STELLINGEN

I

With enhanced intuitive knowledge about shell buckling, the designer should have an improved ability to foresee situations in which buckling might occur and thus modify a design to avoid it.

D. Bushnell, " Computerized buckling analysis of shells", Martinus Nijhoff Publishers, Dordrecht, 1985

II

There is still much to do, our knowledge of shell forms other than spherical and cylindrical is still relatively limited, and the problems of actual loadings, many of which are not easy to handle either theoretically or experimentally, need more attention.

E.E. Sechler, The historical development of shell research and design, in Thin Shell Structures, Edited by Y.C. Fung and E.E. Sechler, Prentice-Hall, Inc., 1974

III

The state-of-the art of all the current shell design manuals still adheres to the so-called "Lower Bound Design Philosophy" that has already been in use for over 50 years. ...... As for the design of conical shells, the design codes and recommendations are usually based on an approximate theory, i.e., the theory of equivalent cylindrical shells, where the selections of some appropriate geometries are normally based on engineering judgments that have not been well-justified on theoretical grounds, due to the fact that conical shells present more difficulties in both theoretical analysis and manufacturing processes.

G.Q. Zhang and J. Arbocz, Initial postbuckling analysis of anisotropic conical shells, paper presented in the 34th SDM conference, La Jolla, USA, 1993.

IV

Initial results indicate that in order to obtain reliable results for the bifurcation buckling analysis of anisotropic conical shells, enforcing the boundary condition rigorously and using the nonlinear prebuckling solution are indeed a must.

G.Q. Zhang and J. Arbocz, Buckling of anisotropic conical shells, paper presented in the 18th IUTAM conference, Haifa, Israel, 1992.

V

Anisotropic conical shells are generally imperfection sensitive. Thus to obtain a reliable result for the stability behavior of anisotropic conical shells it is necessary to carry out the initial postbuckling and imperfection sensitivity analyses after the critical bifurcation buckling load has been found.

G.Q. Zhang and J. Arbocz, Initial postbuckling analysis of anisotropic conical shells, paper presented in the 34th SDM conference, La Jolla, USA, 1993.

VI

Unfortunately, the cone-like coffee beakers are never under such load that instability could occur. Otherwise the present theory and programs might be used to provide some design criterion.

VII

There is a great doubt about human being's intelligence, if one considers the fact that many intelligent people have been working for years (even life-times) just pursuing one slightly higher "knockdown" factor.

VIII

The world is simple in the sense of similarity, and complicated in the sense of variety.

IX

Basically, it is the same to cook a meal and to conduct a technical research. An experienced "cook" gets delicious "meals".

X

It is known that human being has some unavoidable "moral imperfections" and the society is "imperfection sensitive". Thus, to prevent the "collapse" of a society, everyone has to do his best to keep the "imperfection amplitude" as small as possible and to avoid some dangerous "imperfection forms".

STABILITY ANALYSIS OF ANISOTROPIC CONICAL SHELLS

G.Q. ZHANG

STABILITY ANALYSIS OF ANISOTROPIC CONICAL SHELLS

PROEFSCHRIFT

Ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus,
Prof. Drs. P.A. Schenck,
in het openbaar te verdedigen
ten overstaan van een commissie
aangewezen door het College van Dekanen
op 17 juni 1993 te 14.00 uur

door

Guo-Qi Zhang

geboren te Beijing, China

Master of Science
in Aerospace Engineering

DELFT UNIVERSITY PRESS/1993

Dit proefschrift is goedgekeurd door de promotor:
Prof. dr. J. Arbocz

Published and distributed by:

Delft University Press
Stevinweg 1
2628 CN Delft
The Netherlands

Tel: + 31 15 783254
Fax: + 31 15 781661

ISBN 90-6275-876-2/CIP

Printed in The Netherlands.

ERRATA

On p. 68	first line of Eq. (3.10): " s_2 " should be " s^2 "
On p. 102	all " W " in Table 4.1 should be " $\tilde{W}$ "
On p. 119	second line from above " Liberescu " should be " Librescu "
On p. 166	in Fig. 6.3 " symmetric " should be " asymmetric "
On p. 201	line 8 from above " Eq. (4.39) " should be " Eq. (4.38) "

To my parents and my wife

CONTENTS

SUMMARY

Due to the rapid development of composite materials anisotropic conical shells are widely employed in aerospace, offshore, chemical, and civil engineering fields as well as in many other industries. In practice it happens frequently that the loss of stability is one of the important failure phenomena, thus the stability behavior may dictate the choice of some of the critical dimensions of the structure. This implies that in the design of anisotropic conical shells it is imperative to employ an accurate and reliable stability criterion for the different loading cases. For shells in general all the current design manuals adhere to the so-called 'Lower Bound Design Philosophy' that has already been in use for over 50 years. This criterion recommends the use of an empirical 'knockdown' factor, which is so chosen that when it is multiplied with the (classical) buckling load of the perfect shell a 'lower bound' to all available experimental data is obtained. As for the design of conical shells, the design codes and recommendations are usually based on an approximate theory called the theory of equivalent cylindrical shells, where the selection of some appropriate geometric parameters is normally based on engineering judgment that has not been well-justified on theoretical grounds. Obviously, weight - critical applications cannot afford such excessive conservatism. Therefore, there is a practical reason for trying to obtain an 'Improved Shell Design Criterion' for conical shells.

Theoretically, to be able to derive an 'Improved Shell Design Criterion', a thorough understanding of the nonlinear stability behavior is crucial, whereby the analysis of some aspects of the nonlinear behavior of thin shell structures is still one of the main challenges for both engineers and applied mathematicians.

Finally, compared with what has been done for anisotropic cylindrical shells, less theoretical and experimental effort has been put into the stability analysis of anisotropic conical shells, and few results have been published. This is partly due to the fact that

composite conical shells present more difficulties in both theoretical analysis and manufacturing processes. Thus the practical importance, the theoretical challenge and a lack of available results have prompted the present systematical analysis of the stability behavior of anisotropic conical shells.

In this investigation, a rigorous treatment of the bifurcation buckling and initial postbuckling behavior of perfect and imperfect anisotropic conical shells is presented, which includes the effects of boundary conditions and nonlinear prebuckling deformations. This analysis is based on nonlinear Donnell-type strain-displacement relations and uses an extension of Koiter's initial postbuckling theory.

To carry out the above analysis, the nonlinear Donnell-type anisotropic shell equations in terms of the radial displacement W and the Airy stress function F are derived via the stationary potential energy criterion. A complete set of boundary conditions, which partially or completely satisfies Seide's geometric constraint, is also formulated in terms of W and F. By using the perturbation technique proposed by Koiter, three sets of differential equations governing the prebuckling, buckling and postbuckling problems, and the corresponding boundary conditions are obtained. Koiter's initial postbuckling theory is put into a form suitable for the investigation of initial imperfection sensitivity of anisotropic conical shells.

To solve the resulting boundary value problems, the nonlinear prebuckling solution is assumed to be axisymmetric, while the circumferential dependence of the buckling and postbuckling equations is eliminated by Fourier decomposition. The resulting sets of ordinary differential equations are solved numerically via the 'Parallel Shooting Method', whereby the accuracy of the solution is controlled by a user chosen round-off error-bound. Two kinds of shooting schemes are employed in the numerical solution of the buckling problem. One takes the eigenvalue as one of the unknowns in the iteration step, the other uses a generalized Stodola-type method. The specific boundary conditions are enforced rigorously not only in the prebuckling but also in the buckling and postbuckling states. As an option, rigorous, linear, or membrane-like prebuckling solutions can be used.

Calculations are presented for various truncated conical shells covering a wide range of geometries. Comparisons with results of other investigations indicate that the present theory and computer programs can correctly and adequately predict the instability of conical shells. The effects of different shell geometries, prebuckling solutions, boundary conditions, material properties, and stacking sequences on the buckling and initial postbuckling behavior of conical shells are studied.

As a preliminary effort to account for the effect of physical imperfections consisting of variations of the wall thickness and the stiffness coefficients that occur when laminated conical shells are made by filament winding, some simplified formulae for equivalent constant wall thickness and stiffness coefficients are suggested. Numerical studies are carried out to verify the validity of these formulae.

SAMENVATTING

Als gevolg van de snelle ontwikkeling van composietmaterialen worden anisotrope conische schalen vaak toegepast in de lucht- en ruimtevaart, offshore, chemische techniek, civiele techniek en op vele andere gebieden. In de praktijk is instabiliteit dikwijls een van de belangrijkste verschijnselen waardoor de constructie kan bezwijken, waarmee het stabiliteitsgedrag de keuze van kritieke afmetingen van de constructie kan bepalen. Om deze reden is het in het ontwerp van anisotrope conische schalen noodzakelijk om een nauwkeurig en betrouwbaar stabiliteitscriterium voor de verschillende belastingsgevallen toe te passen. De huidige ontwerp-handboeken houden voor schalen nog steeds vast aan de zogenaamde 'Lower Bound Design Philosophy' die al meer dan 50 jaar in gebruik is. Dit criterium beveelt het gebruik van een empirisch bepaalde 'knockdown' factor aan, die, vermenigvuldigd met de (klassieke) kniklast van de perfecte schaal, een ondergrens aangeeft voor alle beschikbare experimentele resultaten. De ontwerpvoorschriften en aanbevelingen voor conische schalen zijn gewoonlijk gebaseerd op een benaderingstheorie, de theorie van equivalente cylinderschalen. Hierbij is de keuze van de geschikte geometrische parameters normaalgesproken gebaseerd op technisch inzicht en ervaring zonder een goede theoretische rechtvaardiging. Het is duidelijk dat men zich voor toepassingen waarin een laag gewicht vereist is een dergelijk extreem conservatisme niet kan veroorloven. Het is daarom van praktisch belang om een verbeterd schaalontwerpcriterium voor conische schalen op te stellen.

Om dit verbeterde schaalontwerpcriterium te kunnen opstellen, is een zeer goed begrip van het niet-lineaire stabiliteitsgedrag van cruciaal belang. De analyse van het niet-lineaire gedrag van dunwandige schaalconstructies vormt nog steeds een belangrijke uitdaging in zowel de techniek als de toegepaste wiskunde.

Tenslotte is er voor anisotrope conische schalen minder theoretisch en experimenteel onderzoek naar het stabiliteitsgedrag gedaan dan voor cylinderschalen, en zijn er weinig resultaten voor conische schalen gepubliceerd. Dit is gedeeltelijk een gevolg van het feit dat conische composietschalen moeilijker te fabriceren en te analyseren zijn. Het praktisch belang, de theoretische uitdaging en een gebrek aan beschikbare resultaten vormden zo de aanzet voor de systematische analyse van het stabiliteitsgedrag van anisotrope conische schalen die in dit proefschrift gepresenteerd wordt.

In dit onderzoek wordt een exacte behandeling van het bifurcatie knikgedrag en het initiële naknikgedrag van perfecte en imperfecte anisotrope conische schalen gepresenteerd, waarin het effect van randvoorwaarden en van een niet-lineaire grondtoestand in rekening wordt gebracht. De analyse is gebaseerd op niet-lineaire rek-verplaatsingsrelaties van het Donnell-type en maakt gebruik van een uitbreiding van Koiter's initiële naknik-theorie.

Om de bovengenoemde analyse uit te voeren, worden de niet-lineaire schaalvergelijkingen van het Donnell-type voor een anisotrope schaal, met de radiale verplaatsing W en een Airy spanningsfunctie F als onbekenden, afgeleid via het principe van stationaire potentiële energie. Een complete set randvoorwaarden die geheel of gedeeltelijk aan Seide's geometrische voorwaarde voldoet, wordt ook geformuleerd in W en F. Door gebruik te maken van de door Koiter aangegeven perturbatietechniek, worden drie stelsels differentiaalvergelijkingen, die de grondtoestand, knik, en het naknikgedrag beschrijven, verkregen, en tevens de corresponderende randvoorwaarden. Koiter's initiële naknik-theorie wordt omgewerkt tot een vorm die geschikt is om de imperfectiegevoeligheid van anisotrope conische schalen te onderzoeken.

Om de resulterende randvoorwaarde problemen op te lossen, wordt de niet-lineaire grondtoestand axiaal-symmetrisch verondersteld, terwijl de afhankelijkheid in omtreksrichting voor de knik- en naknikvergelijkingen wordt geëlimineerd door middel van een Fourier ontwikkeling. De resulterende stelsels gewone differentiaalvergelijkingen worden numeriek opgelost via de 'Parallel Shooting Method', waarbij de nauwkeurigheid van de oplossing gecontroleerd wordt via een door de gebruiker gekozen tolerantie voor de afbreekfout. Twee 'shooting' methodes zijn toegepast in de oplossing van het knikprobleem. De eerste gebruikt de eigenwaarde als onbekende in een iteratiestap, de tweede gebruikt een gegeneraliseerde Stodola methode. Aan de randvoorwaarden wordt exact voldaan, niet alleen in de grondtoestand, maar ook in de knik- en nakniktoestand. Naar keuze kan de exacte, lineaire, of membraan-oplossing voor de grondtoestand worden gebruikt.

Berekeningen worden gepresenteerd voor verscheidene afgeknotte conische schalen voor een breed gebied van geometrieën. Vergelijking met resultaten uit andere onderzoeken geeft aan dat de gebruikte theorie en computerprogramma's correct en adequaat het instabiliteitsgedrag van conische schalen kunnen beschrijven. Het effect van verschillende schaalgeometrieën, oplossingen voor de grondtoestand, randvoorwaarden, materiaaleigenschappen en laminaatopbouw op het knikgedrag en het initiële naknikgedrag zijn bestudeerd.

Als een eerste poging om het effect in rekening te brengen van een fysische imperfectie, bestaande uit variaties van de wanddikte en van de stijfheidscoëfficienten, die optreden wanneer gelamineerde conische schalen worden gewikkeld, worden enige simpele formules voor een equivalente constante wanddikte en equivalente constante stijfheidscoëfficienten voorgesteld. Numerieke studies zijn uitgevoerd om de geldigheid van deze formules te verifiëren.

NOMENCLATURE

In the following some common symbols used in this dissertation are listed. To keep the symbols close to the international conventions, some of them may have different meanings in different chapters. In that case they are explained as they appear.

a, b	Initial postbuckling coefficients defined by Eqs. (7.12) and (7.13)
A_{ij}^*	Semi-inverted extensional stiffness matrix
B_{ij}^*	Semi-inverted bending-stretching coupling matrix
c^2	$= 3(1-\upsilon^2)$
D_{ij}^*	Semi-inverted flexural stiffness matrix
E	Arbitrarily chosen reference Young's modulus
E_s	Young's modulus in meridional direction
E_θ	Young's modulus in circumferential direction
f_o	Prebuckling Airy stress function defined by Eq. (3.50)
f_1, f_2	Buckling Airy stress functions defined by Eq. (3.59)
f_α, f_β, f_γ	Postbuckling Airy stress functions defined by Eq. (3.69)
$\tilde{F}$	Airy stress function defined by Eqs. (3.10)
F	Transformed Airy stress function defined By Eqs. (3.32)
$F^{(o)}$, $F^{(1)}$, $F^{(2)}$	Zeroth order, first order and second order fields, respectively
G	Shear modulus
M_x, M_y, M_{xy}	Moment resultants of cylindrical shells
M_s, M_θ, $M_{s\theta}$	Moment resultants of conical shells
n_1	Number of full waves in the circumferential direction
N_x, N_y, N_{xy}	Stress resultants of cylindrical shells
N_s, N_θ, $N_{s\theta}$	Stress resultants of conical shells
p	Hydrostatic pressure

$\bar{p}$	Nondimensional hydrostatic pressure ($\bar{p} = pcs_1{}^2\sin^2\alpha_o/(Et^2\cos^2\alpha_o)$)
P	Axial compression load
Q_{ij}	Specially orthotropic lamina stiffness matrix
$\bar{Q}_{ij}$	Generally orthotropic lamina stiffness matrix
R_1	Radius at the small edge of the conical shell
R_2	Radius at the large edge of the conical shell
$\hat{R}$	$= \lambda/\bar{p}$
s	Coordinate in meridional direction
s_1	Distance from the vertex to the small edge of the conical shell
s_2	Distance from the vertex to the large edge of the conical shell
t	Shell wall-thickness
t_k	Thickness of the k^{th} layer
V_m	Volume fraction of the matrix
V_f	Volume fraction of the fiber
w_o	Prebuckling radial displacement function defined by Eq. (3.49)
w_1, w_2	Buckling radial displacement functions defined by Eq. (3.60)
w_α, w_β, w_γ	Postbuckling radial displacement functions defined by Eq. (3.68)
$\tilde{W}$	Radial displacement (positive inward)
W	Transformed radial displacement defined by Eq. (3.32)
$\bar{W}$	Initial radial imperfection (positive inward)
$\hat{W}$	Shape of the initial imperfection defined by Eq. (7.33)
$W^{(o)}$, $W^{(1)}$, $W^{(2)}$	Zeroth order, first order and second order fields, respectively
z_o	$= \ln(s_2/s_1)$
α_o	Semi-vertex angle of a conical shell
α, β	Imperfection form factors defined by Eqs. (7.19) and (7.20)
$\bar{\beta}$	Initial winding angle defined by Eq. (2.11)
ε_{ij}	Strain components referred to an arbitrary coordinate
ε_s, ε_θ, $\gamma_{s\theta}$	Strain components of conical shell
χ_s, χ_θ, $\chi_{s\theta}$	Curvature components of conical shell
ε_x, ε_y, γ_{xy}	Strain components of cylindrical shell
χ_x, χ_y, χ_{xy}	Curvature components of cylindrical shell
$\bar{\theta}$	$= \theta\sin\alpha_o$
λ	Nondimensional axial load parameter ($\lambda = Pc/(2\pi Et^2\cos^2\alpha_o)$)
λ_a	Reliability based 'knockdown' factor
Λ	Nondimensional variable load factor
Λ_c	Nondimensional variable load factor evaluated at the bifurcation point

Λ_s	Nondimensional variable load factor evaluated at the limit point
$\mu^{(k)}$	Eigenvalue estimate
ν	Arbitrarily chosen reference Poisson's ratio
ξ	Perturbation parameter
$\bar{\xi}$	Amplitude of the initial imperfection
ρ_s	Normalized variable load parameter ($\rho_s = \Lambda_s/\Lambda_c$)
ϕ	$= f_o' + f_o$
Φ_1	Thickness variation factor
ψ	$= w_o' + w_o$

Introduction

Conical shells are important structural elements for aerospace, civil, offshore, chemical and many other engineering fields and industries. In aerospace engineering they are used as engine frame, adapter sections of multistage rockets and are proposed as truncated-cone nose cap configuration for planetary re-entry vehicles. In civil engineering they are widely used as chimney stacks, shafts of overhead tanks and for foundations of tall structures. In offshore engineering they are used as transition elements of jacket legs between two cylindrical sections of different diameters. In chemical engineering they are used as reactors, tanks, silos and end-closures in pressure vessels, etc.. After all, the conical shell is a primary structural element for many practical applications. In most of the applications, however, the conical shells employed are thin-walled since these structural configurations exhibit favorable strength over weight ratios.

The most attractive properties of composite materials are the high strength-to-weight ratio and the high stiffness-to-weight ratio. Those added to their excellent fatigue strength, ease of formability, wide range of operating temperature (thermoplastic resins), negative or low coefficient of thermal expansion, high damping, resistance to corrosion, and their capability and flexibility of being tailored for a required application result in materials with seemingly unlimited potentials. In recent years, due to the rapid development of various kinds of composite materials, there has been a continuous increase of applications of composite materials for different shell structures[1-4]. Often encountered among these applications are anisotropic conical shells.

It has been observed that, as for the cases of other thin shell structures, the loss of stability by buckling of anisotropic conical shells is one of the important and crucial failure phenomena which may lead to a disaster. This implies that the stability behavior may dictate the choice of some of the critical dimensions of the structures. Therefore, in the design of anisotropic conical shells it is imperative to employ an accurate and reliable stability criterion for the different loading cases.

While considering the design criteria for buckling of shell structures, one observes that all the current design manuals[5-6] adhere to the so-called 'Lower Bound Design Philosophy' that has been in use for over 50 years. This criterion recommends the use of an empirical 'knockdown' factor, which is so chosen that when it is multiplied with the (classical) buckling load of the perfect shell, a 'lower bound' to all available experimental data is obtained (see Fig. 1). As for the design of conical shells, the design codes and recommendations are usually based on an approximate theory[7] called the theory of equivalent cylindrical shells, where the selection of some appropriate geometries is normally based on engineering judgment that has not been well-justified on theoretical grounds. Obviously, weight - critical applications cannot afford such excessive conservatism. Therefore, in recent years much effort[8-19] has been spent on the development of 'Improved Shell Design Criteria'.

Unfortunately, most of the effort so far has been directed to cylindrical shells, and less theoretical and experimental effort has been addressed to the stability analysis of anisotropic conical shells. This is partly due to the fact that anisotropic conical shells present more difficulties in both theoretical analysis and manufacturing processes. However, it is felt that a design criterion for anisotropic conical shells should be included in 'Improved Shell Design Criteria'.

Therefore, the central goal of present investigation is the development of 'Improved Shell Design Criteria' for the buckling sensitive anisotropic conical shells, by carrying out a systematical study of their buckling and initial postbuckling behavior.

To set up 'Improved Shell Design Criteria' one must have a thorough understanding for the buckling phenomena of thin-walled shell structures, and distinguish between collapse at the maximum point in a load versus deflection curve and bifurcation buckling. To obtain the critical load levels one can carry out an asymptotic analysis or a general nonlinear analysis.

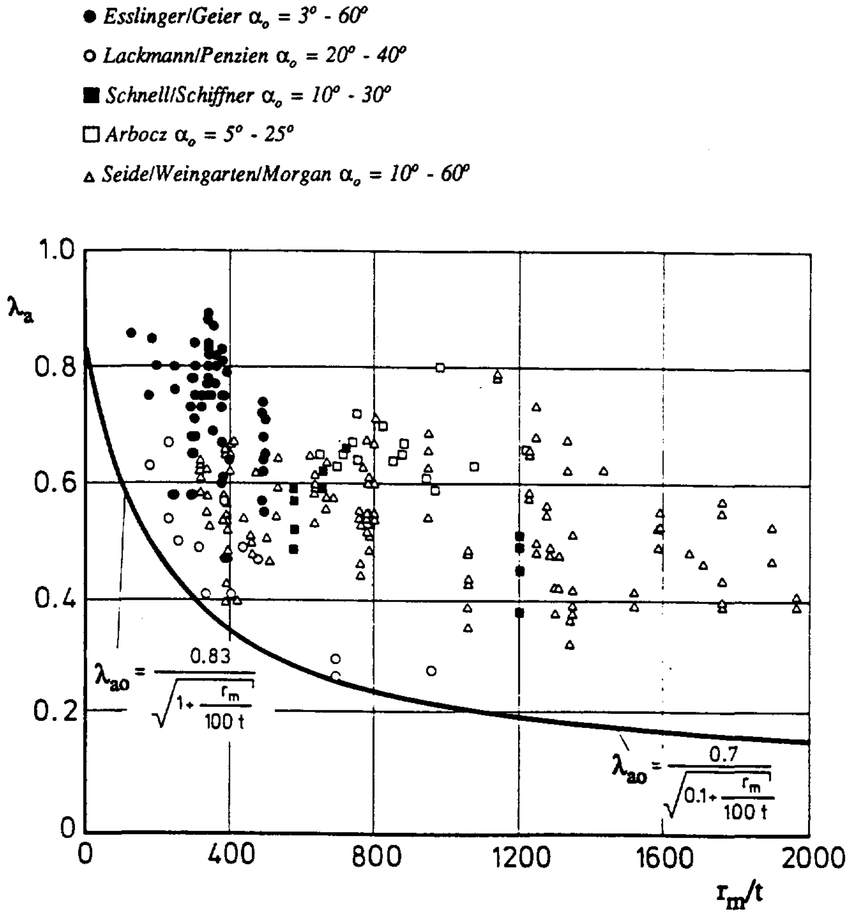


Fig. 1 *Knockdown factors for conical shells subjected to axial compression obtained at different institutes*[28]

As explained by Arbocz[20], applying the asymptotic analysis to an axially compressed perfect anisotropic shell, initially the buckling displacement W_b will be identically zero until the bifurcation load λ_c at point B has been reached (see Fig. 2). Following bifurcation the initial failure of the perfect structure will be characterized by a rapidly growing asymmetric deformation along the path BD with a decreasing axial load λ.

On the other hand, if one employs the general nonlinear analysis the axially compressed perfect anisotropic shell deforms axisymmetrically along the path OA (see Fig. 3) until a maximum (or limit) load λ_L is reached at point A. However, here a bifurcation point B lies between O and A. Thus, once the bifurcation load λ_c has been reached, the initial failure of the perfect structure will be characterized by a rapidly growing asymmetric deformation along the path BD with a decreasing axial load λ. Thus, in this case, the collapse load of the perfect structure λ_L is of no engineering significance.

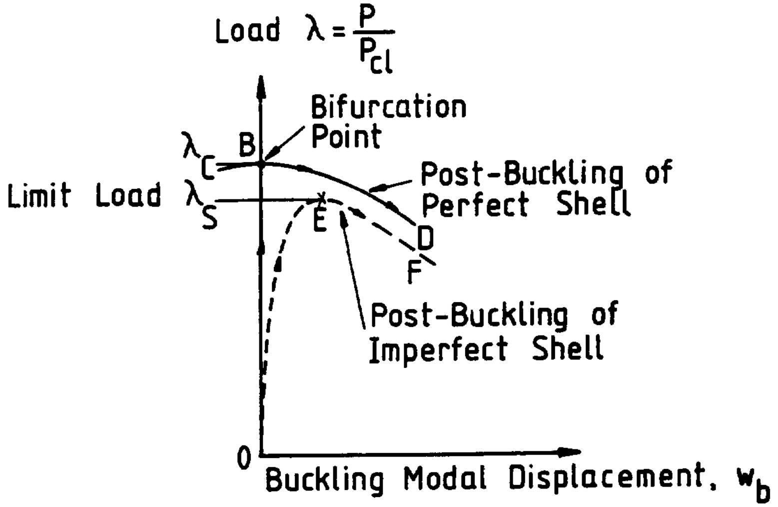


Fig. 2 *Bifurcation point and limit point via asymptotic analysis*[20]

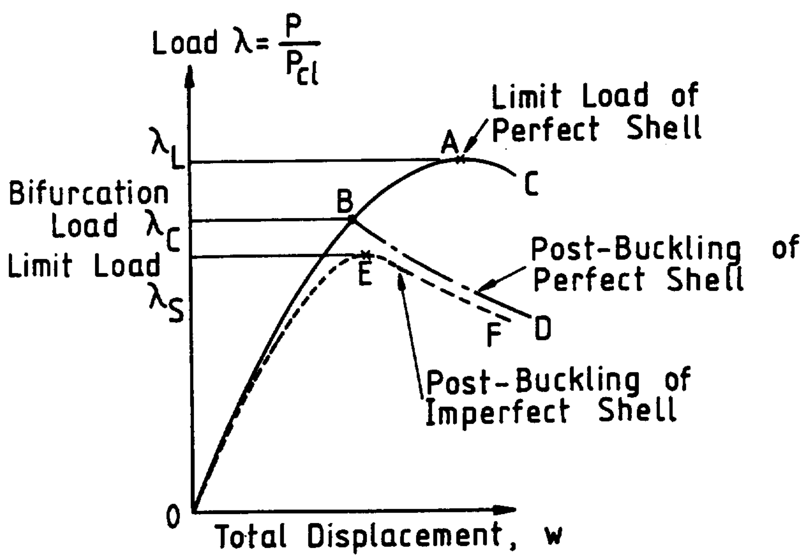


Fig. 3 *Bifurcation point and limit point via nonlinear analysis*[20]

In the case of real shells, which contain unavoidable initial imperfections, for both approaches the structural response will follow a fundamental path OEF, with failure occurring as a "snapthrough" at point E at the limit (collapse) load λ_s. In this case there are no bifurcation points between O and E. However, considering Figures 2 and 3, one can state that if there are no significant prebuckling load redistributions then the bifurcation model often leads to a good approximation of the failure load and mode, especially in the cases involving significant pre-bifurcation symmetries.

If one considers the stability analysis of conical shells, some comprehensive reviews concerning the buckling behavior of various conical shells can be found in the literature. Seide[21] has examined the state-of-art for isotropic conical shells of constant thickness and concluded that there were still numerous unanswered and bypassed questions. Kobayashi[22] has tabulated and summarized some existing formulae and solutions of buckling of conical shells from eighty-three published papers and reports. Singer[23] has summarized the extensive work on stiffened and unstiffened conical shells performed at Technion, Israel Institute of Technology. The Column Research Committee of Japan[24] has presented about thirty-five results on the solutions of different kinds of conical shells under different loading cases, based on more than ninety references. Their results cover stiffened and unstiffened, orthotropic and sandwich conical shells. Singer and Baruch[25] have reviewed many of their contributions on the use of stiffeners to carry out structural optimization for elastic stability of conical shells. Schlünz[26] has presented 300 references with abstracts for the stability of thin conical and spherical shells covering the time interval mainly from 1961 to 1970. Ellinas et al.[27] have summarized some results of buckling of isotropic conical shells from several design codes and recommendations. Esslinger et al.[28] have summarized many experimental results obtained at DFVLR for the stability of isotropic conical shells. Weingarten and Seide[7] have recommended practices for predicting buckling of uniform stiffened and unstiffened conical shells under various types of static loadings, and suggested procedures that yield conservative estimates of buckling loads. Sullins et al.[29] have presented some design recommendations for the general instability of truncated sandwich conical shells. Other reviews for the stability analysis of conical shells may also exist.

Despite of the existence of many published results for the stability analysis of conical shells, most of the results are for isotropic conical shells with greatly simplified theoretical treatments.

The earlier investigators were inclined to ignore the nonlinear terms of the strain and displacement relations, which leads to the approximate theories such as those of Hoff[30], Mushtari and Sachenkov[31], Seide[32-34], Schnell[35] and others. Employing these linearized forms of the strain-displacement relations, as is usual for the stability analysis of other thin shells, one obtains theoretical results which are in qualitative, but not quantitative, agreement with experimental results, especially in the case of axial load and inner pressure[21,36,37]. Sometimes these theoretical predictions bore little relationship to the behavior of the actual structural configuration or even yield erroneous results. This initial general approach for the buckling analysis of conical shells is known as the classical stability theory. At first, the discrepancy between the theoretical predictions based on the classical stability theory and the experimental observations was attributed to the inelastic behavior exhibited by the materials from which the real structures were constructed. Further study showed, however, that inelastic behavior was only one of several possible causes of the deviation between theory and experiment. In some cases it was discovered that certain nonlinear terms in the strain and displacement relations should be retained since the buckling process inherently involves moderately large displacement. Later, extensive investigations have been carried out based on various kinds of nonlinear strain and displacement relations. Seide[38,39] developed a more accurate Donnell-type theory by retaining certain terms omitted by Hoff and Singer[40] in the strain and displacement relations. This Donnell-type theory has been widely used by many investigators, such as Seide[41], Arbocz[42], Singer and Baruch[43], Ender[44] and others, and yielded improved theoretical results.

As the knowledge of stability theory grew, one discovered that the influence of boundary conditions on the buckling loads of shell structures can be also important[45-49]. Singer[50] studied the effect of axial constraint on the stability behavior of conical shells under hydrostatic pressure. His results suggest that the effect is usually small. Later, Singer[51] presented results for a clamped isotropic conical shell under hydrostatic pressure, which can be compared with results for a cone with simply supported edges. The difference in the boundary conditions for the two cases is that simply supported edges impose no restraint against rotation or displacements in a direction parallel to the generator, while clamped edges impose infinite restraint against these displacements. The results presented indicate increases in buckling pressure of approximately 50 percent or more, as compared to cones with simply supported boundary conditions. Thurston[52] presented limited results which show that replacing $N_s = 0$ by $u = 0$ in the boundary conditions the buckling load will increase roughly 45 percent. Refs. [34,53] presented experimental data that suggests edge rotation can be significant. Baruch et al.[54] systematically studied the effect of four sets of in-plane boundary conditions on the buckling behavior of isotropic conical shells under hydrostatic pressure. In their analysis, the first two stability equations are solved by the assumed displacements, while the third is solved by the Galerkin procedure. The boundary conditions are satisfied with four unknown coefficients in the expressions for u and v. It was found that, as in the cases of cylindrical shells, the effect of axial constraint is of primary importance for conical shells under hydrostatic pressure, whereas the circumferential restraint has little influence. The effect of the four possible in-plane boundary conditions on the buckling behavior of isotropic conical shells under axial compression, was also studied by Baruch et al.[55] using a method similar to the one used in Ref. [54]. It was found that under axial compression both circumferential and axial restraints are of primary importance. In the cases SS1 ($N_{s\theta} = N_s = 0$) and SS3 ($v = N_s = 0$), axisymmetric buckling mode dominates and $\lambda = 0.5$. The axisymmetric buckling mode in the SS3 case involves large axial translations. When these are prevented, only the classical modes with $\lambda = 1$ can occur. In the SS2 case ($N_{s\theta} = u = 0$) an asymmetric buckling mode occurs and $\lambda = 0.5$. In the SS4 case ($v = u = 0$) the buckling mode is again asymmetric but with $\lambda = 1$.

So far most of the results obtained for the stability analysis of isotropic conical shells are still limited to the simplified theoretical treatment, because in all the analyses the membrane-like prebuckling solution was normally used to search for the buckling solutions. It has been found that the accuracy of solutions is often questionable if one uses the membrane-like prebuckling solutions. For cylindrical shells using membrane prebuckling solutions implies that one relaxes completely the supports in the prebuckling range and thus assumes that the prebuckling stresses and deformations are constant[49,56]. For most practical cylindrical shell structures, however, some measure of radial support is present from the beginning of loading so that, prior to buckling, complicated axisymmetric deformations and stresses are present which modify the loading-shortening behavior of the cylinder and influence its buckling load. This influence is especially noticeable for cylindrical shells in axial compression and for short cylindrical shells under external lateral pressure. Unfortunately, compared with what has been known for cylindrical shells, only a few results can be found in the literature where the influence of prebuckling solutions on the buckling behavior of isotropic conical shells is considered. Famili[57] studied the asymmetric buckling behavior of truncated and complete conical shells under uniform hydrostatic pressure by means of finite difference analysis, taking into account the large deformation in the prebuckling state. Kobayashi[58] carried out an analytical investigation to determine the effect of prebuckling deformation on the compressive buckling load of truncated conical shells with clamped boundary conditions by using a finite difference approach. His limited results indicate that the decrease in the buckling load from the classical result due to prebuckling deformation is somewhat greater for conical shells than for cylindrical shells.

Besides the above explanations for the discrepancies between theoretical and experimental results, there is now a general agreement that initial imperfections, i.e., small accidental deviations from the assumed initial shape of the structures, are the principal cause of the disagreement. Moreover, it was pointed out by Masur and Schreger[59] that even if the structure is 'perfect' in its initial unloaded state, the loading process itself may cause deformations which in their effect on the stability are similar to initial imperfections. A major contribution to the present understanding of the role of initial imperfections was made by Koiter[60,61]. Excellent reviews of Koiter's theory and of the many applications of it to buckling of monocoque and stiffened elastic and elastic-plastic shells are given by Hutchinson and Koiter[62], Tvergaard[63], and Budiansky and Hutchinson[64]. The theory itself is reiterated in some detail by Budiansky[65], Seide[66], and Arbocz[67] and extended to dynamic buckling by Budiansky and Hutchinson[68].

The purpose of Koiter's theory is twofold: determine the lowest bifurcation point on the equilibrium path and ascertain the sensitivity of the maximum load-carrying capacity of the structure to initial geometric imperfections. This approach focuses attention on initial postbuckling behavior and provides a theory that is exact in the asymptotic sense, i.e., exact at the bifurcation point itself and a close approximation for postbuckling configurations near the bifurcation point. When the initial portion of the secondary path has a positive slope, considerable postbuckling strength can be developed by the structure, and loss of stability on the primary path does not result in structural collapse. When the initial portion of the secondary path has a negative slope, on the other hand, the buckling is precipitous and the magnitude of the critical load is subject to the influence of initial imperfections.

Since Koiter first presented his theory, numerous results of applications of his theory to various kinds of shell stability problems have been published. Most of the results, however, are for different kinds of cylindrical and spherical shells[69], few attention has been paid to the conical shell problems, even for isotropic conical shells. Among the few research effort and results, Akkas[70] studied the buckling and initial postbuckling behavior of isotropic spherical and conical caps under axisymmetric ring loads and uniform pressure with clamped boundary conditions, based on the formulation of Fitch[71]. His results show that the conical caps under the types of loads considered are generally imperfection-sensitive. Later, Akkas[72] extended his previous theory to the buckling and initial postbuckling problems of shallow spherical and conical sandwich shells. The corresponding numerical results show that the buckling and initial postbuckling behavior of the sandwich cap is similar to that of the isotropic cap. The influence of initial imperfections on the stability behavior of conical shells has also been studied by some investigators using different theories and methods. Arbocz[73] did some experimental investigations for the buckling load of isotropic conical shells, where the effect of an initial imperfection with an axially symmetric shape was also investigated. Arbocz and Klompe[74] carried out some initial imperfection surveys on conical shells at the HOECHST AG in 1983. Cooper and Dexter[75] presented some results for the effect of local imperfections on the critical buckling load of an axially compressed isotropic conical shell. Their results show that the buckling load obtained from a bifurcation buckling analysis is highly dependent on the circumferential arc length of the imperfection type studied there. As the circumferential arc length of the imperfection is increased, a reduction of up to 50 percent of the buckling load of the perfect shell can occur. Shiau et al.[76] presented some results on the dynamic stability of a truncated isotropic conical shell with some geometric imperfections under a wide variety of dynamic loading conditions. It was found that conical shells are less sensitive to initial imperfections than the cylindrical shells. Vandepitte et al.[77] presented some results of an experimental investigation of the buckling of hydrostatically loaded isotropic conical shells. In their investigation the initial imperfections were measured, and the safety factor accounting for the imperfection sensitivity was suggested. Bermus et al.[78] studied the influence of initial imperfections on the buckling of orthotropic truncated spherical and conical shells under axial compression. The few results presented there show that the small imperfections can lower the buckling loads of perfect shells.

Since weight is of critical importance in aerospace applications, there is considerable interest in the design of conical shells utilizing some forms of stiffening, such as rings and stringers. As a consequence, several investigations have been carried out for the stability analysis of stiffened conical shells. Singer et al.[79] studied the buckling of conical shells stiffened with rings and stringers by treating the stiffened shells as equivalent orthotropic shells. However, since the effect of the eccentricity of stiffeners cannot be studied by the classical orthotropic approach, and in the case of longitudinal stiffening the classical orthotropic approach is limited to stringers whose effective moment of inertia is proportional to the distance from the vertex, a more accurate method which considers the separate distributed stiffness of the rings and stringers was suggested by Baruch and Singer[80,81]. The results obtained by using this method reveal that stiffener eccentricity may be significant, and external stringers yield higher critical pressure than internal stringers. Baruch et al.[82] further extended the previous method to conical shells with non-uniformly spaced stiffeners. Theoretical results indicate that for hydrostatic pressure loading, rings are the most efficient stiffeners. On account of the cone geometry, equally spaced rings divide a conical shell into 'sub-shells' of unequal local buckling strength, while unequal spacings may result in 'sub-shells' of equal local buckling strength. By changing the ring spacings and the stiffener cross-sections structural optimization can be achieved. Samuelson[83] suggested an approximate method for the analysis of stiffened conical shells by defining some equivalent cylindrical shells for stiffened cones. More recently, Baruch, Arbocz and Zhang[84] suggested the geodesic stiffening and other non-traditional stiffener arrangements for the stiffening of conical shells. At about the same time, Gendron et al.[85] considered the geodetically stiffened composite cylindrical shells. The motivations to present the non-traditional stiffening concept are mainly due to structural optimization and the rapid development of composite material techniques which provide the possibilities for the realization of this concept. However, more effort is needed to study the effect of non-traditional stiffeners on the stability behavior of conical shells.

The number of theories and results for the stability analysis of orthotropic conical shells are limited. Serpico[86] and Singer[87] derived the Donnell-type governing equations for classical orthotropic conical shells by assuming that the orthotropic character of the shell is described by only distinguishing between Young's moduli and Poisson's ratios in two mutually orthogonal directions. Singer et al.[79] studied the buckling behavior of classical orthotropic conical shells under external pressure by satisfying slightly relaxed boundary conditions. Dixon[88] considered the buckling of classical orthotropic truncated conical shells with elastic edge restraint subjected to lateral pressure and axial load. Schiffner[89] carried out perhaps the most elegant theoretical investigation for the buckling of classical orthotropic conical shells, at that time. In his analysis, besides using membrane-like prebuckling solution, he also presented results where the axisymmetric prebuckling solution was first solved rigorously by a finite difference method, and then substituted into the buckling equations which were also solved by the finite difference method. In the solutions of both prebuckling and buckling states two sets of boundary conditions were rigorously satisfied. What is more, the effect of a specific kind of axisymmetric imperfection on the buckling load was also studied, and the results showed a decrease of buckling load of isotropic conical shells under combined axial load and inner pressure due to the present of the geometric imperfection. Dumir and Khatri[90] presented results for axisymmetric static and dynamic buckling analysis of classical orthotropic truncated shallow conical cap with a clamped edge. More recently, Zhang, Baruch and Arbocz[91,92] investigated the possibilities of producing classical orthotropic conical shells by the different manufacturing processes currently available. As distinguished from the classical orthotropic conical shells, a so-called quasi-orthotropic conical shell and its corresponding constitutive relations are presented, which are more representative for some manufacturing processes. For mainly academic interest the buckling loads of classical orthotropic conical shells under axial compression are calculated. Buckling loads lower than those obtained from another literature source[93] are found.

There are some results for the stability analysis of sandwich conical shells. Among the existing results are those of Akkas[72], Seide[33], Cohen[94,95], Bert et al.[96], Struk[97], Anderson et al.[98] and others. Since the present analysis is not going to cover the sandwich conical shells, no detailed reviews for the buckling of sandwich conical shells will be presented here.

There are a few theories and results for the stability analysis of anisotropic conical shells published in the open literature, while numerous studies have been performed and published on anisotropic cylindrical shells. Shul'ga et al.[99] studied the buckling problem of multilayer orthotropic conical shells under axial compression by employing the membrane-like prebuckling solution. Limited results presented there show that the buckling mode is asymmetric, and using the C1, C2, SS1 and SS2 boundary conditions will lead to considerable reduction (about 50%) for the buckling load compared with using the C3, C4, SS3 and SS4 boundary conditions. Other investigations for anisotropic, especially for laminated conical shells are confined to the vibration analysis by employing the linear strain and displacement relations[100-103]. No results have been published for the initial postbuckling behavior and imperfection sensitivity of anisotropic conical shells, as far as the author knows.

Thus the practical importance, the theoretical challenge and a lack of available results prompted the present work involving a systematic analysis of the stability behavior of anisotropic conical shells.

As part of the effort concerning the development of 'Improved Shell Design Criteria', in the following a rigorous treatment of the bifurcation buckling and initial postbuckling behavior of perfect and imperfect anisotropic conical shells is presented. This analysis includes the effects of boundary conditions and nonlinear prebuckling deformations, based on nonlinear Donnell-type strain-displacement relations and an extension of Koiter's initial postbuckling theory. To carry out the above analysis, the nonlinear Donnell-type anisotropic shell equations in terms of the radial displacement W and the Airy stress function F are used. A complete set of boundary conditions, which partially or completely satisfies

Seide's geometric constraint, is also formulated in terms of W and F. By using the perturbation technique proposed by Koiter, three sets of differential equations governing the behavior of the prebuckling, buckling and postbuckling problems, and the corresponding boundary conditions, are obtained. Koiter's initial postbuckling theory[60,104] is rederived and cast into a form suitable for the investigation of initial imperfection sensitivity of anisotropic conical shells. In searching for the solutions, the prebuckling state is assumed to be axisymmetric, while the circumferential dependence of the buckling and postbuckling equations is eliminated by Fourier decomposition. The resulting sets of ordinary differential equations are solved numerically via the 'Parallel Shooting Method'[105-107], whereby the accuracy of the solution is controlled by a user chosen round-off error-bound. Specific boundary conditions are enforced rigorously not only in the prebuckling but also in the buckling and postbuckling states. As an option, rigorous, linear, or membrane-like prebuckling solutions can be used.

Chapter 1
Kinematic Relations

1.1 Introduction

Since shell configurations of different wall constructions have been widely used as structural elements, various geometrically linear shell theories, which can be used to solve these shell problems, have been extensively investigated and widely employed[108-110].

However, there still exist many other shell problems, such as the problems involving large displacements and rotations, loss of stability, dependence of external forces on deformation, in which it is currently recognized that the nonlinear effects play an important role and must be taken into account by using geometrically nonlinear shell theories instead of the linear theories. Therefore, much attention has been devoted to setting up various geometrically nonlinear shell theories[111-119], and a vast number of numerical applications based on these theories have proven their validity within the bounds given by the respective underlying assumptions.

In the following, both linear and nonlinear thin shell theories based on Kirchhoff-Love assumptions are reviewed and discussed. This effort is not motivated by an inability of the existing theories to solve the present problem, but rather by a desire to define and to choose a theory that is characterized by simplicity, consistency, clarity and accuracy for the stability analysis of anisotropic conical shells.

1.2 General Kinematic Relations

According to the three dimensional elasticity theory, the strain components referred to an arbitrary orthogonal coordinate system (α_1, α_2, α_3) can be written as[112]:

$$
\begin{aligned}
\varepsilon_{11} &= e_{11} + \frac{1}{2}[\, e_{11}^2 + (\frac{1}{2}e_{12}+\omega_3)^2 + (\frac{1}{2}e_{13}-\omega_2)^2\,] \\
\varepsilon_{22} &= e_{22} + \frac{1}{2}[\, e_{22}^2 + (\frac{1}{2}e_{12}-\omega_3)^2 + (\frac{1}{2}e_{23}+\omega_1)^2] \\
\varepsilon_{33} &= e_{33} + \frac{1}{2}[e_{33}^2 + (\frac{1}{2}e_{13}+\omega_2)^2 + (\frac{1}{2}e_{23}-\omega_1)^2\,] \\
\varepsilon_{12} &= e_{12} + e_{11}(\frac{1}{2}e_{12}-\omega_3) + e_{22}(\frac{1}{2}e_{12}+\omega_3) + (\frac{1}{2}e_{13}-\omega_2)(\frac{1}{2}e_{23}+\omega_1) \\
\varepsilon_{13} &= e_{13} + e_{11}(\frac{1}{2}e_{13}+\omega_2) + e_{33}(\frac{1}{2}e_{13}-\omega_2) + (\frac{1}{2}e_{12}+\omega_3)(\frac{1}{2}e_{23}-\omega_1) \\
\varepsilon_{23} &= e_{23} + e_{22}(\frac{1}{2}e_{23}-\omega_1) + e_{33}(\frac{1}{2}e_{23}+\omega_1) + (\frac{1}{2}e_{12}-\omega_3)(\frac{1}{2}e_{13}+\omega_2)
\end{aligned}
\tag{1.1}
$$

Here the parameters e_{ij} and ω_i are given by

$$
\begin{aligned}
e_{11} &= \frac{1}{H_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{H_1H_2}\frac{\partial H_1}{\partial \alpha_2}v + \frac{1}{H_1H_3}\frac{\partial H_1}{\partial \alpha_3}w \\
e_{22} &= \frac{1}{H_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{H_2H_3}\frac{\partial H_2}{\partial \alpha_3}w + \frac{1}{H_1H_2}\frac{\partial H_2}{\partial \alpha_1}u \\
e_{33} &= \frac{1}{H_3}\frac{\partial w}{\partial \alpha_3} + \frac{1}{H_1H_3}\frac{\partial H_3}{\partial \alpha_1}u + \frac{1}{H_2H_3}\frac{\partial H_3}{\partial \alpha_2}v \\
e_{12} &= \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_1}(\frac{v}{H_2}) + \frac{H_1}{H_2}\frac{\partial}{\partial \alpha_2}(\frac{u}{H_1}) \\
e_{13} &= \frac{H_1}{H_3}\frac{\partial}{\partial \alpha_3}(\frac{u}{H_1}) + \frac{H_3}{H_1}\frac{\partial}{\partial \alpha_1}(\frac{w}{H_3}) \\
e_{23} &= \frac{H_3}{H_2}\frac{\partial}{\partial \alpha_2}(\frac{w}{H_3}) + \frac{H_2}{H_3}\frac{\partial}{\partial \alpha_3}(\frac{v}{H_2})
\end{aligned}
\tag{1.2}
$$

$$2\omega_1 = \frac{1}{H_2H_3}\left[\frac{\partial}{\partial\alpha_2}(H_3w) - \frac{\partial}{\partial\alpha_3}(H_2v)\right]$$

$$2\omega_2 = \frac{1}{H_1H_3}\left[\frac{\partial}{\partial\alpha_3}(H_1u) - \frac{\partial}{\partial\alpha_1}(H_3w)\right]$$

$$2\omega_3 = \frac{1}{H_1H_2}\left[\frac{\partial}{\partial\alpha_1}(H_2v) - \frac{\partial}{\partial\alpha_2}(H_1u)\right]$$

where the Lame coefficients H_1, H_2, and H_3 are

$$H_1 = \sqrt{(\frac{\partial x}{\partial\alpha_1})^2 + (\frac{\partial y}{\partial\alpha_1})^2 + (\frac{\partial z}{\partial\alpha_1})^2}$$

$$H_2 = \sqrt{(\frac{\partial x}{\partial\alpha_2})^2 + (\frac{\partial y}{\partial\alpha_2})^2 + (\frac{\partial z}{\partial\alpha_2})^2} \qquad (1.3)$$

$$H_3 = \sqrt{(\frac{\partial x}{\partial\alpha_3})^2 + (\frac{\partial y}{\partial\alpha_3})^2 + (\frac{\partial z}{\partial\alpha_3})^2}$$

To derive the theory of deformation of thin shells, one can use the coordinate system shown in Figure 1.1. The coordinates α_1, α_2 and z form a triply orthogonal system.

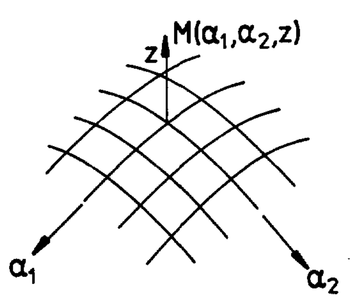


Fig. 1.1 *An arbitrary triply orthogonal coordinate system*

The positions of points on the middle surface of the shell are determined by the Gaussian curvilinear coordinates α_1 and α_2 of the surface. The Lame coefficients corresponding to this curvilinear system are denoted by A_1 and A_2, and the principal radii of curvature by R_1 and R_2.

Thus the Lame coefficients of above triply orthogonal system are simplified as

$$
\begin{aligned}
H_1 &= A_1\left(1+\frac{z}{R_1}\right) \\
H_2 &= A_2\left(1+\frac{z}{R_2}\right) \\
H_3 &= 1
\end{aligned}
\tag{1.4}
$$

where A_1, A_2, R_1 and R_2 must satisfy the Gauss-Codazzi relations of surface theory:

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right) = \frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1} \\
&\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right) = \frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2} \\
&\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right) = -\frac{A_1A_2}{R_1R_2}
\end{aligned}
\tag{1.5}
$$

Substituting Eqs. (1.4) into Eqs. (1.2), one obtains, with the help of Eqs. (1.5), the relations

$$
\begin{aligned}
e_{11} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v+\frac{w}{R_1}\right) \\
e_{22} &= \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial v}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u+\frac{w}{R_2}\right) \\
e_{12} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u\right) + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v\right)
\end{aligned}
$$

$$e_{1z} = \frac{\partial u}{\partial z} + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right) \tag{1.6}$$

$$e_{2z} = \frac{\partial v}{\partial z} + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)$$

$$e_{zz} = \frac{\partial w}{\partial z}$$

$$2\omega_1 = -\frac{\partial v}{\partial z} + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)$$

$$2\omega_2 = \frac{\partial u}{\partial z} - \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)$$

$$2\omega_z = \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right) - \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)$$

Here u, v, w are displacements of an arbitrary point in the directions of α_1, α_2, and z, respectively.

Equations (1.1) and (1.6) form the general kinematic relations of thin shells which, however, may be too complicated to be used directly for the solutions of some specific shell problems. Nevertheless, they establish the foundations for the following simplifications which will lead to many practical and well-known linear and nonlinear shell theories.

1.3 Some Linear Thin Shell Theories

By summarizing certain aspects of the widely used linear thin shell theories based on Love's first approximation, and analyzing where the differences originate, one expects to obtain better understanding of the nature of various nonlinear thin shell theories.

Love's first approximation

Love suggested four assumptions for the derivation of the classical thin shell theory of small displacement. They are

- The thickness of the shell is small compared with the other dimensions, for example, the smallest radius of curvature of the middle surface of the shell. This defines what is meant by 'thin shells'.

- Strains and displacements are sufficiently small, so that the quantities of second and higher-order magnitudes in the strain-displacement relations may be neglected in comparison with the first-order terms. This permits one to refer all calculations to the original configuration of the shell, and ensures that the differential equations will be linear.

- The transverse normal stress is small compared with the other normal stress components, and may be neglected.

- Normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension.

More explanations of these assumptions can be found in Refs. [109,110].

General linear kinematic relations - the relations of Flügge and Novozhilov

According to the second assumption of Love's first approximation, the linear kinematic relations can be obtained by neglecting all the nonlinear terms in Eqs. (1.1) under the additional restriction that the rotations of material fibers be also small everywhere. They are given as

$$
\begin{aligned}
\varepsilon_{11} &= e_{11} \\
\varepsilon_{22} &= e_{22} \\
\varepsilon_{12} &= e_{12} \\
\varepsilon_{1z} &= e_{1z} \\
\varepsilon_{2z} &= e_{2z} \\
\varepsilon_{zz} &= e_{zz}
\end{aligned}
\qquad (1.7)
$$

where e_{ij} can be found in Eqs. (1.6).

Further, to satisfy the Kirchhoff hypothesis, the class of displacements is restricted to the following linear relations

$$
\begin{aligned}
u(\alpha_1, \alpha_2, z) &= \hat{u}(\alpha_1, \alpha_2) + z\chi(\alpha_1, \alpha_2) \\
v(\alpha_1, \alpha_2, z) &= \hat{v}(\alpha_1, \alpha_2) + z\psi(\alpha_1, \alpha_2) \qquad (1.8) \\
w(\alpha_1, \alpha_2, z) &= \hat{w}(\alpha_1, \alpha_2)
\end{aligned}
$$

where $\hat{u}$, $\hat{v}$ and $\hat{w}$ are the components of displacements at the middle surface in the α_1, α_2 and normal directions, respectively; χ and ψ are the rotations of the normal to the middle surface during deformation about the α_1 and α_2 axes, respectively, i.e.,

$$
\begin{aligned}
\chi &= \frac{\partial u(\alpha_1, \alpha_2, z)}{\partial z} \\
\psi &= \frac{\partial v(\alpha_1, \alpha_2, z)}{\partial z}
\end{aligned}
\qquad (1.9)
$$

Substituting Eqs. (1.8) into ε_{1z}, ε_{2z}, and ε_{zz} and letting $\varepsilon_{1z} = 0$, $\varepsilon_{2z} = 0$, $\varepsilon_{zz} = 0$, one obtains

$$
\begin{aligned}
\chi &= \frac{\hat{u}}{R_1} - \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} \\
\psi &= \frac{\hat{v}}{R_2} - \frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2}
\end{aligned}
\qquad (1.10)
$$

Thus the behavior of a thin shell can be described with sufficient accuracy by the behavior of the shell middle surface.

Substituting Eqs. (1.8) into Eqs. (1.7a, b, c) yields

$$
\varepsilon_{11} = \frac{1}{(1+\frac{z}{R_1})}(\hat{\varepsilon}_{11} + z\chi_{11})
$$

$$\varepsilon_{22} = \frac{1}{(1+\frac{z}{R_2})}(\hat{\varepsilon}_{22} + z\chi_{22}) \tag{1.11}$$

$$\varepsilon_{12} = \frac{1}{(1+\frac{z}{R_1})(1+\frac{z}{R_2})}\left[(1-\frac{z^2}{R_1R_2})\,\hat{\varepsilon}_{12} + z(1+\frac{z}{2R_1}+\frac{z}{2R_2})\,\chi_{12}\right]$$

where $\hat{\varepsilon}_{11}$, $\hat{\varepsilon}_{22}$ and $\hat{\varepsilon}_{12}$ are the normal and shear strains in the middle surface ($z = 0$) given by

$$\begin{aligned}
\hat{\varepsilon}_{11} &= \frac{1}{A_1}\frac{\partial\hat{u}}{\partial\alpha_1} + \frac{\hat{v}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{\hat{w}}{R_1} \\
\hat{\varepsilon}_{22} &= \frac{\hat{u}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial\hat{v}}{\partial\alpha_2} + \frac{\hat{w}}{R_2} \\
\hat{\varepsilon}_{12} &= \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\hat{u}}{A_1}) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\hat{v}}{A_2})
\end{aligned} \tag{1.12}$$

and χ_{11} and χ_{22} are the mid-surface changes in curvature and χ_{12} is the mid-surface twist, given by

$$\begin{aligned}
\chi_{11} &= \frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\psi \\
\chi_{22} &= \frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\chi \\
\chi_{12} &= \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\chi}{A_1}) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\psi}{A_2}) + \frac{1}{R_1}(\frac{1}{A_2}\frac{\partial\hat{u}}{\partial\alpha_2} - \frac{\hat{v}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}) \\
&\quad + \frac{1}{R_2}(\frac{1}{A_1}\frac{\partial\hat{v}}{\partial\alpha_1} - \frac{\hat{u}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2})
\end{aligned} \tag{1.13}$$

Equations (1.11) are the general linear kinematic relations used by Flügge, Novozhilov and others.

Relations of Love and Timoshenko

If in Eqs. (1.11) the terms z/R_1, z/R_2 and their products are neglected as compared to unity, one obtains

$$\begin{aligned} \varepsilon_{11} &= \hat{\varepsilon}_{11} + z\chi_{11} \\ \varepsilon_{22} &= \hat{\varepsilon}_{22} + z\chi_{22} \\ \varepsilon_{12} &= \hat{\varepsilon}_{12} + z\chi_{12} \end{aligned} \qquad (1.14)$$

where $\hat{\varepsilon}_{ij}$ and χ_{ij} have the forms given by Eqs. (1.12) and (1.13).

Equations (1.14) are the ones used by Love and Timoshenko.

Relations of Reissner and Naghdi

If in Eqs. (1.7) terms like z/R_1 and z/R_2 are neglected as compared with unity, one obtains

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \\ \varepsilon_{22} &= \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \\ \varepsilon_{12} &= \left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u\right) + \left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} v\right) \end{aligned} \qquad (1.15)$$

Then, substituting Eqs. (1.8) into Eqs. (1.15), one obtains the total strains expressed by the mid-surface parameters, which differ from Eqs. (1.14) only in that χ_{12} is given by

$$\chi_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\chi}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\psi}{A_2}\right) \qquad (1.16)$$

These are the kinematic relations used by Reissner and Naghdi.

Relations of Donnell and Mushtari

If in Eqs. (1.14), the $\hat{e}_{ij}$ are still given by Eqs. (1.12), while in the expressions for χ_{ij} given by Eqs. (1.13) the tangential displacements and their derivatives are neglected, then the Donnell and Mushtari kinematic relations are obtained. They are

$$\begin{aligned}
\chi_{11} &= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\hat{w}}{\partial\alpha_1}\right) - \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\hat{w}}{\partial\alpha_2} \\
\chi_{22} &= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\hat{w}}{\partial\alpha_2}\right) - \frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\hat{w}}{\partial\alpha_1} \qquad (1.17) \\
\chi_{12} &= -\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_2^2}\frac{\partial\hat{w}}{\partial\alpha_2}\right) - \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_1^2}\frac{\partial\hat{w}}{\partial\alpha_1}\right)
\end{aligned}$$

Relations of Vlasov

Noticing that for a thin shell z/R_i ($i = 1, 2$) is always less than unity, $1/(1+z/R_i)$ can be expanded into a geometric series; i.e.,

$$\frac{1}{\left(1+\frac{z}{R_i}\right)} = \sum_{n=0}^{\infty}\left(-\frac{z}{R_i}\right)^n, \qquad i=1,2 \qquad (1.18)$$

Substituting Eqs. (1.8) and (1.18) into Eqs. (1.7a,b,c), and neglecting the second and higher-order terms of z, one obtains kinematic relations similar to Eqs. (1.14) except that the ones for χ_{ij} become

$$\chi_{11} = \frac{1}{A_1}\frac{\partial \chi}{\partial \alpha_1} + \frac{\psi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{1}{R_1}\left(\frac{1}{A_1}\frac{\partial \hat{u}}{\partial \alpha_1} + \frac{\hat{v}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\hat{w}}{R_1}\right)$$

$$\chi_{22} = \frac{1}{A_2}\frac{\partial \psi}{\partial \alpha_2} + \frac{\chi}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} - \frac{1}{R_2}\left(\frac{1}{A_2}\frac{\partial \hat{v}}{\partial \alpha_2} + \frac{\hat{u}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\hat{w}}{R_2}\right) \qquad (1.19)$$

$$\chi_{12} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\hat{u}}{A_1}\right) - \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\hat{v}}{A_2}\right)\right]$$

$$- \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2^2}\frac{\partial \hat{w}}{\partial \alpha_2}\right) - \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_1^2}\frac{\partial \hat{w}}{\partial \alpha_1}\right)$$

whereas the $\hat{\varepsilon}_{ij}$ are the same as those given by Eqs. (1.12).

Relations of Sanders

Sanders' kinematic relations can be obtained by adding the correction factor

$$\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\frac{1}{2A_1A_2}\left(\frac{\partial A_2 \hat{v}}{\partial \alpha_1} - \frac{\partial A_1 \hat{u}}{\partial \alpha_2}\right)$$

to the χ_{12} expression of Reissner et al.. This addition is needed in order to eliminate the non-zero χ_{12} arising from rigid body rotation.

Modified Donnell relations

In Ref. [119] modified Donnell-type relations are suggested by neglecting the tangential displacements and their derivatives in the mid-surface changes in curvature and twist from Vlasov's theory. Thus

$$\chi_{11} = -\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1}\right) - \frac{1}{A_1 A_2^2}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{1}{R_1^2}\hat{w}$$

$$\chi_{22} = -\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2}\right) - \frac{1}{A_1^2 A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial \hat{w}}{\partial \alpha_1} - \frac{1}{R_2^2}\hat{w} \qquad (1.20)$$

where $\hat{\varepsilon}_{ij}$ and χ_{12} are the same as those of Donnell's relations.

Discussions

- Because of the linearization of Eqs. (1.1), it is possible to express Kirchhoff assumptions by Eqs. (1.8).

- In all the results discussed above, there are two types of expressions representing the total strains. They are summarized in Table 1.1.

Table 1.1 *Total strains at any point of a shell*

Theory	ε_{11} , ε_{22}	ε_{12}
Flügge, Novozhilov	$\frac{1}{(1+\frac{z}{R_1})}(\hat{\varepsilon}_{11} + z\chi_{11})$ $\frac{1}{(1+\frac{z}{R_2})}(\hat{\varepsilon}_{22} + z\chi_{22})$	$\frac{1}{(1+\frac{z}{R_1})(1+\frac{z}{R_2})}[(1-\frac{z^2}{R_1R_2})\hat{\varepsilon}_{12}$ $+ z(1+\frac{z}{2R_1}+\frac{z}{2R_2})\chi_{12}]$
Love, Timoshenko, Reissner, Naghdi, Sanders, Vlasov, Donnell, Mushtari, Modified Donnell	$\hat{\varepsilon}_{11} + z\chi_{11}$ $\hat{\varepsilon}_{22} + z\chi_{22}$	$\hat{\varepsilon}_{12} + z\chi_{12}$

It can easily be seen that the expressions for the mid-surface strains $\hat{\varepsilon}_{ij}$ are the same according to all the theories presented above. They are given by Eqs. (1.12).

- The changes in curvature and twist of the mid-surface are summarized in Tables 1.2 and 1.3 respectively.

Table 1.2 Changes in curvature of the mid-surface

Theory	χ_{11}	χ_{22}
Flügge, Novozhilov, Love, Timoshenko, Reissner, Naghdi, Sanders	$\frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1} + \frac{\psi}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}$	$\frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{\chi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}$
Vlasov	$\frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1} + \frac{\psi}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} - \frac{1}{R_1}(\frac{1}{A_1}\frac{\partial\hat{u}}{\partial\alpha_1} + \frac{\hat{v}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{\hat{w}}{R_1})$	$\frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{\chi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} - \frac{1}{R_2}(\frac{1}{A_2}\frac{\partial\hat{v}}{\partial\alpha_2} + \frac{\hat{u}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \frac{\hat{w}}{R_2})$
Donnell, Mushtari	$-\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{1}{A_1}\frac{\partial\hat{w}}{\partial\alpha_1}) - \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\hat{w}}{\partial\alpha_2}$	$-\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{1}{A_2}\frac{\partial\hat{w}}{\partial\alpha_2}) - \frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\hat{w}}{\partial\alpha_1}$
Modified Donnell	$-\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{1}{A_1}\frac{\partial\hat{w}}{\partial\alpha_1}) - \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\hat{w}}{\partial\alpha_2} - \frac{1}{R_1^2}\hat{w}$	$-\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{1}{A_2}\frac{\partial\hat{w}}{\partial\alpha_2}) - \frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\hat{w}}{\partial\alpha_1} - \frac{1}{R_2^2}\hat{w}$

Table 1.3 Change in twist of the mid-surface

Theory	χ_{12}
Flügge, Novozhilov, Timoshenko, Love	$\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\chi}{A_1}) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\psi}{A_2}) + \frac{1}{R_1}(\frac{1}{A_2}\frac{\partial\hat{u}}{\partial\alpha_2} - \frac{\hat{v}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}) + \frac{1}{R_2}(\frac{1}{A_1}\frac{\partial\hat{v}}{\partial\alpha_1} - \frac{\hat{u}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2})$
Reissner, Naghdi	$\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\chi}{A_1}) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\psi}{A_2})$
Vlasov	$(\frac{1}{R_1} - \frac{1}{R_2})[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\hat{u}}{A_1}) - \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\hat{v}}{A_2})] - \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{1}{A_2^2}\frac{\partial\hat{w}}{\partial\alpha_2}) - \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{1}{A_1^2}\frac{\partial\hat{w}}{\partial\alpha_1})$
Sanders	$\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\chi}{A_1}) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\psi}{A_2}) + \frac{1}{2A_1A_2}(\frac{1}{R_2} - \frac{1}{R_1})(\frac{\partial A_2\hat{v}}{\partial\alpha_1} - \frac{\partial A_1\hat{u}}{\partial\alpha_2})$
Mushtari, Donnell, Modified Donnell	$-\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{1}{A_2^2}\frac{\partial\hat{w}}{\partial\alpha_2}) - \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{1}{A_1^2}\frac{\partial\hat{w}}{\partial\alpha_1})$

It is found that there is a partial agreement among the different theories for the expressions of the mid-surface curvature changes χ_{11}, χ_{22}. The exceptions are the Donnell-type expressions which differ from others because of neglecting terms containing the tangential displacements $\hat{u}$ and $\hat{v}$, and Vlasov's expressions differ from the others because of replacing $1/(1+z/R_i)$ by its series expansion in the derivation.

However, there is a widespread disagreement concerning the proper form for the mid-surface change in twist χ_{12}. The Donnell-type expression differs from the others for the same reasons given in the discussion of χ_{11} and χ_{22}. The Reissner-Naghdi expression differs from the others because of neglecting z/R_i compared with unity at an earlier stage in the derivation than the Flügge et al.'s derivation. Sanders' expression is derived through modifying Reissner-Naghdi formula by adding a term to eliminate the non-zero χ_{12} arising from rigid body rotation. All these theories except those of Donnell and Reissner-Naghdi give no deformation due to rigid body motion.

- All the above theories are based on Love's first approximation. But in many cases, these assumptions may not yield satisfactory results. In such cases some of them must be discarded. For example, to broaden the scope of the theory by including the effects of transverse normal and shear deformations, the assumptions 3 and 4 must be discarded. The thin shell theories without Love's first approximation will not be discussed here despite their importance for certain applications. Nevertheless, all the theories discussed before are linear and they can only be applied to some particular class of shell problems, while the nonlinear theories, which embrace all elastic deformation problems, will be discussed below.

1.4 Some Nonlinear Thin Shell Theories

Up to now, the search for appropriate geometrically nonlinear small strain Kirchhoff-Love type theories, which are able to describe the large deflection and stability behavior of thin elastic shells, has already led to many successful formulations. Some of them are presented in Refs. [111-119]. In the following, the Novozhilov, Sanders and Donnell-type nonlinear thin shell theories are rederived from the three dimensional elasticity formulation, and their differences are reviewed. As mentioned earlier, this effort is not motivated by an inability of the existing theories to solve the present problem, but rather by a desire to choose a theory that is characterized by simplicity, consistency, clarity and accuracy for the present investigation.

Kirchhoff assumptions

According to Eqs. (1.1) and (1.6), the Kirchhoff hypothesis $\varepsilon_{1z} = \varepsilon_{2z} = \varepsilon_{zz} = 0$ can be formulated analytically as follows;

$$\begin{aligned}
\varepsilon_{1z} = \frac{\partial u}{\partial z} &+ \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right) + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v + \frac{w}{R_1}\right)\frac{\partial u}{\partial z} \\
&+ \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)\frac{\partial v}{\partial z} + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)\frac{\partial w}{\partial z} = 0 \\
\varepsilon_{2z} = \frac{\partial v}{\partial z} &+ \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right) + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u + \frac{w}{R_2}\right)\frac{\partial v}{\partial z} \\
&+ \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)\frac{\partial u}{\partial z} + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)\frac{\partial w}{\partial z} = 0 \\
\varepsilon_{zz} = \frac{\partial w}{\partial z} &+ \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] = 0
\end{aligned} \tag{1.21}$$

Meanwhile one can expand the displacement components into power series;

$$\begin{aligned}
u &= u(\alpha_1, \alpha_2, 0) + \left(\frac{\partial u}{\partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^2 u}{\partial z^2}\right)_0 z^2 + \dots \\
v &= v(\alpha_1, \alpha_2, 0) + \left(\frac{\partial v}{\partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^2 v}{\partial z^2}\right)_0 z^2 + \dots \\
w &= w(\alpha_1, \alpha_2, 0) + \left(\frac{\partial w}{\partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^2 w}{\partial z^2}\right)_0 z^2 + \dots
\end{aligned} \tag{1.22}$$

Retaining in the series only the first two terms and introducing the notations

$$u(\alpha_1, \alpha_2, 0) = \hat{u}, \quad v(\alpha_1, \alpha_2, 0) = \hat{v}, \quad w(\alpha_1, \alpha_2, 0) = \hat{w}$$
$$(\frac{\partial u}{\partial z})_0 = \chi, \quad (\frac{\partial v}{\partial z})_0 = \psi, \quad (\frac{\partial w}{\partial z})_0 = \phi \tag{1.23}$$

one obtains the following expressions for the displacements

$$u = \hat{u}(\alpha_1, \alpha_2) + z\chi(\alpha_1, \alpha_2)$$
$$v = \hat{v}(\alpha_1, \alpha_2) + z\psi(\alpha_1, \alpha_2) \tag{1.24}$$
$$w = \hat{w}(\alpha_1, \alpha_2) + z\phi(\alpha_1, \alpha_2)$$

Notice that Eq. (1.24c) differs from Eq. (1.8c) because of using the nonlinear relations of Eqs. (1.21).

Substituting Eqs. (1.24) into Eqs. (1.21) and ordering by powers of z, one obtains five equations, two of them, however, are implied by one of the other three. Hence, just three equations are independent, namely

$$\chi^2 + \psi^2 + (1+\phi)^2 = 1$$
$$(1+\phi)(\frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} - \frac{\hat{u}}{R_1}) + (\frac{1}{A_2}\frac{\partial \hat{u}}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\hat{v})\psi$$
$$+ (1 + \frac{1}{A_1}\frac{\partial \hat{u}}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\hat{v} + \frac{\hat{w}}{R_1})\chi = 0 \tag{1.25}$$
$$(1+\phi)(\frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{\hat{v}}{R_2}) + (\frac{1}{A_1}\frac{\partial \hat{v}}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\hat{u})\chi$$
$$+ (1 + \frac{1}{A_2}\frac{\partial \hat{v}}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\hat{u} + \frac{\hat{w}}{R_2})\psi = 0$$

Simultaneous solution of the last two equations yields

$$\chi = \frac{\hat{\alpha}_{31}}{\hat{\alpha}_{33}}(1+\phi)$$
$$\psi = \frac{\hat{\alpha}_{32}}{\hat{\alpha}_{33}}(1+\phi) \tag{1.26}$$

where

$$\hat{\alpha}_{31} = -\,\hat{e}_{13}(1+\hat{e}_{22}) + \hat{e}_{23}\hat{e}_{12}$$
$$\hat{\alpha}_{32} = -\,\hat{e}_{23}(1+\hat{e}_{11}) + \hat{e}_{13}\hat{e}_{21} \tag{1.27}$$
$$\hat{\alpha}_{33} = (1+\hat{e}_{11})(1+\hat{e}_{22}) - \hat{e}_{12}\hat{e}_{21}$$

with

$$\hat{e}_{11} = \frac{1}{A_1}\frac{\partial \hat{u}}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\hat{v} + \frac{\hat{w}}{R_1}$$
$$\hat{e}_{22} = \frac{1}{A_2}\frac{\partial \hat{v}}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\hat{u} + \frac{\hat{w}}{R_2}$$
$$\hat{e}_{21} = \frac{1}{A_1}\frac{\partial \hat{v}}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\hat{u}$$
$$\hat{e}_{13} = \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} - \frac{\hat{u}}{R_1} \tag{1.28}$$
$$\hat{e}_{12} = \frac{1}{A_2}\frac{\partial \hat{u}}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\hat{v}$$
$$\hat{e}_{23} = \frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{\hat{v}}{R_2}$$

Substituting Eqs. (1.26) into the first of Eqs. (1.25), one obtains

$$\phi = \frac{\hat{\alpha}_{33}}{\sqrt{\hat{\alpha}_{31}^2 + \hat{\alpha}_{32}^2 + \hat{\alpha}_{33}^2}} - 1 \tag{1.29}$$

Here it must be mentioned that[112]

$$\sqrt{\alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2} = \frac{1 + \nabla}{1 + E_\zeta} \tag{1.30}$$

where ∇ is the relative change in volume and E_ζ is the relative elongation of a line element of the shell after the deformation.

Substitution in the above equation of $\hat{\alpha}_{ij}$ for α_{ij}, which corresponds to the replacement of u, v, w by $\hat{u}$, $\hat{v}$, $\hat{w}$, is equivalent to considering ∇ and E_ζ on the mid-surface of the shell. Hence, if the shears and elongations are neglected in comparison with unity, one obtains

$$\sqrt{\hat{\alpha}_{31}^2 + \hat{\alpha}_{32}^2 + \hat{\alpha}_{33}^2} \approx 1 \tag{1.31}$$

Thus

$$\phi \approx \hat{\alpha}_{33} - 1 = \hat{e}_{11} + \hat{e}_{22} + \hat{e}_{11}\hat{e}_{22} - \hat{e}_{12}\hat{e}_{21} \tag{1.32}$$

Substituting Eq. (1.32) into Eqs. (1.26), one obtains

$$\begin{aligned} \chi &= - \hat{e}_{13}(1+\hat{e}_{22}) + \hat{e}_{23}\hat{e}_{12} \\ \psi &= - \hat{e}_{23}(1+\hat{e}_{11}) + \hat{e}_{13}\hat{e}_{21} \end{aligned} \tag{1.33}$$

Equations (1.32), (1.33) and (1.24) express the displacements of an arbitrary point of the shell in terms of the displacements of the corresponding point of the mid-surface, which will serve as the basis for the following derivations.

Relations of Novozhilov

Substituting Eqs. (1.4) and (1.5) into Eqs. (1.1), one obtains

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v + \frac{w}{R_1}\right) \\
&+ \frac{1}{2}\frac{1}{(1+\frac{z}{R_1})^2}\left[\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v + \frac{w}{R_1}\right)^2\right. \\
&\left. + \left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right)^2 + \left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)^2\right] \\
\varepsilon_{22} &= \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u + \frac{w}{R_2}\right) \\
&+ \frac{1}{2}\frac{1}{(1+\frac{z}{R_2})^2}\left[\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)^2\right. \qquad (1.34) \\
&\left. + \left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u + \frac{w}{R_2}\right)^2 + \left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)^2\right] \\
\varepsilon_{12} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right) + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right) \\
&+ \frac{1}{(1+\frac{z}{R_1})(1+\frac{z}{R_2})}\left[\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v + \frac{w}{R_1}\right)\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)\right. \\
&+ \left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right)\left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u + \frac{w}{R_2}\right) \\
&\left. + \left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)\right]
\end{aligned}$$

In the case of thin shells, one can set $1+z/R_i \approx 1$. Further substituting Eqs. (1.24) into Eqs. (1.34), one obtains the Novozhilov nonlinear kinematic relations as

$$\begin{aligned} \varepsilon_{11} &= \hat{\varepsilon}_{11} + z\chi_{11} + z^2\rho_{11} \\ \varepsilon_{22} &= \hat{\varepsilon}_{22} + z\chi_{22} + z^2\rho_{22} \\ \varepsilon_{12} &= \hat{\varepsilon}_{12} + z\chi_{12} + z^2\rho_{12} \end{aligned} \tag{1.35}$$

where $\hat{\varepsilon}_{ij}$ are the elongations and shear of the mid-surface of the shell. They are given by

$$\begin{aligned} \hat{\varepsilon}_{11} &= \hat{e}_{11} + \frac{1}{2}[\hat{e}_{11}^2 + \hat{e}_{12}^2 + \hat{e}_{13}^2] \\ \hat{\varepsilon}_{22} &= \hat{e}_{22} + \frac{1}{2}[\hat{e}_{21}^2 + \hat{e}_{22}^2 + \hat{e}_{23}^2] \\ \hat{\varepsilon}_{12} &= \hat{e}_{12} + \hat{e}_{21} + \hat{e}_{11}\hat{e}_{21} + \hat{e}_{22}\hat{e}_{12} + \hat{e}_{13}\hat{e}_{23} \end{aligned} \tag{1.36}$$

The parameters χ_{11}, χ_{22} and χ_{12}, which characterize the variations of the curvature and the twist of the mid-surface induced by the deformation, are given by

$$\begin{aligned} \chi_{11} &= (1+\hat{e}_{11})k_{11} + \hat{e}_{12}k_{12} + \hat{e}_{13}k_{13} \\ \chi_{22} &= (1+\hat{e}_{22})k_{22} + \hat{e}_{21}k_{21} + \hat{e}_{23}k_{23} \\ \chi_{12} &= k_{21}(1+\hat{e}_{11}) + k_{12}(1+\hat{e}_{22}) + k_{11}\hat{e}_{21} + k_{22}\hat{e}_{12} + k_{13}\hat{e}_{23} + k_{23}\hat{e}_{13} \end{aligned} \tag{1.37}$$

where

$$\begin{aligned} k_{11} &= \frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1} + \frac{\psi}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{\phi}{R_1} \\ k_{22} &= \frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{\chi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \frac{\phi}{R_2} \\ k_{21} &= \frac{1}{A_2}\frac{\partial\chi}{\partial\alpha_2} - \frac{\psi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \end{aligned}$$

$$k_{12} = \frac{1}{A_1}\frac{\partial \psi}{\partial \alpha_1} - \frac{\chi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \tag{1.38}$$

$$k_{13} = \frac{1}{A_1}\frac{\partial \phi}{\partial \alpha_1} - \frac{\chi}{R_1}$$

$$k_{23} = \frac{1}{A_2}\frac{\partial \phi}{\partial \alpha_2} - \frac{\psi}{R_2}$$

The above Novozhilov theory is valid for the problems with small strains and arbitrary rotations. The presence of the parameters ρ_{11}, ρ_{22} and ρ_{12} in Eqs. (1.35) indicates that the linear law of variation of displacements through the thickness of the shell can result in a nonlinear variation in the strain components. However, for small elongations and shears ρ_{11}, ρ_{22} and ρ_{12} are always negligible. Thus, it is not necessary to present the detailed expressions for ρ_{11}, ρ_{22} and ρ_{12}.

Relations of Sanders

Sanders' nonlinear kinematic relations can be obtained by neglecting some second-order terms from Eqs. (1.36), and also retaining only the linear terms of Eqs. (1.37). They are

$$\hat{\varepsilon}_{11} = \hat{e}_{11} + \frac{1}{2}\hat{e}_{13}^2$$

$$\hat{\varepsilon}_{22} = \hat{e}_{22} + \frac{1}{2}\hat{e}_{23}^2 \tag{1.39}$$

$$\hat{\varepsilon}_{12} = \hat{e}_{12} + \hat{e}_{21} + \hat{e}_{13}\hat{e}_{23}$$

and

$$\chi_{11} = \frac{1}{A_1}\frac{\partial \chi}{\partial \alpha_1} + \frac{\psi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\chi_{22} = \frac{1}{A_2}\frac{\partial \psi}{\partial \alpha_2} + \frac{\chi}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{1.40}$$

$$\chi_{12} = k_{21} + k_{12}$$

where

$$\chi = -\hat{e}_{13}$$
$$\psi = -\hat{e}_{23} \qquad (1.41)$$

The above Sanders' relations corresponds to small strains, moderate rotations about in-plane axes and small rotation about the normal.

Relations of Donnell

Equations (1.39)-(1.41) can be further simplified so as to yield the so-called Donnell-type equations for quasi-shallow shells. The simplifications are based on the consideration that the terms containing $\hat{u}$ and $\hat{v}$ in Eqs. (1.41) are of negligible influence for shell segments that are almost flat and for shells whose displacement components in the deformed configuration are rapidly varying functions of the shell coordinates. Neglecting $\hat{u}$ and $\hat{v}$ in Eqs. (1.41) yields

$$\chi = -\hat{e}_{13} = -\frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1}$$
$$\psi = -\hat{e}_{23} = -\frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2} \qquad (1.42)$$

Substituting Eqs. (1.42) into Eqs. (1.39)-(1.41) yields the widely used expressions for the quasi-shallow shells. The nonlinear Donnell-type kinematic relations are perhaps the most popular theory for many engineering analysis of shells. Most of the published results for the stability analysis of conical shells are based on the Donnell-type theory. Many previous results obtained by using this theory and the comparisons with those from other theories[120] show that the Donnell theory can provide sufficiently accurate results for many engineering applications. The main reason for the wide employment of the Donnell-type theory is that a stress function can be introduced, which leads to a reduction in the number of dependent variables in the analysis. Thus high computational efficiency can be achieved. However, one must keep in mind that the Donnell-type theory is not accurate enough when applied to shells which buckle in an almost inextensional mode.

1.5 Discussions and Conclusion

In this chapter, both the linear and the nonlinear thin shell theories based on Kirchhoff-Love assumptions are reviewed and discussed. As mentioned earlier, this effort is not motivated by an inability of the existing theories to solve the present problem, but rather by a desire to choose and to define a theory that is characterized by simplicity, consistency, clarity and accuracy for the stability analysis of anisotropic conical shells.

As suggested by Naghdi[113] and Bushnell[121], the differences attributable to retention of z/R_i are of little importance for most engineering problems, and it is best to choose the simplest theory in this regard.

All the simplifications and assumptions used in the derivation of the kinematic relations should be kept consistent with those which will be used in the derivation of constitutive relations.

For the nonlinear theories of Sanders and Donnell, the nonlinear terms are only retained in the strain-displacement relations, but not in the curvature-displacement relations. The physical explanation for this simplification is given by Bushnell[121] as follows: If a thin shell deflects a large amount, let us say with an amplitude many times the thickness, the strains are usually small even though the deflections are rather large. Hence, the linear terms in the strain-displacement relations will tend to cancel each other, and the nonlinear terms will become significant for much smaller displacements than they would have if the linear terms had not been canceled. The linear terms in the expressions for the change in curvature, however, do not tend to cancel, and the wall rotations must be large indeed before nonlinear terms have to be included in these expressions.

Many previous results obtained by using the nonlinear Donnell-type theory and the comparisons with those from other theories[120] show that the Donnell theory can yield sufficiently accurate results for many engineering applications, including the case of conical shells. Besides, compared with other nonlinear theories, one can see that the Donnell theory is the simplest one. Therefore, it is decided to use the nonlinear Donnell-type theory in the present stability analysis of anisotropic conical shells.

Chapter 2
Constitutive Relations

2.1 Introduction

In recent years, with the rapid development of various types of advanced composite materials and their corresponding manufacturing techniques[122-125], there has been a continuous increase of applications of composite materials for different shell structures. Often encountered among these applications are various kinds of conical shells with different wall constructions. Whereas it is known that the successful employment of composite materials depends on many factors, two of them, the manufacturing process and the method of analysis can be pinpointed as the principal ones. However, while the importance of fabrication process and method of analysis is well recognized, few verifications can be found in the open literature for correlations between the specific manufacturing process and the corresponding method of analysis. In other words, many analyses were based on rather rough modelling and simplifications, some of them have even been performed on the idealized structures which cannot be practically realized. As an example, most of the published theoretical results for the stability analysis of laminated conical shells are based on the simplification of using classical lamination theory without considering the specific techniques by which the conical shells are built[84]. The reasons for this inconsistency between manufacturing processes and methods of analysis are mainly due to the difficulties of deriving a theoretical model for certain manufacturing processes and the indifference of some theoretical analyses as to the real manufacturing processes.

In an effort of trying to correlate the specific manufacturing processes with appropriate methods of analysis and to obtain more accurate modelling and analysis results for real structures, in the following the constitutive relations of conical shells with different wall constructions are reviewed, derived and discussed for some commonly used manufacturing processes[91].

2.2 Orthotropic Conical Shells

Among various kinds of anisotropic materials and structures, the cases with orthotropic behavior take a special place for both manufacturing processes and theoretical analysis. The literature connected with the orthotropic plates and cylindrical shells is enormous and will not be cited here. Some results for the behavior of 'orthotropic' conical shells have also been published, see Serpico[86], Singer[23,79,87], Dixon[88], Schiffner[89], Dumir et al.[90], Librescu[126,127], Weingarten and Seide[7] and more recently Tong, Tabarrok and Wang[93]. Despite the continuous interest for the analysis of 'orthotropic' conical shells, it is found that there are some gaps between the theoretical analysis and the possible manufacturing processes. Some basic questions, such as, "what is an orthotropic conical shell" or "how could one build it" are still not clearly answered. In the following some 'orthotropic' constitutive relations of conical shells are presented, and the attention is addressed to the relations of these constitutive equations with the possible fabrication processes in practice.

2.2.1 Classical definition of orthotropic conical shells

Before considering the orthotropic constitutive relations of conical shells, let us first recall the definitions for the orthotropic cylindrical shells. Generally speaking, there exist two kinds of orthotropic cylindrical shells. One of them is the classical orthotropic cylindrical shell, its constitutive equations[109,128] are formulated as Eqs. (2.1). Another is called modified orthotropic cylindrical shell, its constitutive equations[2,129] are given as Eqs. (2.2).

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ (M_{xy}+M_{yx})/2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ (M_{xy}+M_{yx})/2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & C_{14} & C_{15} & 0 \\ C_{12} & C_{22} & 0 & C_{24} & C_{25} & 0 \\ 0 & 0 & C_{33} & 0 & 0 & C_{36} \\ C_{14} & C_{24} & 0 & C_{44} & C_{45} & 0 \\ C_{15} & C_{25} & 0 & C_{45} & C_{55} & 0 \\ 0 & 0 & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{2.2}$$

where the notation and sign convention of the cylindrical shell and its force and moment resultants are shown in Figures 2.1 and 2.2.

