal{N} \setminus \{1\}. \tag{18}$$

In addition, the power flow equations must be satisfied; disregarding the power losses, these equations can be formulated for DC grids as

$$\sum_{r \in \mathcal{N}} V_i(h)(V_i(h) - V_r(h)) Y_{i,r} = P_i(h) + e_i(h)\ \forall i \in \mathcal{N},\ \forall h \in \mathcal{H} \tag{19}$$

where $Y_{i,r}$ is the element (i, r) of the conductance matrix, defined as

$$Y_{i,r} = \begin{cases} \sum_{h \neq r} Y_{n,h} & \text{if } i = r \\ -Y_{i,r} & \text{if } i \neq r \end{cases} \quad \forall (i, r) \in \mathcal{E} \tag{20}$$

and $Y_{i,r} = Y_{r,i}$ is the line conductance between buses i and r. Summing up, all users are subject to a global nonconvex feasible set defined as

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^{2HN} \,|\, (16) - (19) \text{ hold} \right\}. \tag{21}$$

As regards cost functions, we assume that the control objective of the noncooperative game is to increase the microgrid predictability through a liberalized market where the cost of energy is proportional to the microgrid's energy mismatch. Hence, the cost function of each independent active user comprehends the cost of energy, which is proportional to the microgrid's power mismatch, and generation costs as follows:

$$f_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{h \in \mathcal{H}\Delta \mathrm{h}} \left(\kappa_i \sum_{j \in \mathcal{N} \setminus \{1\}} \left(\mathbf{e}_j(h) - \mathbf{P}_j(h)\right) \mathbf{e}_i(h) + \eta_i \mathbf{e}_i(h) \right) \forall i \in \mathcal{N} \tag{22}$$

where $\kappa_i > 0$ are the pricing coefficients. Note that for the slack bus we have $\kappa_1 = 0$, since we assumed that in the slack bus, we can only inject power from the main grid.

The proposed microgrid model comprehends physical constraints, and thus, can be seen as a noncooperative version of the OPF problem to determine the best operating point with respect to power losses and energy production costs. The aforementioned game, defined by the cost functions f_i in (22) and by the shared feasible set $\mathcal{X}$ in (21), straightforwardly verifies Assumptions 1 and 2, falling exactly in our proposed game-theoretic framework. Hence, if we can prove that the set (21) is proximally smooth we can compute a vLGNE employing (14) or (15).

Proposition 2: The (nonconvex) set defined in (21) is proximally smooth. ∎

We emphasize that our goal is the noncooperative control of multiple agents in DC microgrid, striving to reach a local equilibrium within a game. This involves solving N interdependent optimization problems with different cost functions, as denoted by (22). Notably, this approach differs from the classical OPF problem, which typically considers a single cost function for the overall system.

B. Numerical Experiments

In this section, we show the performance of (14) and (15) on the noncooperative power flow. Note that in this section all the quantities are expressed in the per-unit system p.u. For the case study, we employ a test microgrid depicted in Fig. 5. The microgrid comprises ten buses, which are connected through nine branches in a radial topological distribution, as commonly occurs in low-voltage distribution networks.

We consider a control horizon including $H = 24$ time slots of $\Delta \mathrm{h} = 1$ h and we assume for all buses, except the slack bus $i = 1$, a net power profile randomly generated. The slack bus can buy energy from the main grid and inject it in the microgrid with a price coefficient $\eta_1 = 1$ financial units. In the remaining nine buses, the distribution grid has diesel generators whose

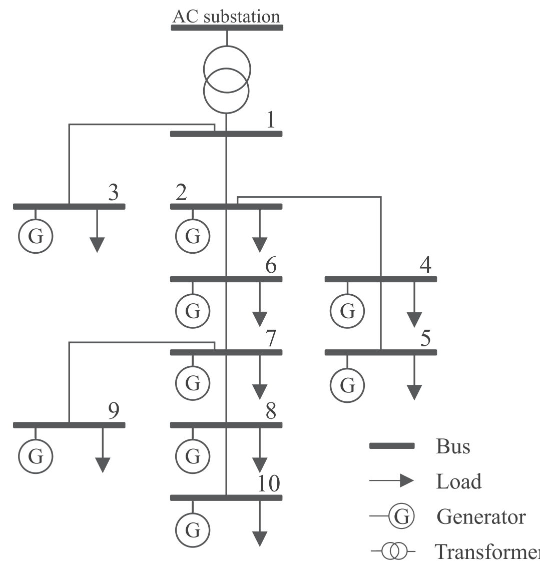


Fig. 5. Scheme of the simulated DC microgrid.

TABLE I
PARAMETERS OF THE DIFFERENT GENERATORS

i	$e_i^{\min}$ (p.u.)	$e_i^{\max}$ (p.u.)	η_i (financial units)
2	0	3.50	0.35
3	0	1.30	0.60
4	0	2.00	0.55
5	0	2.00	0.50
6	0	2.00	0.70
7	0	2.00	0.35
8	0	2.00	0.70
9	0	1.00	0.41
10	0	4.50	0.45

parameters are indicated in Table I. As for the voltage magnitude, we set $V_{\mathrm{ref}} = 1$ p.u., $V_i^{\min} = 0$ p.u., and $V_i^{\max} = 2$ p.u..

The step coefficient for (14) and (15) is $\gamma = 0.01$. All software simulations are conducted in the MATLAB 2020a environment on a laptop with a 1.3-GHz Intel Corei7 CPU with 8-GB RAM memory. Specifically, we have employed a standard interior-point algorithm, which is suitable for our problem, thanks to Assumptions 1 and 2.

In the rest of this section, we first evaluate the algorithms' performance by analyzing the impact of the initial state variation on the output of the algorithms. Consequently, the results of the noncooperative approach are compared with those obtained by a standard version of the OPF.

