Single-qubit dynamics

Determining the density matrix of a qubit in closed and open quantum systems when considering free evolution and weak measurements

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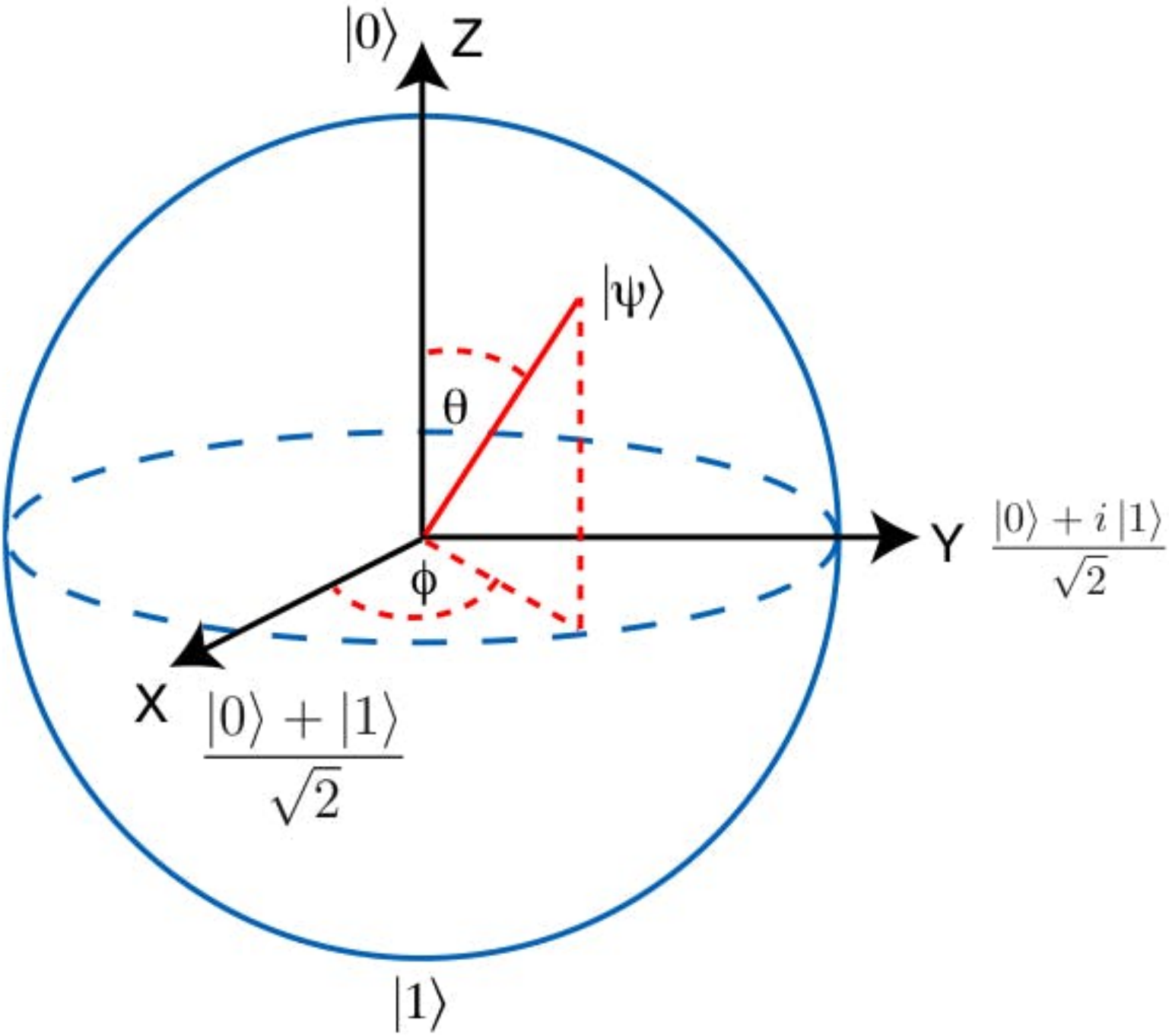

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by

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Abstract

Quantum technology is evolving faster than ever. Currently, all eyes are on the quantum computer, the promising computer that can solve problems which are unsolvable for regular computers. In order to understand this new technology, it is necessary to understand the qubit, the basic unit of quantum information. This can be done by means of the density matrix: a mathematical representation of the state of a system. The aim of this thesis is to find the density matrix of a single qubit in closed (isolated) and open quantum systems. In the case of a closed system, an alternating sequence of two processes with different Hamiltonians is considered, which both last a fixed amount of time after one another. These systems have been solved using a direct formula for a 2×2 matrix with distinct eigenvalues raised to the power n for any $n \in \mathbb{N}$. In the case of an open system, dissipation is taken into account compared to a single process in a closed system. These open systems have been solved numerically using the Lindblad master equation and the spin-boson model to model the environment as a bath of bosons. However, the use of multiple approximations and assumptions questions the validity of the results for 'strong' interactions. Suggestions for further research include investigating quantitatively when the Lindblad equation is valid to use and solve open quantum systems using different models for the environment.

Contents

1 Introduction

Nowadays, most people cannot imagine life without a computer or without the internet. Though computers have already evolved from huge inefficient machines to powerful supercomputers, humanity still demands more computation power as time passes. As regular computers have trouble keeping up with the current challenges the world is facing, a new kind of technology is required: the quantum computer.

The quantum computer is the new promising computer that should be able to outperform the classical computer based on properties like computation power and computation speed. Whereas the classical computer consists of bits, the quantum computer consists of quantum bits (qubits), the basic unit of quantum information. Being able to utilize the concepts from quantum mechanics allows the quantum computer to outperform the classical computer. However, because of the current state of development of quantum computers, the current quantum computers are too small to outperform classical computers, but this will change in the future. This is the reason that quantum computers or, more specifically, qubits are the subject of a lot of papers at the moment, since qubits are the fundamentals of quantum computers.

As the fundamentals of quantum computers, understanding qubits is extremely important for a variety of purposes. Applications include subjects in quantum mechanics like quantum information theory [7], but also less obvious branches ranging from artificial intelligence [11] to finance [4]. In order to advance in these areas, it is necessary to get a better understanding of qubits and the effect of manipulations on them. This can be done by means of the density matrix, the mathematical representation of the state of a qubit. After all, the density matrix contains all the information obtained on a qubit.

It is clear that qubits have a lot of applications and this is the motivation for this research which will be described in this paragraph. Consider closed and open quantum systems with a particle with spin 1/2, which represents a qubit, and a detector with coordinate q and Gaussian initial state. A closed/isolated quantum system is a system where the environment is neglected; these systems are easier to solve than open systems, but are less realistic. On the other hand, open quantum systems take dissipation into account, the loss of energy towards the environment. This results in open quantum systems being more realistic, but at the same time becoming almost impossible to solve exactly.

The qubit can undergo two different kind of transitions: free evolution and weak measurements done by a detector. Free evolution is described by the Hamiltonian $H_0 = B\sigma_z$ where B denotes an external magnetic field. A weak measurement is a quantum measurement where the observer obtains little information about a system, while also disturbing the system a little [2]. This is described by the Hamiltonian $H_{int} = g(t)q\sigma_x$ where $g(t)$ denotes the interaction strength between the qubit and the detector. Furthermore, σ_z and σ_x are Pauli spin matrices. The research question is then as follows: what is the density matrix of the qubit in different closed and open quantum systems.

This thesis is a follow-up research on the works of K. Kramers who focused on observing the quantum Zeno effect and determining the density matrix in simpler systems [8]. Kramers' thesis formed the start of the measurement set-up and some of their results have been used to expand to more complicated systems. Moreover, this thesis does not consider the quantum Zeno effect, but adds open quantum systems which have not been considered before.

The following structure has been used in this thesis. Chapter 2 provides some preliminary mathematics which are related to quantum mechanics and this thesis and therefore necessary to understand.

Chapter 3 elaborates on some concepts of quantum mechanics and results from the works of Kramers that are used in this thesis. Chapters 4 and 5 provide the results on the density matrix of a qubit in closed and open quantum systems respectively which are also accompanied by visual representations in the form of graphs of the expectation values $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$. Thereafter, chapter 6 provides a discussion on the outcomes that have been presented in the previous chapter and chapter 7 will summarize the results in a clear answer to the main question and suggests possibilities for further research. Finally, there are also two appendices: appendix A contains a formal proof of an important formula, which is explained in the last section of chapter 2. Appendix B provides the Python code that has been used to generate the graphical results in chapter 4 and 5.

This research has been done as a bachelor thesis to complete the bachelors of science Applied Physics and Applied Mathematics at the Delft University of Technology.

2

Preliminary mathematics

This chapter discusses some fundamental mathematical concepts that form the basis of quantum mechanics which are thus essential to understand.

2.1. Hilbert space

One of the most important mathematical concepts that paves the way for quantum mechanics is the Hilbert space:

Definition. A Hilbert space $\mathcal{H}$ is a complex vector space on which an inner product is defined that is also a complete metric space.

Hilbert spaces play a very important role in quantum mechanics, since the pure state of a quantum system is represented by a unit vector in a Hilbert space. In addition, mixed states are represented by a matrix which can be calculated from these vectors; the difference between pure and mixed states is explained further in section 3.1. The notation used for these vectors is $|\psi\rangle$, called a "ket", whereas $\langle\psi|$, called a "bra", is the hermitian conjugate of a ket. The inner product of two vectors $|\psi\rangle$ and $|\phi\rangle$ is then denoted by $\langle\psi|\phi\rangle$.

Most quantum systems have an associated Hilbert space in the form of $\mathbb{C}^n$ where $n \in \mathbb{N} \cup \{\infty\}$. However, this thesis focusses on two-level systems (which are explained further in section 3.2) which have an associated Hilbert space $\mathbb{C}^2$.

2.2. Tensor product

When discussing composite quantum systems, it is necessary to introduce the tensor product. Consider two subsystems A and B respectively associated with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, then the Hilbert space $\mathcal{H}$ of the total system consisting of these two subsystems is given by

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.1}$$

The same principle can be applied to any composition of a finite amount of subsystems. Furthermore, the same can be true for the state vectors of the systems; when the two subsystems are prepared in the states $|\psi\rangle_A$ and $|\phi\rangle_B$ and the total state is given by $|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$, then the state $|\Psi\rangle$ is called separable. On the contrary, when $|\Psi\rangle$ cannot be written as a single tensor product of $|\psi\rangle_A$ and $|\psi\rangle_B$, but only as a linear combination, $|\Psi\rangle$ is called entangled; a phenomenon that is not used in this thesis, for further reading see A. Einstein, B. Podolsky and N. Rosen [3].

While the tensor product can be determined of any two vector spaces, it is solely used on matrices in this thesis; for this reason, a simplified definition of the tensor product will be considered:

Definition. Consider a $m \times n$ matrix X and a $p \times q$ matrix Y, the tensor product $X \otimes Y$ is then a

$mp \times nq$ matrix given by:

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \dots & x_{1n}Y \\ \vdots & \ddots & \vdots \\ x_{m1}Y & \dots & x_{mn}Y \end{bmatrix}. \tag{2.2}$$

2.3. Partial trace

Before going into the partial trace, consider the regular trace of a $n \times n$ matrix X which is defined as the sum of its diagonal elements:

$$\mathrm{Tr}\, X = \sum_{i=1}^{n} x_{ii} \tag{2.3}$$

where Tr denotes the trace. Similar to the tensor product, the partial trace is a powerful tool when discussing composite quantum systems. Consider two square matrices X on $\mathcal{H}_A$ and Y on $\mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces. If the tensor product of X and Y on $\mathcal{H}_A \otimes \mathcal{H}_B$ is known, taking the partial trace over one of the matrices yields the other matrix multiplied by some factor:

$$\begin{aligned} \mathrm{Tr}_B(X \otimes Y) &= X\, \mathrm{Tr}(Y) \\ \mathrm{Tr}_A(X \otimes Y) &= \mathrm{Tr}(X) Y. \end{aligned} \tag{2.4}$$

In the more complicated case of a linear map C on $\mathcal{H}_A \otimes \mathcal{H}_B$, a few more steps are required. Let $\{|a_i\rangle\}$ be a basis of $\mathcal{H}_A$ and $\{|b_i\rangle\}$ be a basis of $\mathcal{H}_B$. Any linear map C on $\mathcal{H}_A \otimes \mathcal{H}_B$ can then be decomposed as

$$C = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|. \tag{2.5}$$

The general partial traces can then be computed as:

$$\begin{aligned} \mathrm{Tr}_B(C) &= \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j|\, \mathrm{Tr}(|b_k\rangle\langle b_l|) = \sum_{ijk} c_{ijkk} |a_i\rangle\langle a_j| \langle b_k|b_k\rangle \\ \mathrm{Tr}_A(C) &= \sum_{ijkl} c_{ijkl}\, \mathrm{Tr}(|a_i\rangle\langle a_j|) |b_k\rangle\langle b_l| = \sum_{ikl} c_{iikl} \langle a_i|a_i\rangle |b_k\rangle\langle b_l| \end{aligned} \tag{2.6}$$

where $\sum_{ijkl}$ denotes the sum over all possible combinations of i, j, k and l.