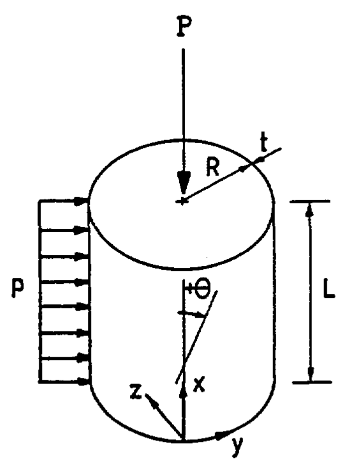


Fig. 2.1 *Notation and sign convention of cylindrical shell*

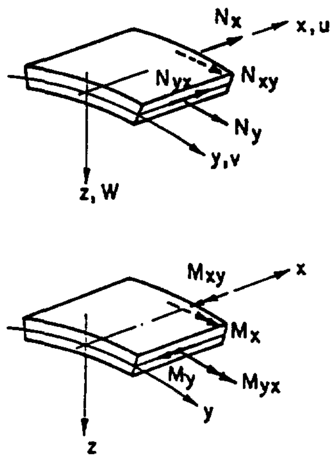


Fig. 2.2 *Definition of stress and moment resultants of cylindrical shell*

It can be seen that the constitutive relations of modified orthotropic cylindrical shells differ from the classical ones by including the coupling terms between extensional forces and curvature change and between bending moments and extensional strain in its formulation. Thus, the modified orthotropic cylindrical shells have more general meanings than the classical one. One must emphasize, however, that in Eqs. (2.1)-(2.2) the coefficients C_{ij} are usually assumed to be constants.

The classical orthotropic cylindrical shells can be built by using some kind of single layer homogenous orthotropic materials, or by using some laminate exactly symmetric about its mid-surface[1], while the modified orthotropic cylindrical shells cover cylindrical sheets stiffened by closely-spaced circular rings or longitudinal stringers, fiber-reinforced shells, corrugated-skin constructions, etc.[129]. That is to say, different manufacturing processes lead to different formulations of the constitutive relations. Therefore, it is necessary to know the specific manufacturing process in order to choose the appropriate constitutive relations.

For conical shells the situations are much more complicated. From a literature study it was found that in most of the theoretical analyses of orthotropic conical shells little attention has been paid to the relation between the modelling of constitutive relations and the real manufacturing processes used to build the shells. There is even no clear definition for the 'orthotropic' constitutive relations of conical shells which are properly related to certain manufacturing processes in practice. The constitutive equations[87,89,126-128] of the classical orthotropic conical shells are given as Eqs. (2.3). Here again, the C_{ij} coefficients are assumed to be constants.

The corresponding notation and sign convention, and the stress and moment resultants are shown in Figures 2.3 and 2.4.

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_s \\ M_\theta \\ (M_{s\theta}+M_{s\theta})/2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \\ \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.3}$$

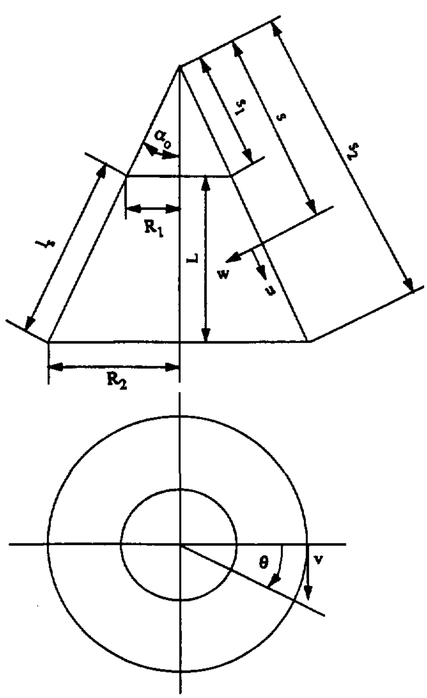


Fig. 2.3 Notation and sign convention of a conical shell

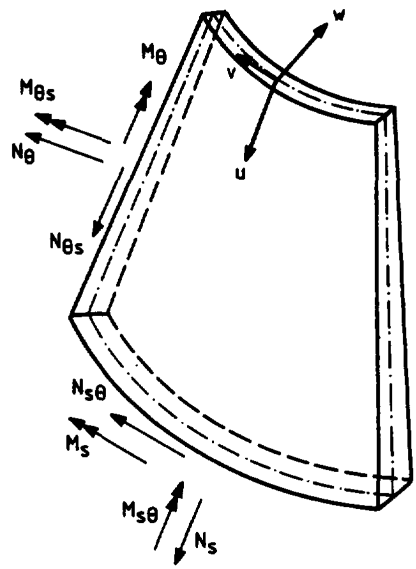


Fig. 2.4 *Definition of stress and moment resultants of a conical shell*

2.2.2 Modifications of the constitutive relations of classical orthotropic conical shells

Equations (2.3) are employed by nearly all researchers for the analysis of orthotropic conical shells. In some of the analyses the orthotropic character of the shell was described by only distinguishing between Young's modulus and Poisson's ratio in two mutually orthogonal directions without considering the practical possibilities to build the shells[7,86,93], while in other analyses the orthotropic character was described by some greatly simplified modelling of stiffeners[88-90]. Hence, one can conclude that the constitutive relations of classical orthotropic conical shells expressed by Eqs. (2.3) have only academic value, which can hardly be achieved in the real manufacturing environment. Therefore, it is imperative to modify the above unrealistic orthotropic constitutive relations and to define an alternative one which is closer to the manufacturing processes used.

The first modification was suggested by Baruch, Singer and Harari[82] for ring and stringer stiffened isotropic conical shells. Later, it was rewritten in the matrix form by Baruch, Arbocz and Zhang[84] as

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_s \\ M_\theta \\ (M_{s\theta}+M_{s\theta})/2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & B_{22} & 0 & D_{21} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \\ \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.4}$$

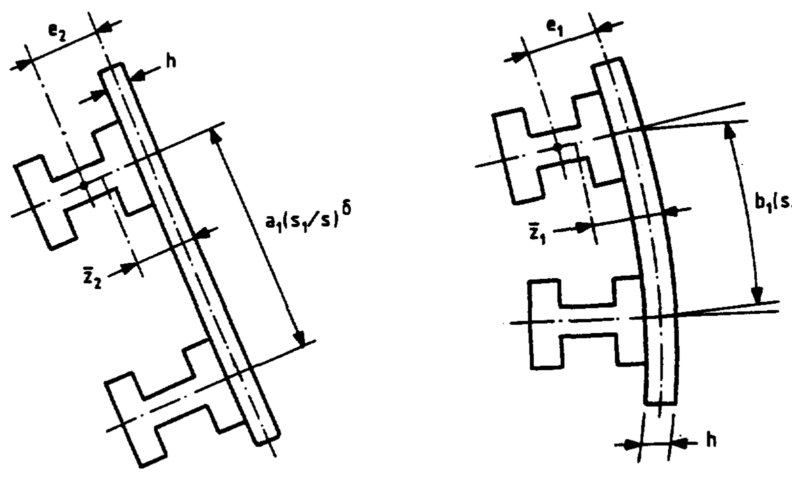
where

$$\begin{aligned}
A_{11} &= Eh/(1-\nu^2) + (s_1/s)(E_1A_1/b_1) = A_{11}(s) \\
A_{12} &= A_{21} = \nu Eh/(1-\nu^2) \\
A_{22} &= Eh/(1-\nu^2) + (s/s_1)^\delta (E_2A_2/a_1) = A_{22}(s) \\
A_{66} &= Eh/2(1+\nu) \\
B_{11} &= (s_1/s)(E_1A_1e_1/b_1) = B_{11}(s) \\
B_{22} &= (s/s_1)^\delta (E_2A_2e_2/a_1) = B_{22}(s) \\
D_{11} &= D + (s_1/s)(E_1I_{01}/b_1) = D_{11}(s) \\
D_{12} &= D_{21} = \nu D \\
D_{22} &= D + (s/s_1)^\delta (E_2I_{02}/a_1) = D_{22}(s) \\
D_{66} &= D(1-\nu)/2 + (1/4)[(s_1/s)(G_1I_{t1}/b_1) + (s/s_1)^\delta (G_2I_{t2}/a_1)] = D_{66}(s)
\end{aligned} \tag{2.5}$$

where A_1 and A_2 are the cross-sectional areas of the stringers and the rings respectively; a_1 is the distance between the rings at s_1; b_1 is the distance between the stringers at s_1; e_1 is the distance of the centroid of the stringer cross section from the shell mid-surface; e_2 is the distance of the centroid of the ring cross section from the shell mid-surface. E, E_1, and E_2 are the moduli of elasticity of the sheet and the stiffeners respectively; G_1 and G_2 are the shear moduli of the stiffeners and h is the thickness of the sheet. I_{o1} and I_{o2} are the moments of inertia of stringer and ring cross sections, respectively, about the line of reference; I_{t1} and I_{t2} are the torsion constants of the stiffener-cross sections; ν is the

Poisson's ratio of the sheet; and δ is a parameter introduced in Ref. [82] for optimization purposes (see also Fig. 2.5).

The description for the stiffeners is shown in Figure 2.5.



detail of rings detail of stringers

Fig. 2.5 Description of stiffeners on the surface of a conical shell

Following the definition for the modified orthotropic cylindrical shell, the material properties of the stiffened isotropic conical shells with constitutive relations given by Equations (2.4) are also orthotropic. Considering the fact that the stiffness coefficients of the stiffened conical shell are functions of the longitudinal coordinate s, which is not necessarily the case for stiffened cylindrical shells, one can classify the constitutive relations which have the form similar to Eqs. (2.2), but with variable stiffness coefficients, as quasi-orthotropic. However, if the shell is short or the semi-vertex angle is small, the variable stiffness coefficients can be taken as constants, approximately.

Further modification for the orthotropic constitutive relations of conical shells is presented by Baruch, Arbocz and Zhang[84] via studying the constitutive relations of laminated conical shells made by filament winding. According to their results, the additions to the stiffness coefficients [A], [B] and [D] due to the stiffeners of a non-traditionally stiffened conical shell can be written as

$$[A]_a = 2a_{11}J_2$$
$$[B]_a = 2b_{11}J_2 \qquad (2.6)$$
$$[D]_a = 2d_{11}J_2 + 2d_{66}L_2$$

where

$$a_{11} = E_{st}A_{st}/b$$
$$b_{11} = E_{st}A_{st}e/b$$
$$d_{11} = E_{st}I_{ost}/b \qquad (2.7)$$
$$d_{66} = (1/4)G_{st}I_{tst}/b$$

and G_{st} is the shear modulus of elasticity of the stiffener; A_{st} is the cross-section area of the stiffener; e is the distance of the centroid of the stiffener cross-section from the surface of reference; I_{ost} is the moment of inertia of the stiffener cross-section about the reference line; I_{tst} is the torsion constant of the stiffener cross-section; b is the distance between the stiffeners at s.

The balanced matrices J_2 and L_2 are given by

$$J_2 = \begin{bmatrix} C^4 & C^2S^2 & 0 \\ C^2S^2 & S^4 & 0 \\ 0 & 0 & C^2S^2 \end{bmatrix}$$
$$L_2 = \begin{bmatrix} 4C^2S^2 & -4C^2S^2 & 0 \\ -4C^2S^2 & 4C^2S^2 & 0 \\ 0 & 0 & (C^2-S^2)^2 \end{bmatrix} \qquad (2.8)$$

where

$$C = \cos\beta$$
$$S = \sin\beta \qquad (2.9)$$

Notice that if the inclination angle β of the stiffener is a function of the coordinate s, then the stiffness coefficients of a non-traditionally stiffened conical shells are also functions of the coordinate s. Therefore, the corresponding constitutive equations of this kind of structure are also quasi-orthotropic. However, as suggested by Ref. [84], if a_{11}, b_{11}, d_{11} and β are made to be constants, a classical orthotropic behavior can be expected.

Based on the same idea of using non-traditional stiffeners, the orthotropic conical shells can also be built in a thermoplastic composite by non-geodetic filament winding. The filament winding method and its application for conical shells will be discussed in the next section.

Van Rijn suggested another method, called prepreg cone segment method, for the manufacturing of an orthotropic conical shell[130]. His basic idea consists of assembling a cone out of several prepreg segments. The segments with variable fibre angles are produced and placed on the mandrel with a small overlap at the joints. The detailed description for the manufacturing process, productional and computational remarks related to this method are given in Ref. [130].

2.2.3 Discussions and conclusion

To summarize briefly, the constitutive relations of classical orthotropic conical shells given by Eqs. (2.3) have mainly academic value. Till now, they can hardly be realized by practical manufacturing processes. The quasi-orthotropic constitutive relations given by Eqs. (2.4) and Eqs. (2.6), however, are more representative of the corresponding manufacturing processes. The stiffness coefficients in Eqs. (2.4) and Eqs. (2.6) can be built as constants in real manufacturing processes by employing some special techniques. Thus the quasi-orthotropic constitutive equations of conical shells become equivalent with those of modified orthotropic cylindrical shells.

2.3 Laminated Conical Shells Made by Filament Winding

The rapid development and wide applications of composite materials have generated a number of possible fabrication processes. A summary of the existing fabrication techniques can be found in Refs. [3,124,125]. Among these fabrication processes filament winding offers some distinct advantages over other processes. From practical considerations the primary advantages are[124,131]: low cost, highly repetitive nature of fiber placement, capable of using continuous fibers over a whole component area, suitable also for large structures and automated production and the possibility for using high fiber volume. What is more, filament winding provides possibility and flexibility for optimal structural design upon which the successful implementation of a composite material in a specific structural application is often hinged. From theoretical point of view, the analysis and conclusions for laminated conical shells made by filament winding are also valid for the laminated conical shells made by other continuous fiber reinforcement processes. Therefore, in the following attention is focused on the modelling of constitutive relations of laminated conical shells made by filament winding.

2.3.1 Filament winding processes and winding patterns

Filament winding is a reinforced plastic process that employs a series of continuous resin impregnated fibers applied to a rotating mandrel in a predetermined geometrical relationship under controlled tension. Classical filament winding applies a series of reinforcements, roving, drawn through a resin bath mounted on a traversing carriage. While the mandrel rotates about its central axis, the carriage traverses a number of circuits from end to end of the mandrel. The desired wall thickness is built up on the mandrel. The composite is then cured on the mandrel, and at the end of the cure cycle, the mandrel is extracted from the wound product[132]. Since it is possible to align the reinforcement along the direction of high stress, the strength of filaments can be utilized in an efficient manner. Besides, filament winding offers also a wide variety of materials available to the design engineers. The simplest form of filament winding is shown in Figure 2.6, the typical manufacturing flow diagram for filament winding is shown in Figure 2.7 and the possible winding patterns are summarized in Figure 2.8.

In Figure 2.8 V_m is the volume fraction of the matrix, V_f is the volume fraction of the fiber, t is the wall thickness of the shell at s, β is the winding angle at s, n is the number of fibers in one cross-section of the shell.

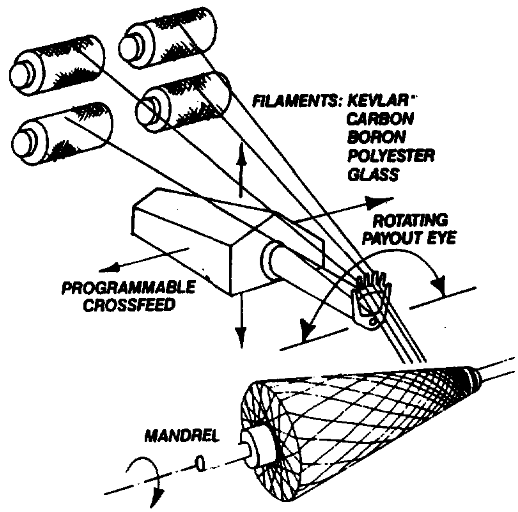


Fig. 2.6 *Filament winding process*[133]

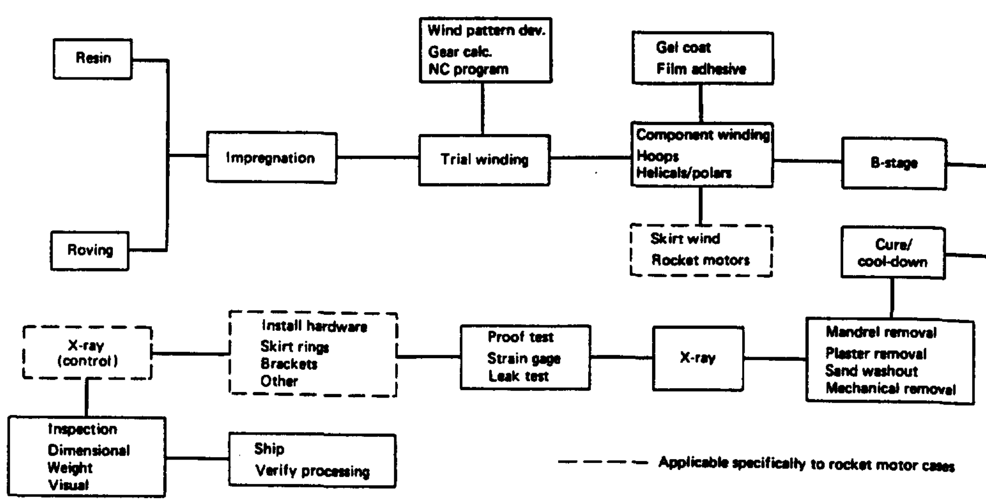


Fig. 2.7 *Typical manufacturing flow diagram for filament winding*[125]

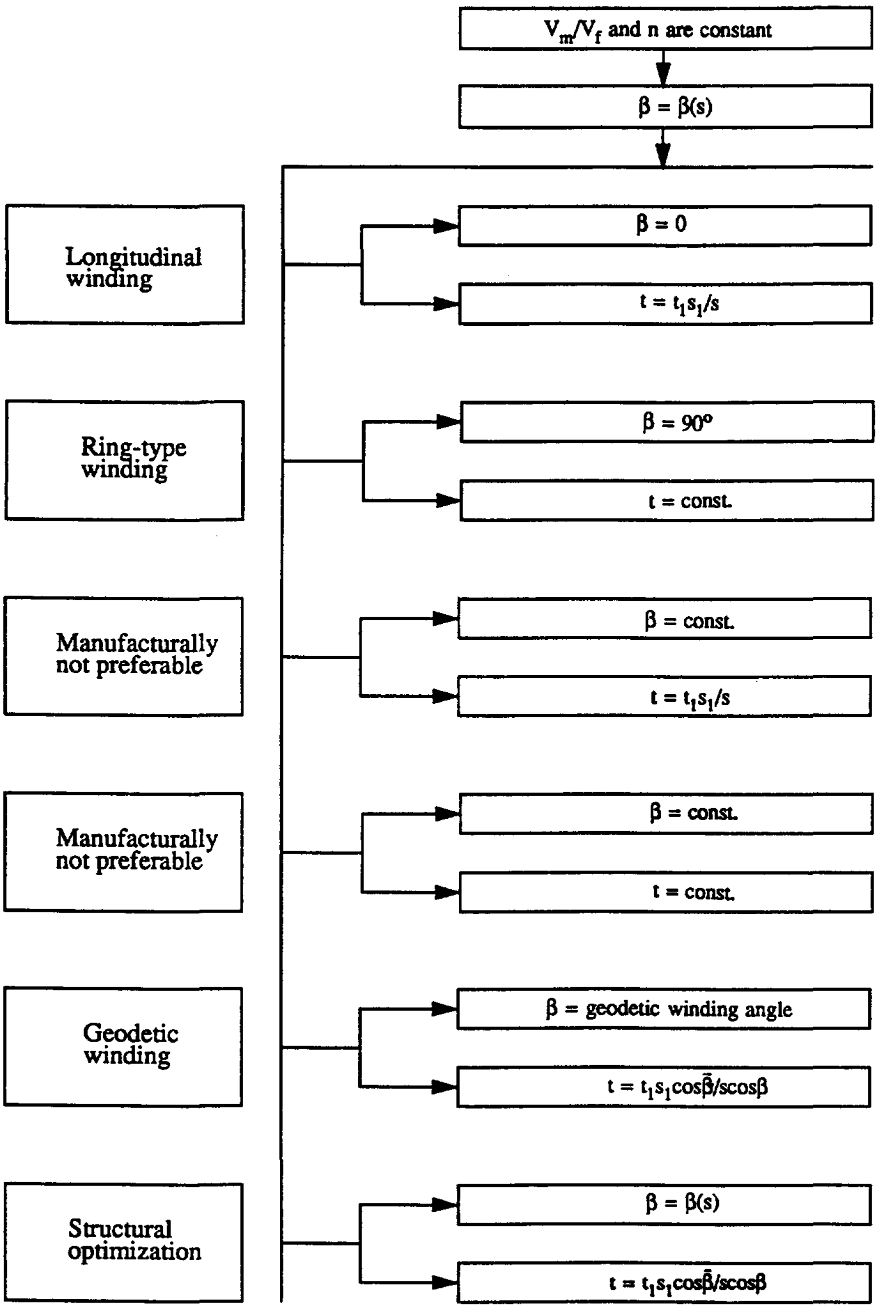


Fig. 2.8 Possible winding patterns

From manufacturing considerations the state of art of the filament winding is such that among these possible winding patterns, geodetic winding is the most favorable one[133-136]. This is due to the fact that the fiber slippage can be automatically prevented if the fibers are wound along the geodetic lines. However, with the development of filament winding techniques and advanced winding machines, other non-geodetic patterns are also becoming possible[137-144] so as to meet the need for structural optimization. The equations describing the different winding patterns and the fiber positions on the surface of the conical shell have been derived in Ref. [84].

2.3.2 Variable wall thickness and modelling of the mandrel

Unlike the case of filament-wound cylindrical shells, where most fabrication processes can result in nominally uniform wall thickness and constant stiffness properties, for conical shells the opposite is true. Unless special techniques are employed, the wall thickness and the stiffness properties of the filament-wound conical shells will depend on the longitudinal coordinate s[84,145,146]. In the following the thickness of filament-wound conical shells will be formulated with respect to the general winding pattern, while the change of winding angles corresponding to different winding patterns has already been derived systematically by Baruch, Arbocz and Zhang[84].

Variable wall thickness

From Figure 2.9 one can see that

$$\cos\beta = a/a_1(\beta) \tag{2.10}$$

$$n a_1(\bar{\beta}) = 2R_1 t_1 \pi \tag{2.11}$$

where a is the cross-sectional area of an undeformed fiber, a_1 is the projection of area a in the circumferential direction at s, $\bar{\beta}$ is the initial winding angle at $s = s_1$, R_1 is the average radius at $s = s_1$, t_1 is the wall thickness at $s = s_1$, and n is the number of fibers in one cross-section of the shell.

For any cross-section of a conical shell at s one has

$$na_1(\beta) = 2Rt\pi \tag{2.12}$$

where β is the winding angle at s, R is the average radius at s, t is the wall thickness at s.

In the winding process it is assumed that every fiber brings with itself the same amount of matrix along the fiber direction. This means that V_m/V_f and the apparent moduli of the lamina are constants at any cross-section of the shell, while the thickness is variable. Thus one can further obtain

$$t = s_1 t_1 \cos\bar{\beta} / (s \cos\beta) \tag{2.13}$$

Equation (2.13) is suitable for any winding pattern except for ring-type winding.

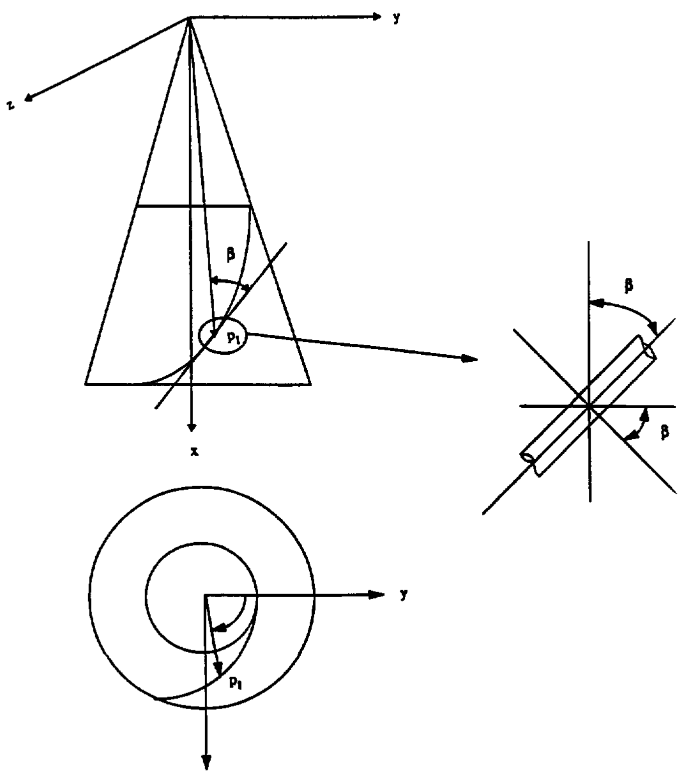


Fig. 2.9 A fiber wound on the surface of conical shell

Equation (2.13) can be rewritten as

$$t = t_1 \Phi_1 \tag{2.14}$$

where the thickness variation factor Φ_1 is given by

$$\Phi_1 = s_1 \cos\bar{\beta} / s \cos\beta \tag{2.15}$$

It is known that for geodetic winding the variable winding angle is given by

$$\beta = \text{Arcsin}(s_1 \sin\bar{\beta}/s) \tag{2.16}$$

Substituting Eq. (2.16) into Eq. (2.15) yields the thickness variation factor for geodetic winding as

$$\Phi_1 = s_1 \cos\bar{\beta} / \sqrt{s^2 - (s_1 \sin\bar{\beta})^2} \tag{2.17}$$

which is shown in Figure 2.10 for given initial winding angle $\bar{\beta}$ and s_1/s ratios.

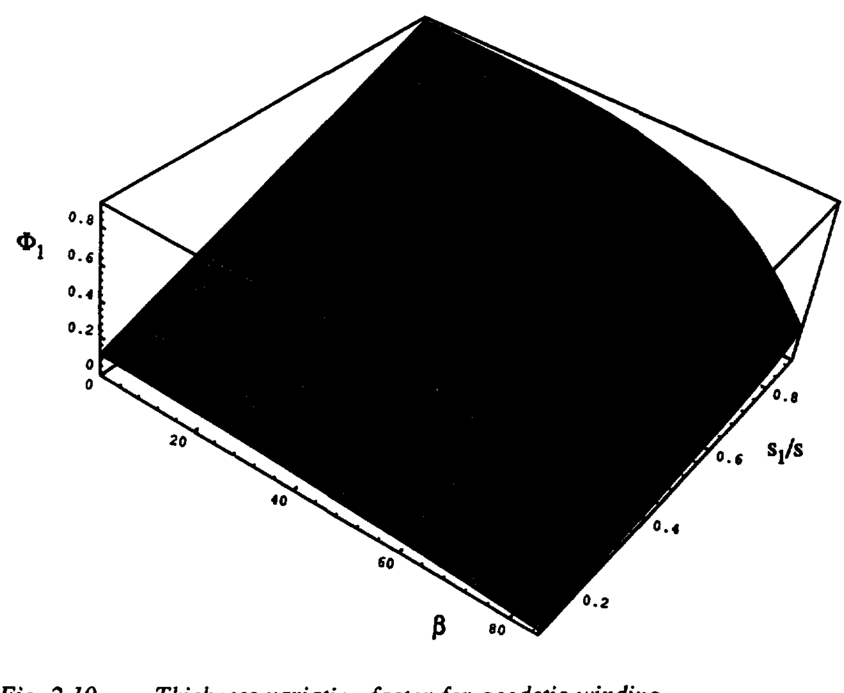


Fig. 2.10 Thickness variation factor for geodetic winding

It can be seen that employing the geodetic winding one can expect a dramatic thickness change along the longitudinal direction of the conical shell for certain initial winding angles and geometric parameters. For other winding patterns except for the ring-type winding, thickness variation can also be expected, and the corresponding thickness variation factors can be obtained by substituting the corresponding $\beta(s)$ function into Equation (2.15).

Notice that with the following 3 assumptions

- the mandrel is specially designed as to give a conical-type mid-surface for the wanted conical shell,
- s is measured along the mid-surface of the cone, and the influence of the changing thickness on coordinate s is neglected,
- each layer follows the same formula of thickness variation,

one obtains the full thickness of the shell as

$$t = \sum_{k=1}^{N} t_k = t(s) \tag{2.18}$$

where t_k is the wall thickness of the k^{th}-layer at s, and N is the total number of layers.

Modelling of the mandrel

As can be seen from Eq. (2.18), the wall thickness of the conical shell is a function of s. This thickness variation can be also regarded as an initial geometric imperfection[84]. To minimize its influence on the behavior of the shell the rigid mandrel should be designed as

$$r_m = s\sin\alpha_o - t/2 \tag{2.19}$$

where r_m is the running radius of the mandrel.

It should be emphasized, however, that during the winding process some additional imperfections for the mandrel and the shell may occur due to thermochemical, thermal and mechanical loads.

2.3.3 Variable stiffness coefficients

In the previous section the thickness variations of the laminated conical shells made by filament winding are formulated as functions of the longitudinal coordinate s. In the following the variable stiffness coefficients of laminated conical shells will be derived based on the variable wall thickness.

The constitutive equations for the k^{th} homogeneous orthotropic lamina are assumed to be

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ (1/2)\gamma_{12} \end{bmatrix}_k \tag{2.20}$$

where

$$\begin{aligned} Q_{11} &= E_{11}/(1-\upsilon_{12}\upsilon_{21}) \\ Q_{22} &= E_{22}/(1-\upsilon_{12}\upsilon_{21}) \\ Q_{12} &= \upsilon_{21}E_{11}/(1-\upsilon_{12}\upsilon_{21}) = \upsilon_{12}E_{22}/(1-\upsilon_{12}\upsilon_{21}) \\ Q_{66} &= G_{12} \end{aligned} \tag{2.21}$$

Notice that there are 4 elastic constants E_{11}, E_{22}, ν_{12} and G_{12}. Since the stiffness matrix must be symmetrical therefore $\nu_{12}E_{22} = \nu_{21}E_{11}$. Thus the fifth elastic constant ν_{21} can be expressed in terms of the other constants.

For filamentary materials either with unidirectional fibers or woven fibers, the above constitutive relations can always be used. In other word, macroscopically, the composites are assumed to be homogeneously idealized orthotropic materials, and thus one does not account for the details of fiber-resin geometry and interaction.

Normally, the lamina principal axes (1,2) do not coincide with the reference axes of the shell wall (s,θ). Thus the constitutive equations for each individual lamina must be transformed to the shell wall reference axes in order to be able to determine the shell wall (or laminate) constitutive equations. This transformation yields

$$\begin{bmatrix} \sigma_s \\ \sigma_\theta \\ \tau_{s\theta} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix}_k \tag{2.22}$$

where

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}C^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}S^4 \\ \bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})C^2S^2 + Q_{12}(C^4+S^4) \\ \bar{Q}_{22} &= Q_{11}S^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}C^4 \\ \bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})C^2S^2 + Q_{66}(C^4+S^4) \\ \bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})C^3S + (Q_{12}-Q_{22}+2Q_{66})CS^3 \\ \bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})CS^3 + (Q_{12}-Q_{22}+2Q_{66})C^3S \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} C &= \cos\beta_k \\ S &= \sin\beta_k \end{aligned} \tag{3.24}$$

where β_k is the variable winding angle of the k^{th}-layer.

The $[\bar{Q}]$ matrix is now fully populated and it appears that there are 6 elastic constants. However, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are merely linear combinations of the 4 basic elastic constants and are not independent. Corresponding to different winding patterns, the $[\bar{Q}]$ matrix will change along with the change of the winding angles.

Recalling the Kirchhoff-Love hypothesis for a thin shell, one can write the total strains at any layer in terms of the strains and curvature of the mid-surface as

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix}_k = \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} + z \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.25}$$

or

$$[\varepsilon]_k = [\varepsilon] + z[\kappa] \tag{2.26}$$

Substituting Eq. (2.25) into Eq. (2.22) yields

$$\begin{bmatrix} \sigma_s \\ \sigma_\theta \\ \tau_{s\theta} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.27}$$

or

$$[\sigma]_k = [\bar{Q}]_k [\varepsilon] + z[\bar{Q}]_k [\kappa] \tag{2.28}$$

This expression can be used to compute the stress at any location for any of the k laminae.

Considering the thin-walled conical shell, where z/R_i can be neglected as compared to 1 according to Donnell-type theory, the stress and moment resultants acting at the shell midsurface are obtained by integration of the stresses in each layer (or lamina) through the laminate thickness as

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \begin{bmatrix} \sigma_s \\ \sigma_\theta \\ \tau_{s\theta} \end{bmatrix}_k dz \tag{2.29}$$

$$\begin{bmatrix} M_s \\ M_\theta \\ \dfrac{M_{s\theta}+M_{\theta s}}{2} \end{bmatrix} = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \begin{bmatrix} \sigma_s \\ \sigma_\theta \\ \tau_{s\theta} \end{bmatrix}_k z dz \tag{2.30}$$

where N is the total number of laminae, and the position of the k^{th} lamina is defined by

$t_{k-1} < z < t_k$.

Substituting Eq. (2.27) into Eqs. (2.29)-(2.30) yields

$$\begin{aligned} \begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} = \sum_{k=1}^{N} \{ & \int_{t_{k-1}}^{t_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} dz \\ & + \int_{t_{k-1}}^{t_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} z dz \; \} \end{aligned} \tag{2.31}$$

$$\begin{aligned} \begin{bmatrix} M_s \\ M_\theta \\ \dfrac{(M_{s\theta}+M_{\theta s})}{2} \end{bmatrix} = \sum_{k=1}^{N} \{ & \int_{t_{k-1}}^{t_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} z dz \\ & + \int_{t_{k-1}}^{t_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} z^2 dz \; \} \end{aligned} \tag{2.32}$$

Notice that in the above integrations, [ε] and [κ] are not functions of z, and within any layer (from t_{k-1} to t_k) the $[\bar{Q}]$ matrix is also not a function of z. Thus Eqs. (2.31)-(2.32) can be rewritten as

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} = \sum_{k=1}^{N} \{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} \int_{t_{k-1}}^{t_k} dz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \int_{t_{k-1}}^{t_k} z dz \} \tag{2.33}$$

$$\begin{bmatrix} M_s \\ M_\theta \\ \frac{(M_{s\theta}+M_{\theta s})}{2} \end{bmatrix} = \sum_{k=1}^{N} \{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} \int_{t_{k-1}}^{t_k} z dz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \int_{t_{k-1}}^{t_k} z^2 dz \} \tag{2.34}$$

Furthermore, since [ε] and [κ] are not functions of z, Eqs. (2.33)-(2.34) can be reduced to the following forms

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.35}$$

or

$$[N] = [A][\varepsilon] + [B][\kappa] \tag{2.36}$$

$$\begin{bmatrix} M_s \\ M_\theta \\ \dfrac{M_{s\theta}+M_{\theta s}}{2} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \end{bmatrix} \tag{2.37}$$

or

$$[M] = [B][\varepsilon] + [D][\kappa] \tag{2.38}$$

where

$$[A] = \sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k (t_k - t_{k-1}) \tag{2.39}$$

$$[B] = \frac{1}{2} \sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k (t_k^2 - t_{k-1}^2) \tag{2.40}$$

$$[D] = \frac{1}{3} \sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k (t_k^3 - t_{k-1}^3) \tag{2.41}$$

After partial inversion, Eqs. (2.35) and (2.37) become

$$\begin{bmatrix} \varepsilon \\ M \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} \begin{bmatrix} N \\ \kappa \end{bmatrix} \tag{2.42}$$

where

$$\begin{aligned} [A^*] &= [A]^{-1} \\ [B^*] &= -\,[A]^{-1}[B] \\ [C^*] &= -\,[B^*]^T \\ [D^*] &= [D] - [B][A]^{-1}[B] \end{aligned} \tag{2.43}$$

It is seen that the [A] matrix is the extensional stiffness matrix relating the in-plane stress resultants (N's) to the mid-surface strains (ε's) and the [D] matrix is the flexural stiffness matrix relating the moment resultants (M's) with the curvatures (κ's). Since the [B] matrix relates M's to ε's and N's to κ's, it is called the bending-stretching coupling matrix. It should be noted that a laminated structure can have bending-stretching coupling even if all laminae are isotropic, for example, a laminate composed of one lamina of steel and another of polyester. In fact, only when the structure is exactly symmetric about its mid-surface are all of the [B] components equal to zero, and this requires symmetry in laminae properties, orientation, and location from the mid-surface.

Besides, stretching-shearing coupling occurs when A_{16} and A_{26} are non-zero. Twisting-stretching coupling and bending-shearing coupling occur when B_{16} and B_{26} terms are non-zero, and bending-twisting coupling comes from non-zero values of the D_{16} and D_{26} terms. Usually the 16 and 26 terms are avoided by proper stacking sequences, but there could be some structural applications where these effects could be used as an advantage, such as in aeroelastic tailoring.

Finally, it should be emphasized here that unlike the case of laminated cylindrical shells, where winding angle β_k and the resulting wall thickness can be taken as constant within one lamina, the winding angle and wall thickness of laminated conical shell usually depend on the winding processes used and will vary with the longitudinal coordinate s. This variation is caused by the inherent geometry of a conical shell. This special conical geometry dictates the reinforcement trace in which the fiber orientation relative to the cone axis changes as the fiber is wound. Therefore, the [A], [B] and [D] matrixes are functions of s.

By substituting the variable winding angle and the variable wall thickness of certain winding pattern into Eqs. (2.39)-(2.41), one obtains the variable stiffness coefficients of a laminated conical shell made by filament winding.

2.3.4 Equivalent wall thickness and stiffness coefficients

There exists some theoretical and practical interest for the non-geodetic winding processes in recent years[143,144] from optimization considerations. However, up till now, geodetic winding is still the most favorable process because of ease of manufacture. As mentioned earlier, for laminated conical shells made by geodetic winding process, the wall thickness and the stiffness matrixes [A], [B] and [D] are functions of the shell coordinates. With some special manufacturing constraints, they can be made functions of the longitudinal coordinate s only.

Generally speaking, there are two ways to calculate the stiffness coefficients and the wall thickness of laminated conical shells made by geodetic filament winding, i.e., the exact method and the approximate method. For the exact method, the stiffness coefficients and wall thickness are calculated by employing the actual variable winding angle. Hence the resulting stiffness properties and wall thickness will be functions of the longitudinal coordinate s. This, finally, will lead to a set of rather involved differential equations governing the stability behavior of laminated conical shells. For the approximate method, the variable stiffness properties and wall thickness are approximately taken as constants, as it was always done in the open literature. However, the constant stiffness properties and wall thickness are not randomly chosen, but are based on some kind of averaging method. This approximation will result in a great simplification of the stability analysis, and moreover, makes it possible to obtain some design formulae for engineering reference within the required error bound.

Equivalent cylindrical mean

Taking the conical shell as an equivalent cylindrical shell, one can obtain the average radius for the equivalent cylindrical shell as[27]

$$R_{av} = (R_1 + R_2)/2\cos\alpha_o \qquad (2.44)$$

For the equivalent cylindrical shell geodetic winding will yield constant winding angle as

$$\beta_{av} = \mathrm{Arcsin}((2s_1\cos\alpha_o \sin\bar{\beta})/(s_1 + s_2)) \tag{2.45}$$

thus the constant wall thickness will be

$$\begin{aligned} t_{av} &= s_1 t_1 \cos\bar{\beta} / (s_{av}\cos\beta_{av}) \\ &= 2t_1\cos\bar{\beta}\cos\alpha_o / \sqrt{(1 + s_2/s_1)^2 - (2\cos\alpha_o \sin\bar{\beta})^2} \end{aligned} \tag{2.46}$$

Geometric mean

By taking the average thickness as the geometric mean of the thicknesses at the two ends of the shell, one obtains

$$t_{av} = \sqrt{t_1 t_2} = t_1 \sqrt{(s_1\cos\bar{\beta}) / \sqrt{s_2^2 - (s_1\sin\bar{\beta})^2}} \tag{2.47}$$

and the average s_{av} is

$$s_{av} = s_1\sqrt{(t_1\cos\bar{\beta} / t_2)^2 + \sin^2\bar{\beta}} \tag{2.48}$$

thus the average winding angle is

$$\beta_{av} = \mathrm{Arcsin}\,(1/\sqrt{1 + (t_1 \cot\bar{\beta} / t_2)^2}\ ) \tag{2.49}$$

Once the equivalent constant wall thickness and constant winding angle are known, one can calculate easily the stiffness coefficients according to Eqs. (2.39)-(2.41).

2.3.5 Discussions and conclusion

The state of art of the filament winding is such that the fibers are usually positioned on the geodetic paths of the conical shell. For the purpose of optimization, other non-geodetic windings are also theoretically interesting and becoming practically possible. For all these cases, however, the stiffness coefficients and wall thickness obtained for conical shells are at least functions of the longitudinal coordinate s. This fact must be taken into account in the analysis of laminated conical shells made by filament winding.

Nevertheless, as an approximation equivalent constant stiffness coefficients and wall thickness can be obtained from some kind of averaging method. Two of them are suggested here, the equivalent cylindrical mean and the equivalent geometric mean, which will be used for the stability analysis of laminated conical shells. Besides, it should be noted that in the present analysis the classical laminate theory, i.e., no transverse shear deformation or transverse normal stress, is employed.

The formulae derived to calculate the stiffness coefficients of laminated conical shells made by filament winding are also valid for other manufacturing processes which are based on continuous fiber reinforcement.

Chapter 3
Governing Equations of Anisotropic Conical Shells

3.1 Introduction

Based on the considerations described earlier it is decided to use the nonlinear Donnell-type strain-displacement relations and the constitutive equations of classical laminate theory for the stability analysis of anisotropic conical shells. In the following the nonlinear Donnell-type governing equations of anisotropic conical shells in terms of the radial displacement W and the Airy stress function F are derived via the stationary potential energy criterion[147]. The resulting nonlinear partial differential equations with variable coefficients are transformed to some simpler forms via the transformation suggested by Mushtari and Sachenkov[31]. By using the perturbation technique proposed by Koiter[61], three sets of partial differential equations governing the behavior of the prebuckling, buckling and postbuckling problems, are obtained. Further, by assuming axisymmetric solutions for the prebuckling equations and employing Fourier decomposition for the circumferential dependencies of the buckling and postbuckling equations, the three sets of partial differential equations can all be reduced to ordinary differential equations.

3.2 Original Governing Equations

In the following the governing equations, i.e., the equilibrium equation and the compatibility equation in the conventional coordinate system (s and θ) are derived. Next the correctness of these equations is verified.

3.2.1 Equilibrium equations

Substituting $\alpha_1 = s$, $\alpha_2 = \theta$, $R_1 \rightarrow \infty$, $R_2 = s\tan\alpha_o$, $A_1 = 1$, and $A_2 = s\sin\alpha_o$ into Eqs. (1.39)-(1.40) and (1.42), and including the initial geometric imperfections $\bar{W}$, one obtains the Donnell-type kinematic relations of imperfect conical shells as[89,147,148]

$$
\begin{aligned}
\varepsilon_s &= u_{,s} + (\tilde{W}_{,s})^2/2 + \tilde{W}_{,s}\bar{W}_{,s} \\
\varepsilon_\theta &= (u - \tilde{W}\cot\alpha_o)/s + v_{,\bar{\theta}}/s + (\tilde{W}_{,\bar{\theta}})^2/2s^2 + \tilde{W}_{,\bar{\theta}}\bar{W}_{,\bar{\theta}}/s^2 \\
\gamma_{s\theta} &= v_{,s} - v/s + u_{,\bar{\theta}}/s + \tilde{W}_{,s}\tilde{W}_{,\bar{\theta}}/s + (\bar{W}_{,\bar{\theta}}\tilde{W}_{,s} + \bar{W}_{,s}\tilde{W}_{,\bar{\theta}})/s
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\kappa_s &= -\tilde{W}_{,ss} \\
\kappa_\theta &= -(\tilde{W}_{,s}/s + \tilde{W}_{,\bar{\theta}\bar{\theta}}/s^2) \\
\kappa_{s\theta} &= -2(\tilde{W}_{,s\bar{\theta}}/s - \tilde{W}_{,\bar{\theta}}/s^2)
\end{aligned}
\tag{3.2}
$$

where u, v, $\tilde{W}$ are the displacements along s, θ and z directions respectively, $\bar{W}$ is the initial geometric imperfection, and $\bar{\theta} = \theta\sin\alpha_o$.

The total potential energy of a conical shell is given by

$$
\Pi = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 \tag{3.3}
$$

where the strain energy is

$$
\Pi_1 = \frac{1}{2}\int_{\theta_1}^{\theta_2}\int_{s_1}^{s_2}[N_s\varepsilon_s + N_\theta\varepsilon_\theta + N_{s\theta}\gamma_{s\theta} + M_s\kappa_s + M_\theta\kappa_\theta + M_{s\theta}\kappa_{s\theta}]\, s\, ds\, d\bar{\theta} \tag{3.4}
$$

the work done by the edge loads at the edges s = constant is

$$
\Pi_2 = -\int_{\theta_1}^{\theta_2}\{s(\bar{N}_s u + \bar{N}_{s\theta} v + \bar{M}_s\tilde{W}_{,s} + \bar{M}_{s\theta}\tilde{W}_{,\bar{\theta}}/s + \bar{Q}_s\tilde{W})\}\Big|_{s_1}^{s_2} d\bar{\theta} \tag{3.5}
$$

the work done by the edge loads at the edges θ = constant is

$$\Pi_3 = -\int_{s_1}^{s_2} \{\bar{N}_\theta v + \bar{N}_{s\theta} u + \bar{M}_\theta \tilde{W},_{\bar{\theta}}/s + \bar{M}_{s\theta}\tilde{W},_s + \bar{Q}_\theta \tilde{W}\} \Big|_{\theta_1}^{\theta_2} ds \tag{3.6}$$

and the work done by external pressure is

$$\Pi_4 = -\int_{\theta_1}^{\theta_2}\int_{s_1}^{s_2} p\tilde{W} s\, ds\, d\bar{\theta} \tag{3.7}$$

The equilibrium equations can be established by minimizing the above total potential energy of the conical shell. Namely, by taking the first variation of the potential energy expression, eliminating variations of derivatives of the displacements by integration by parts, regrouping and letting the first variation of potential energy vanish, one obtains the following equilibrium equations

$$\begin{aligned}
&N_\theta - N_{s\theta},_{\bar{\theta}} - (sN_s),_s = 0\\
&N_{s\theta} + N_{\theta},_{\bar{\theta}} + (sN_{s\theta}),_s = 0\\
&[\, sN_s(\tilde{W},_s+\bar{W},_s) + N_{s\theta}(\tilde{W},_{\bar{\theta}}+\bar{W},_{\bar{\theta}})],_s + [\, N_\theta(\tilde{W},_{\bar{\theta}}+\bar{W},_{\bar{\theta}})/s\\
&\qquad + N_{s\theta}(\tilde{W},_s+\bar{W},_s)\,],_{\bar{\theta}} + N_\theta \cot\alpha_o + (sM_s),_{ss} - M_\theta,_s + (M_\theta/s),_{\bar{\theta}\bar{\theta}}\\
&\qquad + [\,(M_{s\theta}+M_{\theta s}),_{\bar{\theta}}\,]/s + (M_{s\theta}+M_{\theta s}),_{s\bar{\theta}} + p\,s = 0
\end{aligned} \tag{3.8}$$

with boundary conditions at the edges s = constant;

$$\begin{aligned}
&N_s = \bar{N}_s \quad \text{or} \quad \delta u = 0\\
&N_{s\theta} = \bar{N}_{s\theta} \quad \text{or} \quad \delta v = 0\\
&M_s = \bar{M}_s \quad \text{or} \quad \delta\tilde{W},_s = 0\\
&N_s(\tilde{W},_s+\bar{W},_s)\, s + N_{s\theta}(\tilde{W},_{\bar{\theta}}+\bar{W},_{\bar{\theta}}) - M_\theta + (M_s s),_s\\
&\qquad + (M_{\theta s}+M_{s\theta}),_{\bar{\theta}} = s\bar{Q}_s + \bar{M}_{s\theta,\bar{\theta}} \quad \text{or} \quad \delta\tilde{W} = 0
\end{aligned} \tag{3.9}$$

For the closed conical shell ($0 \leq \bar{\theta} \leq 2\pi\sin\alpha_o$), which is the case here, the boundary conditions at the edges $\bar{\theta}$ = constant are replaced by the periodicity condition with the period of $2\pi\sin\alpha_o$.

The Airy stress function for the solution of the conical shell problem can be defined as

$$\begin{aligned} N_s &= \frac{1}{s}\tilde{F}_{,s} + \frac{1}{s_2}\tilde{F}_{,\bar{\theta}\bar{\theta}} \\ N_\theta &= \tilde{F}_{,ss} \\ N_{s\theta} &= \frac{1}{s^2}\tilde{F}_{,\bar{\theta}} - \frac{1}{s}\tilde{F}_{,\bar{\theta}s} \end{aligned} \qquad (3.10)$$

Substituting Eqs. (3.10) into Eqs. (3.8), the first two of Eqs. (3.8) are identically satisfied. With the help of Eqs. (2.42) the third one (the out-of-plane equilibrium equation) becomes

$$L_{B_1^*}(\tilde{F}) + L_{B_3^*}(\tilde{F}) - L_{D^*}(\tilde{W}) = L_{NL}(\tilde{F}, \tilde{W} + \bar{W}) + s^3\tilde{F}_{,ss}\cot\alpha_o + ps^4 \qquad (3.11)$$

where the linear differential operators are

$$\begin{aligned} L_{B_1^*}(\ ) = {} & B_{21}^*(\ )_{,ssss}\, s^4 + (2B_{26}^*-B_{61}^*)(\ )_{,sss\bar{\theta}}\, s^3 + (B_{11}^*+B_{22}^*-2B_{66}^*)(\ )_{,ss\bar{\theta}\bar{\theta}}\, s^2 \\ & + (2B_{16}^*-B_{62}^*)(\ )_{,s\bar{\theta}\bar{\theta}\bar{\theta}}\, s + B_{12}^*(\ )_{,\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}} + (B_{62}^*-2B_{16}^*)(\ )_{,\bar{\theta}\bar{\theta}\bar{\theta}} \\ & - B_{12}^*(\ )_{,ss}\, s^2 + B_{12}^*(\ )_{,s}\, s \end{aligned} \qquad (3.12)$$

$$\begin{aligned} L_{B_3^*}(\ ) = {} & (2B_{21}^*+B_{11}^*-B_{22}^*)(\ )_{,sss}\, s^3 + (2B_{26}^*+2B_{16}^*+B_{61}^*+B_{62}^*)(\ )_{,ss\bar{\theta}}\, s^2 \\ & + 2\,(B_{66}^*-B_{11}^*)(\ )_{,s\bar{\theta}\bar{\theta}}\, s + 2\,(B_{11}^*-B_{66}^*+B_{12}^*)(\ )_{,\bar{\theta}\bar{\theta}} \\ & - 2\,(B_{62}^*+B_{61}^*)(\ )_{,s\bar{\theta}}\, s + 2\,(B_{62}^*+B_{61}^*)(\ )_{,\bar{\theta}} \end{aligned} \qquad (3.13)$$

$$
\begin{aligned}
L_{D^*}(\ ) = & - D_{11}^*(\ )_{,ssss}\, s^4 - 4D_{16}^*(\ )_{,sss\bar\theta}\, s^3 - 2\,(2D_{66}^*+D_{12}^*)(\ )_{,ss\bar\theta\bar\theta}\, s^2 \\
& - 4D_{26}^*(\ )_{,s\bar\theta\bar\theta\bar\theta}\, s - D_{22}^*(\ )_{,\bar\theta\bar\theta\bar\theta\bar\theta} - 2D_{11}^*(\ )_{,sss}\, s^3 + 2\,(2D_{66}^*+D_{12}^*)(\ )_{,s\bar\theta\bar\theta}\, s \\
& + 4D_{26}^*(\ )_{,\bar\theta\bar\theta\bar\theta} - 4\,(D_{26}^*+D_{16}^*)(\ )_{,s\bar\theta}\, s - D_{22}^*(\ )_{,ss}\, s^2 \\
& - 2\,(2D_{66}^*+D_{22}^*+D_{12}^*)(\ )_{,\bar\theta\bar\theta} + 4(D_{26}^*+D_{16}^*)(\ )_{,\bar\theta} - D_{22}^*(\ )_{,s}\, s
\end{aligned}
\tag{3.14}
$$

and the nonlinear differential operator is

$$
\begin{aligned}
L_{NL}(X,Y) = & \; s^3\,[X_{,ss}(Y_{,s} + Y_{,\bar\theta\bar\theta}/s) + Y_{,ss}(X_{,s} + X_{,\bar\theta\bar\theta}/s) \\
& - 2(sX_{,s\bar\theta} - X_{,\bar\theta})\,(sY_{,s\bar\theta} - Y_{,\bar\theta})/\,s^3\,]
\end{aligned}
\tag{3.15}
$$

Commas in the subscripts denote repeated partial differentiation with respect to the independent variables following the comma.

3.2.2 Compatibility equation

To obtain the second partial differential equation involving the dependent variables W and F one uses the necessary and sufficient conditions for the existence of valid solutions of anisotropic conical shells. That is, the compatibility condition[126,149], which guarantees that the mid-surface of the shell remains continuous after deformation.

Eliminating u and v from Eqs. (3.1) one obtains

$$
\begin{aligned}
& s\gamma_{s\theta,\bar\theta s} + \gamma_{s\theta,\bar\theta} - \varepsilon_{s,\bar\theta\bar\theta} + s\varepsilon_{s,s} - 2s\varepsilon_{\theta,s} - s^2\varepsilon_{\theta,ss} \\
& \quad = \tilde W_{,\bar\theta}\,(2\tilde W_{,\bar\theta s}/s + 2\bar W_{,\bar\theta s}/s - \tilde W_{,\bar\theta}/s^2 - 2\bar W_{,\bar\theta}/s^2) + 2\bar W_{,\bar\theta}\tilde W_{,\bar\theta s}/s \\
& \quad + s\tilde W_{,s}(\tilde W_{,ss}+\bar W_{,ss}) + s\tilde W_{,ss}\bar W_{,s} + \tilde W_{,\bar\theta\bar\theta}(\tilde W_{,ss}+\bar W_{,ss}) \\
& \quad + \bar W_{,\bar\theta\bar\theta}\tilde W_{,ss} - \tilde W_{,s\bar\theta}(\tilde W_{,s\bar\theta}+2\bar W_{,s\bar\theta}) + s\tilde W_{,ss}\cot\alpha_o
\end{aligned}
\tag{3.16}
$$

Notice that since Eq. (3.16) is exclusively deduced on kinematic considerations, the compatibility equation is independent of the anisotropic character of the structure material.

Further, substituting Eqs. (3.2), (3.10) and (2.42) into the left hand side of Eq. (3.16) one obtains

$$L_{A^*}(\tilde{F}) - L_{B_1^*}(\tilde{W}) - L_{B_2^*}(\tilde{W}) = -L_{NL}(\tilde{W}, \tilde{W} + 2\bar{W})/2 - s^3 \tilde{W}_{,ss} \cot\alpha_o \tag{3.17}$$

where the two new linear differential operators are

$$\begin{aligned} L_{A^*}(\;) = {} & A_{22}^*(\;)_{,ssss} s^4 - 2A_{26}^*(\;)_{,sss\bar{\theta}} s^3 + (A_{66}^*+2A_{12}^*)(\;)_{,ss\bar{\theta}\bar{\theta}} s^2 \\ & - 2A_{16}^*(\;)_{,s\bar{\theta}\bar{\theta}\bar{\theta}} s + A_{11}^*(\;)_{,\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}} + 2A_{22}^*(\;)_{,sss} s^3 - (2A_{12}^*+A_{66}^*)(\;)_{,s\bar{\theta}\bar{\theta}} s \\ & + 2A_{16}^*(\;)_{,\bar{\theta}\bar{\theta}\bar{\theta}} - 2(A_{26}^*+A_{16}^*)(\;)_{,s\bar{\theta}} s - A_{11}^*(\;)_{,ss} s^2 \\ & + (A_{66}^*+2A_{12}^*+2A_{11}^*)(\;)_{,\bar{\theta}\bar{\theta}} + 2(A_{26}^*+A_{16}^*)(\;)_{,\bar{\theta}} + A_{11}^*(\;)_{,s} s \end{aligned} \tag{3.18}$$

$$\begin{aligned} L_{B_2^*}(\;) = {} & (2B_{21}^*-B_{11}^*+B_{22}^*)(\;)_{,sss} s^3 - (2B_{26}^*+2B_{16}^*+B_{61}^*+B_{62}^*)(\;)_{,ss\bar{\theta}} s^2 \\ & + 2(B_{66}^*-B_{22}^*)(\;)_{,s\bar{\theta}\bar{\theta}} s + 2(B_{22}^*-B_{66}^*+B_{12}^*)(\;)_{,\bar{\theta}\bar{\theta}} \\ & + 4(B_{26}^*+B_{16}^*)(\;)_{,s\bar{\theta}} s - 4(B_{26}^*+B_{16}^*)(\;)_{,\bar{\theta}} \end{aligned} \tag{3.19}$$

Equations (3.11) and (3.17) are nonlinear partial differential equations with two unknowns $\tilde{W}$ and $\tilde{F}$. These equations, together with the appropriate boundary conditions, govern the behavior of the imperfect anisotropic conical shells

- In the prebuckling stress and deformation state.
- At the bifurcation point or limit point (if there is one).
- In the postbuckling stress and deformation state.

3.2.3 *Checking the correctness of the governing equations*

The correctness of Eqs. (3.11) and (3.17) can be verified by the following considerations.

Degeneration to anisotropic cylindrical shells

By letting the cone semi-vertex angle α_o approach zero, and the distance from the vertex s approach infinity, while the projection of this distance on a plane perpendicular to the cone axis (s $\sin\alpha_o$) approaches a constant value R, Eqs. (3.11) and (3.17) reduce to the following equations

$$L_{A_c^*}(\tilde{F}) - L_{B_c^*}(\tilde{W}) = - \tilde{W}_{,xx}/R - L_{NL_c}(\tilde{W}, \tilde{W} + 2\bar{W})/2 \tag{3.20}$$

$$L_{B_c^*}(\tilde{F}) + L_{D_c^*}(\tilde{W}) = \tilde{F}_{,xx}/R + L_{NL_c}(\tilde{W}, \tilde{W} + \bar{W}) + p \tag{3.21}$$

where the differential operators are

$$\begin{aligned} L_{A_c^*}(\ ) = {} & A_{22}^*(\ )_{,xxxx} - 2A_{26}^*(\ )_{,xxxy} + (2A_{12}^*+A_{66}^*)(\ )_{,xxyy} \\ & - 2A_{16}^*(\ )_{,xyyy} + A_{11}^*(\ )_{,yyyy} \end{aligned} \tag{3.22}$$

$$\begin{aligned} L_{B_c^*}(\ ) = {} & B_{21}^*(\ )_{,xxxx} + (2B_{26}^*-B_{61}^*)(\ )_{,xxxy} + (B_{11}^*+B_{22}^*-2B_{66}^*)(\ )_{,xxyy} \\ & + (2B_{16}^*-B_{62}^*)(\ )_{,xyyy} + B_{12}^*(\ )_{,yyyy} \end{aligned} \tag{3.23}$$

$$\begin{aligned} L_{D_c^*}(\ ) = {} & D_{11}^*(\ )_{,xxxx} + 4D_{16}^*(\ )_{,xxxy} + 2\,(2D_{66}^*+D_{12}^*)(\ )_{,xxyy} \\ & + 4D_{26}^*(\ )_{,xyyy} + D_{22}^*(\ )_{,yyyy} \end{aligned} \tag{3.24}$$

$$L_{NL_c}(S,T) = S_{,xx}T_{,yy} - 2S_{,xy}T_{,xy} + S_{,yy}T_{,xx} \tag{3.25}$$

It is easy to verify that Eqs. (3.20)-(3.25) are the governing equations of an anisotropic cylindrical shell of radius R derived by Arbocz[15].

Degeneration to orthotropic conical shells

By setting the stiffness coefficients [B] and A_{16}, A_{26}, D_{16}, and D_{26} equal to zero, Eqs. (3.11) and (3.17) become

$$L_{A_o^*}(\tilde{F}) = - L_{NL}(\tilde{W}, \tilde{W} + 2\bar{W})/2 - s^3\tilde{W}_{,ss}\cot\alpha_o \tag{3.26}$$

$$L_{D_o^*}(\tilde{W}) = -L_{NL}(\tilde{F}, \tilde{W} + \bar{W}) - s^3 \tilde{F},_{ss}\cot\alpha_o - p\,s^4 \tag{3.27}$$

where the differential operators are

$$\begin{aligned} L_{A_o^*}(\ ) = {} & A_{22}^*(\ ),_{ssss} s^4 + (A_{66}^*+2A_{12}^*)(\ ),_{ss\bar\theta\bar\theta} s^2 + A_{11}^*(\ ),_{\bar\theta\bar\theta\bar\theta\bar\theta} \\ & + 2A_{22}^*(\ ),_{sss}s^3 - (2A_{12}^*+A_{66}^*)(\ ),_{s\bar\theta\bar\theta} s \\ & - A_{11}^*(\ ),_{ss} s^2 + (A_{66}^*+2A_{12}^*+2A_{11}^*)(\ ),_{\bar\theta\bar\theta} + A_{11}^*(\ ),_s s \end{aligned} \tag{3.28}$$

$$\begin{aligned} L_{D_o^*}(\ ) = {} & - D_{11}^*(\ ),_{ssss} s^4 - 2\,(2D_{66}^*+D_{12}^*)(\ ),_{ss\bar\theta\bar\theta} s^2 \\ & - D_{22}^*(\ ),_{\bar\theta\bar\theta\bar\theta\bar\theta} - 2D_{11}^*(\ ),_{sss}s^3 + 2\,(2D_{66}^*+D_{12}^*)(\ ),_{s\bar\theta\bar\theta} s \\ & + D_{22}^*(\ ),_{ss} s^2 - 2\,(2D_{66}^*+D_{22}^*+D_{12}^*)(\ ),_{\bar\theta\bar\theta} - D_{22}^*(\ ),_s s \end{aligned} \tag{3.29}$$

Notice that Eqs. (3.26)-(3.29) are exactly the same as the governing equations of the classical orthotropic conical shells given by Schiffner[89].

Checking by the Reduce-based package GEACS

By using the Computer Algebra system REDUCE, a symbolic package GEACS1, Generating Equations for Anisotropic Conical Shells, was written to derive the governing differential equations of anisotropic conical shells (see Appendix A1.1 for the source program). The computer generated equations (see Appendix A1.2) are exactly the same as Eqs. (3.11) and (3.17) derived by hand.

3.2.5 Discussions

Some observations show that

- Unlike the case of anisotropic cylindrical shells where there exists only one linear differential operator $L_{B^*}(\ )$ related to the stiffness coefficients [B], there exist three different linear differential operators, $L_{B_1^*}(\ )$, $L_{B_2^*}(\ )$ and $L_{B_3^*}(\ )$ in the governing equations of anisotropic conical shells. However, one can see that before introducing the anisotropic constitutive relations, the linear part of the compatibility equation does have the same form as the linear part of the out-of-plane equilibrium equation, i.e.,

$$L(A, B, C) = (s\,A_{,\bar{\theta}})_{,s} - (s^2 B_{,s})_{,s} - C_{,\bar{\theta}\bar{\theta}} + s\,C_{,s} \tag{3.30}$$

Substituting $A = \gamma_{s\theta}$, $B = \varepsilon_\theta$ and $C = \varepsilon_s$ into Eq. (3.30) yields the linear part of the compatibility equation, while substituting $A = (M_{s\theta} + M_{\theta s})$, $B = - M_s$ and $C = - M_\theta$ into Eq. (3.30) yields the linear part of the equilibrium equation.

Therefore, the anisotropic characteristics and the geometrical properties are the reasons for the differences of relevant linear differential operators between the cylindrical shells and conical shells.

- It can be seen that the resulting governing equations for anisotropic conical shells, which are nonlinear and also have variable coefficients, are more complicated than those of cylindrical shells, although the relatively simple Donnell-type kinematic relations and constant stiffness coefficients are employed. It is also seen that if one employs the constitutive equations with variable stiffness coefficients in the derivations instead of the constant ones, the resulting governing equations will be very involved leading to extreme complications even for numerical solutions. Thus it was decided to use the constant coefficient constitutive equations. In Appendix A1.3 the governing equations expressed in terms of $\tilde{W}$ and $\tilde{F}$ are derived for anisotropic conical shells with variable stiffness coefficients via the REDUCE based package GEACS1. The governing equations expressed in terms of the displacements u, v, $\tilde{W}$ for anisotropic conical shells with variable stiffness coefficients can be obtained via the REDUCE-based package GEACS2 (see Appendix A1.4 for the source program). It is known that there exist 81, 72 and 339 terms involved in the resulting three equilibrium equations respectively, and they will not be presented here. The correctness of these equations can be verified by comparing them with those obtained via other Computer Algebra systems, such as MAPLE and MATHEMATICA[150-152].

3.3 Transformed Governing Equations for Perfect Shells

To simplify the rather complicated nonlinear governing equations with variable coefficients, one can employ the following transformation suggested by Mushtari and Sachenkov[31] and followed by von Ender[44], Dixon[88], Schiffner[89], and others, namely

$$s = s_1 e^{z} \tag{3.31}$$

and

$$\begin{aligned} F(z,\bar{\theta}) &= e^{-z}\tilde{F}(s,\bar{\theta}) \\ W(z,\bar{\theta}) &= e^{-z}\tilde{W}(s,\bar{\theta}) \end{aligned} \tag{3.32}$$

Notice that $s = s_1$ corresponds to $z = 0$, and $s = s_2$ corresponds to $z = \ln(s_2/s_1) = z_o$.

The following nondimensional stiffness parameters are also used

$$\bar{A}^*_{ij} = EtA^*_{ij} \quad ; \quad \bar{B}^*_{ij} = (2c/t)B^*_{ij} \quad ; \quad \bar{D}^*_{ij} = (4c^2/Et^3)D^*_{ij} \tag{3.33}$$

where

$$c^2 = 3(1-\upsilon^2) \tag{3.34}$$

By carrying out the above transformations and eliminating the geometric imperfections, Eqs. (3.11) and (3.17) become

$$L_{\bar{B}_1^*}(F) - L_{\bar{B}_2^*}(F) - L_{\bar{D}^*}(W) = L_{NL_2}(F,W) + s_1 e^{z}\cot\alpha_o(F_{,zz}+F_{,z}) + e^{3z}s_1^4\, p \tag{3.35}$$

$$L_{\bar{A}^*}(F) - L_{\bar{B}_1^*}(W) - L_{\bar{B}_2^*}(W) = -L_{NL_2}(W,W)/2 - s_1 e^{z}\cot\alpha_o(W_{,zz}+W_{,z}) \tag{3.36}$$

where the linear differential operators are

$$L_{\bar{A}^*}(\ ) = [\bar{A}^*_{22}(\ )_{,zzzz} - 2\bar{A}^*_{26}(\ )_{,zzz\bar{\theta}} + (2\bar{A}^*_{12}+\bar{A}^*_{66})(\ )_{,zz\bar{\theta}\bar{\theta}} - 2\bar{A}^*_{16}(\ )_{,z\bar{\theta}\bar{\theta}\bar{\theta}} + \bar{A}^*_{11}(\ )_{,\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}} - (\bar{A}^*_{22}+\bar{A}^*_{11})(\ )_{,zz} - 2\bar{A}^*_{16}(\ )_{,z\bar{\theta}} + 2\bar{A}^*_{11}(\ )_{,\bar{\theta}\bar{\theta}} + \bar{A}^*_{11}(\ )]/Et \tag{3.37}$$

$$L_{\bar{B}_1^*}(\ ) = t\,[\bar{B}^*_{21}(\ )_{,zzzz} + (2\bar{B}^*_{26}-\bar{B}^*_{61})(\ )_{,zzz\bar{\theta}} + (\bar{B}^*_{11}+\bar{B}^*_{22}-2\bar{B}^*_{66})(\ )_{,zz\bar{\theta}\bar{\theta}} + (2\bar{B}^*_{16}-\bar{B}^*_{62})(\ )_{,z\bar{\theta}\bar{\theta}\bar{\theta}} + \bar{B}^*_{12}(\ )_{,\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}} - (\bar{B}^*_{12}+\bar{B}^*_{21})(\ )_{,zz} + (2\bar{B}^*_{16}-\bar{B}^*_{62})(\ )_{,z\bar{\theta}} + 2\bar{B}^*_{12}(\ )_{,\bar{\theta}\bar{\theta}} + \bar{B}^*_{12}(\ )\,]/2c \tag{3.38}$$

$$L_{\bar{B}_2^*}(\ ) = t\,[\,(\bar{B}^*_{22}-\bar{B}^*_{11})(\ )_{,zzz} - (2\bar{B}^*_{26}+2\bar{B}^*_{16}+\bar{B}^*_{61}+\bar{B}^*_{62})(\ )_{,zz\bar{\theta}} + (\bar{B}^*_{11}-\bar{B}^*_{22})(\ )_{,z\bar{\theta}\bar{\theta}} + (\bar{B}^*_{11}-\bar{B}^*_{22})(\ )_{,z}\,]/2c \tag{3.39}$$

$$L_{\bar{D}^*}(\ ) = E\,t^3\,[\,-\bar{D}^*_{11}()_{,zzzz} - 4\bar{D}^*_{16}(\ )_{,zzz\bar{\theta}} - 2\,(2\bar{D}^*_{66}+\bar{D}^*_{12})(\ )_{,zz\bar{\theta}\bar{\theta}} - 4\bar{D}^*_{26}(\ )_{,z\bar{\theta}\bar{\theta}\bar{\theta}} - \bar{D}^*_{22}(\ )_{,\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}} + (\bar{D}^*_{11}+\bar{D}^*_{22})(\ )_{,zz} - 4\bar{D}^*_{26}(\ )_{,z\bar{\theta}} - 2\bar{D}^*_{22}(\ )_{,\bar{\theta}\bar{\theta}} - \bar{D}^*_{22}(\ )\,]/4c^2 \tag{3.40}$$

and the nonlinear differential operators is

$$L_{NL_2}(X,Y) = e^{z}[\,X_{,zz}(Y_{,z}+Y_{,\bar{\theta}\bar{\theta}}+Y) + Y_{,zz}(X_{,z}+X_{,\bar{\theta}\bar{\theta}}+X) + X_{,z}(Y+Y_{,\bar{\theta}\bar{\theta}}) + Y_{,z}(X+X_{,\bar{\theta}\bar{\theta}}) + 2X_{,z}Y_{,z} - 2X_{,z\bar{\theta}}Y_{,z\bar{\theta}}\,] \tag{3.41}$$

The advantages of introducing Eqs. (3.31) and (3.32) are obvious. First of all, the variable coefficients on the left hand sides of Eqs. (3.11) and (3.17) become constants. Secondly, the linear differential operators $L_{B_2^*}(\ )$ and $L_{B_3^*}(\ )$ have combined to one as $L_{\bar{B}_2^*}(\ )$. These transformations have led to some significant simplifications for the original governing equations of anisotropic conical shells, thus making it easier to carry out the classic linearized small deflection analysis of anisotropic conical shells. Therefore, Eqs. (3.35) and (3.36) will be used in the following analysis.

Further, assuming that the eigenvalue problem for the buckling load will yield a unique buckling mode $W^{(1)}$ with the associated stress function $F^{(1)}$, a solution, to be valid in the initial postbuckling regime, is sought in the form of the following asymptotic expansions[15]

$$\begin{aligned} \Lambda/\Lambda_c &= 1 + a\xi + b\xi^2 + \dots \\ W &= W^{(o)} + \xi W^{(1)} + \xi^2 W^{(2)} + \dots \\ F &= F^{(o)} + \xi F^{(1)} + \xi^2 F^{(2)} + \dots \end{aligned} \tag{3.42}$$

where ξ is a scalar parameter which tends to zero when Λ approaches the buckling load Λ_c. $W^{(1)}$ will be normalized with respect to the shell thickness t and $W^{(2)}$ is orthogonal to $W^{(1)}$ in some appropriate sense[71].

A formal substitution of this expansion into the nonlinear governing equations (3.35) and (3.36) generates a sequence of equations for the functions appearing in the expansions.

Governing equations of the 0th-order state (Prebuckling problem)

$$\begin{aligned} & L_{\bar{B}_1^*}(F^{(o)}) - L_{\bar{B}_2^*}(F^{(o)}) - L_{\bar{D}^*}(W^{(o)}) \\ & \qquad = L_{NL_2}(F^{(o)}, W^{(o)}) + s_1 e^{z} \cot\alpha_o (F_{,zz}^{(o)} + F_{,z}^{(o)}) + e^{3z} s_1^4 p \end{aligned} \tag{3.43}$$

$$\begin{aligned} & L_{\bar{A}^*}(F^{(o)}) - L_{\bar{B}_1^*}(W^{(o)}) - L_{\bar{B}_2^*}(W^{(o)}) \\ & \qquad = - L_{NL_2}(W^{(o)}, W^{(o)})/2 - s_1 e^{z} \cot\alpha_o (W_{,zz}^{(o)} + W_{,z}^{(o)}) \end{aligned} \tag{3.44}$$

Governing equations of the 1st-order state (Buckling problem)

$$\begin{aligned} & L_{\bar{B}_1^*}(F^{(1)}) - L_{\bar{B}_2^*}(F^{(1)}) - L_{\bar{D}^*}(W^{(1)}) \\ & \qquad = L_{NL_2}(F^{(o)}, W^{(1)}) + L_{NL_2}(W^{(o)}, F^{(1)}) + s_1 e^{z} \cot\alpha_o (F_{,zz}^{(1)} + F_{,z}^{(1)}) \end{aligned} \tag{3.45}$$

$$L_{\bar{A}^*}(F^{(1)}) - L_{\bar{B}_1^*}(W^{(1)}) - L_{\bar{B}_2^*}(W^{(1)})$$
$$= - L_{NL_2}(W^{(o)}, W^{(1)}) - s_1 e^z \cot\alpha_o (W_{,zz}^{(1)} + W_{,z}^{(1)}) \tag{3.46}$$

Governing equations of the 2^nd^-order state (Postbuckling problem)

$$L_{\bar{B}_1^*}(F^{(2)}) - L_{\bar{B}_2^*}(F^{(2)}) - L_{\bar{D}^*}(W^{(2)})$$
$$= L_{NL_2}(F^{(o)}, W^{(2)}) + L_{NL_2}(W^{(o)}, F^{(2)}) + L_{NL_2}(F^{(1)}, W^{(1)}) \tag{3.47}$$
$$+ s_1 e^z \cot\alpha_o (F_{,z}^{(2)} + F_{,zz}^{(2)})$$

$$L_{\bar{A}^*}(F^{(2)}) - L_{\bar{B}_1^*}(W^{(2)}) - L_{\bar{B}_2^*}(W^{(2)})$$
$$= - L_{NL_2}(W^{(o)}, W^{(2)}) - L_{NL_2}(W^{(1)}, W^{(1)})/2 \tag{3.48}$$
$$- s_1 e^z \cot\alpha_o (W_{,z}^{(2)} + W_{,zz}^{(2)})$$

The correctness of Eqs. (3.35)-(3.36) and (3.43)-(3.48) is confirmed by comparing them with those obtained via the REDUCE-based package GEACS3 (see Appendix A1.5 for the source program).

Equations (3.43)-(3.48) are the governing partial differential equations. Together with some appropriate boundary conditions, they determine the behavior of anisotropic conical shells in the prebuckling, buckling and postbuckling states, respectively.

3.4 Reduction to Ordinary Differential Equations

All the governing equations derived in previous sections are partial differential equations. To search for the solutions of these equations one can use one of the standard two-dimensional discretization methods. However, these two dimensional discretization methods often lead to a large set of equations and need long computing times, especially if accurate results are required. Based on some existing experimental and theoretical observations it is known that the original partial differential equations considered here can be further reduced to ordinary differential equations. This will lead to a much smaller set of equations than that of the standard two dimensional methods.

3.4.1 Ordinary differential equations for prebuckling problem

Since we are considering the conical shells with axisymmetric loading and boundary conditions, the prebuckling deformation is also axisymmetric. The condition of axisymmetric stress distribution indicates that the prebuckling Airy stress function must be also a function of z alone (see Eqs. (3.10)). Thus one can assume

$$W^{(o)}(z,\bar{\theta}) = tw_o(z) \tag{3.49}$$

$$F^{(o)}(z,\bar{\theta}) = (Et^2 s_1 \sin\alpha_o/c) f_o(z) \tag{3.50}$$

Substituting Eqs. (3.49) and (3.50) into Eqs. (3.43) and (3.44) and regrouping yields

$$\begin{aligned} &\bar{A}^*_{22} f_o^{iv} - (\bar{A}^*_{11}+\bar{A}^*_{22}) f_o'' + \bar{A}^*_{11} f_o + (t/2s_1\sin\alpha_o)[\,-\bar{B}^*_{21} w_o^{iv} + (\bar{B}^*_{11}-\bar{B}^*_{22}) w_o''' \\ &\quad + (\bar{B}^*_{12}+\bar{B}^*_{21}) w_o'' + (\bar{B}^*_{22}-\bar{B}^*_{11}) w_o' - \bar{B}^*_{12} w_o\,] \\ &\quad = (-e^{z}c/\sin\alpha_o)\{\cot\alpha_o(w_o'+w_o'') + (t/s_1)[\,w_o''(w_o+w_o') + w_o w_o' + w_o'^2\,]\} \end{aligned} \tag{3.51}$$

$$\begin{aligned} &\bar{B}^*_{21} f_o^{iv} + (\bar{B}^*_{11}-\bar{B}^*_{22}) f_o''' - (\bar{B}^*_{12}+\bar{B}^*_{21}) f_o'' + (\bar{B}^*_{22}-\bar{B}^*_{11}) f_o' + \bar{B}^*_{12} f_o \\ &\quad + (t/2s_1\sin\alpha_o)\,[\bar{D}^*_{11} w_o^{iv} - (\bar{D}^*_{11}+\bar{D}^*_{22}) w_o'' + \bar{D}^*_{22} w_o] \\ &\quad = (\,2e^{z} c\, s_1 \cot\alpha_o/t)(f_o''+f_o') + 2e^{z}c\,[f_o''(w_o+w_o') + w_o''(f_o+f_o') \\ &\qquad + f_o' w_o + w_o' f_o + 2w_o' f_o'] + 2e^{3z}\bar{p}\,c\,s_1\cot^2\alpha_o/(t\sin\alpha_o) \end{aligned} \tag{3.52}$$

It can be seen that the above axisymmetric prebuckling equations are coupled, nonlinear ordinary differential equations. This is unlike the case of anisotropic cylindrical shells, where the prebuckling governing equations are linear[67].

3.4.2 Ordinary differential equations for buckling problem

It is known that when the conical shell is supporting the critical load (at the bifurcation point) there exist at least two equilibrium positions: one axisymmetric (identical with the prebuckling stress and deformation state) and one or more asymmetric (beginning of buckling). At the bifurcation point infinitesimal external disturbances will make the prebuckling state unstable and the shell snaps over into an asymmetric equilibrium state.

Since the coefficients of Eqs. (3.45) and (3.46) are functions of z only, a solution is possible by means of separation of variable, whereby

$$F_n^{(1)}(z,\bar{\theta}) = (Et^2 s_1 \sin\alpha_o/c)\, f_n(z)\, e^{in\bar{\theta}} \tag{3.53}$$

$$W_n^{(1)}(z,\bar{\theta}) = t\, w_n(z)\, e^{in\bar{\theta}} \tag{3.54}$$

where $n = n_1/\sin\alpha_o$, $c^2 = 3(1-\nu^2)$, and n_1 is the number of waves in the circumferential direction.

Equations (3.53) and (3.54) satisfy the continuity requirement, that is,

$$F_n^{(1)}(z,\ \bar{\theta}) = F_n^{(1)}(z,\ \bar{\theta}+2\pi\sin\alpha_o)\ , \quad W_n^{(1)}(z,\ \bar{\theta}) = W_n^{(1)}(z,\ \bar{\theta}+2\pi\sin\alpha_o)$$

Without any loss of generality, these orthogonal solutions can be normalized and hence constitute a complete set of eigenfunctions for the problem. Therefore, the solution of Eqs. (3.45) and (3.46) can be represented by the following expansions

$$F^{(1)}(z,\bar{\theta}) = \sum_{n=0}^{\infty} F_n^{(1)}(z,\bar{\theta}) \tag{3.55}$$

$$W^{(1)}(z,\bar{\theta}) = \sum_{n=0}^{\infty} W_n^{(1)}(z,\bar{\theta}) \tag{3.56}$$

However, only real solutions are of interest. The real parts of the solutions for $F_n^{(1)}$ and $W_n^{(1)}$ are then

$$F_n^{(1)}(z,\bar{\theta}) = (Et^2 s_1 \sin\alpha_o/c)\,[\,f_{1n}(z)\cos n\bar{\theta} + f_{2n}(z)\sin n\bar{\theta}\,] \tag{3.57}$$

$$W_n^{(1)}(z,\bar{\theta}) = t\,[w_{1n}(z)\cos n\bar{\theta} + w_{2n}(z)\sin n\bar{\theta}] \tag{3.58}$$

To simplify notations, the n^{th} buckling mode is denoted as

$$F^{(1)} = (Et^2 s_1 \sin\alpha_o / c)[f_1(z)\cos n\bar{\theta} + f_2(z)\sin n\bar{\theta}] \tag{3.59}$$

$$W^{(1)} = t[w_1(z)\cos n\bar{\theta} + w_2(z)\sin n\bar{\theta}] \tag{3.60}$$

The buckling mode shape given by the above equations corresponds to a skewed buckling pattern.