1) Impact of the Initial State: The solution of the resulting problem depends on the agents' initial state; hence, we can compute, as for the standard OPF, the regions of convergence for the proposed algorithms, that in our case correspond to the basin of attraction of the different vLGNEs. To this aim, we perform a set of simulations changing the initial voltage magnitude V_i^0 each time and the power injection in each bus e_i^0, sampling them over the interval $[0, 2]$. However, to show the convergence results in a 2-D plot, we set an equal voltage magnitude and power injection for all buses in each simulation. Note that setting different initial conditions for players may

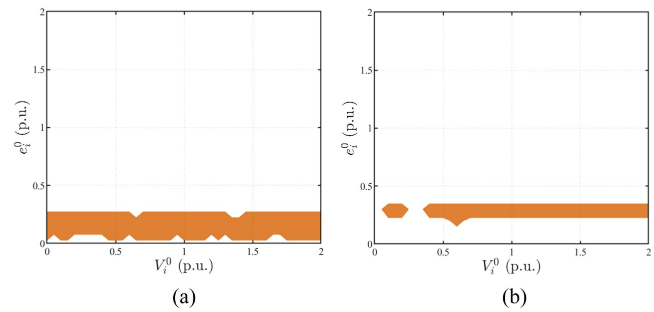


Fig. 6. Convergence regions with respect to the initial voltage magnitude and power injection (both equal for all buses): (a) Algorithm (14) and (b) Algorithm (15). The white and the orange regions correspond to the two equilibria of the noncooperative game.

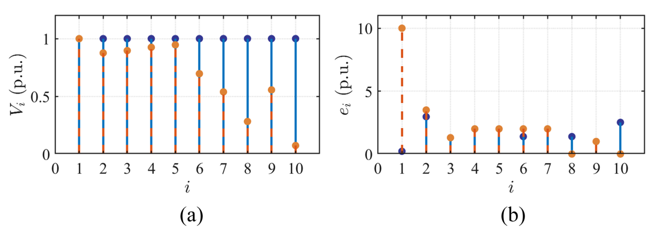


Fig. 7. Results of (14) and (15) achieved by different initial conditions: (a) voltage magnitudes and (b) power injections. The blue and the orange stems represent the two equilibria of the noncooperative game, corresponding to the white and orange area of Fig. 6, respectively.

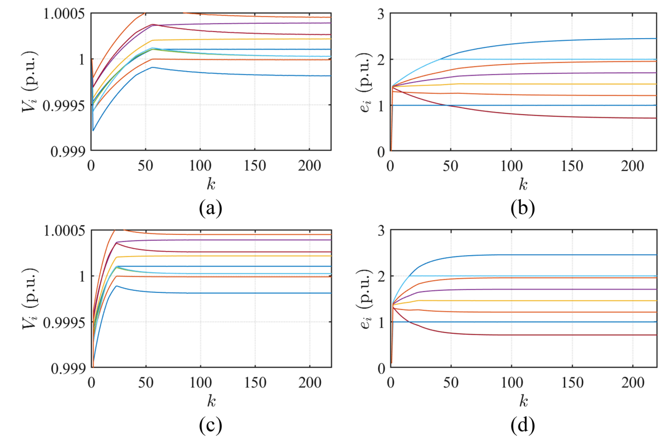


Fig. 8. Convergence of Algorithms (14) and (15) with the flat-start approach: (a) voltage magnitudes with Algorithm (14), (b) power injections with Algorithm (14), (c) voltage magnitudes with Algorithm (15), and (d) power injections with Algorithm (15).

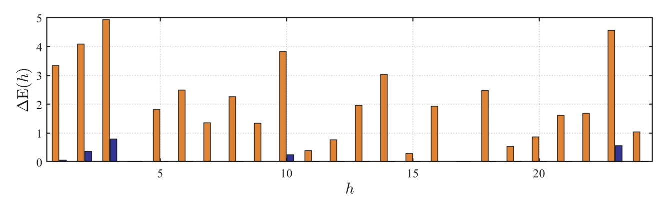


Fig. 9. Power mismatch $\Delta\mathrm{E}(h)$ of the microgrid over the control horizon achieved by the standard OPF (orange bars) and the noncooperative approach (blue bars).

lead to additional equilibria. The results of these simulations are presented in Fig. 6, where we plot the region of convergence with different initial states. As it can be seen, there are two different regions of convergence corresponding to the same number of different vLGNEs, i.e., the white and orange areas. We point out that, interestingly, the equilibria reached with both algorithms are identical.

Let us now analyze in detail these two equilibria reported in Fig. 7(a) and (b), where we plot for each bus $i \in \mathcal{N}$ the voltage magnitude and the power injection, respectively.

Note that the white region of convergence corresponds to the solution with high voltage magnitudes in all buses and balanced power injections for all generators, while the orange region converges to the low voltage solution. The latter solution corresponds to the case where there is a high power injection from the slack bus, and consequently, in order to make feasible this power transfer, a low-voltage magnitude in the remote buses of the microgrid. As for the standard OPF, this low-voltage solution should be avoided, since it is less interesting in terms of power quality requirements. Nevertheless, as for the classical OPF, we can employ the so-called flat start approach, setting all voltage magnitudes equal to 1 p.u. and all power injections equal to zero [56]. In Fig. 8(a) and (b), we show, respectively, the convergence of Algorithm (14) with respect to the voltage magnitudes and the power injections of each bus when this standard approach is used. Furthermore, in Fig. 8(a) and (b), we show the same results when Algorithm (15) is used. From the figure, it is clear that this latter approach requires less iteration to converge.

2) Comparison With the Standard OPF: Having computed the region of convergence for the proposed test microgrid, let us now compare the proposed approach with the standard OPF with the same control goal, i.e., reducing the power gap at the slack bus, and thus, increasing predictability. The standard OPF presents several disadvantages inherited by the centralized architecture that leads to the minimization of the overall microgrid cost, possibly penalizing some users. In fact, to overcome these drawbacks, in our noncooperative framework, several coupled optimization problems are simultaneously solved by all users.

From the results illustrated in Fig. 9, we see that the noncooperative approach is able to reduce the power mismatch $\Delta\mathrm{E}(h) = \sum_{j\in\mathcal{N}\setminus\{1\}}(-\mathbf{P}_j(h) + \mathbf{e}_j(h))$ of the microgrid under 1 p.u. However, as regards the total energy costs of the microgrid, the standard OPF has a total cost of 2872 financial units, while the noncooperative approach requires 3 342 financial units. Therefore, the total cost of the noncooperative approach is 15% higher than the centralized one. In the centralized scheme, the goal is to minimize the overall gap by ignoring any possible inequalities between different users; on the other hand, in the noncooperative case, each user aims at optimizing its objective thus leading to a more fair yet higher cost summation.