2.4. Raising a 2×2 matrix to the power n

Consider a 2×2 matrix X with distinct eigenvalues λ_- and λ_+; then the formula for X^n for any $n \in \mathbb{N}$ is given by:

$$X^n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} X - \lambda_+ \lambda_- \frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} I. \tag{2.7}$$

Here, I is the identity matrix. A proof of this formula is presented in appendix A.

3

Theory on quantum mechanics

This chapter gives an overview of the necessary theory on quantum mechanics aiming to supply the reader with sufficient knowledge in order to understand the concepts used in the thesis.

3.1. Density matrix

The density matrix represents the quantum state of a system; from the density matrix, the probabilities of possible measurement outcomes of the system can be calculated. The motivation for the density matrix is that it can represent mixed states, whereas state vectors can only represent pure states. Mixed states are used when the state of a system is unknown; in that case it is regarded as a statistical ensemble of different states $|\psi_j\rangle$ with corresponding probabilities p_j [5]. The density matrix can then be written as:

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \tag{3.1}$$

where the probabilities p_j satisfy $\sum_j p_j = 1$. Additionally, when the system is in a pure state $|\psi\rangle$, the density matrix is described by a single term:

$$\rho = |\psi\rangle\langle\psi|. \tag{3.2}$$

The density matrix has an important property and gives rise to the definition of the expectation of an observable A, namely:

$$\mathsf{Tr}(\rho) = 1 \tag{3.3}$$

$$\langle A\rangle = \mathsf{Tr}(\rho A). \tag{3.4}$$

The fact that $\mathsf{Tr}(\rho) = 1$, has an important consequence for composite systems: when the density matrix ρ_{AB} of a composite system consisting of subsystems A and B is known, the density matrices of the subsystems can be computed:

$$\begin{aligned} \mathsf{Tr}_B(\rho_{AB}) &= \rho_A \\ \mathsf{Tr}_A(\rho_{AB}) &= \rho_B \end{aligned} \tag{3.5}$$

where ρ_A and ρ_B are the density matrices of the subsystems A and B respectively.

3.2. Qubits

A quantum bit, qubit for short, is the quantum analog of the classical bit. Whereas a classical bit is used to represent information in classical computers, a qubit is used to represent information in quantum computers. A classical bit can only be 0 or 1, but a qubit is in a superposition of 0 and 1, allowing its state to be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{3.6}$$

where $\alpha, \beta \in \mathbb{C}$ and $|0\rangle$ and $|1\rangle$ are the two orthonormal basis states. This representation can be used for any two-level system, where $|\psi\rangle$ is an element of the Hilbert space $\mathbb{C}^2$. The state of a qubit can

also be represented visually by means of the so-called Bloch sphere. The Bloch sphere is a sphere of radius 1 which is used to represent the state of a qubit as a vector in or on this sphere. The Bloch sphere is illustrated in figure 3.1.

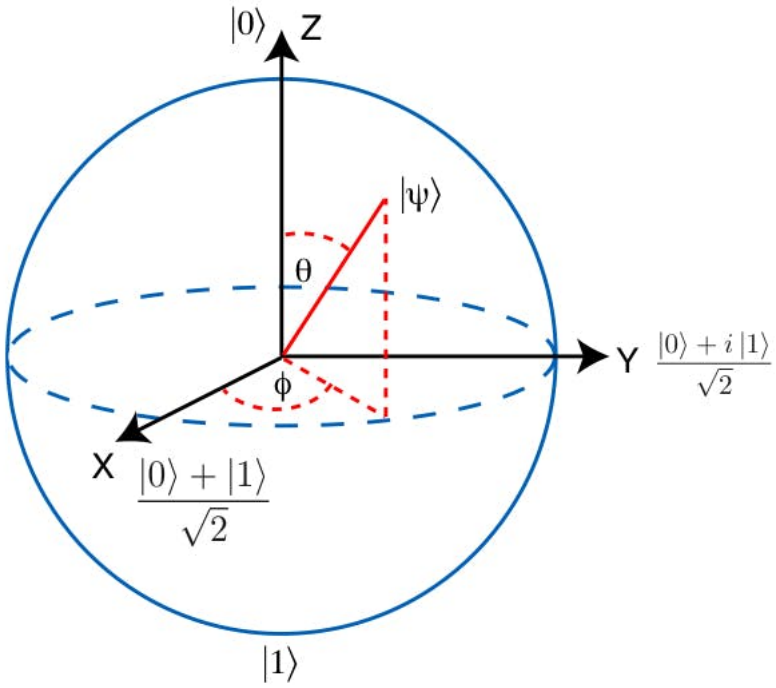


Figure 3.1: The state $|\psi\rangle$ of some qubit which is represented as a vector on the Bloch sphere. $|0\rangle$ and $|1\rangle$ are defined as unit vectors on the z-axis in the positive and negative direction respectively [12].

When using this representation, the so-called Born rule states that, when measuring the qubit, the possible outcomes are $|0\rangle$ and $|1\rangle$ with respective probabilities $|\alpha|^2$ and $|\beta|^2$. This implies that α and β must satisfy the following constraint:

$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.7}$$

3.3. Time evolution

A system's time evolution is described by the time-dependent Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{3.8}$$

with $\hbar$ the reduced Planck constant, $|\psi(t)\rangle$ the state at time t and H the Hamiltonian operator. This differential equation can be solved:

$$|\psi(t)\rangle = \exp(-\frac{i}{\hbar}\int_0^t H dt')|\psi(0)\rangle \tag{3.9}$$

with $|\psi(0)\rangle$ the initial state. Furthermore, if the Hamiltonian is time-independent, the solution can be written as:

$$|\psi(t)\rangle = \exp(-\frac{i}{\hbar}Ht)|\psi(0)\rangle = U(t)|\psi(0)\rangle. \tag{3.10}$$

The operator $U(t) = \exp(-\frac{i}{\hbar}Ht)$ is known as the time-evolution operator. When the time-evolution operator and the initial state are known, the density matrix can be found:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \exp(-\frac{i}{\hbar}Ht)|\psi(0)\rangle\langle\psi(0)|\exp(\frac{i}{\hbar}Ht) = \exp(-\frac{i}{\hbar}Ht)\rho(0)\exp(\frac{i}{\hbar}Ht) \tag{3.11}$$

In the next subsections, three important cases of time evolution will be treated which will be used later on in chapters 4 and 5 [8].

3.3.1. Free evolution

First, consider a qubit that only evolves freely under the influence of a magnetic field. The Hamiltonian is then given by $H_0 = B\sigma_z$ with B the strength of the magnetic field and σ_z a Pauli spin matrix. Using a Taylor series and defining $\alpha \equiv \frac{1}{\hbar}Bt$, the time-evolution operator can be written as:

$$U(t) = \exp(-i\alpha\sigma_z) = \sum_{n=0}^{\infty} \frac{(-i\alpha\sigma_z)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-i\alpha\sigma_z)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\alpha\sigma_z)^{2n+1}}{(2n+1)!}. \tag{3.12}$$

Since $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, it is clear that $\sigma_z^2 = I$, where I is the identity matrix, hence $\sigma_z^{2n} = I$ and $\sigma_z^{2n+1} = \sigma_z$ $\forall n \in \mathbb{N}$. Using these simplifications and recognizing the Taylor series for the hyperbolic sine and cosine, the time-evolution operator then becomes:

$$U(t) = I\sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n}}{(2n)!} + \sigma_z \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n+1}}{(2n+1)!} = I\cosh(-i\alpha) + \sigma_z \sinh(-i\alpha). \tag{3.13}$$

This can then be written as a matrix:

$$U(t) = \begin{pmatrix} \cosh(-i\alpha) + \sinh(-i\alpha) & 0 \\ 0 & \cosh(-i\alpha) - \sinh(-i\alpha) \end{pmatrix} = \begin{pmatrix} \exp(-\frac{i}{\hbar}Bt) & 0 \\ 0 & \exp(\frac{i}{\hbar}Bt) \end{pmatrix}. \tag{3.14}$$

Assuming for example an initial state $|0\rangle = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, the density matrix can then be computed as a function of time:

$$\rho(t) = \frac{1}{2}\begin{pmatrix} \exp(-\frac{i}{\hbar}Bt) & 0 \\ 0 & \exp(\frac{i}{\hbar}Bt) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \exp(\frac{i}{\hbar}Bt) & 0 \\ 0 & \exp(-\frac{i}{\hbar}Bt) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \exp(-2\frac{i}{\hbar}Bt) \\ \exp(2\frac{i}{\hbar}Bt) & 1 \end{pmatrix} \tag{3.15}$$

which clearly satisfies $\mathrm{Tr}(\rho) = 1$ as expected.

3.3.2. Weak measurements

The idea of weak measurements was first introduced in 1988 [1]. A weak measurement is a quantum measurement where the observer obtains little information about a system, while also disturbing the system a little [2]. Consider a detector with coordinate q and Gaussian initial state:

$$\phi(0) = \frac{1}{(\pi\sigma)^{1/4}} \exp(-\frac{(q-q_0)^2}{2\sigma}) \tag{3.16}$$

with q_0 the mean of the detector coordinate q and σ the dispersion. The initial state of the composite system will then be given by the tensor product $|\Phi(0)\rangle = \phi(0) \otimes |\psi(0)\rangle = \phi(0)|\psi(0)\rangle$ where $|\psi(0)\rangle$ is the initial state of the qubit (the tensor product reduces to a regular product since $\phi(0)$ is a number).