Substituting Eqs. (3.59) and (3.60) into Eqs. (3.45) and (3.46), regrouping and equating the coefficients of like trigonometric terms result in the following system of 4 linear homogeneous ordinary differential equations with variable coefficients in terms of f_1, f_2, w_1 and w_2 for each value of n.

$$\begin{aligned}
&\bar{A}^*_{22} f_1^{iv} - [n^2(2\bar{A}^*_{12} + \bar{A}^*_{66}) + (\bar{A}^*_{11} + \bar{A}^*_{22})] f_1'' + (n^2-1)^2 \bar{A}^*_{11} f_1 \\
&\quad - 2n\bar{A}^*_{26} f_2''' + 2n(n^2-1)\bar{A}^*_{16} f_2' - (t/2s_1\sin\alpha_o)[\bar{B}^*_{21} w_1^{iv} + (\bar{B}^*_{22} - \bar{B}^*_{11}) w_1''' \\
&\quad + (2n^2\bar{B}^*_{66} - n^2\bar{B}^*_{22} - n^2\bar{B}^*_{11} - \bar{B}^*_{21} - \bar{B}^*_{12}) w_1'' + (1-n^2)(\bar{B}^*_{11} - \bar{B}^*_{22}) w_1' \\
&\quad + (n^2-1)^2 \bar{B}^*_{12} w_1 + n(2\bar{B}^*_{26} - \bar{B}^*_{61}) w_2''' - n(2\bar{B}^*_{26} + 2\bar{B}^*_{16} + \bar{B}^*_{62} + \bar{B}^*_{61}) w_2'' \\
&\quad + n(1-n^2)(2\bar{B}^*_{16} - \bar{B}^*_{62}) w_2'] + (e^{z} c \cot\alpha_o / \sin\alpha_o)(w_1'' + w_1') \\
&\quad + (e^{z} c\, t/s_1 \sin\alpha_o)[w_1''\psi + w_1'(\psi' + \psi) + (1-n^2) w_1 \psi'] = 0
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
&\bar{A}^*_{22} f_2^{iv} - (2n^2\bar{A}^*_{12} + n^2\bar{A}^*_{66} + \bar{A}^*_{11} + \bar{A}^*_{22}) f_2'' + (n^2-1)^2 \bar{A}^*_{11} f_2 + 2n\bar{A}^*_{26} f_1''' \\
&\quad + 2n(1-n^2)\bar{A}^*_{16} f_1' - (t/2s_1\sin\alpha_o)[\bar{B}^*_{21} w_2^{iv} + (\bar{B}^*_{22} - \bar{B}^*_{11}) w_2''' \\
&\quad + (2n^2\bar{B}^*_{66} - n^2\bar{B}^*_{22} - n^2\bar{B}^*_{11} - \bar{B}^*_{21} - \bar{B}^*_{12}) w_2'' + (1-n^2)(\bar{B}^*_{11} - \bar{B}^*_{22}) w_2' \\
&\quad + (n^2-1)^2 \bar{B}^*_{12} w_2 + n(\bar{B}^*_{61} - 2\bar{B}^*_{26}) w_1''' + n(2\bar{B}^*_{26} + 2\bar{B}^*_{16} + \bar{B}^*_{62} + \bar{B}^*_{61}) w_1'' \\
&\quad + n(n^2-1)(2\bar{B}^*_{16} - \bar{B}^*_{62}) w_1'] + (e^{z} c \cot\alpha_o / \sin\alpha_o)(w_2'' + w_2') \\
&\quad + (e^{z} c\, t/s_1 \sin\alpha_o)[w_2''\psi + w_2'(\psi' + \psi) + (1-n^2) w_2 \psi'] = 0
\end{aligned} \tag{3.62}$$

$$
\begin{aligned}
&\bar{D}^*_{11}w_1^{iv} - (2n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66}+\bar{D}^*_{11}+\bar{D}^*_{22})w_1'' + (n^2-1)^2\bar{D}^*_{22}w_1 + 4n\bar{D}^*_{16}w_2''' \\
&\quad + 4n(1-n^2)\bar{D}^*_{26}w_2' + (2s_1\sin\alpha_o/t)[\bar{B}^*_{21}f_1^{iv} + (\bar{B}^*_{11}-\bar{B}^*_{22})f_1''' \\
&\quad + (2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-n^2\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})f_1'' + (n^2-1)(\bar{B}^*_{11}-\bar{B}^*_{22})f_1' \\
&\quad + (n^2-1)^2\bar{B}^*_{12}f_1 + n(2\bar{B}^*_{26}-\bar{B}^*_{61})f_2''' + n(2\bar{B}^*_{26}+2\bar{B}^*_{16}+\bar{B}^*_{62}+\bar{B}^*_{61})f_2'' \\
&\quad + n(n^2-1)(\bar{B}^*_{62}-2\bar{B}^*_{16})f_2'] - (4c\,e^{z}s_1\sin\alpha_o/t)[(s_1\cot\alpha_o/t)(f_1''+f_1') \\
&\quad + w_1''\varphi + w_1'(\varphi'+\varphi) + (1-n^2)w_1\varphi' + f_1''\psi + f_1'(\psi'+\psi) + (1-n^2)f_1\psi'] = 0
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
&\bar{D}^*_{11}w_2^{iv} - (2n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66}+\bar{D}^*_{11}+\bar{D}^*_{22})w_2'' + (n^2-1)^2\bar{D}^*_{22}w_2 - 4n\bar{D}^*_{16}w_1''' \\
&\quad + 4n(n^2-1)\bar{D}^*_{26}w_1' + (2s_1\sin\alpha_o/t)[\bar{B}^*_{21}f_2^{iv} + (\bar{B}^*_{11}-\bar{B}^*_{22})f_2''' \\
&\quad + (2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-n^2\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})f_2'' + (n^2-1)(\bar{B}^*_{11}-\bar{B}^*_{22})f_2' \\
&\quad + (n^2-1)^2\bar{B}^*_{12}f_2 + n(\bar{B}^*_{61}-2\bar{B}^*_{26})f_1''' - n(2\bar{B}^*_{26}+2\bar{B}^*_{16}+\bar{B}^*_{62}+\bar{B}^*_{61})f_1'' \\
&\quad + n(n^2-1)(2\bar{B}^*_{16}-\bar{B}^*_{62})f_1'] - (4c\,e^{z}s_1\sin\alpha_o/t)[(s_1\cot\alpha_o/t)(f_2''+f_2') \\
&\quad + w_2''\varphi + w_2'(\varphi'+\varphi) + (1-n^2)w_2\varphi' + f_2''\psi + f_2'(\psi'+\psi) + (1-n^2)f_2\psi'] = 0
\end{aligned}
\tag{3.64}
$$

where

$$
\begin{aligned}
\varphi &= f_o + f_o' \\
\psi &= w_o + w_o'
\end{aligned}
\tag{3.65}
$$

3.4.3 Ordinary differential equations for postbuckling problem

Substituting Eqs. (3.49)-(3.50) and (3.59)-(3.60) into Eqs. (3.47) and (3.48) yields

$$
\begin{aligned}
& L_{\bar{B}_1^*}(F^{(2)}) - L_{\bar{B}_2^*}(F^{(2)}) - L_{\bar{D}^*}(W^{(2)}) \\
&= te^{z}[\,\psi'(F^{(2)} + F_{,z}^{(2)}) + \psi(F_{,zz}^{(2)} + F_{,z}^{(2)}) + F_{,\bar{\theta}\bar{\theta}}^{(2)}\psi'\,] + e^{z}s_1\cot\alpha_o\,(F_{,zz}^{(2)} + F_{,z}^{(2)}) \\
&\quad + (e^{z}Et^2 s_1\sin\alpha_o/c)\,[\,\varphi'(W^{(2)} + W_{,z}^{(2)}) + \varphi\,(W_{,zz}^{(2)} + W_{,z}^{(2)}) + \varphi' W_{,\bar{\theta}\bar{\theta}}^{(2)}\,] \\
&\quad + (e^{z}Et^3 s_1\sin\alpha_o/2c)\,[w_1''(f_1'-n^2f_1+f_1) + w_1'(f_1''+2f_1'-n^2f_1+f_1-2n^2f_1') \\
&\quad + w_2''(f_2'-n^2f_2+f_2) + w_2'(f_2''-2n^2f_2'+2f_2'-n^2f_2+f_2) \\
&\quad + (1-n^2)w_1(f_1''+f_1') + (1-n^2)w_2(f_2''+f_2')] \\
&\quad + (e^{z}Et^3 s_1\sin\alpha_o/2c)\,\{\,[\,w_1''(f_1'-n^2f_1+f_1) + w_1'(f_1''+2n^2f_1'+2f_1'-n^2f_1+f_1) \\
&\qquad\qquad - w_2''(f_2'-n^2f_2+f_2) - w_2'(f_2''+2n^2f_2'+2f_2'-n^2f_2+f_2) \\
&\qquad\qquad + (1-n^2)w_1(f_1''+f_1') + (n^2-1)w_2(f_2''+f_2')]\cos 2n\bar{\theta} \\
&\qquad\qquad + [w_1''(f_2'-n^2f_2+f_2) + w_1'(f_2''+2n^2f_2'+2f_2'-n^2f_2+f_2) \\
&\qquad\qquad + w_2''(f_1'-n^2f_1+f_1) + w_2'(f_1''+2n^2f_1'+2f_1'-n^2f_1+f_1) \\
&\qquad\qquad + (1-n^2)w_2(f_1''+f_1') + (1-n^2)w_1(f_2''+f_2')\,]\sin 2n\bar{\theta}\,\}
\end{aligned}
\tag{3.66}
$$

$$
\begin{aligned}
& L_{\bar{A}^*}(F^{(2)}) - L_{\bar{B}_1^*}(W^{(2)}) - L_{\bar{B}_2^*}(W^{(2)}) \\
&= -\,te^{z}[\,\psi'(\,W^{(2)} + W_{,z}^{(2)} + W_{,\bar{\theta}\bar{\theta}}^{(2)}) + \psi(W_{,zz}^{(2)} + W_{,z}^{(2)})\,] \\
&\quad - e^{z}s_1\cot\alpha_o(W_{,zz}^{(2)} + W_{,z}^{(2)}) - (e^{z}t^2/2)\,\{\,[\,w_1''(w_1'-n^2w_1+w_1) \\
&\quad + (1-n^2)(w_1'^2+w_2'^2) + (1-n^2)(w_1'w_1+w_2'w_2) + w_2''(w_2'-n^2w_2+w_2)\,] \\
&\qquad + [w_1''(w_1'-n^2w_1+w_1) + (1+n^2)(w_1'^2-w_2'^2) \\
&\qquad + (1-n^2)(w_1'w_1-w_2'w_2) - w_2''(w_2'-n^2w_2+w_2)\,]\cos 2n\bar{\theta} \\
&\qquad + [\,w_1''(w_2'-n^2w_2+w_2) + w_1'(w_2''+2n^2w_2'+2w_2') \\
&\qquad + (1-n^2)w_2w_1'+(1-n^2)w_1(w_2''+w_2')\,]\sin 2n\bar{\theta}\,\}
\end{aligned}
\tag{3.67}
$$

These equations admit separable solutions of the form

$$W^{(2)} = t[w_\alpha(z) + w_\beta(z)\cos 2n\bar{\theta} + w_\gamma(z)\sin 2n\bar{\theta}] \tag{3.68}$$

$$F^{(2)} = (Et^2 s_1 \sin\alpha_o/c)[f_\alpha(z) + f_\beta(z)\cos 2n\bar{\theta} + f_\gamma(z)\sin 2n\bar{\theta}] \tag{3.69}$$

Substituting, regrouping and equating coefficients of like trigonometric terms yields the following system of 6 linear inhomogeneous ordinary differential equations with variable coefficients

$$\begin{aligned}
&\bar{B}^*_{21} f^{iv}_\alpha + (\bar{B}^*_{11} - \bar{B}^*_{22}) f'''_\alpha - (\bar{B}^*_{12} + \bar{B}^*_{21}) f''_\alpha + (\bar{B}^*_{22} - \bar{B}^*_{11}) f'_\alpha + \bar{B}^*_{12} f_\alpha \\
&\qquad - (t/2s_1 \sin\alpha_o)[\, -\bar{D}^*_{11} w^{iv}_\alpha + (\bar{D}^*_{11} + \bar{D}^*_{22}) w''_\alpha - \bar{D}^*_{22} w_\alpha \,] \\
&= e^{z} c\,[\, 2\psi'(f_\alpha + f'_\alpha) + 2\varphi'(w'_\alpha + w_\alpha) + 2\varphi(w'_\alpha + w''_\alpha) + 2\psi(f''_\alpha + f'_\alpha) \\
&\qquad\qquad + w''_1(f'_1 - n^2 f_1 + f_1) + w'_1(f''_1 - 2n^2 f'_1 + 2f'_1 - n^2 f_1 + f_1) \\
&\qquad\qquad + w''_2(f'_2 - n^2 f_2 + f_2) + w'_2(f''_2 - 2n^2 f'_2 + 2f'_2 - n^2 f_2 + f_2) \\
&\qquad\qquad + (1 - n^2) w_1 (f''_1 + f'_1) + (1 - n^2) w_2 (f''_2 + f'_2) \\
&\qquad\qquad + (2s_1 \cot\alpha_o/t)\,(f''_\alpha + f'_\alpha)\,]
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
&\bar{B}^*_{21} f^{iv}_\beta + (\bar{B}^*_{11} - \bar{B}^*_{22}) f'''_\beta + (8n^2 \bar{B}^*_{66} - 4n^2 \bar{B}^*_{11} - 4n^2 \bar{B}^*_{22} - \bar{B}^*_{12} - \bar{B}^*_{21}) f''_\beta \\
&\qquad + (4n^2 \bar{B}^*_{11} - 4n^2 \bar{B}^*_{22} - \bar{B}^*_{11} + \bar{B}^*_{22}) f'_\beta + (1 - 4n^2)^2\, \bar{B}^*_{12} f_\beta \\
&\qquad + 2n(2\bar{B}^*_{26} - \bar{B}^*_{61}) f'''_\gamma + 2n(\bar{B}^*_{61} + \bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}^*_{26}) f''_\gamma \\
&\qquad + 2n(2\bar{B}^*_{16} - \bar{B}^*_{62} - 8n^2 \bar{B}^*_{16} + 4n^2 \bar{B}^*_{62}) f'_\gamma \\
&\qquad - (t/2s_1 \sin\alpha_o)[\, -\bar{D}^*_{11} w^{iv}_\beta + (\bar{D}^*_{11} + \bar{D}^*_{22} + 8n^2 \bar{D}^*_{12} + 16n^2 \bar{D}^*_{66}) w''_\beta \\
&\qquad - (1 - 4n^2)^2\, \bar{D}^*_{22} w_\beta - 8n\bar{D}^*_{16} w'''_\gamma + 8n(4n^2 - 1)\bar{D}^*_{26} w'_\gamma \,] \\
&= e^{z} c\,[\, 2\psi(f''_\beta + f'_\beta) + 2\psi'(f_\beta + f'_\beta - 4n^2 f_\beta) + 2\varphi(w''_\beta + w'_\beta) + 2\varphi'(w'_\beta + w_\beta - 4n^2 w_\beta) \\
&\qquad\qquad + w''_1(f'_1 - n^2 f_1 + f_1) + w'_1(f''_1 + 2n^2 f'_1 + 2f'_1 - n^2 f_1 + f_1) \\
&\qquad\qquad + w''_2(n^2 f_2 - f_2 - f'_2) + w'_2(n^2 f_2 - f_2 - 2f'_2 - 2n^2 f'_2 - f''_2) \\
&\qquad\qquad + (1 - n^2)(f''_1 w_1 + f'_1 w_1 - f''_2 w_2 - f'_2 w_2) + (2s_1 \cot\alpha_o)/t)\,(f''_\beta + f'_\beta)\,]
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
&\bar{B}^*_{11}f^{iv}_\gamma + (\bar{B}^*_{11}-\bar{B}^*_{22})f'''_\gamma + (8n^2\bar{B}^*_{66}-4n^2\bar{B}^*_{11}-4n^2\bar{B}^*_{22}-\bar{B}^*_{12}-\bar{B}^*_{21})f''_\gamma \\
&\quad + (4n^2\bar{B}^*_{11}-4n^2\bar{B}^*_{22}-\bar{B}^*_{11}+\bar{B}^*_{22})f'_\gamma + (1-4n^2)^2\,\bar{B}^*_{12}f_\gamma \\
&\quad + 2n(\bar{B}^*_{61}-2\bar{B}^*_{26})f'''_\beta - 2n(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})f''_\beta \\
&\quad + 2n(\bar{B}^*_{62}-2\bar{B}^*_{16}+8n^2\bar{B}^*_{16}-4n^2\bar{B}^*_{62})f'_\beta \\
&\quad - (t/2s_1\sin\alpha_o)\,[\,-\,\bar{D}^*_{11}w^{iv}_\gamma + (\bar{D}^*_{11}+\bar{D}^*_{22}+8n^2\bar{D}^*_{12}+16n^2\bar{D}^*_{66})w''_\gamma \\
&\quad - (1-4n^2)^2\,\bar{D}^*_{22}w_\gamma + 8n\bar{D}^*_{16}w'''_\beta - 8n(4n^2-1)\bar{D}^*_{26}w'_\beta\,] \\
&= e^{z}c\,[\,2\psi(f''_\gamma+f'_\gamma) + 2\psi'(f_\gamma+f'_\gamma-4n^2f_\gamma) + 2\varphi(w''_\gamma+w'_\gamma)+2\varphi'(w'_\gamma+w_\gamma-4n^2w_\gamma) \\
&\quad + w''_1(f'_2-n^2f_2+f_2) + w'_1(f''_2+2n^2f'_2+2f'_2-n^2f_2+f_2) \\
&\quad - w''_2(n^2f_1-f_1-f'_1) - w'_2(n^2f_1-f_1-2f'_1-2n^2f'_1-f''_1) \\
&\quad + (1-n^2)(f''_2w_1+f'_2w_1+f''_1w_2+f'_1w_2) + (2s_1\cot\alpha_o/t)(f''_\gamma+f'_\gamma)\,]
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
&\bar{A}^*_{22}f^{iv}_\alpha - (\bar{A}^*_{11}+\bar{A}^*_{22})f''_\alpha + \bar{A}^*_{11}f_\alpha + (t/2s_1\sin\alpha_o)\,[\,-\bar{B}^*_{21}w^{iv}_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{22})w'''_\alpha \\
&\quad + (\bar{B}^*_{12}+\bar{B}^*_{21})w''_\alpha + (\bar{B}^*_{22}-\bar{B}^*_{11})w'_\alpha - \bar{B}^*_{12}w_\alpha\,] \\
&= (-e^{z}ct/2s_1\sin\alpha_o)\,[\,2\psi(w''_\alpha+w'_\alpha) + 2\psi'(w'_\alpha+w_\alpha) + w''_1(w'_1-n^2w_1+w_1) \\
&\quad + w'_1(1-n^2)(w'_1+w_1) + w''_2(w'_2-n^2w_2+w_2) + w'_2(1-n^2)(w'_2+w_2) \\
&\quad + (2s_1\cot\alpha_o/t)(w''_\alpha+w'_\alpha)\,]
\end{aligned} \tag{3.73}$$

$$\bar{A}_{22}^{*}f_{\beta}^{iv} - (8n^2\bar{A}_{12}^{*}+4n^2\bar{A}_{66}^{*}+\bar{A}_{11}^{*}+\bar{A}_{22}^{*})f_{\beta}'' + (1-4n^2)^2\bar{A}_{11}^{*}f_{\beta} - 4n\bar{A}_{26}^{*}f_{\gamma}'''$$
$$+ 4n(4n^2-1)\bar{A}_{16}^{*}f_{\gamma}' + (t/2s_1\sin\alpha_o)[-\bar{B}_{21}^{*}w_{\beta}^{iv} + (\bar{B}_{11}^{*}-\bar{B}_{22}^{*})w_{\beta}'''$$
$$+ (4n^2\bar{B}_{11}^{*}+4n^2\bar{B}_{22}^{*} - 8n^2\bar{B}_{66}^{*}+\bar{B}_{21}^{*}+\bar{B}_{12}^{*})w_{\beta}'' + (4n^2\bar{B}_{11}^{*}-4n^2\bar{B}_{22}^{*}-\bar{B}_{11}^{*}+\bar{B}_{22}^{*})w_{\beta}'$$
$$- (1-4n^2)^2\bar{B}_{12}^{*}w_{\beta} + 2n(\bar{B}_{61}^{*}-2\bar{B}_{26}^{*})w_{\gamma}''' + 2n(\bar{B}_{61}^{*}+\bar{B}_{62}^{*}+2\bar{B}_{16}^{*}+2\bar{B}_{16}^{*})w_{\gamma}'' \quad (3.74)$$
$$+ 2n(\bar{B}_{62}^{*}+8n^2\bar{B}_{16}^{*}-2\bar{B}_{16}^{*}-4n^2\bar{B}_{62}^{*})w_{\gamma}']$$
$$= (-e^{z}ct/2s_1\sin\alpha_o)[2\psi(w_{\beta}''+w_{\beta}') + 2\psi'(w_{\beta}'-4n^2w_{\beta}+w_{\beta}) + w_1''(w_1'-n^2w_1+w_1)$$
$$+ w_1'(n^2w_1'+w_1'-n^2w_1+w_1) + w_2''(n^2w_2-w_2'-w_2) + w_2'(-n^2w_2'-w_2'$$
$$+ n^2w_2-w_2) + (w_{\beta}''+w_{\beta}')(2s_1\cot\alpha_o/t)]$$

$$\bar{A}_{22}^{*}f_{\gamma}^{iv} - (8n^2\bar{A}_{12}^{*}+4n^2\bar{A}_{66}^{*}+\bar{A}_{11}^{*}+\bar{A}_{22}^{*})f_{\gamma}'' + (1-4n^2)^2\bar{A}_{11}^{*}f_{\gamma} + 4n\bar{A}_{26}^{*}f_{\beta}'''$$
$$- 4n(4n^2-1)\bar{A}_{16}^{*}f_{\beta}' + (t/2s_1\sin\alpha_o)[-\bar{B}_{21}^{*}w_{\gamma}^{iv} + (\bar{B}_{11}^{*}-\bar{B}_{22}^{*})w_{\gamma}'''$$
$$+ (4n^2\bar{B}_{11}^{*}+4n^2\bar{B}_{22}^{*} - 8n^2\bar{B}_{66}^{*}+\bar{B}_{21}^{*}+\bar{B}_{12}^{*})w_{\gamma}'' + (4n^2\bar{B}_{11}^{*}-4n^2\bar{B}_{22}^{*}-\bar{B}_{11}^{*}+\bar{B}_{22}^{*})w_{\gamma}'$$
$$- (1-4n^2)^2\bar{B}_{12}^{*}w_{\gamma} + 2n(2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})w_{\beta}''' - 2n(\bar{B}_{61}^{*}+\bar{B}_{62}^{*}+2\bar{B}_{16}^{*}+2\bar{B}_{16}^{*})w_{\beta}'' \quad (3.75)$$
$$- 2n(\bar{B}_{62}^{*}+8n^2\bar{B}_{16}^{*}-2\bar{B}_{16}^{*}-4n^2\bar{B}_{62}^{*})w_{\beta}']$$
$$= (-e^{z}ct/2s_1\sin\alpha_o)[2\psi(w_{\gamma}''+w_{\gamma}') + 2\psi'(w_{\gamma}'-4n^2w_{\gamma}+w_{\gamma}) + w_1''(w_2'-n^2w_2+w_2)$$
$$+ w_1'(w_2''+2n^2w_2'+2w_2'-n^2w_2+w_2) + w_1(1-n^2)(w_2''+w_2')$$
$$+ (w_{\gamma}''+w_{\gamma}')(2s_1\cot\alpha_o)/t]$$

The correctness of Eqs. (3.51)-(3.52), (3.61)-(3.64) and (3.70)-(3.75) is confirmed by comparing them with those obtained via the REDUCE-based package GEACS3.

3.5 Discussions and Conclusion

To summarize briefly, the nonlinear governing equations of anisotropic conical shells in terms of the radial displacement $\tilde{W}$ and the Airy stress function $\tilde{F}$ are derived via the stationary potential energy criterion, based on the nonlinear Donnell-type strain-displacement relations. By employing some special transformations, the original nonlinear partial differential equations are transformed to some simpler forms. Further, using certain perturbation technique three sets of partial differential equations governing the behavior of prebuckling, buckling and postbuckling problems, respectively, are obtained. To reduce these partial differential equations to ordinary differential equations, the nonlinear prebuckling solution is assumed to be axisymmetric, while based on some existing theoretical results and experimental observations the circumferential dependencies of the buckling and postbuckling equations are eliminated by Fourier decomposition.

Two other kinds of governing differential equations for anisotropic conical shells with variable stiffness coefficients, which are more complicated than the one presented previously, were also derived with the help of the Computer Algebra system REDUCE. One is expressed in terms of $\check{\tilde{W}}$ and $\check{\tilde{F}}$, and the other in terms of u, v, W. The purpose of this derivation is to give some impression for the complexity of the corresponding governing differential equations. It is felt that they will be useful for further investigations.

The tedious symbolic and algebraic manipulations in above derivations make it no longer reliable to obtain the governing equations by hand. Thus, the computerized symbolic computations are needed in order to increase the efficiency of the lengthy derivations and guarantee the reliability of the results. By means of the REDUCE-based program GEACS1 and GEACS3 all the derivations involved in this chapter can be performed by computer, and correctness of all these complicated equations is verified by comparing them with either other known results or computer results calculated via other Computer Algebra systems.

Chapter 4
Reduced Boundary Conditions

4.1 Introduction

In many practical situations when dealing with the structural stability problems one must handle the boundary conditions appropriately if one wants to obtain the correct solutions. The proper handling of boundary conditions depends on at least two factors: correct modelling and correct implementation. For correct modelling, first, one has to study carefully the practical boundary conditions in the real structures, and then establish a mathematical model for them. This mathematical model should reproduce as close as possible the corresponding practical boundary conditions. For correct implementation, specifically when one performs a Koiter-type stability analysis of conical shells, one has to use consistent boundary conditions in the prebuckling, buckling and postbuckling states, based on the previously established mathematical model.

As to the boundary conditions imposed on the conical shells, generally speaking, they can be separated into two groups: requirements of geometric compatibility called displacement boundary conditions, and requirements of force equilibrium called force boundary conditions. Although the boundary conditions presented by Eqs. (3.9) provide many possibilities for different combinations of displacement and force boundary conditions, they are not necessarily the correct modelling for some practical boundary conditions. To obtain the correct modelling for practical boundary conditions, additional geometric constraints, such as the one suggested by Seide[32], should be implemented. Besides, before using the boundary conditions to solve the stability problem of conical shells, it is necessary to express them in terms of the functions for W and F. In the following the individual boundary conditions are derived in terms of the assumed functions for the prebuckling, buckling and postbuckling states by satisfying the Seide-type geometric constraint. Then, some appropriate combinations can be chosen, which are known to constitute some fairly good models for some practical boundary conditions.

4.2 Seide's Additional Geometric Constraint

Observations of actual experiments show that in addition to the displacement and force boundary conditions derived by the variational method, there exists always some kind of geometric constraint in the practical experimental setup. Therefore, to obtain an accurate model for the real boundary conditions in the experiment, it is necessary to take these additional geometric constraints into considerations. Seide[32] suggested the following geometric constraint which was also adopted by many others[52,57,89,153]:

$$u - \tilde{W}\cot\alpha_o = 0 \qquad (4.1)$$

This geometric constraint is based on the fact that the edges of the conical shell are usually built into some kind of very stiff endplates (or rigid rings) in the experiment. Equation (4.1) indicates that the total displacement in the horizontal direction vanishes because of the relatively rigid bulkheads. Thus the end with bulkhead can only deform in the vertical direction. It is easy to see that for the case of $u = \tilde{W} = 0$ at the edges this additional geometric constraint vanishes identically. Notice that for cylindrical shells where $\alpha_o = 0$ it becomes $\tilde{W} = 0$.

Imposing this additional geometric constraint at the edges indicates also that some of the boundary conditions given by Eqs. (3.9) are no longer independent of each other. By using $u - \tilde{W}\cot\alpha_o = 0$ while evaluating the first variation of the total potential energy expression, one obtains the modified general boundary conditions as

either

$$u - \tilde{W}\cot\alpha_o = 0$$

and

$$N_{s\theta} = \bar{N}_{s\theta} \quad \text{or} \quad \delta v = 0 \qquad (4.2)$$

$$M_s = \bar{M}_s \quad \text{or} \quad \delta\tilde{W}_{,s} = 0$$

$$H + sN_s\cot\alpha_o = \bar{H} + s\bar{N}_s\cot\alpha_o \quad \text{or} \quad \delta\tilde{W} = 0$$

or

$$u - \tilde{W} \cot\alpha_o = 0$$

and

$$N_{s\theta} = \bar{N}_{s\theta} \quad \text{or} \quad \delta v = 0 \tag{4.3}$$

$$M_s = \bar{M}_s \quad \text{or} \quad \delta \tilde{W}_{,s} = 0$$

$$H \tan\alpha_o + s N_s = \bar{H} \tan\alpha_o + s \bar{N}_s \quad \text{or} \quad \delta u = 0$$

where the barred quantities are prescribed force or moment resultants at the boundary, and H and $\bar{H}$ are the left and right hand sides of the last force boundary condition of Equations (3.9), respectively.

4.3 Reduced Individual Boundary Conditions

Recalling that

$$W(z, \bar{\theta}) = t\, w_o(z) + t\,\xi[\, w_1(z) \cos n\bar{\theta} + w_2(z) \sin n\bar{\theta}] + t\,\xi^2 [\, w_\alpha(z) + w_\beta \cos 2n\bar{\theta} + w_\gamma \sin 2n\bar{\theta}] \tag{4.4}$$

$$F(z, \bar{\theta}) = (E t^2 s_1 \sin\alpha_o / c)\{ f_o(z) + \xi\, [\, f_1(z) \cos n\bar{\theta} + f_2(z) \sin n\bar{\theta}] + \xi^2 [\, f_\alpha(z) + f_\beta \cos 2n\bar{\theta} + f_\gamma \sin 2n\bar{\theta}\,]\} \tag{4.5}$$

the individual boundary conditions can be expressed as follows.

(1) Boundary condition: $v = 0$

Here one must express the condition $v = 0$ in terms of the variables W and F.

Substituting the geometric constraint $u - \tilde{W} \cot\alpha_o = 0$ into the second of Eqs. (3.1), and considering only perfect shells, one obtains

$$\varepsilon_\theta = \frac{1}{s} v_{,\bar{\theta}} + \frac{1}{2s^2}\tilde{W}_{,\bar{\theta}}^2 \tag{4.6}$$

Recalling the fact that if a function $\Omega(x,y)$ in an orthogonal reference frame x, y satisfies the condition

$$\Omega(x, y) = c \qquad \text{at} \qquad x = x_o \tag{4.7}$$

where both C and x_o are constants then

$$\frac{\partial^r}{\partial y^r}\Omega(x, y) = 0 \qquad \text{at} \qquad x = x_o \tag{4.8}$$

for r = 1, 2, 3, ..., one knows that v = 0 at $s = s_1$ (or $s = s_2$) implies that also

$$v_{,\bar{\theta}} = 0 \tag{4.9}$$

at the shell edges.

Thus Eq. (4.6) becomes

$$\varepsilon_\theta = \frac{1}{2s^2}\tilde{W}_{,\bar{\theta}}^2 \tag{4.10}$$

Substituting the semi-inverted constitutive equations and Eqs. (3.2) and (3.10) into Eqs. (4.10), and carrying out the z-transformations yields

$$\begin{aligned} &B_{21}^* W_{,zz} + 2B_{26}^* W_{,z\bar{\theta}} + B_{22}^* W_{,\bar{\theta}\bar{\theta}} + (B_{21}^* + B_{22}^*)W_{,z} + B_{22}^* W \\ &\quad - A_{22}^* F_{,zz} + A_{26}^* F_{,z\bar{\theta}} - A_{12}^* F_{,\bar{\theta}\bar{\theta}} - (A_{12}^* + A_{22}^*)F_{,z} - A_{12}^* F + e^z W_{,\bar{\theta}}^2/2 = 0 \end{aligned} \tag{4.11}$$

Introducing now for W and F the assumed forms of Eqs. (4.4) and (4.5), regrouping by powers of ξ and equating coefficients of like trigonometric terms yields the boundary conditions for the

Prebuckling problem

$$\bar{A}^*_{22}\varphi' + \bar{A}^*_{12}\varphi - (t/2s_1\sin\alpha_o)(\bar{B}^*_{21}\psi' + \bar{B}^*_{22}\psi) = 0 \tag{4.12}$$

Buckling problem

$$\begin{aligned}&\bar{A}^*_{22}f_1'' + (\bar{A}^*_{12}+\bar{A}^*_{22})f_1' + (1-n^2)\bar{A}^*_{12}f_1 - n\bar{A}^*_{26}f_2' \\ &- (t/2s_1\sin\alpha_o)[\bar{B}^*_{21}w_1'' + (\bar{B}^*_{21}+\bar{B}^*_{22})w_1' + (1-n^2)\bar{B}^*_{22}w_1 + 2n\bar{B}^*_{26}w_2'] = 0\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\bar{A}^*_{22}f_2'' + (\bar{A}^*_{12}+\bar{A}^*_{22})f_2' + (1-n^2)\bar{A}^*_{12}f_2 + n\bar{A}^*_{26}f_1' \\ &- (t/2s_1\sin\alpha_o)[\bar{B}^*_{21}w_2'' + (\bar{B}^*_{21}+\bar{B}^*_{22})w_2' + (1-n^2)\bar{B}^*_{22}w_2 - 2n\bar{B}^*_{26}w_1'] = 0\end{aligned} \tag{4.14}$$

Postbuckling problem

$$\begin{aligned}&\bar{A}^*_{22}f_\beta'' + (\bar{A}^*_{12}+\bar{A}^*_{22})f_\beta' + (1-4n^2)\bar{A}^*_{12}f_\beta - 2n\bar{A}^*_{26}f_\gamma' + (t/4s_1\sin\alpha_o)[-2\bar{B}^*_{21}w_\beta'' \\ &\quad - 2(\bar{B}^*_{21}+\bar{B}^*_{22})w_\beta' + 2(4n^2-1)\bar{B}^*_{22}w_\beta - 8n\bar{B}^*_{26}w_\gamma' + e^z c(n^2w_1^2 - w_2^2)] = 0\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\bar{A}^*_{22}f_\gamma'' + (\bar{A}^*_{12}+\bar{A}^*_{22})f_\gamma' + (1-4n^2)\bar{A}^*_{12}f_\gamma + 2n\bar{A}^*_{26}f_\beta' + (t/2s_1\sin\alpha_o)[-\bar{B}^*_{21}w_\gamma'' \\ &\qquad - (\bar{B}^*_{21}+\bar{B}^*_{22})w_\gamma' + (4n^2-1)\bar{B}^*_{22}w_\gamma + 4n\bar{B}^*_{26}w_\beta' + e^z cn^2w_1w_2] = 0\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\bar{A}^*_{22}f_\alpha'' + (\bar{A}^*_{12}+\bar{A}^*_{22})f_\alpha' + \bar{A}^*_{12}f_\alpha - (t/4s_1\sin\alpha_o)[2\bar{B}^*_{21}w_\alpha'' - 2(\bar{B}^*_{21}+\bar{B}^*_{22})w_\alpha' \\ &\qquad + 2\bar{B}^*_{22}w_\alpha + e^z cn^2(w_1^2+w_2^2)] = 0\end{aligned} \tag{4.17}$$

(2) Boundary condition: $u - \tilde{W}\cot\alpha_o = 0$

To express the geometric boundary condition $u - \tilde{W}\cot\alpha_o = 0$ in terms of the variables W and F one substitutes first $u - \tilde{W}\cot\alpha_o = 0$ into the second of Eqs. (3.1), and eliminates u and v from Eqs. (3.1). This yields

$$s\varepsilon_{\theta,s} - \gamma_{s\theta,\bar\theta} - \varepsilon_s = -\tilde W_{,s}\cot\alpha_o - \tilde W_{,\bar\theta}^2/s^2 - \tilde W_{,s}\tilde W_{,\bar\theta\bar\theta}/s - \tilde W_{,s}^2/2 - u_{,\bar\theta\bar\theta}/s \tag{4.18}$$

Recalling the behavior of a function $\Omega(x,y)$ in an orthogonal reference frame one concludes that $u - \tilde W\cot\alpha_o = 0$ at the edges implies that also

$$u_{,\bar\theta\bar\theta} - \tilde W_{,\bar\theta\bar\theta}\cot\alpha_o = 0 \tag{4.19}$$

at the same edges.

Thus Eq. (4.18) becomes

$$\begin{aligned} &s\varepsilon_{\theta,s} - \gamma_{s\theta,\bar\theta} - \varepsilon_s \\ &= -\tilde W_{,s}\cot\alpha - \tilde W_{,\bar\theta}^2/s^2 - \tilde W_{,s}\tilde W_{,\bar\theta\bar\theta}/s - \tilde W_{,s}^2/2 - \tilde W_{,\bar\theta\bar\theta}\cot\alpha/s \end{aligned} \tag{4.20}$$

Equation (4.20) can be further transformed into an new form by introducing the semi-inverted constitutive equations and carrying out the z-transformation.

Introducing for W and F in the assumed forms of Eqs. (4.4) and (4.5), regrouping by powers of ξ and equating coefficients of like trigonometric terms result in the following boundary conditions:

Prebuckling problem

$$\begin{aligned} &\bar A_{22}^* f_o''' - (\bar A_{11}^* + \bar A_{12}^* + \bar A_{22}^*) f_o' - (\bar A_{11}^* + \bar A_{12}^*) f_o \\ &\quad - (t/2s_1\sin\alpha_o)\,[\,\bar B_{21}^* w_o''' + (\bar B_{22}^* - \bar B_{11}^*) w_o'' \\ &\quad - (\bar B_{11}^* + \bar B_{21}^* + \bar B_{12}^*) w_o' - (\bar B_{12}^* + \bar B_{22}^*) w_o\,] \\ &= -(e^{z}ct/2s_1\sin\alpha_o)[\,w_o^2 + 2s_1\cot\alpha_o(w_o' + w_o)/t + 2w_o w_o' + w_o'^2\,] \end{aligned} \tag{4.21}$$

Substituting the periodicity condition (see Appendix A2.1 for details) and Eqs. (3.65) into Eq. (4.21) yields an equation which is exactly the same as Eq. (4.12). This indicates that $u - \tilde W\cot\alpha_o = 0$ gives no new boundary condition for the prebuckling problem.

Buckling problem

$$\begin{aligned}
&\bar{A}^*_{22}f_1''' - (n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66} + \bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})f_1' + (n^2-1)(\bar{A}^*_{11}+\bar{A}^*_{12})f_1 \\
&\quad - 2n\bar{A}^*_{26}f_2'' + n(n^2-1)\bar{A}^*_{16}f_2 - (t/2s_1 \sin\alpha_o)[\,\bar{B}^*_{21}w_1''' + (\bar{B}^*_{22}-\bar{B}^*_{11})w_1'' \\
&\quad + (2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})w_1' + (n^2-1)(\bar{B}^*_{12}+\bar{B}^*_{22})w_1 \\
&\quad + n(2\bar{B}^*_{26}-\bar{B}^*_{61})w_2'' - n(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})w_2' + n(n^2-1)\bar{B}^*_{62}w_2\,] \\
&= -(e^z ct/s_1 \sin\alpha_o)\{w_1\psi + w_1'\psi - n^2w_1\psi + s_1\cot\alpha_o[\,w_1(1-n^2) + w_1'\,]/t\}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\bar{A}^*_{22}f_2''' - (n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66} + \bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})f_2' + (n^2-1)(\bar{A}^*_{11}+\bar{A}^*_{12})f_2 \\
&\quad + 2n\bar{A}^*_{26}f_1'' - n(n^2-1)\bar{A}^*_{16}f_1 - (t/2s_1 \sin\alpha_o)[\,\bar{B}^*_{21}w_2''' + (\bar{B}^*_{22}-\bar{B}^*_{11})w_2'' \\
&\quad + (2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})w_2' + (n^2-1)(\bar{B}^*_{12}+\bar{B}^*_{22})w_2 \\
&\quad - n(2\bar{B}^*_{26}-\bar{B}^*_{61})w_1'' + n(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})w_1' - n(n^2-1)\bar{B}^*_{62}w_1\,] \\
&= -(e^z ct/s_1 \sin\alpha_o)\{w_2\psi + w_2'\psi - n^2w_2\psi + s_1\cot\alpha_o[\,w_2(1-n^2) + w_2'\,]/t\}
\end{aligned} \tag{4.23}$$

Postbuckling problem

$$\begin{aligned}
&\bar{A}^*_{22}f_\beta''' - (4n^2\bar{A}^*_{12}+4n^2\bar{A}^*_{66}+\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})f_\beta' + (4n^2-1)(\bar{A}^*_{11}+\bar{A}^*_{12})f_\beta \\
&\quad - 4n\bar{A}^*_{26}f_\gamma'' + 2n(4n^2-1)\bar{A}^*_{16}f_\gamma + (t/2s_1 \sin\alpha_o)[\,-\bar{B}^*_{21}w_\beta''' - (\bar{B}^*_{22}-\bar{B}^*_{11})w_\beta'' \\
&\quad - (8n^2\bar{B}^*_{66}-4n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})w_\beta' - (4n^2-1)(\bar{B}^*_{12}+\bar{B}^*_{22})w_\beta \\
&\quad - 2n(2\bar{B}^*_{26}-\bar{B}^*_{61})w_\gamma'' + 2n(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})w_\gamma' - 2n(4n^2-1)\bar{B}^*_{62}w_\gamma\,] \\
&= -(e^z ct/4s_1 \sin\alpha_o)[\,w_1'(w_1'-2n^2w_1+2w_1) + w_2'(2n^2w_2-w_2'-2w_2) \\
&\quad + 4(w_\beta-4n^2w_\beta+w_\beta')(\psi+s_1\cot\alpha_o/t) + (1-4n^2)(w_1^2-w_2^2)\,]
\end{aligned} \tag{4.24}$$

$$
\begin{aligned}
&\bar{A}^*_{22} f'''_\gamma - (4n^2\bar{A}^*_{12} + 4n^2\bar{A}^*_{66} + \bar{A}^*_{11} + \bar{A}^*_{12} + \bar{A}^*_{22}) f'_\gamma + (4n^2-1)(\bar{A}^*_{11} + \bar{A}^*_{12}) f_\gamma \\
&\quad + 4n\bar{A}^*_{26} f''_\beta - 2n(4n^2-1)\bar{A}^*_{16} f_\beta + (t/2s_1 \sin\alpha_o)\,[\,-\bar{B}^*_{21} w'''_\gamma - (\bar{B}^*_{22} - \bar{B}^*_{11}) w''_\gamma \\
&\quad - (8n^2\bar{B}^*_{66} - 4n^2\bar{B}^*_{22} - \bar{B}^*_{11} - \bar{B}^*_{21} - \bar{B}^*_{12}) w'_\gamma - (4n^2-1)(\bar{B}^*_{12} + \bar{B}^*_{22}) w_\gamma \\
&\quad + 2n(2\bar{B}^*_{26} - \bar{B}^*_{61}) w''_\beta - 2n(\bar{B}^*_{61} + \bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}^*_{26}) w'_\beta \\
&\quad + 2n(4n^2-1)\bar{B}^*_{62} w_\beta\,] \\
&= -(e^z ct/2s_1 \sin\alpha_o)\,[\,w'_1(w'_2 - n^2 w_2 + w_2) + (1-n^2) w_1 w'_2 \\
&\quad + 2(w_\gamma - 4n^2 w_\gamma + w'_\gamma)(\psi + s_1 \cot\alpha_o/t) + (1-4n^2) w_1 w_2\,]
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&\bar{A}^*_{22} f'''_\alpha - (\bar{A}^*_{11} + \bar{A}^*_{12} + \bar{A}^*_{22}) f'_\alpha - (\bar{A}^*_{11} + \bar{A}^*_{12}) f_\alpha \\
&\quad - (t/2s_1 \sin\alpha_o)\,[\,-\bar{B}^*_{21} w'''_\alpha - (\bar{B}^*_{22} - \bar{B}^*_{11}) w''_\alpha \\
&\quad + (\bar{B}^*_{11} + \bar{B}^*_{21} + \bar{B}^*_{12}) w'_\alpha + (\bar{B}^*_{12} + \bar{B}^*_{22}) w_\alpha\,] \\
&= -(e^z ct/4s_1 \sin\alpha_o)\,[\,w'_1(w'_1 - 2n^2 w_1 + 2w_1) - w'_2(2n^2 w_2 - w'_2 - 2w_2) \\
&\quad + 4\psi(w_\beta + w'_\beta) + (w_1^2 + w_2^2) + (4 s_1 \cot\alpha_o/t)(w'_\alpha + w_\alpha)\,]
\end{aligned}
\tag{4.26}
$$

(3) Boundary condition: $\tilde{W}_{,s} = 0$

Carrying out the z-transformation for $\tilde{W}_{,s} = 0$ yields

$$
W + W_{,z} = 0 \tag{4.27}
$$

Substituting Eq. (4.4) into Eq. (4.27) yields

Prebuckling problem

$$
\psi = w'_o + w_o = 0 \tag{4.28}
$$

Buckling problem

$$w_1' + w_1 = 0$$
$$w_2' + w_2 = 0 \qquad (4.29)$$

Postbuckling problem

$$w_\beta' + w_\beta = 0$$
$$w_\gamma' + w_\gamma = 0 \qquad (4.30)$$
$$w_\alpha' + w_\alpha = 0$$

(4) Boundary condition: $M_s = 0$

Here one must first express M_s in terms of the variables W and F. Using the correspon-ding semi-inverted constitutive equation and carrying out the z-transformation one obtains upon substitution

$$B_{21}^* F_{,zz} - B_{61}^* F_{,z\bar{\theta}} + B_{11}^* F_{,\bar{\theta}\bar{\theta}} + (B_{11}^* + B_{21}^*) F_{,z} + B_{11}^* F$$
$$+ D_{11}^* W_{,zz} + 2D_{16}^* W_{,z\bar{\theta}} + D_{12}^* W_{,\bar{\theta}\bar{\theta}} + (D_{11}^* + D_{12}^*) W_{,z} + D_{12}^* W = 0 \qquad (4.31)$$

Introducing for W and F the assumed perturbation expansions (4.4) and (4.5) and regrouping by powers of ξ yields

Prebuckling problem

$$\bar{D}_{11}^* \psi' + \bar{D}_{12}^* \psi + (2\, s_1 \sin\alpha_o / t)\, (\bar{B}_{21}^* \varphi + \bar{B}_{11}^* \varphi) = 0 \qquad (4.32)$$

Buckling problem

$$\bar{D}_{11}^{*}w_1'' + (\bar{D}_{11}^{*}+\bar{D}_{12}^{*})w_1' + (1-n^2)\bar{D}_{12}^{*}w_1 + 2n\bar{D}_{16}^{*}w_2'$$
$$+ (2s_1\sin\alpha_o/t)[\bar{B}_{21}^{*}f_1'' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*})f_1' + (1-n^2)\bar{B}_{11}^{*}f_1 - n\bar{B}_{61}^{*}f_2'] = 0 \tag{4.33}$$

$$\bar{D}_{11}^{*}w_2'' + (\bar{D}_{11}^{*}+\bar{D}_{12}^{*})w_2' + (1-n^2)\bar{D}_{12}^{*}w_2 - 2n\bar{D}_{16}^{*}w_1'$$
$$+ (2s_1\sin\alpha_o/t)[\bar{B}_{21}^{*}f_2'' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*})f_2' + (1-n^2)\bar{B}_{11}^{*}f_2 + n\bar{B}_{61}^{*}f_1'] = 0 \tag{4.34}$$

Postbuckling problem

$$\bar{D}_{11}^{*}w_\beta'' + (\bar{D}_{11}^{*}+\bar{D}_{12}^{*})w_\beta' + (1-4n^2)\bar{D}_{12}^{*}w_\beta + 4n\bar{D}_{16}^{*}w_\gamma'$$
$$+ (2\,s_1\sin\alpha_o/t)[\bar{B}_{21}^{*}f_\beta'' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*})f_\beta' + (1-4n^2)\bar{B}_{11}^{*}f_\beta - 2n\bar{B}_{61}^{*}f_\gamma'] = 0 \tag{4.35}$$

$$\bar{D}_{11}^{*}w_\gamma'' + (\bar{D}_{11}^{*}+\bar{D}_{12}^{*})w_\gamma' + (1-4n^2)\bar{D}_{12}^{*}w_\gamma - 4n\bar{D}_{16}^{*}w_\beta'$$
$$+ (2\,s_1\sin\alpha_o/t)[\bar{B}_{21}^{*}f_\gamma'' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*})f_\gamma' + (1-4n^2)\bar{B}_{11}^{*}f_\gamma + 2n\bar{B}_{61}^{*}f_\beta'] = 0 \tag{4.36}$$

$$\bar{D}_{11}^{*}w_\alpha'' + (\bar{D}_{11}^{*}+\bar{D}_{12}^{*})w_\alpha' + \bar{D}_{12}^{*}w_\alpha$$
$$+ (2\,s_1\sin\alpha_o/t)[\bar{B}_{21}^{*}f_\alpha'' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*})f_\alpha' + \bar{B}_{11}^{*}f_\alpha] = 0 \tag{4.37}$$

(5) Boundary condition: $H = \bar{H}$

For a perfect conical shell, the last natural boundary condition of Eqs. (4.2) can be written as

$$sN_s\tilde{W}_{,s} + sN_s\cot\alpha_o + N_{s\theta}\tilde{W}_{,\bar{\theta}} + (sM_s)_{,s} - M_\theta + 2M_{s\theta,\bar{\theta}} = s(\bar{Q}_s + \bar{N}_s\cot\alpha_o) \tag{4.38}$$

First substituting the semi-inverted constitutive equations and Equations (4.4) and (4.5) into Equation. (4.38), then carrying out the z-transformation and regrouping by powers of ξ yields the following reduced boundary conditions for the

Prebuckling problem

$$\begin{aligned}
&\bar{B}_{21}^{*}f_o''' + (\bar{B}_{21}^{*}+\bar{B}_{11}^{*}-\bar{B}_{22}^{*})f_o'' + (\bar{B}_{11}^{*}-\bar{B}_{22}^{*}-\bar{B}_{12}^{*})f_o' - \bar{B}_{12}^{*}f_o \\
&\qquad + (t/2s_1\sin\alpha_o)\,[\bar{D}_{11}^{*}w_o''' + \bar{D}_{11}^{*}w_o'' - \bar{D}_{22}^{*}w_o' - \bar{D}_{22}^{*}w_o] \\
&= 2e^{z}c(w_o+w_o')(f_o+f_o') + (2e^{z}cs_1\cot\alpha_o/t)(f_o+f_o') \\
&\qquad + e^{3z}\bar{p}\,cs_1\cot^2\alpha_o/(t\sin\alpha_o) + 2e^{z}\lambda s_1 c\cot^2\alpha_o/(t\sin\alpha_o)
\end{aligned} \tag{4.39}$$

Buckling problem

$$\begin{aligned}
&\bar{D}_{11}^{*}w_1''' + \bar{D}_{11}^{*}w_1'' - (\bar{D}_{22}^{*}+n^2\bar{D}_{12}^{*}+4n^2\bar{D}_{66}^{*})w_1' + (n^2-1)\bar{D}_{22}^{*}w_1 \\
&\qquad + 4n\bar{D}_{16}^{*}w_2'' + 2n\bar{D}_{16}^{*}w_2' + 2n(1-n^2)\bar{D}_{26}^{*}w_2 \\
&\qquad + (2s_1\sin\alpha_o/t)\,[\,\bar{B}_{21}^{*}f_1''' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*}-\bar{B}_{22}^{*})f_1'' \\
&\qquad + (\bar{B}_{11}^{*}+2n^2\bar{B}_{66}^{*}-n^2\bar{B}_{11}^{*}-\bar{B}_{12}^{*}-\bar{B}_{22}^{*})f_1' - n(\bar{B}_{61}^{*}-2\bar{B}_{26}^{*})f_2'' \\
&\qquad + n(\bar{B}_{62}^{*}+2\bar{B}_{16}^{*}+2\bar{B}_{26}^{*})f_2' + 2n(1-n^2)\bar{B}_{16}^{*}f_2 + (n^2-1)\bar{B}_{12}^{*}f_1\,] \\
&= (4e^{z}s_1c\sin\alpha_o/t^2)\{t\,[\,(1-n^2)\psi f_1 + w_1\varphi + w_1'\varphi + f_1'\psi\,] \\
&\qquad + (1-n^2)s_1\cot\alpha_o f_1 + s_1\cot\alpha_o f_1'\}
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
&\bar{D}_{11}^{*}w_2''' + \bar{D}_{11}^{*}w_2'' - (\bar{D}_{22}^{*}+n^2\bar{D}_{12}^{*}+4n^2\bar{D}_{66}^{*})w_2' + (n^2-1)\bar{D}_{22}^{*}w_2 \\
&\qquad - 4n\bar{D}_{16}^{*}w_1'' - 2n\bar{D}_{16}^{*}w_1' - 2n(1-n^2)\bar{D}_{26}^{*}w_1 \\
&\qquad + (2s_1\sin\alpha_o/t)\,[\,\bar{B}_{21}^{*}f_2''' + (\bar{B}_{11}^{*}+\bar{B}_{21}^{*}-\bar{B}_{22}^{*})f_2'' \\
&\qquad + (\bar{B}_{11}^{*}+2n^2\bar{B}_{66}^{*}-n^2\bar{B}_{11}^{*}-\bar{B}_{12}^{*}-\bar{B}_{22}^{*})f_2' + n(\bar{B}_{61}^{*}-2\bar{B}_{26}^{*})f_1'' \\
&\qquad - n(\bar{B}_{62}^{*}+2\bar{B}_{16}^{*}+2\bar{B}_{26}^{*})f_1' - 2n(1-n^2)\bar{B}_{16}^{*}f_1 + (n^2-1)\bar{B}_{12}^{*}f_2\,] \\
&= (4e^{z}s_1c\sin\alpha_o/t^2)\{t\,[\,(1-n^2)\psi f_2 + w_2\varphi + w_2'\varphi + f_2'\psi\,] \\
&\qquad + (1-n^2)s_1\cot\alpha_o f_2 + s_1\cot\alpha_o f_2'\}
\end{aligned} \tag{4.41}$$

Postbuckling problem

$$\bar{D}^*_{11}w'''_\alpha + \bar{D}^*_{11}w''_\alpha - \bar{D}^*_{22}w'_\alpha - \bar{D}^*_{22}w_\alpha + (2s_1\sin\alpha_o/t)\,[\,\bar{B}^*_{21}f'''_\alpha + (\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})f''_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22})f'_\alpha - \bar{B}^*_{12}f_\alpha\,]$$
$$= (2ce^{z}s_1\sin\alpha_o/t)\,[\,w'_1(f'_1-n^2f_1+f_1) + w'_2(f'_2-n^2f_2+f_2) + (1-n^2)(w_1f'_1+w_2f'_2+w_1f_1+w_2f_2) + 2\varphi(w'_\alpha+w_\alpha) + 2\psi(f'_\alpha+f_\alpha) + (2s_1\cot\alpha_o/t)(f'_\alpha+f_\alpha)\,] \quad (4.42)$$

$$\bar{D}^*_{11}w'''_\gamma + \bar{D}^*_{11}w''_\gamma - (4n^2\bar{D}^*_{12} + 16n^2\bar{D}^*_{66} + \bar{D}^*_{22})w'_\gamma + (4n^2-1)\bar{D}^*_{22}w_\gamma - 8n\bar{D}^*_{16}w''_\beta - 4n\bar{D}^*_{16}w'_\beta - 4n(4n^2-1)\bar{D}^*_{26}w_\beta$$
$$+ (2\,s_1\sin\alpha_o/t)\,[\,\bar{B}^*_{21}f'''_\gamma + (\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})f''_\gamma + (\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22}+8n^2\bar{B}^*_{66}-4n^2\bar{B}^*_{11})f'_\gamma + (4n^2-1)\bar{B}^*_{12}f_\gamma$$
$$+ 2n(\bar{B}^*_{61}-2\bar{B}^*_{26})f''_\beta - 2n(\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})f'_\beta + 4n(4n^2-1)\bar{B}^*_{16}f_\beta\,] \quad (4.43)$$
$$= (2ce^{z}s_1\sin\alpha_o/t)\,[\,w'_1(f'_2-n^2f_2+f_2) + w'_2(f'_1-n^2f_1+f_1) + (1-n^2)(w_1f_2+w_2f_1) + (1+n^2)(w_2f'_1+w_1f'_2) + 2\varphi(w'_\gamma+w_\gamma) + 2(f'_\gamma-4n^2f_\gamma+f_\gamma)(\psi+s_1\cot\alpha_o/t)\,]$$

$$\bar{D}^*_{11}w'''_\beta + \bar{D}^*_{11}w''_\beta - (4n^2\bar{D}^*_{12} + 16n^2\bar{D}^*_{66} + \bar{D}^*_{22})w'_\beta + (4n^2-1)\bar{D}^*_{22}w_\beta + 8n\bar{D}^*_{16}w''_\gamma + 4n\bar{D}^*_{16}w'_\gamma + 4n(4n^2-1)\bar{D}^*_{26}w_\gamma$$
$$+ (2\,s_1\sin\alpha_o/t)\,[\,\bar{B}^*_{21}f'''_\beta + (\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})f''_\beta + (\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22}+8n^2\bar{B}^*_{66}-4n^2\bar{B}^*_{11})f'_\beta + (4n^2-1)\bar{B}^*_{12}f_\beta$$
$$- 2n(\bar{B}^*_{61}-2\bar{B}^*_{26})f''_\gamma + 2n(\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})f'_\gamma - 4n(4n^2-1)\bar{B}^*_{16}f_\gamma\,] \quad (4.44)$$
$$= (2ce^{z}s_1\sin\alpha_o/t)\,[\,w'_1(f'_1-n^2f_1+f_1) - w'_2(f'_2-n^2f_2+f_2) + (1-n^2)(w_1f_1-w_2f_2) + (1+n^2)(w_1f'_1-w_2f'_2) + 2\varphi(w'_\beta+w_\beta) + 2(f'_\beta-4n^2f_\beta+f_\beta)(\psi+s_1\cot\alpha_o/t)\,]$$

(6) Boundary condition: $N_{s\theta} = 0$

Expressing $N_{s\theta}$ by the Airy stress function $\tilde{F}$ and carrying out the z-transformation one obtains

$$F_{,\bar{\theta}z} = 0 \tag{4.45}$$

Substituting Eq. (4.5) into Eq. (4.45) yields

Prebuckling problem

Since $N_{s\theta} = 0$ is identically satisfied for the axisymmetric problem, it will yield no new boundary condition for the axisymmetric prebuckling state.

Buckling problem

$$\begin{aligned} f_1' &= 0 \\ f_2' &= 0 \end{aligned} \tag{4.46}$$

Postbuckling problem

$$\begin{aligned} f_\gamma' &= 0 \\ f_\beta' &= 0 \end{aligned} \tag{4.47}$$

(7) Boundary condition: $N_s = \bar{N}_s$

While partially satisfying Seide's geometric constraint, the force boundary condition $N_s = \bar{N}_s$ of Eqs. (3.9) can still be used[89,154]. $\bar{N}_s$ is the known force resultant at the edge of the shell. For axial compression and hydrostatic pressure it is given as

$$\bar{N}_s = -(E\,t^2 \cos^2\alpha_o \cot\alpha_o / c)\,[\,(\lambda/s) + (\bar{p}\,s/2s_1^2)\,] \tag{4.48}$$

where $\lambda = Pc/(2\pi Et^2\cos^2\alpha_o)$ and $\bar{p} = pcs_1{}^2\sin^2\alpha_o/(Et^2\cos^2\alpha_o)$

It is noticed that $\bar{N}_s$ given by Refs. [148,155] is incorrect.

Expressing N_s by Airy stress function F yields

$$F_{,z} + F_{,\bar{\theta}\bar{\theta}} + F + (E\,t^2 s_1 \cos^2\alpha_o \cot\alpha_o / c)\,[\,\lambda \; + \; (e^{2z}\bar{p}/2)\,] = 0 \tag{4.49}$$

Substituting Eq. (4.5) into Eq. (4.49) and regrouping by powers of ξ yields

Prebuckling problem

$$f_o' + f_o + \cos\alpha_o \cot^2\alpha_o \lambda + e^{2z}\cot^2\alpha_o \cos\alpha_o \bar{p}/2 = 0 \tag{4.50}$$

Buckling problem

$$\begin{aligned} f_1' + (1-n^2)f_1 &= 0 \\ f_2' + (1-n^2)f_2 &= 0 \end{aligned} \tag{4.51}$$

Postbuckling problem

$$\begin{aligned} f_\beta' + (1-4n^2)f_\beta &= 0 \\ f_\gamma' + (1-4n^2)f_\gamma &= 0 \\ f_\alpha' + f_\alpha &= 0 \end{aligned} \tag{4.52}$$

(8) Boundary condition: $\tilde{W}$ = const.

To solve for the prebuckling displacement w_o, it is necessary to assume that one of the edges is fixed while another is moveable. Thus one obtains the corresponding boundary condition as

$$\tilde{W} = 0 \qquad (4.53)$$

at the fixed edge.

By considering the assumed perturbation expansion (4.4) for W then it follows by inspection that the reduced boundary conditions are for the

Prebuckling problem

$$w_o = 0 \qquad (4.54)$$

Buckling problem

$$\begin{aligned} w_1 &= 0 \\ w_2 &= 0 \end{aligned} \qquad (4.55)$$

Postbuckling problem

$$\begin{aligned} w_\alpha &= 0 \\ w_\beta &= 0 \\ w_\gamma &= 0 \end{aligned} \qquad (4.56)$$

All the previous derivations of the reduced boundary conditions show that Seide's geometric constraint $u - \tilde{W}\cot\alpha_0 = 0$ can only be introduced in the displacement boundary conditions. It has no influence on the force boundary conditions.

4.4 Combined Boundary Conditions

Based on the previously derived individual boundary conditions some combinations which either completely or partially satisfy Seide's geometric constraint are listed in Table 4.1.

Table 4.1 Summary of combined boundary conditions

Type	Cases	Boundary Conditions			
1	MSS1	W - u $\tan\alpha_o$=0	M_s=0	$N_{s\theta}$=0	H=0
2	MSS2	W - u $\tan\alpha_o$=0	M_s=0	$N_{s\theta}$=0	W=0
3	MSS3 (Famili)[57]	W - u $\tan\alpha_o$=0	M_s=0	v=0	H=0
4	MSS4	W - u $\tan\alpha_o$=0	M_s=0	v=0	W=0
5	SS1	W=0	M_s=0	$N_{s\theta}$=0	N_s=0
6	SS3 (Schiffner)[89]	W=0	M_s=0	v=0	N_s=0
7	SS3 (Seide)[41,161]	W - u $\tan\alpha_o$=0	M_s=0	v=0	N_s=0
8	MC1	W - u $\tan\alpha_o$=0	$W_{,s}$=0	$N_{s\theta}$=0	H=0
9	MC2	W - u $\tan\alpha_o$=0	$W_{,s}$=0	$N_{s\theta}$=0	W=0
10	MC3	W - u $\tan\alpha_o$=0	$W_{,s}$=0	v=0	H=0
11	MC4	W - u $\tan\alpha_o$=0	$W_{,s}$=0	v=0	W=0
12	C1	W=0	$W_{,s}$=0	$N_{s\theta}$=0	N_s=0
13	C3 (Schiffner)[89]	W=0	$W_{,s}$=0	v=0	N_s=0
14	C3 (Seide)[41,161]	W - u $\tan\alpha_o$=0	$W_{,s}$=0	v=0	N_s=0

In the following, the combined boundary conditions are reformulated in some suitable forms which can be used directly in the numerical solution process. All the constants in the reformulated boundary conditions are listed in Appendix A2.2.

4.4.1 Combined boundary conditions for the prebuckling problem

For all the combined boundary conditions listed in Table 4.1, there exist only two kinds of independent combinations for the axisymmetric prebuckling problem. One is called simply supported boundary condition given by

$$\psi' = B_1\psi + B_2\varphi$$
$$\varphi' = B_3\varphi + B_4\psi \tag{4.57}$$

the other is called clamped boundary conditions given by

$$\psi = 0$$
$$\varphi' = B_5\varphi + B_6\psi' \tag{4.58}$$

If the edge is fixed, the additional condition $w_o = 0$ should be used when solving for the prebuckling deformations.

4.4.2 Combined boundary conditions for the buckling problem

The boundary conditions which completely satisfy Seide's geometric constraint are summarized as follows:

MSS-1:

$$f_1' = f_2' = 0$$
$$w_1'' = -B_{91}w_1' + B_{95}w_1 - B_{92}w_2' - B_{93}f_1'' - B_{94}f_1$$
$$w_2'' = -B_{91}w_2' + B_{95}w_2 + B_{92}w_1' - B_{93}f_2'' - B_{94}f_2$$
$$f_1''' = J_{58}f_1'' + J_{59}f_1 + J_{60}f_2'' + J_{61}f_2 + J_{62}w_1' + J_{63}w_1$$
$$+ J_{64}w_2' + J_{65}w_2 + e^z[B_{33}\psi(w_1+w_1') + B_{34}w_1\psi + B_{35}w_1$$
$$+ B_{36}w_1' + B_{29}f_1\psi + B_{30}\varphi(w_1+w_1') + B_{31}f_1]$$

$$f_2''' = J_{58}f_2'' + J_{59}f_2 - J_{60}f_1'' - J_{61}f_1 + J_{62}w_2' + J_{63}w_2$$
$$- J_{64}w_1' - J_{65}w_1 + e^z[B_{33}\psi(w_2+w_2') + B_{34}w_2\psi + B_{35}w_2 \tag{4.59}$$
$$+ B_{36}w_2' + B_{29}f_2\psi + B_{30}\varphi(w_2+w_2') + B_{31}f_2]$$

$$w_1''' = J_{66}w_1' + J_{67}w_1 + J_{68}w_2' + J_{69}w_2 + J_{70}f_1'' + J_{71}f_1 + J_{72}f_2'' + J_{73}f_2 + e^z[B_{49}f_1\psi + B_{50}(w_1+w_1')\varphi + B_{51}f_1 + B_{53}\psi(w_1+w_1') + B_{54}w_1\psi + B_{55}w_1 + B_{56}w_1']$$

$$w_2''' = J_{66}w_2' + J_{67}w_2 - J_{68}w_1' - J_{69}w_1 + J_{70}f_2'' + J_{71}f_2 - J_{72}f_1'' - J_{73}f_1 + e^z[B_{49}f_2\psi + B_{50}(w_2+w_2')\varphi + B_{51}f_2 + B_{53}\psi(w_2+w_2') + B_{54}w_2\psi + B_{55}w_2 + B_{56}w_2']$$

<u>MSS-2</u>:

$$w_1 = w_2 = 0$$

$$f_1' = f_2' = 0$$

$$w_1'' = -B_{91}w_1' - B_{92}w_2' - B_{93}f_1'' - B_{94}f_1$$

$$w_2'' = -B_{91}w_2' + B_{92}w_1' - B_{93}f_2'' - B_{94}f_2 \qquad (4.60)$$

$$f_1''' = J_{52}f_1 + J_{53}f_2'' + J_{54}f_2 + B_6w_1''' + J_{55}w_1' + J_{56}w_2' + J_{57}f_1'' + e^z(B_{60}w_1'\psi + B_{59}w_1')$$

$$f_2''' = J_{52}f_2 - J_{53}f_1'' - J_{54}f_1 + B_6w_2''' + J_{55}w_2' - J_{56}w_1' + J_{57}f_2'' + e^z(B_{60}w_2'\psi + B_{59}w_2')$$

<u>MSS-3</u>:

$$f_1'' = B_7f_1' + B_8f_1 + B_9f_2' + B_4w_1' + B_{10}w_1 + B_{11}w_2'$$

$$f_2'' = B_7f_2' + B_8f_2 - B_9f_1' + B_4w_2' + B_{10}w_2 - B_{11}w_1'$$

$$w_1'' = B_{12}w_1' + B_{13}w_1 + B_{14}w_2' + B_2f_1' + B_{15}f_1 + B_{16}f_2'$$

$$w_2'' = B_{12}w_2' + B_{13}w_2 - B_{14}w_1' + B_2f_2' + B_{15}f_2 - B_{16}f_1'$$

$$f_1''' = J_1 f_1' + J_2 f_1 + J_3 f_2' + J_4 f_2 + J_5 w_1' + J_6 w_1 + J_7 w_2' + J_8 w_2$$
$$+ e^z [B_{29} \psi f_1 + B_{30} (w_1 \varphi + w_1' \varphi + f_1' \psi) + B_{31} f_1 + B_{32} f_1' \quad (4.61)$$
$$+ B_{33} (w_1 \psi + w_1' \psi) + B_{34} w_1 \psi + B_{35} w_1 + B_{36} w_1']$$

$$f_2''' = J_1 f_2' + J_2 f_2 - J_3 f_1' - J_4 f_1 + J_5 w_2' + J_6 w_2 - J_7 w_1' - J_8 w_1$$
$$+ e^z [B_{29} \psi f_2 + B_{30} (w_2 \varphi + w_2' \varphi + f_2' \psi) + B_{31} f_2 + B_{32} f_2'$$
$$+ B_{33} (w_2 \psi + w_2' \psi) + B_{34} w_2 \psi + B_{35} w_2 + B_{36} w_2']$$

$$w_1''' = J_9 w_1' + J_{10} w_1 + J_{11} w_2' + J_{12} w_2 + J_{13} f_1' + J_{14} f_1 + J_{15} f_2' + J_{16} f_2$$
$$+ e^z [B_{49} f_1 \psi + B_{50} (w_1 \varphi + w_1' \varphi + f_1' \psi) + B_{51} f_1 + B_{52} f_1'$$
$$+ B_{53} (w_1 \psi + w_1' \psi) + B_{54} w_1 \psi + B_{55} w_1 + B_{56} w_1']$$

$$w_2''' = J_9 w_2' + J_{10} w_2 - J_{11} w_1' - J_{12} w_1 + J_{13} f_2' + J_{14} f_2 - J_{15} f_1' - J_{16} f_1$$
$$+ e^z [B_{49} f_2 \psi + B_{50} (w_2 \varphi + w_2' \varphi + f_2' \psi) + B_{51} f_2 + B_{52} f_2'$$
$$+ B_{53} (w_2 \psi + w_2' \psi) + B_{54} w_2 \psi + B_{55} w_2 + B_{56} w_2']$$

<u>MSS-4</u>:

$$w_1 = w_2 = 0$$
$$f_1'' = B_7 f_1' + B_8 f_1 + B_9 f_2' + B_4 w_1' + B_{11} w_2'$$
$$f_2'' = B_7 f_2' + B_8 f_2 - B_9 f_1' + B_4 w_2' - B_{11} w_1'$$
$$w_1'' = B_{12} w_1' + B_{14} w_2' + B_2 f_1' + B_{15} f_1 + B_{16} f_2'$$
$$w_2'' = B_{12} w_2' - B_{14} w_1' + B_2 f_2' + B_{15} f_2 - B_{16} f_1' \quad (4.62)$$
$$f_1''' = J_{17} f_1' + J_{18} f_1 + J_{19} f_2' + J_{20} f_2 + B_6 w_1'''$$
$$+ J_{21} w_1' + J_{22} w_2' + e^z (B_{59} w_1' + B_{60} w_1' \psi)$$

$$f_2''' = J_{17}f_2' + J_{18}f_2 - J_{19}f_1' - J_{20}f_1 + B_6w_2'''$$
$$+ J_{21}w_2' - J_{22}w_1' + e^{z}(B_{59}w_2' + B_{60}w_2'\psi)$$

MC-1:

$$f_1' = f_2' = 0$$
$$w_1' = -w_1$$
$$w_2' = -w_2$$
$$w_1''' = B_{37}w_1'' + B_{77}w_1 + B_{38}w_2'' + B_{78}w_2 + B_{39}f_1'' + B_{40}f_2'' + B_{48}f_2$$
$$+ B_{46}f_1 + e^{z}(B_{49}f_1\psi + B_{51}f_1 + B_{54}w_1\psi + B_{58}w_1)$$
$$w_2''' = B_{37}w_2'' + B_{77}w_2 - B_{38}w_1'' - B_{78}w_1 + B_{39}f_2'' - B_{40}f_1'' - B_{48}f_1 \quad (4.63)$$
$$+ B_{46}f_2 + e^{z}(B_{49}f_2\psi + B_{51}f_2 + B_{54}w_2\psi + B_{58}w_2)$$
$$f_1''' = B_{17}f_1'' + B_{22}f_1 + B_{18}f_2'' + B_{24}f_2 + B_{19}w_1'' + B_{20}w_2'' + B_{75}w_1$$
$$+ B_{76}w_2 + e^{z}(B_{34}w_1\psi + B_{57}w_1 + B_{29}f_1\psi + B_{31}f_1)$$
$$f_2''' = B_{17}f_2'' + B_{22}f_2 - B_{18}f_1'' - B_{24}f_1 + B_{19}w_2'' - B_{20}w_1'' + B_{75}w_2$$
$$- B_{76}w_1 + e^{z}(B_{34}w_2\psi + B_{57}w_2 + B_{29}f_2\psi + B_{31}f_2)$$

MC-2:

$$w_1 = w_2 = 0$$
$$f_1' = f_2' = 0$$
$$w_1' = w_2' = 0 \quad (4.64)$$
$$f_1''' = B_{62}f_1 + B_{63}f_2'' + B_{64}f_2 + B_6w_1''' + B_{65}w_1'' + B_{67}w_2''$$
$$f_2''' = B_{62}f_2 - B_{63}f_1'' - B_{64}f_1 + B_6w_2''' + B_{65}w_2'' - B_{67}w_1''$$

MC-3:

$$w_1' = -w_1$$

$$w_2' = -w_2$$

$$f_1'' = B_{69}f_1' + B_{70}f_1 + B_{71}f_2' + B_{72}w_1'' + B_{73}w_1 + B_{74}w_2$$

$$f_2'' = B_{69}f_2' + B_{70}f_2 - B_{71}f_1' + B_{72}w_2'' + B_{73}w_2 - B_{74}w_1 \qquad (4.65)$$

$$f_1''' = J_{23}f_1' + J_{24}f_1 + J_{25}f_2' + J_{26}f_2 + J_{27}w_1'' + J_{28}w_1 + J_{29}w_2'' + J_{30}w_2$$
$$+ e^z[B_{29}f_1\psi + B_{30}f_1'\psi + B_{31}f_1 + B_{32}f_1' + B_{34}w_1\psi + B_{57}w_1]$$

$$f_2''' = J_{23}f_2' + J_{24}f_2 - J_{25}f_1' - J_{26}f_1 + J_{27}w_2'' + J_{28}w_2 - J_{29}w_1'' - J_{30}w_1$$
$$+ e^z[B_{29}f_2\psi + B_{30}f_2'\psi + B_{31}f_2 + B_{32}f_2' + B_{34}w_2\psi + B_{57}w_2]$$

$$w_1''' = J_{31}w_1'' + J_{32}w_1 + J_{33}w_2'' + J_{34}w_2 + J_{35}f_1' + J_{36}f_1 + J_{37}f_2' + J_{38}f_2$$
$$+ e^z[B_{49}f_1\psi + B_{50}f_1'\psi + B_{51}f_1 + B_{52}f_1' + B_{54}w_1\psi + B_{58}w_1]$$

$$w_2''' = J_{31}w_2'' + J_{32}w_2 - J_{33}w_1'' - J_{34}w_1 + J_{35}f_2' + J_{36}f_2 - J_{37}f_1' - J_{38}f_1$$
$$+ e^z[B_{49}f_2\psi + B_{50}f_2'\psi + B_{51}f_2 + B_{52}f_2' + B_{54}w_2\psi + B_{58}w_2]$$

MC-4:

$$w_1 = w_2 = 0$$

$$w_1' = w_2' = 0$$

$$f_1'' = B_{69}f_1' + B_{70}f_1 + B_{71}f_2' + B_{72}w_1''$$

$$f_2'' = B_{69}f_2' + B_{70}f_2 - B_{71}f_1' + B_{72}w_2'' \qquad (4.66)$$

$$f_1''' = J_{39}f_1' + B_{62}f_1 + J_{40}f_2' + J_{41}f_2 + B_6w_1''' + B_{65}w_1'' + J_{48}w_2''$$

$$f_2''' = J_{39}f_2' + B_{62}f_2 - J_{40}f_1' - J_{41}f_1 + B_6w_2''' + B_{65}w_2'' - J_{48}w_1''$$

By following Schiffner's formulation the boundary conditions which partially satisfy Seide's geometric constraint are summarized as follows:

<u>SS-1</u>:

$$\begin{aligned} w_1 &= w_2 = 0 \\ f_1 &= f_2 = 0 \\ f_1' &= f_2' = 0 \\ w_1'' &= -B_{91}w_1' - B_{92}w_2' - B_{93}f_1'' \\ w_2'' &= -B_{91}w_2' + B_{92}w_1' - B_{93}f_2'' \end{aligned} \tag{4.67}$$

<u>SS-3</u>:

$$\begin{aligned} w_1 &= w_2 = 0 \\ f_1' &= (n^2-1)f_1 \\ f_2' &= (n^2-1)f_2 \\ f_1'' &= (1-n^2)f_1 + B_{80}f_2 + B_4w_1' + B_{11}w_2' \\ f_2'' &= (1-n^2)f_2 - B_{80}f_1 + B_4w_2' - B_{11}w_1' \\ w_1'' &= B_{12}w_1' + B_{14}w_2' + B_{81}f_2 \\ w_2'' &= B_{12}w_2' - B_{14}w_1' - B_{81}f_1 \end{aligned} \tag{4.68}$$

C-1:

$$
\begin{aligned}
w_1 &= w_2 = 0 \\
w_1' &= w_2' = 0 \\
f_1 &= f_2 = 0 \\
f_1' &= f_2' = 0
\end{aligned}
\tag{4.69}
$$

C-3:

$$
\begin{aligned}
w_1 &= w_2 = 0 \\
w_1' &= w_2' = 0 \\
f_1' &= (n^2-1)f_1 \\
f_2' &= (n^2-1)f_2 \\
f_1'' &= (1-n^2)f_1 + B_{87}f_2 + B_6 w_1'' \\
f_2'' &= (1-n^2)f_2 - B_{87}f_1 + B_6 w_2''
\end{aligned}
\tag{4.70}
$$

By following Seide's formulation the boundary conditions which inconsistently satisfy Seide's geometric constraint are summarized as follows:

SS-3:

$$
\begin{aligned}
f_1' &= (n^2-1)f_1 \\
f_2' &= (n^2-1)f_2 \\
w_1'' &= B_{12}w_1' + B_{13}w_1 + B_{14}w_2' + B_{81}f_2 \\
w_2'' &= B_{12}w_2' + B_{13}w_2 - B_{14}w_1' - B_{81}f_1 \\
f_1'' &= B_{79}f_1 + B_{80}f_2 + B_4 w_1' + B_{11}w_2' + B_{10}w_1 \\
f_2'' &= B_{79}f_2 - B_{80}f_2 + B_4 w_2' - B_{11}w_1' + B_{10}w_2
\end{aligned}
\tag{4.71}
$$

$$f_1''' = J_{42}f_1 + J_{43}f_2 + B_6w_1''' + J_{21}w_1' + J_{44}w_1 + J_{22}w_2' + J_{45}w_2$$
$$+ e^z[B_{60}\psi(w_1+w_1') + B_{85}w_1\psi + B_{86}w_1 + B_{59}w_1']$$
$$f_2''' = J_{42}f_2 - J_{43}f_1 + B_6w_2''' + J_{21}w_2' + J_{44}w_2 - J_{22}w_1' - J_{45}w_1$$
$$+ e^z[B_{60}\psi(w_2+w_2') + B_{85}w_2\psi + B_{86}w_2 + B_{59}w_2']$$

<u>C-3</u>:

$$w_1' = -w_1$$
$$w_2' = -w_2$$
$$f_1' = (n^2-1)f_1$$
$$f_2' = (n^2-1)f_2$$
$$f_1'' = (1-n^2)f_1 + B_{87}f_2 + B_6w_1'' + B_{73}w_1 + B_{74}w_2$$
$$f_2'' = (1-n^2)f_2 - B_{87}f_1 + B_6w_2'' + B_{73}w_2 - B_{74}w_1 \qquad (4.72)$$
$$f_1''' = J_{46}f_1 + J_{47}f_2 + B_6w_1''' + B_{65}w_1'' + J_{49}w_1 + J_{50}w_2'' + J_{51}w_2$$
$$+ e^z(B_{85}\psi w_1 + B_{90}w_1)$$
$$f_2''' = J_{46}f_2 - J_{47}f_1 + B_6w_2''' + B_{65}w_2'' + J_{49}w_2 - J_{50}w_1'' - J_{51}w_1$$
$$+ e^z(B_{85}\psi w_2 + B_{90}w_2)$$

4.4.3 Combined boundary conditions for the postbuckling problem

Due to the complexity of the combined postbuckling boundary conditions expressed by the assumed functions W and F, only some of the possible combinations which have been used in the numerical examples are given in the following:

<u>MSS-3:</u>

$$\bar{f}'_\alpha = B_3\bar{f}_\alpha + B_4\bar{w}_\alpha + B_{124}(w_1^2+w_2^2)\,e^z$$

$$\bar{w}'_\alpha = B_1\bar{w}_\alpha + B_2\bar{f}_\alpha - B_{127}(w_1^2+w_2^2)\,e^z$$

$$f''_\beta = B_7f'_\beta + B_{105}f'_\gamma + B_{106}f_\beta + B_4w'_\beta + B_{104}w'_\gamma + B_{123}w_\beta + B_{124}e^z(w_2^2-w_1^2)$$

$$f''_\gamma = B_7f'_\gamma - B_{105}f'_\beta + B_{106}f_\gamma + B_4w'_\gamma - B_{104}w'_\beta + B_{123}w_\beta - B_{125}e^z w_1 w_2$$

$$w''_\beta = B_{12}w'_\beta + B_{107}w'_\gamma + B_{126}w_\beta + B_2f'_\beta + B_{109}f'_\gamma + B_{110}f_\beta + B_{127}e^z(w_1^2-w_2^2)$$

$$w''_\gamma = B_{12}w'_\gamma - B_{107}w'_\beta + B_{126}w_\gamma + B_2f'_\gamma - B_{109}f'_\beta + B_{110}f_\gamma + B_{128}e^z w_1 w_2 \tag{4.73}$$

$$\begin{aligned} f'''_\beta = {} & J_{104}f'_\beta + J_{105}f_\beta + J_{106}f'_\gamma + J_{107}f_\gamma + J_{108}w'_\beta + J_{109}w_\beta \\ & + J_{110}w'_\gamma + J_{111}w_\gamma + e^z\{J_{112}(w_1^2-w_2^2) + J_{113}w_1w_2 \\ & + B_{149}[(w'_2+w_2)^2 - (w'_1+w_1)^2] + B_{125}(w'_1w_1-w'_2w_2) + B_{150}\,\psi\, w_\beta \\ & + B_{151}w_\beta + B_{36}w'_\beta + B_{33}\,\psi\, w'_\beta + B_{30}(\varphi\, w'_\beta + \psi\, f'_\beta + \varphi\, w_\beta) + B_{32}f'_\beta \\ & + B_{152}(w'_1f'_1-w'_2f'_2) + B_{153}(w'_1f_1-w'_2f_2 + w_1f_1-w_2f_2) \\ & + B_{154}(f'_1w_1-f'_2w_2) + B_{155}f_\beta + B_{156}\,\psi\, f_\beta\} \end{aligned}$$

$$\begin{aligned} f'''_\gamma = {} & J_{104}f'_\gamma + J_{105}f_\gamma - J_{106}f'_\beta - J_{107}f_\beta + J_{108}w'_\gamma + J_{109}w_\gamma \\ & - J_{110}w'_\beta - J_{111}w_\beta + e^z\{J_{114}(w_1^2-w_2^2) + J_{115}w_1w_2 \\ & + B_{157}(w'_1w'_2+w'_1w_2+w'_2w_1) + B_{125}(w'_1w_2+w'_2w_1) + B_{150}\,\psi\, w_\gamma \\ & + B_{151}w_\gamma + B_{36}w'_\gamma + B_{33}\,\psi\, w'_\gamma + B_{30}(\varphi\, w'_\gamma + \psi\, f'_\gamma + \varphi\, w_\gamma) + B_{32}f'_\gamma \\ & + B_{152}(w'_1f'_2+w'_2f'_1) + B_{153}(w'_1f_2+w'_2f_1 + w_1f_2+w_2f_1) \\ & + B_{154}(f'_1w_2+f'_2w_1) + B_{155}f_\gamma + B_{156}\,\psi\, f_\gamma\} \end{aligned}$$

$$w_\beta''' = J_{116}w_\beta' + J_{117}w_\beta + J_{118}w_\gamma' + J_{119}w_\gamma + J_{120}f_\beta' + J_{121}f_\beta$$
$$+ J_{122}f_\gamma' + J_{123}f_\gamma + e^z\{J_{124}(w_1^2 - w_2^2) + J_{125}w_1w_2$$
$$+ B_{169}[(w_1'+w_1)^2 - (w_2'+w_2)^2] - B_{128}(w_1'w_1 - w_2'w_2) + B_{156}\psi w_\beta$$
$$+ B_{155}w_\beta + B_{32}w_\beta' + B_{30}\psi w_\beta' + B_{50}(\varphi w_\beta' + \psi f_\beta' + \varphi w_\beta) + B_{52}f_\beta'$$
$$+ B_{170}(w_1'f_1' - w_2'f_2') + B_{171}(w_1'f_1 - w_2'f_2 + w_1f_1 - w_2f_2)$$
$$+ B_{172}(f_1'w_1 - f_2'w_2) + B_{177}f_\beta + B_{176}\psi f_\beta\}$$

$$w_\gamma''' = J_{116}w_\gamma' + J_{117}w_\gamma - J_{118}w_\beta' - J_{119}w_\beta + J_{120}f_\gamma' + J_{121}f_\gamma$$
$$- J_{122}f_\beta' - J_{123}f_\beta + e^z\{J_{126}(w_1^2 - w_2^2) + J_{127}w_1w_2$$
$$+ B_{152}w_1'w_2' + B_{153}(w_1'w_2 + w_2'w_1) + B_{156}\psi w_\gamma$$
$$+ B_{155}w_\gamma + B_{56}w_\gamma' + B_{30}\psi w_\gamma' + B_{50}(\varphi w_\gamma' + \psi f_\gamma' + \varphi w_\gamma) + B_{52}f_\gamma'$$
$$+ B_{170}(w_1'f_2' + w_2'f_1') + B_{171}(w_1'f_2 + w_2'f_1 + w_2f_1 + w_1f_2)$$
$$+ B_{172}(f_1'w_2 + f_2'w_1) + B_{177}f_\gamma + B_{176}\psi f_\gamma\}$$

<u>MSS-4</u>:

$$w_\alpha = w_\beta = w_\gamma = 0$$
$$\bar{f}_\alpha' = B_3\bar{f}_\alpha + B_4\bar{w}_\alpha$$
$$\bar{w}_\alpha' = B_2\bar{f}_\alpha + B_1\bar{w}_\alpha$$
$$f_\beta'' = B_4w_\beta' + B_{104}w_\gamma' + B_7f_\beta' + B_{105}f_\gamma' + B_{106}f_\beta \qquad (4.74)$$
$$f_\gamma'' = B_4w_\gamma' - B_{104}w_\beta' + B_7f_\gamma' - B_{105}f_\beta' + B_{106}f_\gamma$$
$$w_\beta'' = B_{12}w_\beta' + B_{107}w_\gamma' + B_{108}f_\beta' + B_{109}f_\gamma' + B_{110}f_\beta$$
$$w_\gamma'' = B_{12}w_\gamma' - B_{107}w_\beta' + B_{108}f_\gamma' - B_{109}f_\beta' + B_{110}f_\gamma$$

$$f_\beta''' = J_{78}f_\beta' + J_{79}f_\beta + J_{80}f_\gamma' + J_{81}f_\gamma + J_{82}w_\beta' + J_{83}w_\gamma' + B_6 w_\beta'''$$
$$- e^z[\,B_{113}(w_1'^2 - w_2'^2) - B_{60}\,\psi\, w_\beta' - B_{59}\, w_\beta'\,]$$

$$f_\gamma''' = J_{78}f_\gamma' + J_{79}f_\gamma - J_{80}f_\beta' - J_{81}f_\beta + J_{82}w_\gamma' - J_{83}w_\beta' + B_6 w_\gamma'''$$
$$- e^z[\,B_{114}w_1'w_2' - B_{60}\,\psi\, w_\gamma' - B_{59}\, w_\gamma'\,]$$

<u>C-1</u>:

$$\begin{aligned} w_\alpha &= w_\alpha' = \bar{w}_\alpha = 0 \\ w_\beta &= w_\beta' = 0 \\ w_\gamma &= w_\gamma' = 0 \\ \bar{f}_\alpha &= 0 \\ f_\beta &= f_\beta' = 0 \\ f_\gamma &= f_\gamma' = 0 \end{aligned} \tag{4.75}$$

<u>MC-2</u>:

$$\begin{aligned} \bar{w}_\alpha &= 0 \\ w_\beta &= w_\beta' = 0 \\ w_\gamma &= w_\gamma' = 0 \\ f_\beta' &= f_\gamma' = 0 \\ \bar{f}_\alpha' &= B_5\bar{f}_\alpha + B_6\bar{w}_\alpha' \end{aligned}$$

$$f_\beta''' = -B_{100}f_\beta + B_{101}f_\gamma'' - B_{102}f_\gamma + B_6 w_\beta''' + B_{65}w_\beta'' - B_{103}w_\gamma''$$
$$- e^z[\,B_{113}(w_1'^2 - w_2'^2) + B_{115}(w_1 w_1' - w_2 w_2') + B_{116}(w_1^2 - w_2^2)\,] \tag{4.76}$$

$$f_\gamma''' = -B_{100}f_\gamma - B_{101}f_\beta'' + B_{102}f_\beta + B_6w_\gamma''' + B_{65}w_\gamma'' + B_{103}w_\beta''$$

$$- e^z[B_{114}w_1'w_2' + B_{115}(w_1'w_2+w_2'w_1) + B_{117}w_1w_2]$$

MC-4:

$$\bar{w}_\alpha = w_\alpha = w_\alpha' = 0$$

$$w_\beta = w_\beta' = 0$$

$$w_\gamma = w_\gamma' = 0$$

$$\bar{f}_\alpha' = B_5\bar{f}_\alpha + B_6\bar{w}_\alpha'$$

$$f_\beta'' = B_{69}f_\beta' - B_{98}f_\beta + B_{63}f_\gamma' + B_6w_\beta'' \tag{4.77}$$

$$f_\gamma'' = B_{69}f_\gamma' - B_{98}f_\gamma - B_{63}f_\beta' + B_6w_\gamma''$$

$$f_\beta''' = J_{74}f_\beta' - B_{100}f_\beta + J_{75}f_\gamma' + J_{76}f_\gamma + B_6w_\beta''' + B_{65}w_\beta'' + J_{77}w_\gamma''$$

$$f_\gamma''' = J_{74}f_\gamma' - B_{100}f_\gamma - J_{75}f_\beta' - J_{76}f_\beta + B_6w_\gamma''' + B_{65}w_\gamma'' - J_{77}w_\beta''$$

4.5 Discussions and Conclusion

To summarize briefly, the displacement boundary conditions derived by variational method are modified to account for Seide's geometric constraint which occurs frequently in practical experiments. The resulting individual boundary conditions are transformed to some new forms in terms of the assumed functions W and F. Some appropriate combinations of the transformed individual boundary conditions are reformulated for the prebuckling, buckling and postbuckling problems, which completely or partially satisfy Seide's geometric constraint. These combinations can be directly used in the numerical solution process.

The correctness of the individual and combined boundary conditions is confirmed by comparing them with those derived by using some Computer Algebra Systems.

As in the case of anisotropic cylindrical shells, the four simply supported and four clamped boundary conditions for the prebuckling state have degenerated into two. One is for simply supported and the other for clamped boundary conditions. Besides, since the stresses and deformations in the axisymmetric prebuckling state satisfy the inhomogeneous boundary conditions, all the derived boundary conditions for the asymmetric buckling state are homogenous.

Chapter 5
Simplified Buckling Analysis of Perfect Anisotropic Conical Shells

5.1 Introduction

It is well-known that the finding of the buckling solutions depends on the knowledge of the axisymmetric stress and deformation states of the unbuckled configuration, i.e., the prebuckling solutions. It has been also known that unlike the case of anisotropic cylindrical shells, where the governing equations for axisymmetric prebuckling state are linear and admit a closed-form solution[67], there is no closed-form solution for the axisymmetric prebuckling state of anisotropic conical shells, due to the nonlinearity of their prebuckling governing equations. This conclusion is true even for isotropic conical shells. Hence, the use of membrane prebuckling stress state and omission of prebuckling deformation terms have been the general practice in buckling analyses of conical shells[21-23,88,93,128]. Employing such simplified prebuckling analysis leads to a significant reduction of the analysis effort and increases the computational efficiency. However, it must be emphasized that, in many practical applications, the predictions obtained from the simplified analysis may not be in fair agreement with experimental results, and they may not be quantitatively accurate enough for certain practical applications. Nevertheless, the simplified solutions can provide at least some qualitative references.

The simplified analysis for isotropic conical shells is well known[58,163]. The simplified analysis for classical orthotropic conical shells was presented by Schiffner[89] in terms of the radial displacement W and Airy stress function F. However, no results can be found in the literature, as far as the author knows, for the simplified buckling analysis of anisotropic conical shells, due to the intractable analytic calculations and algebraic manipulations involved in the analysis. Thanks to the development of various Computer Algebra systems with different sophistication it is possible to carry out the cumbersome algebraic calculations by computer. In the following the simplified buckling analysis of anisotropic conical shells is carried out with the help of some Computer Algebra systems. This simplified analysis is the first step for the rigorous stability analysis of anisotropic conical shells, and the corresponding solution can be used as the starting values in the iteration process of rigorous analysis.

5.2 Membrane-like Prebuckling Solutions

In the study of equilibrium of a shell, if all moment expressions are neglected, the resulting theory is called 'membrane' theory of shells. A shell can be considered to act as a membrane if flexural strains are zero or negligible compared with direct axial strains[126,128,149]. It is apparent that two types of shells comply with this definition of a membrane: (1) shells with bending stiffness sufficiently small so that they are physically incapable of resisting bending and (2) shells that are flexurally stiff but loaded and supported in a manner that avoids the introduction of bending strains. The state of stress in a membrane is referred to as a 'momentless' state. For an absolutely flexible shell, since it offers no resistance to bending, only a momentless state of stress is possible. For shells with finite stiffness, such a sate of stress is only one of the possible stress conditions, and several supplementary conditions relating to the shape of the shell, the character of the load applied, and the support of its edges must be fulfilled.

Although the momentless state of stress condition is a desirable feature in the design of shell structures because it offers the advantage of uniform utilization of the strength capabilities of the shell material, it cannot be realized for many practical laminated composite shell structures. Consequently, the membrane solution techniques for homogeneous shells do not carry over directly for the general laminated anisotropic shells. Ambartsumyan[2] presented methods for the membrane stress analysis of certain laminated shells. However, the type of construction which he considered possesses elastic symmetry about the mid-surface, thus reducing the laminated shell equations and the solution techniques to those of equivalent homogeneous shells. In the following a membrane theory will be presented for laminated anisotropic shells without the restriction of symmetry about the mid-surface.

First, let us recall the procedures used to obtain a membrane solution for a homogeneous shell which were outlined by Gol'denveizer[156], and followed by Liberescu[126] et al.. They can be summarized as follows

A. determination of the solution of the equilibrium equations in terms of the stress resultants.

B. determination of the components of the in-plane strain from the stress resultant-strain equations.

C. determination of the displacements from the strain-displacement equations.

D. determination of the components of the curvature changes with the help of the displacements obtained in step C.

E. determination of the stress couples from the stress couple-curvature change equations.

F. determination of the transverse shear resultants from the moment equilibrium equations.

The first three steps are the primary objectives of membrane stress analysis, i.e., to determine the state of membrane stress, while the remaining steps are performed in order to assess the validity of the membrane theory by examining the magnitude of the stress couples and shear stress resultants.