VII. Conclusion

In this work, we have addressed the problem of solving a class of games with nonconvex coupling constraints. We have defined a novel local equilibrium concept presenting conditions for optimality, existence and local uniqueness. Our technical contributions allow us, under certain hypotheses, to define two iterative schemes with global convergence guarantee toward this novel equilibrium concept.

This work can be extended in several directions, namely, designing fully distributed algorithms for local equilibrium seeking and exploring the Lyapunov theory and potential functions to enhance the characterization of quasi-asymptotic stability in large-scale networks. An additional interesting research line is the alignment of the agents' objective functions toward a social optimum to reconcile their selfish behavior with social optimality.

Appendix A
Basic Concepts on Cones

In the related literature, several different definitions of cones have been proposed, hence, for the sake of clarity, let us here recall some concepts following the convention used in [29] and [57].

First, for a given feasible set composed of a set of equality and inequality constraints:

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} g_m(\mathbf{x}) = 0,\ m \in \mathcal{E} \\ g_m(\mathbf{x}) \leq 0,\ m \in \mathcal{I} \end{array} \right. \tag{23}$$

we define the active set $A(\mathbf{x})$ as the set comprehending all indices $m \in \mathcal{E} \cup \mathcal{I}$ with $g_m(\mathbf{x}) = 0$.

Definition 7 (Linearized feasible directions set (linearized cone)): We define the set of *linearized feasible directions set* $\mathcal{F}(\mathcal{X}, \mathbf{x})$ as the set comprehending all vectors $\mathbf{y} \in \mathbb{R}^n$ such that

$$\begin{gathered} \nabla g_m(\mathbf{x})^\top \mathbf{y} = 0 \,\forall\, m \in \mathcal{E} \\ \nabla g_m(\mathbf{x})^\top \mathbf{y} \leq 0 \,\forall\, m \in \mathcal{I} \cap A(\mathbf{x}). \end{gathered} \tag{24}$$

■

It is easy to see that this is a closed convex nonempty cone and a linear approximation of the feasible set at a generic point $\mathbf{x}$. Moreover, it can be shown that the tangent cone is a subset of the linearized feasible directions set $T(\mathcal{X}, \mathbf{x}) \subset \mathcal{F}(\mathcal{X}, \mathbf{x})$, however, when a quasiregularity constraint qualification holds (e.g., Mangasarian–Fromovitz) these two cones are equal, i.e., $T(\mathcal{X}, \mathbf{x}) = \mathcal{F}(\mathcal{X}, \mathbf{x})$ [58].

Finally, let us define *normal cone* and the *proximal normal cone* as follows.

Definition 8 (Normal cone): Let us consider a nonempty subset $\mathcal{X} \subseteq \mathbb{R}^n$. The *normal cone* of $\mathcal{X}$ in a generic point $\mathbf{x} \in \mathrm{cl}(\mathcal{X})$ is defined as the polar cone of tangent cone to $\mathcal{X}$ in $\mathbf{x}$, that is,

$$\begin{aligned} N(\mathcal{X}, \mathbf{x}) &:= T(\mathcal{X}, \mathbf{x})^\circ = \\ &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^\top \mathbf{d} \leq 0 \,\forall \mathbf{d} \in T(\mathcal{X}, \mathbf{x})\}. \end{aligned} \tag{25}$$

■

Definition 9 (Proximal normal cone): Let us consider a nonempty subset $\mathcal{X} \subseteq \mathbb{R}^n$. The *proximal normal cone* of $\mathcal{X}$ in a generic point $\mathbf{x} \in \mathrm{cl}(\mathcal{X})$ is given by

$$N_{\mathrm{px}}(\mathcal{X}, \mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{x} \in \mathrm{proj}_{\mathcal{X}}(\mathbf{x} + \alpha \mathbf{y})\} \tag{26}$$

where $\alpha > 0$ is a constant. ■

Note that $N(\mathcal{X}, \mathbf{x})$ is always a closed and convex cone while $N_{\mathrm{px}}(\mathcal{X}, \mathbf{x})$ is convex but may not be closed. In general $N_{\mathrm{px}}(\mathcal{X}, \mathbf{x}) \subset N(\mathcal{X}, \mathbf{x})$, however for proximally smooth sets $N_{\mathrm{px}}(\mathcal{X}, \mathbf{x})$ is closed and $N_{\mathrm{px}}(\mathcal{X}, \mathbf{x}) = N(\mathcal{X}, \mathbf{x})$ [45].

Appendix B
Proofs

Proof of Theorem 1: To prove 1) ⇔ 2), we recall that the linearized feasible direction set comprehends only a subset of the original set; thus, we can write the following condition on the cost function of each agent:

$$\begin{aligned} f_i(\mathbf{y}, \mathbf{x}_{-i}^*) &\geq f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) + \\ &\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)^\top (\mathbf{y} - \mathbf{x}_i^*) \;\forall \mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*). \end{aligned} \tag{27}$$

Moreover, since $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$ and due to the definition of normal cone, we have

$$\inf_{\mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*)} \nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)^\top (\mathbf{y} - \mathbf{x}_i^*) \geq 0. \tag{28}$$

Then, it follows that $f_i(\mathbf{y}, \mathbf{x}_{-i}^*) \geq f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$, $\forall \mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*)$. Hence, if this holds for all agents $i \in \mathcal{N}$, then $\mathbf{x}^*$ is an LGNE.

