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An Equity and Foreign Exchange Heston-Hull-White model for Variable Annuities

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"An Equity and Foreign Exchange Heston-Hull-White model for Variable Annuities"

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Abstract

This project aims to develop and validate the Heston-Hull-White model on Variable Annuities. Such a stochastic modelling assumption is crucial in pricing and hedging the long term exotic options. We calibrate the Equity and FX Heston-Hull-White model in the corresponding markets. A novel numerical integration option pricing method-COS method significantly improve this calibration process. From the conditioned calibration, large amounts of scenarios of 6 stock indices and 3 exchange rates are generated based on this hybrid model using Monte Carlo simulations. Finally we compare the Heston-Hull-White model with the Black Scholes model in the scenario-based valuation of the Guaranteed Minimum Withdrawal Benefits to see the impact of the stochastic model.

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Chapter 1

Introduction

A variable annuity (VA) is a contract between an insurer and a policy holder, under which the insurer agrees to make periodic payments or a lump-sum amount at contract maturity to the policy holder in return for a fee. The fee is necessary to cover operational costs as well as the cost of the guarantee that is embedded in a VA contract. When purchasing a VA, a customer gives the insurer an amount that is invested in mutual funds. The value of the investment in mutual funds, called the account value (AV), will vary in time and depends on market factors as well as fees deducted from the AV to finance the guarantee. This is a special feature of VAs: the fee for the option is not paid upfront but periodically deducted from the AV. Furthermore, the guarantee comes in several flavors, generically called Guaranteed Minimum Benefits (GMxB). GMxB's include the Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB) and Guaranteed Minimum Withdrawal Benefit (GMWB). The pricing and risk management of GMxB's is challenging due to the exotic payoffs, the insurance risks such as mortality or longevity risk, and the fact that the pricing of GMxB's is similar to pricing complex options on a basket of funds (bond and equity funds) with tenors often exceeding 10 years which exposes GMxB's to multi-asset risks.

In the Black-Scholes valuation framework, which some insurers still use for the pricing of variable annuities, the underlying assets in the basket are assumed to satisfy geometric Brownian motion with deterministic interest rate and deterministic volatility. Even though the Black-Scholes model suffices for simple vanilla options with short terms to maturity, we expect that accounting for stochastic volatility and stochastic interest rates will improve the pricing and hedging of GMxB's. Incorporating stochastic interest rates and stochastic volatility in pricing and hedging is also more in line with market practice and the observed characteristics of volatility and interest rates. In order to price and hedge with stochastic volatility and interest rates we will apply the combined Heston stochastic volatility and Hull-White stochastic interest rate model. The choice for the Heston-Hull-White model is motivated by the fact that the characteristic function of this model, which is an essential component of all Fourier-based calibration, can be approximated relatively easily. This will lead to better parameter estimation, i.e. better calibration process. Besides, the combined model may be sufficient to capture the interest rate risk and the equity risk. Once calibrated, the GMxB's can be priced in a straightforward manner by Monte Carlo simulation. Hence, the project can be sub-divided into roughly two parts: calibration and pricing.

For pricing purposes brute force Monte Carlo (MC) methods will be employed. This is inevitable due to the complex features of GMxB's. However, using Monte Carlo for calibration will be too computationally expensive and other (semi-analytical) methods must be sought. One semi-analytical and often used calibration technique is the Fast Fourier Transform (FFT), first introduced by Carr & Madan. In this project, however, we will apply the novel Fourier-based method-COS pricing method of Fang & Oosterlee. It has been proved in several papers that the COS-method is even more efficient than the FFT approach.