It is noticed that to be able to follow the above procedures the constitutive equations have to be uncoupled. In order to modify the above method for an arbitrarily laminated shell with coupled constitutive equations, the assumption that all bending moments vanish identically must be made. In this way, the membrane strains may be calculated. However, some generality is surrendered because steps D to F, i.e., an error estimate, cannot be carried out. Some results presented by Dong[157] show that this additional restriction is not overly severe, and the method is applicable to a relatively large class of significant engineering problems.

The determination of the membrane stress resultants for a laminated anisotropic shell is independent of the material properties used to build the shell, because the problems of static equilibrium are the same for both homogeneous and laminated shells. For conical shells under axial compression and hydrostatic pressure the membrane stress resultants are given as[55,89,158]

$$N_s = -\, p\, s \tan\alpha_o/2 \;-\; P/(\pi\, s \sin 2\alpha_o)$$

$$N_\theta = -\, p\, s \tan\alpha_o \qquad (5.1)$$

$$N_{s\theta} = 0$$

Considering the definition for Airy stress function of a conical shell, one obtains

$$\tilde{F}^{(o)} = -\, p\, s^3 \tan\alpha_o/6 \;-\; P\, s/(\pi \sin 2\alpha_o) \qquad (5.2)$$

Carrying out the z-transformation and introducing the following normalized load parameters

$$\bar{p} = p\, c\, s_1^2 \sin^2\alpha_o/(E t^2 \cos^2\alpha_o) \qquad (5.3)$$

$$\lambda = P c/(2\pi E t^2 \cos^2\alpha_o) \qquad (5.4)$$

one can rewrite Eq. (5.2) as

$$F^{(o)} = -\, E t^2 s_1 (e^{2z}\bar{p}/6 + \lambda)/(c \tan\alpha_o) \qquad (5.5)$$

where $c^2 = 3(1-\nu^2)$

With the assumption that the moment resultants vanish identically, the components of the in-plane strains are related to the changes of curvature through Eq. (2.38):

$$[\kappa] = -\, [D]^{-1}[B][\varepsilon] \qquad (5.6)$$

Substituting Eq. (5.6) into Eq. (2.36) yields

$$[N] = [\tilde{A}][\varepsilon] \qquad (5.7)$$

where

$$[\tilde{A}] = [A] \;-\; [B][D]^{-1}[B] \qquad (5.8)$$

The inverse of Eq. (5.7) is

$$[\varepsilon] = [\tilde{B}][N] \tag{5.9}$$

where

$$[\tilde{B}] = [\tilde{A}]^{-1} \tag{5.10}$$

After the in-plane strain components have been determined through Eq. (5.9), the remaining step of the membrane analysis is the integration of these strain components to obtain the displacements via the linear Donnell-type strain-displacement equations. It is obvious that since two of the membrane stress resultants of a conical shell are functions of the longitudinal coordinate s, there exist non-constant prebuckling deformations even if no constraint of deformation is applied at the boundary.

Here, the using of 'membrane-like' prebuckling solution for conical shells instead of using 'membrane' prebuckling solutions is due to the following considerations: For cylindrical shells using Donnell-type theory, the so-called 'membrane' prebuckling solution consists of the membrane stress resultants and the constant deflection due to the Poisson effect. This membrane solution does satisfy the axisymmetric prebuckling governing equations[67] and as a whole is introduced in the buckling equations. For the case of conical shells, however, the situation is different. When talking about using membrane prebuckling solution in the buckling equation, one usually means only the use of membrane stress resultants. The non-constant prebuckling deformations due to the membrane stress resultants are always neglected in the buckling equations. Therefore, as distinguished from the complete membrane prebuckling solution of cylindrical shells, the term 'membrane-like' prebuckling solution is adopted for conical shells.

Notice that the just defined membrane-like prebuckling solution of conical shells satisfies neither the axisymmetric prebuckling governing equations nor the appropriate boundary conditions. Therefore, it is only an approximation for the real prebuckling solution. Nevertheless, using this approximation, one can simplify the solution of the buckling governing equations considerably and obtain some preliminary results.

5.3 Buckling Solutions Using "Membrane-like" Prebuckling Solution

Once the membrane-like prebuckling solution of anisotropic conical shells is known, one can proceed to solve the corresponding buckling problem.

5.3.1 Analytical solution for critical buckling load

By neglecting the effect of prebuckling deformations for the buckling problem, one can rewrite Eqs. (3.45) and (3.46) as

$$\begin{aligned} & L_{\bar{B}_1^*}(F^{(1)}) - L_{\bar{B}_2^*}(F^{(1)}) - L_{\bar{D}^*}(W^{(1)}) \\ & \qquad = L_{NL_2}(F^{(o)},W^{(1)}) + s_1 e^{z} \cot\alpha_o (F^{(1)}_{,zz} + F^{(1)}_{,z}) \end{aligned} \tag{5.11}$$

$$\begin{aligned} & L_{\bar{A}^*}(F^{(1)}) - L_{\bar{B}_1^*}(W^{(1)}) - L_{\bar{B}_2^*}(W^{(1)}) \\ & \qquad = - s_1 e^{z} \cot\alpha_o (W^{(1)}_{,zz} + W^{(1)}_{,z}) \end{aligned} \tag{5.12}$$

where $F^{(o)}$ is the Airy stress function of the membrane-like solution given by Eq. (5.5).

Equations (5.11) and (5.12) can be solved by Galerkin's procedure as follows:

- The compatibility equation (5.12) is solved exactly for the stress function $F^{(1)}$ in terms of the assumed radial displacement $W^{(1)}$. This guarantees that a kinematically admissible displacement field will be associated with the solution of the other equation, i.e., the condition of equilibrium. The form $F^{(1)}$ depends on both the form of $W^{(1)}$ and the right-hand side of Eq. (5.12).
- The equilibrium equation (5.11) is solved approximately by substituting therein $F^{(1)}$ and $W^{(1)}$ and then applying Galerkin's error minimization procedure.

For the anisotropic conical shells, one can assume a skewed buckling pattern as

$$\begin{aligned} \bar{W}^{(1)} &= \bar{A} e^{-z} \sin kz \cos(n\bar{\theta} - \tau_k z) \\ &= \bar{A} e^{-z} [\, \sin(k_1 z + n\bar{\theta}) + \sin(k_2 z - n\bar{\theta}) \,] / 2 \end{aligned} \tag{5.13}$$

where

$$k = m\pi / z_o \tag{5.14}$$

$$n = n_1 / \sin\alpha_o \tag{5.15}$$

$$k_1 = k - \tau_k \tag{5.16}$$

$$k_2 = k + \tau_k \tag{5.17}$$

Here n_1 is the number of circumferential waves, and τ_k[19] is introduced in order to take into account the coupling between the changes of curvature and twist in the bending stiffness matrix [D].

Notice that the above assumed displacement satisfies the prescribed boundary conditions:

$W^{(1)} = 0$ at $z = 0$ and $z = z_o$.

Substituting Eq. (5.13) into Eq. (5.12) yields the following linear homogeneous partial differential equation for $F^{(1)}$

$$\begin{aligned} L_{\bar{A}^*}(\bar{F}^{(1)}) &= \sin(k_1 z + n\bar{\theta})(c_1 + c_2 e^{-z}) + \sin(k_2 z - n\bar{\theta})(c_3 + c_4 e^{-z}) \\ &\quad + \cos(k_1 z + n\bar{\theta})(c_5 + c_6 e^{-z}) + \cos(k_2 z - n\bar{\theta})(c_7 + c_8 e^{-z}) \end{aligned} \tag{5.18}$$

where the coefficients c_i ($i = 1 \sim 8$) are listed in Appendix A3.2.

Using the method of undetermined coefficients one can obtain an exact particular solution for Eq. (5.18) as

$$\begin{aligned} \bar{F}^{(1)} &= \sin(k_1 z + n\bar{\theta})(b_1 + b_2 e^{-z}) + \sin(k_2 z - n\bar{\theta})(b_3 + b_4 e^{-z}) \\ &\quad + \cos(k_1 z + n\bar{\theta})(b_5 + b_6 e^{-z}) + \cos(k_2 z - n\bar{\theta})(b_7 + b_8 e^{-z}) \end{aligned} \tag{5.19}$$

where the coefficients b_i ($i = 1 \sim 8$) are listed in Appendix A3.3.

Substituting now for $F^{(1)}$ and $W^{(1)}$ into the equilibrium equation (5.11) and regrouping yields the 'error' as

$$\begin{aligned}\hat{\varepsilon}(z, \bar{\theta}) = {} & \sin(k_1 z + n\bar{\theta})(d_1 e^{2z} + d_2 e^{z} + d_3 + d_4 e^{-z}) \\ & + \sin(k_2 z - n\bar{\theta})(d_5 e^{2z} + d_6 e^{z} + d_7 + d_8 e^{-z}) \\ & + \cos(k_1 z + n\bar{\theta})(d_9 e^{2z} + d_{10} e^{z} + d_{11} + d_{12} e^{-z}) \\ & + \cos(k_2 z - n\bar{\theta})(d_{13} e^{2z} + d_{14} e^{z} + d_{15} + d_{16} e^{-z})\end{aligned} \tag{5.20}$$

where the coefficients d_i ($i = 1 \sim 16$) are listed in Appendix A3.4.

Next, the 'error' can be minimized with respect to the assumed function by applying Galerkin's procedure. This involves the following integral

$$\int_0^{2\pi \sin\alpha_o} \int_0^{z_o} \hat{\varepsilon}(z, \bar{\theta}) e^{-z} \{\sin(k_1 z + n\bar{\theta}) + \sin(k_2 z - n\bar{\theta})\} \, dz \, d\bar{\theta} = 0 \tag{5.21}$$

For the sake of simplicity the following notation is introduced

$$I_i = \int_0^{2\pi \sin\alpha_o} \int_0^{z_o} C^{(i)} e^{-z} \{\sin(k_1 z + n\bar{\theta}) + \sin(k_2 z - n\bar{\theta})\} \, dz \, d\bar{\theta} \tag{5.22}$$

where $C^{(i)}$ ($i = 1 \sim 16$) are the relevant parts of Eq. (5.20) involving the coefficients of d_i. Details of I_i ($i = 1 \sim 16$) are listed in Appendix A3.5.

Finally, back-substituting the integrated I_i into Eq. (5.21) and regrouping yields the characteristic equation involving the eigenvalue as a function of m, n and τ_k for the anisotropic conical shells

$$\sum_{i=1}^{16} d_i I_i = 0 \tag{5.23}$$

Notice that d_1, d_5, d_9 and d_{13} involve only hydrostatic pressure, while d_3, d_7, d_{11} and d_{15}

involve only the axial compression.

5.3.2 Applications of Computer Algebra systems

Because of the rather cumbersome analytic calculations involved in carrying out Galerkin's method, extensive use is made of Computer Algebra systems. All the symbolic calculations involved in the derivation of Eq. (5.23) can be carried out by the REDUCE-based package GMACS, Galerkin Method for Anisotropic Conical Shells. GMACS consists of five blocks. They are

- GMACS0 is for the derivation of prebuckling and buckling governing equations.
- GMACS1 is for the derivation of coefficients in Eqs. (5.18) and (5.19).
- GMACS2 is for the derivation of coefficients in Eq. (5.20).
- GMACS3 is for the calculations of the integral I_i (i = 1 ~ 16).
- GMACS4 is a sub-block of GMACS3, and is used to carry out some necessary trigonometric simplifications.

The flow chart of GMACS is shown in Figure 5.1. The equivalence between the notations in the text and in GMACS is listed in Table 5.1. The source program of GMACS is given in Appendix A3.1.

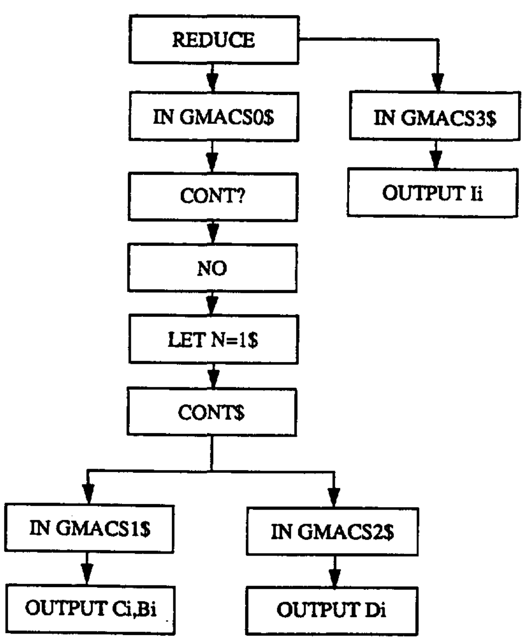


Fig. 5.1 Flow chart of REDUCE-based package GMACS

Table 5.1 Equivalence of notations

In text	In GMACS
$\bar{A}_{ij}^*$ $\bar{B}_{ij}^*$ $\bar{D}_{ij}^*$	aij bij dij
M_s M_θ $M_{s\theta}$	m11 m22 m12
N_s N_θ $N_{s\theta}$	n11 n22 n12
ε_s ε_θ $\varepsilon_{s\theta}$	ep11 ep22 ep12
χ_s χ_θ $\chi_{s\theta}$	k11 k22 k12
$W^{(o)}$ $F^{(o)}$	W(o) F(o)
$W^{(1)}$ $F^{(1)}$	W(1) F(1)
$\bar{\theta}$ α $\bar{A}$ s_1	th al A s1

5.3.2 Checking the correctness of critical load equation

As mentioned earlier, the characteristic equation (5.23) is derived via the Computer Algebra system REDUCE. The correctness of Eq. (5.23) is confirmed by comparing it with results obtained by the Computer Algebra system Maple. In the following comparison will also be made with some known results for classical orthotropic conical shells.

For classical orthotropic conical shells it is found that

$$\begin{aligned} &c_2 = c_4 = c_6 = c_8 = 0 \\ &c_1 = c_3, \quad c_5 = c_7 \\ &b_2 = b_4 = b_6 = b_8 = 0 \\ &b_1 = b_3, \quad b_5 = b_7 \\ &\tau_k = 0 \end{aligned} \tag{5.24}$$

After substitution, one obtains

$$\begin{aligned} &d_1 = d_5, \quad d_3 = d_7, \quad d_4 = d_8, \\ &d_9 = d_{13}, \quad d_{11} = d_{15}, \quad d_{12} = d_{16} \end{aligned} \tag{5.25}$$

Substituting Eqs. (5.25) into Eq. (5.23) yields

$$2I_1d_1 + 2I_3d_3 + 2I_4d_4 + 2I_9d_9 + 2I_{11}d_{11} + 2I_{12}d_{12} + I_2(d_2+d_6) = 0 \tag{5.26}$$

For pure axial compression, Eq. (5.26) becomes

$$2I_3d_3 + 2I_4d_4 + 2I_{11}d_{11} + 2I_{12}d_{12} + I_2(d_2+d_6) = 0 \tag{5.27}$$

For pure hydrostatic pressure, Eq. (5.26) becomes

$$2I_1d_1 + 2I_4d_4 + 2I_9d_9 + 2I_{12}d_{12} + I_2(d_2+d_6) = 0 \tag{5.28}$$

By substituting the coefficients I_i and d_i and regrouping, one can rewrite Eq. (5.26) as

$$\frac{\lambda(1-e^{-z_o})\, s_1 t\,(2k^2+1)}{c\,(4k^2+1)\tan\alpha_o} = \frac{z_o s_1^2 (k^2+1)}{2\tan^2\alpha_o\, \bar{\gamma}^E_{A^*,m,n}} + \frac{(1-e^{-2z_o})\, t^2\, \bar{\gamma}^E_{D^*,m,n}}{16c^2} - \frac{\bar{p}\,(e^{z_o}-1)\, s_1 t\,[\,2(k^2+2n^2) + 1\,]}{2c\,(4k^2+1)\tan\alpha_o} \tag{5.29}$$

where

$$\bar{\gamma}^E_{D^*,m,n} = \bar{D}^*_{11}(k^2-2) + \bar{D}^*_{22}\left[1 + \frac{(1-n^2)^2}{1+k^2}\right] + 2(\bar{D}^*_{12} + 2\bar{D}^*_{66})n^2 \tag{5.30}$$

$$\bar{\gamma}^E_{A^*,m,n} = \bar{A}^*_{11}(1-n^2)^2 + (\bar{A}^*_{11} + \bar{A}^*_{22})k^2 + \bar{A}^*_{22}k^4 + (2\bar{A}^*_{12} + \bar{A}^*_{66})\, n^2k^2 \tag{5.31}$$

By imposing the following restrictions used by Schiffner[89]

$$z_o = \ln(s_2/s_1) \le 1 \quad , \quad (s_2/s_1) \le e \tag{5.32}$$

and

$$k^2 >> 1 \quad , \quad n^2 >> 1 \tag{5.33}$$

Equation (5.29) becomes

$$\lambda = \frac{2c\tan\alpha_o}{(1-e^{-z_o})s_1 t}\{\frac{z_o s_1^2 k^2}{2\tan^2\alpha_o[\bar{A}_{11}^* n^4 + (\bar{A}_{11}^*+\bar{A}_{22}^*)k^2 + \bar{A}_{22}^* k^4 + (2\bar{A}_{12}+\bar{A}_{66}^*)n^2k^2]}$$

$$+\frac{(1-e^{-2z_o})t^2[\bar{D}_{11}^* k^2 + \bar{D}_{22}^*(1+\frac{n^4}{k^2}) + 2(\bar{D}_{12}^* + 2\bar{D}_{66}^*)n^2]}{16c^2} \tag{5.34}$$

$$-\frac{\bar{p}(e^{z_o}-1)s_1 t(k^2+2n^2)}{4ck^2\tan\alpha_o}\}$$

which is identical with Eq. (3.16) of Ref. [89].

Notice that the eigenvalues depend on the wave number m, n_1 and the geometrical parameters of the given conical shell. The classical buckling load is the lowest of all the eigenvalues.

Introducing the nondimensional constant $\hat{R}$ such that $\hat{R} = \lambda/\bar{p}$, then Eq. (5.29) can be written as

$$\lambda + \frac{\bar{p}e^{z_o}[2(k^2+2n^2) + 1]}{2(2k^2+1)} = \hat{R}\bar{p} + \frac{\bar{p}e^{z_o}[2(k^2+2n^2) + 1]}{2(2k^2+1)}$$

$$= \frac{c(4k^2+1)\tan\alpha_o}{(1-e^{-z_o})s_1 t(2k^2+1)}[\frac{z_o s_1^2(k^2+1)}{2\tan^2\alpha_o\bar{\gamma}^E_{A^*,m,n}} + \frac{(1-e^{-2z_o})t^2\bar{\gamma}^E_{D^*,m,n}}{16c^2}] \tag{5.35}$$

Regrouping yields the critical buckling load

$$\bar{p}_{mn} = \frac{1}{\hat{R} + \frac{e^{z_o}[2(k^2+2n^2)+1]}{2(2k^2+1)}}\frac{c(4k^2+1)\tan\alpha_o}{(1-e^{-z_o})s_1 t(2k^2+1)}[\frac{z_o s_1^2(k^2+1)}{2\tan^2\alpha_o\bar{\gamma}^*_{A^*,m,n}} \tag{5.36}$$

$$+\frac{(1-e^{-2z_o})t^2\bar{\gamma}^E_{D^*,m,n}}{16c^2}]$$

For pure axial compression ($\bar{p} = 0$), one obtains the critical buckling load as

$$\lambda_{mn} = \frac{c\,(4k^2+1)\tan\alpha_o}{(1-e^{z_o})\,s_1\,t\,(1+2k^2)}\left[\frac{t^2(1-e^{-2z_o})\,\bar{\gamma}^{E}_{D^*,m,n}}{16\,c^2} + \frac{s_1^2 z_o(1+k^2)}{2\tan^2\alpha_o\,\bar{\gamma}^{E}_{A^*,m,n}}\right] \tag{5.37}$$

For pure hydrostatic pressure ($\lambda = 0$), one obtains the critical buckling load as

$$\bar{p}_{mn} = \frac{c\,(4k^2+1)\tan\alpha_o}{(e^{z_o}-1)\,s_1\,t\,[\,1+2\,(k^2+2n^2)\,]}\left[\frac{t^2(1-e^{-2z_o})\,\bar{\gamma}^{E}_{D^*,m,n}}{8\,c^2} + \frac{s_1^2 z_o(1+k^2)}{\tan^2\alpha_o\,\bar{\gamma}^{E}_{A^*,m,n}}\right] \tag{5.38}$$

5.3.4 Numerical results for critical buckling loads

To confirm the correctness and to ascertain the accuracy of the present theory, comparisons of results from present analysis with those of previous investigations are needed. To carry out the comparisons, however, the analytic solution of Eq. (5.23) should be converted to a numerical code so that one can obtain the critical buckling loads for the different loading cases and shell geometries.

So far the Computer Algebra systems have been used successfully for doing some involved analytic and symbolic calculations. However, being able to carry out lengthy symbolic calculations is not their only advantage. Some of the Computer Algebra systems are also able to generate the numerical codes directly from the final analytic results obtained from symbolic manipulations, such as, REDUCE and MAPLE can convert the symbolic expressions to Fortran codes automatically, Mathematica can generate both Fortran and C codes. With the help of this unique power of Computer Algebra systems, one can easily obtain some numerical codes without much extra programming work, although the resulting numerical codes may be not optimized. This capability of combining some analytic calculations with some numerical computations, not only enlarges the field of possible applications of Computer Algebra systems, but also creates challenges and demands for the development of some new algorithms[159,160].

A Fortran source program SANAC, Simplified ANalysis of Anisotropic Cones[158], is generated for the calculation of critical buckling loads with the help of REDUCE. In the following some numerical results are presented and compared with other known critical buckling loads. Notice that in the following calculations for pure hydrostatic pressure λ = 0; for pure axial compression $\bar{p}$ = 0; for external lateral pressure $P = -\pi p s_1^2 \sin^2\alpha_o$; whereas for combined loading, the nondimensional axial load parameter λ is fixed and the lowest value of $\bar{p}$ is calculated.

Isotropic Conical Shells

Axial compression

In Table 5.2, some critical buckling loads of isotropic conical shells under axial compression obtained from the present theory are compared with those obtained by Baruch et al.[55]. As can be seen the results from the present theory agree well with those of Ref. [55].

Table 5.2 *Classical buckling loads λ of isotropic conical shells under axial compression (R_1/t = 100, ν = 0.3)*

α_o°	l_s/R_1 = 0.2		l_s/R_1 = 0.5	
	Baruch et al.[55]	Present theory	Baruch et al.[55]	Present theory
0.5	1.005 (7)	1.0002 (1,6)	1.002 (8)	1.0017 (2,8)
1	1.005 (7)	1.0002 (1,6)	1.002 (8)	1.0017 (2,8)
2	1.005 (7)	1.0001 (1,6)	1.002 (8)	1.0017 (2,8)
5	1.006 (7)	1.0001 (1,6)	1.002 (8)	1.0017 (2,8)
10	1.007 (7)	0.9999 (1,6)	1.002 (8)	1.0017 (2,8)
30	1.017 (5)	0.9995 (1,4)	1.001 (7)	1.0013 (2,7)
60	1.144 (0)	1.1176 (1,0)	1.044 (7)	1.0015 (1,7)

Notice that the first numbers in brackets indicates the half wave number along the generator, while the second one and the single number in brackets indicate the full wave number along the circumferential direction. l_s is the slant length of the truncated conical shell.

Hydrostatic pressure

Some results of present theory are compared with those obtained by Seide[41], and shown in Table 5.3. As can be seen the agreement is good.

Table 5.3 Hydrostatic buckling pressures of isotropic conical shells

L/R_1	R_1/t	$\alpha_o = 5°$				$\alpha_o = 30°$			
		Seide[41]		Present theory		Seide[41]		Present theory	
		p/E	n	p/E	n	p/E	n	p/E	n
0.5	250	1.950 (-6)	15	1.951 (-6)	15	1.160 (-6)	14	1.175 (-6)	14
	500	3.351 (-7)	18	3.352 (-7)	18	2.004 (-7)	17	2.026 (-7)	17
	700	1.201 (-7)	20	1.201 (-7)	20	7.199 (-8)	19	7.270 (-8)	19
2.0	250	4.161 (-7)	8	4.176 (-7)	8	1.831 (-7)	9	1.984 (-7)	9
	500	7.372 (-8)	10	7.402 (-8)	10	3.217 (-8)	11	3.474 (-8)	11
	750	2.660 (-8)	11	2.670 (-8)	11	1.165 (-8)	13	1.258 (-8)	12

Numbers in brackets denote exponent of factor of 10 by which number should be multiplied.

Combined hydrostatic pressure and axial load

Calculations are also made for isotropic conical shells under combined hydrostatic pressure and axial load. λ is assumed to change from -1.0 to +1.0 by steps of 0.2, while the minimum buckling load p is calculated for the specified λ. Note that the negative values of λ denote axial tension. The results from present theory are compared with those from Ref. [161], and listed in Table 5.4. The agreement is good. Notice that the critical buckling pressure p increases in the presence of axial tension, and decreases as the fixed axial compression λ is increased.

Table 5.4 *Critical buckling loads of isotropic conical shells under combined hydrostatic pressure and axial load ($L/R_1 = 0.5$, $R_1/t = 250$)*

λ	$\alpha_o = 5°$				$\alpha_o = 10°$			
	Seide[161]		Present theory		Seide[161]		Present theory	
	p/E	n	p/E	n	p/E	n	p/E	n
0.8	4.602 (-7)	13	4.582 (-7)	13	4.276 (-7)	13	4.285 (-7)	13
0.6	8.559 (-7)	13	8.566 (-7)	13	8.065 (-7)	13	8.077 (-7)	13
0.4	1.253 (-6)	13	1.254 (-6)	13	1.179 (-6)	14	1.181 (-6)	14
0.2	1.606 (-6)	14	1.605 (-6)	14	1.511 (-6)	14	1.531 (-6)	14
0.0	1.950 (-6)	15	1.951 (-6)	15	1.833 (-6)	15	1.836 (-6)	15
-0.2	2.258 (-6)	15	2.257 (-6)	15	2.126 (-6)	15	2.129 (-6)	15
-0.4	2.556 (-6)	16	2.557 (-6)	16	2.404 (-6)	16	2.408 (-6)	16
-0.6	2.828 (-6)	16	2.829 (-6)	16	2.664 (-6)	16	2.668 (-6)	16
-0.8	3.091 (-6)	17	3.092 (-6)	17	2.908 (-6)	17	2.913 (-6)	17
-1.0	3.333 (-6)	17	3.335 (-6)	17	3.140 (-6)	17	3.145 (-6)	17

λ	$\alpha_o = 20°$				$\alpha_o = 30°$			
	Seide[161]		Present theory		Seide[161]		Present theory	
	p/E	n	p/E	n	p/E	n	p/E	n
0.8	3.587 (-7)	13	3.614 (-7)	13	2.678 (-7)	12	2.711 (-7)	12
0.6	6.733 (-7)	13	6.770 (-7)	13	5.156 (-7)	13	5.230 (-7)	13
0.4	9.849 (-7)	14	9.912 (-7)	14	7.444 (-7)	13	7.537 (-7)	13
0.2	1.260 (-6)	14	1.267 (-6)	14	9.600 (-7)	14	9.741 (-7)	14
0.0	1.529 (-6)	15	1.539 (-6)	15	1.160 (-6)	14	1.176 (-6)	14
-0.2	1.772 (-6)	15	1.782 (-6)	15	1.347 (-6)	15	1.367 (-6)	15
-0.4	2.004 (-6)	16	2.017 (-6)	16	1.523 (-6)	15	1.545 (-6)	15
-0.6	2.220 (-6)	16	2.234 (-6)	16	1.686 (-6)	16	1.712 (-6)	16
-0.8	2.424 (-6)	17	2.440 (-6)	17	1.843 (-6)	16	1.870 (-6)	16
-1.0	2.617 (-6)	17	2.633 (-6)	17	1.987 (-6)	17	2.017 (-6)	17

Table 5.4 continuation

λ	$\alpha_o = 45°$				$\alpha_o = 60°$			
	Seide[161]		Present theory		Seide[161]		Present theory	
	p/E	n	p/E	n	p/E	n	p/E	n
0.8	1.439 (-7)	11	1.480 (-7)	11	5.224 (-8)	10	5.657 (-8)	10
0.6	2.747 (-7)	12	2.841 (-7)	12	9.618 (-8)	10	1.024 (-7)	10
0.4	3.952 (-7)	12	4.070 (-7)	12	1.395 (-7)	11	1.483 (-7)	10
0.2	5.095 (-7)	13	5.283 (-7)	13	1.767 (-7)	11	1.911 (-7)	11
0.0	6.142 (-7)	13	6.344 (-7)	13	2.139 (-7)	11	2.296 (-7)	11
-0.2	7.137 (-7)	14	7.406 (-7)	13	2.467 (-7)	12	2.681 (-7)	11
-0.4	8.051 (-7)	14	8.335 (-7)	14	2.785 (-7)	12	3.018 (-7)	12
-0.6	8.928 (-7)	15	9.261 (-7)	14	3.083 (-7)	13	3.345 (-7)	12
-0.8	9.733 (-7)	15	1.009 (-6)	15	3.358 (-7)	13	3.662 (-7)	13
-1.0	1.052 (-6)	16	1.091 (-6)	15	3.631 (-7)	13	3.944 (-7)	13

Notice that numbers in parentheses denote exponent of factor of 10 by which number should be multiplied.

Classical Orthotropic Conical Shells

Some critical buckling loads for classical orthotropic conical shells under hydrostatic pressure are compared with values published in Ref. [79]. The results are listed in Table 5.5, which show that the solutions from present theory agree approximately with those of Ref. [79]. The differences are probably due to the use of different approximations for the description of the buckling pattern.

Table 5.5 Critical buckling load of classical orthotropic conical shells under hydrostatic pressure (E_s/E_θ =2.591, $\nu_{s\theta}$ = 0.09)

Typical shell	α_o°	R_1/t	L/R_1	t	Singer[79] $(p/E_s)\times10^6$ 1 term	2 term	Present theory $(p/E_s)\times10^6$
1	30	287.95	1	0.1	0.2065	0.2041	0.1695 (1,13)
2	75	556.30	0.518	0.1	0.0145	0.0142	0.01026 (1,14)
3	30	287.95	3	0.1	0.0438	0.0418	0.04279 (1,10)
4	60	168.90	6.178	0.0157	0.0184	0.0133	0.02169 (1,8)
5	75	188.39	5.539	0.0157	0.00601	0.00428	0.00673 (1,7)

Anisotropic Conical Shells

Since no results for anisotropic conical shells are available in the literature, comparison is made with anisotropic cylindrical shells under axial compression using also the simplified Donnell-type theory[162] by taking the semi-vertex angle α_o = 0.1°. The geometric and material properties of the anisotropic cylindrical shell used in the calculations can be found in Table 5.6.

Table 5.6 Booton's anisotropic shell (30°, 0°, -30°), (R_1/t = 100, t = 0.0267 in., $\bar{Z}$ = 400)

$$\begin{bmatrix}\varepsilon_s\\ \varepsilon_\theta\\ \gamma_{s\theta}\end{bmatrix} = \frac{1}{Et}\begin{bmatrix}1.3751 & -0.7582 & 0.\\ -0.7582 & 2.6292 & 0.\\ 0. & 0. & 4.8885\end{bmatrix}\begin{bmatrix}N_s\\ N_\theta\\ N_{s\theta}\end{bmatrix} + \frac{t}{2c}\begin{bmatrix}0. & 0. & 0.1785\\ 0. & 0. & -0.0096\\ 0.7430 & 0.1965 & 0.\end{bmatrix}\begin{bmatrix}\kappa_s\\ \kappa_\theta\\ \kappa_{s\theta}\end{bmatrix}$$

$$\begin{bmatrix}M_s\\ M_\theta\\ \dfrac{M_{s\theta}+M_{\theta s}}{2}\end{bmatrix} = \frac{t}{2c}\begin{bmatrix}0. & 0. & -0.7430\\ 0. & 0. & -0.1965\\ -0.1785 & 0.0096 & 0.\end{bmatrix}\begin{bmatrix}N_s\\ N_\theta\\ N_{s\theta}\end{bmatrix} + D\begin{bmatrix}0.5634 & 0.2214 & 0.\\ 0.2214 & 0.3898 & 0.\\ 0. & 0. & 0.1856\end{bmatrix}\begin{bmatrix}\kappa_s\\ \kappa_\theta\\ \kappa_{s\theta}\end{bmatrix}$$

where $c^2 = 3(1-\nu^2)$, $D = Et^3/4c^2$, $E = 5.83\times10^6$ psi, ν = 0.363.

The critical buckling load parameter λ calculated from the present theory is 0.4072 with five axial half waves and six circumferential full waves. This result is the same as that of Ref. [162]. Besides, it should be mentioned that unlike the cases of isotropic or orthotropic conical shells, the critical buckling load of anisotropic conical shells corresponds to a non-zero τ_k, where in this case $\tau_k = -1.0$.

Summarizing one can state that the present theory yields results that are in fair agreement with those of other investigations. More comparisons and results can be found in Ref. [158].

5.4 On the Behavior of Conical Shells as the Cone Angle Approaches Zero

The buckling load discontinuity of perfect isotropic conical shells with classical SS3 boundary conditions under axial compression was first observed by Baruch, Harari and Singer[55,163]. Although it has been more than twenty years since then, a clear explanation for this unexpected phenomenon is still needed, due to some continuous confusions about this problem in the open literature.

5.4.1 Problem description

It is well-known[164-166] that in the case of simplified analysis for cylindrical shells under axial compression, low buckling loads (half of the classical one) occur for SS1 and SS2 boundary conditions, while for SS3 boundary conditions the critical buckling load is one. However, by using a solution based on displacement formulation in which different boundary conditions are fulfilled in an exact manner, Baruch et al.[163] obtained low critical buckling loads (less than half of the classical value) for isotropic conical shells with the following 'classical simply supported' boundary conditions

$$\text{SS3: } N_s = v = \tilde{W} = M_s = 0$$

The low buckling load corresponds to axisymmetric mode, and occurs even for small cone angle ($\alpha_o = 0.5°$). This result was unexpected, since one would expect that the in-plane boundary effects for a cylindrical shell and for a conical shell with small semi-vertex angle would be similar. Thus, for the SS3 boundary conditions the transition from conical to cylindrical shell is not as smooth as it is for the other boundary conditions. In other words, it seems that there is a discontinuity phenomenon.

In order to verify these unexpected results, the axisymmetric buckling of the axially compressed conical shell was also analyzed by Baruch et al.[163] via two other different methods: a closed-form solution using Hankel functions and a finite difference solution. Calculations by these two alternative methods confirmed the results obtained by the displacement method.

However, the story is not yet finished. It was the believe of the author of this thesis that by making the semi-vertex angle α_o smaller and smaller, finally one would possibly find a "boundary layer" type behavior. In other words, if α_o approaches zero the discontinuity phenomenon may disappear. Therefore, it was decided to reexamine carefully this phenomenon by using the closed-form solution in Hankel function suggested by Baruch et al.[163] with the help of some advanced Computer Algebra systems, such as MAPLE.

5.4.2 Closed-form solution using Hankel functions

Following the procedure suggested by Baruch et al.[163], the analysis is carried out in non-dimensional form, and the non-dimensional quantities are obtained by dividing the original displacements and coordinates by s_1. For example

$$
\begin{aligned}
x &= s / s_1 \\
u^* &= u / s_1 \\
w^* &= \tilde{W} / s_1
\end{aligned}
\tag{5.39}
$$

The equilibrium equations for the axisymmetric buckling mode are obtained from Eq. (3.8) as

$$
\begin{aligned}
&(xN_s)_{,x} = N_\theta \\
&(xM_s)_{,xx} - M_{\theta,x} + s_1 N_\theta \cot\alpha_o + s_1 (xN_{so} w^*_{,x})_{,x} = 0
\end{aligned}
\tag{5.40}
$$

Substituting the first of Eqs. (5.40) into the second and integration yields

$$
(xM_x)_{,x} - M_\theta + s_1 x N_s \cot\alpha_o + s_1 x N_{so} w^*_{,x} = -(D/s_1)K \tag{5.41}
$$

where K is a constant of integration.

Substituting the constitutive relations of isotropic conical shells into the first of Eqs. (5.40) and Eq. (5.41) yields

$$u^*_{,xx} + u^*_{,x}/x - u^*/x^2 + [\, w^*/x^2 - (\nu/x)w^*_{,x}\,]\cot\alpha_o = 0 \tag{5.42a}$$

$$\begin{aligned}(1/x)(xw^*_{,x})_{,xx} - (1/x^2)(xw^*_{,x})_{,x} + (P/\pi\sin2\alpha_o)(s_1/D)(xw^*_{,x}/x^2) \\ + 12\nu\cot^2\alpha_o(s_1/t)^2(w^*/x) - 12(s_1/t)^2\cot\alpha_o(u^*_{,x} + \nu u^*/x) = K/x\end{aligned} \tag{5.42b}$$

where u^* and w^* represent the dimensionless displacements, t is the wall thickness, P is the applied axial load, and

$$D = Et^3/12(1-\nu^2) \tag{5.43}$$

By applying the differential operator[32]

$$x^2\frac{d^2}{dx^2}(\ ) + 3x\frac{d}{dx}(\ )$$

on Eq. (5.42b) and substituting Eq. (5.42a) into the resulting equation yields

$$x^2 f_{,xxxx} + 2x f_{,xxx} + 2k\lambda x f_{,xx} + k^2 f = 0 \tag{5.44}$$

where

$$\begin{aligned} f &= (xw^*_{,x})_{,x} \\ k &= [\,12(1-\nu^2)\,]^{1/2}(s_1/t)\cot\alpha_o \\ \lambda &= P/P_{cl} \\ P_{cl} &= 2E\,t^2\pi\cos^2\alpha_o/\sqrt{3(1-\nu^2)} \end{aligned} \tag{5.45}$$

Notice that Eq. (5.44) is of the 6[th] order in w^*, which can also be written as

$$\left(x\frac{d^2}{dx^2} + b_1\right)\left(x\frac{d^2}{dx^2} + b_2\right) f = 0 \tag{5.46}$$

where

$$\begin{aligned} b_1 &= k\left(\lambda + i\sqrt{1-\lambda^2}\right) \\ b_2 &= k\left(\lambda - i\sqrt{1-\lambda^2}\right) \\ i &= \sqrt{-1} \end{aligned} \tag{5.47}$$

It is assumed that $\lambda<1$.

The solution of Eq. (5.46) is

$$\begin{aligned} f = (x\, w^*_{,x})_{,x} = &- b_1 \left[C_1 (2\sqrt{b_1 x})\, H_1^{(1)}(2\sqrt{b_1 x}) + C_2 (2\sqrt{b_1 x})\, H_1^{(2)}(2\sqrt{b_1 x}) \right] \\ &- b_2 \left[C_3 (2\sqrt{b_2 x})\, H_1^{(1)}(2\sqrt{b_2 x}) + C_4 (2\sqrt{b_2 x})\, H_1^{(2)}(2\sqrt{b_2 x}) \right] \end{aligned} \tag{5.48}$$

where $H_i^{(j)}$ is a Hankel function of kind j and order i.

Further, integrating Eq. (5.48) twice yields

$$\begin{aligned} w^* = &\ C_1\{2H_0^{(1)}(2\sqrt{b_1 x}) + (2\sqrt{b_1 x})H_1^{(1)}(2\sqrt{b_1 x})\} \\ &+ C_2\{2H_0^{(2)}(2\sqrt{b_1 x}) + (2\sqrt{b_1 x})H_1^{(2)}(2\sqrt{b_1 x})\} \\ &+ C_3\{2H_0^{(1)}(2\sqrt{b_2 x}) + (2\sqrt{b_2 x})H_1^{(1)}(2\sqrt{b_2 x})\} \\ &+ C_4\{2H_0^{(2)}(2\sqrt{b_2 x}) + (2\sqrt{b_2 x})H_1^{(2)}(2\sqrt{b_2 x})\} \\ &+ C_5 + C_6 \ln x \end{aligned} \tag{5.49}$$

Notice that Eq. (5.42a) can be rewritten as

$$[x^3(u^*/x)_{,x}]_{,x} = (\nu x w^*_{,x} - w^*)\cot\alpha_o \tag{5.50}$$

Substituting Eq. (5.49) into Eq. (5.50) and carrying out the integration yields

$$\begin{aligned} u^*/\cot\alpha_o = {} & C_1\{4(2\sqrt{b_1x})^{-1}H_1^{(1)}(2\sqrt{b_1x}) + 2\nu H_2^{(1)}(2\sqrt{b_1x})\} \\ & + C_2\{4(2\sqrt{b_1x})^{-1}H_1^{(2)}(2\sqrt{b_1x}) + 2\nu H_2^{(2)}(2\sqrt{b_1x})\} \\ & + C_3\{4(2\sqrt{b_2x})^{-1}H_1^{(1)}(2\sqrt{b_2x}) + 2\nu H_2^{(1)}(2\sqrt{b_2x})\} \\ & + C_4\{4(2\sqrt{b_2x})^{-1}H_1^{(2)}(2\sqrt{b_2x}) + 2\nu H_2^{(2)}(2\sqrt{b_2x})\} \\ & + C_5 + C_6\{\ln x - \nu - \lambda d/2x\} \end{aligned} \tag{5.51}$$

where

$$d = (t/s_1)\tan\alpha_o\sqrt{4(1+\nu)/3(1-\nu)} \tag{5.52}$$

Since here the arguments of the Hankel functions are large, they can be evaluated by using the asymptotic expansions of Watson[167].

In order to avoid the use of complex constants C_1, ..., C_6 the following transformation is used

$$C = T\,B \tag{5.53}$$

where

$$T = \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.54}$$

By satisfying the SS3 boundary conditions one obtains,

$$AB = 0 \tag{5.55}$$

where A and B are real, and the 6x6 terms of the matrix A are functions of λ. λ_{cr} is the buckling load which makes the determinant of A zero.

The calculations were performed by using a MAPLE program with very high accuracy. λ_{cr} was calculated for very small semi-vertex cone angles ($0.5° < \alpha_0 < 0.005°$), and it was found that the buckling load is always very close to 0.5. In other words, for SS3 boundary conditions the transition from conical to cylindrical shell is indeed discontinuous.

5.4.3 Explanation for the discontinuity

The explanation of this unexpected phenomenon was first given by Koiter[168]. He suggested that a rigid body translation permitted by this boundary condition is incorporated in the solutions. Thus, the corresponding buckling mode of a conical shell is nearly identical with the buckling mode of a cylindrical shell with radially free edges, which will certainly lead to a low buckling load[56].

The existence of a rigid body translation can be easily proven from the analytic solution expressed in terms of Hankel functions. From Eqs. (5.49) and (5.51) one of the exact solution reads,

$$\begin{aligned} u^* &= \cot\alpha_0 \, C_5 \\ w^* &= C_5 \end{aligned} \tag{5.56}$$

which clearly represents a rigid body motion of the conical shell. However, it must be emphasized that, as expected, solution (5.56) does not cause any internal forces. It is needed for the fulfillment of the boundary conditions.

In practice, however, the large translations that accompany the low buckling loads with SS3 boundary condition cannot occur. Therefore, the classical SS3 boundary conditions defined here are not an appropriate representation of the boundary conditions that occur in practice for conical shells under axial compression. A modification for the 'classical simply supported' boundary conditions of conical shells was proposed by Seide as type 7 of Table 4.1. Calculations by using this modified SS3 boundary conditions show that, as expected, the corresponding buckling load is about one.

Still another question could be asked, namely, if a very steep conical shell with SS3 boundary conditions can yield a low buckling load, then a cylindrical shell having some initial geometric imperfection in the form of the corresponding conical shell with SS3 boundary conditions would be also expected to yield the same low buckling load. However, analysis shows that this never occurs for a cylindrical shell. Thus, one must conclude that the above mentioned results for conical shells are not realistic. The final elucidation of this question comes from personal communication with Prof. Koiter[169]. It is concluded that although the case of cylindrical shells looks like that of the conical shells, in fact, they are different in the sense of no rigid body translations being permitted in the calculations of imperfect cylindrical shells.

5.5 Discussions and Conclusion

The existence and nonexistence of membrane prebuckling solution for anisotropic conical shells are examined. The results indicate that real membrane solution usually does not exist for the case of anisotropic conical shells.

There exist no closed-form solutions for the nonlinear axisymmetric prebuckling equations of anisotropic conical shells. Also for the linearized prebuckling equations there are no general closed-form solutions available. This is different from the situation of anisotropic cylindrical shells, where the closed-form prebuckling solution is always available because the corresponding prebuckling differential equations with constant coefficients are linear. For the case of perfect isotropic conical shells, there is a closed-form solution for the corresponding linearized prebuckling equations[58,89].

The use of membrane-like prebuckling solution, i.e., membrane prebuckling stress state with omission of prebuckling deformation terms has been the general practice in buckling analyses of conical shells. Employing such simplified analysis will lead to a significant reduction of the analysis effort and increase the computational efficiency, although the corresponding solutions are not always in fair agreement with experimental results. Nevertheless, the corresponding results can provide at least some qualitative references. For more accurate analysis, a rigorous solution of the prebuckling state is needed.

Galerkin's procedure is used to solve analytically for the classical critical buckling load of anisotropic conical shells. To overcome the lengthy analytic and symbolic calculations that occur in Galerkin's procedure, A REDUCE-based program GMACS is developed, which can be used to derive the final algebraic equation of the critical buckling load efficiently. Further, a Fortran program is generated automatically by REDUCE to calculate the classical critical buckling load from Eq. (5.23) for different kinds of conical shells. All the effort indicates that the use of some Computer Algebra systems is vital for the simplified buckling analysis of anisotropic conical shells.

Some numerical results are obtained according to the present theory and compared with those of previous investigations. The agreement is essentially good. Therefore, the correctness of the present theory is insured. The results obtained from present theory can be used as preliminary solutions for practical applications or as starting values for some more rigorous theoretical analysis.

The discontinuity phenomenon in the transition from conical to cylindrical shells under axial compression with classical SS3 boundary conditions is reexamined with the help of analytical solutions derived with MAPLE. The speculation over the existence of some boundary layer type behavior is eliminated.

Chapter 6

Rigorous Bifurcation Buckling Analysis of Anisotropic Conical Shells

6.1 Introduction

In the previous chapter the simplified buckling analysis is carried out by using a membrane-like prebuckling solution and employing Galerkin's procedure. However, as mentioned earlier, it happens frequently that the solutions obtained from the simplified analysis are not in fair agreement with the experimental results. Therefore, as an improvement for the previous simplified analysis, in the following a rigorous treatment for the bifurcation buckling behavior of perfect anisotropic conical shells will be presented. This rigorous bifurcation buckling analysis is carried out by including the effects of boundary conditions and using nonlinear prebuckling deformations, based on the nonlinear Donnell-type theory. The principal objective of the present investigation is twofold: First, the results can be used as an approximation for the actual failure load and mode, and secondly, it is the first step for the initial postbuckling analysis suggested by Koiter[60] to investigate the imperfection sensitivity of the critical buckling load. The final aim of this research is to establish a reliable and improved design criterion for the buckling instability of various kinds of conical shells.

Besides, as a preliminary effort to account for the physical imperfection consisting of variations of the wall thickness and stiffness coefficients that occur when laminated composite conical shells are made by filament winding, numerical studies are also carried out according to the previously suggested formulae (see chapter 2) for the equivalent constant wall thickness and stiffness coefficients.

6.2 Theoretical Analysis

To facilitate the rigorous bifurcation buckling analysis which takes into account the exact form of boundary conditions and nonlinear prebuckling deformations, first, the ordinary differential equations for the prebuckling problem will be further simplified, while the equations for the buckling problem will be regrouped for later uses.

6.2.1 Prebuckling governing equations

Observation shows that the nonlinear differential equations (3.51) and (3.52) can be integrated once, and reduced to two third-order ordinary differential equations as

$$\begin{aligned}&\bar{A}^*_{22}f_o''' + \bar{A}^*_{22}f_o'' - \bar{A}^*_{11}f_o' - \bar{A}^*_{11}f_o - (t/2s_1\sin\alpha_o)[\bar{B}^*_{21}w_o''' \\ &\quad + (\bar{B}^*_{22}+\bar{B}^*_{21}-\bar{B}^*_{11})w_o'' + (\bar{B}^*_{22}-\bar{B}^*_{12}-\bar{B}^*_{11})w_o' - \bar{B}^*_{12}w_o] \\ &= (-e^z c/\sin\alpha_o)[\cot\alpha_o(w_o+w_o') + (t/2s_1)(w_o+w_o')^2] + k_1e^z\end{aligned} \tag{6.1}$$

$$\begin{aligned}&\bar{B}^*_{21}f_o''' + (\bar{B}^*_{21}+\bar{B}^*_{11}-\bar{B}^*_{22})f_o'' + (\bar{B}^*_{11}-\bar{B}^*_{22}-\bar{B}^*_{12})f_o' - \bar{B}^*_{12}f_o \\ &\quad + (t/2s_1\sin\alpha_o)[\bar{D}^*_{11}w_o''' + \bar{D}^*_{11}w_o'' - \bar{D}^*_{22}w_o' - \bar{D}^*_{22}w_o] \\ &= 2e^z c(w_o+w_o')(f_o+f_o') + (2e^z c s_1\cot\alpha_o/t)(f_o+f_o') \\ &\quad + e^{3z}\bar{p}cs_1\cot^2\alpha_o/(t\sin\alpha_o) + k_2e^z\end{aligned} \tag{6.2}$$

where k_1 and k_2 are constants of integration.

By enforcing the periodicity condition (see Appendix A2.1 for details), one obtains

$$k_1 = 0 \tag{6.3}$$

By satisfying the equilibrium condition $H = \bar{H}$ (see Eq. (4.39)) at the boundary, one obtains

$$k_2 = 2\lambda c s_1\cot^2\alpha_o/(t\sin\alpha_o) \tag{6.4}$$

Introducing the new variables defined by Eqs. (3.65), Eqs. (6.1) and (6.2) can be further simplified as

$$\bar{A}^*_{22}\varphi'' - \bar{A}^*_{11}\varphi - (t/2s_1\sin\alpha_o)[\bar{B}^*_{21}\psi'' + (\bar{B}^*_{22}-\bar{B}^*_{11})\psi' - \bar{B}^*_{12}\psi] = (-e^z c/\sin\alpha_o)[\cot\alpha_o\,\psi + (t/2s_1)\psi^2] \tag{6.5}$$

$$\bar{B}^*_{21}\varphi'' + (\bar{B}^*_{11}-\bar{B}^*_{22})\varphi' - \bar{B}^*_{12}\varphi + (t/2s_1\sin\alpha_o)(\bar{D}^*_{11}\psi''-\bar{D}^*_{22}\psi) = 2e^z c\varphi\psi + (2e^z c s_1\cot\alpha_o/t)\varphi + e^{3z}\bar{p}c s_1\cot^2\alpha_o/(t\sin\alpha_o) + 2e^z\lambda c s_1\cot^2\alpha_o/(t\sin\alpha_o) \tag{6.6}$$

Equations (6.5) and (6.6) form a set of two nonlinear, coupled and inhomogeneous ordinary differential equations governing the behavior of anisotropic conical shells in the axisymmetric prebuckling state. Due to the nonlinear nature of the above equations, anything but a numerical solution is out of question.

In order to be able to use the 'shooting method' to solve the nonlinear prebuckling equations, it is necessary, by considering Eqs. (6.5) and (6.6) to eliminate the ϕ'' and ψ'' terms. Some regrouping makes it possible to write the resulting equations as

$$\varphi'' = \hat{c}_1\varphi' + \hat{c}_2\varphi + \hat{c}_3\psi' + \hat{c}_4\psi + \hat{c}_5 e^z\varphi + \hat{c}_6 e^z\psi + \hat{c}_7 e^z\varphi\psi + \hat{c}_8 e^z\psi^2 + \hat{c}_9 e^{3z}\bar{p} + \hat{c}_{10}e^z\lambda \tag{6.7}$$

$$\psi'' = -\hat{c}_1\psi' + \hat{c}_{11}\psi + \hat{c}_{12}\varphi' + \hat{c}_{13}\varphi + \hat{c}_{14}e^z\psi + \hat{c}_{15}e^z\varphi + \hat{c}_{16}e^z\varphi\psi + \hat{c}_{17}e^z\psi^2 + \hat{c}_{18}e^{3z}\bar{p} + \hat{c}_{19}e^z\lambda \tag{6.8}$$

where the coefficients $\hat{c}_i$ (i=1 ~ 19) are listed in Appendix A4.1, the nondimensional load parameters $\bar{p}$ and λ are given by Eqs. (5.3) and (5.4), respectively.

6.2.2 Buckling governing equations

The ordinary differential equations governing the behavior of the buckling state of anisotropic conical shells have been given as Eqs. (3.61)-(3.64). In order to be able to use the 'shooting method' to solve the above differential equations it is necessary, by considering Eqs. (3.61) and (3.63), to eliminate the w_1^{iv} term from Eq. (3.61) and the f_1^{iv} term from Eq. (3.63). Similarly, by considering Eqs. (3.62) and (3.64) one must eliminate the w_2^{iv} term from Eq. (3.62) and the f_2^{iv} term from Eq. (3.64). Finally, some regrouping makes it possible to write the resulting equations as

$$\begin{aligned} w_1^{iv} = {} & c_1 f_1''' + c_2 f_1'' + c_3 f_1' + c_4 f_1 + c_5 f_2''' + c_6 f_2'' + c_7 f_2' + c_8 w_1''' + c_9 w_1'' \\ & + c_{10} w_1' + c_{11} w_1 + c_{12} w_2''' + c_{13} w_2'' + c_{14} w_2' + c_{15} e^z (f_1''+f_1') \\ & + c_{16} e^z (w_1''+w_1') + c_{17} e^z [f_1''\psi + f_1'(\psi'+\psi) + w_1''\varphi + w_1'(\varphi'+\varphi)] \\ & + c_{18} e^z (f_1\psi'+w_1\varphi') + c_{19} e^z [w_1''\psi + w_1'(\psi'+\psi)] + c_{20} e^z w_1 \psi' \end{aligned} \tag{6.9}$$

$$\begin{aligned} w_2^{iv} = {} & c_1 f_2''' + c_2 f_2'' + c_3 f_2' + c_4 f_2 - c_5 f_1''' - c_6 f_1'' - c_7 f_1' + c_8 w_2''' + c_9 w_2'' \\ & + c_{10} w_2' + c_{11} w_2 - c_{12} w_1''' - c_{13} w_1'' - c_{14} w_1' + c_{15} e^z (f_2''+f_2') \\ & + c_{16} e^z (w_2''+w_2') + c_{17} e^z [f_2''\psi + f_2'(\psi'+\psi) + w_2''\varphi + w_2'(\varphi'+\varphi)] \\ & + c_{18} e^z (f_2\psi'+w_2\varphi') + c_{19} e^z [w_2''\psi + w_2'(\psi'+\psi)] + c_{20} e^z w_2 \psi' \end{aligned} \tag{6.10}$$

$$\begin{aligned} f_1^{iv} = {} & - c_8 f_1''' + c_{21} f_1'' - c_{10} f_1' + c_{22} f_1 + c_{23} f_2''' - c_{13} f_2'' + c_{24} f_2' + c_{25} w_1''' \\ & + c_{26} w_1'' + c_{27} w_1' + c_{28} w_1 + c_{29} w_2''' + c_{30} w_2'' + c_{31} w_2' + c_{32} e^z (w_1''+w_1') \\ & + c_{16} e^z (f_1''+f_1') + c_{19} e^z [w_1''\varphi + w_1'(\varphi'+\varphi) + f_1''\psi + f_1'(\psi'+\psi)] \\ & + c_{20} e^z (w_1\varphi'+f_1\psi') + c_{33} e^z [w_1''\psi + w_1'(\psi'+\psi)] + c_{34} e^z w_1 \psi' \end{aligned} \tag{6.11}$$

$$\begin{aligned} f_2^{iv} = {} & - c_8 f_2''' + c_{21} f_2'' - c_{10} f_2' + c_{22} f_2 - c_{23} f_1''' + c_{13} f_1'' - c_{24} f_1' + c_{25} w_2''' \\ & + c_{26} w_2'' + c_{27} w_2' + c_{28} w_2 - c_{29} w_1''' - c_{30} w_1'' - c_{31} w_1' + c_{32} e^z (w_2''+w_2') \\ & + c_{16} e^z (f_2''+f_2') + c_{19} e^z [w_2''\varphi + w_2'(\varphi'+\varphi) + f_2''\psi + f_2'(\psi'+\psi)] \\ & + c_{20} e^z (w_2\varphi'+f_2\psi') + c_{33} e^z [w_2''\psi + w_2'(\psi'+\psi)] + c_{34} e^z w_2 \psi' \end{aligned} \tag{6.12}$$

where the coefficients c_i ($i=1 \sim 34$) are listed in Appendix A4.2.

It can be seen that for every n there is a system of linear ordinary differential equations of the forms represented by Eqs. (6.9)-(6.12) with certain boundary conditions given in chapter 4, which forms an eigenvalue problem. Only for nonzero eigenvalues do there exist nonzero solutions f_1, f_2, w_1 and w_2 which represent the eigenfunctions for the corresponding eigenvalues.

Notice further that the load parameters do not appear explicitly in the buckling equations, but they enter into the buckling equations implicitly via the prebuckling solution. Thus, the buckling equations (6.9)-(6.12) represent a so-called generalized eigenvalue problem.

6.3 Numerical Analysis

Due to the highly nonlinear nature of the two-point boundary value problems governing the axisymmetric prebuckling state and the complicated variable coefficients represented by the prebuckling solution in the linearized stability equations, anything but a numerical solution is out of question. There are many numerical methods which may be used to solve the above problems, for example, the finite difference method, the finite element method and the shooting method. However, considering the successful applications of the shooting method for the stability analysis of various cylindrical shells[67,107,170], it is decided to solve both the prebuckling and buckling problems by employing the 'parallel shooting' method.

As mentioned in Ref. [67], parallel shooting over n-intervals is slower than a coarse standard finite difference or finite element scheme. However, if the length of the intervals of integration is chosen properly so that numerical instability is avoided, then this method gives more accurate results. Also since the step size is changed automatically so as to satisfy the chosen convergence criterion, a single run is sufficient to obtain a converged solution. Thus it is not necessary to repeat the solution with different step sizes to ascertain that a properly converged solution has been found, as is the recommended practice with the standard finite difference or finite element codes.

In the following, two kinds of shooting schemes will be employed in the solution of bifurcation buckling problem. One uses Keller's method[105,107,171] by taking the eigenvalue as one of the unknowns in the iteration step, another uses a generalized version of Stodola's method[67,172,173] of mode iteration. For general descriptions of the 'shooting method' the interested reader should consult Refs. [105,106].

6.3.1 Solution of prebuckling problem

Introducing as a unified variable the 4-dimensional vector $Y^{(o)}$ defined as

$$Y_1^{(o)} = \varphi, \qquad Y_2^{(o)} = \varphi', \qquad Y_3^{(o)} = \psi, \qquad Y_4^{(o)} = \psi' \tag{6.13}$$

the axisymmetric prebuckling governing equations (6.7) and (6.8) become

$$\begin{aligned}\frac{d}{dz}Y_2^{(o)} = {} & \hat{c}_1 Y_2^{(o)} + \hat{c}_2 Y_1^{(o)} + \hat{c}_3 Y_4^{(o)} + \hat{c}_4 Y_3^{(o)} + e^{z}(\hat{c}_5 Y_1^{(o)} + \hat{c}_6 Y_3^{(o)} \\ & + \hat{c}_7 Y_1^{(o)} Y_3^{(o)} + \hat{c}_8 Y_3^{(o)} Y_3^{(o)} + \hat{c}_9 e^{2z}\overline{p} + \hat{c}_{10}\lambda)\end{aligned} \tag{6.14}$$

$$\begin{aligned}\frac{d}{dz}Y_4^{(o)} = {} & - \hat{c}_1 Y_4^{(o)} + \hat{c}_{11} Y_3^{(o)} + \hat{c}_{12} Y_2^{(o)} + \hat{c}_{13} Y_1^{(o)} + e^{z}(\hat{c}_{14} Y_3^{(o)} + \hat{c}_{15} Y_1^{(o)} \\ & + \hat{c}_{16} Y_1^{(o)} Y_3^{(o)} + \hat{c}_{17} Y_3^{(o)} Y_3^{(o)} + \hat{c}_{18} e^{2z}\overline{p} + \hat{c}_{19}\lambda)\end{aligned} \tag{6.15}$$

The above first order, nonlinear and inhomogeneous differential equations governing the axisymmetric prebuckling state can be rewritten as

$$\frac{d}{dz}Y^{(o)} = f^{(o)}(z, Y^{(o)}; \Lambda) \qquad 0 \le z \le z_o \tag{6.16}$$

Neglecting the nonlinear terms in Eqs. (6.14) and (6.15) yields the linearized prebuckling governing equations as

$$\begin{aligned}\frac{d}{dz}Y_2^{(o)} = {} & \hat{c}_1 Y_2^{(o)} + \hat{c}_2 Y_1^{(o)} + \hat{c}_3 Y_4^{(o)} + \hat{c}_4 Y_3^{(o)} + e^{z}(\hat{c}_5 Y_1^{(o)} + \hat{c}_6 Y_3^{(o)} \\ & + \hat{c}_9 e^{2z}\overline{p} + \hat{c}_{10}\lambda)\end{aligned} \tag{6.17}$$

$$\begin{aligned}\frac{d}{dz}Y_4^{(o)} = {} & - \hat{c}_1 Y_4^{(o)} + \hat{c}_{11} Y_3^{(o)} + \hat{c}_{12} Y_2^{(o)} + \hat{c}_{13} Y_1^{(o)} + e^{z}(\hat{c}_{14} Y_3^{(o)} + \hat{c}_{15} Y_1^{(o)} \\ & + \hat{c}_{18} e^{2z}\overline{p} + \hat{c}_{19}\lambda)\end{aligned} \tag{6.18}$$

The linearization of the nonlinear prebuckling equations can be justified according to the following considerations:

- Suppose that one knows a priori that the nonlinear terms are much smaller than the linear ones, then one can immediately neglect all the nonlinear terms. Otherwise
- one can neglect the nonlinear terms by imposing some constraint conditions which ensure that the nonlinear terms are smaller than the linear ones.
- if no constraint conditions are imposed, one can also neglect the nonlinear terms under the conditions that the reliability of the linearized solution can be checked by comparing its theoretical predictions with experimental results.

Together with two set of boundary conditions (simply supported and clamped) given in chapter 4, Eqs. (6.14)-(6.15) and (6.17)-(6.18) can be solved by employing parallel shooting over 2N intervals.

The membrane-like prebuckling solution can also be used in the bifurcation buckling analysis. They are

$$\begin{aligned} Y_1^{(o)} &= -\hat{c}_{20}(e^{2z}\overline{p} + \lambda) \\ Y_3^{(o)} &= e^{z}\hat{c}_{21}\overline{p} - e^{-z}\hat{c}_{22}\lambda \end{aligned} \qquad (6.19)$$

where the coefficients $\hat{c}_i$ (i = 20 ~ 22) are also listed in Appendix A4.1.

6.3.2 Solution of buckling problem via Keller's method

The ordinary differential equations governing the buckling state are given by Eqs. (6.9)-(6.12). Introducing as a unified variable the 16-dimensional vector $Y^{(1)}$ is defined as follows

$$\begin{array}{llll} Y_1^{(1)} = w_1 & Y_5^{(1)} = w_1'' & Y_9^{(1)} = w_2 & Y_{13}^{(1)} = w_2'' \\ Y_2^{(1)} = f_1 & Y_6^{(1)} = f_1'' & Y_{10}^{(1)} = f_2 & Y_{14}^{(1)} = f_2'' \\ Y_3^{(1)} = w_2' & Y_7^{(1)} = w_2''' & Y_{11}^{(1)} = -w_1' & Y_{15}^{(1)} = -w_1''' \\ Y_4^{(1)} = f_2' & Y_8^{(1)} = f_2''' & Y_{12}^{(1)} = -f_1' & Y_{16}^{(1)} = -f_1''' \end{array} \qquad (6.20)$$

then Eqs. (6.9)-(6.12) become

$$\begin{aligned}\frac{d}{dz}Y_7^{(1)} &= c_1Y_8^{(1)} + c_2Y_{14}^{(1)} + c_3Y_4^{(1)} + c_4Y_{10}^{(1)} + c_5Y_{16}^{(1)} - c_6Y_6^{(1)} \\ &+ c_7Y_{12}^{(1)} + c_8Y_7^{(1)} + c_9Y_{13}^{(1)} + c_{10}Y_3^{(1)} + c_{11}Y_9^{(1)} + c_{12}Y_{15}^{(1)} - c_{13}Y_5^{(1)} \\ &+ c_{14}Y_{11}^{(1)} + e^z\{c_{15}(Y_4^{(1)}+Y_{14}^{(1)}) + c_{16}(Y_3^{(1)}+Y_{13}^{(1)}) + c_{17}[Y_{14}^{(1)}Y_3^{(o)} \\ &+ Y_4^{(1)}(Y_3^{(o)}+Y_4^{(o)}) + Y_{13}^{(1)}Y_1^{(o)} + Y_3^{(1)}(Y_1^{(o)}+Y_2^{(o)})] + c_{18}(Y_{10}^{(1)}Y_4^{(o)} \\ &+ Y_9^{(1)}Y_2^{(o)}) + c_{19}[Y_{13}^{(1)}Y_3^{(o)} + Y_3^{(1)}(Y_3^{(o)}+Y_4^{(o)})] + c_{20}Y_9^{(1)}Y_4^{(o)}\}\end{aligned} \tag{6.21}$$

$$\begin{aligned}\frac{d}{dz}Y_8^{(1)} &= -c_8Y_8^{(1)} + c_{21}Y_{14}^{(1)} - c_{10}Y_4^{(1)} + c_{22}Y_{10}^{(1)} + c_{23}Y_{16}^{(1)} + c_{13}Y_6^{(1)} + c_{24}Y_{12}^{(1)} \\ &+ c_{25}Y_7^{(1)} + c_{26}Y_{13}^{(1)} + c_{27}Y_3^{(1)} + c_{28}Y_9^{(1)} + c_{29}Y_{15}^{(1)} - c_{30}Y_5^{(1)} + c_{31}Y_{11}^{(1)} \\ &+ e^z\{c_{32}(Y_3^{(1)}+Y_{13}^{(1)}) + c_{16}(Y_4^{(1)}+Y_{14}^{(1)}) + c_{19}[Y_{13}^{(1)}Y_1^{(o)} + Y_3^{(1)}(Y_1^{(o)}+Y_2^{(o)}) \\ &+ Y_{14}^{(1)}Y_3^{(o)} + Y_4^{(1)}(Y_3^{(o)}+Y_4^{(o)})] + c_{20}(Y_9^{(1)}Y_2^{(o)}+Y_{10}^{(1)}Y_4^{(o)}) \\ &+ c_{33}[Y_{13}^{(1)}Y_3^{(o)} + Y_3^{(1)}(Y_3^{(o)}+Y_4^{(o)})] + c_{34}Y_9^{(1)}Y_4^{(o)}\}\end{aligned} \tag{6.22}$$

$$\begin{aligned}\frac{d}{dz}Y_{15}^{(1)} &= c_1Y_{16}^{(1)} - c_2Y_6^{(1)} + c_3Y_{12}^{(1)} - c_4Y_2^{(1)} - c_5Y_8^{(1)} - c_6Y_{14}^{(1)} - c_7Y_4^{(1)} \\ &+ c_8Y_{15}^{(1)} - c_9Y_5^{(1)} + c_{10}Y_{11}^{(1)} - c_{11}Y_1^{(1)} - c_{12}Y_7^{(1)} - c_{13}Y_{13}^{(1)} - c_{14}Y_3^{(1)} \\ &- e^z\{c_{15}(Y_6^{(1)}-Y_{12}^{(1)}) + c_{16}(Y_5^{(1)}-Y_{11}^{(1)}) + c_{17}[Y_6^{(1)}Y_3^{(o)} - Y_{12}^{(1)}(Y_3^{(o)}+Y_4^{(o)}) \\ &+ Y_5^{(1)}Y_1^{(o)} - Y_{11}^{(1)}(Y_1^{(o)}+Y_2^{(o)})] + c_{18}(Y_2^{(1)}Y_4^{(o)}+Y_1^{(1)}Y_2^{(o)}) \\ &+ c_{19}[Y_5^{(1)}Y_3^{(o)} - Y_{11}^{(1)}(Y_3^{(o)}+Y_4^{(o)})] + c_{20}Y_1^{(1)}Y_4^{(o)}\}\end{aligned} \tag{6.23}$$

$$\frac{d}{dz}Y^{(1)}_{16} = - c_8 Y^{(1)}_{16} - c_{21} Y^{(1)}_6 - c_{10} Y^{(1)}_{12} - c_{22} Y^{(1)}_2 - c_{23} Y^{(1)}_8 + c_{13} Y^{(1)}_{14} - c_{24} Y^{(1)}_4$$
$$+ c_{25} Y^{(1)}_{15} - c_{26} Y^{(1)}_5 + c_{27} Y^{(1)}_{11} - c_{28} Y^{(1)}_1 - c_{29} Y^{(1)}_7 - c_{30} Y^{(1)}_{13} - c_{31} Y^{(1)}_3$$
$$- e^z \{ c_{32}(Y^{(1)}_5 - Y^{(1)}_{11}) + c_{16}(Y^{(1)}_6 - Y^{(1)}_{12}) + c_{19} [Y^{(1)}_5 Y^{(o)}_1 - Y^{(1)}_{11}(Y^{(o)}_1 + Y^{(o)}_2) \qquad (6.24)$$
$$+ Y^{(1)}_6 Y^{(o)}_3 - Y^{(1)}_{12}(Y^{(o)}_3 + Y^{(o)}_4)] + c_{20} (Y^{(1)}_1 Y^{(o)}_2 + Y^{(1)}_2 Y^{(o)}_4)$$
$$+ c_{33} [Y^{(1)}_5 Y^{(o)}_3 - Y^{(1)}_{11}(Y^{(o)}_3 + Y^{(o)}_4)] + c_{34} Y^{(1)}_1 Y^{(o)}_4 \}$$

The above first order, linear and homogenous differential equations can be rewritten as

$$\frac{d}{dz}\boldsymbol{Y}^{(1)} = \boldsymbol{f}^{(1)}(z, \boldsymbol{Y}^{(o)}(z, \Lambda), \boldsymbol{Y}^{(1)}) = \boldsymbol{R}(z, \boldsymbol{Y}^{(o)}) \boldsymbol{Y}^{(1)} \qquad 0 \le z \le z_0$$
$$\boldsymbol{B}^{(1)}_1 \boldsymbol{Y}^{(1)}(z = 0) + \boldsymbol{B}^{(1)}_2 \boldsymbol{Y}^{(1)}(z = z_0) = \boldsymbol{0} \qquad (6.25)$$

where the applied loading consists of axial compression and hydrostatic pressure. They are assumed to have uniform spatial distributions and are divided into a fixed part and a variable part. The magnitude of the variable part is allowed to vary in proportion to a load parameter Λ. This leads to an eigenvalue problem for the critical load Λ_c. In Eq. (6.25) the user can select Λ_c to be either the normalized axial load λ or the normalized hydrostatic pressure $\bar{p}$. The components of the 8×16 matrices $\mathbf{B}_1^{(1)}$ and $\mathbf{B}_2^{(1)}$ depend on the boundary conditions at the shell edges, $\mathbf{Y}^{(o)}$ is the solution of axisymmetric prebuckling problem.

Further

$$\boldsymbol{R}(z, \boldsymbol{Y}^{(o)}) = \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ -\boldsymbol{R}_{12} & \boldsymbol{R}_{11} \end{bmatrix} \qquad (6.26)$$

$$R_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & D_2 & -c_{13} & -c_6 & c_8 & c_1 \\ 0 & 0 & D_7 & D_8 & -c_{30} & c_{13} & c_{25} & -c_8 \end{bmatrix} \tag{6.27}$$

$$R_{12} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ D_3 & D_4 & c_{14} & c_7 & D_5 & D_6 & c_{12} & c_5 \\ D_9 & D_{10} & c_{31} & c_{24} & D_{11} & D_{12} & c_{29} & c_{23} \end{bmatrix} \tag{6.28}$$

where the terms D_i (i=1 ~ 12) are listed in Appendix A4.3.

To make the eigenfunctions unique, within a sign (for simple eigenvalues), one must use some kind of normalization condition. Here

$$\int_0^{z_0} (Y_1^{(1)} Y_1^{(1)} + Y_9^{(1)} Y_9^{(1)}) dz = C \tag{6.29}$$

shall be used, and C is a (positive) constant.

This condition can be written as a differential equation as follows

$$\frac{d}{dz}Y_{17}^{(1)} = Y_1^{(1)}Y_1^{(1)} + Y_9^{(1)}Y_9^{(1)} = f_N(z, Y^{(1)}; \Lambda) \qquad 0 \le z \le z_0 \tag{6.30}$$

with the boundary conditions

$$\begin{aligned} Y_{17}^{(1)}(0) &= 0 \\ Y_{17}^{(1)}(z_o) &= C \end{aligned} \tag{6.31}$$

By adding the above normalization condition to the homogeneous boundary value problem the eigenvalue can be treated as one of the unknown parameters in the iteration scheme[105,107].

The advantage of using the formulations expressed by Eqs. (6.20) for anisotropic conical shells is that together with the corresponding initial conditions the variational equations

$$\boldsymbol{W}' = \boldsymbol{R}\boldsymbol{W} \tag{6.32}$$

have the following property

$$\boldsymbol{W}(z) = \begin{bmatrix} \boldsymbol{W}_{11}(z) & -\boldsymbol{W}_{21}(z) \\ \boldsymbol{W}_{21}(z) & \boldsymbol{W}_{11}(z) \end{bmatrix} \tag{6.33}$$

where $\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{16}]$, and $\mathbf{W}_i(z)$ are submatrixes.

The derivation of Eq. (6.33) is given in Appendix A4.4. For more details of the solution procedure the reader should consult Refs. [105,107].

6.3.3 Solution of buckling problem via Stodola's method

As an alternative for Keller's method, a generalized version of Stodola's method[67,172] will also be used for the calculations of the buckling loads and the corresponding asymmetric buckling modes of anisotropic conical shells.

It has been shown previously that by introducing as a unified variable a 16-dimensional vector $\mathbf{Y}^{(1)}$ defined as follows

$$Y_1^{(1)} = f_1, \quad Y_2^{(1)} = f_2, \quad Y_3^{(1)} = w_1, \quad Y_4^{(1)} = w_2, \qquad (6.34)$$
$$Y_5^{(1)} = f_1', \quad Y_6^{(1)} = f_2', \quad \ldots, \quad Y_{16}^{(1)} = w_2'''$$

one can reduce the system of equations (6.9)-(6.12) to the following nonlinear eigenvalue problem

$$\frac{d}{dz} Y^{(1)} = f^{(1)}(z, Y^{(o)}, Y^{(1)}; \lambda, \bar{p}) \qquad (6.35)$$

$$B_1^{(1)} Y^{(1)}(z = 0) + B_2^{(1)} Y^{(1)}(z = z_o) = 0 \qquad (6.36)$$

where the components of the 8×16 matrices $\mathbf{B}_1^{(1)}$ and $\mathbf{B}_2^{(1)}$ depend on the boundary conditions at the shell edges. Notice that the 4-dimensional vector

$$Y^{(o)} = [\,\varphi,\ \varphi',\ \psi,\ \psi'\,]^T \qquad (6.37)$$

contains the known solution of the prebuckling problem.

Because of the nonlinear dependence of the prebuckling state on the variable load Λ, in general it is necessary to approach the critical eigenvalue (for a given circumferential wave number n_1) by the solution of a sequence of modified (linearized) eigenvalue problems. These equations are obtained by restricting the search for eigenvalue to a sufficiently small neighborhood of an estimate $\Lambda = \Lambda_e$ so that in this neighborhood the functions $\mathbf{Y}^{(o)}$ have a linear dependence on Λ. Setting

$$\Lambda = \Lambda_e + \mu \qquad (6.38)$$

one has to first order in μ

$$Y^{(o)}(\Lambda) = Y^{(o)}(\Lambda_e) + \mu \dot{Y}^{(o)}(\Lambda_e) \tag{6.39}$$

where

$$\dot{Y}^{(o)} = \frac{\partial}{\partial \Lambda} Y^{(o)} \tag{6.40}$$

Substituting this expression into Eqs. (6.35)-(6.36) and using λ as the variable load yields the following modified eigenvalue problem

$$\frac{d}{dz} Y^{(1)} = f^{(1)}(z,\ Y^{(o)},\ Y^{(1)};\ \Lambda_e,\ \bar{p},) + \mu\, g^{(1)}(z,\ \dot{Y}^{(o)},\ Y^{(1)}) \tag{6.41}$$

$$B_1^{(1)} Y^{(1)}(z=0) + B_2^{(1)} Y^{(1)}(z=z_o) = 0 \tag{6.42}$$

Notice that each of the 'effective load terms' is split into a part independent of μ and a second part linear in μ. The iteration equations are obtained by setting μ = 1 in the second parts of the 'effective load terms' and interpreting the buckling mode variables of these parts as being known inputs from the previous iteration. Thus the first parts of the 'effective load terms' become homogenous terms and the second parts become inhomogeneous terms for the equivalent linearized problem of each iteration. Thus one must solve repeatedly

$$\frac{d}{dz} Y^{(k)} = f^{(1)}(z,\ Y^{(o)},\ Y^{(k)};\ \Lambda_e,\ \bar{p},) + g^{(1)}(z,\ \dot{Y}^{(o)},\ Y^{(k-1)}) \tag{6.43}$$

$$B_1^{(1)} Y^{(k)}(z=0) + B_2^{(1)} Y^{(k)}(z=z_o) = 0 \tag{6.44}$$

where

$Y^{(k)}$ = buckling mode of the k^{th} iteration

$Y^{(k-1)}$ = buckling mode of the $(k-1)^{th}$ iteration

After each iteration the corresponding eigenvalue estimate $\mu^{(k)}$ is calculated by evaluating the following Rayleigh quotient

$$\mu^{(k)} = (\sigma^{(k)}, u^{(k)}; \sigma^{(k-1)}, u^{(k-1)}) / (\sigma^{(k)}, u^{(k)}; \sigma^{(k)}, u^{(k)}) \tag{6.45}$$

where the inner products are defined as follows

$$\begin{aligned}(\sigma^{(k)}, u^{(k)}; \sigma^{(k-1)}, u^{(k-1)}) \\ = \mathring{F}^{(o)} * ((W^{(1)})^{(k)}, (W^{(1)})^{(k-1)}) + (F^{(1)})^{(k)} * (\mathring{W}^{(o)}, (W^{(1)})^{(k-1)}) \\ + (F^{(1)})^{(k-1)} * (\mathring{W}^{(o)}, (W^{(1)})^{(k)})\end{aligned} \tag{6.46}$$

where the short hand notation used is

$$\begin{aligned}A*(B,C) = \int_0^{2\pi\sin\alpha_o}\int_0^{z_o} s_1^2 e^{2z} \{(1/e^{z}s_1^4)(A + A_{,z} + A_{,\bar\theta\bar\theta})(B + B_{,z})(C + C_{,z}) \\ + (1/e^{z}s_1^4)(A_{,z} + A_{,zz}) B_{,\bar\theta} C_{,\bar\theta} \\ - (1/s_1^3) A_{,z\bar\theta} [(B + B_{,z})C_{,\bar\theta} + (C + C_{,z})B_{,\bar\theta}]\} d\bar\theta\, dz\end{aligned} \tag{6.47}$$

The iterations are continued until the sum $\Lambda_e^{(k)} + \mu^{(k)}$ remains essentially constant at the value Λ_1. A suitable choice for the sequence $\Lambda_e^{(k)}$ is $\Lambda_e^{(1)} = 0$ and $\Lambda_e^{(k)} = \Lambda_e^{(k-1)} + (1/2)\mu^{(k-1)}$ for $k > 1$, where the 'relaxation factor' 1/2 is inserted in order to assure that at each stage $\Lambda_e^{(k)} < \Lambda_1$. Cohen[172] has shown that in order to insure that the eigenvalues $\mu^{(k)}$ are real it is necessary that $\Lambda_e^{(k)} < \Lambda_1$. For further details of the solution procedure the reader should consult Refs. [67,173].

6.3.4 Problem of starting values

As mentioned earlier, one of the greatest difficulties in implementing the 'shooting method' consists of obtaining a starting estimate of the initial data which is sufficiently close to the exact initial data so that the iteration scheme used to find the solution of the nonlinear problem will converge. In the case of bifurcation buckling analysis of anisotropic conical shells, fortunately, the initial guess for the eigenvalue and eigenmode can be obtained from several simpler analyses. For example, the Schiffner's solutions (Galerkin procedure for an one term deflection function approximately satisfying simply supported boundary conditions for classical orthotropic conical shells), and the simplified solution of anisotropic conical shells (see chapter 5 for details) can be used.

6.3.5 Description of the programs

Based on the successful applications of the parallel shooting method for the stability analysis of anisotropic cylindrical shells[67,107] and the derivations presented earlier, two Fortran programs for the Bifurcation Analysis of Anisotropic Conical Shells (BAAC) were developed. BAAC1 is based on Keller's method, and BAAC2 is based on Stodola's method. Figures 6.1 and 6.2 show the flow diagrams of BAAC1 and BAAC2, respectively.

For both BAAC1 and BAAC2, in order to start the iteration initial guesses for the eigenvalue and eigenmode are obtained from the simpler analysis described earlier.

In BAAC1, the main loop is the Newton iteration to solve the eigenvalue problem. In each iteration step, first the prebuckling state and the prebuckling state differentiated with respect to the load parameter are solved. The converged solutions are used in the integration of the buckling state. In an early stage of the iteration process a damping factor can be used for the corrections to guarantee convergence to the desired root[107].

In BAAC2 there are three main loops. The first and the second loops are the Newton's iterations for solving the prebuckling and bifurcation buckling equations, respectively. The last one is to solve for the corresponding eigenvalue estimate by evaluating the Rayleigh quotient.

For both programs, the accuracy of the solution is controlled by the user chosen round-off error-bound. Because of the inherited characteristics of the 'parallel shooting method' considerable speed up can be achieved by using the vectorization and parallelization facilities on the Convex C3840. Different types of boundary conditions formulated in terms of W and F, which partially or completely satisfy Seide's geometric boundary constraint, can be rigorously enforced. As an option, rigorous, linear or membrane-like prebuckling solutions can be used.

For both programs, the solutions of the associated initial value problems and the corresponding variational equations are done by the library subroutine DEQ from Caltech's Willis Booth Computer Center, which uses an Adams-Moulton predictor-corrector scheme. Starting values are obtained by the method of Runge-Kutta-Gill. The program includes an option with variable interval size and uses automatic truncation error control.

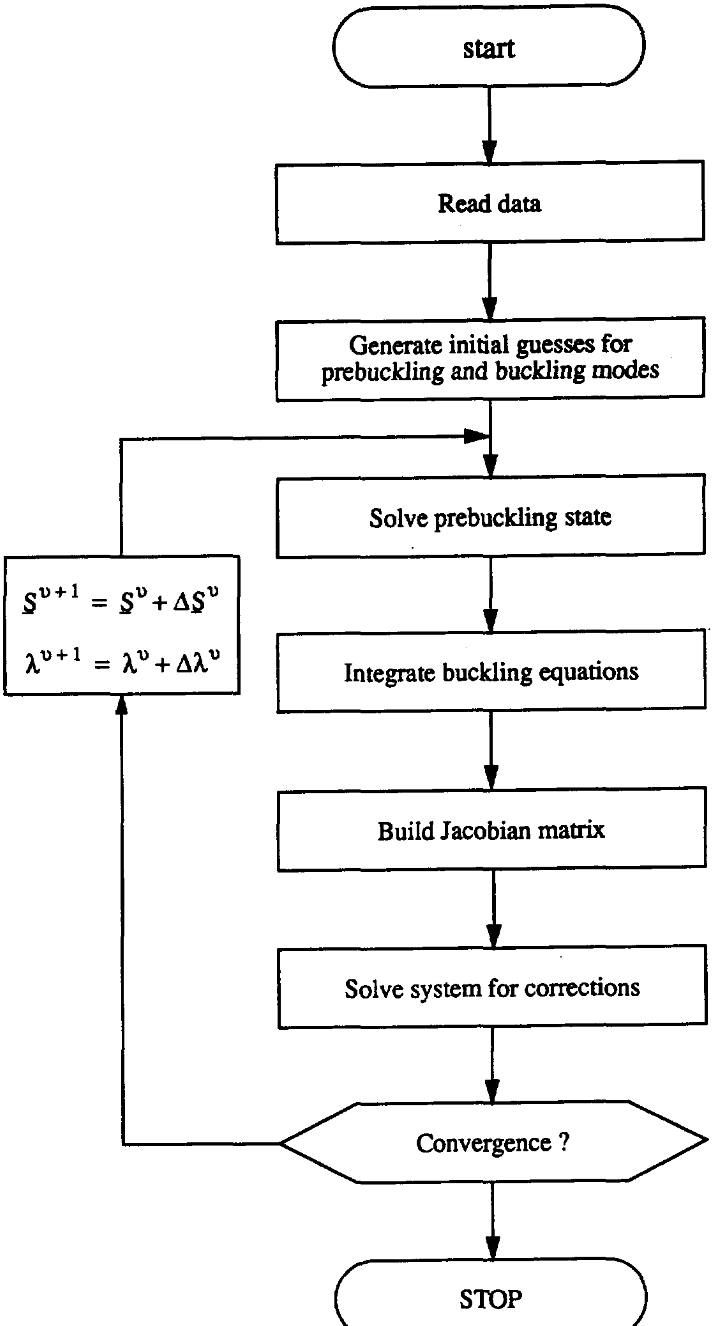


Fig. 6.1 *Flow diagram of BAAC1*

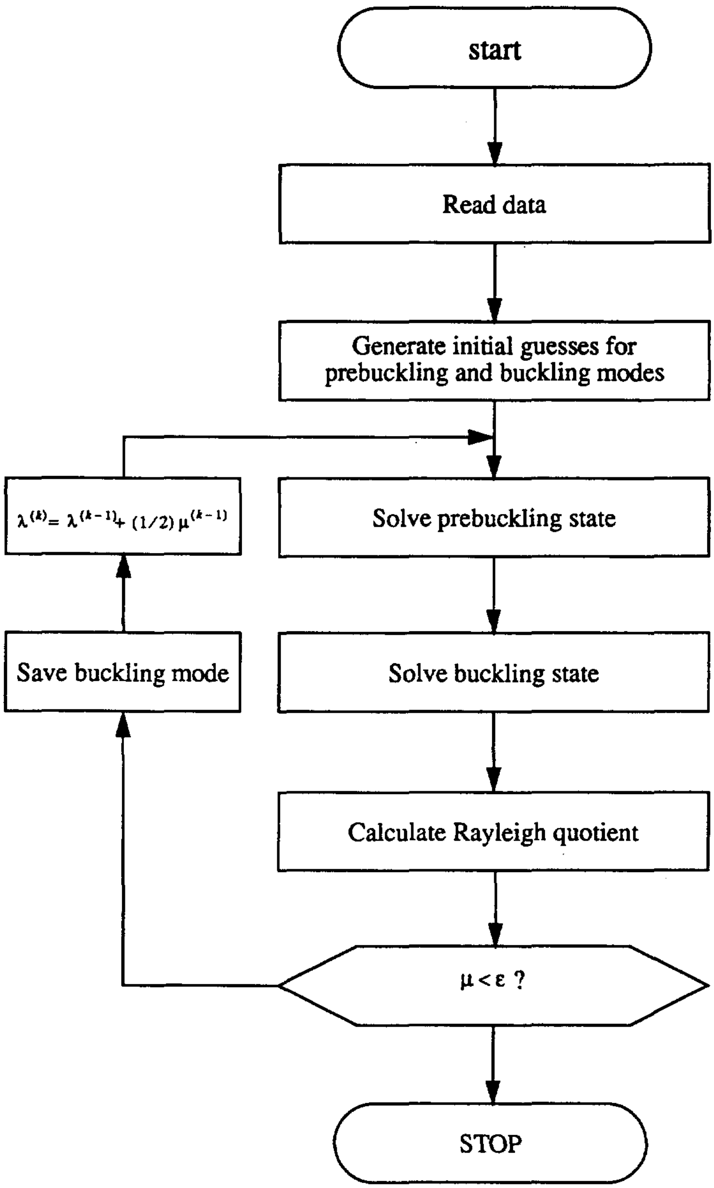


Fig. 6.2 *Flow diagram of BAAC2*

6.4 Numerical Results

Because of the large number of geometric, material and loading parameters involved in this investigation, it is impractical to attempt to present numerical results of a general nature. Rather, it seems more appropriate to present mainly some results for anisotropic conical shells with short or moderate length. However, before proceeding to these results, it is necessary to compare computations of the present investigation with those of previous investigations in order to establish the reliability of the computer programs BAAC1 and BAAC2, described earlier.