To prove 1) ⇔ 3), we recall that or each active constraint $\nabla_{\mathbf{x}_i} g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)^\top \mathbf{y} \leq 0 \,\forall \mathbf{y} \in T(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$. Furthermore, for a generic vector $\mathbf{v}$, under Assumptions 1 and 2, we have that $\mathbf{v} \in N(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$, if and only if there exists a vector $\boldsymbol{\lambda}_i = \mathrm{col}(\lambda_i^1, \ldots, \lambda_i^m, \ldots, \lambda_i^M) \in \mathbb{R}^M$ such that

$$\begin{gathered} \mathbf{v} \in \nabla_{\mathbf{x}_i} g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \boldsymbol{\lambda}_i \\ \mathbf{0}_M \leq \boldsymbol{\lambda}_i \;\perp\; g(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \mathbf{0}_M. \end{gathered} \tag{29}$$

Thus, for each agent $i \in \mathcal{N}$, we can write the KKT conditions in (5) by setting $\mathbf{v} = -\nabla_{\mathbf{x}_i} f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$. Finally, since $\nabla_{\mathbf{x}_i} f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)^\top \mathbf{y} \geq 0$, $\forall \mathbf{y} \in T(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$, we ensure the existence of a nonnegative vector of Lagrange multipliers $\boldsymbol{\lambda}_i = \mathrm{col}(\lambda_i^1, \ldots, \lambda_i^m, \ldots, \lambda_i^M) \in \mathbb{R}^M_{\geq 0}$ satisfying the KKT optimality conditions for each individual optimization problem in (2) if $\mathbf{x}^*$ is an LGNE. ■

Proof of Theorem 2: Under Assumptions 1 and 2, we note that the set $\tilde{\mathcal{X}}(\mathbf{x}^*)$ where we compute the KKT conditions in (5) for the LGNEP is convex. Thus, we can employ [14, Th. 3.1] to conclude the proof. ■

Proof of Lemma 2: Since Assumption 2 holds, $U(\mathcal{X}, r)$ is compact, and thus, $\bar{F}_{U(\mathcal{X},r)}$ is limited. If $\mathbf{x}^* \in \mathcal{X}$, then we have that

$$\begin{aligned} \mathrm{dist}(\mathbf{x}^* - \gamma F(\mathbf{x}^*), \mathcal{X}) &= \min_{\mathbf{y} \in \mathcal{X}} \|\mathbf{x}^* - \gamma F(\mathbf{x}^*) - \mathbf{y}\| \\ &\leq \|\mathbf{x}^* - \gamma F(\mathbf{x}^*) - \mathbf{x}^*\| = \gamma \|F(\mathbf{x}^*)\| \\ &\leq \gamma \bar{F}_{U(\mathcal{X},r)} < \frac{r'}{1 + \bar{F}_{U(\mathcal{X},r)}} \bar{F}_{U(\mathcal{X},r)} < r' \end{aligned} \tag{30}$$

hence, $\mathbf{x}^* - \gamma F(\mathbf{x}^*) \in U(\mathcal{X}, r')$. Due to Lemma 1, $\text{proj}_{\mathcal{X}}(\mathbf{x})$ is a singleton and $N_{\text{px}}(\mathcal{X}, \mathbf{x}^*) = N(\mathcal{X}, \mathbf{x}^*)$, therefore, we can recall Corollary 1 to conclude the proof. ∎

Proof of Proposition 1: From Lemma 2, we have that any fixed point of the forward–backward mapping with $\gamma \in (0, \frac{r'}{1+\bar{F}_{U(\mathcal{X},r)}})$ is a vLGNE. Since Assumptions 1 and 2 hold and since the set is simply connected, we can employ the Lefschetz fixed-point theorem to prove that there exists at least one fixed point [59], [60]. ∎

Proof of Lemma 3: By definition, a QAS equilibrium is a fixed point of the dynamical system's evolution of (14) [61], while by Lemma 2, any fixed point of (14) is a vLGNE. ∎

Proof of Theorem 3: In order to preserve the properties of Lemma 1, let us assume that $\mathbf{x}^0 \in \mathcal{X}$. Consequently, since at each iteration $\mathbf{x}^k \in \mathcal{X}$ and due to Lemma 2, we have that $\mathbf{x}^k - \gamma F(\mathbf{x}^k) \in U(\mathcal{X}, r')$, and thus, Lemma 1 holds. Therefore, we have that

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^*\right\|^2 = \\
&= \left\|\text{proj}_{\mathcal{X}}(\mathbf{x} - \gamma F(\mathbf{x})) - \text{proj}_{\mathcal{X}}(\mathbf{x}^* - \gamma F(\mathbf{x}^*))\right\|^2 \\
&\leq p^2 \left\|(\mathbf{x}^k - \gamma F(\mathbf{x}^k)) - (\mathbf{x}^* - \gamma F(\mathbf{x}^*))\right\|^2 \\
&= p^2 \left\|(\mathbf{x}^k - \mathbf{x}^*) - \gamma(F(\mathbf{x}^k) - F(\mathbf{x}^*))\right\|^2 \\
&= p^2 (\left\|\mathbf{x}^k - \mathbf{x}^*\right\|^2 + \gamma^2 \left\|F(\mathbf{x}^k) - F(\mathbf{x}^*)\right\|^2 \\
&\qquad - 2\gamma (F(\mathbf{x}^k) - F(\mathbf{x}^*))^\top (\mathbf{x}^k - \mathbf{x}^*)) \\
&\leq p^2 (1 - 2\gamma\mu + \gamma^2 \ell^2) \left\|\mathbf{x}^k - \mathbf{x}^*\right\|^2
\end{aligned} \tag{31}$$

where the first inequality holds since for a generic (possibly nonconvex) proximally smooth set the projection is Lipschitz continuous with constant $p = r/_{r-r'}$, while for the last one, we use the strong monotonicity and the Lipschitz continuity of the mapping F. Next we note that, if $\gamma < \frac{\sqrt{\ell^2 + p^2(\mu^2 - \ell^2)} + \mu p}{\ell^2 p}$ with $\mu \geq \ell$, we have that $p^2(1 - 2\gamma\mu + \gamma^2\ell^2) < 1$. Consequently, $\lim_{k\to\infty} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 = 0$, and thus, the evolution of the discrete-time autonomous system (14) converges to a QAS equilibrium point that exists [61]. To conclude the proof we recall Lemma 3. ∎