This thesis is structured as follows: Chapter 2 gives a general introduction into VAs and the payoffs of several GMxB's. In Chapter 3 we will give details on how to use the COS method for option pricing and apply it to the Heston model. Next, in Chapters 4 and 5 the combined Heston-Hull-White model for Equity and FX is discussed. Also, an approximate form of the characteristic function for the combined model is given. The approximation is used in Chapter 6 for the calibration process of the Heston-Hull-White model. Chapter 7 looks at multi-asset Monte Carlo pricing and investigates the impact of the Heston-Hull-White model on the valuation Guaranteed Minimum Withdrawal Benefits. Chapter 8 summarizes the findings of this thesis in a conclusion.

Chapter 2

Variable Annuities

2.1 Introduction

A Variable Annuity (VA) is an investment / insurance product which offers insurance against a possible downturn of the financial market. The insurer agrees to pay a periodic payment to the clients, immediately in the beginning, or at some future date. Due to their advantages in retirement planning, and tax benefits, VAs have proved to be popular insurance products recently as a combination of equity based derivatives and insurance. After getting significant market shares in the US and Japan, the VAs are now starting to show some success in some European markets, too. Additionally, the current financial crisis has increased the clients' interest in investment guarantees, as provided by VAs. So, these contracts represent good possibilities for investors and the crisis made the business model even more convincing.

VAs are very similar to long-maturity exotic financial derivatives, as the guarantees are based on a basket of underlyings (equity, bond and funds). For VAs, however, this guarantee is separate from the underlying investments. There are some other notable differences to basket options. A VA policy combines, for example, financial and insurance risk like surrender, longevity and mortality. For pricing and hedging of VAs, the financial markets as well as the client behavior have to be taken into account.

The best established and most common VA product is by far the Guaranteed Minimum Benefits product. It is typically referred to as GMAB (Accumulation), GMDB (Death), GMWB (Withdrawal) and GMIB (Income). They can provide an additional guaranteed minimum performance level of the underlying, so that also policy-holder and mortality risk need to be considered. Risk management practitioners are thus exposed to multiple sources of risks, some of which are mitigated by the product structuring while others are addressed using a hedging strategy.

2.2 Risk Exposure

VA products can be viewed as unit-linked products with a guarantee and the risk management process is complex. The VA hedging objective is to offset the guarantee variation with a similar hedge variation. More specifically, for any movement of the capital markets, the change of value of the portfolio should coincide with the change of value of the guarantees. The valuation of the guarantee is therefore key during the hedging process. In the case of Guaranteed Minimum Benefits products, the multiple embedded risks have a significant impact on pricing. The major sources of risk include actuarial and financial risk.

The actuarial risk (insurance risk) contains mortality, lapse and surrender risk. Variable Annuities have a death benefit. If the clients die before the insurer has started making payments, the beneficiary is guaranteed to receive a specified amount – typically at least the amount of purchase payments. Consequently, the survival probability has to be added in order to determine the payoff of the product. More advanced mortality stochastic modeling is also needed for these insurance linked products. Compared to the mortality risk, policy-holder risk such as lapse and withdrawal, is difficult to capture and quantify. In the US market, the GMWB product is very popular currently. The GMWB gives the holder the option to withdraw guaranteed periodic amounts up to the value of the initial capital. Better analysis of this policy-holder behavior needs a powerful statistical approach and an appropriate approximation of the correlation with the financial market.

In this project, we are mainly interested in the financial risk, which can be managed by proper hedging instruments in the derivatives market. The main financial risks are as follows:

•**Delta risk:** VA guarantees are similar to selling basket put options. This risk can be hedged at a reasonable price by a put option through the process of Delta hedging. The purpose of Delta hedging is to set the Delta of a portfolio to zero, resulting in the portfolio's value being relatively insensitive to changes in the value of the underlying security. This hedging technique requires frequent updating to capture the convexity of the hedged position.

•**Volatility risk:** The real shape of the equity volatility surface is hard to incorporate, especially for long term financial products, such as VAs. Variance swaps can be used to hedge the risk.

•**Basis risk:** A VA is not intended to take risk embedded in fund returns. Therefore, fund returns need to be hedged ideally by instruments which perfectly replicate the payoff of the funds. Although most of the return can be hedged, there are still residuals which are not replicable by the market indices. This basis risk cannot be hedged.