When a weak measurement is performed on the x-component of the spin, the interaction Hamiltonian becomes $H_{int} = g(t)q\sigma_x$ where $g(t)$ denotes the interaction strength between the qubit and the detector. Taking g independent of time and defining $\beta \equiv \frac{1}{\hbar}gqt$, the time-evolution operator then becomes $U(t) = \exp(-i\beta\sigma_x)$. Since $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ implies that $\sigma_x^2 = I$, the same steps as in the previous subsection can be applied to derive:

$$U(t) = I\cosh(-i\beta) + \sigma_x \sinh(-i\beta). \tag{3.17}$$

This can also be written as a matrix:

$$U(t) = \begin{pmatrix} \cosh(-i\beta) & \sinh(-i\beta) \\ \sinh(-i\beta) & \cosh(-i\beta) \end{pmatrix} = \begin{pmatrix} \cos(\frac{1}{\hbar}gqt) & -i\sin(\frac{1}{\hbar}gqt) \\ -i\sin(\frac{1}{\hbar}gqt) & \cos(\frac{1}{\hbar}gqt) \end{pmatrix} \tag{3.18}$$

where the formulas $\cosh(-ix) = \cos(x)$ and $\sinh(-ix) = -i\sin(x)$ have been used. Assuming an initial state $|\Phi(0)\rangle$ where $|\psi(0)\rangle = |0\rangle = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, the total density matrix ρ_T can be computed in a similar way as in equation 3.15:

$$\begin{aligned} \rho_T(t) &= \frac{1}{2\sqrt{\pi\sigma}} \exp(-\frac{(q-q_0)^2}{\sigma}) \begin{pmatrix} \cos(\frac{1}{\hbar}gqt) & -i\sin(\frac{1}{\hbar}gqt) \\ -i\sin(\frac{1}{\hbar}gqt) & \cos(\frac{1}{\hbar}gqt) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos(\frac{1}{\hbar}gqt) & i\sin(\frac{1}{\hbar}gqt) \\ i\sin(\frac{1}{\hbar}gqt) & \cos(\frac{1}{\hbar}gqt) \end{pmatrix} \\ &= \frac{1}{2\sqrt{\pi\sigma}} \exp(-\frac{(q-q_0)^2}{\sigma}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{3.19}$$

In order to find the density matrix of the qubit, q has to be integrated out as stated in [8]. Since the integral over the real line of the probability density function $\frac{1}{\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})$ with respect to q is equal to 1, the density matrix of the qubit is easily found as:

$$\rho(t) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \tag{3.20}$$

which obviously satisfies $\mathrm{Tr}(\rho) = 1$ as expected.

3.3.3. Weak measurements during free evolution

For the last case, suppose that the processes of the previous subsections are happening simultaneously. This results in the Hamiltonian

$$H = H_0 + H_{int} = B\sigma_z + gq\sigma_x = \begin{pmatrix}B & gq\\ gq & -B\end{pmatrix} \tag{3.21}$$

where g is again taken time-independent. The initial state of the system is, as in the previous subsection, the product $|\Phi(0)\rangle = \phi(0)|\psi(0)\rangle$ where $|\psi(0)\rangle$ is the initial state of the qubit and $\phi(0)$ is given in equation 3.16.

The time-evolution operator then results in $U(t) = \exp(-\frac{i}{\hbar}tH)$ where the exponent will again be expanded using a Taylor expansion and then the same method can be used as in the previous subsections. Note that $H^2 = (B^2+g^2q^2)I$, which implies that $H^{2n} = (B^2+g^2q^2)^n I$ and $H^{2n+1} = (B^2+g^2q^2)^n H$. After rewriting some parts, the time-evolution operator becomes:

$$\begin{aligned}
U(t) &= I\sum_{n=0}^{\infty}(B^2+g^2q^2)^n\frac{(-it/\hbar)^{2n}}{(2n)!} + H\sum_{n=0}^{\infty}(B^2+g^2q^2)^n\frac{(-it/\hbar)^{2n+1}}{(2n+1)!}\\
&= I\sum_{n=0}^{\infty}\frac{(\sqrt{B^2+g^2q^2}\cdot -it/\hbar)^{2n}}{(2n)!} + H\sum_{n=0}^{\infty}\frac{1}{\sqrt{B^2+g^2q^2}}\frac{(\sqrt{B^2+g^2q^2}\cdot -it/\hbar)^{2n+1}}{(2n+1)!}\\
&= I\cosh(-\frac{i}{\hbar}t\sqrt{B^2+g^2q^2}) + H\frac{1}{\sqrt{B^2+g^2q^2}}\sinh(-\frac{i}{\hbar}t\sqrt{B^2+g^2q^2})\\
&\equiv I\cosh(-i\gamma) + H\xi\sinh(-i\gamma)
\end{aligned} \tag{3.22}$$

where in the last line the substitutions $\gamma = \frac{1}{\hbar}t\sqrt{B^2+g^2q^2}$ and $\xi = \frac{1}{\sqrt{B^2+g^2q^2}}$ have been used. The time-evolution operator can then be written as a matrix:

$$U(t) = \begin{pmatrix}\cosh(-i\gamma) + B\xi\sinh(-i\gamma) & gq\xi\sinh(-i\gamma)\\ gq\xi\sinh(-i\gamma) & \cosh(-i\gamma) - B\xi\sinh(-i\gamma)\end{pmatrix}. \tag{3.23}$$

Reverting the substitutions and using $\cosh(-ix) = \cos(x)$ and $\sinh(-ix) = -i\sin(x)$ results in:

$$U(t) = \begin{pmatrix}\cos(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) - i\frac{B}{\sqrt{B^2+g^2q^2}}\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) & -i\frac{gq}{\sqrt{B^2+g^2q^2}}\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2})\\ -i\frac{gq}{\sqrt{B^2+g^2q^2}}\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) & \cos(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) + i\frac{B}{\sqrt{B^2+g^2q^2}}\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2})\end{pmatrix}. \tag{3.24}$$

Assuming an initial state $|\Phi(0)\rangle$ where $|\psi(0)\rangle = |0\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$, the total density matrix ρ_T can be computed in the same way as in equation 3.19:

$$\rho_T(t) = \frac{1}{2\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} \tag{3.25}$$

where

$$\begin{aligned}
\rho_{11} =& 1 + 2\frac{Bgq}{B^2+g^2q^2}\sin^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) \\
\rho_{12} =& \cos^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) + \frac{-B^2+g^2q^2}{B^2+g^2q^2}\sin^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) \\
& - 2i\frac{B}{\sqrt{B^2+g^2q^2}}\cos(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2})\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) \\
\rho_{21} =& \cos^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) + \frac{-B^2+g^2q^2}{B^2+g^2q^2}\sin^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) \\
& + 2i\frac{B}{\sqrt{B^2+g^2q^2}}\cos(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2})\sin(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}) \\
\rho_{22} =& 1 - 2\frac{Bgq}{B^2+g^2q^2}\sin^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2}).
\end{aligned} \tag{3.26}$$

Similar to the previous subsection, q has to be integrated out in order to find the density matrix ρ of the qubit. However, these integrals cannot be solved analytically. Nevertheless, it can be shown that the trace of the resulting density matrix is 1. To this end, define $x(q) \equiv \frac{Bgq}{B^2+g^2q^2}\sin^2(\frac{1}{\hbar}t\sqrt{B^2+g^2q^2})$ such that $\rho_{11} = 1 + 2x(q)$ and $\rho_{22} = 1 - 2x(q)$. This results in:

$$\mathrm{Tr}(\rho) = \int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})(\rho_{11}+\rho_{22})dq = \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})dq = 1 \tag{3.27}$$

There is one exception where the density matrix can be computed analytically. When $\sigma \to 0$, the probability density function $\frac{1}{\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})$ converges towards the Dirac delta peak $\delta(q-q_0)$. The density matrix of the qubit is then equal to the total density matrix, but without the factor $\frac{1}{\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})$ and every other q is replaced with q_0.

3.4. Open quantum systems

An open quantum system is the combination of a quantum system of interest and an environment or bath that interacts with this system, as shown in the figure below.

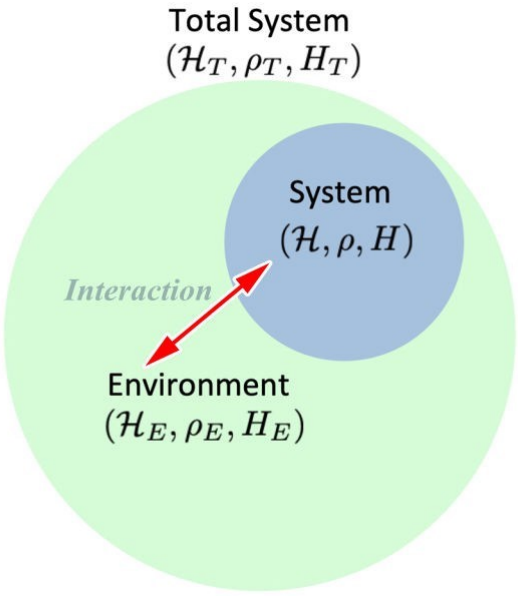


Figure 3.2: A total system (belonging to a Hilbert space $\mathcal{H}_T$, with states described by density matrices ρ_T, and with dynamics determined by a Hamiltonian H_T) divided into the system of interest, "system", and the environment [9].

This leads to dissipation: the loss of energy from the system to the environment. Open quantum systems are more realistic than closed quantum systems, because no quantum system can be completely

isolated. The next two subsections will introduce the general description of an open quantum system and the Lindbladian (an equation that describes the evolution of the density matrix ρ). Finally, the spin-boson model which is a commonly used model to describe dissipation in open quantum systems, is explained in the last subsection.

3.4.1. Describing an open quantum system

An open quantum system is in general described by a Hamiltonian in the form of

$$H_T = H \otimes I_E + I \otimes H_E + gH_I \tag{3.28}$$

where H and H_E are respectively the Hamiltonian of the system and the environment and I_E and I are respectively the identity matrices of the Hilbert spaces $\mathcal{H}_E$ and $\mathcal{H}$ (i.e. $I_E|\psi\rangle = |\psi\rangle \ \forall|\psi\rangle \in \mathcal{H}_E$ and $I|\phi\rangle = |\phi\rangle \ \forall|\phi\rangle \in \mathcal{H}$); H_I is the Hamiltonian for the system-environment interaction and g denotes the interaction strength [9]. Moreover, without loss of generality, the interaction Hamiltonian can be written in the following way:

$$H_I = \sum_i S_i \otimes E_i \tag{3.29}$$

where $S_i \in B(\mathcal{H})$ and $E_i \in B(\mathcal{H}_E)$ with $B(\mathcal{H}) : \mathcal{H} \to \mathcal{H}$ the space of bounded operators acting on the Hilbert space $\mathcal{H}$.

In order to determine the evolution of the total state of the system, the initial state of ρ_T is required. It is assumed to be the tensor product of the initial density matrices $\rho(0)$ of the system of interest and $\rho_E(0)$ of the bath:

$$\rho_T(0) = \rho(0) \otimes \rho_E(0). \tag{3.30}$$

This assumption corresponds to the absence of correlations between the system and the environment. Furthermore, a second assumption is made, where the initial state of the environment is thermal, meaning that it is described by the following density matrix:

$$\rho_E(0) = \frac{1}{Z} \exp(-\frac{H_E}{k_B T}), \ Z = \mathrm{Tr}(-\frac{H_E}{k_B T}) \tag{3.31}$$

where k_B and T respectively denote the Boltzmann constant and the temperature. The evolution of ρ_T is equivalent to equation 3.11, but now with the total Hamiltonian H_T and provided that H_T is time-independent. Finally, $\rho(t)$ can be determined by tracing out the environment of $\rho_T(t)$:

$$\rho(t) = \mathrm{Tr}_E(\rho_T(t)). \tag{3.32}$$

3.4.2. Lindbladian

The Lindblad master equation, also known as the Lindbladian, is an equation that describes the evolution of the density matrix of an open quantum system. The Lindblad equation for an N-dimensional system's density matrix ρ can be written as

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{i=1}^{N^2-1} g_i (L_i \rho(t) L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\}) \tag{3.33}$$

where the Hamiltonian H and the coefficients g_i determine the system dynamics. $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$ are respectively the commutator and the anticommutator and L_i are called the Lindblad or jump operators of the system. The derivation of the Lindbladian will not be reproduced here as it can be found in several sources [9].

The motivation for the Lindbladian is that it allows one to find the density matrix of the system of interest without taking into account the Hamiltonian of the environment; the Lindblad operators are determined from the S_i term in equation 3.29. However, the Lindblad equation is restricted to some more assumptions and approximations besides the one that was already mentioned in equation 3.30:

- First, it is assumed that the interaction is 'weak' and that the environment is 'much larger' than the system of interest. This has two important consequences: the state of the environment does not change significantly and equation 3.30 is valid for all times. These consequences can be summarised in the following equation $\forall t \geq 0$:

$$\rho_T(t) = \rho(t) \otimes \rho_E. \tag{3.34}$$

- Secondly, it is assumed that the time scales of correlation and relaxation of the environment are much smaller than the time scale of the system of interest. As the coupling strength is very weak, this is a reasonable assumption which induces an important approximation in the derivation of the Lindbladian.