6.4.1 Comparisons with previous investigations

In the following results obtained from the present analysis are compared with results of previous investigations. The purpose of this comparison is twofold: (1) to insure that the results of the present investigation agree with results of previous investigations using a similar theory (i.e., Donnell-type theory); and (2) to test the accuracy and reliability of the computer program 'BAAC'. Notice that in the following calculations the nondimensional load parameters $\bar{p}$ and λ given by Eqs. (5.3) and (5.4) are used, and for pure hydrostatic pressure $\lambda = 0$; for pure axial compression $\bar{p} = 0$; for external lateral pressure $P = -\pi p s_1^2 \sin^2\alpha_o$.

Isotropic conical shells

All the cases shown in Table 6.1 are for simply supported (MSS4 and MSS2 boundary conditions) isotropic conical shells with various geometric properties under axial compression. All these cases have been treated previously by Baruch et al.[55] using a Donnell-type theory and membrane-like prebuckling solution. As can be seen, the results of the present investigation using also the membrane-like prebuckling solution are essentially the same as those obtained by Baruch et al.[55]. However, it must be emphasized that the results from SS2 or MSS2 boundary conditions are only for the purpose of comparison, since Donnell type solutions are inaccurate for the practical applications due to only 2 full waves in the circumferential direction.

The cases shown in Table 6.2 are for simply supported isotropic conical shells with various geometric properties under hydrostatic pressure. Cases 1-12 were previously treated by Seide[41] using membrane-like prebuckling solution and the modified simply supported boundary conditions (type 7). With this type of boundary conditions the shell is assumed to be closed by hypothetical bulkheads which are rigid in their own plane, but free to distort out of their plane, and offer no restraint against rotation of the ends of the shell. As can be seen, the results of the present investigations are in excellent agreement with those obtained by Seide. Cases 13-16 were previously studied by Baruch et al.[54] using a Donnell-type theory and membrane-like prebuckling solution. As can be seen, the results of present analysis are almost the same as those obtained by Baruch et al.. Cases 17-18 were first treated by Thurston[52] using a finite-deflection theory. He obtained exact prebuckling stresses and displacement by using Reissner's finite-deflection theory[174]. For the buckling analysis, he used the same theory and truncated Fourier series to permit asymmetric modes. The results shown in Table 6.2 indicate that the present analysis yields nearly the same results as those obtained using a finite-deflection theory.

All the cases shown in Table 6.3 are for simply supported (type 7) isotropic conical shells with various geometric properties subjected to combined hydrostatic pressure and axial load (either compression or tension). They were previously studied by Seide[161] using membrane-like prebuckling solution. Again, it can be seen that the results from the present analysis are in excellent agreement with those obtained by Seide.

To summarize briefly, the results shown in Tables 6.1-6.3 indicate that the present analysis and computer programs adequately predict the instability of isotropic truncated conical shells subjected to axial load and (or) hydrostatic pressure based on the comparison with the results of other investigations.

Table 6.1 Comparison of results of isotropic conical shells under axial compression (R_1/t = 100)

Case	l/R_1	α_o°	Previous investigation [55]		Present investigation			
			B.C.	λ	B.C.	λ		
				membrane prebuckling		membrane prebuckling	linear prebuckling	nonlinear prebuckling
1	0.5	1	SS4	1.002(8)	MSS4	1.0023(8)	1.0468(8)	0.86447(8) 0.86175(9)
2	0.5	2	SS4	1.002(8)	MSS4	1.0023(8)	1.0466(8)	0.86539(8) 0.86245(9)
3	0.5	5	SS4	1.002(8)	MSS4	1.0020(8)	1.0455(8)	0.86935(8) 0.86589(9)
4	0.5	10	SS4	1.002(8)	MSS4	1.0013(8)	1.0432(8)	0.87698(8) 0.87316(9)
5	0.5	30	SS4	1.001(7)	MSS4	0.99937(7)	1.0304(7)	0.93443(8) 0.93505(9)
6	0.5	1	SS2	0.5191(2)	MSS2	0.51899(2)	0.52380(2)	0.53056(2)
7	0.5	2	SS2	0.5193(2)	MSS2	0.51908(2)	0.52390(2)	0.53066(2)
8	0.5	5	SS2	0.5196(2)	MSS2	0.51951(2)	0.52442(2)	0.53114(2)
9	0.5	10	SS2	0.5203(2)	MSS2	0.52029(2)	0.52537(2)	0.53203(2)
10	0.5	30	SS2	0.5203(2)	MSS2	0.51990(2)	0.52602(2)	0.53246(2)
11	0.5	60	SS2	0.4657(2)	MSS2	0.46482(2)	0.47310(2)	0.48331(2)
12	2	1	SS2	0.5070(2)	MSS2	0.50727(2)	0.50764(2)	0.51017(2)
13	2	2	SS2	0.5070(2)	MSS2	0.50714(2)	0.50754(2)	0.51006(2)
14	2	5	SS2	0.5076(2)	MSS2	0.50688(2)	0.50735(2)	0.50988(2)
15	2	10	SS2	0.5075(2)	MSS2	0.50680(2)	0.50738(2)	0.50996(2)
16	2	30	SS2	0.5098(2)	MSS2	0.50904(2)	0.51004(2)	0.51319(2)
17	2	60	SS2	0.5181(2)	MSS2	0.51750(2)	0.51918(2)	0.52432(2)
18	2	80	SS2	0.5415(2)	MSS2	0.53994(2)	0.54301(2)	0.55618(2)
19	0.985	10			MSS4	1.0001(6)		0.84404(9)

Table 6.2 Comparison of results of isotropic conical shells under hydrostatic pressure

Case	L/R_1 [L_s/R_1]	R_1/t	m	α_o°	Previous investigations [41, 52, 54]		Present investigation			
					B.C.	$(p/E) \times 10^m$ membrane [nonlinear] prebuckling	B.C.	$p/E \times 10^m$ membrane prebuckling	$p/E \times 10^m$ linear prebuckling	$p/E \times 10^m$ nonlinear prebuckling
1	1	250	7	5	Type 7	9.024(11)	Type 7	9.0244(11)	8.9863(11)	8.9854(11)
2	1	250	7	10	Type 7	8.254(11)	Type 7	8.2549(11)	8.2167(11)	8.2159(11)
3	1	250	7	20	Type 7	6.578(11)	Type 7	6.5813(11)	6.5468(11)	6.5466(11)
4	1	250	7	30	Type 7	4.781(11)	Type 7	4.7854(11)	4.7591(11)	4.7593(11)
5	1	250	7	45	Type 7	2.351(11)	Type 7	2.3542(11)	2.3410(11)	2.3415(11)
6	1	250	8	60	Type 7	7.507(10)	Type 7	7.5272(10)	7.4846(10)	7.4887(10)
7	2	500	8	5	Type 7	7.372(10)	Type 7	7.3724(10)	7.3709(10)	7.3709(10)
8	2	500	8	10	Type 7	6.383(10)	Type 7	6.3828(10)	6.3812(10)	6.3812(10)
9	2	500	8	20	Type 7	4.696(11)	Type 7	4.6960(11)	4.6947(10)	4.6947(10)
10	2	500	8	30	Type 7	3.217(11)	Type 7	3.2172(11)	3.2161(11)	3.2161(11)
11	2	500	8	45	Type 7	1.451(12)	Type 7	1.4510(12)	1.4504(12)	1.4504(12)
12	2	500	9	60	Type 7	3.940(14)	Type 7	3.9387(14)	3.9352(14)	3.9356(14)
13	[0.5]	100	6	10	SS1	17.70(10)	Type 5	17.1630(10)	16.0213(10)	15.6695(11)
14	[0.5]	100	6	10	SS2	18.98(10)	MSS2	18.4123(10)	18.6747(10)	18.2916(10)
15	[0.5]	100	6	10	SS3	19.40(11)	Type 6	18.8153(11)	17.2162(11)	16.8012(11)
16	[0.5]	100	6	10	SS4	21.65(12)	MSS4	20.9974(12)	20.7093(12)	20.1970(12)
17	2	500	8	10	Type 7	[6.373(10)]	Type 7	6.3829(10)	6.3813(10)	6.3813(10)
18	2	500	8	30	Type 7	[3.202(11)]	Type 7	3.2175(11)	3.2165(11)	3.2165(11)

Remarks:

1. Notice that m is the exponent used in the definition of the eigenvalue.
2. In some cases L_s/R_1 is used instead of L/R_1. These cases are identified by a [ ].

Table 6.3 Comparison of results of isotropic conical shells under combined axial load and hydrostatic pressure (R_1/t = 250, type 7 boundary conditions)

Case	L/R_1	m	λ	α_o°	Previous investigation [161]	Present investigation		
					$(p/E) \times 10^m$	$p/E \times 10^m$		
					membrane prebuckling	membrane prebuckling	linear prebuckling	nonlinear prebuckling
1	0.5	6	0.0	5	1.950(15)	1.9497(15)	1.8960(15)	1.8900(15)
2	0.5	6	0.2	5	1.605(14)	1.6044(14)	1.5742(14)	1.5697(14)
3	0.5	6	0.4	5	1.253(13)	1.2539(13)	1.2554(13)	1.2554(13)
4	0.5	7	0.6	5	8.559(13)	8.5642(13)	8.9519(13)	8.6048(13)
5	0.5	6	-0.2	5	2.258(15)	2.2565(15)	2.1775(15)	2.1857(15)
6	0.5	6	-0.4	5	2.556(16)	2.5556(16)	2.4640(16)	2.5057(16)
7	0.5	6	-0.6	5	2.828(16)	2.8280(16)	2.7165(16)	2.8098(16)
8	0.5	6	0.0	30	1.160(14)	1.1612(14)	1.1254(14)	1.1223(14)
9	0.5	7	0.2	30	9.600(14)	9.6144(14)	9.4292(14)	9.4009(14)
10	0.5	7	0.4	30	7.444(13)	7.4683(13)	7.4690(13)	7.4755(13)
11	0.5	7	0.6	30	5.156(13)	5.1865(13)	5.3928(13)	5.2159(13)
12	0.5	6	-0.2	30	1.347(15)	1.3470(15)	1.3007(15)	1.3055(15)
13	0.5	6	-0.4	30	1.523(15)	1.5233(15)	1.4635(15)	1.4871(15)
14	0.5	6	-0.6	30	1.686(16)	1.6854(16)	1.6201(16)	1.6765(16)
15	1.0	7	-0.2	30	5.526(12)	5.5242(12)	5.4828(12)	5.4884(12)
16	1.0	7	0.2	30	3.932(11)	3.9408(11)	3.9342(11)	3.9367(11)
17	1.0	7	0.2	5	7.528(10)	7.5350(10)	7.5278(10)	7.5282(10)
18	1.0	6	-0.2	5	1.051(11)	1.0515(11)	1.0444(11)	1.0453(11)

Orthotropic conical shells

The cases shown in Table 6.4 are for simply supported (type 6) classical orthotropic conical shells with various geometric properties under hydrostatic pressure. These cases were previously treated by Singer et al.[79] using a Donnell-type theory and membrane-like prebuckling solution. The results obtained from present analysis are about 15%, 10% and 17% lower for cases 1-3 respectively than those obtained by Singer et al.. The higher buckling loads of Singer et al. are probably due to using too few terms in their series type solutions.

Table 6.4 Comparison of results of classical orthotropic conical shells under hydrostatic pressure

Case	l_1/R_1	R_1/t	α_o°	Previous investigation [79]		Present investigation			
				B.C.	$(p/E_s) \times 10^6$ membrane prebuckling	B.C.	$(p/E_s) \times 10^6$ membrane prebuckling	linear prebuckling	nonlinear prebuckling
1	1	287.95	30	SS3	0.2041	Type 6	0.17232(12)	0.16986(12)	0.16984(12)
2	3	287.95	30	SS3	0.0418	Type 6	0.03741(11)	0.03737(11)	0.03737(11)
3	6.18	168.90	60	SS3	0.0122	Type 6	0.01003(13)	0.00996(13)	0.00996(13)

Based on converging polynomial series, which can be used to fulfill the boundary conditions, Tong et al.[93] presented some results for buckling of classical orthotropic conical shells under axial compression with SS3 (type 6) boundary conditions. According to their results shown in Figure 5 of Ref. [93] it appears that they considered only axisymmetric buckling modes, which would be incorrect if one wants to find the minimum buckling loads. Hence, it was decided to investigate this case by using the present theory. In the present investigation membrane-like prebuckling solution is employed, and the results of comparisons are shown in Figure 6.3.

Notice that in Figure 6.3 the notations used are

$$\lambda_{cr} = P_{cr}/P_{cl} \tag{6.48}$$

where P_{cr} is the critical buckling load obtained by the present calculation, and

$$P_{cl} = P_{cyl}\cos^2\alpha_o = 2Et^2\pi\cos^2\alpha_o / \sqrt{3(1-\nu^2)} \tag{6.49}$$

The classical load given by Eq. (6.49) represents the critical buckling load of an isotropic conical shell obtained by Seide[32]. In Eq. (6.49), P_{cyl} is the classical buckling load for a cylindrical shell, t is wall thickness, α_o is the semi-vertex angle of the conical shell, E is the Young's modulus and ν is the Poisson's ratio.

For classical orthotropic conical shells with SS3 boundary conditions, P_{cl} is calculated from Eq. (6.49) with ν replaced by $\nu_{s\theta}$ and E replaced by E_θ (not by E_s as mistakenly stated by Tong et al.). The results given in Figure 6.3 are for $\alpha_o = 10°$. However, calculations for other conical shells with $\alpha_o = 30°$ and $\alpha_o = 45°$ gave results very close to those obtained for $\alpha_o = 10°$.

Using either the BAAC1 or the BAAC2 program, the critical buckling load was calculated for two possible mode shapes: axisymmetric mode for which the number of circumferential waves are zero and asymmetric mode for which the buckling load was minimized with respect to the number of circumferential waves ($n_1 \geq 4$). As one can see from Figure 6.3, for orthotropicity factor E_s/E_θ close to one the critical buckling load obtained corresponds to an axisymmetric mode shape. However, for E_s/E_θ about 12 and higher the mode shape associated with the critical buckling load becomes asymmetric (with number of circumferential wave equal to eight or nine) and this critical buckling load is much lower than the one obtained for the axisymmetric mode shape. From Figure 6.3 it appears that the buckling load calculated by Tong et al. using polynomial series, follows closely the axisymmetric branch (see also Figure 5 of Tong et al.), and hence when E_s/E_θ is greater than 12 their results are much higher than the ones obtained by the present method.

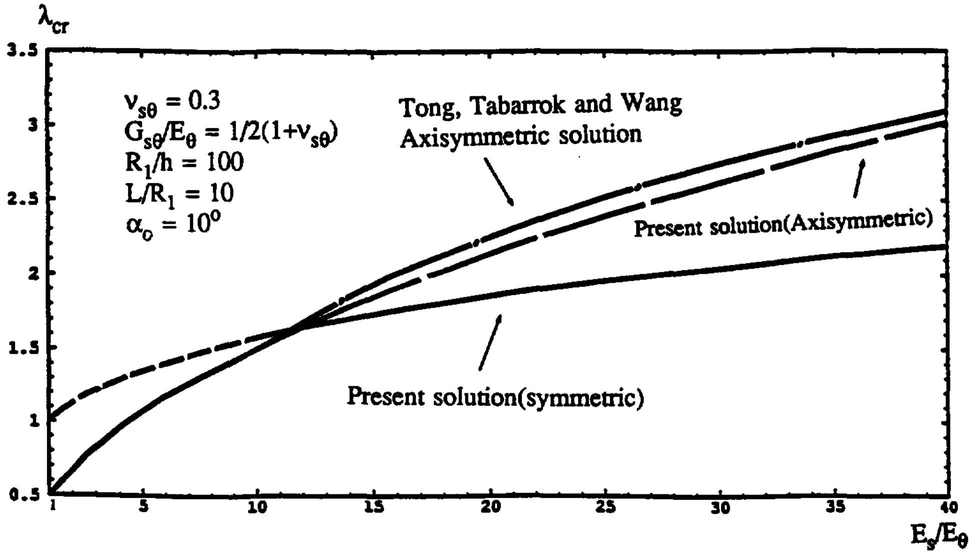


Fig. 6.3 Influence of E_s/E_θ on the ratio λ_{cr} for SS3 boundary conditions

Notice that the difference between buckling loads also exists when E_s / E_θ = 1, which represents the cases of isotropic conical shells. The lower buckling load given by Ref. [93] is due to the fact that a rigid body translation is possible for isotropic conical shells with classical SS3 boundary conditions under axial compression. As it has been explained earlier in chapter 4, for the case of isotropic conical shells under axial compression the classical SS3 boundary conditions are not an appropriate representation of a practical situation because they permit a rigid body translation. As it was concluded, the modified MSS3 boundary conditions given by the present theory are the one to be used.

Anisotropic conical shells

All the cases shown in Table 6.5 are for simply supported (SS4 boundary conditions) regular antisymmetric cross-plied conical shells with various geometric properties and different numbers of orthotropic laminae under axial compression. These cases were previously treated by Tong et al.[175] using Donnell-type theory and membrane-like prebuckling solution. For cases 1-12 the results of present analysis are about 9% lower than those obtained by Tong et al.. For cases 1, 4, 7 and 9 there are differences in the circumferential wave numbers. The higher buckling loads of Ref. [175] are probably due to using SS4 boundary conditions instead of MSS4 in their solutions. For cases 13-15 the conical shells become longer, the influence of the boundary conditions is less important, thus the present results are in fair agreement with those obtained by Tong et al..

The cases shown in Table 6.6 are for simply supported (type 6 and MSS3 boundary conditions) regular antisymmetric cross-plied conical shells with various geometric properties and different numbers of laminae under axial compression. These cases are also compared with the results of Tong et al.[175]. As can be seen, the results of present analysis are in fair agreement with those obtained by Tong et al.. However, there are some differences in the circumferential wave numbers for the cases 7, 9 and 13.

There are no results published for conical shells with general anisotropic properties, as far as the author knows. Thus in the following comparisons are made with anisotropic cylindrical shells. The results of corresponding anisotropic cylindrical shells were previously calculated by Jansen[107] using Donnell-type theory. The anisotropic shells considered are built by three layer glass-epoxy laminae with (30°, 0°, -30°) stacking sequence.

All the results of present analysis in Tables 6.7 and 6.8 are for very steep (α_o = 1°) conical shells with different L/R_1 ratios and boundary conditions under axial compression and hydrostatic pressure, respectively. The material data for these steep anisotropic conical shells are assumed to be the same as those of corresponding anisotropic cylindrical shells. As can be seen, all the results of present analysis are in excellent agreement with those obtained by Jansen[107].

To summarize briefly, the results shown in Tables 6.5-6.8 indicate that based on the comparisons with the results of other investigations the present analysis and computer programs can provide accurate information for the instability of anisotropic conical shells subjected to axial compression and/or hydrostatic pressure.

Table 6.5 Comparison of results of regular antisymmetric cross-plied conical shells under axial compression (R_1/t = 100)

Case	L	N	α_o°	Previous investigation [175]		Present investigation		
				B.C.	λ	B.C.	λ	
					membrane prebuckling		membrane prebuckling	nonlinear prebuckling
1	25.738	2	10	SS4	0.07926(9)	MSS4	0.073895(7)	0.072646(7)
2	25.738	4	10	SS4	0.1101(6)	MSS4	0.10244(6)	0.10504(6)
3	25.738	6	10	SS4	0.1119(6)	MSS4	0.10690(6)	0.10850(6)
4	24.744	2	30	SS4	0.08389(9)	MSS4	0.073814(7)	0.072672(7)
5	24.744	4	30	SS4	0.1106(6)	MSS4	0.10282(6)	0.10516(6)
6	24.744	6	30	SS4	0.1128(6)	MSS4	0.10692(6)	0.10868(6)
7	21.474	2	45	SS4	0.08703(6)	MSS4	0.074382(7)	0.073680(7)
8	21.474	4	45	SS4	0.1150(6)	MSS4	0.10339(6)	0.10682(6)
9	21.474	6	45	SS4	0.1173(5)	MSS4	0.10796(6)	0.11046(6)
10	15.954	2	60	SS4	0.0938(6)	MSS4	0.077978(6)	0.078941(6)
11	15.954	4	60	SS4	0.1226(5)	MSS4	0.10896(5)	0.11385(5)
12	15.954	6	60	SS4	0.1248(5)	MSS4	0.11288(5)	0.11619(5)
13	59.922	2	10	SS4	0.0699(7)	MSS4	0.070477(7)	0.070484(7)
14	59.922	4	10	SS4	0.1021(6)	MSS4	0.10294(6)	0.10356(6)
15	59.922	6	10	SS4	0.1070(6)	MSS4	0.10751(6)	0.10802(6)

where N is the total number of layers.

Table 6.6 Comparison of results of regular antisymmetric cross-plied conical shells under axial compression (R_1/t = 100)

Case	L	N	α_o°	Previous investigation [175]		Present investigation		
				B.C.	λ	B.C.	λ	
					membrane prebuckling		membrane prebuckling	nonlinear prebuckling
1	25.738	2	10	SS3	0.06742(7)	Type 6	0.068083(7)	0.068981(7)
2	25.738	4	10	SS3	0.1016(6)	Type 6	0.10198(6)	0.10392(6)
3	25.738	6	10	SS3	0.1066(6)	Type 6	0.10679(6)	0.10833(6)
4	24.744	2	30	SS3	0.06726(7)	Type 6	0.067599(7)	0.068962(7)
5	24.744	4	30	SS3	0.1018(6)	Type 6	0.10176(6)	0.10390(6)
6	24.744	6	30	SS3	0.1075(6)	Type 6	0.10680(6)	0.10847(6)
7	21.474	2	45	SS3	0.06751(6)	Type 6	0.066618(7)	0.068663(7)
8	21.474	4	45	SS3	0.1054(6)	Type 6	0.10187(6)	0.10491(6)
9	21.474	6	45	SS3	0.1117(5)	Type 6	0.10769(6)	0.11003(6)
10	15.954	2	60	SS3	0.06849(6)	Type 6	0.066248(6)	0.069824(6)
11	15.954	4	60	SS3	0.1111(5)	Type 6	0.10648(5)	0.11027(5)
12	15.954	6	60	SS3	0.1185(5)	Type 6	0.11249(5)	0.11531(5)
13	59.922	2	10	SS3	0.06986(9)	MSS3	0.069347(7)	0.069863(7)
14	59.922	4	10	SS3	0.1017(6)	MSS3	0.10290(6)	0.10350(6)
15	59.922	6	10	SS3	0.1067(6)	MSS3	0.10750(6)	0.10800(6)

where N is the total number of layers.

Table 6.7 Comparison of results with anisotropic cylindrical shells under axial compression (t = 0.0267 in., R_1/t = 100)

Case	L/R_1	Previous investigation [107]			Present investigation		
		B.C.	λ (α_o = 0°)		B.C.	λ (α_o = 1°)	
			membrane prebuckling	nonlinear prebuckling		membrane prebuckling	nonlinear prebuckling
1	0.70711	SS3	0.42465(6)	0.37096(8)	MSS3	0.42475(6)	0.37079(8)
2	0.70711	SS4		0.39349(8)	MSS4		0.39331(8)
3	0.70711	C1		0.41815(7)	MC1		0.41872(7)

Table 6.7 continuation

4	0.70711	C2		0.43689(8)	MC2		0.43736(8)
5	0.70711	C3		0.41993(8)	MC3		0.42047(8)
6	0.70711	C4	0.46257(7)	0.43835(8)	MC4	0.46286(7)	0.43890(8)
7	0.70711	SS3	0.40926(6)	0.39303(7)	MSS3	0.40931(6)	0.39316(7)
8	2.0	SS4		0.40556(8)	MSS4		0.40681(8)
9	2.0	C1		0.40790(6)	MC1		0.40809(6)
10	2.0	C2		0.41178(6)	MC2		0.41185(6)
11	2.0	C3		0.40865(6)	MC3		0.40879(6)
12	2.0	C4	0.41224(6)	0.41194(6)	MC4	0.41233(6)	0.41202(6)

Table 6.8 Comparison of results with anisotropic cylindrical shells under hydrostatic pressure (t = 0.0267 in., R_1/t = 100)

Case	L/R_1	Previous investigation [107]			Present investigation		
		B.C.	$\bar{p}$ ($\alpha_0 = 0°$)		B.C.	$\bar{p}$ ($\alpha_0 = 1°$)	
			membrane prebuckling	nonlinear prebuckling		membrane prebuckling	nonlinear prebuckling
1	0.70711	SS1		0.10382(9)	MSS1		0.10288(9)
2	0.70711	SS2		0.12367(10)	MSS2		0.12245(10)
3	0.70711	SS3	0.11710(10)	0.10985(10)	MSS3	0.11611(10)	0.10882(10)
4	0.70711	SS4		0.13360(11)	MSS4		0.13227(11)
5	0.70711	C1		0.12933(10)	MC1		0.12824(10)
6	0.70711	C2		0.14739(11)	MC2		0.14603(11)
7	0.70711	C3		0.13019(10)	MC3		0.12914(10)
8	0.70711	C4	0.15341(11)	0.14984(11)	MC4	0.15214(11)	0.14853(11)
9	2.0	SS1		0.03646(6)	MSS1		0.035718(6)
10	2.0	SS2		0.04831(7)	MSS2		0.047153(7)
11	2.0	SS3	0.03808(6)	0.03795(6)	MSS3	0.037494(6)	0.037352(6)
12	2.0	SS4		0.05046(7)	MSS4		0.049477(7)
13	2.0	C1		0.04036(7)	MC1		0.039073(7)
14	2.0	C2		0.05141(7)	MC2		0.050406(7)
15	2.0	C3		0.04042(7)	MC3		0.039124(7)
16	2.0	C4	0.05146(7)	0.05188(7)	MC4	0.050501(7)	0.050934(7)

6.4.2 Effect of different prebuckling solutions

Most of the available results for the stability analysis of conical shells are based on the use of membrane-like prebuckling solutions. It is well known[45,56,67] that for cylindrical shells using membrane prebuckling solutions implies that one relaxes completely the supports in the prebuckling range and thus assumes that the prebuckling stresses and deformations are constant. In most practical cylindrical shell structures, however, some measure of radial support is present from the beginning of loading so that, prior to buckling, complicated axisymmetric deformations and stresses are present to modify the loading-shortening behavior of the cylindrical shells and to influence their buckling load. This influence is especially important for cylindrical shells in axial compression and for short cylindrical shells under hydrostatic pressure.

Whereas the effect of prebuckling deformations on the buckling load of cylindrical shells has been investigated extensively, few results have been published for conical shells, and practically none for anisotropic conical shells. Therefore, in the following, the effect of different prebuckling solutions on the buckling behavior of various conical shells under axial compression and hydrostatic pressure will be studied for a wide range of geometries. The emphasis and most of the numerical calculations, however, relate to anisotropic conical shells with short or moderate length.

Isotropic conical shells

First, the effect of different prebuckling solutions on the buckling behavior of isotropic conical shells under axial compression is shown in Table 6.1. As can be seen, for short conical shells ($l_s/R_1 = 0.5$) with MSS4 boundary conditions the buckling loads calculated using rigorous prebuckling solutions are about 10% lower than those calculated with membrane-like prebuckling solutions. However, for short conical shells ($l_s/R_1 = 0.5$) with MSS2 boundary conditions the buckling loads calculated using rigorous prebuckling solutions are about 2% higher than those calculated with membrane-like prebuckling solutions. This result indicates that using membrane-like prebuckling solutions does not always guarantee that the resulting buckling loads are on the safety side (upper bound). This phenomena was also reported by Almroth[49] for isotropic cylindrical shells. Nevertheless, as the shells become longer ($l_s/R_1 = 2$), the difference caused by using different prebuckling solutions will decrease.

Considering the results in Table 6.1, one observes that using linear prebuckling solutions for isotropic conical shells with MSS4 boundary conditions will always result in higher buckling loads than when membrane-like or rigorous prebuckling solutions are employed. Therefore, as in the case of using membrane-like prebuckling solution, using linear prebuckling solutions will not yield conservative buckling loads. Besides, for the case of short conical shells (L/R_1 = 0.5) with MSS4 boundary conditions there exist some closely-spaced eigenvalues with different eigenmodes if the rigorous prebuckling solutions are employed. The buckling modes of case 19 are shown in Figures 6.4 and 6.5 for membrane-like and rigorous prebuckling solutions, respectively. Notice the distinct changes that occur.

The effect of different prebuckling solutions on the buckling behavior of isotropic conical shells under hydrostatic pressure is shown in Table 6.2. As can be seen, for short conical shells (L/R_1 = 0.5) with MSS4 boundary conditions the buckling load calculated using rigorous prebuckling analysis is about 4% lower than those with membrane-like prebuckling solutions, while for types 5 or 6 (SS1 or SS3) boundary conditions the differences are about 10%. For conical shells with L/R_1 = 1 the differences are less than 1%, and shells with L/R_1 = 2 the differences are negligible. The buckling modes of case 1 are shown in Figures 6.6 and 6.7 for membrane-like and rigorous prebuckling solutions, respectively.

The effect of different prebuckling solutions on the buckling behavior of isotropic conical shells under combined axial loads and hydrostatic pressure is shown in Table 6.3. As can be seen, for short conical shells (L/R_1 = 0.5) with type 7 boundary conditions the differences are about 1%, while for shells with moderate length (L/R_1 = 1) the differences are negligible. The buckling modes of cases 17 and 18 are shown in Figures 6.8 and 6.9, respectively. It is found that if the given axial compression (or tension) does not dominate, the resulting buckling modes are similar with those of corresponding conical shells under pure hydrostatic pressure.

To summarize briefly, the results shown in Tables 6.1-6.3 indicate that in order to obtain reliable results for short isotropic conical shells it is essential to employ the rigorous prebuckling solutions in the buckling analysis. For conical shells with moderate length the use of rigorous prebuckling solutions is also recommended since sometimes the accuracy of using membrane-like or linear prebuckling solutions is questionable.

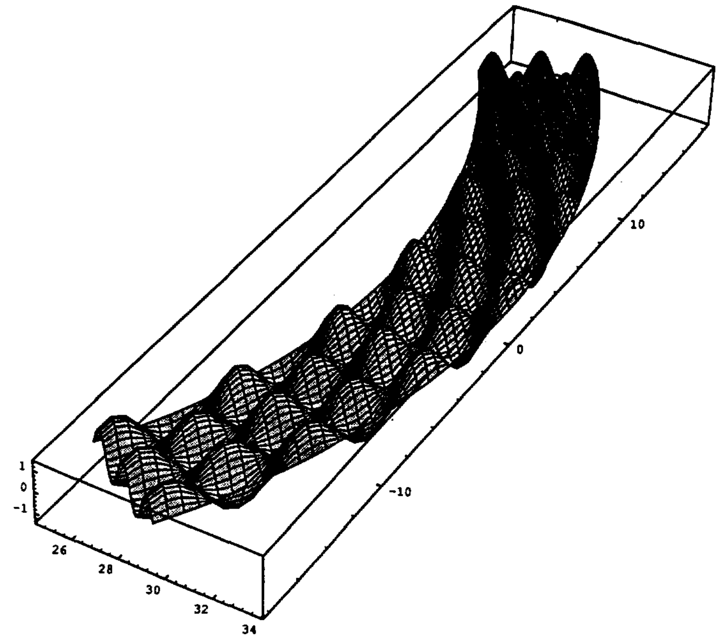


Fig. 6.4 *Buckling mode of isotropic conical shell under axial compression (α_o = 10°; MSS4 boundary conditions; membrane-like prebuckling solution)*

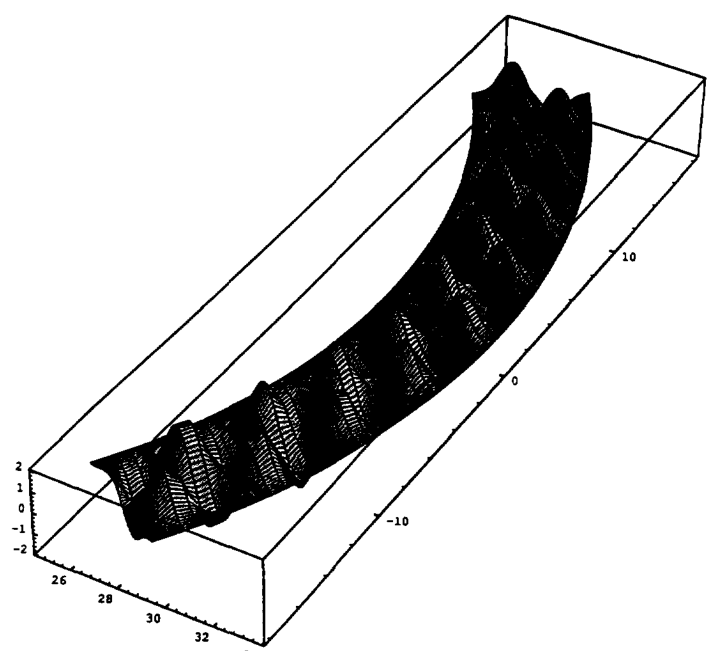


Fig. 6.5 *Buckling mode of isotropic conical shell under axial compression (α_o = 10°; MSS4 boundary conditions; nonlinear prebuckling solution)*

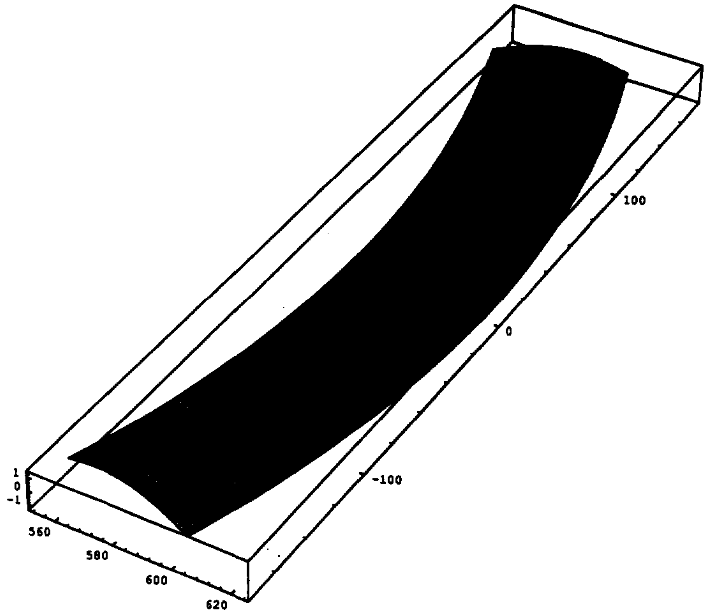


Fig. 6.6 *Buckling mode of isotropic conical shell under hydrostatic pressure ($\alpha_o = 5^o$; SS3 boundary conditions; membrane-like prebuckling solution)*

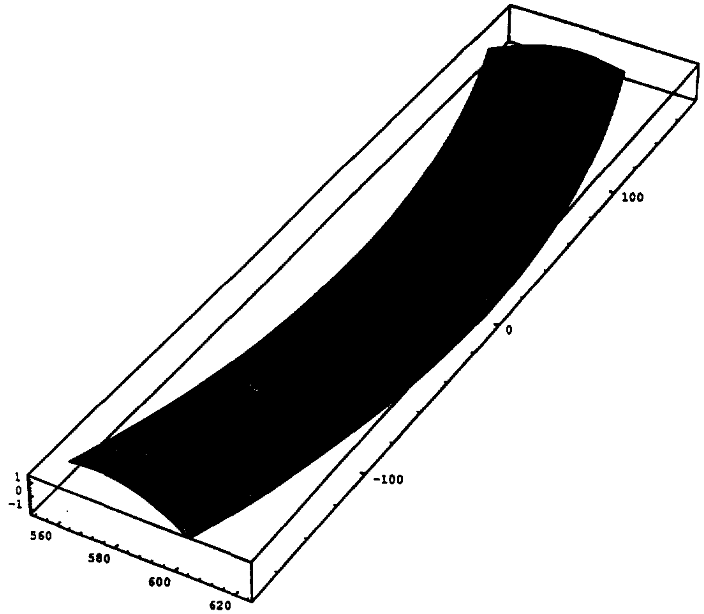


Fig. 6.7 *Buckling mode of isotropic conical shell under hydrostatic pressure ($\alpha_o = 5^o$; SS3 boundary conditions; nonlinear prebuckling solution)*

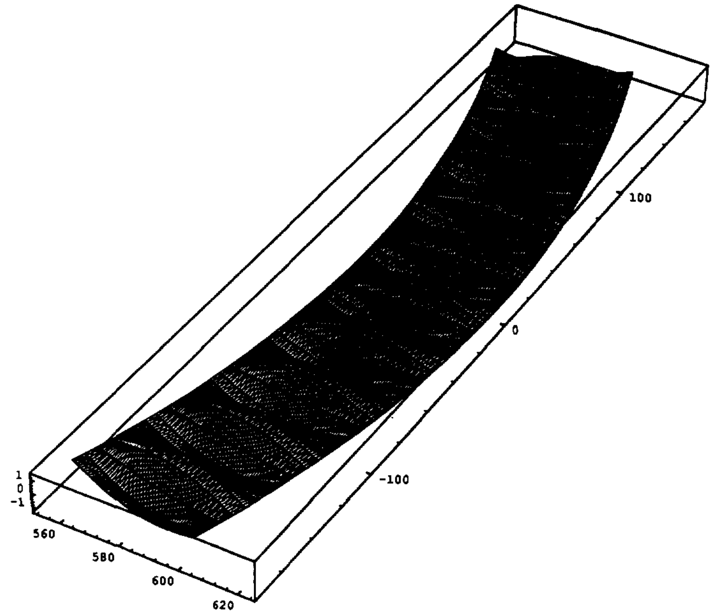


Fig. 6.8 *Buckling mode of isotropic conical shell under combined axial compression and hydrostatic pressure (α_o = 5°; λ = 0.2; SS3 boundary conditions; membrane-like prebuckling solution)*

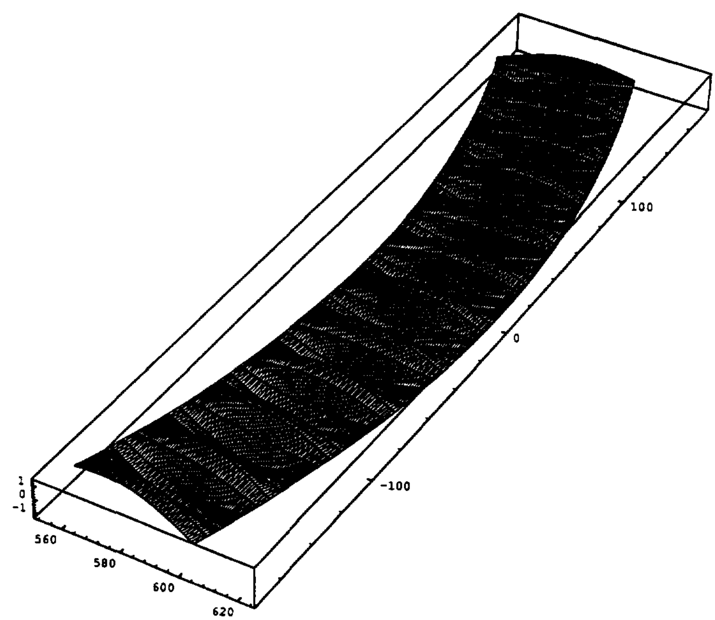


Fig. 6.9 *Buckling mode of isotropic conical shell under combined axial tension and hydrostatic pressure (α_o = 5°; λ = -0.2; SS3 boundary conditions; membrane-like prebuckling solution)*

Classical orthotropic conical shells

The effect of different prebuckling solutions on the buckling behavior of classical orthotropic conical shells under hydrostatic pressure is shown in Table 6.4. As can be seen, the differences between the buckling loads obtained by using different prebuckling solutions become smaller as the shells become longer.

Anisotropic conical shells

As mentioned earlier, most of the published results for buckling analysis of isotropic and classical orthotropic conical shells were based on the use of membrane-like prebuckling solutions. Nevertheless, there are also a few published papers[52,57,58,89] using the linear or nonlinear prebuckling solutions. However, it seems that there are no results published for the influence of different prebuckling solutions on the buckling behavior of anisotropic conical shells.

The effect of different prebuckling solutions on the buckling behavior of anisotropic conical shells made by anti-symmetric cross-ply laminates under axial compression is shown in Tables 6.5 and 6.6 for boundary conditions of MSS3, MSS4 and type 6, respectively. As can be seen, for MSS3 boundary conditions the buckling loads calculated using the rigorous prebuckling solutions are always higher than those with membrane-like prebuckling solutions. For relatively short shells the difference is about 5%, while for longer shells the difference becomes negligible. The buckling modes for case 14 of Table 6.6 are shown in Figures 6.10 and 6.11 for membrane-like and rigorous prebuckling solutions, respectively. The buckling mode for case 13 of Table 6.5 is shown in Figure 6.12.

The effect of different prebuckling solutions on the buckling behavior of anisotropic conical shells with very slight conicity (α_o = 1.0°) is shown in Table 6.7 for axial compression and Table 6.8 for hydrostatic pressure, respectively. As expected, the results of the anisotropic conical shells with very slight conicity are very similar to those of corresponding anisotropic cylindrical shells. As can be found from Table 6.7, the effect of rigorous prebuckling solutions is important for the short conical shells (L/R_1 = 0.70711) under axial compression. The buckling loads are lowered by 13% for MSS3 boundary conditions and by 5% for MC4 boundary conditions, as compared with the results obtained by using membrane-like prebuckling solutions. As the shells become longer (L/R_1 = 2), the decreases are less pronounced (4% for MSS3 and negligible for MC4 boundary conditions). As can be seen from Table 6.8, for short shells (L/R_1 = 0.70711) under hydrostatic pressure, using nonlinear prebuckling solution the buckling loads are lowered by 6% and 2% for MSS3 and MC4 boundary conditions, respectively, as compared with the buckling loads obtained by using membrane-like prebuckling solutions. As the shells become longer (L/R_1 = 2), the effect of rigorous prebuckling solution is less noticeable (0.4% decrease for MSS3 boundary conditions and 1% increase for MC4 boundary conditions). The buckling modes of anisotropic conical shell (L/R_1 = 2, α_o = 4°, R_1/t = 100) with MSS3 boundary conditions under axial compression are shown in Figures 6.13 and 6.14 for membrane-like and rigorous prebuckling solutions, respectively.

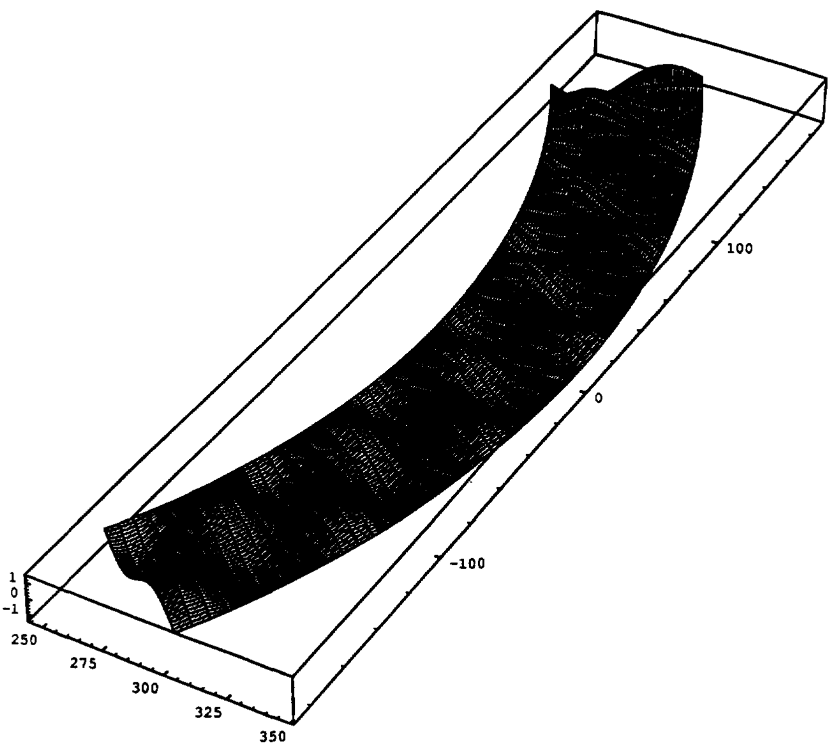


Fig. 6.10 *Buckling mode of antisymmetric cross-ply conical shell under axial compression (α_o = 10°; MSS3 boundary conditions; membrane-like prebuckling solution; four layers)*

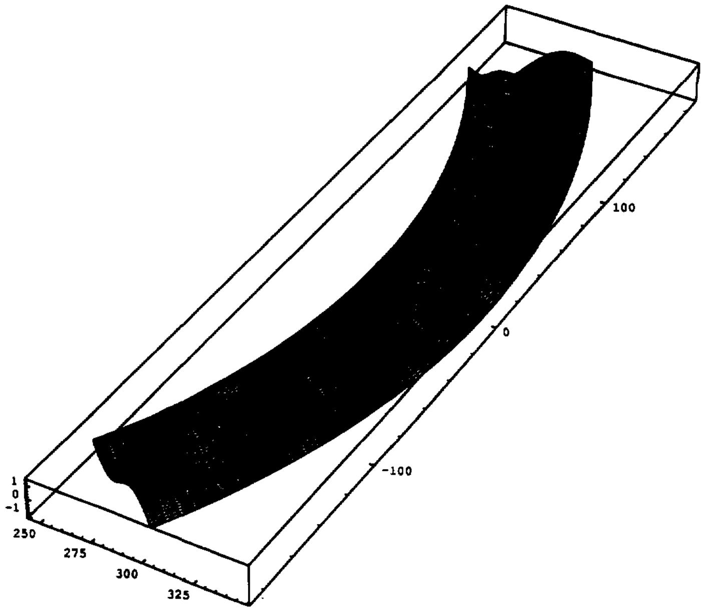


Fig. 6.11 *Buckling mode of antisymmetric cross-ply conical shell under axial compression (α_o = 10^o; MSS3 boundary conditions; nonlinear prebuckling solution; four layers)*

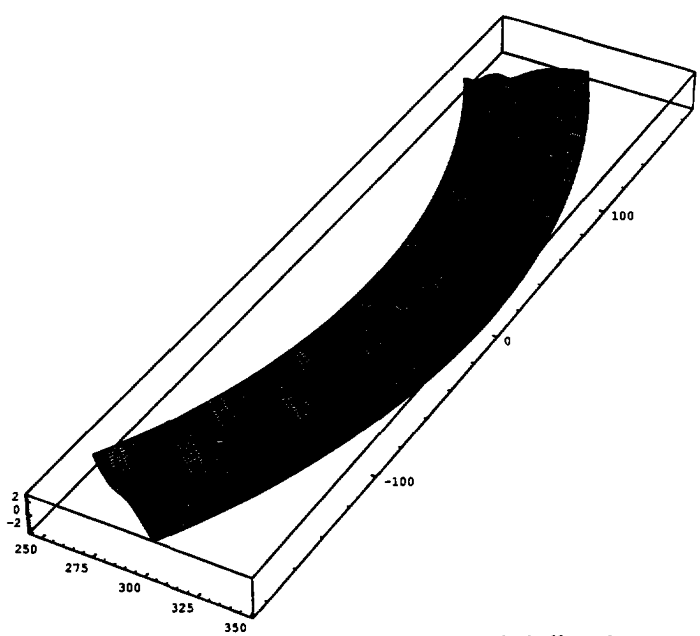


Fig. 6.12 *Buckling mode of antisymmetric cross-ply conical shell under axial compression (α_o = 10^o; MSS4 boundary conditions; membrane-like prebuckling solution; two layers)*

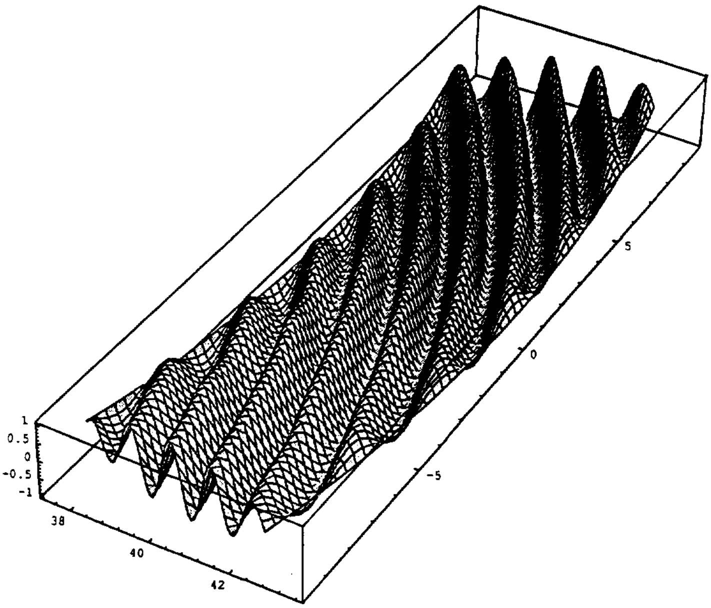


Fig. 6.13 *Buckling mode of anisotropic conical shell under axial compression (α_o = 4°; MSS3 boundary conditions; membrane-like prebuckling solution)*

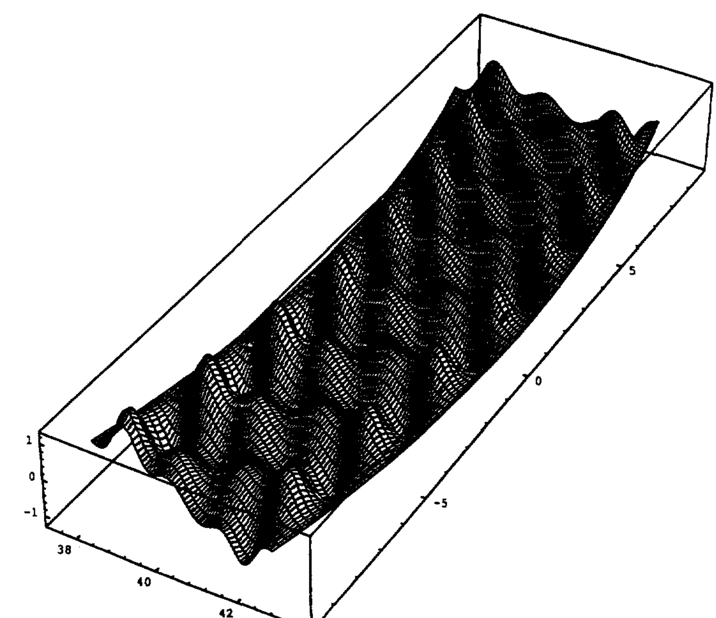


Fig. 6.14 *Buckling mode of anisotropic conical shell under axial compression (α_o = 4°; MSS3 boundary conditions; nonlinear prebuckling solution)*

The effect of different prebuckling solutions on the buckling behavior of anisotropic conical shells is also shown in Table 6.9 for axial compression and Table 6.10 for hydrostatic pressure, respectively. Notice that if the anisotropic conical shells are made by geodetic filament winding, the layer thicknesses and the fiber orientations used are the equivalent constant thicknesses and winding angles which can be calculated according to the formulae given in chapter 2.

As can be seen from Table 6.9, for short conical shells (L/R_1 = 0.70711) with different semi-vertex angle (α_o = 10°, 30°, 45°) the buckling loads are about 5% lower for MC4 boundary conditions and 8% lower for MSS4 boundary conditions, as compared with those obtained by using membrane-like prebuckling solutions. The semi-vertex angle α_o has no important influence on the normalized buckling loads. As the shells become longer (L/R_1 = 2), the differences become smaller (about 1% and 3% lower for MC4 and MSS4 boundary conditions, respectively).

The buckling modes of case 8 in Table 6.9 are shown in Figure 6.15 for membrane-like prebuckling solution and in Figure 6.16 for rigorous prebuckling solution. As can be seen the two modes do not differ much. Figure 6.17 shows the skewed pattern of Figure 6.15. The buckling modes of case 13 in Table 6.9 are shown in Figures 6.18 and 6.19 for membrane-like and rigorous prebuckling solutions, respectively. As can be seen the prebuckling solutions have important influence on the corresponding buckling modes. Using the same circumferential wave number (n_1 = 6) as that of Figure 6.18 and the nonlinear prebuckling solution yields the buckling mode shown in Figure 6.20. As can be seen, Figure 6.20 is similar to Figure 6.19 since for both cases nonlinear prebuckling solutions are employed. The skewed pattern of Figure 6.19 is shown again in Figure 6.21.

As can be seen from Table 6.10, for short conical shells (L/R_1 = 0.70711) with different semi-vertex angles (α_o = 10°, 30°, 45°) the buckling loads using nonlinear prebuckling solutions are about 2.5% and 7% lower for MC4 and MSS3 boundary conditions, respectively, as compared with those obtained from membrane-like prebuckling solutions. As the shells become longer (L/R_1 = 2), the buckling loads are about 1% lower for MSS3 boundary conditions and 1% higher for MC4 boundary conditions if one uses nonlinear prebuckling solutions. The buckling modes of cases 13 and 14 of Table 6.10 are shown in Figures 6.22 and 6.23, respectively. Observation reveals that if the hydrostatic pressure dominates, the buckling modes exhibit very little skewness.

To summarize briefly, the effect of different prebuckling solutions on the buckling behavior is studied for conical shells covering a wide range of geometries with different boundary conditions under different loadings. For some relatively long shells the use of membrane-like prebuckling solutions can often yield satisfactory results. For short conical shells and shells with moderate length membrane-like prebuckling solutions must be used with great cautions, since as is the case for cylindrical shells, the critical buckling loads may be drastically over or under estimated. In these cases, the prebuckling nonlinearity must be taken into account if the critical buckling loads are to be determined with accuracy.

Table 6.9 *Comparison of buckling load λ of anisotropic conical shells using membrane-like and nonlinear prebuckling solutions ($R_1/t=100$, $t=0.0267$ in.)*

Case	L/R_1	α_o°	B.C.	λ	
				membrane prebuckling	nonlinear prebuckling
1	0.70711	10	MC4	0.46491(7)	0.44281(8)
2	0.70711	30	MC4	0.46469(7)	0.44192(8)
3	0.70711	45	MC4	0.46085(7)	0.43364(8)
4	0.70711	10	MSS4	0.42648(6)	0.39163(9)
5	0.70711	30	MSS4	0.42622(6)	0.39149(9)
6	0.70711	45	MSS4	0.42426(6)	0.39433(8)
7	2	10	MC4	0.41447(7)	0.41333(7)
8	2	30	MC4	0.41510(7)	0.41432(7)
9	2	45	MC4	0.41558(7)	0.41421(7)
10	2	10	MSS4	0.41005(6)	0.40246(8)
11	2	30	MSS4	0.41134(7)	0.39970(8)
12	2	45	MSS4	0.41159(6)	0.39689(7)
13	2	30	MSS3	0.41122(6)	0.39104(7)

Table 6.10 *Comparison of buckling load p of anisotropic conical shells using membrane-like and nonlinear prebuckling solutions ($R_1/t=100$, $t=0.0267$ in.)*

Case	L/R_1	α_o°	B.C.	$\bar{p}$	
				membrane prebuckling	nonlinear prebuckling
1	0.70711	10	MSS3	0.10724(10)	0.099758(10)
2	0.70711	30	MSS3	0.08394(10)	0.077603(10)
3	0.70711	45	MSS3	0.06357(10)	0.058968(10)
4	0.70711	10	MC4	0.14048(11)	0.13673(11)
5	0.70711	30	MC4	0.10943(12)	0.10675(12)
6	0.70711	45	MC4	0.082546(11)	0.080966(11)
7	2	10	MSS3	0.029509(7)	0.029383(7)
8	2	30	MSS3	0.018963(8)	0.018830(8)
9	2	45	MSS3	0.012681(9)	0.012532(9)
10	2	10	MC4	0.039995(8)	0.040347(8)
11	2	30	MC4	0.027734(8)	0.028264(8)
12	2	45	MC4	0.019089(8)	0.019603(8)
13	2	5	MSS3	0.033137(7)	0.033024(7)
14	2	5	MC4	0.044850(8)	0.045148(8)

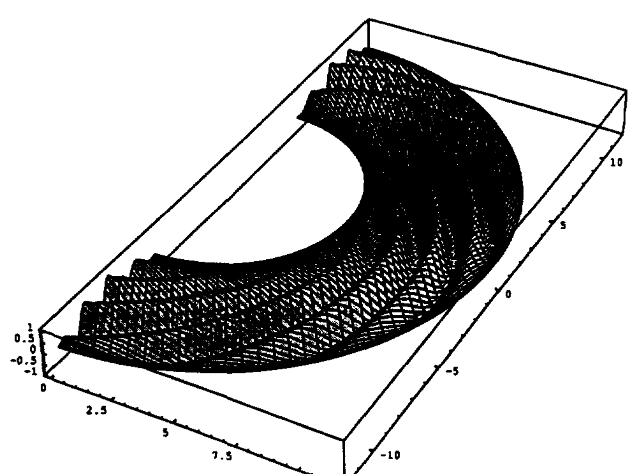


Fig. 6.15 *Buckling mode of anisotropic conical shell under axial compression ($\alpha_o = 30°$; MC4 boundary conditions; membrane-like prebuckling solution)*

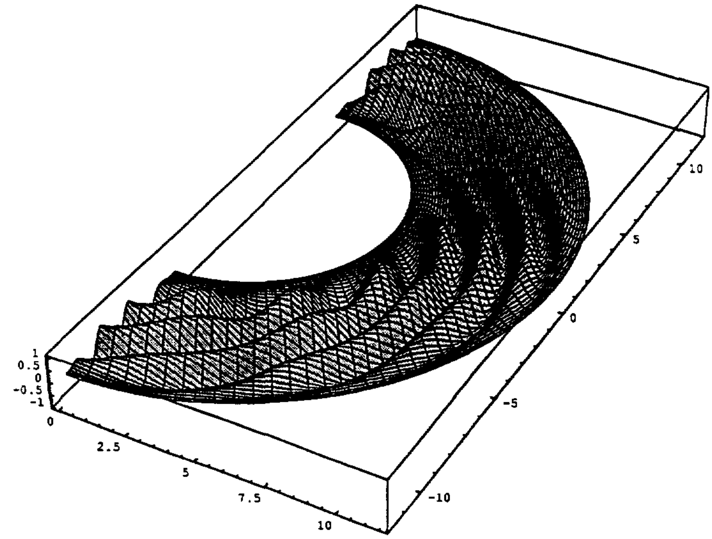


Fig. 6.16 *Buckling mode of anisotropic conical shell under axial compression (α_o = 30°; MC4 boundary conditions; nonlinear prebuckling solution)*

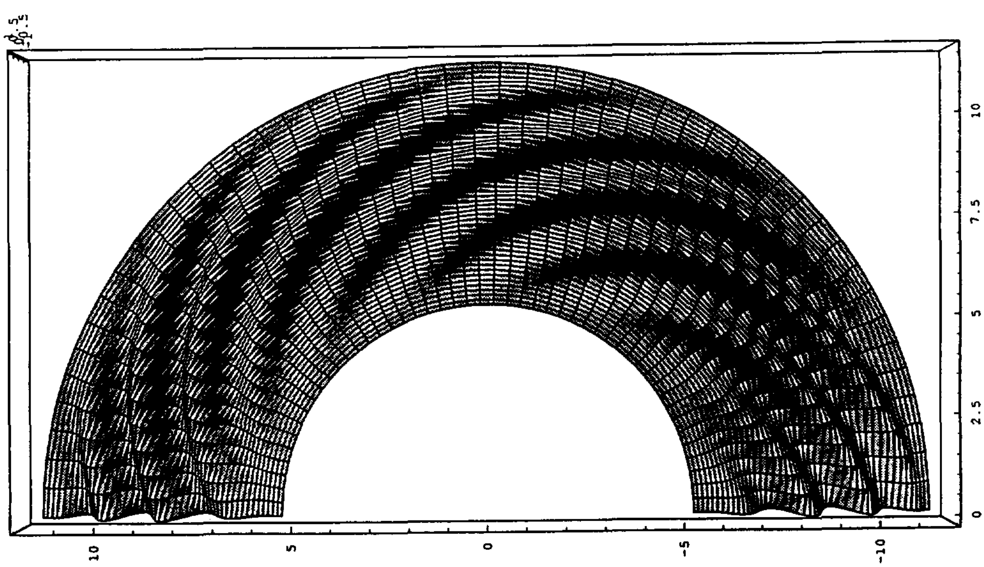


Fig. 6.17 *Another view of the buckling mode of anisotropic conical shell under axial compression (α_o = 30°; MC4 boundary conditions; membrane-like prebuckling solution)*

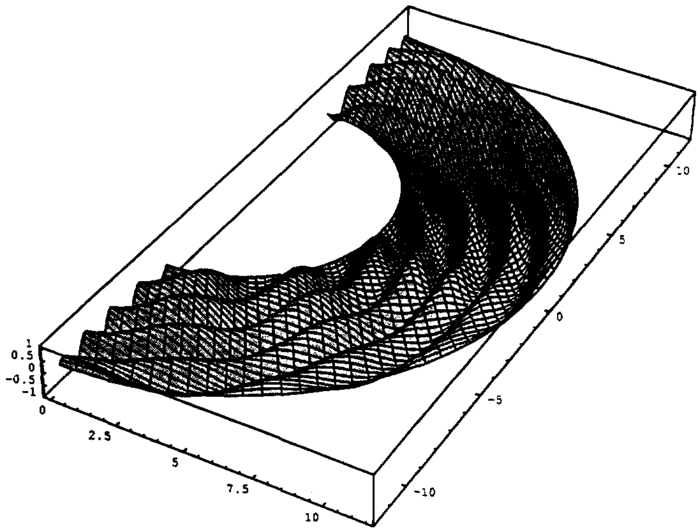


Fig. 6.18 *Buckling mode of anisotropic conical shell under axial compression (α_o = 30°; MSS3 boundary conditions; membrane-like prebuckling solution)*

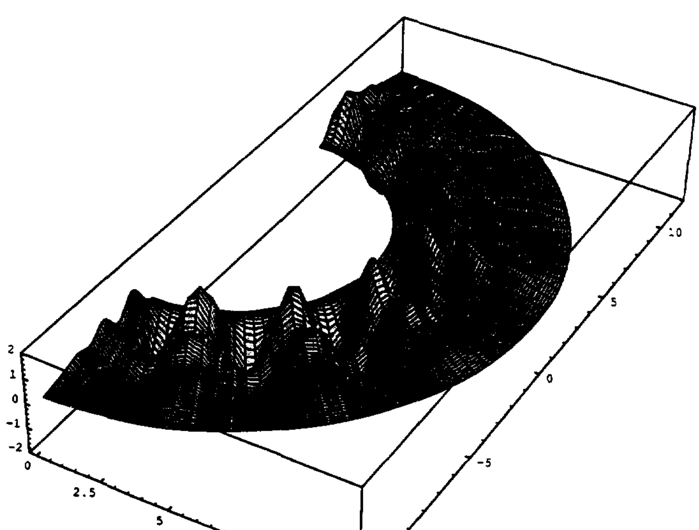


Fig. 6.19 *Buckling mode of anisotropic conical shell under axial compression (α_o = 30°; MSS3 boundary conditions; nonlinear prebuckling solution)*

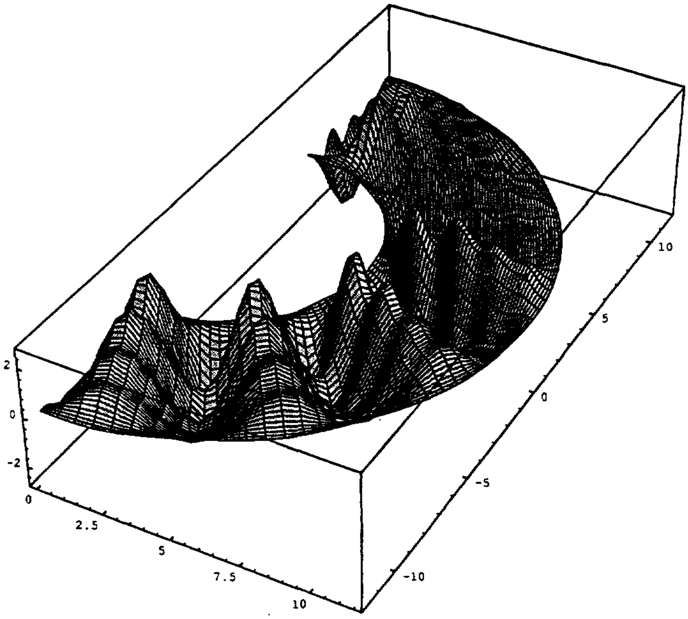


Fig. 6.20 *Buckling mode of anisotropic conical shell under axial compression (α_o = 30°; n_1 = 6; MSS3 boundary conditions; nonlinear prebuckling solution)*

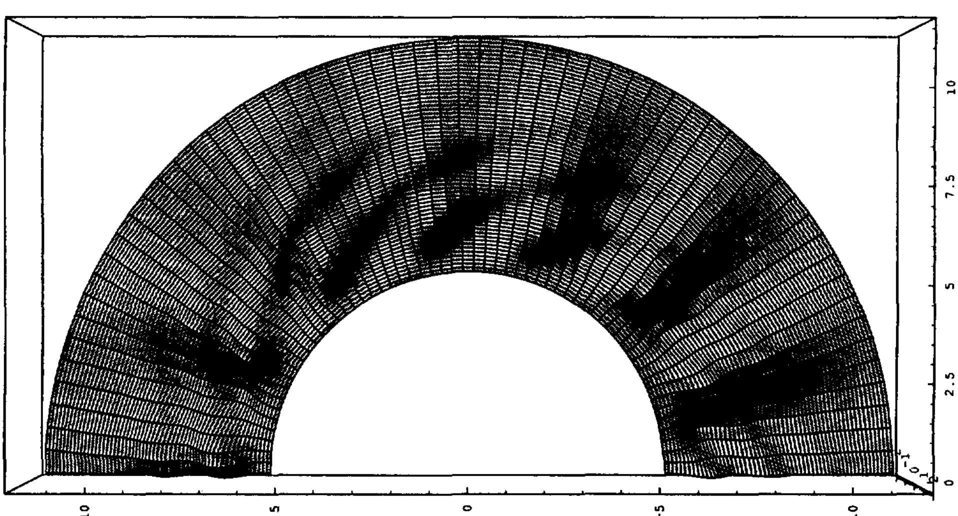


Fig. 6.21 *Another view of the buckling mode of anisotropic conical shell under axial compression (α_o = 30°; n_1 = 7; MSS3 boundary conditions; nonlinear pre-buckling solution)*

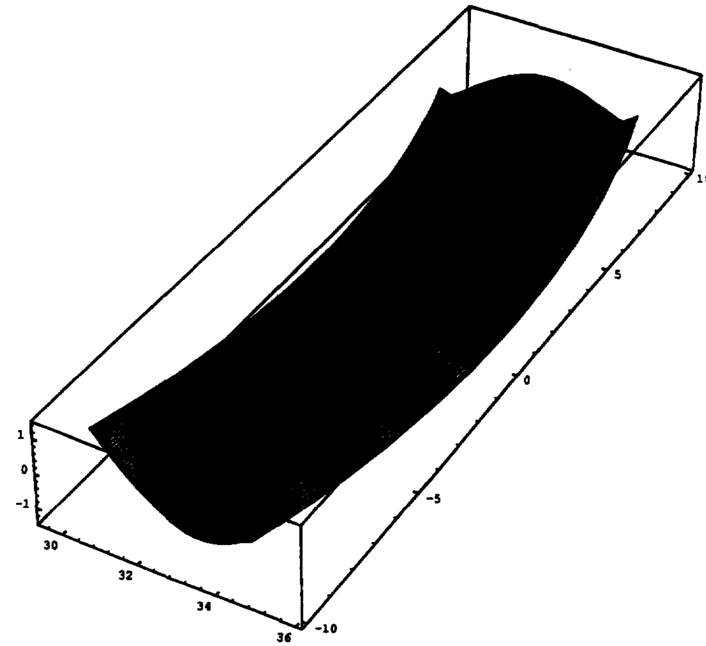


Fig. 6.22 *Buckling mode of anisotropic conical shell under hydrostatic pressure ($\alpha_o = 5^o$; MSS3 boundary conditions; membrane-like prebuckling solution)*

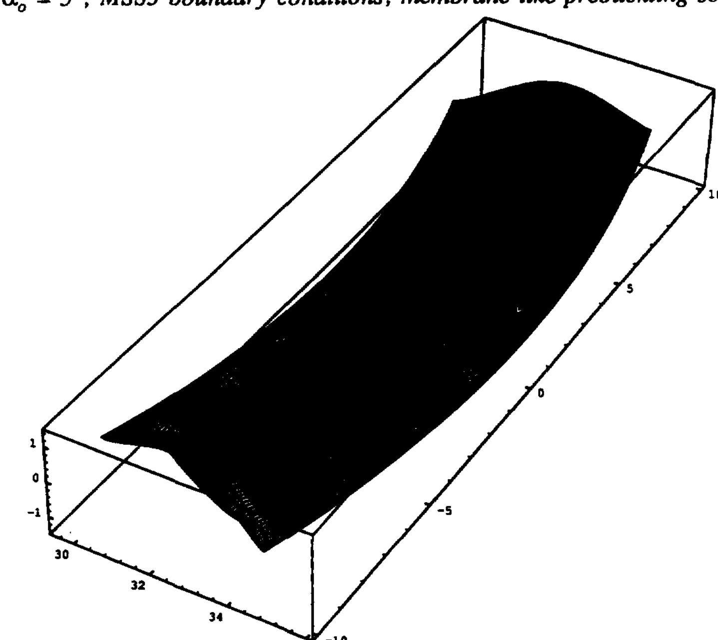


Fig. 6.23 *Buckling mode of anisotropic conical shell under hydrostatic pressure ($\alpha_o = 5^o$; MC4 boundary conditions; membrane-like prebuckling solution)*

6.4.3 Effect of different boundary conditions

It is well known that the load-carrying capacity of cylindrical shells can be influenced greatly by a variety of support conditions[45-49,67,107].

For isotropic conical shells under hydrostatic pressure, Singer[50] studied the effect of axial constraint on the instability behavior. Baruch et al.[54] systematically studied the effect of four sets of in-plane boundary conditions on the buckling behavior. In both investigations, the first two stability equations are solved by the assumed displacement method, while the third one is solved by Galerkin's procedure. The boundary conditions are satisfied by adjusting the 4 unknown coefficients in the expressions for u and v. It was found, as in the case of cylindrical shells, that the effect of axial constraint is of primary importance for isotropic conical shells under hydrostatic pressure whereas the circumferential restraint has little influence.

The effect of four sets of in-plane boundary conditions on the buckling behavior of isotropic conical shells under axial compression and combined axial compression with hydrostatic pressure was also studied by Baruch et al.[55] using a method similar to the one used in Ref. [54]. It was found that under axial compression both circumferential and axial restraints are of primary importance. In the cases SS1 ($N_{s\theta} = N_s = 0$) and SS3 ($v = N_s = 0$), the axisymmetric buckling mode dominates and $\lambda = 0.5$. The axisymmetric buckling mode in the SS3 case involves large axial translation. If this rigid body translation is prevented, only the classical modes with $\lambda = 1$ can occur. In the SS2 case ($N_{s\theta} = u = 0$) an asymmetric buckling mode occurs and $\lambda = 0.5$. In the SS4 case ($v = u = 0$) the buckling mode is again asymmetric but with $\lambda = 1$.

However, in most of the previous investigations involving the effect of boundary conditions membrane-like solution has been used to define the prebuckling state. This is actually an inconsistent assumption with regard to the prebuckling and buckling boundary conditions. Schiffner[89] studied the influence of nonlinear prebuckling solutions on the buckling behavior of isotropic and classical orthotropic conical shells by enforcing the boundary conditions of types 6 (equivalent to SS3) and 13 (equivalent to C3). However, in his solutions Seide's geometric constraint was used only in the prebuckling state and neglected in the buckling state. Hence inconsistency still exists[154]. Famili[57] solved the asymmetric buckling problem of isotropic conical shells under hydrostatic pressure by using rigorous prebuckling solution and enforcing consistently MSS3 boundary conditions. However, to the author's knowledge, there are no publications where the influence of consistent boundary conditions on the buckling behavior of conical shells is systematically investigated. This is especially needed for anisotropic conical shells. Therefore, in the following the effect of different boundary conditions on the buckling behavior of anisotropic conical shells will be investigated using rigorous prebuckling solutions.

Axial compression

The nondimensional buckling loads of anisotropic conical shells (α_o = 30°) under axial compression with different boundary conditions are given in Table 6.11. All these buckling loads are calculated using rigorous prebuckling solutions. As can be seen, the buckling load of simply supported shells is drastically reduced for the weak boundary support $N_{s\theta} = 0$ (v is free). This result has also been reported for anisotropic cylindrical shells by Jansen[107]. However, as suggested by Almroth[47], this kind of weak boundary conditions rarely occur in practice since real shell structures tend to be restrained at the ends by fairly rigid rings or bulkheads. Therefore, the actual boundary conditions are somewhere between clamped and simply supported.

For short conical shells (L/R_1 = 0.70711), the edge constraint in axial direction (u = 0) raises the buckling load by about 5% as compared to the cases in which u is free. The effect of clamping ($\tilde{W}_{,s} = 0$) is seen to be predominant, giving increases by more than 10%. For shells with moderate length (L/R_1 = 2), the increases are less than 3% for u = 0 and less than 4% for $\tilde{W}_{,s} = 0$.

Further, the low buckling load for isotropic conical shells with SS3 boundary conditions does not occur here, since the rigid body translation is eliminated by using MSS3 boundary conditions.

From Table 6.7 it can be seen that all the above conclusions are also valid for anisotropic conical shells with very small semi-vertex angles.

Hydrostatic pressure

The nondimensional buckling loads of anisotropic conical shells (α_o = 30°) under hydrostatic pressure with different boundary conditions are given in Table 6.12. All these buckling loads are obtained using rigorous prebuckling solutions. As can be seen, for short conical shells (L/R_1 = 0.70711) both the axial constraint (u = 0) and the rotational constraint ($\tilde{W}_{,s}$ = 0) increase the buckling loads considerably (by about 10-25%). For shells with moderate length (L/R_1 = 2), the axial constraint has a pronounced influence. The buckling loads corresponding to MSS2 and MSS4 boundary conditions increase about 30% and 33% as compared to those from MSS1 and MSS3 boundary conditions, respectively.

It is also found that for both short shells (L/R_1 = 0.70711) and shells with moderate length (L/R_1 = 2) axial constraint will increase not only the buckling loads but also the wave numbers in the circumferential direction.

To summarize briefly, for short anisotropic conical shells under axial compression the effect of rotational constraint is found to be predominant, while for short anisotropic conical shells under hydrostatic pressure both the axial and the rotational constraints are important. For anisotropic conical shells with moderate length under axial compression, the effect of boundary conditions are less important, while for anisotropic conical shells with moderate length under hydrostatic pressure both the axial and rotational constraints are important, and usually the axial constraint is predominant.

Table 6.11 *Comparison of buckling load λ of anisotropic conical shells using nonlinear prebuckling solution for different boundary conditions*

Case	R_1/t	L/R_1	t	α_o°	B.C.	λ
1	100	0.70711	0.0267	30	MSS3	0.36954(8)
2	100	0.70711	0.0267	30	MSS4	0.39149(9)
3	100	0.70711	0.0267	30	MC1	0.42097(8)
4	100	0.70711	0.0267	30	MC2	0.43965(8)
5	100	0.70711	0.0267	30	MC3	0.42320(8)
6	100	0.70711	0.0267	30	MC4	0.44192(8)
7	100	2	0.0267	30	MSS3	0.39104(7)
8	100	2	0.0267	30	MSS4	0.39970(8)
9	100	2	0.0267	30	MC1	0.40791(7)
10	100	2	0.0267	30	MC2	0.41390(7)
11	100	2	0.0267	30	MC3	0.40954(7)
12	100	2	0.0267	30	MC4	0.41432(7)

Table 6.12 Comparison of buckling load $\bar{p}$ of anisotropic conical shells using nonlinear prebuckling solution for different boundary conditions

Case	R_1/t	L/R_1	t	$\alpha_0°$	B.C.	$\bar{p}$
1	100	0.70711	0.0267	30	MSS1	0.073069(9)
2	100	0.70711	0.0267	30	MSS2	0.086850(10)
3	100	0.70711	0.0267	30	MSS3	0.077603(10)
4	100	0.70711	0.0267	30	MSS4	0.094445(11)
5	100	0.70711	0.0267	30	MC1	0.092078(10)
6	100	0.70711	0.0267	30	MC2	0.010451(11)
7	100	0.70711	0.0267	30	MC3	0.092532(11)
8	100	0.70711	0.0267	30	MC4	0.10675(12)
9	100	2	0.0267	30	MSS1	0.018729(7)
10	100	2	0.0267	30	MSS2	0.024407(8)
11	100	2	0.0267	30	MSS3	0.020536(7)
12	100	2	0.0267	30	MSS4	0.027317(8)
13	100	2	0.0267	30	MC1	0.019888(8)
14	100	2	0.0267	30	MC2	0.025299(9)
15	100	2	0.0267	30	MC3	0.019998(8)
16	100	2	0.0267	30	MC4	0.025607(10)

6.4.4 Effect of different wall constructions

The spectrum of applications for composite materials is growing rapidly because such materials can be tailored to meet design performance specifications at a reduced weight. The most important variables that describe a particular laminated composite structure are the properties of the reinforcing fibers, matrice properties, lamina fiber orientations, lamina thicknesses, lamina stacking sequences, and the number of laminae. As a preliminary step for the design optimization of anisotropic conical shells the effect of different wall constructions on the buckling behavior of conical shells is studied in the following.

Antisymmetric cross-ply laminated conical shells

A regular antisymmetric cross-ply laminate consists of an even number of orthotropic laminae of the same thickness laid on each other with the directions of the stiffest principal material properties alternating at 0 degree and 90 degree to the laminate s axis. For unidirectional fiber-reinforced laminae, the fiber direction alternates at 0 degree and 90 degree to the s axis. Only the coupling stiffnesses B_{11} and B_{22} exist for these cross-ply laminates.

Variation in E_s / E_θ: The buckling loads of two layer antisymmetric cross-ply laminated conical shells (L/R_1 = 0.515) with MSS3 boundary conditions under axial compression are shown in Figure 6.24. Notice that the buckling loads are calculated with both membrane-like and rigorous prebuckling solutions. It can be observed that as in the case of classical orthotropic conical shells the buckling loads increase as E_s / E_θ increases, and they approach a constant when E_s / E_θ is large enough. Besides, similar to the situation of cylindrical shells[45], the buckling load calculated from rigorous prebuckling solution is about 12% lower than that obtained from membrane-like prebuckling solution at E_s / E_θ = 5 and 2% higher at E_s / E_θ = 40.

Variation of number of layers: The buckling loads of antisymmetric cross-ply laminated conical shells with MSS3 boundary conditions under axial compression are shown in Figure 6.25. Notice again that the buckling loads are obtained from nonlinear prebuckling solution. As can be seen, for conical shells with fixed L/R_1 ratio and α_o the buckling loads increase and the corresponding circumferential wave numbers tend to decrease as the number of layers increases. This result is expected since the stiffness coefficients B_{ij} (stretching-bending coupling) decrease rapidly as the number of layer increases.

General laminated conical shells

The results listed in Table 6.13 make a comparison of the buckling loads of anisotropic conical shells with different fiber orientations under hydrostatic pressure possible. Two kinds of boundary conditions, namely, MSS3 and MC4 are considered. The details of shell geometry are given in Figure 6.26. The radius-thickness ratio is held constant (R_1/t = 100). Under such condition, for a given value of L/R_1, varying the cone angle is equivalent to increasing the slant length of the cone and the radius at the large end. As can be seen from Table 6.3, increasing the fiber orientation angles of the antisymmetric angle-ply laminated conical shells from 30° to 60°, the corresponding buckling loads obtained from rigorous prebuckling solutions can increase up to 18%.

With the results shown in Table 6.14 one can compare the buckling loads of two kinds of anisotropic conical shells (symmetric angle-ply and antisymmetric angle-ply) with different boundary conditions under axial compression. Both short shells (L/R_1 = 0.70711) and shells with moderate length ($L/R_1 = 2$) are considered, while the radius-thickness ratio R_1/t and the semi-vertex angle α_0 are fixed. The buckling loads shown in Table 6.14 are obtained from rigorous prebuckling solution. As can be seen, by using the symmetric angle-ply the buckling loads increase about 8% for short shells and 6% for shells with moderate length, as compared with the corresponding anisotropic conical shells made by antisymmetric angle-ply.

It is known that unlike the case of filament-wound cylindrical shells, where most fabrication processes will result in nominally uniform wall thicknesses and stiffness properties, for conical shells the opposite is true[84]. Unless special techniques are employed, the wall thickness and the stiffness properties of laminated conical shells produced by filament winding will depend on the longitudinal coordinate s. To extend the present theory to this manufacturing process, formulae to calculate the equivalent constant stiffness coefficients and the constant wall thickness are suggested in chapter 2. By using these formulae the buckling loads of anisotropic conical shells under axial compression with different boundary conditions are calculated and listed in Tables 6.15-6.17. As can be seen, the physical buckling loads calculated from the equivalent cylindrical mean and geometric mean are very close, while the results from the arithmetic mean are higher. Therefore, as an approximation it seems proper to use the equivalent cylindrical or geometric mean to calculate the stiffness coefficients and wall thicknesses of anisotropic conical shells made by filament winding.

However, it is recommended here that in order to investigate the effect of physical imperfections caused by filament winding more accurately, another kind of nonlinear Donnell-type shell equations which takes into account the variations of wall thicknesses and stiffness properties should be employed in a future theoretical analysis.

To summarize briefly, the above numerical computations show that the buckling behavior of laminated conical shells depends significantly on the different wall constructions. This behavior, naturally, makes the designers' task very difficulty with regards to finding optimum reinforcement architecture or lamination pattern in order to increase the buckling strength to weight ratios.

Table 6.13 Comparison of buckling load $\bar{p}$ of anisotropic conical shells using nonlinear prebuckling solutions for different fiber orientations

Case	R_1/t	L/R_1	t	α_o°	B.C.	$\bar{p}$	
						(30°, 0, -30°)	(60°, 0, -60°)
1	100	0.70711	0.0267	10	MC4	0.13673(11)	0.15773(11)
2	100	0.70711	0.0267	30	MC4	0.10675(12)	0.12311(11)
3	100	0.70711	0.0267	45	MC4	0.080966(11)	0.095033(11)
4	100	0.70711	0.0267	10	MSS3	0.099758(10)	0.11492(10)
5	100	0.70711	0.0267	30	MSS3	0.077603(10)	0.089587(10)
6	100	0.70711	0.0267	45	MSS3	0.058968(10)	0.069635(10)
7	100	2	0.0267	10	MC4	0.040347(8)	0.047616(8)
8	100	2	0.0267	30	MC4	0.028264(8)	0.030087(9)
9	100	2	0.0267	45	MC4	0.019603(8)	0.019971(9)
10	100	2	0.0267	10	MSS3	0.029383(7)	0.034583(7)
11	100	2	0.0267	30	MSS3	0.018830(8)	0.022157(8)
12	100	2	0.0267	45	MSS3	0.012532(9)	0.014659(8)

Table 6.14 Comparison of buckling load λ of anisotropic conical shells for symmetric and antisymmetric angle-ply

Case	R_1/t	L/R_1	t	α_o°	B.C.	λ	
						Symmetric angle-ply (30°, 0, 30°)	Antisymmetric angle-ply (30°, 0, -30°)
1	100	0.70711	0.0267	30	MSS2	0.26759(2)	0.24508(2)
2	100	0.70711	0.0267	30	MSS3	0.39283(8)	0.36954(8)
3	100	0.70711	0.0267	30	MSS4	0.41317(9)	0.39149(9)
4	100	0.70711	0.0267	30	MC1	0.45578(8)	0.42097(8)
5	100	0.70711	0.0267	30	MC2	0.47697(8)	0.43965(8)
6	100	0.70711	0.0267	30	MC3	0.46120(8)	0.42320(8)
7	100	0.70711	0.0267	30	MC4	0.47988(8)	0.44192(8)
8	100	2	0.0267	30	MSS2	0.26240(2)	0.24041(2)
9	100	2	0.0267	30	MSS3	0.41708(8)	0.39104(7)

10	100	2	0.0267	30	MSS4	0.42544(8)	0.39970(8)
11	100	2	0.0267	30	MC1	0.43514(8)	0.40791(7)
12	100	2	0.0267	30	MC2	0.43947(8)	0.41390(7)
13	100	2	0.0267	30	MC3	0.43659(8)	0.40954(7)
14	100	2	0.0267	30	MC4	0.43970(8)	0.41432(7)

Table 6.15 ***Buckling load λ of anisotropic conical shells by equivalent cylindrical mean ($\alpha_o = 30°$, $L/R_1 = 2$, [30°, 0°, -30°])***

case	B.C.	Equivalent cylindrical mean							
		t_{av1}(in.)	t_{av2}(in.)	t_{av3}(in.)	t(in.)	$\beta_{av1}°$	$\beta_{av2}°$	$\beta_{av3}°$	λ
1	MSS3	0.005656	0.006128	0.005656	0.01744	20.149	0	-20.149	0.39511(10) (0.17696×10^5)
2	MSS4	0.005656	0.006128	0.005656	0.01744	20.149	0	-20.149	0.39679(10)
3	MC3	0.005656	0.006128	0.005656	0.01744	20.149	0	-20.149	0.39743(11)
4	MC4	0.005656	0.006128	0.005656	0.01744	20.149	0	-20.149	0.39789(11)

Table 6.16 ***Buckling load λ of anisotropic conical shells by arithmetic mean ($\alpha_o = 30°$, $L/R_1 = 2$, [30°, 0°, -30°])***

case	B.C.	Arithmetic mean							
		t_{av1}(in.)	t_{av2}(in.)	t_{av3}(in.)	t(in.)	$\beta_{av1}°$	$\beta_{av2}°$	$\beta_{av3}°$	λ
1	MSS3	0.006298	0.006515	0.006298	0.01909	18.481	0	-18.481	0.39390(10) (0.19371×10^5)
2	MSS4	0.006298	0.006515	0.006298	0.01909	18.481	0	-18.481	0.39530(10)
3	MC3	0.006298	0.006515	0.006298	0.01909	18.481	0	-18.481	0.39632(10)
4	MC4	0.006298	0.006515	0.006298	0.01909	18.481	0	-18.481	0.39665(10)

Table 6.17 ***Buckling load λ of anisotropic conical shells by geometric mean ($\alpha_o = 30°$, $L/R_1 = 2$, [30°, 0°, -30°])***

case	B.C.	Geometric mean							
		t_{av1}(in.)	t_{av2}(in.)	t_{av3}(in.)	t(in.)	$\beta_{av1}°$	$\beta_{av2}°$	$\beta_{av3}°$	λ
1	MSS3	0.00572	0.00606	0.00572	0.0175	20.36	0	-20.36	0.39502(10) (0.17757×10^5)
2	MSS4	0.00572	0.00606	0.00572	0.0175	20.36	0	-20.36	0.39674(10)
3	MC3	0.00572	0.00606	0.00572	0.0175	20.36	0	-20.36	0.39745(10)
4	MC4	0.00572	0.00606	0.00572	0.0175	20.36	0	-20.36	0.39797(10)

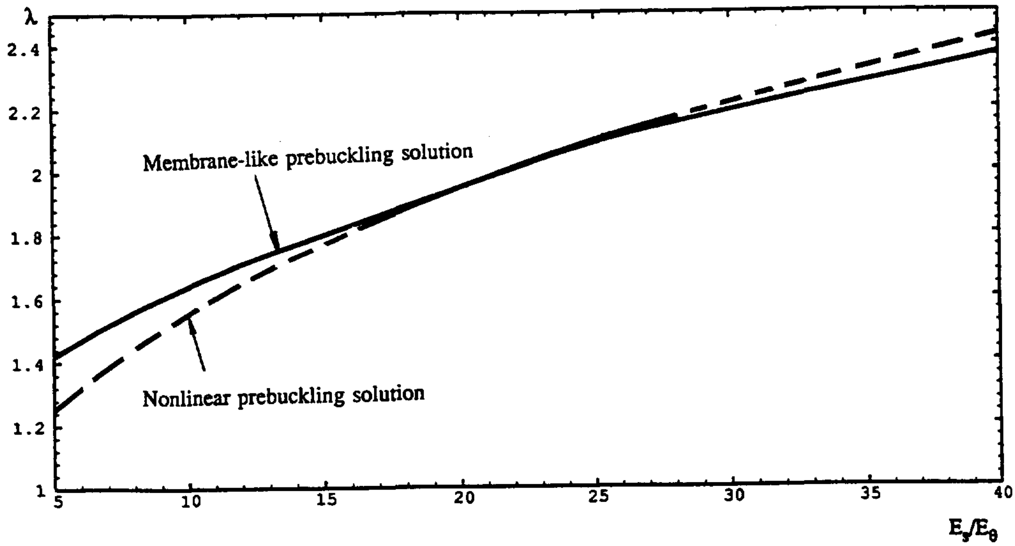


Fig. 6.24 *Buckling loads of antisymmetric cross-ply laminated conical shells as functions of E_s/E_θ*

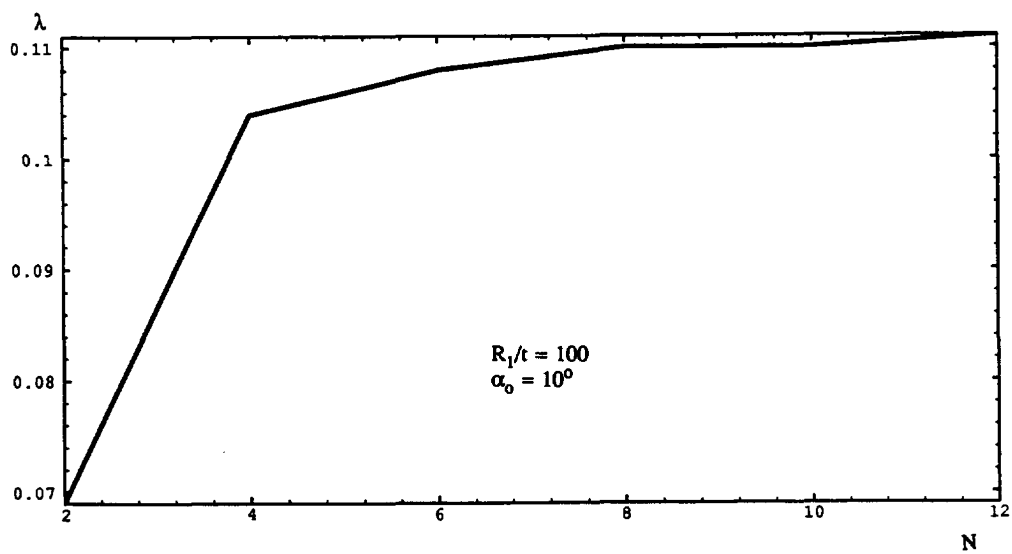


Fig. 6.25 *Buckling loads of antisymmetric cross-ply laminated conical shells as functions of the number of laminae N*

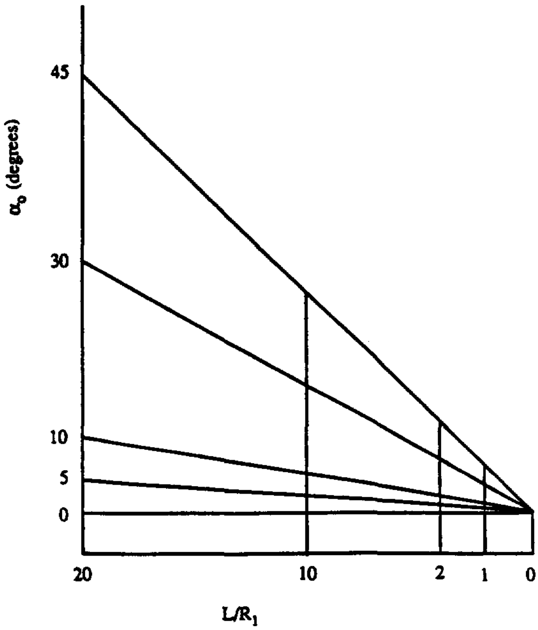


Fig. 6.26 *Some cone geometries*

6.5 Discussions and Conclusion

The nonlinear Donnell-type anisotropic shell equations and fourteen sets of appropriate boundary conditions in terms of radial displacement W and Airy stress function F are used for the bifurcation buckling analysis of anisotropic conical shells.