Proof of Lemma 4: The proof follows directly from Lemma 3. ∎

Proof of Lemma 5: The proof follows directly from Lemma 3. ∎

Proof of Theorem 4: In order to preserve the smoothness properties of Lemma 1, we need to ensure that each iteration is contained in $U(\mathcal{X}, r')$. First, by setting $\gamma \in (0, \frac{r'}{4(1+\bar{F}_{U(\mathcal{X},r)})})$, we note that

$$\begin{aligned}
\text{dist}(\mathbf{y}^k, \mathbf{x}^{k+1}) &\leq \left\|\mathbf{x}^k - \gamma F(\mathbf{x}^k) - \mathbf{x}^k + \gamma F(\mathbf{y}^k)\right\| \\
&\leq \gamma \left\|F(\mathbf{x}^k)\right\| + \gamma \left\|F(\mathbf{y}^k)\right\| \leq 2\gamma \bar{F}_{U(\mathcal{X},r)} \\
&< \frac{2r'}{4(1 + \bar{F}_{U(\mathcal{X},r)})} \bar{F}_{U(\mathcal{X},r)} < \frac{r'}{2}
\end{aligned} \tag{32}$$

where the first inequality holds since the linearized feasible direction set is convex. Hence, assuming that $\mathbf{x}^0 \in \mathcal{X}$ and since $\mathbf{y}^k \in \mathcal{X}$, we ensure that each point $\mathbf{x}^k$ is contained in $U(\mathcal{X}, r'/2)$. Similarly, we have that $\text{dist}(\mathbf{x}^k, \mathbf{x}^k - \gamma F(\mathbf{x}^k)) \leq \gamma \bar{F}_{U(\mathcal{X},r)} \leq r'/2$, and thus, by induction, $\mathbf{x}^k - \gamma F(\mathbf{x}^k)$ is contained in $U(\mathcal{X}, r')$. The same argument applies to $\mathbf{x}^k - \gamma F(\mathbf{y}^k)$.

Next, by defining $\mathbf{z}^k = \mathbf{x}^k - \gamma F(\mathbf{y}^k)$ and $\mathbf{x}^{*k} = \text{proj}_{\tilde{\mathcal{X}}(\mathbf{y}^k)}(\mathbf{x}^*)$, we have that

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*k}\right\|^2 = (\mathbf{x}^{k+1} - \mathbf{z}^k + \mathbf{z}^k - \mathbf{x}^{*k})^\top (\mathbf{x}^{k+1} - \mathbf{z}^k + \mathbf{z}^k - \mathbf{x}^{*k}) \\
&= \left\|\mathbf{z}^k - \mathbf{x}^{*k}\right\|^2 + \left\|\mathbf{z}^k - \mathbf{x}^{k+1}\right\|^2 + 2(\mathbf{x}^{k+1} - \mathbf{z}^k)^\top (\mathbf{z}^k - \mathbf{x}^{*k}).
\end{aligned} \tag{33}$$

Moreover, since $\tilde{\mathcal{X}}(\mathbf{y}^k)$ is convex and $\mathbf{x}^{k+1} = \text{proj}_{\tilde{\mathcal{X}}(\mathbf{y}^k)}(\mathbf{z}^k)$, we have that $2\|\mathbf{z}^k - \mathbf{x}^{k+1}\|^2 + 2(\mathbf{x}^{k+1} - \mathbf{z}^k)^\top (\mathbf{z}^k - \mathbf{x}^{*k}) = 2(\mathbf{z}^k - \mathbf{x}^{k+1})^\top (\mathbf{x}^{*k} - \mathbf{x}^{k+1}) \leq 0$. Hence, we rewrite (33) as

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*k}\right\|^2 \leq \left\|\mathbf{z}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{z}^k - \mathbf{x}^{k+1}\right\|^2 \\
&\leq \left\|\mathbf{x}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^k - \mathbf{x}^{k+1}\right\|^2 + 2\gamma(\mathbf{y}^k - \mathbf{x}^{k+1})^\top F(\mathbf{y}^k)
\end{aligned} \tag{34}$$

where for the last inequality we employ the monotonicity of F. Since $\tilde{\mathcal{X}}(\mathbf{y}^k)$ is convex and $\mathbf{x}^{k+1} \in \tilde{\mathcal{X}}(\mathbf{y}^k)$, we have that $(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top ((\mathbf{x}^k - \gamma F(\mathbf{x}^k)) - \mathbf{y}^k) \leq 0$, and therefore, $(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (\mathbf{x}^k - \gamma F(\mathbf{y}^k) - \mathbf{y}^k) = (\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (\mathbf{x}^k - \gamma F(\mathbf{x}^k) - \mathbf{y}^k) + \gamma(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (F(\mathbf{x}^k) - F(\mathbf{y}^k)) \leq \gamma(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (F(\mathbf{x}^k) - F(\mathbf{y}^k))$. Thus, we can rewrite (34) as

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*k}\right\|^2 \leq \left\|\mathbf{x}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 \\
&\quad - \left\|\mathbf{y}^k - \mathbf{x}^{k+1}\right\|^2 + 2\gamma(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (F(\mathbf{x}^k) - F(\mathbf{y}^k)).
\end{aligned} \tag{35}$$

Furthermore, by employing the Cauchy–Schwarz inequality and the Lipschitz property of F, we have that $2\gamma(\mathbf{x}^{k+1} - \mathbf{y}^k)^\top (F(\mathbf{x}^k) - F(\mathbf{y}^k)) \leq 2\gamma\ell \|\mathbf{x}^{k+1} - \mathbf{y}^k\| \|\mathbf{x}^k - \mathbf{y}^k\|$, and thus,

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*k}\right\|^2 \leq \left\|\mathbf{x}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 - \left\|\mathbf{y}^k - \mathbf{x}^{k+1}\right\|^2 \\
&+ 2\gamma\ell \left\|\mathbf{x}^{k+1} - \mathbf{y}^k\right\| \left\|\mathbf{x}^k - \mathbf{y}^k\right\| \leq \left\|\mathbf{x}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 \\
&\quad - \left\|\mathbf{y}^k - \mathbf{x}^{k+1}\right\|^2 + \gamma^2\ell^2 \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 + \left\|\mathbf{y}^k - \mathbf{x}^{k+1}\right\|^2
\end{aligned} \tag{36}$$

where in the last inequality, we use that $(\gamma\ell\|\mathbf{x}^k - \mathbf{y}^k\| - \|\mathbf{y}^k - \mathbf{x}^{k+1}\|)^2 \geq 0$. Therefore, we have that

$$\begin{aligned}
(1 - \gamma^2\ell^2) \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 &\leq \left\|\mathbf{x}^k - \mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^{k+1} - \mathbf{x}^{*k}\right\|^2 \\
&\leq \left\|\mathbf{x}^k\right\|^2 + \left\|\mathbf{x}^{*k}\right\|^2 - \left\|\mathbf{x}^{k+1}\right\|^2 - \left\|\mathbf{x}^{*k}\right\|^2.
\end{aligned} \tag{37}$$