There are also other approximations which are only relevant in the mathematical derivation and hence are not mentioned here [9].

3.4.3. The spin-boson model

The spin-boson model is used as a model for open quantum systems in order to make it easy to take dissipation into account. It models the environment as a bath of bosons which behave like harmonic oscillators [13]. The Hamiltonian of the spin-boson model is given by:

$$H_T = (\epsilon\sigma_z + \Delta\sigma_x) \otimes I_E + I \otimes \sum_k \hbar\omega_k b_k^\dagger b_k + \sigma_z \otimes \sum_k g_k(b_k^\dagger + b_k) \tag{3.35}$$

where g_k and ω_k respectively denote the interaction strength and angular velocity of boson k. Furthermore, $b_k^\dagger$ and b_k are the creation and annihilation operators respectively, which are not relevant in this thesis. Using this model, the Lindblad equation can actually be solved exactly with $L_i = \sigma_z$. Take for example the Hamiltonian with $\Delta = 0$, then the Lindblad equation reduces to:

$$\begin{aligned}\dot{\rho} &= -\frac{i\epsilon}{\hbar}[\sigma_z, \rho] + g\sigma_z\rho\sigma_z^\dagger - \frac{g}{2}\sigma_z^\dagger\sigma_z\rho - \frac{g}{2}\rho\sigma_z^\dagger\sigma_z \\ &= -\frac{i\epsilon}{\hbar}[\sigma_z, \rho] + g\sigma_z\rho\sigma_z - g\rho\end{aligned} \tag{3.36}$$

where $\sigma_z\sigma_z^\dagger = \sigma_z^2 = I$ has been used. This leads to four ordinary differential equations:

$$\begin{aligned}\dot{\rho_{00}} &= 0 \\ \dot{\rho_{11}} &= 0 \\ \dot{\rho_{01}} &= (-\frac{2i\epsilon}{\hbar} - 2g)\rho_{01} \\ \dot{\rho_{10}} &= (\frac{2i\epsilon}{\hbar} - 2g)\rho_{10}\end{aligned} \tag{3.37}$$

where $\rho = \left(\begin{smallmatrix}\rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11}\end{smallmatrix}\right)$. Assuming an initial state $\rho(0) = \frac{1}{2}\left(\begin{smallmatrix}1 & 1 \\ 1 & 1\end{smallmatrix}\right)$, the density matrix becomes:

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & \exp((-\frac{2i\epsilon}{\hbar} - 2g)t) \\ \exp((\frac{2i\epsilon}{\hbar} - 2g)t) & 1 \end{pmatrix}. \tag{3.38}$$

It is clear that the off-diagonal elements, which are responsible for coherence, decay over time (in quantum mechanics, every object has wave-like properties which allows the introduction of quantum coherence).

4

Density matrices of closed quantum systems

In this chapter, the outcomes of the calculations on closed quantum systems with alternating Hamiltonians will be provided, aiming to answer the research question as stated in the introduction. Multiple graphs of $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ in different situations are also provided as a visual interpretation of the results.

4.1. Results of a series of measurements alternated with free evolution

For the first part, a qubit is considered which alternately is influenced by a magnetic field and a detector. The difficulty in determining the density matrix for general times now lies in the fact that different Hamiltonians alternate. To this end, define the time steps $t_{ev} + t_m$ where t_{ev} is the duration of the qubit evolving under the effect of the Hamiltonian $H_0 = B\sigma_z$ and t_m is the duration of the qubit being measured by the detector which causes an interaction Hamiltonian $H_{int} = g(t)q\sigma_x$. The time steps always start with free evolution and subsequently, the measurement takes place. This results in the following time-evolution operator:

$$\begin{aligned} U(t_{ev}+t_m) &= \exp(-\frac{i}{\hbar}\int_{t_{ev}}^{t_{ev}+t_m} H_{int}dt)\exp(-\frac{i}{\hbar}\int_0^{t_{ev}} H_0 dt) \\ &= \begin{pmatrix} \cos(\frac{1}{\hbar}gqt_m) & -i\sin(\frac{1}{\hbar}gqt_m) \\ -i\sin(\frac{1}{\hbar}gqt_m) & \cos(\frac{1}{\hbar}gqt_m) \end{pmatrix} \begin{pmatrix} \exp(-\frac{i}{\hbar}Bt_{ev}) & 0 \\ 0 & \exp(\frac{i}{\hbar}Bt_{ev}) \end{pmatrix} \\ &= \begin{pmatrix} \exp(-i\alpha)\cos(\beta) & -i\exp(i\alpha)\sin(\beta) \\ -i\exp(-i\alpha)\sin(\beta) & \exp(i\alpha)\cos(\beta) \end{pmatrix} \end{aligned} \tag{4.1}$$

where equations 3.14 and 3.18 and the following substitutions have been used:

$$\alpha \equiv \frac{1}{\hbar}Bt_{ev},\ \beta \equiv \frac{1}{\hbar}gqt_m. \tag{4.2}$$

Assuming for example an initial state of the qubit $|0\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$ and an initial state of the detector as in equation 3.16, the state at time $nt_{ev} + nt_m\ \forall n \in \mathbb{N}$ is given by:

$$|\Phi(nt_{ev}+nt_m)\rangle = \frac{1}{2^{1/2}(\pi\sigma)^{1/4}}\exp(-\frac{(q-q_0)^2}{2\sigma})U^n\begin{pmatrix}1\\1\end{pmatrix}. \tag{4.3}$$

This allows for the calculation of the total density matrix at time $nt_{ev} + nt_m$:

$$\begin{aligned} \rho_T(nt_{ev}+nt_m) &= |\Phi(nt_{ev}+nt_m)\rangle\langle\Phi(nt_{ev}+nt_m)| = \frac{1}{2\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma})U^n\begin{pmatrix}1&1\\1&1\end{pmatrix}U^{n^*} \\ &= \frac{1}{2\sqrt{\pi\sigma}}\exp\left(-\frac{(q-q_0)^2}{\sigma}\right)\begin{pmatrix}(U_{n,00}+U_{n,01})(U^*_{n,00}+U^*_{n,01}) & (U_{n,00}+U_{n,01})(U^*_{n,10}+U^*_{n,11}) \\ (U_{n,10}+U_{n,11})(U^*_{n,00}+U^*_{n,01}) & (U_{n,10}+U_{n,11})(U^*_{n,10}+U^*_{n,11})\end{pmatrix} \end{aligned} \tag{4.4}$$

where $U^n = \left(\begin{smallmatrix} U_{n,00} & U_{n,01} \\ U_{n,10} & U_{n,11} \end{smallmatrix}\right)$ and $\cdot^*$ denotes the conjugate. It is now clear that, in order to find the density matrix, the elements of the matrix U^n are required. This is where equation 2.7 is used; the matrix $U(t_{ev}+t_m)$ from equation 4.1 has two distinct eigenvalues, which implies that U^n can be expressed as a linear combination of U and the identity matrix I. Since $U(t_{ev}+t_m)$ is a 2×2 matrix, the eigenvalues of $U(t_{ev}+t_m)$ can be found:

$$\lambda_\pm = \frac{1}{2}\exp(-i\alpha)\left(\exp(2i\alpha)\cos(\beta) \pm \sqrt{\cos^2(\beta)(1+\exp(2i\alpha))^2 - 4\exp(2i\alpha)} + \cos(\beta)\right). \tag{4.5}$$

This leads to the calculation of the elements of U^n which are all written as fractions:

$$U_{n,00} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}\exp(-i\alpha)\cos(\beta) - \lambda_+\lambda_-\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} = \frac{U_{n,001}}{U_{n,002}} \tag{4.6}$$

$$U_{n,01} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}(-i\exp(i\alpha)\sin(\beta)) = \frac{U_{n,011}}{U_{n,012}} \tag{4.7}$$

$$U_{n,10} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}(-i\exp(-i\alpha)\sin(\beta)) = \frac{U_{n,101}}{U_{n,102}} \tag{4.8}$$

$$U_{n,11} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}\exp(i\alpha)\cos(\beta) - \lambda_+\lambda_-\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} = \frac{U_{n,111}}{U_{n,112}} \tag{4.9}$$

$$\begin{aligned} U_{n,001} =& ((\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta)) - (\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n)2^{-n+1} + 4\cos(\beta)\exp(-i\alpha)((\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) \\ &- \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} + \cos(\beta))^n - (\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) \\ &+ \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} + \cos(\beta))^n)2^{-n} \end{aligned} \tag{4.10}$$

$$\begin{aligned} U_{n,002} =& -2(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} + \cos(\beta)) \\ &+ 2(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} + \cos(\beta)) \end{aligned} \tag{4.11}$$

$$\begin{aligned} U_{n,011} =& -2i\exp(i\alpha)(-(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n + (\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n)2^{-n}\sin(\beta) \end{aligned} \tag{4.12}$$

$$\begin{aligned} U_{n,012} =& \exp(-i\alpha)(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta)) - \exp(-i\alpha)(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta)) \end{aligned} \tag{4.13}$$

$$\begin{aligned} U_{n,101} =& 2i\exp(-i\alpha)((\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n - (\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta))^n)2^{-n}\sin(\beta) \end{aligned} \tag{4.14}$$

$$\begin{aligned} U_{n,102} =& \exp(-i\alpha)(\exp(2i\alpha)\cos(\beta) + \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta)) - \exp(-i\alpha)(\exp(2i\alpha)\cos(\beta) - \sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha) + \cos^2(\beta)(\exp(4i\alpha)+1)} \\ &+ \cos(\beta)) \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
U_{n,111} =&((\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)+\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}\\
&+\cos(\beta))^n(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)-\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}\\
&+\cos(\beta))-(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)+\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}\\
&+\cos(\beta))(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)-\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}\\
&+\cos(\beta))^n)2^{-n+1}+4\cos(\beta)\exp(i\alpha)((\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)\\
&-\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}+\cos(\beta))^n-(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)\\
&+\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}+\cos(\beta))^n)2^{-n}
\end{aligned}
\tag{4.16}
$$

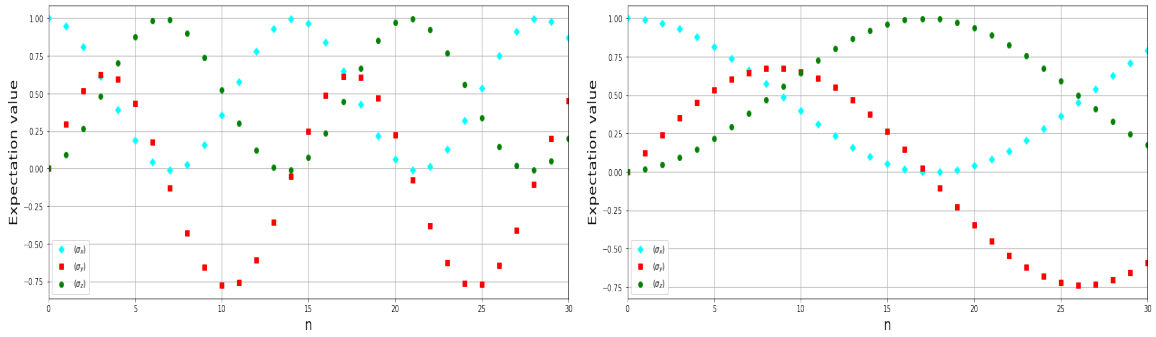
$$
\begin{aligned}
U_{n,112} =&-2(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)+\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}+\cos(\beta))\\
&+2(\exp(-i\alpha))(\exp(2i\alpha)\cos(\beta)-\sqrt{(2\cos^2(\beta)-4)\exp(2i\alpha)+\cos^2(\beta)(\exp(4i\alpha)+1)}+\cos(\beta)).
\end{aligned}
\tag{4.17}
$$

However, in order to find the density matrix of the qubit, q has to be integrated out again. Unfortunately, this is not possible analytically, since β depends on q. Similar to subsection 3.3.3, it is possible when $\sigma \to 0$, which results in $\frac{1}{\sqrt{\pi\sigma}}\exp(-\frac{(q-q_0)^2}{\sigma}) \to \delta(q-q_0)$. The integral is then easy to solve and results in the following density matrix:

$$
\rho(nt_{ev}+nt_m) = \frac{1}{2}\begin{pmatrix}(U'_{n,00}+U'_{n,01})(U'^{*}_{n,00}+U'^{*}_{n,01}) & (U'_{n,00}+U'_{n,01})(U'^{*}_{n,10}+U'^{*}_{n,11})\\ (U'_{n,10}+U'_{n,11})(U'^{*}_{n,00}+U'^{*}_{n,01}) & (U'_{n,10}+U'_{n,11})(U'^{*}_{n,10}+U'^{*}_{n,11})\end{pmatrix} \tag{4.18}
$$

where $\cdot'$ implies that β should be replaced by $\beta' \equiv \frac{1}{\hbar}gq_0t_m$. Note that this formula only holds for discrete time steps; the method to find the density matrix at any time has been explained on page 16 of the report of Kramers [8].