•**Interest risk:** This refers to the effect of changes in the prevailing market rate of interest on bond values. This risk can be approximated by a measure called duration. Currently, the interest rate swap is commonly used to hedge the interest risk.

•**Other risks:** There are some other risks such as credit risk, FX risk and correlation risk. Some can also be perfectly hedged by the strong hedging instruments. Due to the existence of all these risks from the financial market, the modeling of the variable annuity becomes more complicated.

2.3 The GMXB Payoff

The Guaranteed Minimum Benefits are specific kinds of VAs with embedded guarantees offered in the policies. These benefits include both death benefits and living benefits. The first type of product is called the Guaranteed Minimum Death Benefit (GMDB), which offers a guaranteed amount when the policy-holder passes away. Another class, which is typically referred to as a Guaranteed Minimum Living Benefit (GMLB), distinguishes three main products. The Guaranteed Minimum Withdrawal Benefit (GMWB), introduced in the previous section, is one of the products providing living benefits. The most basic GMLB product is the Guaranteed Minimum Accumulation Benefit (GMAB). It is similar to the GMDB except that it offers benefits when the policy-holder is alive at some specific date. The last type of this class is called the Guaranteed Minimum Income Benefit (GMIB). It only guarantees when the account value is annuitized at time T.

Here, we consider a standard situation and present the payoff of these guarantee minimum benefits policies.

•**GMDB:** The death benefit is paid at the death of the policy-holder. The payoff corresponds to the underlying account plus an embedded put option. Therefore, the price of a GMDB at the death time should be

$$GMDB_\tau = P(0,\tau)E^{\mathbb{Q}}[(\max(H - S_\tau, 0) + S_\tau)|\mathcal{F}_0], \tag{2.3.1}$$

where τ is the stochastic death time, $P(0,\tau)$ is the zero coupon bound price S_τ is the account value and H is the guarantee, which, for example, is equal to $S_0e^{g\tau}$, with g the roll-up rate. The present value of the GMDB is given by the expectations under τ:

$$GMDB = E_t^{\mathbb{Q}}[GMDB_\tau|\tau = t] = \int_0^T f(t)P(0,T)E^{\mathbb{Q}}[(\max(H - S_t, 0) + S_t)|\mathcal{F}_0]dt. \tag{2.3.2}$$

Here, $f(t)$ is the instantaneous death rate.

•**GMAB:** The customer of the GMAB is entitled to receive at least the invested notional at maturity. The payoff will be $\max(S_T, S_0)$. So, the price will be

$$GMAB = P(0,T)E^{\mathbb{Q}}[\max(S_T, S_0)|\mathcal{F}_0] = P(0,T)E^{\mathbb{Q}}[\max(S_0 - S_T, 0)|\mathcal{F}_0] + S_0. \tag{2.3.3}$$

This is nothing but the sum of a basic put option on an account value S_t with strike S_0 and the notional. Here, we simply assume that the customer is alive at maturity. In practice, it is conditional on the fact that the policy-holder is alive at time T.

•**GMIB:** The GMIB also offers living benefits. At maturity, the policy-holder can choose, as usual, to obtain the account value (without guarantee) or to annuitize the account value at current market conditions (also without any guarantee). However, the GMIB option offers an additional choice: A policy-holder may annuitize some guaranteed amount at annuitization rates α. The payoff is similar to a GMAB:

$$\begin{aligned} GMIB &= P(0,T)E^{\mathbb{Q}}[\max(\alpha \max(S_T, S_0), S_T)|\mathcal{F}_0] \\ &= \max(1, \frac{\alpha}{\beta})\{P(0,T)E^{\mathbb{Q}}[\max(S_0 - S_T, 0)|\mathcal{F}_0] + S_0\}. \end{aligned} \tag{2.3.4}$$

We remark that it would also be a standard put option without the term β.