Since the sequence $(\mathbf{x}^k)_{k=0}^\infty$ is bounded, by summing up for all integers $K \geq 0$, we get

$$(1 - \gamma^2\ell^2) \sum_{k=0}^{K} \left\|\mathbf{x}^k - \mathbf{y}^k\right\|^2 \leq \left\|\mathbf{x}^0\right\|^2 \tag{38}$$

where the sequence $(\sum_{k=0}^K \|\mathbf{x}^k - \mathbf{y}^k\|^2)_{K\in\mathbb{N}}$ is monotonically increasing and bounded. Therefore, if $\gamma < 1/\ell$, we have that $\lim_{k\to\infty} \|\mathbf{x}^k - \mathbf{y}^k\|^2 = 0$, and by Lemma 4, we have that the evolution of the discrete-time autonomous system (15) converges to a QAS equilibrium (existence follows by [61]). To conclude the proof, we invoke Lemma 5. ■

Proof of Proposition 2: The set in (21) is defined by a set of finitely many equalities and inequalities $m \in \mathcal{M}$ that are continuously differentiable. Hence, when considered individually, they are proximally smooth with constant r_m [41]. It is easy to verify that the intersection of these different constraints is metrically calm with constant $\zeta > 1$ as in [43]. Therefore, by defining $\bar{r}_m = \min_{m \in \mathcal{M}} r_m$, we can verify the proximal smoothness of (21) with constant $r = \bar{r}_m/\zeta M$ (see [43, Proposition 7.4]). ■

References

[1] J. F. Nash et al., "Equilibrium points in N-person games," *Proc. Nat. Acad. Sci. USA*, vol. 36, no. 1, pp. 48–49, 1950.

[2] Q. Lu, S. Lü, and Y. Leng, "A Nash-Stackelberg game approach in regional energy market considering users' integrated demand response," *Energy*, vol. 175, pp. 456–470, 2019.

[3] S. Jošilo and G. Dán, "Selfish decentralized computation offloading for mobile cloud computing in dense wireless networks," *IEEE Trans. Mobile Comput.*, vol. 18, no. 1, pp. 207–220, Jan. 2019.

[4] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Berlin, Germany: Springer, 2007.

[5] L Pavel, "Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach," *IEEE Trans. Autom. Control*, vol. 65, no. 4, pp. 1584–1597, Apr. 2020.

[6] S. Li, W. Zhang, J. Lian, and K. Kalsi, "Market-based coordination of thermostatically controlled loads—Part I: A mechanism design formulation," *IEEE Trans. Power Syst.*, vol. 31, no. 2, pp. 1170–1178, Mar. 2016.

[7] P. Scarabaggio, S. Grammatico, R. Carli, and M. Dotoli, "Distributed demand side management with stochastic wind power forecasting," *IEEE Trans. Control Syst. Technol.*, vol. 30, no. 1, pp. 97–112, Jan. 2022.

[8] L. Han, M. Zhou, W. Jia, Z. Dalil, and X. Xu, "Intrusion detection model of wireless sensor networks based on game theory and an autoregressive model," *Inf. Sci.*, vol. 476, pp. 491–504, 2019.

[9] R. Deng, Z. Yang, J. Chen, N. R. Asr, and M.-Y. Chow, "Residential energy consumption scheduling: A coupled-constraint game approach," *IEEE Trans. Smart Grid*, vol. 5, no. 3, pp. 1340–1350, May 2014.

[10] Y. Li, K. P. Tee, R. Yan, W. L. Chan, and Y. Wu, "A framework of human–robot coordination based on game theory and policy iteration," *IEEE Trans. Robot.*, vol. 32, no. 6, pp. 1408–1418, Dec. 2016.

[11] A. A. Kulkarni and U. V. Shanbhag, "On the variational equilibrium as a refinement of the generalized Nash equilibrium," *Automatica*, vol. 48, no. 1, pp. 45–55, 2012.

[12] G. Belgioioso, W. Ananduta, S. Grammatico, and C. Ocampo-Martinez, "Operationally-safe peer-to-peer energy trading in distribution grids: A game-theoretic market-clearing mechanism," *IEEE Trans. Smart Grid*, vol. 13, no. 4, pp. 2897–2907, Jul. 2022.

[13] G. Belgioioso and S. Grammatico, "Semi-decentralized Nash equilibrium seeking in aggregative games with separable coupling constraints and non-differentiable cost functions," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 400–405, Oct. 2017.

[14] F. Facchinei, A. Fischer, and V. Piccialli, "On generalized Nash games and variational inequalities," *Operations Res. Lett.*, vol. 35, no. 2, pp. 159–164, 2007.

[15] M. R. Baye, G. Tian, and J. Zhou, "Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs," *Rev. Econ. Stud.*, vol. 60, no. 4, pp. 935–948, 1993.

[16] J.-S. Pang and G. Scutari, "Nonconvex games with side constraints," *SIAM J. Optim.*, vol. 21, no. 4, pp. 1491–1522, 2011.

[17] S. Grammatico, "Dynamic control of agents playing aggregative games with coupling constraints," *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4537–4548, Sep. 2017.

[18] G. Belgioioso and S. Grammatico, "Semi-decentralized generalized nash equilibrium seeking in monotone aggregative games," *IEEE Trans. Autom. Control*, vol. 68, no. 1, pp. 140–155, Jan. 2023.

[19] H. Farzaneh, M. Shokri, H. Kebriaei, and F. Aminifar, "Robust energy management of residential nanogrids via decentralized mean field control," *IEEE Trans. Sustain. Energy*, vol. 11, no. 3, pp. 1995–2002, Jul. 2019.