In order to visualise these equations, graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ over time are provided using equation 3.4. These graphs can be made for any value of α and β' which are proportional to respectively t_{ev} and t_m (as stated in equation 4.2). Figure 4.1 contains these graphs for a few cases of equal α and β'; in particular, the cases for $\frac{\pi}{20}$ and $\frac{\pi}{50}$ have been illustrated.



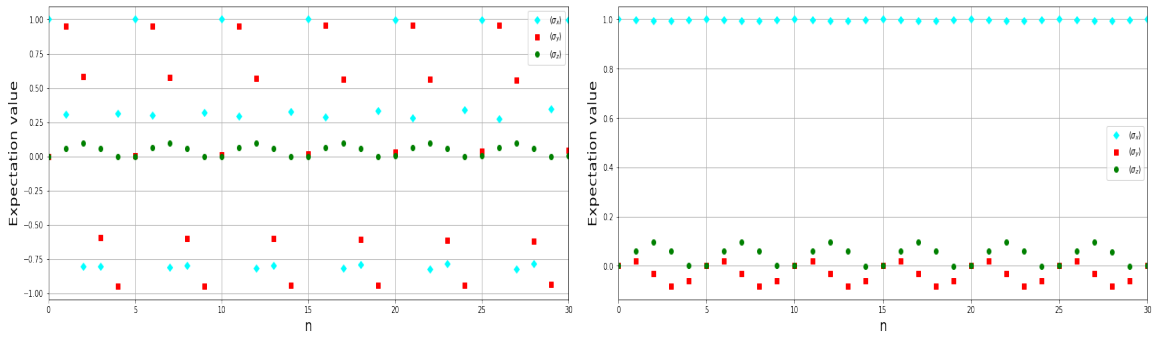
(a) $\alpha = \beta' = \pi/20$.

(b) $\alpha = \beta' = \pi/50$.

Figure 4.1: Graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$ (cyan), $\langle\sigma_y\rangle$ (red) and $\langle\sigma_z\rangle$ (green) over time for different cases of equal $\alpha \equiv \frac{1}{\hbar}Bt_{ev}$ and $\beta' \equiv \frac{1}{\hbar}gq_0t_m$. The horizontal axis contains the time steps n such that the total time $t = nt_{ev} + nt_m$ where t_{ev} is the duration of the qubit experiencing free evolution which is continued by a time period of t_m where weak measurements are being performed on the qubit. The cases that are being considered here are when α and β' take on the values of (a) $\frac{\pi}{20}$ and (b) $\frac{\pi}{50}$.

As the time steps become smaller, the graphs look more like continuous functions. This makes sense since, over time, the expectation values are computed more often. All these graphs also show a clear oscillatory behaviour for any time step.

Besides the cases for equal α and β', another interesting case would be when $t_m << t_{ev}$ or $t_{ev} << t_m$. A few of these cases have been illustrated in figure 4.2 to show these effects.



(a) $\alpha = \pi/5;\ \beta' = \pi/100.$ **(b)** $\alpha = \pi/100;\ \beta' = \pi/5.$

Figure 4.2: Graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$ (cyan), $\langle\sigma_y\rangle$ (red) and $\langle\sigma_z\rangle$ (green) over time for different cases of $\alpha \equiv \frac{1}{\hbar}Bt_{ev}$ and $\beta' \equiv \frac{1}{\hbar}gq_0t_m$ where $t_m << t_{ev}$ or $t_{ev} << t_m$. The horizontal axis contains the time steps n such that the total time $t = nt_{ev} + nt_m$ where t_{ev} is the duration of the qubit experiencing free evolution which is continued by a time period of t_m where weak measurements are being performed on the qubit. The cases that are being considered here are (a) $\alpha = \frac{\pi}{5};\ \beta' = \frac{\pi}{100}$ and (b) $\alpha = \frac{\pi}{100};\ \beta' = \frac{\pi}{5}$.

The subfigures 4.2a and 4.2b show the effects of a small perturbation of respectively weak measurements and free evolution since the difference between α and β' is very large. These graphs can be compared to the cases of only free evolution or only weak measurements. When only free evolution is being applied, the graph looks almost the same as in subfigure 4.2a, but then $\langle\sigma_z\rangle = 0$. Furthermore, the graph corresponding to only weak measurements is very similar to the graph from subfigure 4.2b, but then $\langle\sigma_x\rangle = 1$ and $\langle\sigma_y\rangle = \langle\sigma_z\rangle = 0$.

4.2. Results of a series of measurements during free evolution alternated with only free evolution

For the second part, a qubit is considered which also experiences free evolution while being measured. The time steps are again defined as $t_{ev} + t_m$ where t_{ev} is the duration of the only qubit evolving under the effect of the Hamiltonian $H_0 = B\sigma_z$ and t_m is the duration of the qubit being measured while experiencing free evolution; this process is described by the Hamiltonian $H = H_0 + H_{int} = B\sigma_z + gq\sigma_x$. The time steps always start with only free evolution and thereafter the measurement takes place while the free evolution continues. This results in the following time-evolution operator:

$$\begin{aligned} U(t_{ev}+t_m) &= \exp(-\frac{i}{\hbar}\int_{t_{ev}}^{t_{ev}+t_m} H dt)\exp(-\frac{i}{\hbar}\int_0^{t_{ev}} H_0 dt) \\ &= \begin{pmatrix} \cos(\gamma) - iB\xi\sin(\gamma) & -igq\xi\sin(\gamma) \\ -igq\xi\sin(\gamma) & \cos(\gamma) + iB\xi\sin(\gamma) \end{pmatrix}\begin{pmatrix} \exp(-i\alpha) & 0 \\ 0 & \exp(i\alpha) \end{pmatrix} \\ &= \begin{pmatrix} \exp(-i\alpha)(\cos(\gamma) - iB\xi\sin(\gamma)) & -igq\xi\exp(i\alpha)\sin(\gamma) \\ -igq\xi\exp(-i\alpha)\sin(\gamma) & \exp(i\alpha)(\cos(\gamma) + iB\xi\sin(\gamma)) \end{pmatrix} \end{aligned} \tag{4.19}$$

where equations 3.14 and 3.23 and the following substitutions have been used:

$$\alpha \equiv \frac{1}{\hbar}Bt_{ev},\ \gamma \equiv \frac{1}{\hbar}t_m\sqrt{B^2+g^2q^2},\ \xi \equiv \frac{1}{\sqrt{B^2+g^2q^2}}. \tag{4.20}$$

Assuming the initial state $|\Phi(0)\rangle = \phi(0)|\psi(0)\rangle$ with $\phi(0)$ given as in equation 3.16 and $|\psi(0)\rangle = |0\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$ and using the same method as in the previous section, the total density matrix for general discrete times $nt_{ev} + nt_m\ \forall n \in \mathbb{N}$ can be found using equation 4.4. To this end, it is necessary to find the elements of the matrix $U^n = \binom{U_{n,00}\ U_{n,01}}{U_{n,10}\ U_{n,11}}$ using equation 2.7, hence the eigenvalues of the time-evolution operator are required which can be computed since it is a 2×2 matrix:

$$\lambda_\pm = \frac{1}{2}((-iB\xi\sin(\gamma) \pm \sqrt{D} + \cos(\gamma))\exp(-i\alpha) + \exp(i\alpha)(\cos(\gamma) + iB\xi\sin(\gamma))) \tag{4.21}$$

where

$$D = ((4g^2q^2\xi^2 + 2B^2\xi^2 - 2)\cos^2(\gamma) - 2\xi^2(2g^2q^2 + B^2))\exp(2i\alpha) + ((B^2\xi^2 + 1)\cos^2(\gamma) + 2\,iB\cos(\gamma)\sin(\gamma)\xi - B^2\xi^2)\exp(4i\alpha) + (B^2\xi^2 + 1)\cos^2(\gamma) - 2\,iB\cos(\gamma)\sin(\gamma)\xi - B^2\xi^2. \tag{4.22}$$

This leads to the calculation of the elements of U^n:

$$U_{n,00} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}\exp(-i\alpha)(\cos(\gamma) - iB\xi\sin(\gamma)) - \lambda_+\lambda_-\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} = \frac{U_{n,001}}{2\sqrt{D}} \tag{4.23}$$

$$U_{n,01} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}(-igq\xi\exp(i\alpha)\sin(\gamma)) = \frac{U_{n,011}}{\sqrt{D}} \tag{4.24}$$

$$U_{n,10} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}(-igq\xi\exp(-i\alpha)\sin(\gamma)) = \frac{U_{n,101}}{\sqrt{D}} \tag{4.25}$$

$$U_{n,11} = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}\exp(i\alpha)(\cos(\gamma) + iB\xi\sin(\gamma)) - \lambda_+\lambda_-\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} = \frac{U_{n,111}}{2\sqrt{D}} \tag{4.26}$$

$$U_{n,001} = (2^{-n}\sqrt{D} + \cos(\gamma)(\exp(2i\alpha) - 1)2^{-n} + i(\exp(2i\alpha) + 1)\sin(\gamma)\xi B2^{-n})((-\sqrt{D} + (\cos(\gamma) + iB\xi\sin(\gamma))\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n - (-2^{-n}\sqrt{D} + \cos(\gamma)(\exp(2i\alpha) - 1)2^{-n} + i(\exp(2i\alpha) + 1)\sin(\gamma)\xi B2^{-n})((\sqrt{D} + (\cos(\gamma) + iB\xi\sin(\gamma))\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n \tag{4.27}$$

$$U_{n,011} = igq\xi\exp(2i\alpha)2^{-n}(-((\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n + ((-\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n)\sin(\gamma) \tag{4.28}$$

$$U_{n,101} = igq\xi 2^{-n}(-((\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n + ((-\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n)\sin(\gamma) \tag{4.29}$$

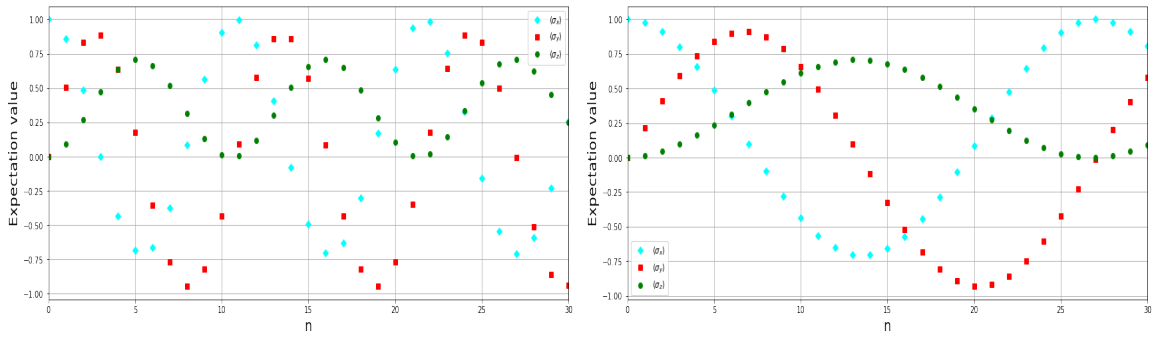
$$U_{n,111} = -(-2^{-n}\sqrt{D} + \cos(\gamma)(\exp(2i\alpha) - 1)2^{-n} + (\exp(2i\alpha) + 1)\sin(\gamma)\xi B2^{-n}i)((-\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n - (2^{-n}\sqrt{D} + \cos(\gamma)(\exp(2i\alpha) - 1)2^{-n} + (\exp(2i\alpha) + 1)\sin(\gamma)\xi B2^{-n}i)((\sqrt{D} + (\cos(\gamma) + B\xi\sin(\gamma)i)\exp(2i\alpha) - iB\xi\sin(\gamma) + \cos(\gamma))\exp(-i\alpha))^n \tag{4.30}$$