•**GMWB:** A GMWB is more complex than the other three contracts. It is a put option attached to an equity-like insurance product. If the account value is higher than the withdrawal amount, there is no liability under the GMWB. However, if the account value reaches zero, the GMWB guarantees all remaining periodic payments. This is similar to a basic put option but with a random exercise time. Milevsky and Salisbury [MS06] propose a pricing formulation of the GMWB with both static and dynamic withdrawals under a constant interest rate. They analyze the fair proportional fees that should be charged on the provision of the guarantee. More specifically, the dynamics of the asset S_t underlying GMWB policy would be:

$$dS_t = rS_t dt + \sigma S_t dB_t^{\mathbb{Q}}. \tag{2.3.5}$$

The sub-account value should incorporate two additional features: Proportional insurance fees q and withdrawals. Therefore, the dynamics of the sub-account value of the GMWB W_t would be in the following form:

$$dW_t = (r - q)W_t dt - Gdt + \sigma W_t dB_t^{\mathbb{Q}}, \tag{2.3.6}$$

where G is the guaranteed withdrawal amount, usually paid annually. If W_t reaches zero, it will remain zero to maturity time. Consequently, the payoff of the GMWB is a collection of residual sub-account values at maturity and guaranteed withdrawals, i.e. the value of the GMWB is:

$$GMWB = P(0,T)E^{\mathbb{Q}}[W_T|\mathcal{F}_0] + \int_0^T P(0,t)Gdt. \tag{2.3.7}$$

2.4 Valuation framework

The pricing of VA policies in general is similar to the pricing of basket options. Currently in industrial practice, an underlying asset in the basket is assumed to satisfy geometric Brownian motion with a deterministic interest rate and a deterministic volatility. For some VA guarantees, those are path-dependent, it is not possible to find a closed-form solution for the fair value. Therefore, Monte Carlo methods are widely used for the valuation of these products.

Blamont and Sagoo [BS09] have given a numerical approach for the pricing of GMWBs. The Monte Carlo method is used there to generate the expected residual sub-account at maturity (a call option with strike zero). The advantage of the Monte Carlo method is that many variables can be considered to be stochastic. The parameters can, in principle, be calibrated from the market and the payoff can be computed according to the product features.

Recently, more involved stochastic models, different from the Black-Scholes model, are proposed to account for stochastic volatility and stochastic interest rate in the real market. However, there is a trade-off between more complex models and goodness of calibration. The more complicated the model gets, the more parameters have to be calibrated, and the more unstable a calibration may become. How to choose an appropriate model is crucial for pricing and hedging of VA policies. In this project, we aim to find a comprehensive model which can be calibrated relatively easily and, at the same time, is able to produce a satisfactory fit to the market for long term. Although such models cannot substitute the capital market experience, they may give a better mathematical insight in the market's behavior. More details will be explained in the chapters to follow.

Chapter 3

COS pricing method

We will consider hybrid stochastic models in this MSc thesis, like the Heston Hull-White model. For such hybrid models highly efficient computational techniques are needed for efficient calibration. Numerical integration based on Fourier transform methods represent such efficient procedures. These techniques can be further sub-divided into different types based on the Fourier method employed. For example, the Carr-Madan Fast-Fourier transform method is a popular numerical integration procedure, used by several investment banks. Fang and Oosterlee [FO08] have proposed a novel method, which is called the COS pricing method. This method can also be used to value plain vanilla and some exotic options. The COS method has been proved to be superior to some other Fourier methods. In this MSc project we will use the COS method to value VA products under hybrid stochastic models. We will focus on the Heston Hull-White model, because its characteristic function can be approximated accurately.

3.1 Recovery of the density function via Cosine Expansion

First we will show the main idea of the COS method, i.e. how to use the Fourier cosine series expansion to approximate an integral. Details can be found in Fang and Oosterlee [FO08].