[20] D. Gadjov and L. Pavel, "Single-timescale distributed GNE seeking for aggregative games over networks via forward–backward operator splitting," *IEEE Trans. Autom. Control*, vol. 66, no. 7, pp. 3259–3266, Jul. 2021.

[21] C. Biasi et al., "Weak local Nash equilibrium," *Topological Methods Nonlinear Anal.*, vol. 41, no. 2, pp. 409–419, 2013.

[22] C. Alós-Ferrer and A. B. Ania, "Local equilibria in economic games," *Econ. Lett.*, vol. 70, no. 2, pp. 165–173, 2001.

[23] B. Cornet and M.-O. Czarnecki, "Existence of generalized equilibria," *Nonlinear Analysis, Theory, Methods Appl.*, vol. 44, no. 5, pp. 555–574, 2001.

[24] J.-S. Pang and G. Scutari, "Joint sensing and power allocation in nonconvex cognitive radio games: Quasi-Nash equilibria," *IEEE Trans. Signal Process.*, vol. 61, no. 9, pp. 2366–2382, May 2013.

[25] X. Huang, B. Beferull-Lozano, and C. Botella, "Quasi-Nash equilibria for non-convex distributed power allocation games in cognitive radios," *IEEE Trans. Wireless Commun.*, vol. 12, no. 7, pp. 3326–3337, Jul. 2013.

[26] L. J. Ratliff, S. A. Burden, and S. S. Sastry, "On the characterization of local Nash equilibria in continuous games," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2301–2307, Aug. 2016.

[27] F. Facchinei and S. Sagratella, "On the computation of all solutions of jointly convex generalized nash equilibrium problems," *Optim. Lett.*, vol. 5, no. 3, pp. 531–547, 2011.

[28] D. P. Bertsekas, "Nonlinear programming," *J. Oper. Res. Soc.*, vol. 48, no. 3, pp. 334–334, 1997.

[29] K. Miettinen and M. M. Mäkelä, *Tangent and Normal Cones in Nonconvex Multiobjective Optimization*. Berlin, Germany: Springer, 2000.

[30] L. F. Bueno, G. Haeser, and F. N. Rojas, "Optimality conditions and constraint qualifications for generalized Nash equilibrium problems and their practical implications," *SIAM J. Optim.*, vol. 29, no. 1, pp. 31–54, 2019.

[31] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[32] B. Franci and S. Grammatico, "Stochastic generalized Nash equilibrium seeking in merely monotone games," *IEEE Trans. Autom. Control*, vol. 67, no. 8, pp. 3905–3919, Aug. 2022.

[33] J. B. Rosen, "Existence and uniqueness of equilibrium points for concave N-person games," *Econometrica, J. Econometric Soc.*, vol. 33, pp. 520–534, 1965.

[34] J.-S. Pang and M. Fukushima, "Quasi-variational inequalities, generalized nash equilibria, and multi-leader-follower games," *Comput. Manage. Sci.*, vol. 2, no. 1, pp. 21–56, 2005.

[35] H. Federer, "Curvature measures," *Trans. Amer. Math. Soc.*, vol. 93, no. 3, pp. 418–491, 1959.

[36] A. Shapiro, "Existence and differentiability of metric projections in hilbert spaces," *SIAM J. Optim.*, vol. 4, no. 1, pp. 130–141, 1994.

[37] J.-P. Vial, "Strong and weak convexity of sets and functions," *Math. Operations Res.*, vol. 8, no. 2, pp. 231–259, 1983.

[38] R. Poliquin, R. Rockafellar, and L. Thibault, "Local differentiability of distance functions," *Trans. Amer. Math. Soc.*, vol. 352, no. 11, pp. 5231–5249, 2000.

[39] S. Adly, F. Nacry, and L. Thibault, "New metric properties for prox-regular sets," *Math. Program.*, vol. 189, no. 1, pp. 7–36, 2021.

[40] F. H. Clarke, R. Stern, and P. Wolenski, "Proximal smoothness and the lower-C2 property," *J. Convex Anal.*, vol. 2, no. 1-2, pp. 117–144, 1995.

[41] S. Adly, F. Nacry, and L. Thibault, "Preservation of prox-regularity of sets with applications to constrained optimization," *SIAM J. Optim.*, vol. 26, no. 1, pp. 448–473, 2016.

[42] S. Adly, F. Nacry, and L. Thibault, "Prox-regularity approach to generalized equations and image projection," *ESAIM, Control, Optimisation Calculus Variations*, vol. 24, no. 2, pp. 677–708, 2018.

[43] F. Bernard, L. Thibault, and N. Zlateva, "Prox-regular sets and epigraphs in uniformly convex banach spaces: Various regularities and other properties," *Trans. Amer. Math. Soc.*, vol. 363, no. 4, pp. 2211–2247, 2011.

[44] A. Canino, "On p-convex sets and geodesics," *J. Differ. Equ.*, vol. 75, no. 1, pp. 118–157, 1988.

[45] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178. Berlin, Germany: Springer, 2008.

[46] D. P. Palomar and Y. C. Eldar, *Convex Optimization in Signal Processing and Communications*. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[47] G. M. Korpelevich, "The extragradient method for finding saddle points and other problems," *Ekonomie i Mathematik Metody*, vol. 12, pp. 747–756, 1976.

[48] P. Scarabaggio, R. Carli, and M. Dotoli, "A game-theoretic control approach for the optimal energy storage under power flow constraints in distribution networks," in *Proc. IEEE 16th Int. Conf. Automat. Sci. Eng.*, 2020, pp. 1281–1286.