Again, q has to be integrated out in order to find the density matrix of the qubit at time $nt_{ev} + nt_m$, which is once again only possible when $\sigma \to 0$. The density matrix of the qubit at time $nt_{ev} + nt_m$ then becomes

$$\rho(nt_{ev} + nt_m) = \frac{1}{2}\begin{pmatrix}(U'_{n,00} + U'_{n,01})(U'^*_{n,00} + U'^*_{n,01}) & (U'_{n,00} + U'_{n,01})(U'^*_{n,10} + U'^*_{n,11}) \\ (U'_{n,10} + U'_{n,11})(U'^*_{n,00} + U'^*_{n,01}) & (U'_{n,10} + U'_{n,11})(U'^*_{n,10} + U'^*_{n,11})\end{pmatrix} \tag{4.31}$$

where $\cdot'$ implies that γ should be replaced by γ', ξ should be replaced by ξ' and D should be replaced by D'; these accents indicate that every q in these formulas is replaced by q_0. Note that this formula only holds for discrete time steps; the method to find the density matrix at any time is explained on page 26 of the report of Kramers [8].

Similar to the previous section, graphs will be provided of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ over time using equation 3.4. As it does not matter what value the constants B, g and q_0 take, these are all chosen to be 1, from which it follows that $\xi' = \frac{1}{\sqrt{2}}$. Figure 4.3 contains the graphs for the cases where α and γ' are both equal to $\frac{\pi}{20}$ and $\frac{\pi}{50}$.

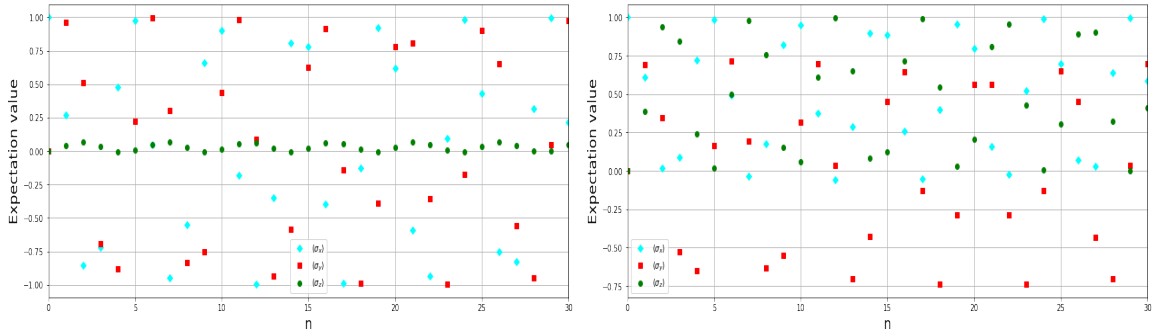


(a) $\alpha = \gamma' = \pi/20$.

(b) $\alpha = \gamma' = \pi/50$.

Figure 4.3: Graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$ (cyan), $\langle\sigma_y\rangle$ (red) and $\langle\sigma_z\rangle$ (green) over time for different cases of equal $\alpha \equiv \frac{1}{\hbar}Bt_{ev}$ and $\gamma' \equiv \frac{1}{\hbar}t_m\sqrt{B^2+g^2q^2}$. The horizontal axis contains the time steps n such that the total time $t = nt_{ev} + nt_m$ where t_{ev} is the duration of the qubit experiencing free evolution which is continued by a time period of t_m where weak measurements are being performed on the qubit while the qubit is experiencing free evolution. The cases that are being considered here are when α and γ' take on the values of (a) $\frac{\pi}{20}$ and (b) $\frac{\pi}{50}$.

Furthermore, the cases $t_m << t_{ev}$ and $t_{ev} << t_m$ will be analysed again. Some examples of these situations are illustrated in figure 4.4.



(a) $\alpha = \pi/5; \gamma' = \pi/100$.

(b) $\alpha = \pi/100; \gamma' = \pi/5$.

Figure 4.4: Graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$ (cyan), $\langle\sigma_y\rangle$ (red) and $\langle\sigma_z\rangle$ (green) over time for different cases of $\alpha \equiv \frac{1}{\hbar}Bt_{ev}$ and $\gamma' \equiv \frac{1}{\hbar}t_m\sqrt{B^2+g^2q^2}$ where $t_m << t_{ev}$ or $t_{ev} << t_m$. The horizontal axis contains the time steps n such that the total time $t = nt_{ev} + nt_m$ where t_{ev} is the duration of the qubit experiencing free evolution which is continued by a time period of t_m where weak measurements are being performed on the qubit while the qubit is experiencing free evolution. The cases that are being considered here are (a) $\alpha = \frac{\pi}{5}; \gamma' = \frac{\pi}{100}$ and (b) $\alpha = \frac{\pi}{100}; \gamma' = \frac{\pi}{5}$.

These graphs show very similar behaviour to the graphs from the previous subsection. The only difference this time is that subfigure 4.4b is still very hectic in comparison with subfigure 4.2b. This makes sense since both subfigure 4.2a and 4.4a almost neglect the part where weak measurements are being performed. This results in both graphs being very similar to the case of only free evolution. On the other hand, both subfigure 4.2b and 4.4b almost neglect the part where only free evolution takes place. In the previous section, this results in weak measurements dominating the experiment. However, in this section, this results in weak measurements during free evolution dominating the experiment, which explains the difference.

5

Density matrices of open quantum systems

In this chapter, the outcomes of the calculations on open quantum systems with one continuous Hamiltonian will be provided, aiming to answer the research question as stated in the introduction. Multiple graphs of $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ in different situations are also provided as a visual interpretation of the results.

When considering open quantum systems, finding an exact solution for the density matrix is often not possible, which necessitates the use of numerical methods to solve the system dynamics. The Python library QuTiP [6] is used to solve these processes using the Lindbladian and the results of the different situations will be compared with section 3.3, where the corresponding closed systems have been discussed.

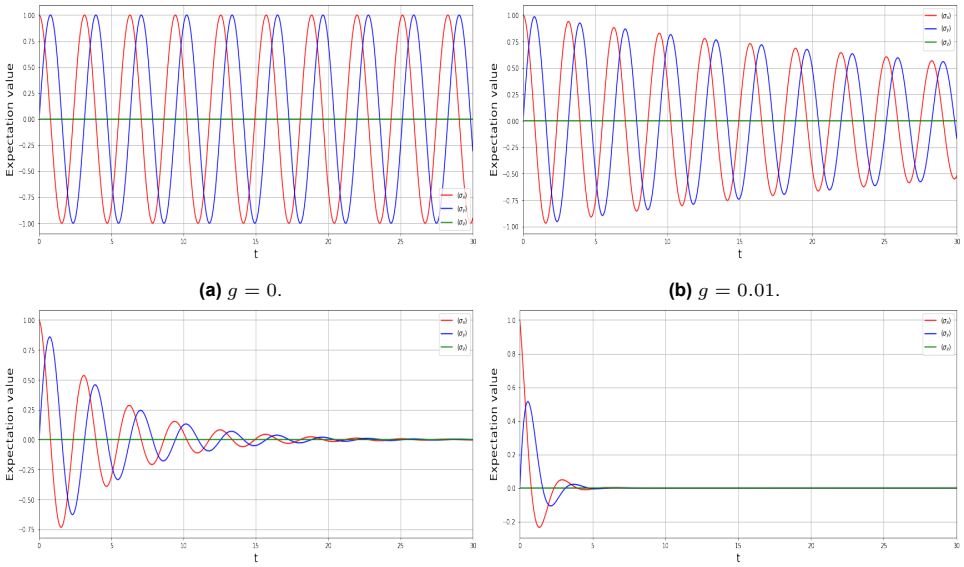
When using QuTiP, only 2 inputs are necessary. The first one is the Hamiltonian that describes the qubit alone (the first term in equation 3.28), which determines the first term in the Lindbladian. If there was no environment, i.e. $g_i = 0$, the Lindbladian would reduce to the von Neumann equation which describes the evolution of the density matrix in a closed quantum system. Secondly, a list of the collapse operators C_i is necessary, defined as $C_i = \sqrt{g_i}L_i$. These operators fully describe the interaction of the system of interest with the environment. Using the spin-boson model, the Lindblad operators are equal to the Pauli spin matrix σ_z. As mentioned in subsection 3.4.3, the spin-boson model actually allows an exact calculation of the density matrix using the Lindbladian, but here QuTiP will be used to demonstrate how the density matrix can be found numerically when the used model has no exact solution. The used Python code can be found in appendix B.

5.1. Results of a Hamiltonian of the form $H = a\sigma_z$ where $a \neq 0$

First, consider the case where the system Hamiltonian is in the form $H = a\sigma_z$ where $a \neq 0$. Solving the corresponding Lindblad equations from equation 3.33 yields the graphs presented in figure 5.1 on the next page. In these graphs, it can be seen that $\langle\sigma_z\rangle = 0$ for all times and $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ show oscillatory behaviour, but converge to 0 when $g > 0$. The value for a is chosen to be 1, which does not change the result since a only determines the frequency of the oscillations.

5.2. Results of a Hamiltonian of the form $H = b\sigma_x$ where $b \neq 0$

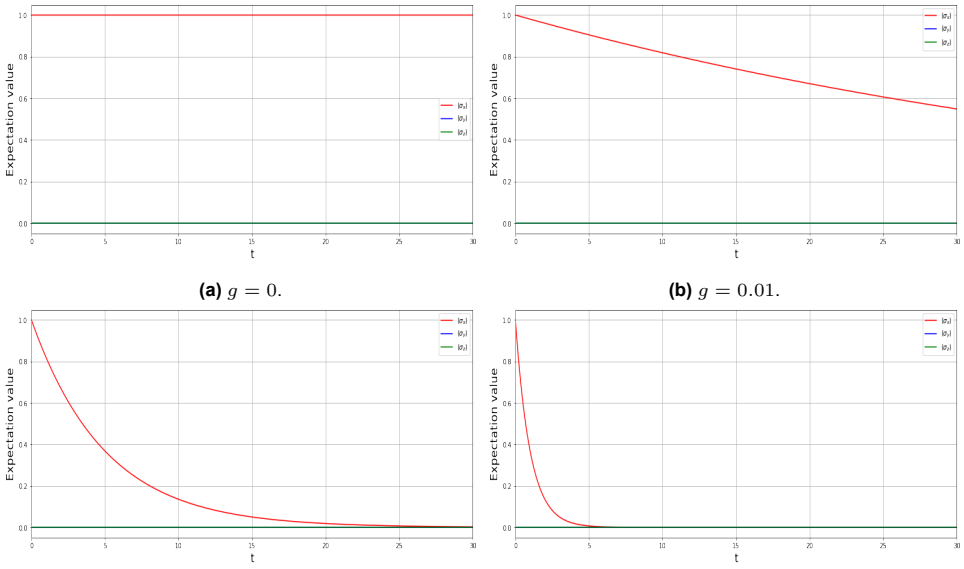
Next, consider the case where the system Hamiltonian is in the form $H = b\sigma_x$ where $b \neq 0$. Solving the corresponding Lindblad equations yields the graphs presented in figure 5.2 on the next page. In these graphs, it can be seen that $\langle\sigma_y\rangle = \langle\sigma_z\rangle = 0$ for all times and $\langle\sigma_x\rangle = 1$ for all times when $g = 0$, but decays to 0 when $g > 0$. The value for b is chosen to be 1, which does not change the result since the graph does not change for different values of b.