Both the nonlinear ordinary differential equations for prebuckling state and the linear ordinary differential equations with variable coefficients for buckling state can be solved via the 'parallel shooting' method by using either Keller's or Stodola's method. Using Keller's method the eigenvalue is taken as an unknown in the iteration scheme. The eigenvalue and corresponding eigenmode can be obtained simultaneously by adding a normalization condition to the nonlinear eigenvalue equations. Using Stodola's method the eigenmodes are solved by an iteration process, while the eigenvalue corrections are calculated by the Rayleigh quotient.

Extensive numerical studies have been carried out for different types of conical shells. First, the influences of different prebuckling solutions and different boundary conditions on the critical buckling load are studied for anisotropic conical shells with a wide range of geometries. Initial results indicate that in order to obtain reliable solutions for anisotropic conical shells, the use of nonlinear prebuckling solution and enforcing the consistent boundary conditions rigorously is indeed a must.

Secondly, to account for the variable stiffness coefficients and the variations in wall thickness that occur when laminated composite conical shells are made by filament winding, numerical results are presented for buckling loads by using the simplified formulae for the equivalent constant stiffness coefficients and wall thickness. The validity and importance of using proper simplification formulae are verified.

Chapter 7
Initial Postbuckling Analysis of Perfect and Imperfect Anisotropic Conical Shells

7.1 Introduction

A rigorous treatment of the bifurcation buckling of anisotropic conical shells was presented in the previous chapter. It is known that the bifurcation buckling analysis represents a search for the load at which the equilibrium of a structure ceases to be stable. Unfortunately, it yields no information about the ultimate load carrying capacity of the structure. In order to find out whether in practice the critical load of the structure is degraded by the presence of the unavoidable initial imperfections, it is necessary to determine the characteristics of the post-bifurcation path in the neighborhood of the bifurcation point. By employing a perturbation technique Koiter[60] developed a general theory for the initial postbuckling behavior and the imperfection sensitivity of elastic structures, which relies on the principle of stationary potential energy. Later, Budiansky and Hutchinson[68] derived equivalent results, which are more convenient for applications, by writing the field equations directly in variational form with the aid of the principle of virtual work. However, all these formulations are restricted to structures whose prebuckling behavior is linearly proportional to the applied load. Fitch[71] and Cohen[104] extended Budiansky-Hutchinson's approach to account for the nonlinear prebuckling behavior. Arbocz[67] further extended Koiter's and Cohen's asymptotic formulations to anisotropic shell structures under combined loadings. All these theoretical efforts have inspired extensive and successful applications of Koiter's theory for many practical structural problems. An excellent review of the applications is given by Bushnell[176].

Although the studies of the initial postbuckling behavior and imperfection sensitivity of shell structures are of considerable theoretical and practical interest, few results have been published for anisotropic conical shells. In the following the initial postbuckling coefficients 'a' and 'b' and the imperfection form factors 'α' and 'β' are rederived in forms suitable for conical shells. Then numerical investigations for the initial postbuckling behavior and imperfection sensitivity of anisotropic conical shells are carried out based on the extended Koiter-type formulae and Donnell-type nonlinear theory in terms of the radial displacement W and Airy stress function F.

7.2 Theoretical Analysis

To carry out an initial postbuckling analysis, the six ordinary differential equations governing the behavior of anisotropic conical shells in the initial postbuckling state presented by Eqs. (3.70)-(3.75) have to be further simplified and regrouped. Further, despite the existence of formulae of the initial postbuckling coefficients and imperfection form factors for shells of revolution, cylindrical shells, spherical shells[60,67,71,177,178] there are few available results which can be directly used for anisotropic conical shells. Therefore, the equations defining the initial postbuckling coefficients and the imperfection form factors are derived for the applications of anisotropic conical shells.

7.2.1 Postbuckling governing equations

Equations (3.70) and (3.73) can be integrated once respectively and yield

$$\begin{aligned}
&\bar{A}^*_{22}f'''_\alpha + \bar{A}^*_{22}f''_\alpha - \bar{A}^*_{11}f'_\alpha - \bar{A}^*_{11}f_\alpha + (t/2s_1\sin\alpha_o)\,[\,-\bar{B}^*_{21}w'''_\alpha \\
&\qquad + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w''_\alpha + (\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w'_\alpha + \bar{B}^*_{12}w_\alpha\,] \\
&= -(e^{z}ct/4s_1\sin\alpha_o)\,[\,4\psi(w'_\alpha+w_\alpha) + w'_1(w'_1-2n^2w_1+2w_1) \qquad (7.1)\\
&\qquad + w'_2(w'_2-2n^2w_2+2w_2) + (1-n^2)(w_1^2+w_2^2) \\
&\qquad + (4s_1\cot\alpha_o/t)(w'_\alpha+w_\alpha)\,] + c_1
\end{aligned}$$

$$\bar{D}^*_{11}w'''_\alpha + \bar{D}^*_{11}w''_\alpha - \bar{D}^*_{22}w'_\alpha - \bar{D}^*_{22}w_\alpha + (2s_1\sin\alpha_o/t)\,[\,\bar{B}^*_{21}f'''_\alpha$$

$$+ (\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})f''_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22})f'_\alpha - \bar{B}^*_{12}f_\alpha\,]$$

$$= (2ce^{z}s_1\sin\alpha_o/t)\,[\,w'_1(f'_1-n^2f_1+f_1) + w'_2(f'_2-n^2f_2+f_2) \qquad (7.2)$$

$$+ (1-n^2)(w_1f'_1+w_2f'_2+w_1f_1+w_2f_2) + 2\varphi(w'_\alpha+w_\alpha)$$

$$+ 2\psi(f'_\alpha+f_\alpha) + (2s_1\cot\alpha_o/t)(f'_\alpha+f_\alpha)\,] + c_2$$

where c_1 and c_2 are constants of integration. By enforcing the periodicity condition (see Appendix A2.1 for details) c_1 is known to be zero. By enforcing the boundary equilibrium condition $H = \bar{H}$ (see Eq. (4.39)), c_2 is also equal to zero.

Introducing the following transformations

$$\bar{f}_\alpha = f'_\alpha + f_\alpha \qquad (7.3)$$

$$\bar{w}_\alpha = w'_\alpha + w_\alpha$$

Equations (7.1) and (7.2) can be further simplified yielding

$$\bar{A}^*_{22}\bar{f}''_\alpha - \bar{A}^*_{11}\bar{f}_\alpha + (t/2s_1\sin\alpha_o)\,[\,-\bar{B}^*_{21}\bar{w}''_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{22})\bar{w}'_\alpha + \bar{B}^*_{12}\bar{w}_\alpha\,]$$

$$= -(e^{z}ct/4s_1\sin\alpha_o)\,[\,4\psi\,\bar{w}_\alpha + w'_1(w'_1-2n^2w_1+2w_1) \qquad (7.4)$$

$$+ w'_2(w'_2-2n^2w_2+2w_2) + (1-n^2)(w_1^2+w_2^2) + (4s_1\cot\alpha_o/t)\,\bar{w}_\alpha\,]$$

$$\bar{D}^*_{11}\bar{w}''_\alpha - \bar{D}^*_{22}\bar{w}_\alpha + (2s_1\sin\alpha_o/t)\,[\,\bar{B}^*_{21}\bar{f}''_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{22})\bar{f}'_\alpha - \bar{B}^*_{12}\bar{f}_\alpha\,]$$

$$= (2ce^{z}s_1\sin\alpha_o/t)\,[\,w'_1(f'_1-n^2f_1+f_1) + w'_2(f'_2-n^2f_2+f_2) \qquad (7.5)$$

$$+ (1-n^2)(w_1f'_1+w_2f'_2+w_1f_1+w_2f_2) + 2\varphi\,\bar{w}_\alpha + 2\psi\,\bar{f}_\alpha + (2s_1\cot\alpha_o/t)\,\bar{f}_\alpha\,]$$

Further, in order to be able to use the 'shooting method' to solve the governing equations of the 2nd order state it is necessary, by considering Eqs. (3.71) and (3.74), to eliminate the w_β^{iv} term from Eq. (3.71) and the f_β^{iv} term from Eq. (3.74). Similarly, by considering Eqs. (3.72) and (3.75) one must eliminate the w_γ^{iv} term from Eq. (3.72) and the f_γ^{iv} term from Eq. (3.75). Next by considering Eqs. (7.4) and (7.5) one can eliminate the $\bar{w}_\alpha''$ term from Eq. (7.4) and the $\bar{f}_\alpha''$ term from Eq. (7.5). Finally, some regrouping makes it possible to write the resulting equations as

$$
\begin{aligned}
w_\gamma^{iv} = {} & D_1 w_\gamma''' + D_{37} w_\gamma'' + +D_{16} w_\gamma' - D_{38} w_\gamma - D_{39} w_\beta''' - D_{19} w_\beta'' - D_{40} w_\beta' \\
& + D_4 f_\gamma''' + D_{41} f_\gamma'' + D_{42} f_\gamma' + D_{43} f_\gamma - D_{44} f_\beta''' + D_{45} f_\beta'' - D_{46} f_\beta' \\
& + e^z \{ D_6 [\psi'(w_\gamma' + w_\gamma) + \psi(w_\gamma'' + w_\gamma')] + D_7 [w_2(w_1'' + w_1') + w_1(w_2'' + w_2')] \\
& + D_8(w_2' w_1'' + w_2'' w_1') + D_{34} w_1' w_2' - D_{35} \psi' w_\gamma + D_9(w_\gamma'' + w_\gamma') \\
& + D_{10} [\psi'(f_\gamma' + f_\gamma) + \varphi'(w_\gamma' + w_\gamma) + \varphi(w_\gamma'' + w_\gamma') + \psi(f_\gamma'' + f_\gamma')] \\
& - D_{48}(\psi' f_\gamma + \varphi' w_\gamma) + D_{13}(w_1'' f_2' + w_1' f_2'' + w_2'' f_1' + w_2' f_1'') \\
& + D_{49} [f_2(w_1'' + w_1') + f_1(w_2'' + w_2') + w_2(f_1'' + f_1') + w_1(f_2'' + f_2')] \\
& + D_{50}(w_1' f_2' + w_2' f_1') + D_{11}(f_\gamma'' + f_\gamma') \}
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
w_\beta^{iv} = {} & D_1 w_\beta''' + D_{37} w_\beta'' + +D_{16} w_\beta' - D_{38} w_\beta + D_{39} w_\gamma''' + D_{19} w_\gamma'' + D_{40} w_\gamma' \\
& + D_4 f_\beta''' + D_{41} f_\beta'' + D_{42} f_\beta' + D_{43} f_\beta + D_{44} f_\gamma''' - D_{45} f_\gamma'' + D_{46} f_\gamma' \\
& + e^z \{ D_6 [\psi'(w_\beta' + w_\beta) + \psi(w_\beta'' + w_\beta')] + D_7 [w_1(w_1'' + w_1') - w_2(w_2'' + w_2')] \\
& + D_8(w_1' w_1'' - w_2'' w_2') + D_{47}(w_1'^2 - w_2'^2) - D_{35} \psi' w_\beta + D_9(w_\beta'' + w_\beta') \\
& + D_{10} [\psi'(f_\beta' + f_\beta) + \varphi'(w_\beta' + w_\beta) + \varphi(w_\beta'' + w_\beta') + \psi(f_\beta'' + f_\beta')] \\
& - D_{48}(\psi' f_\beta + \varphi' w_\beta) + D_{13}(w_1'' f_1' + w_1' f_1'' - w_2'' f_2' - w_2' f_2'') \\
& + D_{49} [f_1(w_1'' + w_1') - f_2(w_2'' + w_2') + w_1(f_1'' + f_1') - w_2(f_2'' + f_2')] \\
& + D_{50}(w_1' f_1' - w_2' f_2') + D_{11}(f_\beta'' + f_\beta') \}
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
f_\gamma^{iv} = & - D_1 f_\gamma''' - D_{15} f_\gamma'' - D_{16} f_\gamma' - D_{17} f_\gamma + D_{18} f_\beta''' + D_{19} f_\beta'' + D_{20} f_\beta' \\
& - D_{21} w_\gamma''' - D_{22} w_\gamma'' - D_{23} w_\gamma' - D_{24} w_\gamma + D_{25} w_\beta''' + D_{26} w_\beta'' + D_{27} w_\beta' \\
& - e^z \{ D_{28} [\psi'(w_\gamma' + w_\gamma) + \psi(w_\gamma'' + w_\gamma')] - D_{29} \psi' w_\gamma + D_{30} [w_2 (w_1'' + w_1') \\
& + w_1 (w_2'' + w_2')] + D_{36} w_1' w_2' + D_{32} (w_2' w_1'' + w_2'' w_1') + D_{33} (w_\gamma'' + w_\gamma') \\
& - D_6 [\psi'(f_\gamma' + f_\gamma) + \varphi'(w_\gamma' + w_\gamma) + \varphi(w_\gamma'' + w_\gamma') + \psi(f_\gamma'' + f_\gamma')] \\
& - D_7 [f_2 (w_1'' + w_1') + f_1 (w_2'' + w_2') + w_2 (f_1'' + f_1') + w_1 (f_2'' + f_2')] \\
& - D_{34} (w_1' f_2' + w_2' f_1') + D_{35} (\psi' f_\gamma + \varphi' w_\gamma) \\
& - D_8 (w_1'' f_2' + w_1' f_2'' + w_2'' f_1' + w_2' f_1'') - D_9 (f_\gamma'' + f_\gamma') \}
\end{aligned}
\tag{7.8}
$$

$$
\begin{aligned}
f_\beta^{iv} = & - D_1 f_\beta''' - D_{15} f_\beta'' - D_{16} f_\beta' - D_{17} f_\beta - D_{18} f_\gamma''' - D_{19} f_\gamma'' - D_{20} f_\gamma' \\
& - D_{21} w_\beta''' - D_{22} w_\beta'' - D_{23} w_\beta' - D_{24} w_\beta - D_{25} w_\gamma''' - D_{26} w_\gamma'' - D_{27} w_\gamma' \\
& - e^z \{ D_{28} [\psi'(w_\beta' + w_\beta) + \psi(w_\beta'' + w_\beta')] - D_{29} \psi' w_\beta + D_{30} [w_1 (w_1'' + w_1') \\
& - w_2 (w_2'' + w_2')] + D_{31} (w_1'^2 - w_2'^2) + D_{32} (w_1' w_1'' - w_2'' w_2') + D_{33} (w_\beta'' + w_\beta') \\
& - D_6 [\psi'(f_\beta' + f_\beta) + \varphi'(w_\beta' + w_\beta) + \varphi(w_\beta'' + w_\beta') + \psi(f_\beta'' + f_\beta')] \\
& + D_7 [f_2 (w_2'' + w_2') - f_1 (w_1'' + w_1') + w_2 (f_2'' + f_2') - w_1 (f_1'' + f_1')] \\
& + D_{34} (w_2' f_2' - w_1' f_1') + D_{35} (\psi' f_\beta + \varphi' w_\beta) \\
& - D_8 (w_1'' f_1' - w_2'' f_2' + w_1' f_1'' - w_2' f_2'') - D_9 (f_\beta'' + f_\beta') \}
\end{aligned}
\tag{7.9}
$$

$$
\begin{aligned}
\bar{w}_\alpha'' = & \ D_1 \bar{w}_\alpha' + D_3 \bar{w}_\alpha + D_4 \bar{f}_\alpha' + D_5 \bar{f}_\alpha \\
& + e^z \{ D_6 \psi \bar{w}_\alpha + D_{51} (w_1^2 + w_2^2) + D_{52} (w_1'^2 + w_2'^2) + D_7 (w_1' w_1 + w_2' w_2) \\
& + D_9 \bar{w}_\alpha + D_{13} (w_1' f_1' + w_1' f_1 + w_2' f_2' + w_2' f_2 + f_1' w_1 + f_2' w_2 + w_1 f_1 + w_2 f_2) \\
& - D_{14} (w_1' f_1 + w_2' f_2 + f_1' w_1 + f_2' w_2 + w_1 f_1 + w_2 f_2) + D_{11} \bar{f}_\alpha + D_{10} (\varphi \bar{w}_\alpha + \psi \bar{f}_\alpha) \}
\end{aligned}
\tag{7.10}
$$

$$
\begin{aligned}
\bar{f}''_\alpha = & -D_1\bar{f}'_\alpha + D_{53}\bar{f}_\alpha - D_{21}\bar{w}'_\alpha + D_{54}\bar{w}_\alpha \\
& - e^z\{D_{28}\psi\bar{w}_\alpha + D_{55}(w_1^2+w_2^2) + D_{56}(w_1'^2+w_2'^2) + D_{30}(w_1'w_1+w_2'w_2) \\
& + D_{33}\bar{w}_\alpha - D_8(w_1'f_1+w_2'f_2'+w_2'f_2+w_1f_1'+w_2f_2'+w_1f_1+w_2f_2+w_1'f_1') \\
& + D_{57}(w_1'f_1+w_2'f_2+f_1'w_1+f_2'w_2+w_1f_1+w_2f_2) - D_9\bar{f}_\alpha - D_6(\varphi\bar{w}_\alpha+\psi\bar{f}_\alpha)\}
\end{aligned}
\tag{7.11}
$$

The constants D_i (i = 1 ~ 56) are listed in Appendix A5.1.

This set of inhomogeneous differential equations with variable coefficients together with some appropriate boundary conditions is sufficient to yield the required information about the initial postbuckling behavior and the associated imperfection sensitivity of anisotropic conical shells.

7.2.2 Postbuckling coefficients and imperfection form factors

For perfect shells one is interested in the variation of $\Lambda(\xi)$ with ξ in the vicinity of $\Lambda = \Lambda_c$. Near the bifurcation point Λ_c the asymptotic expansion given by Eqs. (3.42) is valid. The derivation of postbuckling coefficients 'a' and 'b' for anisotropic cylindrical shells can be found in Ref. [67], while the derivation of those for anisotropic conical shells will be carried out in the following by a REDUCE-based program GEPCAC, GEnerating Postbuckling Coefficients of Anisotropic Conical Shells (see Appendix A5.2 for the source program). After extensive symbolic calculations the initial postbuckling coefficients are obtained as

$$
a = -(3/2\Lambda_c\hat{\Delta})F^{(1)} * (W^{(1)}, W^{(1)}) \tag{7.12}
$$

$$
\begin{aligned}
b = & -(1/\Lambda_c\hat{\Delta})\,[2F^{(1)} * (W^{(1)}, W^{(2)}) + F^{(2)} * (W^{(1)}, W^{(1)}) \\
& + a\Lambda_c\Pi_1 + (1/2)(a\Lambda_c)^2\Pi_2\,]
\end{aligned}
\tag{7.13}
$$

where

$$
\hat{\Delta} = 2F^{(1)} * (\dot{W}_c,\ W^{(1)}) + \dot{F}_c * (W^{(1)},\ W^{(1)}) \tag{7.14}
$$

$$\Pi_1 = F^{(1)} * (\dot{W}_c, W^{(2)}) + F^{(2)} * (\dot{W}_c, W^{(1)}) + \dot{F}_c * (W^{(1)}, W^{(2)}) \tag{7.15}$$

$$\Pi_2 = 2F^{(1)} * (\ddot{W}_c, W^{(1)}) + \ddot{F}_c * (W^{(1)}, W^{(1)}) \tag{7.16}$$

$$(\dot{\ })_c = \frac{\partial}{\partial \Lambda}(\ )_c \tag{7.17}$$

The subscript $(\ )_c$ denotes the fact that the prebuckling solution is evaluated at the bifurcation point. $F^{(i)}$ and $W^{(i)}$ ($i = 1 \sim 2$) are the solutions after the z-transformation. The short-hand notation used is given by Eq. (6.47).

For imperfect shells the variation of $\Lambda(\xi, \bar{\xi})$ in the vicinity of the bifurcation point $\Lambda = \Lambda_c$ is given by the following asymptotic expansion[67,104] (see also Figure 7.1)

$$(\Lambda-\Lambda_c)\xi = a\Lambda_c\xi^2 + b\Lambda_c\xi^3 + \ldots - \alpha\Lambda_c\bar{\xi} - \beta(\Lambda-\Lambda_c)\bar{\xi} + 0(\xi\bar{\xi}) \tag{7.18}$$

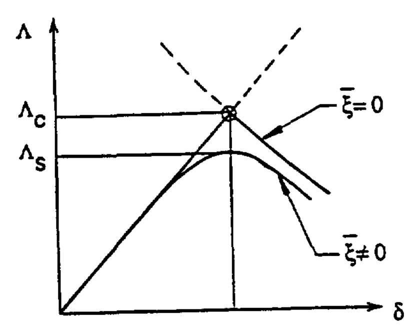


Fig. 7.1 Equilibrium paths of perfect and imperfect systems[67]

The imperfection form factors 'α' and 'β' are derived by GEPCAC as

$$\alpha = (1/\Lambda_c \hat{\Delta})\,[\,F_c * (\hat{W}, W^{(1)}) + F^{(1)} * (\hat{W}, W_c)\,] \tag{7.19}$$

$$\beta = (1/\hat{\Delta})\{\dot{F}_c * (\hat{W}, W^{(1)}) + F^{(1)} * (\hat{W}, \dot{W}_c) + \Pi_3 - \alpha\Lambda_c[\,(1/2)\Pi_4 + \Pi_5\,]\} \tag{7.20}$$

where

$$\begin{aligned}
\Pi_3 &= (\dot{W}_c, W^{(1)}) * (\hat{W}, W_c) \\
\Pi_4 &= 2F^{(1)} * (\ddot{W}_c, W^{(1)}) + \ddot{F}_c * (W^{(1)}, W^{(1)}) \\
\Pi_5 &= (\dot{W}_c, W^{(1)}) * (\dot{W}_c, W^{(1)})
\end{aligned} \tag{7.21}$$

in which

$$\begin{aligned}
(A, B) * (C, D) = &\int_0^{2\pi \sin\alpha_o} \int_0^{z_o} (e^{(2z)}/s_1^2)\,\{\,\{A_{11}(A+A_{,z})(B+B_{,z}) \\
&+ A_{12}A_{,\bar{\theta}}B_{,\bar{\theta}} + A_{16}[\,(A+A_{,z})B_{,\bar{\theta}} + (B+B_{,z})A_{,\bar{\theta}}\,](C+C_{,z})(D+D_{,z})\} \\
&+ \{A_{12}(A+A_{,z})(B+B_{,z}) + A_{22}A_{,\bar{\theta}}B_{,\bar{\theta}} + A_{26}[\,(A+A_{,z})B_{,\bar{\theta}} + (B+B_{,z})A_{,\bar{\theta}}\,]\,C_{,\bar{\theta}}D_{\bar{\theta}}\} \\
&+ \{A_{16}(A+A_{,z})(B+B_{,z}) + A_{26}A_{,\bar{\theta}}B_{,\bar{\theta}} + A_{66}[(A+A_{,z})B_{,\bar{\theta}} + (B+B_{,z})A_{,\bar{\theta}}\,] \\
&[\,(C+C_{,z})D_{,\bar{\theta}} + (D+D_{,z})C_{,\bar{\theta}}\,]\}\,\}\,dz\,d\bar{\theta}
\end{aligned} \tag{7.22}$$

The initial imperfection is assumed to be

$$\bar{W} = \bar{\xi}\hat{W} \tag{7.23}$$

where $\hat{W}$ represents the shape of the initial imperfection and $\bar{\xi}$ is the imperfection amplitude. Notice that if the initial imperfection is assumed to be affine to the buckling mode then

$$\hat{W} = W^{(1)} \tag{7.24}$$

Substituting the axisymmetric prebuckling expression and the Fourier decompositions for the buckling and postbuckling states into the Eqs. (7.12)-(7.13) and (7.20)-(7.21), and carrying out the lengthy integration in the θ direction by GEPCAC yields

$$
\begin{aligned}
a &= 0 \\
b &= -\Pi_b / (\Lambda_c \hat{\Delta}) \\
\alpha &= \Pi_\alpha / (\Lambda_c \hat{\Delta}) \\
\beta &= \{\Pi_\beta + \Pi_3 - \alpha\Lambda_c[\,(1/2)\Pi_4 + \Pi_5\,]\} / \hat{\Delta}
\end{aligned}
\tag{7.25}
$$

where

$$
\begin{aligned}
\hat{\Delta} = {} & (Et^4\pi \sin\alpha_o / cs_1) \int_0^{z_o} e^{z} \{ n^2\phi'(w_1^2 + w_2^2) - 2n^2\psi (f_1' w_1 + f_2' w_2) \\
& + \phi [\,(w_1' + w_1)^2 + (w_2' + w_2)^2\,] + 2\psi [\,f_1' w_1' + f_1' w_1 + f_2' w_2' + f_2' w_2 \\
& + (1 - n^2)(w_1' f_1 + w_2' f_2 + f_1 w_1 + f_2 w_2)\,] \}\, dz
\end{aligned}
\tag{7.26}
$$

$$
\begin{aligned}
\Pi_4/2 = {} & (Et^4\pi \sin\alpha_o / 2cs_1) \int_0^{z_o} e^{z} \{ n^2\ddot{\phi}'(w_1^2 + w_2^2) - 2n^2\ddot{\psi} (f_1' w_1 + f_2' w_2) \\
& + \ddot{\phi} [\,(w_1' + w_1)^2 + (w_2' + w_2)^2\,] + 2\ddot{\psi} [\,f_1' w_1' + f_1' w_1 + f_2' w_2' + f_2' w_2 \\
& + (1 - n^2)(w_1' f_1 + w_2' f_2 + f_1 w_1 + f_2 w_2)\,] \}\, dz
\end{aligned}
\tag{7.27}
$$

$$
\begin{aligned}
\Pi_5 = {} & (Et^5\pi / s_1^2) \int_0^{z_o} \psi^2 e^{2z} \{ n^2 \bar{A}_{66} (w_1^2 + w_2^2) \\
& + 2n\bar{A}_{16} (w_1' w_2 - w_2' w_1) + \bar{A}_{11} [\,(w_1' + w_1)^2 + (w_2' + w_2)^2\,] \}\, dz
\end{aligned}
\tag{7.28}
$$

$$
\begin{aligned}
\Pi_b = (Et^4\pi\sin\alpha_o/2cs_1)\int_0^{z_o} e^{z}\, \{ \\
& 2w_\beta\,[\,2n^2f_1''w_1 + f_1'(-2n^2w_1'+w_1'+n^2w_1+w_1) - 2n^2f_2''w_2 \\
& \quad + f_2'(2n^2w_2'-w_2'-n^2w_2-w_2) + (1-n^2)w_1'f_1 \\
& \quad - (1-n^2)w_2'f_2 + (1-n^2)f_1w_1 - (1-n^2)f_2w_2\,] \\
& + 2w_\gamma\,[\,2n^2f_1''w_2 + f_1'(-2n^2w_2'+w_2'+n^2w_2+w_2) + 2n^2f_2''w_1 \\
& \quad + f_2'(-2n^2w_1'+w_1'+n^2w_1+w_1) + (1-n^2)f_2w_1' \\
& \quad + (1-n^2)w_2'f_1 + (1-n^2)(f_1w_2+f_2w_1)\,] \\
& + 4\overline{w}_\alpha\,[\,f_1'w_1' + (1-n^2)f_1'w_1 + f_2'w_2' + (1-n^2)f_2'w_2 \\
& \quad + (1-n^2)w_1'f_1 + (1-n^2)w_2'f_2 + (1-n^2)(f_1w_1+f_2w_2)\,] \\
& + f_\beta\,[\,w_1'^2(1-4n^2) + 2w_1'w_1(1-4n^2) - w_2'^2(1-4n^2) \\
& \quad - 2w_2'w_2(1-4n^2) + w_1^2(1-4n^2) - w_2^2(1-4n^2)\,] \\
& + 2f_\gamma\,[\,w_1'w_2'(1-4n^2) + w_1'w_2(1-4n^2) + w_2'w_1(1-4n^2) + w_1w_2(1-4n^2)\,] \\
& + 2\overline{f}_\alpha\,[\,(w_1'+w_1)^2 + (w_2'+w_2)^2\,] \\
& + 2w_\beta'\,[\,f_1'w_1' + (1+n^2)f_1'w_1 - f_2'w_2' - (1+n^2)f_2'w_2 + (1-n^2)w_1'f_1 \\
& \quad - (1-n^2)w_2'f_2 + (1-n^2)w_1f_1 - (1-n^2)w_2f_2\,] \\
& + 2w_\gamma'\,[\,f_1'w_2' + (1+n^2)f_1'w_2 + f_2'w_1' + (1+n^2)f_2'w_1 \\
& \quad + (1-n^2)w_1'f_2 + (1-n^2)w_2'f_1 + (1-n^2)(f_1w_2+f_2w_1)\,] \\
& + f_\beta'\,[\,w_1'^2 + 2(1-2n^2)w_1'w_1 - 2(1-2n^2)w_2'w_2 - w_2'^2 \\
& \quad + (1-5n^2)w_1^2 - (1-5n^2)w_2^2\,] \\
& + 2f_\gamma'\,[\,w_1'w_2' + (1-2n^2)w_1'w_2 + (1-2n^2)w_2'w_1 + (1-5n^2)w_1w_2\,] \\
& + 2n^2\overline{f}_\alpha'(w_1^2+w_2^2) + n^2f_\beta''(w_2^2-w_1^2) - 2n^2f_\gamma''w_1w_2\,\}\,dz
\end{aligned}
\tag{7.29}
$$

$$\Pi_\alpha = (Et^4\pi \sin\alpha_o/s_1c)\int_0^{z_o} e^z\, \{\, n^2\varphi'(w_1\hat{w}_1 + w_2\hat{w}_2) - n^2\psi(f_1'\hat{w}_1 + f_2'\hat{w}_2)$$
$$+ \psi\,[\, f_1'\hat{w}_1' + f_1'\hat{w}_1 + f_2'\hat{w}_2' + f_2'\hat{w}_2 + (1-n^2)\hat{w}_1'f_1$$
$$+ (1-n^2)\hat{w}_2'f_2 + (1-n^2)(f_2\hat{w}_2 + f_1\hat{w}_1)\,] \tag{7.30}$$
$$+ \varphi\,[\, w_1'\hat{w}_1' + w_1'\hat{w}_1 + w_2'\hat{w}_2' + w_2'\hat{w}_2 + \hat{w}_1'w_1$$
$$+ \hat{w}_2'w_2 + w_1\hat{w}_1 + w_2\hat{w}_2\,]\,\}\, dz$$

$$\Pi_\beta = (Et^4\pi \sin\alpha_o/s_1c)\int_0^{z_o} e^z\, \{\, n^2\phi'(w_1\hat{w}_1 + w_2\hat{w}_2) - n^2\Psi(f_1'\hat{w}_1 + f_2'\hat{w}_2)$$
$$+ \Psi\,[\, f_1'\hat{w}_1' + f_1'\hat{w}_1 + f_2'\hat{w}_2' + f_2'\hat{w}_2 + (1-n^2)\hat{w}_1'f_1$$
$$+ (1-n^2)\hat{w}_2'f_2 + (1-n^2)(f_2\hat{w}_2 + f_1\hat{w}_1)\,] \tag{7.31}$$
$$+ \phi\,[\, w_1'\hat{w}_1' + w_1'\hat{w}_1 + w_2'\hat{w}_2' + w_2'\hat{w}_2 + \hat{w}_1'w_1$$
$$+ \hat{w}_2'w_2 + w_1\hat{w}_1 + w_2\hat{w}_2\,]\,\}\, dz$$

$$\Pi_3 = (Et^5\pi/s_1^2)\int_0^{z_o} \Psi\,\psi\, e^{2z}\, \{\, n^2\bar{A}_{66}(w_1\hat{w}_1 + w_2\hat{w}_2)$$
$$+ n\bar{A}_{16}\,(w_1'\hat{w}_2 - w_2'\hat{w}_1 + \hat{w}_1'w_2 - \hat{w}_2'w_1) \tag{7.32}$$
$$+ \bar{A}_{11}\,[\, w_1'(\hat{w}_1' + \hat{w}_1) + w_2'(\hat{w}_2' + \hat{w}_2) + w_1(\hat{w}_1' + \hat{w}_1) + w_2(\hat{w}_2' + \hat{w}_2)\,]\,\}\, dz$$

where $$\bar{W} = t\bar{\xi}(\hat{w}_1\cos n\bar{\theta} + \hat{w}_2\sin n\bar{\theta}) \tag{7.33}$$

It is known that for many practical applications where a unique buckling mode is associated with the lowest buckling load and the buckling and initial postbuckling behaviors are symmetric with respect to the buckling displacement, i.e., the buckling modal displacement $W^{(1)}$ and stress function $F^{(1)}$ are harmonic with respect to the circumferential coordinate, the first postbuckling coefficient 'a' is identically equal to zero. In this case the sign of 'b' determines whether the load initially increases or decreases after bifurcation. A positive 'b' factor characterizes a bifurcation path for which an increase in load is required for further deformation and signifies that the transition from axisymmetric to asymmetric behavior occurs smoothly without a sudden loss in load carrying capacity. Thus the corresponding imperfect structure is said to be imperfection-insensitive. On the other hand, a negative 'b' factor characterizes a bifurcation path for which a reduction in load is required in order to maintain the equilibrium of the system and signifies that the transition from axisymmetric to asymmetric behavior occurs with a sharp snap-buckling. Thus the corresponding imperfect structure is said to be imperfection-sensitive.

Knowing only whether the structure is imperfection-sensitive or not is not enough. The ultimate aim of imperfection sensitivity analysis is to determine the maximum load-carrying capacity. As can be seen from Figure 7.1 the buckling load of the imperfect structure Λ_s occurs at the 'limit point' of the prebuckling state. If the limit point is close enough to the bifurcation point then Λ_s, the maximum load that the structure can support prior to buckling, can also be evaluated from Eq. (7.18) by maximizing Λ with respect to ξ. In the case of 'a' equal to zero using Eq. (7.18) to maximize Λ with respect to ξ leads to the following modified Koiter formula[177]

$$(1-\rho_s)^{3/2} = (3/2)\sqrt{-3\alpha^2 b}\,[\,1-(\beta/\alpha)(1-\rho_s)\,]\,|\bar{\xi}| \tag{7.34}$$

where $\rho_s = \Lambda_s/\Lambda_c$ and $\bar{\xi}$ is the normalized amplitude of the initial imperfection. It should be emphasized that in all cases presented, $\bar{\xi}$ has been normalized with respect to the shell thickness.

In practice, however, the design engineers must obtain an estimate of the 'knockdown' factor with which they have to multiply the buckling load prediction of the perfect structure in order to arrive at the safe allowable load level. With the help of Eq. (7.34) such an estimate can be computed if besides the imperfection sensitivity coefficient $\bar{b} = \alpha^2 b$ one also knows the size of the amplitude $\bar{\xi}$ of the imperfection. For such cases the predictions of Eq. (7.34) are conveniently summarized in Figure 7.2.

Notice that in the above analysis only the deterministic imperfections are considered. In the realistic situation, however, the imperfections are known to be stochastic rather than deterministic properties. Therefore, to obtain more accurate results one has to establish the characteristic initial imperfection distribution that a given fabrication process is likely to produce, and then to combine this information with some kind of statistical analysis of both the initial imperfections and the corresponding critical loads, a kind of statistical imperfection-sensitivity analysis[179].

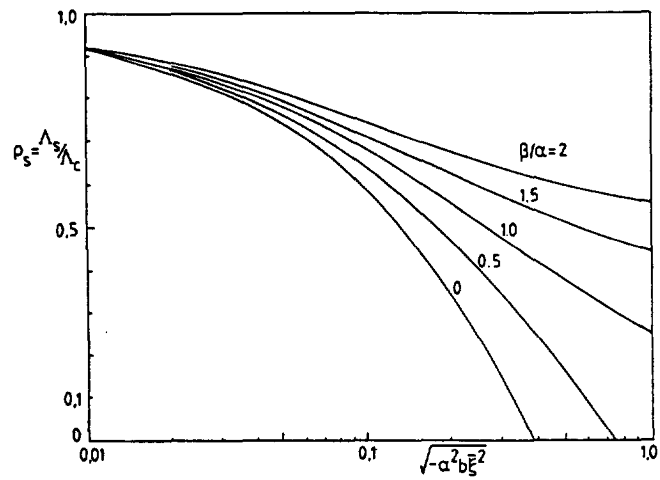


Fig. 7.2 *Estimate of critical loads for imperfection sensitive structures*[177]

7.3 Numerical Analysis and Corresponding Program

Introducing as a unified variable the 20-dimensional vector $\mathrm{Y}^{(2)}$ defined as follows

$$\begin{aligned}
&Y_1^{(2)} = f_\beta, \quad Y_2^{(2)} = f_\gamma, \quad Y_3^{(2)} = \bar{w}_\alpha, \quad Y_4^{(2)} = w_\beta, \quad Y_5^{(2)} = w_\gamma, \\
&Y_6^{(2)} = f_\beta', \quad Y_7^{(2)} = f_\gamma', \quad Y_8^{(2)} = \bar{w}_\alpha', \quad Y_9^{(2)} = w_\beta', \quad Y_{10}^{(2)} = w_\gamma', \\
&Y_{11}^{(2)} = f_\beta'', \quad Y_{12}^{(2)} = f_\gamma'', \quad Y_{13}^{(2)} = \bar{f}_\alpha, \quad Y_{14}^{(2)} = w_\beta'', \quad Y_{15}^{(2)} = w_\gamma'', \\
&Y_{16}^{(2)} = f_\beta''', \quad Y_{17}^{(2)} = f_\gamma''', \quad Y_{18}^{(2)} = \bar{f}_\alpha', \quad Y_{19}^{(2)} = w_\beta''', \quad Y_{20}^{(2)} = w_\gamma'''
\end{aligned} \tag{7.35}$$

then the system of equations (7.6)-(7.11) can be reduced to the following inhomogeneous 2-point boundary value problem

$$\frac{d}{dz} Y^{(2)} = f^{(2)}(z, Y^{(o)}, Y^{(2)}; \lambda, \bar{p}) + \hat{f}(z, Y^{(1)}) \tag{7.36}$$

$$B_1^{(2)} Y^{(2)}(z=0) + B_2^{(2)} Y^{(2)}(z=z_o) = B_3^{(2)} \tag{7.37}$$

where the components of the boundary matrices $\mathbf{B}_1^{(2)}$, $\mathbf{B}_2^{(2)}$ and $\mathbf{B}_3^{(2)}$ depend on the boundary conditions at the shell edges. Notice that here the 4-dimensional vector $\mathbf{Y}^{(o)}$ contains the known solution of the prebuckling problem and that the 16-dimensional vector $\mathbf{Y}^{(1)}$ is the eigenvector of the buckling problem.

Due to the complicated functions of z represented by the known prebuckling and buckling solutions anything but a numerical solution of this linear, inhomogeneous 2-point boundary value problem is out of question. Because of earlier successful experiences with the 'shooting method' applied for the prebuckling and buckling problems it was decided to solve the postbuckling problem also by the numerical technique known as 'parallel shooting over n-intervals'.

After the solutions of the buckling and postbuckling problems have been obtained, one must evaluate the integrals involved in the definition of the postbuckling coefficients 'a' and 'b', and of the imperfection form factors 'α' and 'β'. It has been shown in Ref. [67] that it is advantageous to evaluate the above integrals by solving initial value problems rather than using numerical integration schemes. The same approach is used here.

The program SAAC, Stability Analysis of Anisotropic Conical Shells, is written for the buckling and initial postbuckling analysis of anisotropic conical shells. SAAC is based on ANILISA[67] and BAAC2, in which maximum 40 intervals can be used in the 'shooting' procedure and different kinds of boundary conditions specified at the two edges can be rigorously enforced. As an option, membrane-like, linear and nonlinear prebuckling solutions can be used.

7.4 Numerical Results

To test the accuracy and the reliability of the present formulations for conical shells and the corresponding program SAAC, the results obtained from present theory are compared with those of previous investigations. Among these comparisons is Booton's glass/epoxy (30°, 0°, -30°) composite cylindrical shell calculated via ANILISA[67]. Its geometric and material data are given in Table 5.6. Some mode shapes are depicted in Figures 7.3-7.6, while the results of comparisons are listed in Table 7.1. As can be seen the agreement is good.

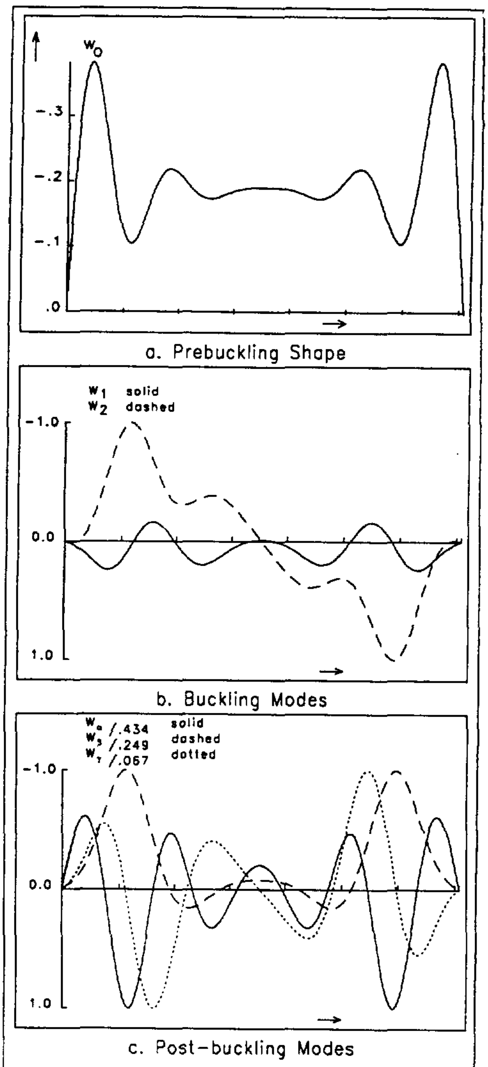


Fig. 7.3 Mode shapes of Booton-type glass-epoxy cylindrical shell (30°, 0°, -30°) under axial compression with SS3 boundary conditions using nonlinear prebuckling solution

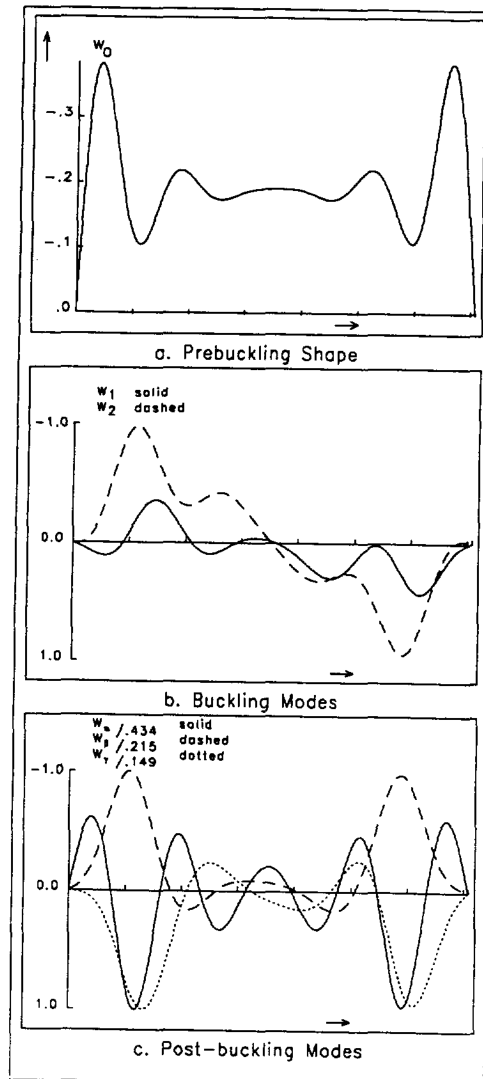


Fig. 7.4 Mode shapes of Booton-type glass-epoxy conical shell (α_o = 0.2°) under axial compression with MSS3 boundary conditions using nonlinear prebuckling solution

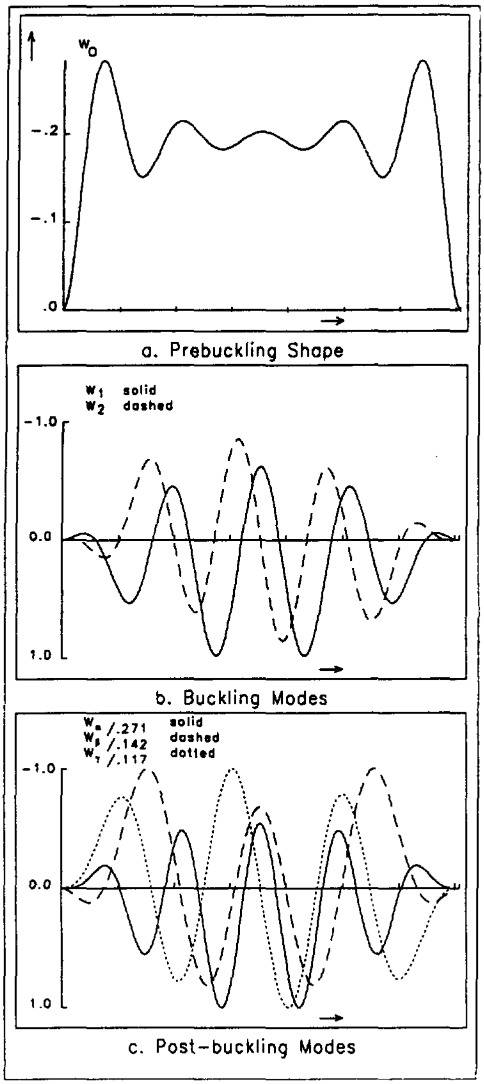


Fig. 7.5 Mode shapes of Booton-type glass-epoxy cylindrical shell (30°, 0°, -30°) under axial compression with C4 boundary conditions using nonlinear prebuckling solution

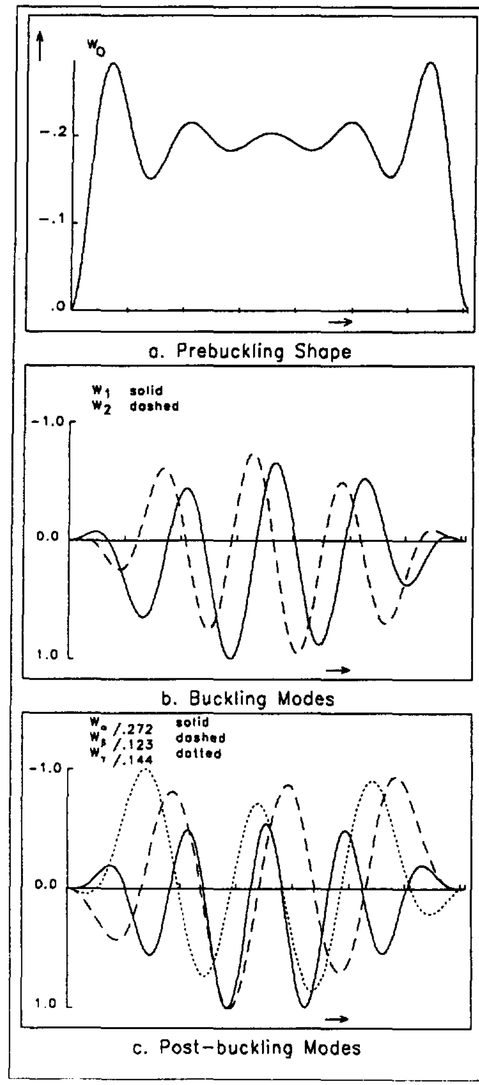


Fig. 7.6 Mode shapes of Booton-type glass-epoxy conical shell (α_o = 0.2°) under axial compression with MC4 boundary conditions using nonlinear prebuckling solution

Table 7.1 Comparison of results with a composite cylindrical shell under axial compression

Cylindrical Shells					Conical Shells (α_0 = 0.2°)				
B.C.	λ	b	α	β	B.C.	λ	b	α	β
SS3	0.39303(7)	-0.36634	0.48043	-0.04943	MSS3	0.39306(7)	-0.36399	0.48166	-0.048798
SS4	0.40556(8)	-0.31596	0.40169	-0.17038	MSS4	0.40577(8)	-0.29423	0.39841	-0.18756
C1	0.40790(6)	-0.28671	0.87486	0.33970	C1	0.40794(6)	-0.29375	0.87457	0.33535
C2	0.41178(6)	-0.025474	0.97729	0.99583	MC2	0.41179(6)	-0.026089	0.97694	0.99475
C4	0.41194(6)	-0.034472	0.97320	0.97137	MC4	0.41195(6)	-0.034937	0.97294	0.97093

Notice that the numbers in the brackets indicate the wave numbers in the circumferential direction.

When one considers the imperfection sensitivity of anisotropic conical shells, there are innumerous combinations with different geometric, material and loading parameters possible and one can choose from different types of boundary conditions. However, in the following, the initial imperfection sensitivity computations are limited to a few cases of interest, based on the assumption that the shapes of the initial imperfections are affine to the corresponding buckling modes. In all cases only the imperfection sensitivity of the lowest buckling load is calculated, and the nonlinear prebuckling solution is employed.

First, the influence of wall constructions on the imperfection sensitivity of three-layer Khot-type conical shells (α_0 = 10°) with fiber orientations in outer, middle and inner layers denoted by (-θ°, θ°, 0°) is studied. The shell is made of either glass-epoxy or boron-epoxy composites. The corresponding elastic constants are listed in Table 7.2[19]. In this study only the fiber orientations are varied. The effect of fiber orientations on the b, α and β factors of the aforementioned conical shells with MSS3 boundary conditions under axial compression is shown in Table 7.3 and Figure 7.7.

Table 7.2 Elastic constants of Khot-type composite conical shells

Constants \ Material	glass-epoxy	boron-epoxy
E_{11}	7.5 x 10^6 psi	40 x 10^6 psi
E_{22}	3.5 x 10^6 psi	4.5 x 10^6 psi
ν_{12}	0.25	0.25
G	1.25 x 10^6 psi	1.5 x 10^6 psi
ν_{21}	ν_{12} E_{22} / E_{11}	ν_{12} E_{22} / E_{11}

where R_1 = 6.0 inches, t = 0.036 inch, L = 12.5 inches.

Table 7.3 Effect of fiber orientations on the α and β factors of Khot-type conical shells (α_o = 10^o) with MSS3 boundary conditions under axial compression

Material	θ°	λ	α	β	$\alpha^2 b$	β/α
Boron-epoxy	10	0.17830 (11)	1.0027	1.0024	-0.033548	0.99962
	20	0.21724 (16)	0.99885	0.99697	-0.041183	0.99812
	30	0.22722 (15)	0.99273	0.98355	-0.043874	0.99075
	40	0.233604(16)	0.99955	0.99772	-0.037644	0.99817
	50	0.23063 (16)	1.0032	1.0037	-0.028567	1.0005
	60	0.19303 (13)	1.0069	1.0090	-0.023651	1.0022
	70	0.15956 (12)	1.0038	1.0059	-0.017310	1.0021
	80	0.14243 (12)	1.0018	1.0042	-0.013813	1.0023
Glass-epoxy	10	0.51014 (12)	0.87342	0.56772	-0.49278	0.65000
	20	0.51915 (13)	0.84710	0.47124	-0.41052	0.55630
	30	0.52836 (13)	0.82534	0.43087	-0.34115	0.52205
	40	0.53667 (13)	0.87531	0.60593	-0.28898	0.69225
	50	0.54023 (13)	0.95402	0.86047	-0.23464	0.90194
	60	0.53574 (13)	0.98378	0.95377	-0.16190	0.96949
	70	0.519361(12)	0.98557	0.95724	-0.20782	0.97125
	80	0.50804 (11)	0.99078	0.97230	-0.11995	0.98135

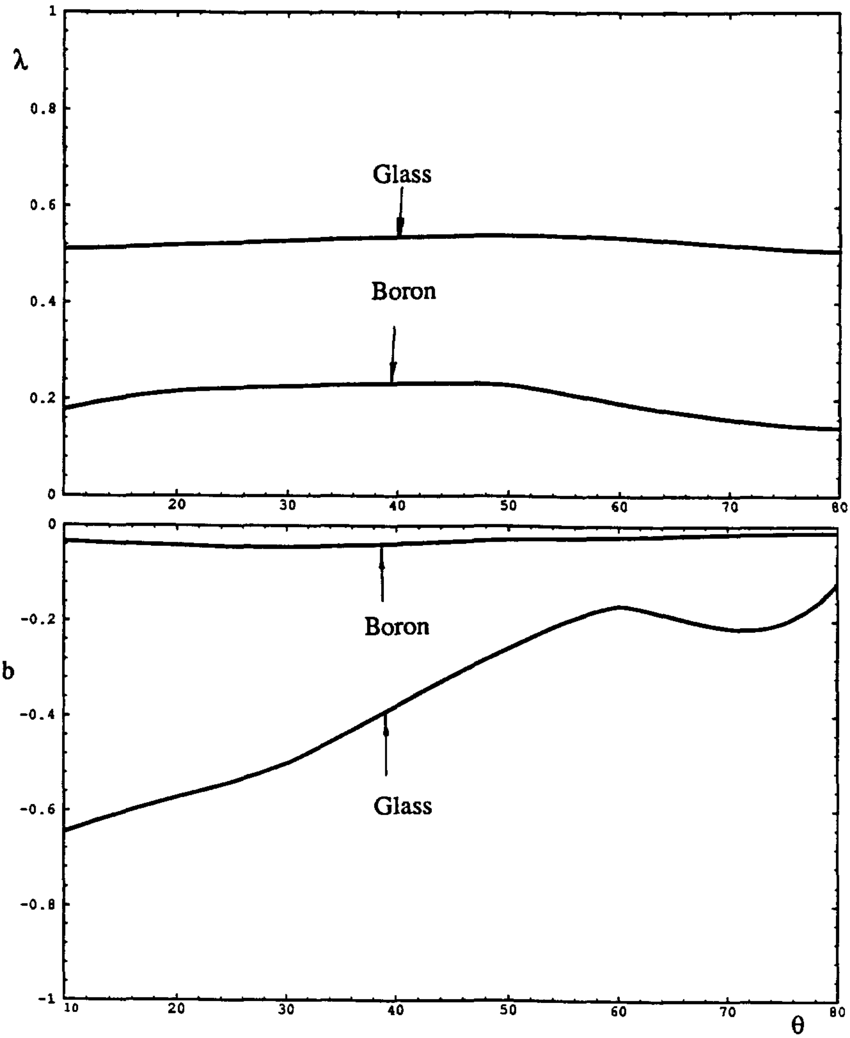


Fig. 7.7 *Effect of fiber orientation on the b factor of Khot-type conical shells (α_o = 10°) with MSS3 boundary conditions under axial compression*

From Figure 7.7 one can see that the magnitude of the b factor is much greater for glass-epoxy shells than for boron-epoxy shells. This indicates that the glass-epoxy shells are more imperfection sensitive than the boron-epoxy shells. This conclusion agrees with that of composite cylindrical shells[17,19].

From Table 7.3 it can be seen that for boron-epoxy shells the imperfection form factors α and β are almost constant, while for glass-epoxy shells α and β factors change as θ changes.

Calculations are also carried out for Khot-type (-40°, 40°, 0°) glass-epoxy shells under hydrostatic pressure with MSS3 boundary conditions. The results of Khot-type shells are listed in Table 7.4. The imperfection sensitivity of the Khot-type anisotropic shells is shown in Figure 7.8. The corresponding mode shapes are shown in Figures 7.9-7.12. It is found that the anisotropic conical shells with small semi-vertex angles have about the same imperfection sensitivities as the corresponding cylindrical shell. For increasing of the cone angle, the imperfection sensitivity will increase.

Table 7.4 Results of Khot-type cylindrical and conical anisotropic shells under hydrostatic pressure

Results \ Shells	Cylindrical shell (α_o = 0°)	Conical shell (α_o=0.1°)	Conical shell (α_o=10°)	Conical shell (α_o=30°)
p	0.031571(7)	0.031489(7)	0.024699(8)	0.0157214(9)
b	-0.024938	-0.025604	-0.028759	-0.04195
α	0.99960	0.99960	0.99958	0.99972
β	0.99962	0.99961	0.99960	0.99982
$\alpha^2 b$	-0.024914	-0.025584	-0.028735	-0.041922
β/α	1.000	1.000	1.000	1.000

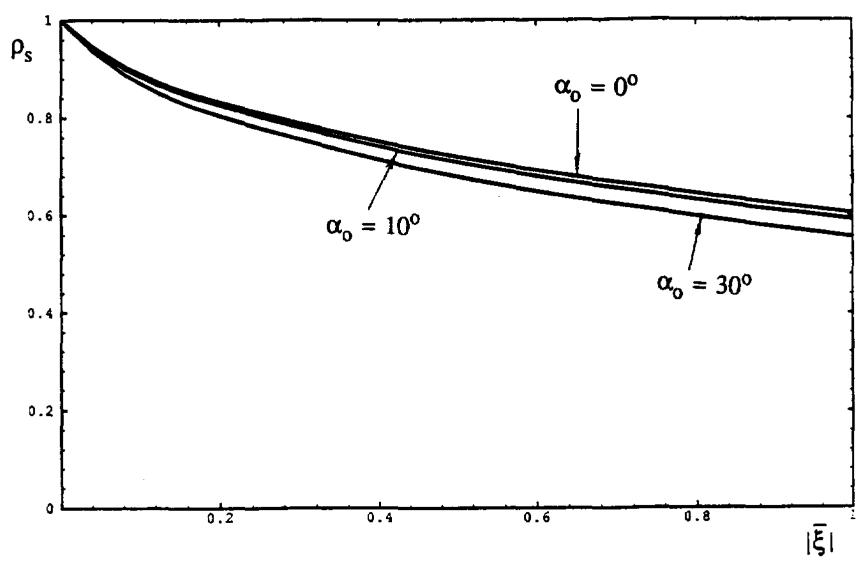


Fig. 7.8 Imperfection sensitivities of Khot-type glass-epoxy shells (-40°, 40°, 0°) with MSS3 boundary conditions under hydrostatic pressure

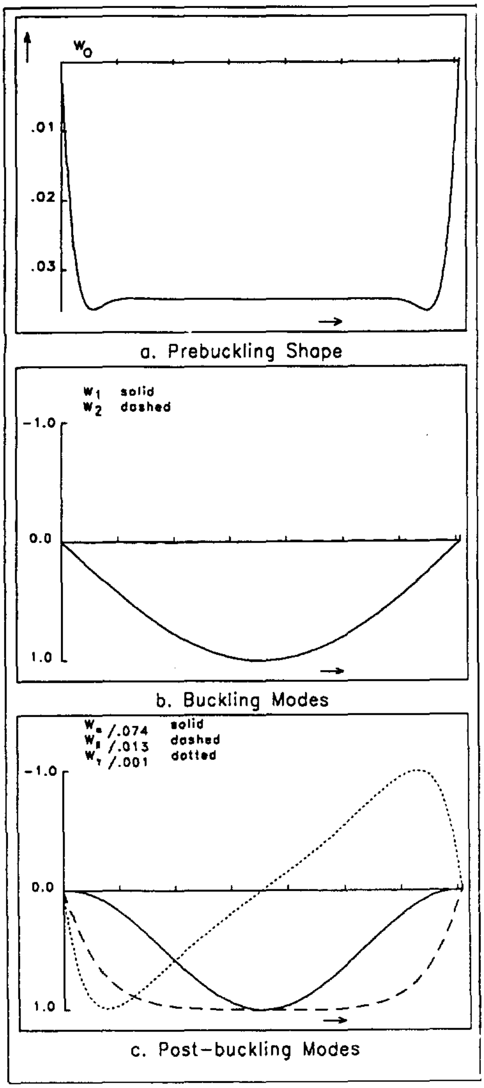


Fig.7.9 Mode shapes of Khot-type glass-epoxy cylindrical shell (-40°, 40°, 0°) under hydrostatic pressure with SS3 boundary conditions using nonlinear prebuckling solution

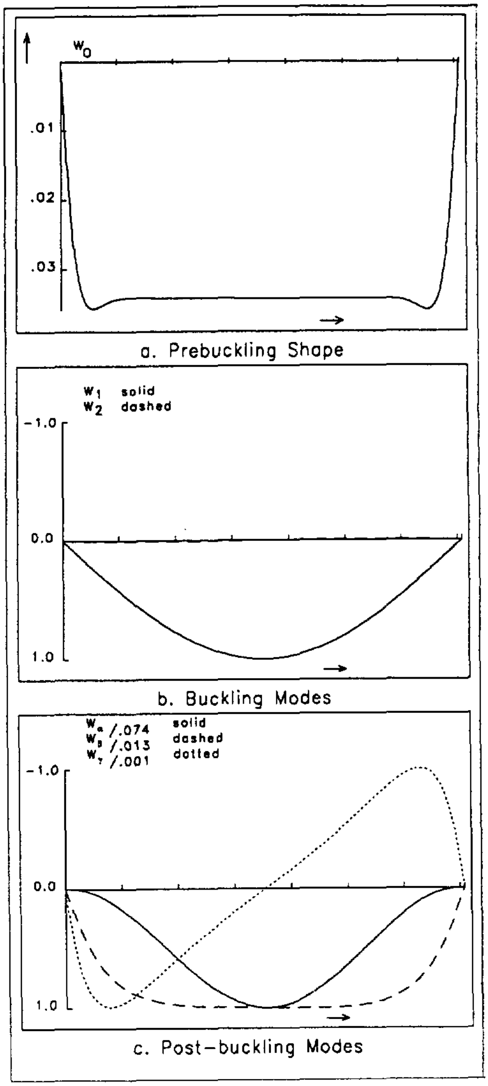


Fig.7.10 Mode shapes of Khot-type glass-epoxy conical shell (α_o = 0.1°) under hydrostatic pressure with MSS3 boundary conditions using nonlinear prebuckling solution

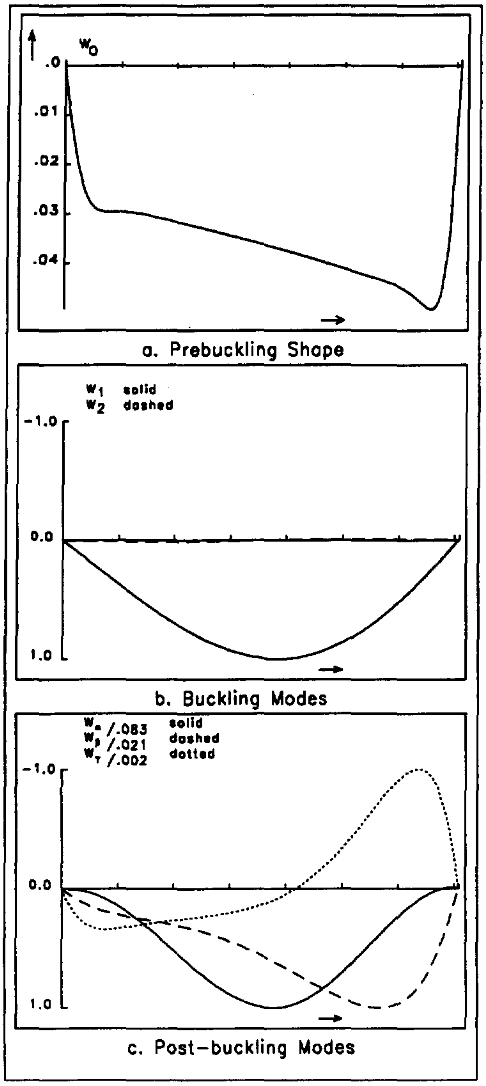


Fig.7.11 Mode shapes of Khot-type glass-epoxy conical shell (α_o = 10°) under hydrostatic pressure with MSS3 boundary conditions using nonlinear prebuckling solution

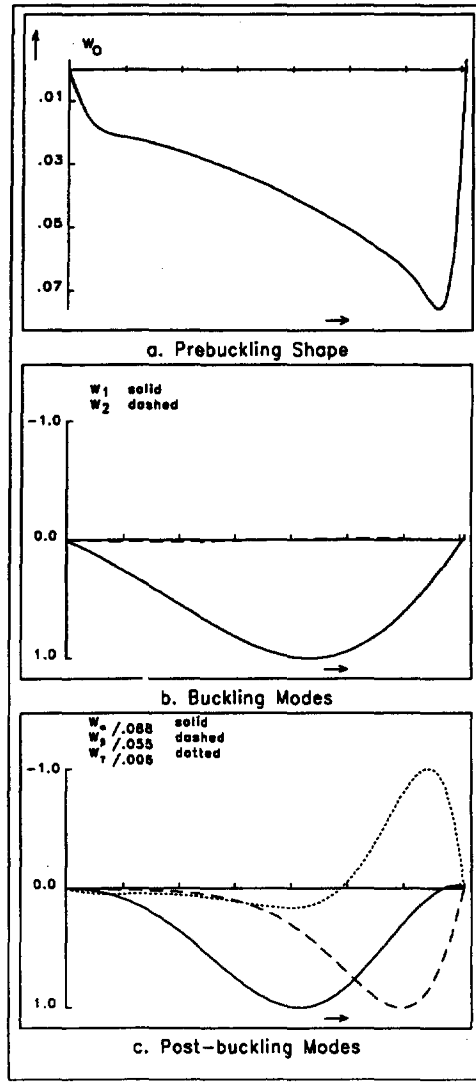


Fig.7.12 Mode shapes of Khot-type glass-epoxy conical shell (α_o = 30°) under hydrostatic pressure with MSS3 boundary conditions using nonlinear prebuckling solution

Further calculations are made to study the effect of different fiber orientations on the b, α and β factors of Booton-type anisotropic conical shells (α_0 = 30°) with MSS3 and MC4 boundary conditions under axial compression. The effects of fiber orientation on the b factor and on the α and β factors are shown in Figure 7.13 and Table 7.5. The influence of boundary conditions on the imperfection sensitivity of the lowest buckling load of the axially compressed Booton-type glass-epoxy conical shell (30°, 0°, -30°) is shown in Figure 7.14.

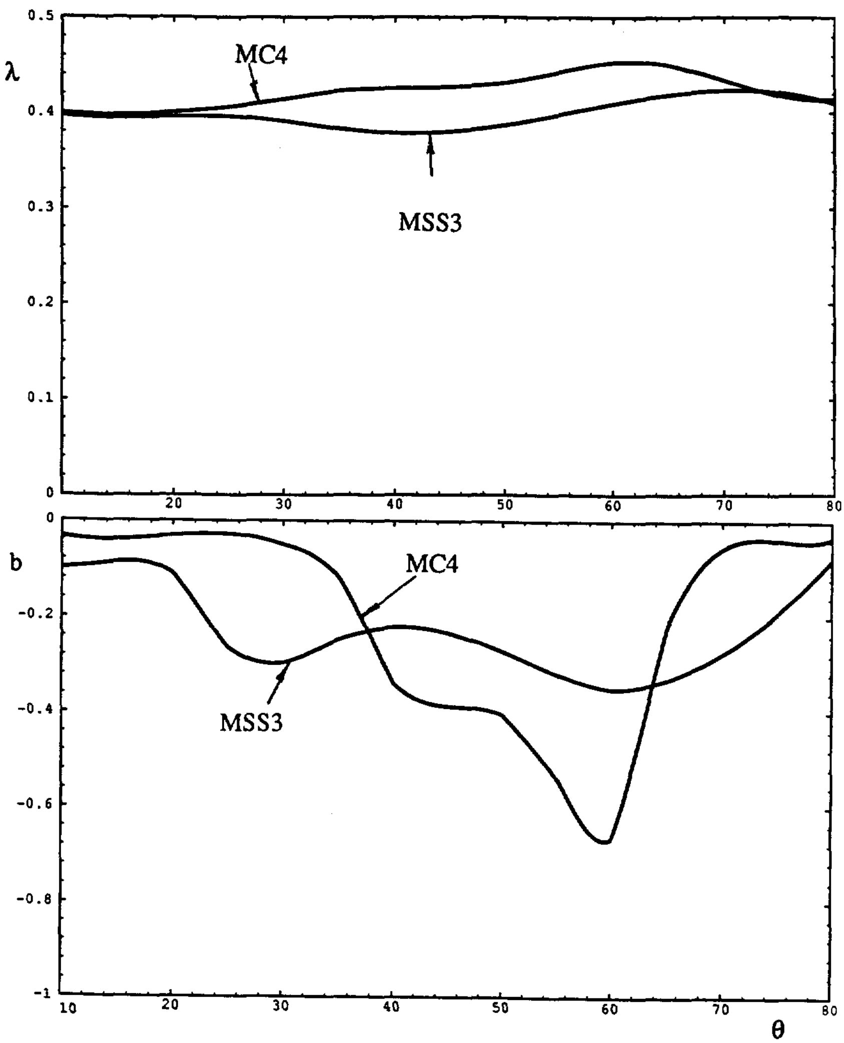


Fig. 7.13 *Effect of fiber orientation on the b factor of Booton-type anisotropic conical shells (α_o = 30°) with MSS3 and MC4 boundary conditions under axial compression*

Table 7.5 *Effect of fiber orientations on the α and β factors of Booton-type anisotropic conical shells under axial compression ($L/R_1 = 2$, $\alpha_o = 30^o$)*

B.C.	θ°	λ	α	β	$\alpha^2 b$	β/α
MSS3	10, 0, -10	0.39719 (10)	0.97040	0.92499	-0.094014	0.95320
	20, 0, -20	0.39590 (8)	0.92794	0.83154	-0.095481	0.89612
	25, 0, -25	0.39628 (7)	0.74148	0.43607	-0.145130	0.58811
	30, 0, -30	0.39104 (7)	0.46346	-0.015277	-0.063670	-0.03296
	35, 0, -35	0.38325 (7)	0.33271	-0.071518	-0.027227	-0.21495
	40, 0, -40	0.37884 (7)	0.27947	-0.054826	-0.017200	-0.19618
	50, 0, -50	0.38697 (8)	0.27061	-0.11328	-0.019787	-0.41861
	60, 0, -60	0.41020 (8)	0.36750	-0.22256	-0.047326	-0.60560
	70, 0, -70	0.42396 (10)	0.79387	0.33795	-0.173950	0.42570
	80, 0, -80	0.41014 (10)	0.97272	0.91835	-0.074945	0.94411
MC4	10, 0, -10	0.39992 (9)	0.99486	0.98630	-0.032651	0.99140
	20, 0, -20	0.40046 (9)	0.98800	0.96965	-0.030765	0.98143
	25, 0, -25	0.40620 (8)	0.98294	0.97100	-0.027416	0.98785
	30, 0, -30	0.41432 (7)	0.96204	0.95240	-0.047103	0.98999
	35, 0, -35	0.42338 (6)	0.89801	1.1179	-0.093904	1.2449
	40, 0, -40	0.42581 (8)	0.26971	-1.4930	-0.024643	-5.5354
	50, 0, -50	0.43173 (9)	0.20226	-1.5272	-0.016533	-7.5506
	55, 0, -55	0.44280 (9)	0.27855	-1.6160	-0.041362	-5.8015
	60, 0, -60	0.45348 (9)	0.46452	-1.1044	-0.14369	-2.3775
	65, 0, -65	0.45055 (10)	0.85576	0.34236	-0.16130	0.40007
	70, 0, -70	0.43441 (10)	0.97187	0.89617	-0.052901	0.92211
	80, 0, -80	0.41515 (9)	0.99370	0.98150	-0.033686	0.98772

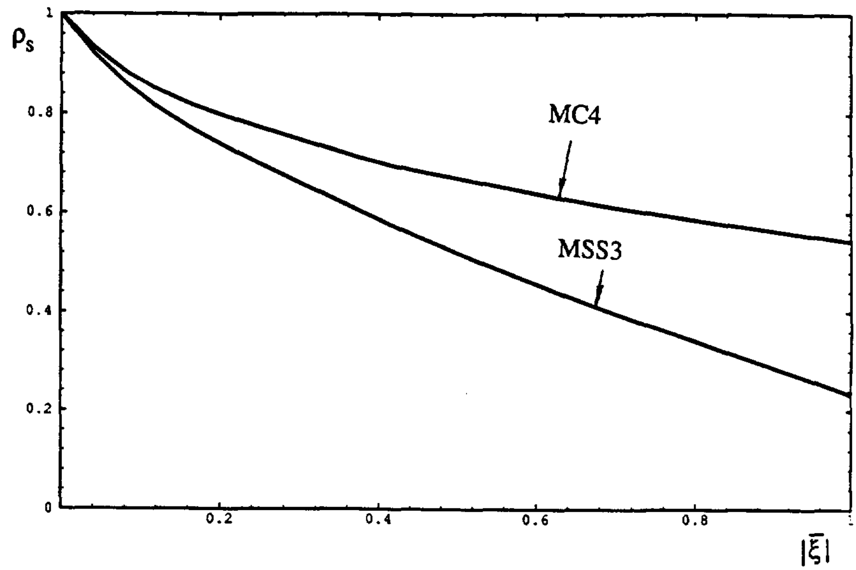


Fig. 7.14 *Imperfection sensitivity of Booton-type anisotropic conical shells (α_o = 30°) with MSS3 and MC4 boundary conditions under axial compression*

From Figure 7.13 one can see that an increase in bifurcation buckling load is accompanied by an increase in imperfection sensitivity. This indicates that a design with higher buckling load is not always the ideal one if one accounts for the possible greater imperfection sensitivity. Therefore, to obtain an optimum design one has to balance these two aspects.

From Figure 7.14 it is seen that the boundary conditions have significant influence on the imperfection sensitivity of Booton-type anisotropic conical shells.

From Table 7.5 one can seen that the imperfection form factors α and β are functions of the fiber orientation θ.

The effect of different fiber orientations on the b, α and β factors of Booton-type anisotropic conical shells (α_o = 30°) with MSS3 and MC4 boundary conditions under hydrostatic pressure is shown in Table 7.6. The imperfection sensitivities of Booton-type anisotropic conical shells (30°, 0°, -30°) with MSS3 boundary conditions under hydrostatic pressure is shown in Figure 7.15.

Table 7.6 *Effect of fiber orientations on b, α and β factors of anisotropic conical shells under hydrostatic pressure ($L/R_1 = 2$, $\alpha_o = 30^o$)*

B.C.	$\theta°$	$\bar{p}$	b	α	β	$\alpha^2 b$	β/α
MSS3	10, 0, -10	0.017684 (8)	-0.081284	0.99858	0.99900	-0.081053	1.0004
	20, 0, -20	0.018220 (8)	-0.078693	0.99890	0.99941	-0.078521	1.0005
	30, 0, -30	0.018830 (8)	-0.071244	0.99926	0.99983	-0.071138	1.0006
	40, 0, -40	0.019551 (8)	-0.062296	0.99955	1.0001	-0.062241	1.0006
	50, 0, -50	0.020552 (8)	-0.055561	0.99971	1.0002	-0.055528	1.0005
	60, 0, -60	0.022157 (8)	-0.051548	0.99971	1.0001	-0.051518	1.0004
	70, 0, -70	0.024328 (7)	-0.052262	0.99997	1.0003	-0.052258	1.0003
	80, 0, -80	0.026986 (7)	-0.047507	0.99986	1.0002	-0.047493	1.0003
MC4	10, 0, -10	0.022913 (9)	-0.083189	1.0076	1.0080	-0.084452	1.0004
	20, 0, -20	0.024275(10)	-0.085151	1.0076	1.0082	-0.086447	1.0006
	30, 0, -30	0.025607(10)	-0.085127	1.0083	1.0091	-0.086550	1.0007
	40, 0, -40	0.026924 (9)	-0.084816	1.0091	1.0098	-0.086367	1.0007
	50, 0, -50	0.028226 (9)	-0.074544	1.0077	1.0082	-0.075691	1.0005
	60, 0, -60	0.030087 (9)	-0.065261	1.0063	1.0067	-0.066085	1.0004
	70, 0, -70	0.032737 (9)	-0.058179	1.0055	1.0059	-0.058824	1.0004
	80, 0, -80	0.035000 (8)	-0.051761	1.0053	1.0056	-0.052313	1.0003

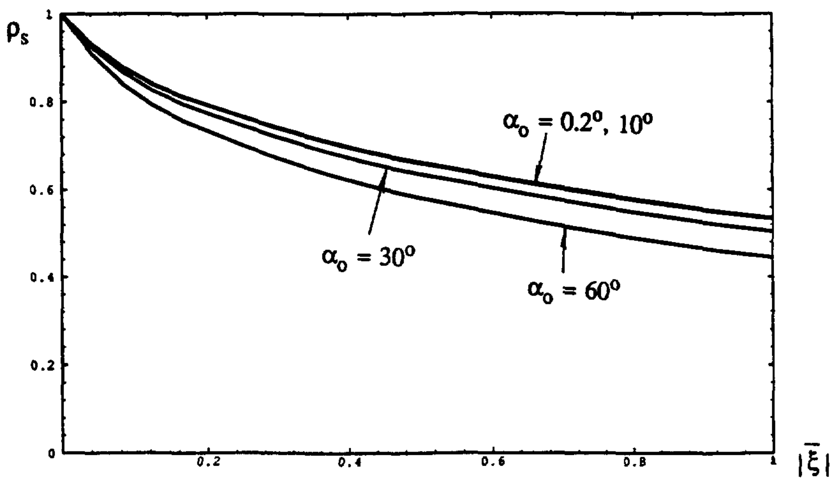


Fig. 7.15 *Imperfection sensitivities of Booton-type glass-epoxy conical shells (30°, 0°, -30°) with MSS3 boundary conditions under hydrostatic pressure*

From Table 7.6 one can see that the buckling load $\bar{p}$ increases as θ increases. This is attributed to the increase of stiffening in the circumferential direction. Unlike the case of conical shells under axial compression the magnitudes of b factor and $\alpha^2 b$ decrease as $\bar{p}$ increases for conical shells with both MSS3 and MC4 boundary conditions under hydrostatic pressure. Again, as in the case of Khot-type anisotropic shells, when increasing the cone angle, also the imperfection sensitivity will increase.

The effect of variations of shell geometry on the imperfection sensitivity of Booton-type anisotropic conical shells (30°, 0°, -30°) with MSS3 boundary conditions under axial compression is shown in Figure 7.16 and Table 7.7. As can be seen the conical shells having smaller semi-vertex angles are more imperfection sensitive than the ones with larger semi-vertex angles. As expected, if L/R_1 is fixed, with the increase of semi-vertex angle the shell becomes flat, and the magnitude of b-factor decreases.

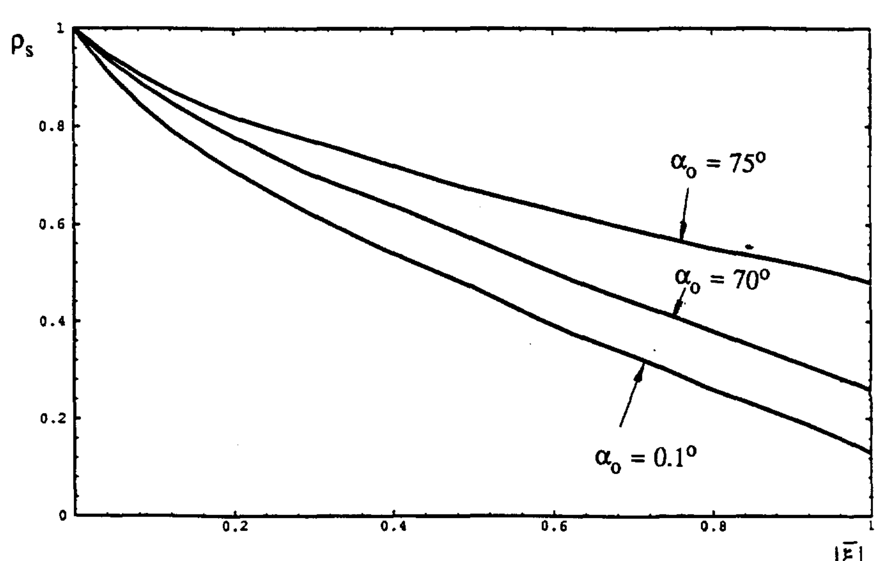


Fig. 7.16 *Imperfection sensitivities of Booton-type anisotropic conical shells having different geometries with MSS3 boundary conditions under axial compression*

Table 7.7 *Effect of shell geometry on b, α and β factors of Booton-type conical shells with MSS3 boundary conditions under axial compression (L/R_1 = 2)*

Semi-vertex angle \ Results	λ	b	α	β
0.2°	0.39306 (7)	-0.36399	0.48166	-0.04880
1.0°	0.39316 (7)	-0.34496	0.48689	-0.04720
2.0°	0.39328 (7)	-0.32602	0.49381	-0.04727
3.0°	0.39301 (8)	-0.35766	0.45919	-0.10239
4.0°	0.39266 (8)	-0.35684	0.46339	-0.10723
5.0°	0.39236 (8)	-0.36251	0.46698	-0.11297
6.0°	0.39212 (8)	-0.37249	0.47000	-0.11917
7.0°	0.39193 (8)	-0.37533	0.47250	-0.12547
8.0°	0.39179 (8)	-0.36646	0.47450	-0.13161
9.0°	0.39170 (8)	-0.35760	0.47604	-0.13736
10.0°	0.39164 (8)	-0.35003	0.47717	-0.14260
12.0°	0.39162 (8)	-0.33454	0.47823	-0.15115
15.0°	0.39174 (8)	-0.31995	0.47720	-0.15870
20.0°	0.39209 (8)	-0.30429	0.46978	-0.15843
25.0°	0.39198 (7)	-0.29805	0.47088	-0.032023
30.0°	0.39104 (7)	-0.29643	0.46346	-0.015277
35.0°	0.39011 (7)	-0.29276	0.45485	-0.015640
40.0°	0.38928 (7)	-0.28931	0.44373	-0.029359
45.0°	0.38862 (7)	-0.28560	0.42978	-0.052876
50.0°	0.38736 (6)	-0.27707	0.43723	-0.058773
55.0°	0.38523 (6)	-0.26770	0.42206	-0.085705
60.0°	0.38356 (6)	-0.25780	0.40320	-0.11250
65.0°	0.38284 (6)	-0.24923	0.37554	-0.15150
70.0°	0.37921 (5)	-0.23036	0.36922	-0.19572
75.0°	0.37743 (5)	-0.21383	0.31464	0.005348

7.5 Discussions and Conclusion

By rederiving Koiter's asymptotic formulae in forms suitable for conical shells, the initial postbuckling behavior and imperfection sensitivity of anisotropic conical shells are studied whereby the deterministic initial imperfection in the shape of the critical buckling mode is used. The results obtained from present theory show that

1). within the context of Koiter's initial postbuckling theory the computer program SAAC can be used successfully to investigate the imperfection sensitivity of the buckling load of anisotropic conical shells taking into account the effect of different boundary conditions and of nonlinear prebuckling solution.
2). anisotropic conical shells are generally imperfection sensitive. Thus, to obtain a reliable result for the stability behavior of anisotropic conical shells it is necessary to carry out the initial postbuckling and imperfection sensitivity analyses after the bifurcation buckling load has been found.

Despite the above results, there are still some unanswered problems. To know the extent to which buckling can be expected to be gradual or sudden, one has to calculate the postbuckling variation of the applied variable load Λ with the generalized displacement; to obtain more realistic and accurate results for the effect of initial imperfection, one has to establish the characteristic initial imperfection distribution that a given fabrication process is likely to produce, and then to combine this information with some kind of statistical analysis of both the initial imperfections and the corresponding critical loads; to ascertain that the buckling load obtained is the minimum, one has to study the possible nonlinear modal interaction. Finally, one has to ascertain the range of validity of the asymptotic theory presented by comparing its predictions with the results obtained by solving the nonlinear problem directly.