[49] J. A. Momoh, R. Adapa, and M. El-Hawary, "A review of selected optimal power flow literature to 1993. I. nonlinear and quadratic programming approaches," *IEEE Trans. Power Syst.*, vol. 14, no. 1, pp. 96–104, Feb. 1999.
[50] M. Niu, C. Wan, and Z. Xu, "A review on applications of heuristic optimization algorithms for optimal power flow in modern power systems," *J. Modern Power Syst. Clean Energy*, vol. 2, no. 4, pp. 289–297, 2014.
[51] B. Houska and Y. Jiang, "Distributed optimization and control with ALADIN," in *Recent Advances in Model Predictive Control, Theory, Algorithms, and Applications*, vol. 485. Cham, Switzerland: Springer, 2021, pp. 135–163.
[52] A. Engelmann, T. Mühlpfordt, Y. Jiang, B. Houska, and T. Faulwasser, "Distributed ac optimal power flow using ALADIN," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 5536–5541, 2017.
[53] I. Atzeni, L. G. Ordóñez, G. Scutari, D. P. Palomar, and J. R. Fonollosa, "Demand-side management via distributed energy generation and storage optimization," *IEEE Trans. Smart Grid*, vol. 4, no. 2, pp. 866–876, Jun. 2013.
[54] A. Kargarian et al., "Toward distributed/decentralized DC optimal power flow implementation in future electric power systems," *IEEE Trans. Smart Grid*, vol. 9, no. 4, pp. 2574–2594, Jul. 2018.
[55] L. Yang, J. Luo, Y. Xu, Z. Zhang, and Z. Dong, "A distributed dual consensus ADMM based on partition for DC-DOPF with carbon emission trading," *IEEE Trans. Ind. Inform.*, vol. 16, no. 3, pp. 1858–1872, Mar. 2020.
[56] A. Vinkovic and R. Mihalic, "A current-based model of an IPFC for Newton–Raphson power flow," *Electric Power Syst. Res.*, vol. 79, no. 8, pp. 1247–1254, 2009.
[57] J. Jahn, *Tangent Cones*. Berlin, Germany: Springer, 2020.
[58] J. Nocedal and S. Wright, *Numerical Optimization*. Berlin, Germany: Springer, 2006.
[59] S. Lefschetz, "Intersections and transformations of complexes and manifolds," *Trans. Amer. Math. Soc.*, vol. 28, no. 1, pp. 1–49, 1926.
[60] R. Brown, "On a homotopy converse to the Lefschetz fixed point theorem," *Pacific J. Math.*, vol. 17, no. 3, pp. 407–411, 1966.
[61] P. Glendinning, *Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations*. Cambridge, U.K.: Cambridge Univ. Press, 1994.

Paolo Scarabaggio (Member, IEEE) received the Ph.D. degree in electrical and information engineering from Politecnico di Bari, Bari, Italy, in 2022.

In 2019, he was a Visiting Student with the Delft Center for Systems and Control, Technical University of Delft, Delft, The Netherlands. He is currently an Assistant Professor (RTDA) with the Decision and Control Laboratory, Politecnico di Bari. He has authored more than 20 printed international publications. His research interests include modeling, optimization, game theory, and control of complex multiagent systems, with application in energy distribution systems, and social networks.

Dr. Scarabaggio was the recipient of the 2022 IEEE CSS Italy Best Young Author Journal Paper Award.

Raffaele Carli (Senior Member, IEEE) received the Laurea degree in electronic engineering (Hons.) and the Ph.D. degree in electrical and information engineering from Politecnico di Bari, Bari, Italy, in 2002 and 2016, respectively.

From 2003 to 2004, he was a Reserve Officer with Italian Navy. From 2004 to 2012, he worked as a System and Control Engineer and Technical Manager for a space and defense multinational company. He is currently an Assistant Professor in automatic control with Politecnico di Bari. He is the author of more than 100 printed international publications. His research interests include the formalization, simulation, and implementation of decision and control systems, as well as modeling and optimization of complex systems.

Dr. Carli is an Associate Editor for IEEE Transactions on Automation Science and Engineering and the IEEE Transactions on Systems, Man, and Cybernetics. He was a member of the International Program Committee of more than 30 international conferences and a Guest Editor for special issues in international journals.

Sergio Grammatico (Senior Member, IEEE) received the bachelor's degree in computer engineering and the master's and Ph.D. degrees in automatic control from the University of Pisa, Pisa, Italy, in 2008, 2009, and 2013, respectively.

From 2012 to 2017, he held research positions with the University of California Santa Barbara, Santa Barbara, CA, USA, with ETH Zürich, Zürich, Switzerland, and with TU Eindhoven, The Netherlands. He is currently an Associate Professor with the Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands. His research interests include multiagent systems, dynamic game theory, and optimal control.

Prof. Grammatico was the recipient of the 2021 Roberto Tempo Best Paper Award. He is currently an Associate Editor for IEEE Transactions on Automatic Control and *IFAC Automatica*.

Mariagrazia Dotoli (Fellow, IEEE) received the Laurea degree in electronic engineering (Hons.) and the Ph.D. degree in electrical engineering from Politecnico di Bari, Bari, Italy, in 1995 and 1999, respectively.

She has been a Visiting Scholar with the Paris 6 University and with the Technical University of Denmark. She has been the Vice Rector for research with Politecnico di Bari and a member elect of the Academic Senate. She is currently a Full Professor in automatic control with Politecnico di Bari, which she joined in 1999. She is author of more than 200 publications, including one textbook (in Italian) and more than 90 international journal papers. Her research interests include modeling, identification, management, control, and diagnosis of discrete event systems, manufacturing systems, logistics systems, traffic networks, smart grids, and networked systems.

Prof. Dotoli was the Cochairman of the Training and Education Committee of ERUDIT, the European Commission network of excellence for fuzzy logic and uncertainty modeling in information technology, and was key node representative of EUNITE, the European Network of excellence on Intelligent Technologies. She is an Expert Evaluator of the European Commission since the 6th Framework Programme. She is a Senior Editor for IEEE Transactions on Automation Science and Engineering and an Associate Editor for the IEEE Transactions on Systems, Man, and Cybernetics. She is the General Chair of the 2024 IEEE Conference on Automation Science and Engineering. She was the General Chair of the 2021 Mediterranean Conference on Control and Automation, the Program Chair of the 2020 IEEE Conference on Automation Science and Engineering, the Program Cochair of the 2017 IEEE Conference on Automation Science and Engineering, the Workshop and Tutorial Chair of the 2015 IEEE Conference on Automation Science and Engineering, the Special Session Cochair of the 2013 IEEE Conference on Emerging Technology and Factory Automation, and the Chair of the National Committee of the 2009 IFAC Workshop on Dependable Control of Discrete Systems. She was a Member of the International Program Committee of more than 80 international conferences.

Open Access provided by 'Politecnico di Bari' within the CRUI CARE Agreement