(a) $g = 0.$ **(b)** $g = 0.01.$

(c) $g = 0.1.$ **(d)** $g = 0.5.$

Figure 5.1: Graphs of the expectation values of the Pauli spin matrices $\langle \sigma_x \rangle$ (red), $\langle \sigma_y \rangle$ (blue) and $\langle \sigma_z \rangle$ (green) over time for different values of the interaction strength g in an open quantum system (only the first graph corresponds to a closed quantum system). The system is solved using the Lindblad equation with system Hamiltonian $H = \sigma_z$ and collapse operator $\sqrt{g}\sigma_z$. The cases that are being considered here are (a) $g = 0$, (b) $g = 0.01$, (c) $g = 0.1$ and (d) $g = 0.5$.



(a) $g = 0.$ **(b)** $g = 0.01.$

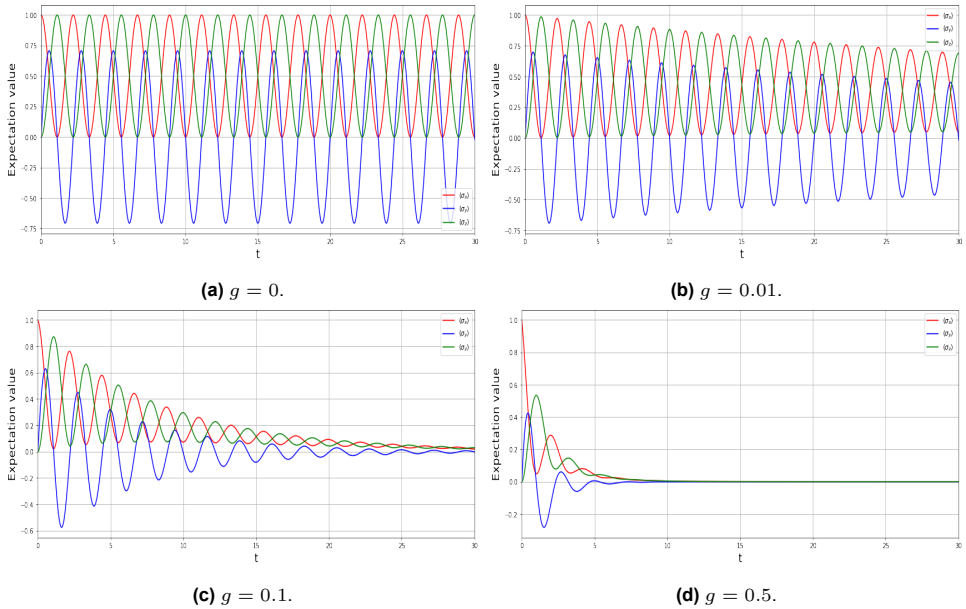
(c) $g = 0.1.$ **(d)** $g = 0.5.$

Figure 5.2: Graphs of the expectation values of the Pauli spin matrices $\langle \sigma_x \rangle$ (red), $\langle \sigma_y \rangle$ (blue) and $\langle \sigma_z \rangle$ (green) over time for different values of the interaction strength g in an open quantum system (only the first graph corresponds to a closed quantum system). The system is solved using the Lindblad equation with system Hamiltonian $H = \sigma_x$ and collapse operator $\sqrt{g}\sigma_z$. The cases that are being considered here are (a) $g = 0$, (b) $g = 0.01$, (c) $g = 0.1$ and (d) $g = 0.5$.

There is a clear difference between these two sections. Figure 5.1 shows a oscillatory behaviour while figure 5.2 shows a very simple behaviour. However, this is as expected since the spin matrices σ_x and σ_z have different effects on the qubit and this is also in accordance with the theory from subsections 3.3.1 and 3.3.2.

5.3. Results of a Hamiltonian of the form $H = a\sigma_z + b\sigma_x$ where $a, b \neq 0$

Finally, consider the case where the system Hamiltonian is in the form $H = a\sigma_z + b\sigma_x$ where $a, b \neq 0$. Solving the corresponding Lindblad equations yields the graphs presented in figure 5.3.



(a) $g = 0$. **(b)** $g = 0.01$. **(c)** $g = 0.1$. **(d)** $g = 0.5$.

Figure 5.3: Graphs of the expectation values of the Pauli spin matrices $\langle\sigma_x\rangle$ (red), $\langle\sigma_y\rangle$ (blue) and $\langle\sigma_z\rangle$ (green) over time for different values of the interaction strength g in an open quantum system (only the first graph corresponds to a closed quantum system). The system is solved using the Lindblad equation with system Hamiltonian $H = \sigma_z + \sigma_x$ and collapse operator $\sqrt{g}\sigma_z$. The cases that are being considered here are (a) $g = 0$, (b) $g = 0.01$, (c) $g = 0.1$ and (d) $g = 0.5$.

In these graphs, it can be seen that all expectation values show oscillatory behaviour and decay to 0 when $g > 0$. The values for a and b are chosen to be 1; changing these values can change the amplitude, frequency and position of the oscillations, but it does not change the process.

The graphs in these sections corresponding to $g = 0$, should be exactly the same as when $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ are computed using equation 3.4 and the density matrices from subsections 3.3.1, 3.3.2 and 3.3.3.

5.4. Deriving a density matrix from $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$

In the last three sections, the focus was on finding the graphs for $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$, but one question remains: how can one find the density matrix of such a system? Writing $\rho = \left(\begin{smallmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{smallmatrix}\right)$ and using equations 3.3 and 3.4 for the spin matrices, a system of 4 equations can be found:

$$\rho_{00} + \rho_{11} = 1 \tag{5.1}$$

$$\rho_{01} + \rho_{10} = \langle\sigma_x\rangle \tag{5.2}$$

$$i(\rho_{01} - \rho_{10}) = \langle\sigma_y\rangle \tag{5.3}$$

$$\rho_{00} - \rho_{11} = \langle\sigma_z\rangle \tag{5.4}$$

which can be solved:

$$\rho = \begin{pmatrix} \frac{1+\langle\sigma_z\rangle}{2} & \frac{\langle\sigma_x\rangle - i\langle\sigma_y\rangle}{2} \\ \frac{\langle\sigma_x\rangle + i\langle\sigma_y\rangle}{2} & \frac{1-\langle\sigma_z\rangle}{2} \end{pmatrix}. \tag{5.5}$$

This shows that the density matrix of the system can be computed when the formulas for $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ are known (which can be determined from the graphs).

6

Discussion

This chapter will provide a critical reflection on the obtained results and will elaborate on their significance.

In the case of closed quantum systems, the total density matrices of the system (the qubit and the detector) were found succesfully, but when it was required to integrate over q, the problem arises that this is not possible analytically. Evaluating the density matrix of the qubit was therefore only possible in the specific case $\sigma \to 0$ which narrows down the applicability of these results. Solving these integrals in general will require the use of numerical methods.

Looking into the figures for equal time steps, it is clear that the expectation values of the Pauli spin matrices look more like sinusoids for smaller time scales; this is as expected, since the values are evaluated more quickly after another. Besides the chosen values of $\frac{\pi}{5}$, $\frac{\pi}{10}$, $\frac{\pi}{20}$ and $\frac{\pi}{50}$, one can of course choose any value, but the behaviour that the graphs show will remain the same. For the graphs where $t_m << t_{ev}$ or $t_{ev} << t_m$, it is expected that those are very similar to respectively $t_m = 0$ and $t_{ev} = 0$ which is also verified.

Next, in the case of open quantum systems, one should be more careful when validating the results. Since the Lindblad equation and a model for the environment are used, a lot of approximations and assumptions have been made. Especially the first assumption for the Lindblad equation is restricting: it assumes that the interaction is 'weak' and that the environment is 'much larger' than the system of interest. Unfortunately, there is no useful quantitative description of these assumptions, which makes it harder to validate results. Analysing for example the graphs in chapter 5 for the cases $g = 0.5$, it is clear that the interaction is already much stronger compared to the other cases. This implies the possibility that it is actually not correct to use the Lindblad equation here, which limits the applicability of this method, since it only works for values of g under a certain unknown threshold. Only the case $g = 0$ (which is equivalent to a closed system since there is no interaction) is definitely correct and can be compared with the equations presented in chapter 3.

7

Conclusion

The aim of this thesis was to find the density matrix of a single qubit in closed and open quantum systems. In the case of a closed system, a sequence of two alternating processes corresponding to different Hamiltonians is considered where both processes take a fixed amount of time. In the case of open systems, where dissipation is taken into account, only one continuous process is considered with a fixed Hamiltonian.

The difficulty in the case of closed systems was mainly due to massive calculations, but this was solved by using a formula for a 2×2 matrix with 2 distinct eigenvalues raised to the power n for any $n \in \mathbb{N}$. This allowed for the computation of the density matrix in the case of any sequence of alternating processes at discrete time steps. Since these equations are very long, they are not reproduced here.

On the other hand, it is not even possible to solve open systems analytically due to the complexity of these calculations. Open quantum systems can be described by the Lindblad master equation, which can be solved numerically. However, this does require a model to describe the environment. Here, the spin-boson model has been used, which models the environment as a bath of bosons which behave like harmonic oscillators. Using this model, it is possible to solve open systems numerically, which resulted in graphs of the expectation values of the Pauli spin matrices. Combining these results with the fact that $\mathrm{Tr}(\rho) = 1$, resulted in a method to find the density matrix of the qubit.

Reviewing the results and the course of the research, some improvements will be suggested for follow-up research. Starting with the case of closed quantum systems, there have been multiple integrals which could only be solved analytically in a specific case; to gain more insights in these integrals for more general cases, one would need to use a numerical method to solve these. Furthermore, it could be an interesting attempt to solve even more difficult systems; consider for example alternating processes with different time periods for every time step.

In the case of open quantum systems, there are many possibilities for follow-up research. Starting with the Lindblad master equation, which depends on multiple assumptions and approximations, it could be interesting to work out the equation for systems where not all of those assumptions and approximations are viable. On the other hand, one can also attempt to solve an open quantum system without using the Lindblad equation. Furthermore, the spin-boson model is not the only model that is used to model dissipation; another commonly used model is the Caldeira-Leggett model, where the environment is modeled as an infinite sum of harmonic oscillators. Lastly, instead of only energy loss from the system to the environment, one can also take into account energy flowing from the environment to the system, making the system even more complex and allowing the introduction of new models.

Finally, the first assumption of the Lindblad equation which says that the interaction should be 'weak' needs a more quantitative description. This would facilitate determining whether a certain interaction strength allows the use of the Lindblad equation to determine the density matrix.

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A

The proof of equation 2.7

Theorem. Let A be a 2 by 2 matrix with distinct eigenvalues λ_- and λ_+ and. Then, $\forall n \in \mathbb{N}$, the formula for A^n is given by:

$$A^n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} A - \lambda_+ \lambda_- \frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} I \tag{A.1}$$

with I the 2 by 2 identity matrix.