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Appendix 1
For Chapter Three

A1.1 REDUCE-based Package GEACS1

```
comment this is for the derivation of the Donnell-type nonlinear
governing equations of imperfect anisotropic conical shells in
terms of W and F with both variable and constant stiffness
coefficients (GEACS1)$

depend f, s,th$
depend w,s,th$
depend wb,s,th$
depend u,s,th$
depend v,s,th$
depend a11,s$
depend a12,s$
depend a22,s$
depend a16,s$
depend a26,s$
depend a66,s$
depend b11,s$
depend b12,s$
depend b21,s$
depend b22,s$
depend b16,s$
depend b26,s$
depend b62,s$
depend b61,s$
depend b66,s$
depend d11,s$
depend d12,s$
depend d22,s$
depend d16,s$
depend d26,s$
depend d66,s$

operator k11,k22,k12,n11,n22,n12,m11,m22,m12,ep11,ep22,ep12$
let k11=-df(w,s,2),
   k22=-(df(w,s)/s+df(w,th,2)/(s^2)),
   k12=-(df(w,s,th)/s-df(w,th)/(s^2)),
   n11=df(f,s)/s+df(f,th,2)/(s^2),
   n22=df(f,s,2),
   n12=df(f,th)/(s^2)-df(f,s,th)/s,
   m11=-(b11*n11+b21*n22+b61*n12)
      +(d11*k11+d12*k22+d16*k12*2),
   m22=-(b12*n11+b22*n22+b62*n12)
      +(d12*k11+d22*k22+d26*k12*2),
   m12=-(b16*n11+b26*n22+b66*n12)
      +(d16*k11+d26*k22+d66*k12*2),
   ep11=(a11*n11+a12*n22+a16*n12)
      +(b11*k11+b12*k22+b16*k12*2),
   ep22=(a12*n11+a22*n22+a26*n12)
      +(b21*k11+b22*k22+b26*k12*2),
   ep12=(a16*n11+a26*n22+a66*n12)
      +(b61*k11+b62*k22+b66*k12*2)$

% The out-of-plane equilibrium equation of the imperfect shell is;
equi:=df(s*n11*(df(w,s)+df(wb,s)),s)+n22*cot(al)+
    df(n22*(df(w,th)+df(wb,th))/s,th)+
    df(n12*(df(w,th)+df(wb,th)),s)+
    df(n12*(df(w,s)+df(wb,s)),th)+
    df(s*m11,s,2)-df(m22,s)+
    df(m22/s,th,2)+2*df(m12/s,th)+2*df(m12,s,th)+p*s;
length(ws);

left:=df(ep12,s,th)*s+df(ep12,th)-df(ep11,th,2)+df(ep11,s)*s
-2*df(ep22,s)*s-df(ep22,s,2)*(s^2)$

begin scalar ep11,ep22,ep12$
ep11:=df(u,s)+(df(w,s)^2)/2+df(wb,s)*df(w,s)$
ep22:=(u-w*cot(al))/s+df(v,th)/s+((df(w,th)/s)^2)/2
+df(w,th)*df(wb,th)/(s^2)$
ep12:=df(v,s)-v/s+df(u,th)/s+df(w,s)*df(w,th)/s
+(df(wb,th)*df(w,s)+df(wb,s)*df(w,th))/s$
right:=df(ep12,s,th)*s+df(ep12,th)-df(ep11,th,2)+s*df(ep11,s)
-2*s*df(ep22,s)-df(ep22,s,2)*s^2$
return right
end$

% The compatibility equation of the imperfect shell is;
comp:=(s^2)*(left-right);
length(ws);

% The governing equations with constant stiffness coefficients;
let df(a11,s)=0,df(a12,s)=0,df(a16,s)=0,df(a22,s)=0,df(a26,s)=0,
df(a66,s)=0,df(b11,s)=0,df(b12,s)=0,df(b16,s)=0,df(b21,s)=0,
df(b22,s)=0,df(b26,s)=0,df(b61,s)=0,df(b62,s)=0,df(b66,s)=0,
df(d11,s)=0,df(d12,s)=0,df(d16,s)=0,df(d22,s)=0,df(d26,s)=0,
df(d66,s)=0$

equili:=equi;
length(ws);
compa:=comp;
length(ws);
end;
```

A1.2 Equations (3.11) and (3.17) Generated by GEACS1

The equilibrium equation is:

```
 - 2*DF(F,S,TH,3)*S*B16 + DF(F,S,TH,3)*S*B62 + 2*DF(F,S,TH,2)*S*B11
                                                   2
 - 2*DF(F,S,TH,2)*S*B66 - 2*DF(F,S,TH)*DF(W,S,TH)*S
                                                       2
 + 2*DF(F,S,TH)*DF(W,TH)*S - 2*DF(F,S,TH)*DF(WB,S,TH)*S
 + 2*DF(F,S,TH)*DF(WB,TH)*S + 2*DF(F,S,TH)*S*B62 + 2*DF(F,S,TH)*S*B61
              4                       3                     3
 - DF(F,S,4)*S *B21 - 2*DF(F,S,3,TH)*S *B26 + DF(F,S,3,TH)*S *B61
              3                    3                  3
 - DF(F,S,3)*S *B11 - 2*DF(F,S,3)*S *B21 + DF(F,S,3)*S *B22
                   2                       2
 - DF(F,S,2,TH,2)*S *B11 - DF(F,S,2,TH,2)*S *B22
                    2                       2
 + 2*DF(F,S,2,TH,2)*S *B66 - 2*DF(F,S,2,TH)*S *B16
                   2                     2                     2
 - 2*DF(F,S,2,TH)*S *B26 - DF(F,S,2,TH)*S *B62 - DF(F,S,2,TH)*S *B61
                      3                        2
 + DF(F,S,2)*DF(W,S)*S + DF(F,S,2)*DF(W,TH,2)*S
                       3                         2
 + DF(F,S,2)*DF(WB,S)*S + DF(F,S,2)*DF(WB,TH,2)*S
                      3             2                          3
 + DF(F,S,2)*COT(AL)*S + DF(F,S,2)*S *B12 + DF(F,S)*DF(W,S,2)*S
                       3
 + DF(F,S)*DF(WB,S,2)*S - DF(F,S)*S*B12 - DF(F,TH,4)*B12
                                                             2
 + 2*DF(F,TH,3)*B16 - DF(F,TH,3)*B62 + DF(F,TH,2)*DF(W,S,2)*S
                          2
 + DF(F,TH,2)*DF(WB,S,2)*S - 2*DF(F,TH,2)*B11 - 2*DF(F,TH,2)*B12
 + 2*DF(F,TH,2)*B66 + 2*DF(F,TH)*DF(W,S,TH)*S - 2*DF(F,TH)*DF(W,TH)
 + 2*DF(F,TH)*DF(WB,S,TH)*S - 2*DF(F,TH)*DF(WB,TH) - 2*DF(F,TH)*B62
 - 2*DF(F,TH)*B61 - 4*DF(W,S,TH,3)*S*D26 + 2*DF(W,S,TH,2)*S*D12
 + 4*DF(W,S,TH,2)*S*D66 - 4*DF(W,S,TH)*S*D16 - 4*DF(W,S,TH)*S*D26
              4                       3                    3
 - DF(W,S,4)*S *D11 - 4*DF(W,S,3,TH)*S *D16 - 2*DF(W,S,3)*S *D11
                    2                        2
 - 2*DF(W,S,2,TH,2)*S *D12 - 4*DF(W,S,2,TH,2)*S *D66
              2
 + DF(W,S,2)*S *D22 - DF(W,S)*S*D22 - DF(W,TH,4)*D22
 + 4*DF(W,TH,3)*D26 - 2*DF(W,TH,2)*D12 - 2*DF(W,TH,2)*D22
                                                           4
 - 4*DF(W,TH,2)*D66 + 4*DF(W,TH)*D16 + 4*DF(W,TH)*D26 + P*S  = 0
```

The compatibility equation is:

```
2*DF(F,S,TH,3)*S*A16 + 2*DF(F,S,TH,2)*S*A12 + DF(F,S,TH,2)*S*A66
                                                        4
 + 2*DF(F,S,TH)*S*A16 + 2*DF(F,S,TH)*S*A26 - DF(F,S,4)*S *A22
                   3                    3
 + 2*DF(F,S,3,TH)*S *A26 - 2*DF(F,S,3)*S *A22
                    2                       2                  2
 - 2*DF(F,S,2,TH,2)*S *A12 - DF(F,S,2,TH,2)*S *A66 + DF(F,S,2)*S *A11
 - DF(F,S)*S*A11 - DF(F,TH,4)*A11 - 2*DF(F,TH,3)*A16
 - 2*DF(F,TH,2)*A11 - 2*DF(F,TH,2)*A12 - DF(F,TH,2)*A66
 - 2*DF(F,TH)*A16 - 2*DF(F,TH)*A26 + 2*DF(W,S,TH,3)*S*B16
 - DF(W,S,TH,3)*S*B62 - 2*DF(W,S,TH,2)*S*B22 + 2*DF(W,S,TH,2)*S*B66
             2  2
 + DF(W,S,TH) *S - 2*DF(W,S,TH)*DF(W,TH)*S
                               2
 + 2*DF(W,S,TH)*DF(WB,S,TH)*S - 2*DF(W,S,TH)*DF(WB,TH)*S
                                                        4
 + 4*DF(W,S,TH)*S*B16 + 4*DF(W,S,TH)*S*B26 + DF(W,S,4)*S *B21
                   3                     3                  3
 + 2*DF(W,S,3,TH)*S *B26 - DF(W,S,3,TH)*S *B61 - DF(W,S,3)*S *B11
                3                  3                       2
 + 2*DF(W,S,3)*S *B21 + DF(W,S,3)*S *B22 + DF(W,S,2,TH,2)*S *B11
                   2                        2
 + DF(W,S,2,TH,2)*S *B22 - 2*DF(W,S,2,TH,2)*S *B66
                   2                       2
 - 2*DF(W,S,2,TH)*S *B16 - 2*DF(W,S,2,TH)*S *B26
                 2                     2                          3
 - DF(W,S,2,TH)*S *B62 - DF(W,S,2,TH)*S *B61 - DF(W,S,2)*DF(W,S)*S
                         2                      3
 - DF(W,S,2)*DF(W,TH,2)*S - DF(W,S,2)*DF(WB,S)*S
                          2                    3             2
 - DF(W,S,2)*DF(WB,TH,2)*S - DF(W,S,2)*COT(AL)*S - DF(W,S,2)*S *B12
                       3
 - DF(W,S)*DF(WB,S,2)*S + DF(W,S)*S*B12 + DF(W,TH,4)*B12
                                                              2
 - 2*DF(W,TH,3)*B16 + DF(W,TH,3)*B62 - DF(W,TH,2)*DF(WB,S,2)*S
                                                                    2
 + 2*DF(W,TH,2)*B12 + 2*DF(W,TH,2)*B22 - 2*DF(W,TH,2)*B66 + DF(W,TH)
 - 2*DF(W,TH)*DF(WB,S,TH)*S + 2*DF(W,TH)*DF(WB,TH) - 4*DF(W,TH)*B16
 - 4*DF(W,TH)*B26 = 0
```

A1.3 Governing Equations of Conical Shells with Variable Stiffness Coefficients Derived by GEACS1

The equilibrium equation is:

```
- 2*DF(F,S,TH,3)*S*B16 + DF(F,S,TH,3)*S*B62
                            2                                2
- 2*DF(F,S,TH,2)*DF(B11,S)*S  + 2*DF(F,S,TH,2)*DF(B66,S)*S
+ 2*DF(F,S,TH,2)*S*B11 - 2*DF(F,S,TH,2)*S*B66
                            2
- 2*DF(F,S,TH)*DF(W,S,TH)*S  + 2*DF(F,S,TH)*DF(W,TH)*S
                             2
- 2*DF(F,S,TH)*DF(WB,S,TH)*S  + 2*DF(F,S,TH)*DF(WB,TH)*S
                           2                          2
- 2*DF(F,S,TH)*DF(B16,S)*S  - DF(F,S,TH)*DF(B62,S)*S
                            3                            2
+ DF(F,S,TH)*DF(B61,S,2)*S  - 2*DF(F,S,TH)*DF(B61,S)*S
                                                          4
+ 2*DF(F,S,TH)*S*B62 + 2*DF(F,S,TH)*S*B61 - DF(F,S,4)*S *B21
                   3                      3
- 2*DF(F,S,3,TH)*S *B26 + DF(F,S,3,TH)*S *B61
                           4              3                   3
- 2*DF(F,S,3)*DF(B21,S)*S  - DF(F,S,3)*S *B11 - 2*DF(F,S,3)*S *B21
              3                    2                        2
+ DF(F,S,3)*S *B22 - DF(F,S,2,TH,2)*S *B11 - DF(F,S,2,TH,2)*S *B22
                     2                                   3
+ 2*DF(F,S,2,TH,2)*S *B66 - 2*DF(F,S,2,TH)*DF(B26,S)*S
                              3                     2
+ 2*DF(F,S,2,TH)*DF(B61,S)*S  - 2*DF(F,S,2,TH)*S *B16
                   2                     2                    2
- 2*DF(F,S,2,TH)*S *B26 - DF(F,S,2,TH)*S *B62 - DF(F,S,2,TH)*S *B61
                        3                          2
+ DF(F,S,2)*DF(W,S)*S  + DF(F,S,2)*DF(W,TH,2)*S
                         3                          2
+ DF(F,S,2)*DF(WB,S)*S  + DF(F,S,2)*DF(WB,TH,2)*S
                            3                          4
- 2*DF(F,S,2)*DF(B11,S)*S  - DF(F,S,2)*DF(B21,S,2)*S
                            3                        3
- 2*DF(F,S,2)*DF(B21,S)*S  + DF(F,S,2)*DF(B22,S)*S
                       3              2                          3
+ DF(F,S,2)*COT(AL)*S  + DF(F,S,2)*S *B12 + DF(F,S)*DF(W,S,2)*S
                         3                          3
+ DF(F,S)*DF(WB,S,2)*S  - DF(F,S)*DF(B11,S,2)*S
                       2
+ DF(F,S)*DF(B12,S)*S  - DF(F,S)*S*B12 - DF(F,TH,4)*B12
- 2*DF(F,TH,3)*DF(B16,S)*S + 2*DF(F,TH,3)*B16 - DF(F,TH,3)*B62
                          2                            2
+ DF(F,TH,2)*DF(W,S,2)*S  + DF(F,TH,2)*DF(WB,S,2)*S
                            2
- DF(F,TH,2)*DF(B11,S,2)*S  + 2*DF(F,TH,2)*DF(B11,S)*S
+ DF(F,TH,2)*DF(B12,S)*S - 2*DF(F,TH,2)*DF(B66,S)*S
- 2*DF(F,TH,2)*B11 - 2*DF(F,TH,2)*B12 + 2*DF(F,TH,2)*B66
+ 2*DF(F,TH)*DF(W,S,TH)*S - 2*DF(F,TH)*DF(W,TH)
+ 2*DF(F,TH)*DF(WB,S,TH)*S - 2*DF(F,TH)*DF(WB,TH)
                                                      2
+ DF(F,TH)*DF(B62,S)*S - DF(F,TH)*DF(B61,S,2)*S
+ 2*DF(F,TH)*DF(B61,S)*S - 2*DF(F,TH)*B62 - 2*DF(F,TH)*B61
                                                         2
- 4*DF(W,S,TH,3)*S*D26 - 2*DF(W,S,TH,2)*DF(D12,S)*S
                              2
- 4*DF(W,S,TH,2)*DF(D66,S)*S  + 2*DF(W,S,TH,2)*S*D12
                                                         3
+ 4*DF(W,S,TH,2)*S*D66 - 2*DF(W,S,TH)*DF(D16,S,2)*S

                            2
+ 4*DF(W,S,TH)*DF(D16,S)*S  - 4*DF(W,S,TH)*S*D16
                                   4                     3
- 4*DF(W,S,TH)*S*D26 - DF(W,S,4)*S *D11 - 4*DF(W,S,3,TH)*S *D16
                          4                  3
- 2*DF(W,S,3)*DF(D11,S)*S  - 2*DF(W,S,3)*S *D11
                     2                       2
- 2*DF(W,S,2,TH,2)*S *D12 - 4*DF(W,S,2,TH,2)*S *D66
                               3                            4
- 6*DF(W,S,2,TH)*DF(D16,S)*S  - DF(W,S,2)*DF(D11,S,2)*S
                            3                        3
- 2*DF(W,S,2)*DF(D11,S)*S  - DF(W,S,2)*DF(D12,S)*S
              2                               3                          2
+ DF(W,S,2)*S *D22 - DF(W,S)*DF(D12,S,2)*S  + DF(W,S)*DF(D22,S)*S
- DF(W,S)*S*D22 - DF(W,TH,4)*D22 - 2*DF(W,TH,3)*DF(D26,S)*S
                                                    2
+ 4*DF(W,TH,3)*D26 - DF(W,TH,2)*DF(D12,S,2)*S
+ 2*DF(W,TH,2)*DF(D12,S)*S + DF(W,TH,2)*DF(D22,S)*S
+ 4*DF(W,TH,2)*DF(D66,S)*S - 2*DF(W,TH,2)*D12 - 2*DF(W,TH,2)*D22
                                                 2
- 4*DF(W,TH,2)*D66 + 2*DF(W,TH)*DF(D16,S,2)*S
- 4*DF(W,TH)*DF(D16,S)*S - 2*DF(W,TH)*DF(D26,S)*S + 4*DF(W,TH)*D16
                          4
+ 4*DF(W,TH)*D26 + P*S = 0
```

The compatibility equation is:

```
                                                        2
2*DF(F,S,TH,3)*S*A16 - 2*DF(F,S,TH,2)*DF(A12,S)*S
                                  2
 - DF(F,S,TH,2)*DF(A66,S)*S  + 2*DF(F,S,TH,2)*S*A12
                                                    3
 + DF(F,S,TH,2)*S*A66 + DF(F,S,TH)*DF(A26,S,2)*S

                            2
 - 2*DF(F,S,TH)*DF(A26,S)*S  + 2*DF(F,S,TH)*S*A16
                                4                     3
 + 2*DF(F,S,TH)*S*A26 - DF(F,S,4)*S *A22 + 2*DF(F,S,3,TH)*S *A26
                              4                3
 - 2*DF(F,S,3)*DF(A22,S)*S  - 2*DF(F,S,3)*S *A22

                       2                       2
 - 2*DF(F,S,2,TH,2)*S *A12 - DF(F,S,2,TH,2)*S *A66
                                3                          3
 + 3*DF(F,S,2,TH)*DF(A26,S)*S  - DF(F,S,2)*DF(A12,S)*S
                              4                          3
 - DF(F,S,2)*DF(A22,S,2)*S  - 2*DF(F,S,2)*DF(A22,S)*S
               2                           2                            3
 + DF(F,S,2)*S *A11 + DF(F,S)*DF(A11,S)*S  - DF(F,S)*DF(A12,S,2)*S
 - DF(F,S)*S*A11 - DF(F,TH,4)*A11 + DF(F,TH,3)*DF(A16,S)*S
 - 2*DF(F,TH,3)*A16 + DF(F,TH,2)*DF(A11,S)*S
                             2
 - DF(F,TH,2)*DF(A12,S,2)*S  + 2*DF(F,TH,2)*DF(A12,S)*S
 + DF(F,TH,2)*DF(A66,S)*S - 2*DF(F,TH,2)*A11 - 2*DF(F,TH,2)*A12
                                                                    2
 - DF(F,TH,2)*A66 + DF(F,TH)*DF(A16,S)*S - DF(F,TH)*DF(A26,S,2)*S
 + 2*DF(F,TH)*DF(A26,S)*S - 2*DF(F,TH)*A16 - 2*DF(F,TH)*A26
 + 2*DF(W,S,TH,3)*S*B16 - DF(W,S,TH,3)*S*B62
                                 2                                  2
 + 2*DF(W,S,TH,2)*DF(B22,S)*S  - 2*DF(W,S,TH,2)*DF(B66,S)*S
                                                             2  2
 - 2*DF(W,S,TH,2)*S*B22 + 2*DF(W,S,TH,2)*S*B66 + DF(W,S,TH) *S
                                                             2
 - 2*DF(W,S,TH)*DF(W,TH)*S + 2*DF(W,S,TH)*DF(WB,S,TH)*S
                                                            2
 - 2*DF(W,S,TH)*DF(WB,TH)*S - 2*DF(W,S,TH)*DF(B16,S)*S
                                3                             2
 + 2*DF(W,S,TH)*DF(B26,S,2)*S  - 4*DF(W,S,TH)*DF(B26,S)*S
                             2
 - DF(W,S,TH)*DF(B62,S)*S  + 4*DF(W,S,TH)*S*B16 + 4*DF(W,S,TH)*S*B26
              4                       3                     3
 + DF(W,S,4)*S *B21 + 2*DF(W,S,3,TH)*S *B26 - DF(W,S,3,TH)*S *B61
                              4               3                 3
 + 2*DF(W,S,3)*DF(B21,S)*S  - DF(W,S,3)*S *B11 + 2*DF(W,S,3)*S *B21
              3                       2                         2
 + DF(W,S,3)*S *B22 + DF(W,S,2,TH,2)*S *B11 + DF(W,S,2,TH,2)*S *B22
                      2                                  3
 - 2*DF(W,S,2,TH,2)*S *B66 + 4*DF(W,S,2,TH)*DF(B26,S)*S
                                3                      2
 - DF(W,S,2,TH)*DF(B61,S)*S  - 2*DF(W,S,2,TH)*S *B16
                     2                    2                     2
 - 2*DF(W,S,2,TH)*S *B26 - DF(W,S,2,TH)*S *B62 - DF(W,S,2,TH)*S *B61
                          3                           2
 - DF(W,S,2)*DF(W,S)*S  - DF(W,S,2)*DF(W,TH,2)*S

                           3                              2
 - DF(W,S,2)*DF(WB,S)*S  - DF(W,S,2)*DF(WB,TH,2)*S
                            3                             4
 - DF(W,S,2)*DF(B11,S)*S  + DF(W,S,2)*DF(B21,S,2)*S
                              3                            3
 + 2*DF(W,S,2)*DF(B21,S)*S  + 2*DF(W,S,2)*DF(B22,S)*S
                          3                2                             3
 - DF(W,S,2)*COT(AL)*S  - DF(W,S,2)*S *B12 - DF(W,S)*DF(WB,S,2)*S
                          2                           3
 - DF(W,S)*DF(B12,S)*S  + DF(W,S)*DF(B22,S,2)*S  + DF(W,S)*S*B12
 + DF(W,TH,4)*B12 - DF(W,TH,3)*DF(B62,S)*S - 2*DF(W,TH,3)*B16
                                                 2
 + DF(W,TH,3)*B62 - DF(W,TH,2)*DF(WB,S,2)*S  - DF(W,TH,2)*DF(B12,S)*S
                                  2
 + DF(W,TH,2)*DF(B22,S,2)*S  - 2*DF(W,TH,2)*DF(B22,S)*S
 + 2*DF(W,TH,2)*DF(B66,S)*S + 2*DF(W,TH,2)*B12 + 2*DF(W,TH,2)*B22
                                    2
 - 2*DF(W,TH,2)*B66 + DF(W,TH)  - 2*DF(W,TH)*DF(WB,S,TH)*S
 + 2*DF(W,TH)*DF(WB,TH) + 2*DF(W,TH)*DF(B16,S)*S
                                  2
 - 2*DF(W,TH)*DF(B26,S,2)*S  + 4*DF(W,TH)*DF(B26,S)*S
 - 4*DF(W,TH)*B16 - 4*DF(W,TH)*B26 = 0
```

A1.4 REDUCE-based Package GEACS2

```
comment This is for the derivation of the nonlinear Donnell-type
governing equations of imperfect anisotropic conical shells in
terms of u, v, w with variable stiffness coefficients (GEACS2)$

depend w,s,th$
depend wb,s,th$
depend u,s,th$
depend v,s,th$
depend a11,s$
depend a12,s$
depend a22,s$
depend a16,s$
depend a26,s$
depend a66,s$
depend b11,s$
depend b12,s$
depend b22,s$
depend b16,s$
depend b26,s$
depend b66,s$
depend d11,s$
depend d12,s$
depend d22,s$
depend d16,s$
depend d26,s$
depend d66,s$

operator k11,k22,k12,ep11,ep12,ep22$
let k11=-df(w,s,2),
   k22=-(df(w,s)/s+df(w,th,2)/(s^2)),
   k12=-(df(w,s,th)/s-df(w,th)/(s^2)),
   ep11=df(u,s)+df(w,s)^2/2+df(w,s)*df(wb,s),
   ep12=df(v,s)-v/s+df(u,th)/s+df(w,s)*df(w,th)/s+
       (df(w,s)*df(wb,th)+df(w,th)*df(wb,s))/s,
   ep22=(u-w*cot(al))/s+df(v,th)/s+df(w,th)^2/(2*s^2)+
       df(w,th)*df(wb,th)/(s^2),
   m11=(b11*ep11+b12*ep22+b16*ep12)
      +(d11*k11+d12*k22+d16*k12*2),
   m22=(b12*ep11+b22*ep22+b26*ep12)
      +(d12*k11+d22*k22+d26*k12*2),
   m12=(b16*ep11+b26*ep22+b66*ep12)
      +(d16*k11+d26*k22+d66*k12*2),
   n11=(a11*ep11+a12*ep22+a16*ep12)
       +(b11*k11+b12*k22+b16*k12*2),
   n22=(a12*ep11+a22*ep22+a26*ep12)
       +(b12*k11+b22*k22+b26*k12*2),
   n12=(a16*ep11+a26*ep22+a66*ep12)
       +(b16*k11+b26*k22+b66*k12*2)$

comment The governing equations of imperfect anisotropic
conical shells with variable stiffness coefficients are;

%The first equilibrium equation is;
equi1:=n22-df(n12,th)-df(s*n11,s);
length(ws);

%The second equilibrium equation is;
equi2:=n12+df(n22,th)+df(s*n12,s);
length(ws);

% The out-of-plane equilibrium equation is;
equi3:=df(s*n11*(df(w,s)+df(wb,s)),s)+n22*cot(al)+
     df(n22*(df(w,th)+df(wb,th))/s,th)+
     df(n12*(df(w,th)+df(wb,th)),s)+
     df(n12*(df(w,s)+df(wb,s)),th)+
     df(s*m11,s,2)-df(m22,s)+
     df(m22/s,th,2)+2*df(m12/s,th)+
     2*df(m12,s,th)+p*s;
length(ws);
end;
```

A1.5 REDUCE-based Package GEACS3

```
comment This is for the derivation of the transformed governing
equations of anisotropic conical shells with constant stiffness
coefficients for the problems of prebuckling, buckling and
postbuckling (GEACS3)$

operator k11,k22,k12,n11,n22,n12,m11,m22,m12$
depend w,z,th$
depend f,z,th$
depend wb,z,th$
depend u,z,th$
depend v,z,th$
let k11=-(df(w,z,2)-df(w,z))/(s1*e^z)^2,
         k22=-(df(w,z)+df(w,th,2))/(s1*e^z)^2,
         k12=-(df(w,z,th)-df(w,th))/(s1*e^z)^2,
         n11=(df(f,z)+df(f,th,2))/(s1*e^z)^2,
         n22=(df(f,z,2)-df(f,z))/(s1*e^z)^2,
         n12=(df(f,th)-df(f,z,th))/(s1*e^z)^2,
         m11=-(b11*n11+b21*n22+b61*n12)*t/(2*c)
            +(d11*k11+d12*k22+d16*k12*2)*g*t^3/(4*c^2),
         m22=-(b12*n11+b22*n22+b62*n12)*t/(2*c)
            +(d12*k11+d22*k22+d26*k12*2)*g*t^3/(4*c^2),
         m12=-(b16*n11+b26*n22+b66*n12)*t/(2*c)
            +(d16*k11+d26*k22+d66*k12*2)*g*t^3/(4*c^2),
         ep11=(a11*n11+a12*n22+a16*n12)/(g*t)
            +(b11*k11+b12*k22+b16*k12*2)*t/(2*c),
         ep22=(a12*n11+a22*n22+a26*n12)/(g*t)
            +(b21*k11+b22*k22+b26*k12*2)*t/(2*c),
         ep12=(a16*n11+a26*n22+a66*n12)/(t*g)
            +(b61*k11+b62*k22+b66*k12*2)*t/(2*c)$

%The out-of-plane equilibrium equation of imperfect shell is;
on list$
equi:=sub(g=e,s1*(s1*e^z)^2*
sub(w=w*e^z,f=f*e^z,wb=wb*e^z,
df(n11*(df(w,z)+df(wb,z)),z)/(s1*e^z)
+n22*cot(al)
+df(n22*(df(w,th)+df(wb,th))/(s1*e^z),th)
+df(n12*(df(w,th)+df(wb,th)),z)/(s1*e^z)
+df(n12*(df(w,z)+df(wb,z))/(s1*e^z),th)
+(df(m11*s1*e^z,z,2)-df(m11*s1*e^z,z))/(s1*e^z)^2
-df(m22,z)/(s1*e^z)
+df(m22/(s1*e^z),th,2)
+2*df(m12/(s1*e^z),th)
+2*df(m12,z,th)/(s1*e^z)
+p*s1*e^z))$
length(ws)$

left:=sub(f=f*e^z,w=w*e^z,
df(ep12,z,th)
+df(ep12,th)
-df(ep11,th,2)
+df(ep11,z)
-2*df(ep22,z)
-df(ep22,z,2)
+df(ep22,z))$
 begin scalar ep11,ep22,ep12$
```

```
ep11:=df(u,z)/(s1*e^z)+((df(w,z)/(s1*e^z))^2)/2
+df(w,z)*df(wb,z)/(s1*e^z)^2$
ep22:=(u-w*cot(al))/(s1*e^z)+df(v,th)/(s1*e^z)
+((df(w,th)/(s1*e^z))^2)/2+df(w,th)
*df(wb,th)/(s1*e^z)^2$
ep12:=df(v,z)/(s1*e^z)-(v-df(u,th))/(s1*e^z)
+(df(w,z)*df(w,th)+df(wb,th)*df(w,z)+df(wb,z)*df(w,th))
/(s1*e^z)^2$

right:=sub(w=w*e^z,wb=wb*e^z,
df(ep12,z,th)
+df(ep12,th)
-df(ep11,th,2)
+df(ep11,z)
-2*df(ep22,z)
-df(ep22,z,2)
+df(ep22,z))$
return right
end;

% The compatibility equation of imperfect shell is;
comp:=sub(g=e,(s1^2)*(left-right)*e^z)$
length(ws)$

let wb=0$
%The equilibrium equation of perfect shell (EQ. (3.35)) is;
equi$
length(ws)$
%The compatibility equation of perfect shell (EQ. (3.36)) is;
comp$
length(ws)$

for i:=0:2 do << depend w(i),z,th; depend f(i),z,th>>$
let w=w(0)+l*w(1)+w(2)*l^2,f=f(0)+l*f(1)+f(2)*l^2$
equi$
comp$

for o:=0:2 do
<<es(o):=coeffn(equi,l,o);
co(o):=coeffn(comp,l,o)>>$

%The partial differential equations for prebuckling problem are;
%EQ. (3.43) is;
equili(0):=es(0)$
length(ws)$
%EQ. (3.44) is;
compat(0):=co(0)$
length(ws)$

%The partial differential equations for buckling problem are;
%EQ. (3.45) is;
equili(1):=es(1)$
length(ws)$
%EQ. (3.46) is;
compat(1):=co(1)$
length(ws)$

%The partial differential equations for postbuckling problem are;
%EQ. (3.47) is;
equili(2):=es(2)$
length(ws)$
%EQ. (3.48) is;
compat(2):=co(2)$
length(ws)$

%The axisymmetric prebuckling equation are;
let df(w(0),th)=0,df(f(0),th)=0$
```

```
%EQ. (3.51) is;
equili(0):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,equili(0))$
length(ws);
%EQ. (3.52) is;
compat(0):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,compat(
0))$
length(ws);

%Asumming W(1) and F(1) have the following forms;
let w(1)=t*(w1*cos(n*th)+w2*sin(n*th)),
f(1)=e*t^2*s1*sin(al)*(f1*cos(n*th)+f2*sin(n*th))/c$
        depend w1,z$
        depend w2,z$
        depend f1,z$
        depend f2,z$

%Then the PDEs of the buckling state can be transformed as;
equili(1):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,equili(1))$
length(ws);
compat(1):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,compat(
1))$
length(ws);

% Eqs. (3.61-64) are;
eq1:=coeffn(equili(1),cos(n*th),1)$
length(ws);
eq2:=coeffn(equili(1),sin(n*th),1)$
length(ws);
eq3:=coeffn(compat(1),cos(n*th),1)$
length(ws);
eq4:=coeffn(compat(1),sin(n*th),1)$
length(ws);

let
cos(n*th)*sin(n*th)=sin(2*n*th)/2,sin(n*th)^2=(1-cos(2*n*th))/2,
cos(n*th)^2=(1+cos(2*n*th))/2$

%The postbuckling governing equations are;
equili(2):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,es(2))$
length(ws);
compat(2):=sub(w(0)=t*w(0),f(0)=e*t^2*s1*sin(al)*f(0)/c,co(2))$
length(ws);

depend kerci(0),z$
depend fi(0),z$
let df(w(0),z)=kerci(0)-w(0),df(w(0),z,2)=df(kerci(0),z)-df(w(0),z)$
let df(f(0),z)=fi(0)-f(0),df(f(0),z,2)=df(fi(0),z)-df(f(0),z)$

% EQ. (3.65) is;
equili2m:=equili(2)$
length(ws);
%EQ. (3.66) is;
compat2m:=compat(2)$
length(ws);

%Asumming w(2) and F(2) have the following forms;
let w(2)=t*(wa+wp*cos(2*n*th)+wr*sin(2*n*th)),
f(2)=e*t^2*s1*sin(al)*(fa+fp*cos(2*n*th)+fr*sin(2*n*th))/c$
      depend wa,z$
      depend wp,z$
      depend wr,z$
      depend fa,z$
      depend fp,z$
      depend fr,z$

%The two postbuckling partial differential equations are;
equili2:=equili2m$
```

```
length(ws);
compat2:=compat2m$
length(ws);

%The six ordinary postbuckling differential equations are;
eq5:=4*c^2*coeffn(coeffn(equili2,cos(2*n*th),0),sin(2*n*th),0)
/(e*t^2)$
length(ws);
eq6:=4*c^2*coeffn(equili2,cos(2*n*th),1)/(e*t^2)$
length(ws);
eq7:=4*c^2*coeffn(equili2,sin(2*n*th),1)/(e*t^2)$
length(ws);
eq8:=-2*c*coeffn(coeffn(compat2,cos(2*n*th),0),sin(2*n*th),0)/t$
length(ws);
eq9:=-2*c*coeffn(compat2,cos(2*n*th),1)/t$
length(ws);
eq10:=-2*c*coeffn(compat2,sin(2*n*th),1)/t$
length(ws);

%The regrouped ODE's which can be directly used for 'shooting'
are;

neweq2:=a22*eq6/t+eq9*b21$
length(ws);
neweq3:=b21*eq6/t-eq9*d11$
length(ws);
neweq4:=eq7*b21/t-eq10*d11$
length(ws);
neweq5:=eq7*a22/t+eq10*b21$
length(ws);
eq11:=a22*df(fa,z,3)+a22*df(fa,z,2)-a11*df(fa,z)-a11*fa
+(t/(2*sin(al)*s1))*(-b21*df(wa,z,3)+(b11-b21-b22)*df(wa,z,2)
+(b11+b12-b22)*df(wa,z)+b12*wa)+(c*t*e^z/(4*sin(al)*s1))*(
4*kerci(0)*(df(wa,z)+wa)+(1-n^2)*(w1^2+w2^2)+(4*cot(al)*s1/t)
*(df(wa,z)+wa)+df(w1,z)*(df(w1,z)-2*n^2*w1+2*w1)+df(w2,z)*(
df(w2,z)-2*n^2*w2+2*w2))$
eq12:=d11*df(wa,z,3)+d11*df(wa,z,2)-d22*df(wa,z)-d22*wa
      +(2*s1*sin(al)/t)*(b21*df(fa,z,3)+(b11+b21-b22)*df(fa,z,2)
      +(b11-b12-b22)*df(fa,z)-b12*fa)-(2*s1*sin(al)*c*e^z/t)*(
      df(w1,z)*(df(f1,z)-n^2*f1+f1)+df(w2,z)*(df(f2,z)
      -n^2*f2+f2)+(1-n^2)*(w1*df(f1,z)+w2*df(f2,z)+w1*f1
      +w2*f2)+2*fi(0)*(df(wa,z)+wa)+2*kerci(0)*(df(fa,z)+fa)+
      (2*cot(al)*s1/t)*(df(fa,z)+fa))$
depend fz,z$
depend wz,z$
let df(fa,z)=fz-fa,df(wa,z)=wz-wa$
eq11$
length(ws);
eq12$
length(ws);
neweq1:=eq11*b21*2*s1*sin(al)/t-eq12*a22$
length(ws);
neweq6:=eq11*d11+eq12*t*b21/(2*s1*sin(al))$
length(ws);
solve(neweq2,df(wp,z,4));
length(ws);
solve(neweq3,df(fp,z,4));
length(ws);
solve(neweq4,df(fr,z,4));
length(ws);
solve(neweq5,df(wr,z,4));
length(ws);
solve(neweq1,df(wz,z,2));
length(ws);
solve(neweq6,df(fz,z,2));
length(ws);
end;
```

Appendix 2
For Chapter Four

A2.1 Periodicity Condition

The equation describing the periodicity requirement of a conical shell is more complicated than that of a cylindrical shell because of the complexity of its nonlinear Donnell-type strain-displacement relations.

From Eqs. (3.1) one can solve for $v_{,\bar{\theta}s}$ as

$$v_{,\bar{\theta}s} = \varepsilon_{\theta} + s\varepsilon_{\theta,s} - \varepsilon_{s} + \tilde{W}_{,s}^{2}/2 + \tilde{W}_{,s}\cot\alpha_{o} + (\tilde{W}_{,\bar{\theta}}/s)^{2}/2 - \tilde{W}_{,\bar{\theta}}\tilde{W}_{,\bar{\theta}s}/s \qquad \text{(A2.1)}$$

Thus if the solution is to satisfy the periodicity requirement, then by definition

$$\int_{0}^{2\pi\sin\alpha_{o}} v_{,\bar{\theta}s}\, d\bar{\theta} = 0 \qquad \text{(A2.2)}$$

Introducing the semi-inverted constitutive equations into Eq. (A2.1) yields

$$v,_{\bar{\theta}s} = (t/2s_1^2 c e^z)\{t[-\bar{B}^*_{21}\psi'' + (\bar{B}^*_{11}-\bar{B}^*_{22})\psi' + \bar{B}^*_{12}\psi]$$

$$+ 2s_1\sin\alpha_o(\bar{A}^*_{22}\varphi''-\bar{A}^*_{11}\varphi) + e^z c(t\psi^2+2s_1\cot\alpha_o\psi)\}$$

$$+(t/2s_1^2 c e^z)\xi\{\{t[-\bar{B}^*_{21}w_1''' + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w_1''$$

$$+ (n^2\bar{B}^*_{22}+\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w_1' + (1-n^2)\bar{B}^*_{12}w_1 - 2n\bar{B}^*_{26}w_2'' + 2n\bar{B}^*_{16}w_2']$$

$$+ 2s_1\sin\alpha_o[\bar{A}^*_{22}f_1''' + \bar{A}^*_{22}f_1'' - (\bar{A}^*_{11}+n^2\bar{A}^*_{12})f_1' + (n^2-1)\bar{A}^*_{11}f_1 - n\bar{A}^*_{26}f_2''$$

$$+ n\bar{A}^*_{16}f_2'] + 2e^z c[s_1\cot\alpha_o(w_1'+w_1) + t\psi(w_1'+w_1)]\}\cos n\bar{\theta}$$

$$+\{t[-\bar{B}^*_{21}w_2''' + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w_2''$$

$$+ (n^2\bar{B}^*_{22}+\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w_2' + (1-n^2)\bar{B}^*_{12}w_2 + 2n\bar{B}^*_{26}w_1'' - 2n\bar{B}^*_{16}w_1']$$

$$+ 2s_1\sin\alpha_o[\bar{A}^*_{22}f_2''' + \bar{A}^*_{22}f_2'' - (\bar{A}^*_{11}+n^2\bar{A}^*_{12})f_2' + (n^2-1)\bar{A}^*_{11}f_2 + n\bar{A}^*_{26}f_1''$$

$$- n\bar{A}^*_{16}f_1'] + 2e^z c[s_1\cot\alpha_o(w_2'+w_2) + t\psi(w_2'+w_2)]\}\sin n\bar{\theta}\}$$

$$+(t/4s_1^2 c e^z)\xi^2\{\{2t[-\bar{B}^*_{21}w_\alpha''' + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w_\alpha'' + (\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w_\alpha' + \bar{B}^*_{12}w_\alpha]$$

$$+ 4s_1\sin\alpha_o[\bar{A}^*_{22}f_\alpha''' + \bar{A}^*_{22}f_\alpha'' - \bar{A}^*_{11}f_\alpha' - \bar{A}^*_{11}f_\alpha] + e^z ct[w_1'(w_1'-2n^2w_1+2w_1)$$

$$+ w_2'(w_2'-2n^2w_2+2w_2) + (1-n^2)(w_1^2+w_2^2) + 4\psi(w_\alpha+w_\alpha')$$

$$+ (4s_1\cot\alpha_o/t)(w_\alpha'+w_\alpha)]\}$$

$$+\{2t[-\bar{B}^*_{21}w_\beta''' + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w_\beta''$$

$$+ (4n^2\bar{B}^*_{22}+\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w_\beta' + (1-4n^2)\bar{B}^*_{12}w_\beta - 4n\bar{B}^*_{26}w_\gamma'' + 4n\bar{B}^*_{16}w_\gamma']$$

$$+ 4s_1\sin\alpha_o[\bar{A}^*_{22}f_\beta''' + \bar{A}^*_{22}f_\beta'' - (4n^2\bar{A}^*_{12}+\bar{A}^*_{11})f_\beta' + (4n^2-1)\bar{A}^*_{11}f_\beta - 2n\bar{A}^*_{26}f_\gamma''$$

$$- 2n\bar{A}^*_{16}f_\gamma'] + e^z c[t(w_1'^2+2n^2w_1'w_1+2w_1'w_1-w_2'^2-2n^2w_2'w_2-2w_2'w_2+n^2w_1^2-n^2w_2^2$$

$$+w_1^2-w_2^2) + 4s_1\cot\alpha_o(w_\beta'+w_\beta) + 4t\psi(w_\beta'+w_\beta)]\}\cos 2n\bar{\theta}\}$$

$$+2\{t[-\bar{B}^*_{21}w'''_\gamma + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w''_\gamma$$

$$+ (4n^2\bar{B}^*_{22}+\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w'_\gamma + (1-4n^2)\bar{B}^*_{12}w_\gamma + 4n\bar{B}^*_{26}w''_\beta - 4n\bar{B}^*_{16}w'_\beta]$$

$$+ 2s_1\sin\alpha_o[\bar{A}^*_{22}f'''_\gamma + \bar{A}^*_{22}f''_\gamma - (4n^2\bar{A}^*_{12}+\bar{A}^*_{11})f'_\gamma + (4n^2-1)\bar{A}^*_{11}f_\gamma + 2n\bar{A}^*_{26}f''_\beta$$

$$- 2n\bar{A}^*_{16}f'_\beta] + e^{z}c[t(w'_1w'_2+n^2w'_1w_2+w'_1w_2+n^2w'_2w_1+w'_2w_1+n^2w_1w_2+w_1w_2)$$

$$+ 2s_1\cot\alpha_o(w'_\gamma+w_\gamma) + 2t\psi(w'_\gamma+w_\gamma)]\}\sin 2n\bar{\theta}\}$$

$$+ (t^2/2s_1^2)\xi^3\{[w'_1(2w'_\alpha+w'_\beta-2n^2w'_\beta+2w_\alpha+w_\beta) + w'_2(w'_\gamma-2n^2w_\gamma+w_\gamma)$$

$$+ (1-2n^2)(w'_\beta w_1+w'_\gamma w_2+w_1w_\beta+w_2w_\gamma) + 2w_1(w'_\alpha+w_\alpha)]\cos n\bar{\theta}$$

$$+[w'_1(w'_\gamma-2n^2w_\gamma+w_\gamma) + w'_2(2w'_\alpha-w'_\beta+2n^2w_\beta+2w_\alpha-w_\beta)$$

$$+ (2n^2-1)(w'_\beta w_2+w_\beta w_2-w'_\gamma w_1-w_\gamma w_1) + 2w_2(w'_\alpha+w_\alpha)]\sin n\bar{\theta}$$

$$+[w'_1(w'_\beta+2n^2w_\beta+w_\beta) - w'_2(w'_\gamma+2n^2w_\gamma+w_\gamma)$$

$$+ (1+2n^2)(w'_\beta w_1-w'_\gamma w_2+w_1w_\beta-w_2w_\gamma]\cos 3n\bar{\theta}$$

$$+[w'_1(w'_1+2n^2w_\gamma+w_\gamma) + w'_2(w'_\beta+2n^2w_\beta+w_\beta)$$

$$+ (1+2n^2)(w'_\beta w_2+w'_\gamma w_1+w_1w_\gamma+w_2w_\beta)]\sin 3n\bar{\theta}\}$$

$$+ (t^2/4s_1^2)\xi^4\{[2w'_\alpha(w'_\alpha+2w_\alpha)+w'_\beta(w'_\beta-8n^2w_\beta+2w_\beta) + w'_\gamma(w'_\gamma-8n^2w_\gamma+2w_\gamma)$$

$$+ (1-4n^2)(w^2_\beta+w^2_\gamma) + 2w^2_\alpha] + 4(w'_\alpha+w_\alpha)(w'_\beta+w_\beta)\cos 2n\bar{\theta}$$

$$+ 4(w'_\alpha+w_\alpha)(w'_\gamma+w_\gamma)\sin 2n\bar{\theta} + [w'_\beta(w'_\beta+8n^2w_\beta+2w_\beta) \qquad \text{(A2.3)}$$

$$- w'_\gamma(w'_\gamma+8n^2w_\gamma+2w_\gamma) + (1+4n^2)(w^2_\beta-w^2_\gamma)]\cos 4n\bar{\theta}$$

$$+ 2[w'_\beta(w'_\gamma+4n^2w_\gamma+w_\gamma) + (1+4n^2)(w'_\gamma+w_\gamma)w_\beta]\sin 4n\bar{\theta}\} + \ldots$$

Substituting Eq. (A2.3) into Eq. (A2.2) and carrying out the integration yields

$$
\begin{aligned}
&(t/2s_1^2ce^{z})\underline{\{t[-\bar{B}^*_{21}\psi'' + (\bar{B}^*_{11}-\bar{B}^*_{22})\psi' + \bar{B}^*_{12}\psi]}\\
&\qquad\underline{+ 2s_1\sin\alpha_o(\bar{A}^*_{22}\varphi''-\bar{A}^*_{11}\varphi) + e^{z}c(t\psi^2+2s_1\cot\alpha_o\psi)\}}\\
&+(t/4s_1^2ce^{z})\xi^2\underline{\{2t[-\bar{B}^*_{21}w'''_\alpha + (\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{22})w''_\alpha + (\bar{B}^*_{11}+\bar{B}^*_{12}-\bar{B}^*_{22})w'_\alpha}\\
&\qquad\underline{+ \bar{B}^*_{12}w_\alpha] + 4s_1\sin\alpha_o[\bar{A}^*_{22}f'''_\alpha + \bar{A}^*_{22}f''_\alpha - \bar{A}^*_{11}f'_\alpha-\bar{A}^*_{11}f_\alpha]}\\
&\qquad\underline{+ e^{z}ct[w_1'(w_1'-2n^2w_1+2w_1) + w_2'(w_2'-2n^2w_2+2w_2)}\\
&\qquad\underline{+ (1-n^2)(w_1^2+w_2^2) + 4\psi(w_\alpha+w'_\alpha) + (4s_1\cot\alpha_o/t)(w'_\alpha+w_\alpha)]\}}\\
&+(t^2/4s_1^2)\xi^4\{2w'_\alpha(w'_\alpha+2w_\alpha) + w'_\beta(w'_\beta-8n^2w_\beta+2w_\beta) + w'_\gamma(w_\gamma-8n^2w_\gamma+2w_\gamma)\\
&\qquad + (1-4n^2)(w_\beta^2+w_\gamma^2) + 2w_\alpha^2\} = 0
\end{aligned}
\qquad (A2.4)
$$

Notice that the underline terms vanish identically since they are equal to equations (6.1) and (7.1), respectively, with the constants $k_1 = c_1 = 0$. Thus the periodicity condition is satisfied up to and including terms of the order ξ^3.

A2.2 Constants Used in Boundary Conditions

$B_1 = -(\bar{D}^*_{12}\bar{A}^*_{22}+\bar{B}^*_{22}\bar{B}^*_{21})/\Delta$

$B_2 = 2s_1\sin\alpha_o(\bar{A}^*_{12}\bar{B}^*_{21}-\bar{A}^*_{22}\bar{B}^*_{11})/t\Delta$

$B_3 = -(\bar{A}^*_{12}\bar{D}^*_{11}+\bar{B}^*_{21}\bar{B}^*_{11})/\Delta$

$B_4 = t(\bar{D}^*_{11}\bar{B}^*_{22}-\bar{D}^*_{12}\bar{B}^*_{21})/(2s_1\sin\alpha_o\Delta)$

$B_5 = -\bar{A}^*_{12}/\bar{A}^*_{22}$

$B_6 = t\bar{B}^*_{21}/(2s_1\sin\alpha_o\bar{A}^*_{22})$

$B_7 = -[\bar{D}^*_{11}(\bar{A}^*_{12}+\bar{A}^*_{22}) + \bar{B}^*_{21}(\bar{B}^*_{11}+\bar{B}^*_{21})]/\Delta$

$B_8 = (n^2-1)(\bar{A}^*_{12}\bar{D}^*_{11}+\bar{B}^*_{11}\bar{B}^*_{21})/\Delta$

$B_9 = n(\bar{A}^*_{26}\bar{D}^*_{11}+\bar{B}^*_{61}\bar{B}^*_{21})/\Delta$

$B_{10} = t(n^2-1)(\bar{D}^*_{12}\bar{B}^*_{21}-\bar{B}^*_{22}\bar{D}^*_{11})/(2s_1\sin\alpha_o\Delta)$

$B_{11} = tn(\bar{D}^*_{11}\bar{B}^*_{26}-\bar{D}^*_{16}\bar{B}^*_{21})/(s_1\sin\alpha_o\Delta)$

$B_{12} = -[\bar{A}^*_{22}(\bar{D}^*_{11}+\bar{D}^*_{12}) + \bar{B}^*_{21}(\bar{B}^*_{21}+\bar{B}^*_{22})]/\Delta$

$B_{13} = (n^2-1)(\bar{D}^*_{12}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{B}^*_{22})/\Delta$

$B_{14} = -2n(\bar{D}^*_{16}\bar{A}^*_{22}+\bar{B}^*_{26}\bar{B}^*_{21})/\Delta$

$B_{15} = 2(n^2-1)s_1\sin\alpha_o(\bar{B}^*_{11}\bar{A}^*_{22}-\bar{A}^*_{12}\bar{B}^*_{21})/(t\Delta)$

$B_{16} = 2ns_1\sin\alpha_o(\bar{B}^*_{61}\bar{A}^*_{22}-\bar{A}^*_{26}\bar{B}^*_{21})/(t\Delta)$

$B_{17} = \bar{B}^*_{21}(\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21})/\Delta$

$B_{18} = n[\bar{B}^*_{21}(\bar{B}^*_{61}-2\bar{B}^*_{26}) + 2\bar{D}^*_{11}\bar{A}^*_{26}]/\Delta$

$B_{19} = t\bar{D}^*_{11}(\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21})/(2s_1\sin\alpha_o\Delta)$

$B_{20} = tn[\bar{D}^*_{11}(2\bar{B}^*_{26}-\bar{B}^*_{61})-4\bar{D}^*_{16}\bar{B}^*_{21}]/(2s_1\sin\alpha_o\Delta)$

$B_{21} = [\bar{D}^*_{11}(n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66}+\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})$
$\qquad -\bar{B}^*_{21}(\bar{B}^*_{11}+2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22})]/\Delta$

$B_{22} = (1-n^2)[\bar{B}^*_{12}\bar{B}^*_{21}+\bar{D}^*_{11}(\bar{A}^*_{11}+\bar{A}^*_{12})]/\Delta$

$B_{23} = -n\bar{B}^*_{21}(\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})/\Delta$

$B_{24} = n(n^2-1)(2\bar{B}^*_{16}\bar{B}^*_{21}-\bar{D}^*_{11}\bar{A}^*_{16})/\Delta$

$B_{25} = t[\bar{B}^*_{21}(\bar{D}^*_{22}+n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66})$
$\qquad + \bar{D}^*_{11}(2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})/(2s_1\sin\alpha_o\Delta)$

$B_{26} = t(n^2-1)[\, \bar{D}^*_{11}(\bar{B}^*_{12}+\bar{B}^*_{22})-\bar{D}^*_{22}\bar{B}^*_{21} \,]/(2s_1\sin\alpha_o\Delta)$

$B_{27} = -nt[\, \bar{D}^*_{11}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26}) + 2\bar{D}^*_{16}\bar{B}^*_{21} \,]/(2s_1\sin\alpha_o\Delta)$

$B_{28} = tn(n^2-1)(2\bar{D}^*_{26}\bar{B}^*_{21}+\bar{B}^*_{62}\bar{D}^*_{11})/(2s_1\sin\alpha_o\Delta)$

$B_{29} = 2(1-n^2)c\bar{B}^*_{21}/\Delta$

$B_{30} = 2c\bar{B}^*_{21}/\Delta$

$B_{31} = 2(1-n^2)cs_1\cot\alpha_o\,\bar{B}^*_{21}/(t\Delta)$

$B_{32} = 2s_1c\cot\alpha_o\,\bar{B}^*_{21}/(t\Delta)$

$B_{33} = -\bar{D}^*_{11}ct/(s_1\sin\alpha_o\,\Delta)$

$B_{34} = \bar{D}^*_{11}ctn^2/(s_1\sin\alpha_o\,\Delta)$

$B_{35} = -\bar{D}^*_{11}c\cot\alpha_o\,(1-n^2)/(\sin\alpha_o\,\Delta)$

$B_{36} = -\bar{D}_{11}c\cot\alpha_o\,/\,(\sin\alpha_o\Delta)$

$B_{37} = -\,[\, \bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^*_{21}(\bar{B}^*_{22}-\bar{B}^*_{11}) \,]/\Delta$

$B_{38} = -\,n[\, 4\bar{D}^*_{16}\bar{A}^*_{22} + \bar{B}^*_{21}(2\bar{B}^*_{26}-\bar{B}^*_{61}) \,]/\Delta$

$B_{39} = 2s_1\sin\alpha_o\,\bar{A}^*_{22}(\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21})/(t\Delta)$

$B_{40} = 2ns_1\sin\alpha_o\,[\, \bar{A}^*_{22}(\bar{B}^*_{61}-2\bar{B}^*_{26})-2\bar{A}^*_{26}\bar{B}^*_{21})/(t\Delta)$

$B_{41} = [\, \bar{A}^*_{22}(\bar{D}^*_{22}+n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66})$
$\quad -\bar{B}^*_{21}(2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12}) \,]/\Delta$

$B_{42} = (1-n^2)\,[\, \bar{B}^*_{21}(\bar{B}^*_{12}+\bar{B}^*_{22})+\bar{D}^*_{22}\bar{A}^*_{22} \,]/\Delta$

$B_{43} = n\,[\, \bar{B}^*_{21}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})-2\bar{D}^*_{16}\bar{A}^*_{22} \,]/\Delta$

$B_{44} = n(1-n^2)\,[\, \bar{B}^*_{21}\bar{B}^*_{62}-2\bar{D}^*_{26}\bar{A}^*_{22} \,]/\Delta$

$B_{45} = -2s_1\sin\alpha_o\,[\, \bar{A}^*_{22}(\bar{B}^*_{11}+2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{11}-\bar{B}^*_{12}-\bar{B}^*_{22})$
$\quad + \bar{B}^*_{21}(n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66}+\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22}) \,]/(t\Delta)$

$B_{46} = 2s_1\sin\alpha_o\,(1-n^2)\,[\, \bar{A}^*_{22}\bar{B}^*_{12}-\bar{B}^*_{21}(\bar{A}^*_{11}+\bar{A}^*_{12})/(t\Delta)$

$B_{47} = -\,2s_1\sin\alpha_o\,n\bar{A}^*_{22}\,(\, \bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26} \,)/(t\Delta)$

$B_{48} = 2n(n^2-1)s_1\sin\alpha_o\,(2\bar{B}^*_{16}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{A}^*_{16})/(t\Delta)$

$B_{49} = 4c(1-n^2)s_1\sin\alpha_o\,\bar{A}^*_{22}/(t\Delta)$

$B_{50} = 4cs_1\sin\alpha_o\,\bar{A}^*_{22}/(t\Delta)$

$B_{51} = 4c(1-n^2)s_1^2\cos\alpha_o\,\bar{A}^*_{22}/(t^2\Delta)$

$B_{52} = 4cs_1^2\cos\alpha_o\,\bar{A}^*_{22}/(t^2\Delta)$

$B_{53} = 2c\,\bar{B}^*_{21}/\Delta$

$B_{54} = -\,2n^2c\,\bar{B}^*_{21}/\Delta$

$B_{55} = 2c(1-n^2)s_1\cot\alpha_o\,\bar{B}^*_{21}/(t\Delta)$

$B_{56} = 2cs_1\cot\alpha_o\,\bar{B}^*_{21}/(t\Delta)$

$B_{57} = n^2c\cot\alpha_o\,\bar{D}^*_{11}/(\sin\alpha_o\,\Delta)$

$B_{58} = -\,2n^2cs_1\cot\alpha_o\,\bar{B}^*_{21}/(t\Delta)$

$B_{59} = -\,c\cot\alpha_o\,/(\sin\alpha_o\,\bar{A}^*_{22})$

$B_{60} = -\,ct\,/(s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{61} = (\,n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66}+\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})/\bar{A}^*_{22}$

$B_{62} = (1-n^2)(\bar{A}^*_{11}+\bar{A}^*_{12})/\bar{A}^*_{22}$

$B_{63} = 2n\bar{A}^*_{26}/\bar{A}^*_{22}$

$B_{64} = n(1-n^2)\bar{A}^*_{16}/\bar{A}^*_{22}$

$B_{65} = t(\bar{B}^*_{22}-\bar{B}^*_{11})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{66} = t(2n^2\bar{B}^*_{66}-n^2\bar{B}^*_{22}-\bar{B}^*_{11}-\bar{B}^*_{21}-\bar{B}^*_{12})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{67} = tn(2\bar{B}^*_{26}-\bar{B}^*_{61})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{68} = -\,tn(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{69} = -\,(\bar{A}^*_{12}+\bar{A}^*_{22})/\bar{A}^*_{22}$

$B_{70} = (n^2-1)\bar{A}^*_{12}/\bar{A}^*_{22}$

$B_{71} = n\bar{A}^*_{26}/\bar{A}^*_{22}$

$B_{72} = t\bar{B}^*_{21}/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{73} = -\,t(\bar{B}^*_{21}+n^2\bar{B}^*_{22})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{74} = -\,2nt\bar{B}^*_{26}/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$

$B_{75} = t\,[\, \bar{D}^*_{11}(n^2\bar{B}^*_{12}+2n^2\bar{B}^*_{22}-2n^2\bar{B}^*_{66}+\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})$
$\quad -\,\bar{B}^*_{21}(n^2\bar{D}^*_{22}+n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66}) \,]/(2s_1\sin\alpha_o\,\Delta)$

$B_{76} = tn\,[\, \bar{D}^*_{11}(\bar{B}^*_{61}+2\bar{B}^*_{16}+2\bar{B}^*_{26}+n^2\bar{B}^*_{62})$
$\quad +\,\bar{B}^*_{21}(2\bar{D}^*_{16}+2n^2\bar{D}^*_{26}-2\bar{D}^*_{26}) \,]/(2s_1\sin\alpha_o\,\Delta)$

$B_{77} = -\,[\, \bar{A}^*_{22}(n^2\bar{D}^*_{22}+n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66})$
$\quad +\,\bar{B}^*_{21}(n^2\bar{B}^*_{12}+2n^2\bar{B}^*_{22}-2n^2\bar{B}^*_{66}+\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22}) \,]/\Delta$

$B_{78} = -n\,[\, \bar{A}^*_{22}(2\bar{D}^*_{26}-2\bar{D}^*_{16}-2n^2\bar{D}^*_{26})$
$\quad +\,\bar{B}^*_{21}(n^2\bar{B}^*_{62}+\bar{B}^*_{61}+2\bar{B}^*_{16}+2\bar{B}^*_{26}) \,]/\Delta$

$B_{79} = (1-n^2)(\bar{A}^*_{22}\bar{D}^*_{11}+\bar{B}^{*\,2}_{21})/\Delta$

$$B_{80} = n(n^2-1)(\bar{D}^*_{11}\bar{A}^*_{26}+\bar{B}^*_{61}\bar{B}^*_{21})/\Delta$$

$$B_{81} = 2n(1-n^2)s_1\sin\alpha_o\,(\bar{A}^*_{26}\bar{B}^*_{21}-\bar{A}^*_{22}\bar{B}^*_{61})/(t\Delta)$$

$$B_{82} = (n^2-1)(n^2\bar{A}^*_{12}+n^2\bar{A}^*_{66}+\bar{A}^*_{22})/\bar{A}^*_{22}$$

$$B_{83} = t(n^2-1)(\bar{B}^*_{12}+\bar{B}^*_{22})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{84} = tn(n^2-1)\bar{B}^*_{62}/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{85} = ctn^2/(s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{86} = -\,c(1-n^2)\cot\alpha_o\,/\,(\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{87} = n(n^2-1)\bar{A}^*_{26}/\bar{A}^*_{22}$$

$$B_{88} = t(n^2\bar{B}^*_{12}+2n^2\bar{B}^*_{22}-2n^2\bar{B}^*_{66}+\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{89} = t(n^3\bar{B}^*_{62}+n\bar{B}^*_{61}+2n\bar{B}^*_{16}+2n\bar{B}_{26})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{90} = cn^2\cot\alpha_o\,/\,(\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{91} = (\bar{D}^*_{11}+\bar{D}^*_{12})/\bar{D}^*_{11}$$

$$B_{92} = 2n\bar{D}^*_{16}/\bar{D}^*_{11}$$

$$B_{93} = 2s_1\sin\alpha_o\,\bar{B}^*_{21}/t\bar{D}^*_{11}$$

$$B_{94} = 2(1-n^2)s_1\sin\alpha_o\,\bar{B}^*_{11}/t\bar{D}^*_{11}$$

$$B_{95} = (n^2-1)\bar{D}^*_{12}/\bar{D}^*_{11}$$

$$B_{96} = 4n\bar{D}^*_{16}/\bar{D}^*_{11}$$

$$B_{97} = \bar{D}^*_{12}/\bar{D}^*_{11}$$

$$B_{98} = (1-4n^2)\bar{A}^*_{12}/\bar{A}^*_{22}$$

$$B_{99} = (4n^2\bar{A}^*_{12}+4n^2\bar{A}^*_{66}+\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22})/\bar{A}^*_{22}$$

$$B_{100} = (4n^2-1)(\bar{A}^*_{11}+\bar{A}^*_{12})/\bar{A}^*_{22}$$

$$B_{101} = 4n\bar{A}^*_{26}/\bar{A}^*_{22}$$

$$B_{102} = 2n(4n^2-1)\bar{A}^*_{16}/\bar{A}^*_{22}$$

$$B_{103} = tn(\bar{B}^*_{61}-2\bar{B}^*_{26})/(s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{104} = 2nt\,(\bar{D}^*_{11}\bar{B}^*_{26}-\bar{D}^*_{16}\bar{B}^*_{21})/(s_1\sin\alpha_o\Delta)$$

$$B_{105} = 2n\,(\bar{B}^*_{21}\bar{B}^*_{61}+\bar{D}^*_{11}\bar{A}^*_{26})/\Delta$$

$$B_{106} = (4n^2-1)\,(\bar{B}^*_{11}\bar{B}^*_{21}+\bar{D}^*_{11}\bar{A}^*_{12})/\Delta$$

$$B_{107} = -\,4n(\bar{D}^*_{16}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{B}^*_{26})/\Delta$$

$$B_{108} = 2s_1\sin\alpha_o(\bar{B}^*_{21}\bar{A}^*_{12}-\bar{B}^*_{11}\bar{A}^*_{22})/(t\Delta)$$

$$B_{109} = 4ns_1\sin\alpha_o(\bar{B}^*_{61}\bar{A}^*_{22}-\bar{B}^*_{21}\bar{A}^*_{26})/(t\Delta)$$

$$B_{110} = 2(4n^2-1)s_1\sin\alpha_o(\bar{B}^*_{11}\bar{A}^*_{22}-\bar{B}^*_{21}\bar{A}^*_{12})/(t\Delta)$$

$$B_{111} = t(4n^2\bar{B}^*_{22}-8n^2\bar{B}^*_{66}+\bar{B}^*_{11}+\bar{B}^*_{12}+\bar{B}^*_{21})/(2s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{112} = nt(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26})/(s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{113} = tc/(4s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{114} = tc/(2s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{115} = tc(1-n^2)/(2s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{116} = tc(1-4n^2)/(4s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{117} = tc(1-4n^2)/(2s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{118} = (\bar{A}^*_{11}+\bar{A}^*_{12})/\bar{A}^*_{22}$$

$$B_{119} = t(\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{120} = 2s_1\sin\alpha_o(1-4n^2)\bar{B}^*_{11}/t\bar{D}^*_{11}$$

$$B_{121} = 2s_1\sin\alpha_o\bar{B}^*_{11}\,/\,t\bar{D}^*_{11}$$

$$B_{122} = t(\bar{B}^*_{12}+\bar{B}^*_{22})/(2s_1\sin\alpha_o\bar{A}^*_{22})$$

$$B_{123} = t(1-4n^2)\,(\bar{D}^*_{11}\bar{B}^*_{22}-\bar{D}^*_{12}\bar{B}^*_{21})/(2s_1\sin\alpha_o\Delta)$$

$$B_{124} = (ctn^2\bar{D}^*_{11})/(4s_1\sin\alpha_o\Delta)$$

$$B_{125} = (ctn^2\bar{D}^*_{11})/(2s_1\sin\alpha_o\Delta)$$

$$B_{126} = (4n^2-1)\,(\bar{B}^*_{21}\bar{B}^*_{22}+\bar{D}^*_{12}\bar{A}^*_{22})/\Delta$$

$$B_{127} = (cn^2\bar{B}^*_{21})/(2\Delta)$$

$$B_{128} = (cn^2\bar{B}^*_{21})/\Delta$$

$$B_{129} = t\bar{B}^*_{22}\,/\,(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{130} = -\,(4n^2-1)\,(\bar{B}^{*2}_{21}+\bar{D}^*_{11}\bar{A}^*_{22})/\Delta$$

$$B_{131} = 2n(4n^2-1)\,(\bar{B}^*_{21}\bar{B}^*_{61}+\bar{D}^*_{11}\bar{A}^*_{26})/\Delta$$

$$B_{132} = 4n(4n^2-1)s_1\sin\alpha_o\,(\bar{B}^*_{61}\bar{A}^*_{22}-\bar{B}^*_{21}\bar{A}^*_{26})/t\Delta$$

$$B_{133} = (4n^2-1)(4n^2\bar{A}^*_{12}+4n^2\bar{A}^*_{66}+\bar{A}^*_{22})/\bar{A}^*_{22}$$

$$B_{134} = t(1-4n^2)(\bar{B}^*_{12}+\bar{B}^*_{22})/(2s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{135} = nt(1-4n^2)\bar{B}^*_{62}/(s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{136} = (4n^2-1)c\cot\alpha_o\,/\,(\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{137} = (4n^2-1)tc/(s_1\sin\alpha_o\,\bar{A}^*_{22})$$

$$B_{138} = [\,\bar{D}^*_{11}(\bar{A}^*_{11}+\bar{A}^*_{12}+\bar{A}^*_{22}) + \bar{B}^*_{21}(\bar{B}^*_{12}+\bar{B}^*_{22}-\bar{B}^*_{11}) + 4n^2\bar{D}^*_{11}(\bar{A}^*_{12}+\bar{A}^*_{66}) + 4n^2\bar{B}^*_{21}(\bar{B}_{11}-2\bar{B}^*_{66})\,]/\Delta$$

$$B_{139} = (1-4n^2)(\bar{D}^*_{11}\bar{A}^*_{11} + \bar{D}^*_{11}\bar{A}^*_{12} + \bar{B}^*_{12}\bar{B}^*_{21})/\Delta$$

$$B_{140} = 2n(1-4n^2)(\bar{D}^*_{11}\bar{A}^*_{16} - 2\bar{B}^*_{21}\bar{B}^*_{16})/\Delta$$

$$B_{141} = 2n(\bar{B}^*_{21}\bar{B}^*_{61} - 2\bar{B}^*_{21}\bar{B}^*_{26} + 2\bar{D}^*_{11}\bar{A}^*_{26})/\Delta$$

$$B_{142} = -2n\bar{B}^*_{21}(\bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}_{26})/\Delta$$

$$B_{143} = t\bar{D}^*_{11}(\bar{B}^*_{22} - \bar{B}^*_{21} - \bar{B}_{11})/(2s_1\sin\alpha_o\Delta)$$

$$B_{144} = t[\bar{D}^*_{11}(n^2\bar{B}^*_{12}+2n^2\bar{B}^*_{22}-2n^2\bar{B}^*_{66}+\bar{B}^*_{11}+\bar{B}^*_{21}-\bar{B}^*_{22})$$
$$- \bar{B}^*_{21}(n^2\bar{D}^*_{22}+n^2\bar{D}^*_{12}+4n^2\bar{D}^*_{66})]/(2s_1\sin\alpha_o\Delta)$$

$$B_{145} = t(1-4n^2)(\bar{B}^*_{21}\bar{D}^*_{22}-\bar{D}^*_{11}\bar{B}^*_{12}-\bar{D}^*_{11}\bar{B}^*_{22})/(2s_1\sin\alpha_o\Delta)$$

$$B_{146} = 2tn(2\bar{D}^*_{11}\bar{B}^*_{26} - \bar{D}^*_{11}\bar{B}^*_{61} - 4\bar{D}^*_{16}\bar{B}^*_{21})/(2s_1\sin\alpha_o\Delta)$$

$$B_{147} = -2tn(2\bar{D}^*_{16}\bar{B}^*_{21} + \bar{D}^*_{11}\bar{B}^*_{61} + \bar{D}^*_{11}\bar{B}^*_{62}$$
$$+ 2\bar{D}^*_{11}\bar{B}^*_{16} + 2\bar{D}^*_{11}\bar{B}^*_{26})/(2s_1\sin\alpha_o\Delta)$$

$$B_{148} = 2nt(4n^2-1)(\bar{B}^*_{62}\bar{D}^*_{11} + 2\bar{B}^*_{21}\bar{D}^*_{26})/(2s_1\sin\alpha_o\Delta)$$

$$B_{149} = ct\bar{D}^*_{11}/(4s_1\sin\alpha_o\Delta)$$

$$B_{150} = ct\bar{D}^*_{11}(4n^2-1)/(s_1\sin\alpha_o\Delta)$$

$$B_{151} = c\cot\alpha_o\bar{D}^*_{11}(4n^2-1)/(\sin\alpha_o\Delta)$$

$$B_{152} = \bar{B}^*_{21}c/\Delta$$

$$B_{153} = \bar{B}^*_{21}c(1-n^2)/\Delta$$

$$B_{154} = \bar{B}^*_{21}c(1+n^2)/\Delta$$

$$B_{155} = 2cs_1\cot\alpha_o(1-4n^2)\bar{B}^*_{21}/(t\Delta)$$

$$B_{156} = 2c(1-4n^2)\bar{B}^*_{21}/\Delta$$

$$B_{157} = -ct\bar{D}^*_{11}/(2s_1\sin\alpha_o\Delta)$$

$$B_{158} = ct\bar{D}^*_{11}(4n^2-1)/(2s_1\sin\alpha_o\Delta)$$

$$B_{159} = [\bar{D}^*_{22}\bar{A}^*_{22} + \bar{B}^*_{21}(\bar{B}^*_{12}+\bar{B}^*_{21}+\bar{B}^*_{11})$$
$$+ 4n^2(4\bar{D}^*_{66}\bar{A}^*_{22}+\bar{D}^*_{12}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{B}^*_{22}-2\bar{B}^*_{21}\bar{B}_{66})]/\Delta$$

$$B_{160} = (1-4n^2)(\bar{D}^*_{22}\bar{A}^*_{22} + \bar{B}^*_{21}\bar{B}^*_{22} + \bar{B}^*_{21}\bar{B}^*_{12})/\Delta$$

$$B_{161} = 2n(\bar{B}^*_{21}\bar{B}^*_{61} - 2\bar{B}^*_{21}\bar{B}^*_{26} - 4\bar{D}^*_{16}\bar{A}^*_{22})/\Delta$$

$$B_{162} = 2n[\bar{B}^*_{21}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}^*_{26}) - 2\bar{D}^*_{16}\bar{A}^*_{22}]/\Delta$$

$$B_{163} = 2n(1-4n^2)(\bar{B}^*_{21}\bar{B}^*_{62} - 2\bar{D}^*_{26}\bar{A}^*_{22})/\Delta$$

$$B_{164} = 2s_1\sin\alpha_o[\bar{A}^*_{22}(4n^2\bar{B}^*_{11}-8n^2\bar{B}^*_{66}-\bar{B}^*_{11}-\bar{B}^*_{21}+\bar{B}^*_{12}+\bar{B}^*_{22})$$
$$- \bar{B}^*_{21}(4n^2\bar{A}^*_{12}+4n^2\bar{A}^*_{66}+\bar{A}^*_{12}+\bar{A}^*_{11})]/(t\Delta)$$

$$B_{165} = 2s_1\sin\alpha_o(1-4n^2)(\bar{B}^*_{12}\bar{A}^*_{22} - \bar{B}^*_{21}\bar{A}^*_{12} - \bar{B}^*_{21}\bar{A}^*_{11})/(t\Delta)$$

$$B_{166} = 2ns_1\sin\alpha_o(\bar{B}^*_{61}\bar{A}^*_{22} - 2\bar{B}^*_{21}\bar{A}^*_{26} - 2\bar{B}^*_{26}\bar{A}^*_{22})/(t\Delta)$$

$$B_{167} = 4ns_1\sin\alpha_o\bar{A}^*_{22}(\bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}^*_{26})/(t\Delta)$$

$$B_{168} = 4ns_1\sin\alpha_o(4n^2-1)(\bar{B}^*_{21}\bar{A}^*_{16} + 2\bar{B}^*_{16}\bar{A}^*_{22})/(t\Delta)$$

$$B_{169} = \bar{B}^*_{21}c/(2\Delta)$$

$$B_{170} = 2s_1c\sin\alpha_o\bar{A}^*_{22}/(t\Delta)$$

$$B_{171} = 2s_1c(1-n^2)\sin\alpha_o\bar{A}^*_{22}/(t\Delta)$$

$$B_{172} = 2s_1c(1+n^2)\sin\alpha_o\bar{A}^*_{22}/(t\Delta)$$

$$B_{173} = -16s_1^2cn^2\cos\alpha_o\bar{A}^*_{22}/(t^2\Delta)$$

$$B_{174} = -16s_1cn^2\sin\alpha_o\bar{A}^*_{22}/(t\Delta)$$

$$B_{175} = c(1-4n^2)\bar{B}^*_{21}/\Delta$$

$$B_{176} = 4cs_1\sin\alpha_o(1-4n^2)\bar{A}^*_{22}/(t\Delta)$$

$$B_{177} = 4cs_1^2\cos\alpha_o(1-4n^2)\bar{A}^*_{22}/(t^2\Delta)$$

where $\Delta = \bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^{*2}_{21}$

$$J_1 = B_{21} + B_{17}B_7 - B_{18}B_9 + B_{19}B_2 - B_{20}B_{16}$$

$$J_2 = B_{22} + B_{17}B_8 + B_{19}B_{15}$$

$$J_3 = B_{23} + B_{17}B_9 + B_{18}B_7 + B_{19}B_{16} + B_{20}B_2$$

$$J_4 = B_{24} + B_{18}B_8 + B_{20}B_{15}$$

$$J_5 = B_{25} + B_{17}B_4 - B_{18}B_{11} + B_{19}B_{12} - B_{20}B_{14}$$

$$J_6 = B_{26} + B_{17}B_{10} + B_{19}B_{13}$$

$$J_7 = B_{27} + B_{17}B_{11} + B_{18}B_4 + B_{19}B_{14} + B_{20}B_{12}$$

$$J_8 = B_{28} + B_{18}B_{10} + B_{20}B_{13}$$

$$J_9 = B_{41} + B_{37}B_{12} - B_{38}B_{14} + B_{39}B_4 - B_{40}B_{11}$$

$$J_{10} = B_{42} + B_{37}B_{13} + B_{39}B_{10}$$

$$J_{11} = B_{43} + B_{37}B_{14} + B_{38}B_{12} + B_{39}B_{11} + B_{40}B_4$$

$$J_{12} = B_{44} + B_{38}B_{13} + B_{40}B_{10}$$

$$J_{13} = B_{45} + B_{37}B_2 - B_{38}B_{16} + B_{39}B_7 - B_{40}B_9$$

$$J_{14} = B_{46} + B_{37}B_{15} + B_{39}B_8$$

$$J_{15} = B_{47} + B_{37}B_{16} + B_{38}B_2 + B_{39}B_9 + B_{40}B_7$$

$$J_{16} = B_{48} + B_{38}B_{15} + B_{40}B_8$$

$J_{17} = B_{61} - B_{63}B_9 + B_{65}B_2 - B_{67}B_{16}$

$J_{18} = B_{62} + B_{65}B_{15}$

$J_{19} = B_{63}B_7 + B_{65}B_{16} + B_{67}B_2$

$J_{20} = B_{64} + B_{67}B_{15} + B_{63}B_8$

$J_{21} = B_{66} - B_{63}B_{11} + B_{65}B_{12} - B_{67}B_{14}$

$J_{22} = B_{68} + B_{63}B_4 + B_{65}B_{14} + B_{67}B_{12}$

$J_{23} = B_{21} + B_{17}B_{69} - B_{18}B_{71}$

$J_{24} = B_{22} + B_{17}B_{70}$

$J_{25} = B_{23} + B_{17}B_{71} + B_{18}B_{69}$

$J_{26} = B_{24} + B_{18}B_{70}$

$J_{27} = B_{19} + B_{17}B_{72}$

$J_{28} = B_{75} + B_{17}B_{73} - B_{18}B_{74}$

$J_{29} = B_{20} + B_{18}B_{72}$

$J_{30} = B_{76} + B_{18}B_{73} + B_{17}B_{74}$

$J_{31} = B_{37} + B_{39}B_{72}$

$J_{32} = B_{77} + B_{39}B_{73} - B_{40}B_{74}$

$J_{33} = B_{38} + B_{40}B_{72}$

$J_{34} = B_{78} + B_{39}B_{74} + B_{40}B_{73}$

$J_{35} = B_{45} + B_{39}B_{69} - B_{40}B_{71}$

$J_{36} = B_{46} + B_{39}B_{70}$

$J_{37} = B_{47} + B_{39}B_{71} + B_{40}B_{69}$

$J_{38} = B_{48} + B_{40}B_{70}$

$J_{39} = B_{61} - B_{63}B_{71}$

$J_{40} = B_{63}B_{69}$

$J_{41} = B_{64} + B_{63}B_{70}$

$J_{42} = B_{82} - B_{63}B_{80} - B_{67}B_{81}$

$J_{43} = B_{64} + B_{63}B_{79} + B_{65}B_{81}$

$J_{44} = B_{83} + B_{65}B_{13}$

$J_{45} = B_{84} + B_{63}B_{10} + B_{67}B_{13}$

$J_{46} = B_{82} - B_{63}B_{87}$

$J_{47} = B_{64} + (1-n^2)B_{63}$

$J_{48} = B_{67} + B_{63}B_{72}$

$J_{49} = B_{88} - B_{63}B_{74}$

$J_{50} = B_{67} + B_{63}B_6$

$J_{51} = B_{89} + B_{63}B_{73}$

$J_{52} = B_{62} - B_{65}B_{94}$

$J_{53} = B_{63} - B_{67}B_{93}$

$J_{54} = B_{64} - B_{67}B_{94}$

$J_{55} = B_{66} - B_{65}B_{91} + B_{67}B_{92}$

$J_{56} = B_{68} - B_{65}B_{92} - B_{67}B_{91}$

$J_{57} = - B_{65}B_{93}$

$J_{58} = B_{17} - B_{19}B_{93}$

$J_{59} = B_{22} - B_{19}B_{94}$

$J_{60} = B_{18} - B_{20}B_{93}$

$J_{61} = B_{24} - B_{20}B_{94}$

$J_{62} = B_{25} - B_{19}B_{91} + B_{20}B_{92}$

$J_{63} = B_{26} + B_{19}B_{95}$

$J_{64} = B_{27} - B_{19}B_{92} - B_{20}B_{91}$

$J_{65} = B_{28} + B_{20}B_{95}$

$J_{66} = B_{41} - B_{37}B_{91} + B_{38}B_{92}$

$J_{67} = B_{42} + B_{37}B_{95}$

$J_{68} = B_{43} - B_{37}B_{92} - B_{38}B_{91}$

$J_{69} = B_{44} + B_{38}B_{95}$

$J_{70} = B_{39} - B_{37}B_{93}$

$J_{71} = B_{46} - B_{37}B_{94}$

$J_{72} = B_{40} - B_{38}B_{93}$

$J_{73} = B_{48} - B_{38}B_{94}$

$J_{74} = B_{99} - B_{101}B_{63}$

$J_{75} = B_{101}B_{69}$

$J_{76} = - B_{102} - B_{101}B_{98}$

$J_{77} = B_{101}B_6 - B_{103}$

$J_{78} = B_{99} - B_{101}B_{105} + B_{65}B_{108} + B_{103}B_{109}$

$J_{79} = - B_{100} + B_{65}B_{110}$

$J_{80} = B_{101}B_7 + B_{65}B_{109} - B_{103}B_{108}$

$J_{81} = - B_{102} + B_{101}B_{106} - B_{103}B_{110}$

$J_{82} = - B_{111} - B_{101}B_{104} + B_{65}B_{12} + B_{103}B_{107}$

$J_{83} = - B_{112} + B_{101}B_4 + B_{65}B_{107} - B_{103}B_{12}$

$J_{84} = - B_{100} - B_{65}B_{120}$

$J_{85} = B_{101} + B_{103}B_{93}$

$J_{86} = - B_{102} + B_{103}B_{120}$

$J_{87} = - B_{65}B_{93}$

$J_{88} = - B_{111} - B_{65}B_{91} - B_{103}B_{96}$

$J_{89} = - B_{112} + B_{103}B_{91} - B_{65}B_{96}$

$J_{90} = 1 + B_{119}B_{93}$

$J_{91} = B_{118} + B_{119}B_{121}$

$J_{92} = - B_{122} + B_{119}B_{97}$

$J_{93} = B_{133} - B_{101}B_{131} + B_{103}B_{132}$

$J_{94} = - B_{102} + B_{101}B_{130} + B_{65}B_{132}$

$J_{97} = - B_{134} + B_{65}B_{126}$

$J_{98} = - B_{135} + B_{101}B_{123} - B_{103}B_{126}$

$J_{99} = - B_{101}B_{125} - B_{103}B_{128}$

$J_{100} = B_{65}B_{127} - B_{116}$

$J_{102} = - B_{117} + B_{65}B_{128}$

$J_{103} = B_{103}B_{127} + B_{101}B_{124}$

$J_{104} = B_{17}B_7 + B_{138} - B_{141}B_{105} + B_{143}B_2 - B_{146}B_{109}$

$J_{105} = B_{17}B_{106} + B_{139} + B_{143}B_{110}$

$J_{106} = B_{17}B_{105} + B_{142} + B_{141}B_7 + B_{143}B_{109} + B_{146}B_2$

$J_{107} = B_{141}B_{106} + B_{140} + B_{146}B_{110}$

$J_{108} = B_{17}B_4 + B_{144} - B_{141}B_{104} + B_{143}B_{12} - B_{146}B_{107}$

$J_{109} = B_{17}B_{123} + B_{145} + B_{143}B_{126}$

$J_{110} = B_{17}B_{104} + B_{147} + B_{141}B_4 + B_{143}B_{107} + B_{146}B_{12}$

$J_{111} = B_{141}B_{123} + B_{148} + B_{146}B_{126}$

$J_{112} = B_{143}B_{127} + B_{34} - B_{17}B_{124}$

$J_{113} = B_{146}B_{128} - B_{141}B_{125}$

$J_{114} = B_{141}B_{124} - B_{146}B_{127}$

$J_{115} = B_{143}B_{128} + B_{158} - B_{17}B_{125}$

$J_{116} = B_{37}B_{12} + B_{159} - B_{161}B_{107} + B_{39}B_4 - B_{166}B_{104}$

$J_{117} = B_{37}B_{126} + B_{160} + B_{39}B_{123}$

$J_{118} = B_{37}B_{107} + B_{162} + B_{161}B_{12} + B_{39}B_{104} + B_{166}B_4$

$J_{119} = B_{161}B_{126} + B_{163} + B_{166}B_{123}$

$J_{120} = B_{37}B_2 + B_{164} - B_{161}B_{109} + B_{39}B_7 - B_{166}B_{105}$

$J_{121} = B_{37}B_{110} + B_{165} + B_{39}B_{106}$

$J_{122} = B_{37}B_{109} + B_{167} + B_{161}B_2 + B_{39}B_{105} + B_{166}B_7$

$J_{123} = B_{161}B_{110} + B_{168} + B_{166}B_{106}$

$J_{124} = - B_{124}B_{39} + B_{54} + B_{37}B_{127}$

$J_{125} = B_{161}B_{128} - B_{166}B_{125}$

$J_{126} = - B_{161}B_{127} + B_{166}B_{124}$

$J_{127} = B_{37}B_{128} + B_{175} - B_{39}B_{125}$

Appendix 3
For Chapter Five

A3.1 REDUCE-based Package GMACS

GMACS0

```
comment This is for the derivation of the Donnell-type
governing  equations of the anisotropic conical shells
for the simplified buckling solution$

operator k11,k22,k12,n11,n22,n12,m11,m22,m12$
depend w,z,th$
depend f,z,th$
depend wb,z,th$
depend u,z,th$
depend v,z,th$
let k11=-(df(w,z,2)-df(w,z))/(s1*e^z)^2,
        k22=-(df(w,z)+df(w,th,2))/(s1*e^z)^2,
        k12=-(df(w,z,th)-df(w,th))/(s1*e^z)^2,
        n11=(df(f,z)+df(f,th,2))/(s1*e^z)^2,
        n22=(df(f,z,2)-df(f,z))/(s1*e^z)^2,
        n12=(df(f,th)-df(f,z,th))/(s1*e^z)^2,
        m11=-(b11*n11+b21*n22+b61*n12)*t/(2*c)
          +(d11*k11+d12*k22+d16*k12*2)*g*t^3/(4*c^2),
        m22=-(b12*n11+b22*n22+b62*n12)*t/(2*c)
          +(d12*k11+d22*k22+d26*k12*2)*g*t^3/(4*c^2),
        m12=-(b16*n11+b26*n22+b66*n12)*t/(2*c)
          +(d16*k11+d26*k22+d66*k12*2)*g*t^3/(4*c^2),
        ep11=(a11*n11+a12*n22+a16*n12)/(g*t)
          +(b11*k11+b12*k22+b16*k12*2)*t/(2*c),
        ep22=(a12*n11+a22*n22+a26*n12)/(g*t)
          +(b21*k11+b22*k22+b26*k12*2)*t/(2*c),
        ep12=(a16*n11+a26*n22+a66*n12)/(t*g)
          +(b61*k11+b62*k22+b66*k12*2)*t/(2*c)$

%The out-of-plane equilibrium equation of imperfect shell is;
%on list$
equi:=sub(g=e,s1*(s1*e^z)^2*
sub(w=w*e^z,f=f*e^z,wb=wb*e^z,
df(n11*(df(w,z)+df(wb,z)),z)/(s1*e^z)
+n22*cot(al)
+df(n22*(df(w,th)+df(wb,th))/(s1*e^z),th)
+df(n12*(df(w,th)+df(wb,th)),z)/(s1*e^z)
+df(n12*(df(w,z)+df(wb,z))/(s1*e^z),th)
+(df(m11*s1*e^z,z,2)-df(m11*s1*e^z,z))/(s1*e^z)^2
-df(m22,z)/(s1*e^z)
+df(m22/(s1*e^z),th,2)
+2*df(m12/(s1*e^z),th)
+2*df(m12,z,th)/(s1*e^z)
+p*s1*e^z))$
length(ws);

left:=sub(f=f*e^z,w=w*e^z,
df(ep12,z,th)
+df(ep12,th)
-df(ep11,th,2)
+df(ep11,z)
-2*df(ep22,z)
-df(ep22,z,2)
+df(ep22,z))$

 begin scalar ep11,ep22,ep12$

 ep11:=df(u,z)/(s1*e^z)+((df(w,z)/(s1*e^z))^2)/2
 +df(w,z)*df(wb,z)/(s1*e^z)^2$

 ep22:=(u-w*cot(al))/(s1*e^z)+df(v,th)/(s1*e^z)
 +((df(w,th)/(s1*e^z))^2)/2+df(w,th)
 *df(wb,th)/(s1*e^z)^2$

 ep12:=df(v,z)/(s1*e^z)-(v-df(u,th))/(s1*e^z)
 +(df(w,z)*df(w,th)+df(wb,th)*df(w,z)+df(wb,z)*df(w,th))
 /(s1*e^z)^2$

 right:=sub(w=w*e^z,wb=wb*e^z,
 df(ep12,z,th)
 +df(ep12,th)
 -df(ep11,th,2)
 +df(ep11,z)
 -2*df(ep22,z)
 -df(ep22,z,2)
 +df(ep22,z))$
 return right
 end;

% The compatibility equation of imperfect shell is;
comp:=sub(g=e,(s1^2)*(left-right)*e^z)$
length(ws);

let wb=0;
%Input N--the highest order of the perturbation;
pause;

for i:=0:n do << depend w(i),z,th; depend f(i),z,th>>$
w:=for i:=0:n sum w(i)*l^i$
f:=for i:=0:n sum f(i)*l^i$
equi$
```

```
comp$

for o:=0:n do
<<es(o):=coeffn(equi,l,o);
co(o):=coeffn(comp,l,o)>>$

%The prebuckling governing equations are;
equili(0):=es(0)$
length(ws);
compat(0):=co(0)$
length(ws);

let w(0)=0,f(0)=-e*s1*t^2*(km*e^(2*z)/6+lm)/(tan(al)*c)$
%The buckling governing equations are;
equili(1):=es(1)$
length(ws);

let w(1)=a*e^(-z)*(sin(k1*z+n1*th)+sin(k2*z-n1*th))/2$
compat(1):=co(1)$
length(ws);

end;
```

GMACS1

```
for all a,b let
sin(a)*cos(a)=sin(2*a)/2,
sin(a)^2=(1-cos(2*a))/2,
cos(a)^2=(1+cos(2*a))/2,
cos(a)*cos(b)=(cos(a+b)+cos(a-b))/2,
cos(a)*sin(b)=(sin(a+b)+sin(b-a))/2,
sin(a)*sin(b)=(cos(a-b)-cos(a+b))/2$
compat(1)$
%Multipling e^z in the following is a trick$
%to solve for Ci$
coe(1):=e^z*coeffn(compat(1),sin(k1*z+n1*th),1)$
coe(2):=e^z*coeffn(compat(1),sin(-n1*th+z*k2),1)$
coe(3):=e^z*coeffn(compat(1),cos(k1*z+n1*th),1)$
coe(4):=e^z*coeffn(compat(1),cos(n1*th-z*k2),1)$

c1:=coeffn(coe(1),e^z,1);
c2:=coeffn(coe(1),e^z,0);
c3:=coeffn(coe(2),e^z,1);
c4:=coeffn(coe(2),e^z,0);
c5:=coeffn(coe(3),e^z,1);
c6:=coeffn(coe(3),e^z,0);
c7:=coeffn(coe(4),e^z,1);
c8:=coeffn(coe(4),e^z,0);

%ope=L(f(1));
ope:=-(compat(1)-coe(1)*sin(k1*z+n1*th)
*e^(-z)-coe(2)*sin(-n1*th+z*k2)
*e^(-z)-coe(3)*cos(k1*z+n1*th)*e^(-z)-coe(4)
*cos(n1*th-z*k2)*e^(-z))$

ope1:=sub(f(1)=sin(k1*z+n1*th)*(b1+b2*e^(-z))
+sin(k2*z-n1*th)*(b3+b4*e^(-z))+cos(k1*z+n1*th)
*(b5+b6*e^(-z))+cos(n1*th-z*k2)
*(b7+b8*e^(-z)),ope)$

coe1(1):=coeffn(ope1,sin(k1*z+n1*th),1)$
coe1(2):=coeffn(ope1,sin(k2*z-n1*th),1)$
coe1(3):=coeffn(ope1,cos(k1*z+n1*th),1)$
coe1(4):=coeffn(ope1,cos(n1*th-k2*z),1)$

%check:=ope1-coe1(1)*sin(k1*z+n1*th)
-coe1(2)*sin(k2*z-n1*th)-i
%coe1(3)%*cos(k1*z+n1*th)-coe1(4)*cos(n1*th-k2*z)$

coe11(1):=coeffn(coe1(1),b1,1)$
coe11(2):=coeffn(coe1(2),b3,1)$
coe11(3):=coeffn(coe1(3),b5,1)$
coe11(4):=coeffn(coe1(4),b7,1)$

b1:=c1/coe11(1)$
b3:=c3/coe11(2)$
b5:=c5/coe11(3)$
b7:=c7/coe11(4)$

%Now to solve for b2,b4,n6,b8$

ce(1):=(coe1(1)-coe11(1)*b1)*e^z$
ce(2):=(coe1(2)-coe11(2)*b3)*e^z$
ce(3):=(coe1(3)-coe11(3)*b5)*e^z$
ce(4):=(coe1(4)-coe11(4)*b7)*e^z$


mid1:=solve(lst(ce(1)-c2,ce(3)-c6),b2,b6)$
b2:=part(part(part(mid1,1),1),2)$
b6:=part(part(part(mid1,1),2),2)$
mid2:=solve(lst(ce(2)-c4,ce(4)-c8),b4,b8)$
b4:=part(part(part(mid2,1),1),2)$
b8:=part(part(part(mid2,1),2),2)$
end;
```

GMACS2

```
w(1):=a*e^(-z)*(sin(k1*z+n1*th)+sin(k2*z-n1*th))/2$
f(1):=sin(k1*z+n1*th)*(b1+b2*e^(-z))+sin(k2*z-n1*th)
*(b3+b4*e^
(-z))+cos(k1*z+n1*th)*(b5+b6*e^(-z))+cos(n1*th-k2*z)
*(b7+b8*e^(-z))$
let tan(al)*cot(al)=1;

equili(1)$
coe(1):=e^z*(coeffn(equili(1),sin(k1*z+n1*th),1))$
coe(2):=e^z*(coeffn(equili(1),sin(k2*z-n1*th),1))$
coe(3):=e^z*(coeffn(equili(1),cos(k1*z+n1*th),1))$
coe(4):=e^z*(coeffn(equili(1),cos(k2*z-n1*th),1))$

d1:=coeffn(coe(1),e^z,3);
d2:=coeffn(coe(1),e^z,2);
d3:=coeffn(coe(1),e^z,1);
d4:=coeffn(coe(1),e^z,0);

d5:=coeffn(coe(2),e^z,3);
d6:=coeffn(coe(2),e^z,2);
d7:=coeffn(coe(2),e^z,1);
d8:=coeffn(coe(2),e^z,0);

d9:=coeffn(coe(3),e^z,3);
d10:=coeffn(coe(3),e^z,2);
d110:=coeffn(coe(3),e^z,1);
d120:=coeffn(coe(3),e^z,0);


d13:=coeffn(coe(4),e^z,3);
d14:=coeffn(coe(4),e^z,2);
d15:=coeffn(coe(4),e^z,1);
d160:=coeffn(coe(4),e^z,0);
end;
```

GMACS3

```
for all a,b let
sin(a)^2=(1-cos(2*a))/2,
cos(a)^2=(1+cos(2*a))/2,
cos(a)*sin(a)=sin(2*a)/2,
sin(a)*sin(b)=(cos(a-b)-cos(a+b))/2,
cos(a)*cos(b)=(cos(a+b)+cos(a-b))/2,
sin(a)*cos(b)=(sin(a+b)+sin(a-b))/2$
procedure ddint(f);
begin scalar m1,m2,m3$
m1:=int(int(f*(sin(k1*z+n1*th)+sin(z*k2-n1*th))
*e^(-z),z),th)$
let k1=(m*pi/z0)-tu,k2=(m*pi/z0)+tu,n1=n/sin(al)$
m2:=sub(z=z0,m1)-sub(z=0,m1)$
m3:=sub(th=2*pi*sin(al),m2)-sub(th=0,m2)$
return m3
end;
i1:=ddint(sin(k1*z+n1*th)*e^(2*z))$
i2:=ddint(sin(n1*th+k1*z)*e^(z))$
i3:=ddint(sin(k1*z+n1*th))$
i4:=ddint(sin(k1*z+n1*th)*e^(-z))$

i5:=ddint(sin(k2*z-n1*th)*e^(2*z))$
i6:=ddint(sin(k2*z-n1*th)*e^(z))$
i7:=ddint(sin(k2*z-n1*th))$
i8:=ddint(sin(k2*z-n1*th)*e^(-z))$

i9:=ddint(cos(k1*z+n1*th)*e^(2*z))$
i10:=ddint(cos(k1*z+n1*th)*e^(z))$
i11:=ddint(cos(k1*z+n1*th))$
i12:=ddint(cos(k1*z+n1*th)*e^(-z))$

i13:=ddint(cos(k2*z-n1*th)*e^(2*z))$
i14:=ddint(cos(k2*z-n1*th)*e^(z))$
i15:=ddint(cos(k2*z-n1*th))$
i16:=ddint(cos(k2*z-n1*th)*e^(-z))$

in gmacs4$
%on factor$
i1;
i2;
i3;
i4;
i5;
i6;
i7;
i8;
i9;
i10;
i11;
i12;
i13;
i14;
i15;
i16;

end;
```

GMACS4

```
comment This is for the further manipulations of the
inte(i)$
for all x,y,z let
sin(x+y)=sin(x)*cos(y)+sin(y)*cos(x),
sin(x-y)=sin(x)*cos(y)-sin(y)*cos(x),
sin(x+y+z)=sin(x)*cos(y)*cos(z)+sin(y)*cos(x)*cos(z)
+sin(z)*cos(x)*cos(y)-sin(x)*sin(y)*sin(z),
cos(x+y+z)=cos(x)*cos(y)*cos(z)-cos(x)*sin(y)*sin(z)
-sin(x)*cos(y)*sin(z)-sin(x)*sin(y)*cos(z),
cos(x+y)=cos(x)*cos(y)-sin(x)*sin(y),
cos(x-y)=cos(x)*cos(y)+sin(x)*sin(y),
sin(4*n*pi)=0,
sin(2*m*pi)=0,
cos(4*n*pi)=1,
cos(2*m*pi)=1$
end;
```

A3.2 Coefficients of Eq. (5.18)

```
                    2
c1 =  COT(AL)*A*S1*K1  / 2

            4      3         3        2       2       2
c2 = (A*T*(N *B12 - N *B62*K1 + 2*N *B16*K1 + N *B11*K1  - 2*N *B12
         2          2         2          2       2           3
     + N *B22*K1  - 2*N *B22 - 2*N *B66*K1  + 2*N *B66 - N*B61*K1
                                                3
     + N*B61*K1 - N*B62*K1 - 6*N*B16*K1 + 2*N*B26*K1  - 10*N*B26*K1
                2        4            2          2            2
     - 3*B11*K1  + B21*K1  - 5*B21*K1  + B12*K1  + 3*B22*K1 ))/(4*C)

                    2
c3 =  COT(AL)*A*S1*K2  / 2

            4      3         3        2       2       2
c4 = (A*T*(N *B12 + N *B62*K2 - 2*N *B16*K2 + N *B11*K2  - 2*N *B12
         2          2         2          2       2           3
     + N *B22*K2  - 2*N *B22 - 2*N *B66*K2  + 2*N *B66 + N*B61*K2
                                                3
     - N*B61*K2 + N*B62*K2 + 6*N*B16*K2 - 2*N*B26*K2  + 10*N*B26*K2
                2        4            2          2            2
     - 3*B11*K2  + B21*K2  - 5*B21*K2  + B12*K2  + 3*B22*K2 ))/(4*C)

c5 =  COT(AL)*A*S1*K1 / 2

              3       3      2           2              2
c6 = (A*T*( - N *B62 + 2*N *B16 + N *B11*K1 + 3*N *B22*K1 - 4*N *B66*K1
                    2           2            2
     - 2*N*B61*K1  + N*B62*K1  + 2*N*B16*K1  - 4*N*B16
                    2              3                    3
     + 8*N*B26*K1  - 4*N*B26 + B11*K1  - 2*B11*K1 + 4*B21*K1
                                         3
     - 2*B21*K1 + 2*B12*K1 - B22*K1  + 2*B22*K1))/(4*C)

      COT(AL)*A*S1*K2
c7 = -----------------
             2

           3       3       2           2              2
c8 = (A*T*(N *B62 - 2*N *B16 + N *B11*K2 + 3*N *B22*K2 - 4*N *B66*K2
                    2           2            2
     + 2*N*B61*K2  - N*B62*K2  - 2*N*B16*K2  + 4*N*B16
                    2              3                    3
     - 8*N*B26*K2  + 4*N*B26 + B11*K2  - 2*B11*K2 + 4*B21*K2
                                         3
     - 2*B21*K2 + 2*B12*K2 - B22*K2  + 2*B22*K2))/(4*C)
```

A3.3 Coefficients of Eq. (5.19)

```
                 4       3            2        2        2    2      2
b1 = (E*T*C1)/(N *A11 - 2*N *A16*K1 - 2*N *A11 + 2*N *A12*K1  + N *A66*K1
                                  3         2                 4
          + 2*N*A16*K1 - 2*N*A26*K1  + A11*K1  + A11 + A22*K1
                    2
          + A22*K1 )

                 4       3            2        2        2    2      2
b3 = (E*T*C3)/(N *A11 + 2*N *A16*K2 - 2*N *A11 + 2*N *A12*K2  + N *A66*K2
                                  3         2                 4
          - 2*N*A16*K2 + 2*N*A26*K2  + A11*K2  + A11 + A22*K2
                    2
          + A22*K2 )

                 4       3            2        2        2    2      2
b5 = (E*T*C5)/(N *A11 - 2*N *A16*K1 - 2*N *A11 + 2*N *A12*K1  + N *A66*K1
                                  3         2                 4
          + 2*N*A16*K1 - 2*N*A26*K1  + A11*K1  + A11 + A22*K1
                    2
          + A22*K1 )

                 4       3            2        2        2    2      2
b7 = (E*T*C7)/(N *A11 + 2*N *A16*K2 - 2*N *A11 + 2*N *A12*K2  + N *A66*K2
                                  3         2                 4
          - 2*N*A16*K2 + 2*N*A26*K2  + A11*K2  + A11 + A22*K2
                    2
          + A22*K2 )

            4           3               3           2
b2 = (E*T*(N *A11*C2 - 2*N *A16*K1*C2 - 2*N *A16*C6 - 2*N *A11*C2
         2         2            2                2
     + 2*N *A12*K1 *C2 + 4*N *A12*K1*C6 - 2*N *A12*C2
        2         2          2                2
     + N *A66*K1 *C2 + 2*N *A66*K1*C6 - N *A66*C2 + 2*N*A16*K1*C2
                              3                  2
     + 2*N*A16*C6 - 2*N*A26*K1 *C2 - 6*N*A26*K1 *C6 + 6*N*A26*K1*C2
                            2                            4
     + 2*N*A26*C6 + A11*K1 *C2 + 2*A11*K1*C6 + A22*K1 *C2
                  3               2
     + 4*A22*K1 *C6 - 5*A22*K1 *C2 - 2*A22*K1*C6))/Denb26

           8    2
Denb26 = (N *A11
      7                  6    2      6                2      6
   - 4*N *A11*A16*K1 - 4*N *A11  + 4*N *A11*A12*K1  - 4*N *A11*A12
      6           2       6                 6    2  2      6    2
   + 2*N *A11*A66*K1  - 2*N *A11*A66 + 4*N *A16 *K1  + 4*N *A16
       5                  5             3     5
   + 12*N *A11*A16*K1 - 4*N *A11*A26*K1  + 12*N *A11*A26*K1
      5           3       5                5             3
   - 8*N *A12*A16*K1  - 8*N *A12*A16*K1 - 4*N *A16*A66*K1
      5                  4    2  2     4    2      4            2
   - 4*N *A16*A66*K1 + 2*N *A11 *K1  + 4*N *A11  - 8*N *A11*A12*K1
      4                4            4     4            2
   + 8*N *A11*A12 + 2*N *A11*A22*K1  - 10*N *A11*A22*K1 -...
```

continues on p.261

continuation from p.260

```
     4           2       4                4     2   4         4     2   2
- 4*N *A11*A66*K1  + 4*N *A11*A66 + 4*N *A12 *K1  + 8*N *A12 *K1

     4    2        4                4       4              2      4
+ 4*N *A12  + 4*N *A12*A66*K1  + 8*N *A12*A66*K1  + 4*N *A12*A66
     4    2   2        4     2        4              4            4
- 8*N *A16 *K1  - 8*N *A16  + 8*N *A16*A26*K1  - 8*N *A16*A26
   4    2   4        4    2   2      4     2         3              3
+ N *A66 *K1  + 2*N *A66 *K1  + N *A66  - 4*N *A11*A16*K1
      3              3                  3      3
- 16*N *A11*A16*K1 + 8*N *A11*A26*K1  - 24*N *A11*A26*K1
     3             3      3                3              5
+ 8*N *A12*A16*K1  + 8*N *A12*A16*K1 - 8*N *A12*A26*K1
      3             3      3                3              5
- 16*N *A12*A26*K1  - 8*N *A12*A26*K1 - 4*N *A16*A22*K1
     3             3      3                3              3
+ 4*N *A16*A22*K1  + 8*N *A16*A22*K1 + 4*N *A16*A66*K1
     3             3      5         3              3
+ 4*N *A16*A66*K1 - 4*N *A26*A66*K1  - 8*N *A26*A66*K1
     3                2    2   2        2              4
- 4*N *A26*A66*K1 - 4*N *A11 *K1  + 4*N *A11*A12*K1
      2           2       2              4      2              2
+ 12*N *A11*A12*K1  - 4*N *A11*A22*K1  + 20*N *A11*A22*K1
     2              4       2              2      2              6
+ 2*N *A11*A66*K1  + 6*N *A11*A66*K1  + 4*N *A12*A22*K1
     2              4       2              2      2    2   2        2    2
+ 8*N *A12*A22*K1  + 4*N *A12*A22*K1  + 4*N *A16 *K1  + 4*N *A16
     2              4         2             6
- 8*N *A16*A26*K1  + 8*N *A16*A26 + 2*N *A22*A66*K1
     2              4       2              2      2    2   6
+ 4*N *A22*A66*K1  + 2*N *A22*A66*K1  + 4*N *A26 *K1
      2    2   4       2    2   2       2    2                    3
+ 12*N *A26 *K1  + 12*N *A26 *K1  + 4*N *A26  + 4*N*A11*A16*K1
                                    5                   3
+ 8*N*A11*A16*K1 - 4*N*A11*A26*K1  - 12*N*A11*A26*K1
                                    5                   3
+ 8*N*A11*A26*K1 + 4*N*A16*A22*K1  - 4*N*A16*A22*K1
                                    7                   5
- 8*N*A16*A22*K1 - 4*N*A22*A26*K1  - 16*N*A22*A26*K1
                    3                     2   4        2   2
- 20*N*A22*A26*K1  - 8*N*A22*A26*K1 + A11 *K1  + 4*A11 *K1
                6                4                 2         2   8
+ 2*A11*A22*K1  + 6*A11*A22*K1  - 8*A11*A22*K1  + A22 *K1
       2   6        2   4        2   2
+ 6*A22 *K1  + 9*A22 *K1  + 4*A22 *K1 )

             4               3              3              2
b6 = (E*T*(N *A11*C6 - 2*N *A16*K1*C6 + 2*N *A16*C2 - 2*N *A11*C6
            2        2               2                 2
     + 2*N *A12*K1 *C6 - 4*N *A12*K1*C2 - 2*N *A12*C6
          2        2            2               2
     + N *A66*K1 *C6 - 2*N *A66*K1*C2 - N *A66*C6 + 2*N*A16*K1*C6
                                  3                2
     - 2*N*A16*C2 - 2*N*A26*K1 *C6 + 6*N*A26*K1 *C2 + 6*N*A26*K1*C6
                                   2                          4
     - 2*N*A26*C2 + A11*K1 *C6 - 2*A11*K1*C2 + A22*K1 *C6
                     3                2
     - 4*A22*K1 *C2 - 5*A22*K1 *C6 + 2*A22*K1*C2))/Denb26
```

```
              4               3               3              2
b4 = (E*T*(N *A11*C4 + 2*N *A16*K2*C4 + 2*N *A16*C8 - 2*N *A11*C4

           2       2       2                  2
     + 2*N *A12*K2 *C4 + 4*N *A12*K2*C8 - 2*N *A12*C4
          2       2       2                  2
     + N *A66*K2 *C4 + 2*N *A66*K2*C8 - N *A66*C4 - 2*N*A16*K2*C4
                                   3                2
     - 2*N*A16*C8 + 2*N*A26*K2 *C4 + 6*N*A26*K2 *C8 - 6*N*A26*K2*C4
                              2                          4
     - 2*N*A26*C8 + A11*K2 *C4 + 2*A11*K2*C8 + A22*K2 *C4
                     3                2
     + 4*A22*K2 *C8 - 5*A22*K2 *C4 - 2*A22*K2*C8))/Denb48

            8    2
Denb48 = (N *A11
       7                  6    2         6              2        6
   + 4*N *A11*A16*K2 - 4*N *A11  + 4*N *A11*A12*K2  - 4*N *A11*A12
       6              2        6             6    2   2       6    2
   + 2*N *A11*A66*K2  - 2*N *A11*A66 + 4*N *A16 *K2  + 4*N *A16
        5                5                3      5
   - 12*N *A11*A16*K2 + 4*N *A11*A26*K2  - 12*N *A11*A26*K2
       5              3       5                5              3
   + 8*N *A12*A16*K2  + 8*N *A12*A16*K2 + 4*N *A16*A66*K2
       5                4    2   2       4       2         4            2
   + 4*N *A16*A66*K2 + 2*N *A11 *K2  + 4*N *A11  - 8*N *A11*A12*K2
       4              4                  4      4              2
   + 8*N *A11*A12 + 2*N *A11*A22*K2  - 10*N *A11*A22*K2
       4              2       4                4    2   4        4    2   2
   - 4*N *A11*A66*K2  + 4*N *A11*A66 + 4*N *A12 *K2  + 8*N *A12 *K2
       4    2        4                4              2       4
   + 4*N *A12  + 4*N *A12*A66*K2  + 8*N *A12*A66*K2  + 4*N *A12*A66
       4    2   2        4     2        4              4            4
   - 8*N *A16 *K2  - 8*N *A16  + 8*N *A16*A26*K2  - 8*N *A16*A26
     4    2   4        4    2   2      4     2         3              3
   + N *A66 *K2  + 2*N *A66 *K2  + N *A66  + 4*N *A11*A16*K2
        3                3               3      3
   + 16*N *A11*A16*K2 - 8*N *A11*A26*K2  + 24*N *A11*A26*K2
       3              3       3                3              5
   - 8*N *A12*A16*K2  - 8*N *A12*A16*K2 + 8*N *A12*A26*K2
        3             3       3                3              5
   + 16*N *A12*A26*K2  + 8*N *A12*A26*K2 + 4*N *A16*A22*K2
       3              3       3                3              3
   - 4*N *A16*A22*K2  - 8*N *A16*A22*K2 - 4*N *A16*A66*K2
       3                3       5         3              3
   - 4*N *A16*A66*K2 + 4*N *A26*A66*K2  + 8*N *A26*A66*K2
       3                2    2   2        2              4
   + 4*N *A26*A66*K2 - 4*N *A11 *K2  + 4*N *A11*A12*K2
        2           2       2              4      2              2
   + 12*N *A11*A12*K2  - 4*N *A11*A22*K2  + 20*N *A11*A22*K2
       2              4       2              2      2              6
   + 2*N *A11*A66*K2  + 6*N *A11*A66*K2  + 4*N *A12*A22*K2
       2              4       2              2      2    2   2        2    2
   + 8*N *A12*A22*K2  + 4*N *A12*A22*K2  + 4*N *A16 *K2  + 4*N *A16
       2              4         2             6
   - 8*N *A16*A26*K2  + 8*N *A16*A26 + 2*N *A22*A66*K2
       2              4       2              2      2    2   6
   + 4*N *A22*A66*K2  + 2*N *A22*A66*K2  + 4*N *A26 *K2 +...
```

continues on p. 262

continuation from p. 261

```
+ 12*N^2*A26^2*K2^4 + 12*N^2*A26^2*K2^2 + 4*N^2*A26^2 - 4*N*A11*A16*K2^3
- 8*N*A11*A16*K2 + 4*N*A11*A26*K2^5 + 12*N*A11*A26*K2^3
- 8*N*A11*A26*K2 - 4*N*A16*A22*K2^5 + 4*N*A16*A22*K2^3
+ 8*N*A16*A22*K2 + 4*N*A22*A26*K2^7 + 16*N*A22*A26*K2^5
+ 20*N*A22*A26*K2^3 + 8*N*A22*A26*K2 + A11^2*K2^4 + 4*A11^2*K2^2
+ 2*A11*A22*K2^6 + 6*A11*A22*K2^4 - 8*A11*A22*K2^2 + A22^2*K2^8
+ 6*A22^2*K2^6 + 9*A22^2*K2^4 + 4*A22^2*K2^2 )

b8 = (E*T*(N^4*A11*C8 + 2*N^3*A16*K2*C8 - 2*N^3*A16*C4 - 2*N^2*A11*C8
   + 2*N^2*A12*K2^2*C8 - 4*N^2*A12*K2*C4 - 2*N^2*A12*C8
   + N^2*A66*K2^2*C8 - 2*N^2*A66*K2*C4 - N^2*A66*C8 - 2*N*A16*K2*C8
   + 2*N*A16*C4 + 2*N*A26*K2^3*C8 - 6*N*A26*K2^2*C4 - 6*N*A26*K2*C8
   + 2*N*A26*C4 + A11*K2^2*C8 - 2*A11*K2*C4 + A22*K2^4*C8
   - 4*A22*K2^3*C4 - 5*A22*K2^2*C8 + 2*A22*K2*C4))/Denb48
```

A3.4 Coefficients of Eq. (5.20)

```
D1 =   A*E*T^2*S1*KM*(2*N^2 + K1^2) / 4*TAN(AL)*C

D2 =   - (S1*K1)*(K1*B1 + B5) / TAN(AL)

D3 = ( - TAN(AL)*N^4*T*B12*B1 + TAN(AL)*N^3*T*B62*K1*B1
     - 2*TAN(AL)*N^3*T*B16*K1*B1 - TAN(AL)*N^2*T*B11*K1^2*B1
     + TAN(AL)*N^2*T*B11*K1*B5 - 2*TAN(AL)*N^2*T*B12*B1
     - TAN(AL)*N^2*T*B22*K1^2*B1 - TAN(AL)*N^2*T*B22*K1*B5
     + 2*TAN(AL)*N^2*T*B66*K1^2*B1 + TAN(AL)*N*T*B61*K1^3*B1
     - TAN(AL)*N*T*B61*K1^2*B5 - TAN(AL)*N*T*B62*K1^2*B5
     - TAN(AL)*N*T*B62*K1*B1 - 2*TAN(AL)*N*T*B16*K1^2*B5
     + 2*TAN(AL)*N*T*B16*K1*B1 - 2*TAN(AL)*N*T*B26*K1^3*B1
     - 2*TAN(AL)*N*T*B26*K1^2*B5 - TAN(AL)*T*B11*K1^3*B5
     - TAN(AL)*T*B11*K1*B5 - TAN(AL)*T*B21*K1^4*B1
     - TAN(AL)*T*B21*K1^2*B1 - TAN(AL)*T*B12*K1^2*B1
     - TAN(AL)*T*B12*B1 + TAN(AL)*T*B22*K1^3*B5
     + TAN(AL)*T*B22*K1*B5 + A*E*T^2*S1*LM*K1^2 - 2*C*S1*K1^2*B2
     + 2*C*S1*K1*B6)/(2*TAN(AL)*C)

D4 = (T*( - A*E*N^4*T^2*D22 - 4*A*E*N^3*T^2*D26*K1 - 2*A*E*N^2*T^2*D12*K1^2
     + 2*A*E*N^2*T^2*D12 + 2*A*E*N^2*T^2*D22 - 4*A*E*N^2*T^2*D66*K1^2
     + 4*A*E*N^2*T^2*D66 - 4*A*E*N*T^2*D16*K1^3
     + 12*A*E*N*T^2*D16*K1 + 4*A*E*N*T^2*D26*K1 - A*E*T^2*D11*K1^4
     + 5*A*E*T^2*D11*K1^2 - A*E*T^2*D22*K1^2 - 4*C*N^4*B12*B2
     + 4*C*N^3*B62*K1*B2 - 4*C*N^3*B62*B6 - 8*C*N^3*B16*K1*B2
     + 8*C*N^3*B16*B6 - 4*C*N^2*B11*K1^2*B2 + 12*C*N^2*B11*K1*B6
     + 8*C*N^2*B11*B2 + 8*C*N^2*B12*B2 - 4*C*N^2*B22*K1^2*B2
     + 4*C*N^2*B22*K1*B6 + 8*C*N^2*B66*K1^2*B2 - 16*C*N^2*B66*K1*B6
     - 8*C*N^2*B66*B2 + 4*C*N*B61*K1^3*B2 - 16*C*N*B61*K1^2*B6
     - 20*C*N*B61*K1*B2 + 8*C*N*B61*B6 - 4*C*N*B62*K1^2*B6
     - 12*C*N*B62*K1*B2 + 8*C*N*B62*B6 - 8*C*N*B16*K1^2*B6
     - 8*C*N*B16*K1*B2 - 8*C*N*B26*K1^3*B2 + 16*C*N*B26*K1^2*B6
     + 8*C*N*B26*K1*B2 - 4*C*B11*K1^3*B6 - 12*C*B11*K1^2*B2
     + 8*C*B11*K1*B6 - 4*C*B21*K1^4*B2 + 16*C*B21*K1^3*B6
     + 20*C*B21*K1^2*B2 - 8*C*B21*K1*B6 - 4*C*B12*K1^2*B2
     + 8*C*B12*K1*B6 + 4*C*B22*K1^3*B6 + 12*C*B22*K1^2*B2
     - 8*C*B22*K1*B6))/(8*C^2)

D5     A*E*T^2*S1*KM*(2*N^2 + K2^2) / 4*TAN(AL)*C
```

```
D6 =   - (S1*K2)*(K2*B3 + B7) / TAN(AL)

                  4             3
D7 = ( - TAN(AL)*N *T*B12*B3 - TAN(AL)*N *T*B62*K2*B3
                   3                   2         2
     + 2*TAN(AL)*N *T*B16*K2*B3 - TAN(AL)*N *T*B11*K2 *B3
                 2                      2
     + TAN(AL)*N *T*B11*K2*B7 + 2*TAN(AL)*N *T*B12*B3
                 2        2              2
     - TAN(AL)*N *T*B22*K2 *B3 - TAN(AL)*N *T*B22*K2*B7
                   2        2                          3
     + 2*TAN(AL)*N *T*B66*K2 *B3 - TAN(AL)*N*T*B61*K2 *B3
                            2                         2
     + TAN(AL)*N*T*B61*K2 *B7 + TAN(AL)*N*T*B62*K2 *B7
                                                     2
     + TAN(AL)*N*T*B62*K2*B3 + 2*TAN(AL)*N*T*B16*K2 *B7
                                                       3
     - 2*TAN(AL)*N*T*B16*K2*B3 + 2*TAN(AL)*N*T*B26*K2 *B3
                              2                     3
     + 2*TAN(AL)*N*T*B26*K2 *B7 - TAN(AL)*T*B11*K2 *B7
                                                 4
     - TAN(AL)*T*B11*K2*B7 - TAN(AL)*T*B21*K2 *B3
                          2                    2
     - TAN(AL)*T*B21*K2 *B3 - TAN(AL)*T*B12*K2 *B3
                                              3
     - TAN(AL)*T*B12*B3 + TAN(AL)*T*B22*K2 *B7
                                 2          2             2
     + TAN(AL)*T*B22*K2*B7 + A*E*T *S1*LM*K2  - 2*C*S1*K2 *B4

     + 2*C*S1*K2*B8)/(2*TAN(AL)*C)

                4  2            3  2               2  2         2
D8 = (T*( - A*E*N *T *D22 + 4*A*E*N *T *D26*K2 - 2*A*E*N *T *D12*K2
              2  2            2  2            2  2          2
     + 2*A*E*N *T *D12 + 2*A*E*N *T *D22 - 4*A*E*N *T *D66*K2
            2  2             2        3
     + 4*A*E*N *T *D66 + 4*A*E*N*T *D16*K2
                2                 2              2        4
     - 12*A*E*N*T *D16*K2 - 4*A*E*N*T *D26*K2 - A*E*T *D11*K2
            2        2          2        2        4
     + 5*A*E*T *D11*K2  - A*E*T *D22*K2  - 4*C*N *B12*B4
            3                 3               3
     - 4*C*N *B62*K2*B4 + 4*C*N *B62*B8 + 8*C*N *B16*K2*B4
            3              2         2           2
     - 8*C*N *B16*B8 - 4*C*N *B11*K2 *B4 + 12*C*N *B11*K2*B8
            2              2                2         2
     + 8*C*N *B11*B4 + 8*C*N *B12*B4 - 4*C*N *B22*K2 *B4
            2                 2         2            2
     + 4*C*N *B22*K2*B8 + 8*C*N *B66*K2 *B4 - 16*C*N *B66*K2*B8
            2                        3                   2
     - 8*C*N *B66*B4 - 4*C*N*B61*K2 *B4 + 16*C*N*B61*K2 *B8
                                                          2
     + 20*C*N*B61*K2*B4 - 8*C*N*B61*B8 + 4*C*N*B62*K2 *B8
```

```
                                                              2
     + 12*C*N*B62*K2*B4 - 8*C*N*B62*B8 + 8*C*N*B16*K2 *B8
                                       3                   2
     + 8*C*N*B16*K2*B4 + 8*C*N*B26*K2 *B4 - 16*C*N*B26*K2 *B8
                                  3                 2
     - 8*C*N*B26*K2*B4 - 4*C*B11*K2 *B8 - 12*C*B11*K2 *B4
                               4                 3
     + 8*C*B11*K2*B8 - 4*C*B21*K2 *B4 + 16*C*B21*K2 *B8
                  2                                 2
     + 20*C*B21*K2 *B4 - 8*C*B21*K2*B8 - 4*C*B12*K2 *B4
                                  3                 2
     + 8*C*B12*K2*B8 + 4*C*B22*K2 *B8 + 12*C*B22*K2 *B4
                              2
     - 8*C*B22*K2*B8))/(8*C )

              2
D9 =   - A*E*T *S1*KM*K1 / 4*TAN(AL)*C

D10 =  S1*K1*( - K1*B5 + B1) / TAN(AL)

                   4             3
D11 = ( - TAN(AL)*N *T*B12*B5 + TAN(AL)*N *T*B62*K1*B5
                   3                   2         2
     - 2*TAN(AL)*N *T*B16*K1*B5 - TAN(AL)*N *T*B11*K1 *B5
                 2                      2
     - TAN(AL)*N *T*B11*K1*B1 + 2*TAN(AL)*N *T*B12*B5
                 2        2              2
     - TAN(AL)*N *T*B22*K1 *B5 + TAN(AL)*N *T*B22*K1*B1
                   2        2                          3
     + 2*TAN(AL)*N *T*B66*K1 *B5 + TAN(AL)*N*T*B61*K1 *B5
                            2                         2
     + TAN(AL)*N*T*B61*K1 *B1 + TAN(AL)*N*T*B62*K1 *B1
                                                     2
     - TAN(AL)*N*T*B62*K1*B5 + 2*TAN(AL)*N*T*B16*K1 *B1
                                                       3
     + 2*TAN(AL)*N*T*B16*K1*B5 - 2*TAN(AL)*N*T*B26*K1 *B5
                              2                     3
     + 2*TAN(AL)*N*T*B26*K1 *B1 + TAN(AL)*T*B11*K1 *B1
                                                 4
     + TAN(AL)*T*B11*K1*B1 - TAN(AL)*T*B21*K1 *B5
                          2                    2
     - TAN(AL)*T*B21*K1 *B5 - TAN(AL)*T*B12*K1 *B5
                                              3
     - TAN(AL)*T*B12*B5 - TAN(AL)*T*B22*K1 *B1
                                 2                        2
     - TAN(AL)*T*B22*K1*B1 + A*E*T *S1*LM*K1 - 2*C*S1*K1 *B6

     - 2*C*S1*K1*B2)/(2*TAN(AL)*C)
```

```
                    3  2              2  2
D12 = (T*( - 2*A*E*N *T *D26 - 2*A*E*N *T *D12*K1
                      2  2                   2       2
             - 4*A*E*N *T *D66*K1 - 6*A*E*N*T *D16*K1
                        2                2              2       3
             + 2*A*E*N*T *D16 + 2*A*E*N*T *D26 - 2*A*E*T *D11*K1
                    2               2               4
             + A*E*T *D11*K1 - A*E*T *D22*K1 - 2*C*N *B12*B6
                    3                  3               3
             + 2*C*N *B62*K1*B6 + 2*C*N *B62*B2 - 4*C*N *B16*K1*B6
                    3               2       2           2
             - 4*C*N *B16*B2 - 2*C*N *B11*K1 *B6 - 6*C*N *B11*K1*B2
                    2               2               2       2
             + 4*C*N *B11*B6 + 4*C*N *B12*B6 - 2*C*N *B22*K1 *B6
                    2                  2       2
             - 2*C*N *B22*K1*B2 + 4*C*N *B66*K1 *B6
                    2                  2                      3
             + 8*C*N *B66*K1*B2 - 4*C*N *B66*B6 + 2*C*N*B61*K1 *B6
                            2
             + 8*C*N*B61*K1 *B2 - 10*C*N*B61*K1*B6 - 4*C*N*B61*B2
                            2
             + 2*C*N*B62*K1 *B2 - 6*C*N*B62*K1*B6 - 4*C*N*B62*B2
                            2                                    3
             + 4*C*N*B16*K1 *B2 - 4*C*N*B16*K1*B6 - 4*C*N*B26*K1 *B6
                            2                                  3
             - 8*C*N*B26*K1 *B2 + 4*C*N*B26*K1*B6 + 2*C*B11*K1 *B2
                          2                                4
             - 6*C*B11*K1 *B6 - 4*C*B11*K1*B2 - 2*C*B21*K1 *B6
                          3                 2
             - 8*C*B21*K1 *B2 + 10*C*B21*K1 *B6 + 4*C*B21*K1*B2
                          2                                3
             - 2*C*B12*K1 *B6 - 4*C*B12*K1*B2 - 2*C*B22*K1 *B2
                          2                          2
             + 6*C*B22*K1 *B6 + 4*C*B22*K1*B2))/(4*C )

                2
D13 =    - A*E*T *S1*KM*K2 / 4*TAN(AL)*C

D14 =   S1*K2*( - K2*B7 + B3) / TAN(AL)
                   4                     3
D15 = ( - TAN(AL)*N *T*B12*B7 - TAN(AL)*N *T*B62*K2*B7
                      3                       2         2
         + 2*TAN(AL)*N *T*B16*K2*B7 - TAN(AL)*N *T*B11*K2 *B7
                    2                         2
         - TAN(AL)*N *T*B11*K2*B3 + 2*TAN(AL)*N *T*B12*B7
                    2         2               2
         - TAN(AL)*N *T*B22*K2 *B7 + TAN(AL)*N *T*B22*K2*B3
                      2         2                         3
         + 2*TAN(AL)*N *T*B66*K2 *B7 - TAN(AL)*N*T*B61*K2 *B7
                              2                         2
         - TAN(AL)*N*T*B61*K2 *B3 - TAN(AL)*N*T*B62*K2 *B3
                                                        2
         + TAN(AL)*N*T*B62*K2*B7 - 2*TAN(AL)*N*T*B16*K2 *B3
                                                          3
         - 2*TAN(AL)*N*T*B16*K2*B7 + 2*TAN(AL)*N*T*B26*K2 *B7
                               2                        3
         - 2*TAN(AL)*N*T*B26*K2 *B3 + TAN(AL)*T*B11*K2 *B3
                                                  4
         + TAN(AL)*T*B11*K2*B3 - TAN(AL)*T*B21*K2 *B7
                            2                       2
         - TAN(AL)*T*B21*K2 *B7 - TAN(AL)*T*B12*K2 *B7
                                               3
         - TAN(AL)*T*B12*B7 - TAN(AL)*T*B22*K2 *B3
                                       2                     2
         - TAN(AL)*T*B22*K2*B3 + A*E*T *S1*LM*K2 - 2*C*S1*K2 *B8

         - 2*C*S1*K2*B4)/(2*TAN(AL)*C)

                 3  2              2  2                 2  2
D16 = (T*(2*A*E*N *T *D26 - 2*A*E*N *T *D12*K2 - 4*A*E*N *T *D66*K2
                        2       2            2                2
             + 6*A*E*N*T *D16*K2  - 2*A*E*N*T *D16 - 2*A*E*N*T *D26
                      2       3       2               2
             - 2*A*E*T *D11*K2  + A*E*T *D11*K2 - A*E*T *D22*K2
                    4               3                  3
             - 2*C*N *B12*B8 - 2*C*N *B62*K2*B8 - 2*C*N *B62*B4
                    3                  3               2       2
             + 4*C*N *B16*K2*B8 + 4*C*N *B16*B4 - 2*C*N *B11*K2 *B8
                    2                  2               2
             - 6*C*N *B11*K2*B4 + 4*C*N *B11*B8 + 4*C*N *B12*B8
                    2       2           2
             - 2*C*N *B22*K2 *B8 - 2*C*N *B22*K2*B4
                    2       2           2                  2
             + 4*C*N *B66*K2 *B8 + 8*C*N *B66*K2*B4 - 4*C*N *B66*B8
                            3                  2
             - 2*C*N*B61*K2 *B8 - 8*C*N*B61*K2 *B4 + 10*C*N*B61*K2*B8
                                           2
             + 4*C*N*B61*B4 - 2*C*N*B62*K2 *B4 + 6*C*N*B62*K2*B8
                                           2
             + 4*C*N*B62*B4 - 4*C*N*B16*K2 *B4 + 4*C*N*B16*K2*B8
                            3                  2
             + 4*C*N*B26*K2 *B8 + 8*C*N*B26*K2 *B4 - 4*C*N*B26*K2*B8
                          3                2
             + 2*C*B11*K2 *B4 - 6*C*B11*K2 *B8 - 4*C*B11*K2*B4
                          4                3                 2
             - 2*C*B21*K2 *B8 - 8*C*B21*K2 *B4 + 10*C*B21*K2 *B8
                                          2
             + 4*C*B21*K2*B4 - 2*C*B12*K2 *B8 - 4*C*B12*K2*B4
                          3                2
             - 2*C*B22*K2 *B4 + 6*C*B22*K2 *B8 + 4*C*B22*K2*B4))/(4
             2
           *C )
```

A3.5 Integration Results of Eq. (5.22)

```
                2   3   Z0
     4*SIN(AL)*M *PI *(E    - 1)
I1 = ----------------------------
              2   2     2
          4*M *PI  + Z0

I2 = SIN(AL)*PI*Z0

                2   3   Z0
     4*SIN(AL)*M *PI *(E    - 1)
I3 = ----------------------------
       Z0      2   2     2
      E  *(4*M *PI  + Z0 )

              2   3   2*Z0
     SIN(AL)*M *PI *(E      - 1)
I4 = ----------------------------
          2*Z0   2   2     2
       2*E    *(M *PI  + Z0 )

               2   3   Z0
    4*SIN(AL)*M *PI *(E    - 1)
I5 = ----------------------------
              2   2     2
          4*M *PI  + Z0

I6 = SIN(AL)*PI*Z0

                2   3   Z0
     4*SIN(AL)*M *PI *(E    - 1)
I7 = ----------------------------
       Z0      2   2     2
      E  *(4*M *PI  + Z0 )

             2   3   2*Z0
    SIN(AL)*M *PI *(E      - 1)
I8 = ----------------------------
        2*Z0   2   2     2
     2*E    *(M *PI  + Z0 )

                      2          Z0
    2*SIN(AL)*M*PI *Z0*( - E    + 1)
I9 = ---------------------------------
                 2   2     2
             4*M *PI  + Z0

I10 = 0

                        2        Z0
       2*SIN(AL)*M*PI *Z0*(E    - 1)
I11 = -----------------------------
        Z0      2   2     2
       E  *(4*M *PI  + Z0 )

                     2        2*Z0
      SIN(AL)*M*PI *Z0*(E      - 1)
I12 = -----------------------------
          2*Z0   2   2     2
       2*E    *(M *PI  + Z0 )

                        2          Z0
       2*SIN(AL)*M*PI *Z0*( - E    + 1)
I13 = ----------------------------------
                  2   2     2
              4*M *PI  + Z0

I14 = 0

                        2        Z0
       2*SIN(AL)*M*PI *Z0*(E    - 1)
I15 = -----------------------------
        Z0      2   2     2
       E  *(4*M *PI  + Z0 )

                     2        2*Z0
      SIN(AL)*M*PI *Z0*(E      - 1)
I16 = -----------------------------
          2*Z0   2   2     2
       2*E    *(M *PI  + Z0 )
```

Appendix 4
For Chapter Six

A4.1 Constants used in the axisymmetric prebuckling equations

$\hat{C}_1 = (\bar{B}^*_{22} - \bar{B}^*_{11})\bar{B}^*_{21}/\Delta$

$\hat{C}_2 = (\bar{B}^*_{12}\bar{B}^*_{21} + \bar{A}^*_{11}\bar{D}^*_{11})/\Delta$

$\hat{C}_3 = (t/2s_1\sin\alpha_o)(\bar{B}^*_{22} - \bar{B}^*_{11})\bar{D}^*_{11}/\Delta$

$\hat{C}_4 = (t/2s_1\sin\alpha_o)(\bar{D}^*_{22}\bar{B}^*_{21} - \bar{D}^*_{11}\bar{B}^*_{12})/\Delta$

$\hat{C}_5 = 2cs_1\cot\alpha_o\bar{B}^*_{21}/(t\Delta)$

$\hat{C}_6 = -\,c\cot\alpha_o\bar{D}^*_{11}/(\sin\alpha_o\Delta)$

$\hat{C}_7 = 2c\bar{B}^*_{21}/\Delta$

$\hat{C}_8 = -\,c\bar{D}^*_{11}t/(2s_1\sin\alpha_o\Delta)$

$\hat{C}_9 = cs_1\cot^2\alpha_o\bar{B}^*_{21}/(t\sin\alpha_o\Delta)$

$\hat{C}_{10} = 2cs_1\cot^2\alpha_o\bar{B}^*_{21}/(t\sin\alpha_o\Delta)$

$\hat{C}_{11} = (\bar{A}^*_{22}\bar{D}^*_{22} + \bar{B}^*_{12}\bar{B}^*_{21})/\Delta$

$\hat{C}_{12} = (2s_1\sin\alpha_o/t)(\bar{B}^*_{22} - \bar{B}^*_{11})\bar{A}^*_{22}/\Delta$

$\hat{C}_{13} = (2s_1\sin\alpha_o/t)(\bar{A}^*_{22}\bar{B}^*_{12} - \bar{A}^*_{11}\bar{B}^*_{21})/\Delta$

$\hat{C}_{14} = 2cs_1\cot\alpha_o\bar{B}^*_{21}/(t\Delta)$

$\hat{C}_{15} = 4cs_1^2\cos\alpha_o\bar{A}^*_{22}/(t^2\Delta)$

$\hat{C}_{16} = 4cs_1\sin\alpha_o\bar{A}^*_{22}/(t\Delta)$

$\hat{C}_{17} = c\bar{B}^*_{21}/\Delta$

$\hat{C}_{18} = 2cs_1^2\cot^2\alpha_o\bar{A}^*_{22}/(t^2\Delta)$

$\hat{C}_{19} = 4cs_1^2\cot^2\alpha_o\bar{A}^*_{22}/(t^2\Delta)$

$\hat{C}_{20} = \cos\alpha_o/\sin^2\alpha_o$

$\hat{C}_{21} = 2(\bar{A}^*_{22} - \bar{A}^*_{11}/4)/c$

$\hat{C}_{22} = \bar{A}^*_{11}/c$

where $\Delta = \bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^{*2}_{21}$

A4.2 Constants used in the asymmetric buckling equations

$C_1 = 2\bar{A}^*_{22}(\bar{B}^*_{22} - \bar{B}^*_{11})s_1\sin\alpha_o/t\Delta$

$$C_2 = 2[n^2(\bar{B}^*_{11}\bar{A}^*_{22} - 2\bar{B}^*_{21}\bar{A}^*_{12} - \bar{B}^*_{21}\bar{A}^*_{66} + \bar{B}^*_{22}\bar{A}^*_{22} - 2\bar{B}^*_{66}\bar{A}^*_{22}) + \bar{B}^*_{12}\bar{A}^*_{22} - \bar{B}^*_{21}\bar{A}^*_{11}]\,s_1\sin\alpha_o/t\Delta$$

$C_3 = 2(n^2-1)\bar{A}^*_{22}(\bar{B}^*_{22} - \bar{B}^*_{11})s_1\sin\alpha_o/t\Delta$

$C_4 = 2(n^2-1)(\bar{B}^*_{21}\bar{A}^*_{11} - \bar{B}^*_{12}\bar{A}^*_{22})s_1\sin\alpha_o/t\Delta$

$C_5 = 2n(\bar{B}^*_{61}\bar{A}^*_{22} - 2\bar{B}^*_{26}\bar{A}^*_{22} - 2\bar{B}^*_{21}\bar{A}^*_{26})s_1\sin\alpha_o/t\Delta$

$C_6 = -2n\bar{A}^*_{22}(\bar{B}^*_{61} + \bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}^*_{26})s_1\sin\alpha_o/t\Delta$

$C_7 = 2n(n^2-1)(2\bar{B}^*_{21}\bar{A}^*_{16} - \bar{B}^*_{62}\bar{A}^*_{22} + 2\bar{B}^*_{16}\bar{A}^*_{22})s_1\sin\alpha_o/t\Delta$

$C_8 = \bar{B}^*_{21}(\bar{B}^*_{11} - \bar{B}^*_{22})/\Delta$

$$C_9 = [n^2(\bar{B}^*_{11}\bar{B}^*_{21} + \bar{B}^*_{21}\bar{B}^*_{22} - 2\bar{B}^*_{21}\bar{B}^*_{66} + 2\bar{D}^*_{12}\bar{A}^*_{22} + 4\bar{D}^*_{66}\bar{A}^*_{22}) + \bar{B}^{*2}_{21} + \bar{B}^*_{12}\bar{B}^*_{21} + \bar{D}^*_{11}\bar{A}^*_{22} + \bar{D}^*_{22}\bar{A}^*_{22}]/\Delta$$

$C_{10} = (n^2-1)(\bar{B}^*_{11}\bar{B}^*_{21} - \bar{B}^*_{21}\bar{B}^*_{22})/\Delta$

$C_{11} = -\,(n^2-1)^2(\bar{B}^*_{21}\bar{B}^*_{12} + \bar{D}^*_{22}\bar{A}^*_{22})/\Delta$

$C_{12} = n(\bar{B}^*_{21}\bar{B}^*_{61} - 2\bar{B}^*_{21}\bar{B}^*_{26} - 4\bar{D}^*_{16}\bar{A}^*_{22})/\Delta$

$C_{13} = n\bar{B}^*_{21}(\bar{B}^*_{61} + \bar{B}^*_{62} + 2\bar{B}^*_{16} + 2\bar{B}^*_{26})/\Delta$

$C_{14} = n(n^2-1)(4\bar{D}^*_{26}\bar{A}^*_{22} + 2\bar{B}^*_{21}\bar{B}^*_{16} - \bar{B}^*_{21}\bar{B}^*_{62})/\Delta$

$C_{15} = 4cs_1^2 \sin\alpha_o \bar{A}^*_{22}/(t^2\Delta\tan\alpha_o)$

$C_{16} = 2cs_1\bar{B}^*_{21}/(t\Delta\tan\alpha_o)$

$C_{17} = 4cs_1\sin\alpha_o\bar{A}^*_{22}/t\Delta$

$C_{18} = 4(1-n^2)cs_1\sin\alpha_o\bar{A}^*_{22}/t\Delta$

$C_{19} = 2c\bar{B}^*_{21}/\Delta$

$C_{20} = 2c(1-n^2)\bar{B}^*_{21}/\Delta$

$C_{21} = [\,\bar{D}^*_{11}(2n^2\bar{A}^*_{12} + n^2\bar{A}^*_{66} + \bar{A}^*_{11} + \bar{A}^*_{22})$
$\quad - \bar{B}^*_{21}(2n^2\bar{B}^*_{66} - n^2\bar{B}^*_{11} - n^2\bar{B}^*_{22} - \bar{B}^*_{21} - \bar{B}^*_{12})\,]/\Delta$

$C_{22} = -(n^2-1)^2(\bar{B}^*_{12}\bar{B}^*_{21} + \bar{D}^*_{11}\bar{A}^*_{11})/\Delta$

$C_{23} = [\,2n\bar{A}^*_{26}\bar{D}^*_{11} - n\bar{B}^*_{21}(2\bar{B}^*_{26} - \bar{B}^*_{61})\,]/\Delta$

$C_{24} = -n(n^2-1)[\,\bar{B}^*_{21}(\bar{B}^*_{62} - 2\bar{B}^*_{16}) + 2\bar{A}^*_{16}\bar{D}^*_{11}\,]/\Delta$

$C_{25} = \bar{D}^*_{11}(\bar{B}^*_{22} - \bar{B}^*_{11})t/(2\Delta s_1\sin\alpha_o)$

$C_{26} = [\,\bar{B}^*_{21}(2n^2\bar{D}^*_{12} + 4n^2\bar{D}^*_{66} + \bar{D}^*_{22})$
$\quad + \bar{D}^*_{11}(2n^2\bar{B}^*_{66} - n^2\bar{B}^*_{22} - n^2\bar{B}^*_{11} - \bar{B}^*_{12})\,]\,t/(2\Delta s_1\sin\alpha_o)$

$C_{27} = \bar{D}^*_{11}(\bar{B}^*_{22} - \bar{B}^*_{11})(n^2-1)t/(2\Delta s_1\sin\alpha_o)$

$C_{28} = (n^2-1)^2 t(\bar{D}^*_{11}\bar{B}^*_{12} - \bar{B}^*_{21}\bar{D}^*_{22})/(2s_1\Delta\sin\alpha_o)$

$C_{29} = nt(2\bar{D}^*_{11}\bar{B}^*_{26} - \bar{D}^*_{11}\bar{B}^*_{61} - 4\bar{B}^*_{21}\bar{D}^*_{16})/(2s_1\sin\alpha_o\Delta)$

$C_{30} = -nt\bar{D}^*_{11}(2\bar{B}^*_{26} + 2\bar{B}^*_{16} + \bar{B}^*_{62} + \bar{B}_{61})/(2s_1\sin\alpha_o\Delta)$

$C_{31} = (n^2-1)t[\,n\bar{D}^*_{11}(\bar{B}^*_{62} - 2\bar{B}^*_{16}) + 4n\bar{B}^*_{21}\bar{D}^*_{26}\,]/(2s_1\sin\alpha_o\Delta)$

$C_{32} = -\bar{D}^*_{11}c\cot\alpha_o/\Delta\sin\alpha_o$

$C_{33} = -\bar{D}^*_{11}ct/(\Delta s_1\sin\alpha_o)$

$C_{34} = -\bar{D}^*_{11}ct(1-n^2)/(\Delta s_1\sin\alpha_o)$

where $\Delta = \bar{B}^{*2}_{21} + \bar{D}^*_{11}\bar{A}^*_{22}$

A4.3 Constants used in Eq. (6.26)

$D_1 = C_{10} + C_{16}e^z + C_{17}e^z(Y_1^{(o)} + Y_2^{(o)}) + C_{19}e^z(Y_3^{(o)} + Y_4^{(o)})$

$D_2 = C_3 + C_{15}e^z + C_{17}e^z(Y_3^{(o)} + Y_4^{(o)})$

$D_3 = C_{11} + C_{18}e^zY_2^{(o)} + C_{20}e^zY_4^{(o)}$

$D_4 = C_4 + C_{18}e^zY_4^{(o)}$

$D_5 = C_9 + C_{16}e^z + C_{17}e^zY_1^{(o)} + C_{19}e^zY_3^{(o)}$

$D_6 = C_2 + C_{15}e^z + C_{17}e^zY_3^{(o)}$

$D_7 = C_{27} + C_{32}e^z + C_{19}e^z(Y_1^{(o)} + Y_2^{(o)}) + C_{33}e^z(Y_3^{(o)} + Y_4^{(o)})$

$D_8 = -C_{10} + C_{16}e^z + C_{19}e^z(Y_3^{(o)} + Y_4^{(o)})$

$D_9 = C_{28} + C_{20}e^zY_2^{(o)} + C_{34}e^zY_4^{(o)}$

$D_{10} = C_{22} + C_{20}e^zY_4^{(o)}$

$D_{11} = C_{26} + C_{32}e^z + C_{19}e^zY_1^{(o)} + C_{33}e^zY_3^{(o)}$

$D_{12} = C_{21} + C_{16}e^z + C_{19}e^zY_3^{(o)}$

A4.4 Properties of W(z)

The initial value problem for W(z) can be written in the following partitioned form

$$\begin{bmatrix} W'_{11}(z) & W'_{12}(z) \\ W'_{21}(z) & W'_{22}(z) \end{bmatrix} = \begin{bmatrix} R_{11}(z) & R_{12}(z) \\ -R_{12}(z) & R_{11}(z) \end{bmatrix} \begin{bmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{bmatrix}$$

with the initial conditions (if multiple shooting is employed)

$$\begin{bmatrix} W_{11}[z(2j)] & W_{12}[z(2j)] \\ W_{21}[z(2j)] & W_{22}[z(2j)] \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where z(2j) are the starting points for the integration (j = 1, ..., N-1, if 2N is the number of intervals). For edge intervals (j = 0 or j = N) the analysis is similar.

One now can obtain two uncoupled initial value problems

$$W'_{11} = R_{11} W_{11} + R_{12} W_{21}$$

$$W'_{21} = -R_{12} W_{11} + R_{11} W_{21}$$

$$W_{11}[z(2j)] = I$$

$$W_{21}[z(2j)] = 0$$

and

$$W'_{22} = R_{12} W_{12} + R_{11} W_{22}$$

$$W'_{12} = R_{11} W_{12} + R_{12} W_{22}$$

$$W_{22}[z(2j)] = I$$

$$W_{12}[z(2j)] = 0$$

Comparing the two initial value problems, it is found that

$$W_{11} = W_{22}$$

and

$$W_{21} = -W_{12}$$

Consequently, W(z) can be written as follows:

$$W(z) = \begin{bmatrix} W_{11}(z) & -W_{21}(z) \\ W_{21}(z) & W_{11}(z) \end{bmatrix}$$

Therefore only eight solutions of the variational equations are required.

Appendix 5
For Chapter Seven

A5.1 Constants used in Eqs. (7.6)-(7.11)

$$D_1 = \bar{B}^*_{21}(\bar{B}^*_{11}-\bar{B}^*_{22})/\Delta$$

$$D_2 = (\bar{B}^{*2}_{21}+\bar{B}^*_{21}\bar{B}^*_{12}+\bar{D}^*_{11}\bar{A}^*_{22}+\bar{D}^*_{22}\bar{A}^*_{22})/\Delta$$

$$D_3 = (\bar{B}^*_{21}\bar{B}^*_{12}+\bar{D}^*_{22}\bar{A}^*_{22})/\Delta$$

$$D_4 = 2s_1 \sin\alpha_o\, \bar{A}^*_{22}(\bar{B}^*_{22}-\bar{B}^*_{11})/\Delta$$

$$D_5 = 2s_1 \sin\alpha_o(\bar{A}^*_{22}\bar{B}^*_{12}-\bar{B}^*_{21}\bar{A}^*_{11})/t\Delta$$

$$D_8 = 2c\bar{B}^*_{21}/\Delta$$

$$D_7 = (1-n^2)c\bar{B}^*_{21}/\Delta$$

$$D_6 = c\bar{B}^*_{21}/\Delta$$

$$D_9 = 2cs_1\, \bar{B}^*_{21}\cot\alpha_o/t\Delta$$

$$D_{10} = 4cs_1\bar{A}^*_{22}\sin\alpha_o/t\Delta$$

$$D_{11} = 4cs_1^2\bar{A}^*_{22}\sin\alpha_o \cot\alpha_o/t^2\Delta$$

$$D_{12} = 4(1-n^2)cs_1\,\bar{A}^*_{22}\sin\alpha_o/t\Delta$$

$$D_{13} = 2cs_1\,\bar{A}^*_{22}\sin\alpha_o/t\Delta$$

$$D_{14} = 2cs_1 n^2\bar{A}^*_{22}\sin\alpha_o/t\Delta$$

$$\begin{aligned} D_{15} = [\,&4n^2(2\bar{B}^*_{21}\bar{B}^*_{66}-\bar{B}^*_{21}\bar{B}^*_{22}-\bar{B}^*_{11}\bar{B}^*_{21} \\ &-2\bar{D}^*_{11}\bar{A}^*_{12}-\bar{D}^*_{11}\bar{A}^*_{66}) - \bar{B}^{*2}_{21} \\ &- \bar{B}^*_{21}\bar{B}^*_{12} - \bar{D}^*_{11}\bar{A}^*_{11} - \bar{D}^*_{11}\bar{A}^*_{22}\,]/\Delta \end{aligned}$$

$$D_{16} = (1-4n^2)\bar{B}^*_{21}(\bar{B}^*_{22}-\bar{B}^*_{11})/\Delta$$

$$D_{17} = (1-4n^2)^2(\bar{D}^*_{11}\bar{A}^*_{11}+\bar{B}^*_{21}\bar{B}^*_{12})/\Delta$$

$$D_{18} = 2n(2\bar{B}^*_{21}\bar{B}^*_{26}-\bar{B}^*_{21}\bar{B}^*_{61}-2\bar{D}^*_{11}\bar{A}^*_{26})/\Delta$$

$$D_{19} = 2n\bar{B}^*_{21}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}_{26})/\Delta$$

$$\begin{aligned} D_{20} = 2n(&4n^2\bar{B}^*_{21}\bar{B}^*_{62}-8n^2\bar{B}^*_{21}\bar{B}^*_{16}+8n^2\bar{D}^*_{11}\bar{A}^*_{16} \\ &-\bar{B}^*_{21}\bar{B}^*_{62}+2\bar{B}^*_{21}\bar{B}^*_{16}-2\bar{D}^*_{11}\bar{A}^*_{16})/\Delta \end{aligned}$$

$$D_{21} = t\bar{D}^*_{11}(\bar{B}^*_{11}-\bar{B}^*_{22})/(2s_1 \sin\alpha_o\Delta)$$

$$\begin{aligned} D_{22} = t(&4n^2\bar{B}^*_{11}\bar{D}^*_{11}-8n^2\bar{B}^*_{21}\bar{D}^*_{12}-16n^2\bar{B}^*_{21}\bar{D}^*_{66} \\ &+4n^2\bar{D}^*_{11}\bar{B}^*_{22}-8n^2\bar{D}^*_{11}\bar{B}^*_{66} \\ &-\bar{B}^*_{21}\bar{D}^*_{22}+\bar{D}^*_{11}\bar{B}^*_{12})/(2s_1 \sin\alpha_o\Delta) \end{aligned}$$

$$D_{23} = t(1-4n^2)\bar{D}^*_{11}(\bar{B}^*_{22}-\bar{B}^*_{11})/(2s_1 \sin\alpha_o\Delta)$$

$$D_{24} = t(1-4n^2)^2(\bar{B}^*_{21}\bar{D}^*_{22}-\bar{B}^*_{12}\bar{D}^*_{11})/(2s_1 \sin\alpha_o\Delta)$$

$$D_{25} = tn(4\bar{B}^*_{21}\bar{D}^*_{16}+\bar{B}^*_{61}\bar{D}^*_{11}-2\bar{D}^*_{11}\bar{B}^*_{26})/(s_1 \sin\alpha_o\Delta)$$

$$D_{26} = tn\bar{D}^*_{11}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}_{26})/(s_1 \sin\alpha_o\Delta)$$

$$\begin{aligned} D_{27} = tn(&8n^2\bar{D}^*_{11}\bar{B}^*_{16}-16n^2\bar{B}^*_{21}\bar{D}^*_{26} \\ &-4n^2\bar{D}^*_{11}\bar{B}^*_{62}+4\bar{B}^*_{21}\bar{D}^*_{26}+\bar{D}^*_{11}\bar{B}^*_{62} \\ &-2\bar{D}^*_{11}\bar{B}^*_{16})/(s_1 \sin\alpha_o\Delta) \end{aligned}$$

$$D_{28} = ct\bar{D}^*_{11}/(s_1 \sin\alpha_o \Delta)$$

$$D_{29} = 4ctn^2\bar{D}^*_{11}/(s_1 \sin\alpha_o \Delta)$$

$$D_{30} = ct(1-n^2)\bar{D}^*_{11}/(2s_1 \sin\alpha_o \Delta)$$

$$D_{31} = ct(1+n^2)\bar{D}^*_{11}/(2s_1 \sin\alpha_o \Delta)$$

$$D_{32} = ct\bar{D}^*_{11}/(2s_1 \sin\alpha_o \Delta)$$

$$D_{33} = c\cot\alpha_o\bar{D}^*_{11}/(\sin\alpha_o \Delta)$$

$$D_{34} = 2(1+n^2)c\bar{B}^*_{21}/\Delta$$

$$D_{35} = 8n^2c\bar{B}^*_{21}/\Delta$$

$$D_{36} = ct(1+n^2)\bar{D}^*_{11}/(s_1 \sin\alpha_o \Delta)$$

$$\begin{aligned} D_{37} = {} & (4n^2\bar{B}^*_{11}\bar{B}^*_{21}+4n^2\bar{B}^*_{21}\bar{B}^*_{22}-8n^2\bar{B}^*_{21}\bar{B}^*_{66} \\ & +8n^2\bar{D}^*_{12}\bar{A}^*_{22}+16n^2\bar{D}^*_{66}\bar{A}^*_{22} \\ & +\bar{B}^{*2}_{21}+\bar{B}^*_{21}\bar{B}^*_{12}+\bar{D}^*_{11}\bar{A}^*_{22}+\bar{D}^*_{22}\bar{A}^*_{22})/\Delta \end{aligned}$$

$$D_{38} = (1-4n^2)^2(\bar{D}^*_{22}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{B}^*_{12})/\Delta$$

$$D_{39} = 2n(\bar{B}^*_{21}\bar{B}^*_{61}-2\bar{B}^*_{21}\bar{B}^*_{26}-4\bar{D}^*_{16}\bar{A}^*_{22})/\Delta$$

$$\begin{aligned} D_{40} = {} & 2n(8n^2\bar{B}^*_{21}\bar{B}^*_{16}-4n^2\bar{B}^*_{21}\bar{B}^*_{62}+16n^2\bar{D}^*_{26}\bar{A}^*_{22} \\ & +\bar{B}^*_{21}\bar{B}^*_{62}-2\bar{B}^*_{21}\bar{B}^*_{16}-4\bar{D}^*_{26}\bar{A}^*_{22})/\Delta \end{aligned}$$

$$\begin{aligned} D_{41} = {} & 2s_1 \sin\alpha_o (4n^2\bar{B}^*_{11}\bar{A}^*_{22}-8n^2\bar{B}^*_{21}\bar{A}^*_{12} \\ & -4n^2\bar{B}^*_{21}\bar{A}^*_{66}+4n^2\bar{B}^*_{22}\bar{A}^*_{22}-8n^2\bar{B}^*_{66}\bar{A}^*_{22} \\ & -\bar{B}^*_{21}\bar{A}^*_{11}+\bar{B}^*_{12}\bar{A}^*_{22})/(t\Delta) \end{aligned}$$

$$D_{42} = 2s_1 \sin\alpha_o (1-4n^2)\bar{A}^*_{22}(\bar{B}^*_{11}-\bar{B}^*_{22})/(t\Delta)$$

$$D_{43} = 2s_1 \sin\alpha_o (1-4n^2)^2(\bar{B}^*_{21}\bar{A}^*_{11}-\bar{B}^*_{12}\bar{A}^*_{22})/(t\Delta)$$

$$D_{44} = 4ns_1 \sin\alpha_o (\bar{B}^*_{61}\bar{A}^*_{22}-2\bar{B}^*_{21}\bar{A}^*_{26}-2\bar{B}^*_{26}\bar{A}^*_{22})/(t\Delta)$$

$$D_{45} = 4ns_1 \sin\alpha_o \bar{A}^*_{22}(\bar{B}^*_{61}+\bar{B}^*_{62}+2\bar{B}^*_{16}+2\bar{B}_{26})/(t\Delta)$$

$$\begin{aligned} D_{46} = {} & 4ns_1 \sin\alpha_o (1-4n^2)(\bar{B}^*_{62}\bar{A}^*_{22} \\ & -2\bar{B}^*_{16}\bar{A}^*_{22}-2\bar{B}^*_{21}\bar{A}^*_{16})/(t\Delta) \end{aligned}$$

$$D_{47} = c(1+n^2)\bar{B}^*_{21}/\Delta$$

$$D_{48} = 16cn^2s_1 \sin\alpha_o \bar{A}^*_{22}/(t\Delta)$$

$$D_{49} = 2(1-n^2)cs_1 \sin\alpha_o \bar{A}^*_{22}/(t\Delta)$$

$$D_{50} = 4(1+n^2)cs_1 \sin\alpha_o \bar{A}^*_{22}/(t\Delta)$$

$$D_{51} = c(1-n^2)\bar{B}^*_{21}/(2\Delta)$$

$$D_{52} = c\bar{B}^*_{21}/(2\Delta)$$

$$D_{53} = (\bar{B}^*_{21}\bar{B}^*_{12}+\bar{D}^*_{11}\bar{A}^*_{11})/\Delta$$

$$D_{54} = t(\bar{D}^*_{22}\bar{B}^*_{21}-\bar{D}^*_{11}\bar{B}^*_{12})/(2s_1 \sin\alpha_o \Delta)$$

$$D_{55} = ct(1-n^2)\bar{D}^*_{11}/(4s_1 \sin\alpha_o \Delta)$$

$$D_{56} = ct\bar{D}^*_{11}/(4s_1 \sin\alpha_o \Delta)$$

A5.2 GEPCAC

```
comment This is for the derivation of the initial postbuckling
coefficients of anisotropic conical shells, namely, a, b, alpha,
beta with the imperfection taken the general form of
Wimpf=ker*Wcapa=ker*(wi1*cos(n*th)+wi2*sin(n*th))$

depend f(0),z,lm$
depend f(1),z,th$
depend f(2),z,th$
depend w(0),z,lm$
depend w(1),z,th$
depend w(2),z,th$
depend wi,z,th$
depend wb(0),s,lm$
depend wb(1),s,th$
depend wb(2),s,th$
depend wis,s,th$
depend f0,z,lm$
depend f1,z$
depend f2,z$
depend fa,z$
depend fp,z$
depend fr,z$
depend w0,z,lm$
depend w1,z$
depend w2,z$
depend wi1,z$
depend wi2,z$
depend wa,z$
depend wp,z$
depend wr,z$

%This is for the coordinate transformations$
%wb(i)=e^z*w(i),wis=e^z*wi,fb(i)=e^z*f(i), s=s1*e^z$
%d()/ds=(d()/dz)*dz/ds=(d()/dz)/(s1*e^z)$
%where wb means W(s,th), wis means W(s,th)-imperfection,
%wi means W(z,th)-imperfection, fb means F(s,th)$

for all k let
ns(k)=(f(k)+df(f(k),z)+df(f(k),th,2))/(s1^2*e^z),
nth(k)=(df(f(k),z,2)+df(f(k),z))/(s1^2*e^z),
nst(k)=-df(f(k),z,th)/(s1^2*e^z),
%wb(k)=e^z*w(k),
```

```
df(wb(k),s)=(df(w(k),z)+w(k))/s1,
df(wb(k),th)=df(w(k),th)/s1$
let df(wis,s)=(df(wi,z)+wi)/s1, df(wis,th)=df(wi,th)/s1$


for all a, b let
sin(a)^2=(1-cos(2*a))/2,
cos(a)^2=(1+cos(2*a))/2,
sin(a)*sin(b)=-(cos(a+b)-cos(a-b))/2,
cos(a)*cos(b)=(cos(a+b)+cos(a-b))/2,
sin(a)*cos(b)=(sin(a+b)+sin(a-b))/2$


let
%si(al)=sin(al)$
f(0)=e*t^2*s1*si(al)*f0/c,
f(1)=e*t^2*s1*si(al)*(f1*cos(n*th)+f2*sin(n*th))/c,
f(2)=e*t^2*s1*si(al)*(fa+fp*cos(2*n*th)+fr*sin(2*n*th))/c,
w(0)=t*w0,
w(1)=t*(w1*cos(n*th)+w2*sin(n*th)),
%wi is the general imperfection mode,if one lets wi1 equal to w1 and
%wi2 equal to w2 then the imperfection is affine to the buckling mode;
wi=t*(wi1*cos(n*th)+wi2*sin(n*th)),
w(2)=t*(wa+wp*cos(2*n*th)+wr*sin(2*n*th))$

%The integration function of the numerator of the factor a$
aeq:=ns(1)*df(wb(1),s)^2+nth(1)*df(wb(1),th)^2+
2*nst(1)*df(wb(1),s)*df(wb(1),th)$
length(ws);

%The integration function of the numerator of the factor b$
beq1:=ns(1)*df(wb(1),s)*df(wb(2),s)$
length(ws);
beq2:=nth(1)*df(wb(1),th)*df(wb(2),th)$
length(ws);
beq3:=nst(1)*(df(wb(1),s)*df(wb(2),th)+df(wb(1),th)*df(wb(2),s))$
length(ws);
beq4:=ns(2)*df(wb(1),s)^2$
length(ws);
beq5:=nth(2)*df(wb(1),th)^2$
length(ws);
beq6:=nst(2)*df(wb(1),s)*df(wb(1),th)$
length(ws);

%The integration function of the numerator of the factor alpha$
aleq1:=ns(0)*df(wb(1),s)*df(wis,s)$
length(ws);
aleq2:=nth(0)*df(wb(1),th)*df(wis,th)$
length(ws);
aleq3:=nst(0)*(df(wb(1),s)*df(wis,th)+df(wb(1),th)*df(wis,s))$
length(ws);
aleq4:=ns(1)*df(wis,s)*df(wb(0),s)$
length(ws);
aleq5:=nth(1)*df(wis,th)*df(wb(0),th)$
length(ws);
aleq6:=nst(1)*(df(wis,s)*df(wb(0),th)+df(wis,th)*df(wb(0),s))$
length(ws);

%The integration function of the numerator of the factor bata$
baeq1:=df(ns(0),lm)*df(wb(1),s)*df(wis,s)$
length(ws);
baeq2:=df(nth(0),lm)*df(wb(1),th)*df(wis,th)$
length(ws);
baeq3:=df(nst(0),lm)*(df(wb(1),s)*df(wis,th)+df(wb(1),th)
*df(wis,s))$
length(ws);
baeq4:=ns(1)*df(wis,s)*df(wb(0),s,lm)$
length(ws);
baeq5:=nth(1)*df(wis,th)*df(wb(0),th,lm)$
length(ws);
baeq6:=nst(1)*(df(wis,s)*df(wb(0),th,lm)+df(wis,th)
*df(wb(0),s,lm))$
length(ws);

%II3;
baeq8:=e*t*(a11*df(wb(0),s,lm)*df(wb(1),s)+
       a12*df(wb(0),th,lm)*df(wb(1),th)+
       a16*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*df(wb(0),s)*df(wis,s)$
length(ws);
baeq9:=e*t*(a12*df(wb(0),s,lm)*df(wb(1),s)+
       a22*df(wb(0),th,lm)*df(wb(1),th)+
       a26*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*df(wb(0),th)*df(wis,th)$
length(ws);
baeq10:=e*t*(a16*df(wb(0),s,lm)*df(wb(1),s)+
       a26*df(wb(0),th,lm)*df(wb(1),th)+
       a66*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*(df(wb(0),th)*df(wis,s)+
       df(wb(0),s)*df(wis,th))$
length(ws);


%II4;
baeq11:=ns(1)*df(wb(1),s)*df(wb(0),s,lm,2)$
length(ws);
baeq12:=nth(1)*df(wb(1),th)*df(wb(0),th,lm,2)$
length(ws);
baeq13:=nst(1)*df(wb(1),th)*df(wb(0),s,lm,2)$
length(ws);
baeq14:=nst(1)*df(wb(1),s)*df(wb(0),th,lm,2)$
length(ws);
baeq15:=df(ns(0),lm,2)*df(wb(1),s)^2$
length(ws);
baeq16:=df(nth(0),lm,2)*df(wb(1),th)^2$
length(ws);
baeq17:=df(nst(0),lm,2)*df(wb(1),s)*df(wb(1),th)$
length(ws);

%II5;
baeq18:=e*t*(a11*df(wb(0),s,lm)*df(wb(1),s)+
       a12*df(wb(0),th,lm)*df(wb(1),th)+
       a16*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*df(wb(0),s,lm)*df(wb(1),s)$
length(ws);
baeq19:=e*t*(a12*df(wb(0),s,lm)*df(wb(1),s)+
       a22*df(wb(0),th,lm)*df(wb(1),th)+
       a26*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*df(wb(0),th,lm)*df(wb(1),th)$
length(ws);
baeq20:=e*t*(a16*df(wb(0),s,lm)*df(wb(1),s)+
       a26*df(wb(0),th,lm)*df(wb(1),th)+
       a66*(df(wb(0),s,lm)*df(wb(1),th)+
       df(wb(0),th,lm)*df(wb(1),s)))*(df(wb(0),th,lm)*df(wb(1),s)
       +df(wb(0),s,lm)*df(wb(1),th))$


%The integration function of the denominator $
neq1:=ns(1)*df(wb(1),s)*df(wb(0),s,lm)$
length(ws);
neq2:=nth(1)*df(wb(1),th)*df(wb(0),th,lm)$
length(ws);
```

```
neq3:=nst(1)*df(wb(1),th)*df(wb(0),s,lm)$
length(ws);
neq4:=nst(1)*df(wb(1),s)*df(wb(0),th,lm)$
length(ws);
neq5:=df(ns(0),lm)*df(wb(1),s)^2$
length(ws);
neq6:=df(nth(0),lm)*df(wb(1),th)^2$
length(ws);
neq7:=df(nst(0),lm)*df(wb(1),s)*df(wb(1),th)$
length(ws);

%The integration of the numerator of the factor a$
ain:=int(aeq,th)$

%The integration of the numerator of the factor b$
bin1:=int(beq1,th)$
bin2:=int(beq2,th)$
bin3:=int(beq3,th)$
bin4:=int(beq4,th)$
bin5:=int(beq5,th)$
bin6:=int(beq6,th)$

%The integration of the numerator of the factor alpha$
alin1:=int(aleq1,th)$
alin2:=int(aleq2,th)$
alin3:=int(aleq3,th)$
alin4:=int(aleq4,th)$
alin5:=int(aleq5,th)$
alin6:=int(aleq6,th)$


%The integration of the numerator of the factor bata$
bain1:=int(baeq1,th)$
bain2:=int(baeq2,th)$
bain3:=int(baeq3,th)$
bain4:=int(baeq4,th)$
bain5:=int(baeq5,th)$
bain6:=int(baeq6,th)$
bain8:=int(baeq8,th)$
bain9:=int(baeq9,th)$
bain10:=int(baeq10,th)$
bain11:=int(baeq11,th)$
bain12:=int(baeq12,th)$
bain13:=int(baeq13,th)$
bain14:=int(baeq14,th)$
bain15:=int(baeq15,th)$
bain16:=int(baeq16,th)$
bain17:=int(baeq17,th)$
bain18:=int(baeq18,th)$
bain19:=int(baeq19,th)$
bain20:=int(baeq20,th)$

%The integration  of the denominator $
nin1:=int(neq1,th)$
nin2:=int(neq2,th)$
nin3:=int(neq3,th)$
nin4:=int(neq4,th)$
nin5:=int(neq5,th)$
nin6:=int(neq6,th)$
nin7:=int(neq7,th)$

let
cos(2*n*pi)=1,
cos(4*n*pi)=1,
cos(6*n*pi)=1,
cos(8*n*pi)=1,
sin(n*pi)=0,
sin(2*n*pi)=0,
sin(4*n*pi)=0,
sin(6*n*pi)=0,
sin(8*n*pi)=0$

on list$

%The  numerator of the factor a$
ain1:=sub(th=2*pi,ain)-sub(th=0,ain)$
length(ws);

%The six components of numerator of factor b$
bin1f:=sub(th=2*pi,bin1)-sub(th=0,bin1)$
length(ws);
bin2f:=sub(th=2*pi,bin2)-sub(th=0,bin2)$
length(ws);
bin3f:=sub(th=2*pi,bin3)-sub(th=0,bin3)$
length(ws);
bin4f:=sub(th=2*pi,bin4)-sub(th=0,bin4)$
length(ws);
bin5f:=sub(th=2*pi,bin5)-sub(th=0,bin5)$
length(ws);
bin6f:=sub(th=2*pi,bin6)-sub(th=0,bin6)$
length(ws);

depend fi(0),z,lm$
depend kerci(0),z,lm$
let df(f0,z)=fi(0)-f0,df(f0,z,2)=df(fi(0),z)-df(f0,z),
   df(w0,z)=kerci(0)-w0,df(w0,z,2)=df(kerci(0),z)-df(w0,z)$

%The seven components of numerator of factor alpha$
alin1f:=sub(th=2*pi,alin1)-sub(th=0,alin1)$
length(ws);
alin2f:=sub(th=2*pi,alin2)-sub(th=0,alin2)$
length(ws);
alin3f:=sub(th=2*pi,alin3)-sub(th=0,alin3)$
length(ws);
alin4f:=sub(th=2*pi,alin4)-sub(th=0,alin4)$
length(ws);
alin5f:=sub(th=2*pi,alin5)-sub(th=0,alin5)$
length(ws);
alin6f:=sub(th=2*pi,alin6)-sub(th=0,alin6)$
length(ws);


%The twenty components of numerator of factor bata$
bain1f:=sub(th=2*pi,bain1)-sub(th=0,bain1)$
length(ws);
bain2f:=sub(th=2*pi,bain2)-sub(th=0,bain2)$
length(ws);
bain3f:=sub(th=2*pi,bain3)-sub(th=0,bain3)$
length(ws);
bain4f:=sub(th=2*pi,bain4)-sub(th=0,bain4)$
length(ws);
bain5f:=sub(th=2*pi,bain5)-sub(th=0,bain5)$
length(ws);
bain6f:=sub(th=2*pi,bain6)-sub(th=0,bain6)$
length(ws);
bain8f:=sub(th=2*pi,bain8)-sub(th=0,bain8)$
length(ws);
bain9f:=sub(th=2*pi,bain9)-sub(th=0,bain9)$
length(ws);
bain10f:=sub(th=2*pi,bain10)-sub(th=0,bain10)$
length(ws);
bain11f:=sub(th=2*pi,bain11)-sub(th=0,bain11)$
length(ws);
bain12f:=sub(th=2*pi,bain12)-sub(th=0,bain12)$
```

```
length(ws);
bain13f:=sub(th=2*pi,bain13)-sub(th=0,bain13)$
length(ws);
bain14f:=sub(th=2*pi,bain14)-sub(th=0,bain14)$
length(ws);
bain15f:=sub(th=2*pi,bain15)-sub(th=0,bain15)$
length(ws);
bain16f:=sub(th=2*pi,bain16)-sub(th=0,bain16)$
length(ws);
bain17f:=sub(th=2*pi,bain17)-sub(th=0,bain17)$
length(ws);
bain18f:=sub(th=2*pi,bain18)-sub(th=0,bain18)$
length(ws);
bain19f:=sub(th=2*pi,bain19)-sub(th=0,bain19)$
length(ws);
bain20f:=sub(th=2*pi,bain20)-sub(th=0,bain20)$
length(ws);

%The seven components of denominator $
nin1f:=sub(th=2*pi,nin1)-sub(th=0,nin1)$
length(ws);
nin2f:=sub(th=2*pi,nin2)-sub(th=0,nin2)$
length(ws);
nin3f:=sub(th=2*pi,nin3)-sub(th=0,nin3)$
length(ws);
nin4f:=sub(th=2*pi,nin4)-sub(th=0,nin4)$
length(ws);
nin5f:=sub(th=2*pi,nin5)-sub(th=0,nin5)$
length(ws);
nin6f:=sub(th=2*pi,nin6)-sub(th=0,nin6)$
length(ws);
nin7f:=sub(th=2*pi,nin7)-sub(th=0,nin7)$
length(ws);

depend fz,z$
depend wz,z$
let df(fa,z)=fz-fa,df(wa,z)=wz-wa$
%The total numerator of factor b is$
%llb,
numb:=(2*(bin1f+bin2f+bin3f)+bin4f+bin5f+2*bin6f)
*2*e^z*c*s1^3/(si(al)*e*t^4*pi);
length(ws);
numb1:=coeffn(numb,wp,1);
length(ws);
numb2:=coeffn(numb,wr,1);
length(ws);
numb3:=coeffn(numb,wz,1);
length(ws);
numb4:=coeffn(numb,fp,1);
length(ws);
numb5:=coeffn(numb,fr,1);
length(ws);
numb6:=coeffn(numb,fz,1);
length(ws);
numb7:=coeffn(numb,df(wp,z),1);
length(ws);
numb8:=coeffn(numb,df(wr,z),1);
length(ws);
numb9:=coeffn(numb,df(wz,z),1);
length(ws);
numb10:=coeffn(numb,df(fp,z),1);
length(ws);
numb11:=coeffn(numb,df(fr,z),1);
length(ws);
numb12:=coeffn(numb,df(fz,z),1);
length(ws);
numb13:=coeffn(numb,df(fp,z,2),1);
length(ws);
numb14:=coeffn(numb,df(fr,z,2),1);
length(ws);

%The total numerator of factor alpha is$
%llalpha,
numalpha:=alin1f+alin2f+alin3f+alin4f+alin5f+alin6f;
length(ws);

%The total numerator of factor beta are$
%llbeta,
numbeta:=bain1f+bain2f+bain3f+bain4f+bain5f+bain6f;
length(ws);
%ll3,
numbeta3:=bain8f+bain9f+bain10f;
length(ws);
%ll4/2,
numbeta4:=(2*(bain11f+bain12f+bain13f+bain14f)+
bain15f+bain16f+2*bain17f)/2;
length(ws);
%ll5,
numbeta5:=bain18f+bain19f+bain20f;
length(ws);

%The total denominator is$
den:=2*(nin1f+nin2f+nin3f+nin4f)+nin5f+nin6f+2*nin7f;
length(ws);
```

ACKNOWLEDGEMENTS

The work presented in this dissertation was performed at the Department of Structure, Stress and Vibration, Faculty of Aerospace Engineering, Delft University of Technology, the Netherlands.

The author would like to express his grateful thanks to his advisor, Prof. dr. J. Arbocz for his valuable supervision, stimulating suggestions and financial support for this promotion work.

The author would also like to express his appreciations to all his colleagues of Vakgroep C and friends who have contributed in one way or another to the completion of this thesis, in particular to Mrs. Annemarie van Lienden-Datema for her years long help, to his ex-roommate Ir. J.M.A.M. Hol, his present roommates Ir. E. L. Jansen, Ir. B. Knops, Ir. E. Deerenberg, for their various contributions to this thesis, to dr. E. Riks, Ir. van Rijn for their helpful discussions.

The author wishes to thank Prof. dr. M. Baruch from Technion, Israel, for his stimulation during his stay in Delft, to Prof. dr. W.T. Koiter and Prof. dr. K. Schiffner for their constructive discussions.

The author is much indebted to his wife, Suping, for her sustained support and encouragement, especially for her skillful and patient typing of all the boring formulae of this thesis during her pregnancy. Finally, the author has to thank his son, Luke, for his inspiration. Right after he saw an echo picture of his son, the author solved a problem in the post-buckling analysis which had bothered him for months.

CURRICULUM VITAE

Guo-Qi Zhang was born in Beijing, P.R. China, on November 24, 1959. In 1975 he graduated from high school. From 1976 to 1977 he worked at Shaanxi Aircraft Manufacturing Company. From 1977 he studied mechanical engineering at Shaanxi Institute of Mechanical Engineering (S.I.M.E). After receiving the Bachelor diploma (B.Sc.) in the beginning of 1982, he worked as a lecturer at the Department of Mechanical Engineering, S.I.M.E.. From 1984 to 1985 he worked as an engineer in Luoyang Machine Building Company. From 1985 to 1987 he studied Computational and Solid Mechanics at the Faculty of Aerospace Engineering, Northwestern Polytechnical University (N.P.U.). After receiving the Master diploma (M.Sc.) in 1987, he worked as a lecturer at N.P.U.. Since April of 1988, he is working at the Department of Structures, Strength and Vibration, Faculty of Aerospace Engineering of Technical University of Delft, for his Doctor diploma in the field of stability of shell structures under the supervision of Prof. dr. J. Arbocz. This dissertation is part of his results.