Proof [10]**.** First, it's necessary to write A^2 as a linear combination of A and I; to this end, write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then:

$$\begin{aligned} A^2 &= \begin{pmatrix} a^2 + bc & ab + db \\ ac + dc & d^2 + bc \end{pmatrix} = \begin{pmatrix} a^2 + ad - ad + bc & ab + db \\ ac + dc & d^2 + ad - ad + bc \end{pmatrix} \\ &= \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} - \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (a+d)A - (ad - bc)I \end{aligned} \tag{A.2}$$

Next up, the eigenvalues of A can be found by solving

$$Det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = 0 \tag{A.3}$$

Solving this yields the following eigenvalues:

$$\lambda_\pm = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} \tag{A.4}$$

Now it can be noted that $a + d = \lambda_\pm + \lambda_\mp$ and $ad - bc = \lambda_\pm \lambda_\mp$ and this leads to the formula for A^2:

$$A^2 = (\lambda_\pm + \lambda_\mp)A - \lambda_\pm \lambda_\mp I \tag{A.5}$$

For the rest of the proof, a new matrix will be defined:

$$B_\pm \equiv \mp(A - \lambda_\pm I) \tag{A.6}$$

and then it is clear that

$$\lambda_\mp B_\pm + \lambda_\pm B_\mp = (\pm\lambda_\pm \mp \lambda_\mp)A \iff A = \frac{\lambda_\mp B_\pm + \lambda_\pm B_\mp}{\pm\lambda_\pm \mp \lambda_\mp} = \frac{\lambda_-}{\lambda_+ - \lambda_-} B_+ + \frac{\lambda_+}{\lambda_+ - \lambda_-} B_- \tag{A.7}$$

In the next two steps, the formula for A^2 from equation A.5 will be used to compute $B_\pm B_\pm$ and $B_\pm B_\mp$:

$$\begin{aligned} B_\pm B_\pm &= (\mp(A - \lambda_\pm I))(\mp(A - \lambda_\pm I)) = A^2 - 2\lambda_\pm A + \lambda_\pm^2 I \\ &= (\lambda_\pm + \lambda_\mp)A - \lambda_\pm \lambda_\mp I - 2\lambda_\pm A + \lambda_\pm^2 I \\ &= (\lambda_\mp - \lambda_\pm)A - (\lambda_\pm \lambda_\mp - \lambda_\pm^2)I = (\lambda_\mp - \lambda_\pm)(A - \lambda_\pm I) \\ &= \mp(\lambda_\mp - \lambda_\pm)(\mp(A - \lambda_\pm I)) = (\lambda_+ - \lambda_-)B_\pm \\ B_\pm B_\mp &= (\mp(A - \lambda_\pm I))(\pm(A - \lambda_\mp I)) = -(A^2 - (\lambda_\pm + \lambda_\mp)A + \lambda_\pm \lambda_\mp I) \\ &= -((\lambda_\pm + \lambda_\mp)A - \lambda_\pm \lambda_\mp I - (\lambda_\pm + \lambda_\mp)A + \lambda_\pm \lambda_\mp I) = 0 \end{aligned} \tag{A.8}$$

Subsequently, as $B_{\pm}B_{\pm} = (\lambda_+ - \lambda_-)B_{\pm}$, it can be shown using induction that, $\forall n \in \mathbb{N}$, the following holds:

$$B_{\pm}^n = (\lambda_+ - \lambda_-)^{n-1} B_{\pm} \tag{A.9}$$

Furthermore, since $B_{\pm}B_{\mp} = 0$, equation A.7 can be raised to the power n $\forall n \in \mathbb{N}$:

$$A = \frac{\lambda_-^n}{(\lambda_+ - \lambda_-)^n} B_+^n + \frac{\lambda_+^n}{(\lambda_+ - \lambda_-)^n} B_-^n \tag{A.10}$$

It is clear that equation A.9 can be applied to equation A.10 to yield the following result:

$$A = \frac{\lambda_-^n}{\lambda_+ - \lambda_-} B_+ + \frac{\lambda_+^n}{\lambda_+ - \lambda_-} B_- \tag{A.11}$$

Then finally, $B_+ = -(A - \lambda_+ I)$ and $B_- = A - \lambda_- I$ can be substituted back into equation A.11 to end up with the desired result:

$$\begin{aligned} A &= \frac{\lambda_-^n}{\lambda_+ - \lambda_-}(-A + \lambda_+ I) + \frac{\lambda_+^n}{\lambda_+ - \lambda_-}(A - \lambda_- I) \\ &= \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} A - \lambda_+ \lambda_- \frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} I \end{aligned} \tag{A.12}$$

□

B

Python code

B.1. Closed quantum systems

The Python code below has been used to plot the expectation values of the spin operators over time for closed quantum systems as described in chapter 4. The values of a and b can be changed depending on the time steps. All constants are chosen to be 1.

```
import numpy as np
import matplotlib.pyplot as plt

a = np.pi/5
b = np.pi/5
K1 = 1/2*np.exp(-1j*a)*(np.exp(2j*a)*np.cos(b) + np.sqrt(np.cos(b)**2*(1 + np.exp(2j*a))**2 - 4*np.exp(2j*a)) + np.cos(b))
L1 = 1/2*np.exp(-1j*a)*(np.exp(2j*a)*np.cos(b) - np.sqrt(np.cos(b)**2*(1 + np.exp(2j*a))**2 - 4*np.exp(2j*a)) + np.cos(b))

n_vals = np.arange(0,31,1)
x_vals = np.array([])
y_vals = np.array([])
z_vals = np.array([])

for n in range(len(n_vals)):
    U_n01 = (K1**n - L1**n)/(K1 - L1)*-1j*np.exp(a*1j)*np.sin(b)
    U_n10 = (K1**n - L1**n)/(K1 - L1)*-1j*np.exp(-a*1j)*np.sin(b)
    U_n00 = (K1**n - L1**n)*np.exp(-1j*a)*np.cos(b)/(K1 - L1) - K1*L1*(K1**(n - 1) - L1**(n - 1))/(K1 - L1)
    U_n11 = (K1**n - L1**n)*np.exp(a*1j)*np.cos(b)/(K1 - L1) - K1*L1*(K1**(n - 1) - L1**(n - 1))/(K1 - L1)
    rho11 = 1/2*(U_n00 + U_n01)*np.conj(U_n00 + U_n01)
    rho12 = 1/2*(U_n00 + U_n01)*np.conj(U_n10 + U_n11)
    rho21 = 1/2*(U_n10 + U_n11)*np.conj(U_n00 + U_n01)
    rho22 = 1/2*(U_n10 + U_n11)*np.conj(U_n10 + U_n11)
    sigmax = rho12 + rho21
    sigmay = (rho12 - rho21)*1j
    sigmaz = rho11 - rho22
    x_vals = np.append(x_vals, sigmax)
    y_vals = np.append(y_vals, sigmay)
    z_vals = np.append(z_vals, sigmaz)

plt.figure(figsize = (14,6))
plt.plot(n_vals, x_vals, 'D', c='cyan')
plt.plot(n_vals, y_vals, 's', c='r')
plt.plot(n_vals, z_vals, 'o', c='g')
plt.xlabel("n", fontsize=18)
plt.ylabel("Expectation value", fontsize=18)
plt.xlim(n_vals[0], n_vals[-1])
plt.legend((r"$\langle \sigma_x \rangle $", r"$\langle \sigma_y \rangle $", r"$\langle \sigma_z \rangle $"))
plt.grid()
plt.show()
```

```
40
41 B = 1
42 q_0 = 1
43 g = 1
44 d = 1/np.sqrt(B**2+g**2*q_0**2)
45 E = ((4*d**2*g**2*q_0**2 + 2*B**2*d**2 - 2)*(np.cos(b))**2 - 2*d**2*(2*g**2*q_0**2 + B**2))*
       np.exp(2j*a) + ((B**2*d**2 + 1)*(np.cos(b))**2 + 2j*B*np.cos(b)*np.sin(b)*d - B**2*d**2)*
       np.exp(4j*a) + (B**2*d**2 + 1)*(np.cos(b))**2 - 2j*B*np.cos(b)*np.sin(b)*d - B**2*d**2
46 K1 = ((-B*d*np.sin(b)*1j + np.sqrt(E) + np.cos(b))*np.exp(-1j*a))/2 + np.exp(1j*a)*(np.cos(b)
       + B*d*np.sin(b)*1j)/2
47 L1 = ((-B*d*np.sin(b)*1j - np.sqrt(E) + np.cos(b))*np.exp(-1j*a))/2 + np.exp(1j*a)*(np.cos(b)
       + B*d*np.sin(b)*1j)/2
48
49 n_vals = np.arange(0,31,1)
50 x_vals = np.array([])
51 y_vals = np.array([])
52 z_vals = np.array([])
53
54 for n in range(len(n_vals)):
55     U_n01 = (K1**n - L1**n)/(K1 - L1)*-1j*g*q_0*d*np.exp(a*1j)*np.sin(b)
56     U_n10 = (K1**n - L1**n)/(K1 - L1)*-1j*g*q_0*d*np.exp(-a*1j)*np.sin(b)
57     U_n00 = (K1**n - L1**n)*np.exp(-1j*a)*(np.cos(b)-1j*B*d*np.sin(b))/(K1 - L1) - K1*L1*(K1
           **(n - 1) - L1**(n - 1))/(K1 - L1)
58     U_n11 = (K1**n - L1**n)*np.exp(1j*a)*(np.cos(b)+1j*B*d*np.sin(b))/(K1 - L1) - K1*L1*(K1
           **(n - 1) - L1**(n - 1))/(K1 - L1)
59     rho11 = 1/2*(U_n00 + U_n01)*np.conj(U_n00 + U_n01)
60     rho12 = 1/2*(U_n00 + U_n01)*np.conj(U_n10 + U_n11)
61     rho21 = 1/2*(U_n10 + U_n11)*np.conj(U_n00 + U_n01)
62     rho22 = 1/2*(U_n10 + U_n11)*np.conj(U_n10 + U_n11)
63     sigmax = rho12 + rho21
64     sigmay = (rho12 - rho21)*1j
65     sigmaz = rho11 - rho22
66     x_vals = np.append(x_vals, sigmax)
67     y_vals = np.append(y_vals, sigmay)
68     z_vals = np.append(z_vals, sigmaz)
69
70 plt.figure(figsize = (14,6))
71 plt.plot(n_vals, x_vals, 'D', c='cyan')
72 plt.plot(n_vals, y_vals, 's', c='r')
73 plt.plot(n_vals, z_vals, 'o', c='g')
74 plt.xlabel("n", fontsize=18)
75 plt.ylabel("Expectation value", fontsize=18)
76 plt.xlim(n_vals[0], n_vals[-1])
77 plt.legend((r"$\langle \sigma_x \rangle $", r"$\langle \sigma_y \rangle $", r"$\langle \
       sigma_z \rangle $"))
78 plt.grid()
79 plt.show()
```

B.2. Open quantum systems

The Python code below has been used to plot the expectation values of the spin operators over time for open quantum systems as described in chapter 5 [6]. The values of a and b can be changed depending on the desired hamiltonian defined in line 7. Furthermore, the value of g determines the strength of the coupling to the environment where $g = 0$ describes a closed quantum system.

```
1  import numpy as np
2  import matplotlib.pyplot as plt
3  from qutip import *
4
5  def qubit_integrate(a, b, g):
6
7      H = a * sigmaz() + b * sigmax() #system hamiltonian
8
9      c_ops = [np.sqrt(g) * (sigmaz())] #collapse operators
10
11     e_ops = [sigmax(), sigmay(), sigmaz()]
12
13     output = mesolve(H, psi0, tlist, c_ops, e_ops)
14
```

```
    return output.expect[0], output.expect[1], output.expect[2]

a = 1
b = 1
g = 0.1

psi0 = (basis(2,0)+basis(2,1)).unit() #intial state

tlist = np.linspace(0,15,200)

sx1, sy1, sz1 = qubit_integrate(a, b, g)

fig, ax = plt.subplots(figsize=(12,6))
ax.plot(tlist, np.real(sx1), 'r')
ax.plot(tlist, np.real(sy1), 'b')
ax.plot(tlist, np.real(sz1), 'g')
#plt.axhline(np.real(sx1[-1]), linestyle='dashed')
ax.set_xlim(tlist[0], tlist[-1])
ax.grid()
ax.legend((r"$\langle S_x \rangle $", r"$\langle S_y \rangle $", r"$\langle S_z \rangle $"))
ax.set_xlabel('Time')
ax.set_ylabel('Expectation value');
```