For functions supported on interval $[0, \pi]$, the cosine expansion reads

$$f(\theta) = \sum{}' A_k \cos(k\theta). \tag{3.1.1}$$

Here $A_k = \frac{2}{\pi}\int_0^\pi f(\theta)\cos(k\theta)d\theta$, and $\sum'$ indicates that the first term in the summation is weighted by one-half. If we transform variables as follows:

$$\theta = \frac{x-a}{b-a}\pi, \quad x = \frac{b-a}{\pi}\theta + a, \tag{3.1.2}$$

the Fourier cosine expansion of any functions supported on a different finite interval, say $[a, b] \in \mathbb{R}$, can be obtained

$$f(x) = \sum_{k=0}^{\infty}{}' A_k \cos(k\pi \frac{x-a}{b-a}) \tag{3.1.3}$$

with

$$A_k = \frac{2}{b-a} \int_a^b f(x) \cos(k\pi \frac{x-a}{b-a}) dx. \tag{3.1.4}$$

A sufficient wide interval $[a, b]$ can in some instances be used to accurately approximate an infinite integration range, i.e.

$$\phi_1(\omega) := \int_a^b e^{i\omega x} f(x) dx \approx \int_{\mathbb{R}} e^{i\omega x} f(x) dx = \phi(\omega). \tag{3.1.5}$$

So, we have:

$$A_k \equiv \frac{2}{b-a} Re\{\phi_1(\frac{k\pi}{b-a}) \cdot \exp(-i\frac{ka\pi}{b-a})\}. \tag{3.1.6}$$

If we now define

$$F_k = \frac{2}{b-a} Re\{\phi(\frac{k\pi}{b-a}) \cdot \exp(-i\frac{ka\pi}{b-a})\} \tag{3.1.7}$$

and use the approximation $A_k \approx F_k$, to obtain the series expansion of $f(x)$ on $[a, b]$:

$$f_1(x) = \sum_{k=0}^{\infty}{}' F_k \cos(k\pi \frac{x-a}{b-a}) \tag{3.1.8}$$

A further approximation can be made by truncating the series summation

$$f_2(x) = \sum_{k=0}^{N-1}{}' F_k \cos(k\pi \frac{x-a}{b-a}) \tag{3.1.9}$$

Formula (3.1.9) gives a highly accurate approximation for function $f(x) \in \mathbb{R}$ for an appropriate value of the number of cosine terms, N. Here, we approximate the probability density function which is used in the pricing formula. For some models, this density function is not available in closed-form and a possible technique to approximate it is by formula (3.1.9), since the characteristic function is often known analytically, or can be computed easily. This is the essence of the COS method.

3.2 General COS pricing method

Pricing European options by numerical integration techniques is via the risk-neutral valuation formula:

$$v(x, t_0) = e^{-r\triangle t} E^{\mathbb{Q}}[v(y, T)|x] = e^{-r\triangle t} \int_{\mathbb{R}} v(y, T) f(y|x) dy. \tag{3.2.1}$$

Suppose we have determined values a and b to approximate (3.2.1) by

$$v_1(x, t_0) = e^{-r\triangle t} \int_a^b v(y, T) f(y|x) dy. \tag{3.2.2}$$

Using (3.1.9) gives:

$$v_1(x, t_0) = e^{-r\triangle t} \int_a^b v(y, T) \sum_{k=0}^{\infty}{}' A_k(x) \cos(k\pi \frac{y-a}{b-a}) dy, \tag{3.2.3}$$

with

$$A_k(x) := \frac{2}{b-a} \int_a^b f(y|x) \cos(k\pi \frac{y-a}{b-a}) dy. \tag{3.2.4}$$

More specifically, we approximate by:

$$A_k(x) \approx \frac{2}{b-a} Re\{\phi(\frac{k\pi}{b-a}) \cdot \exp(-i\frac{ka\pi}{b-a})\}. \tag{3.2.5}$$

By interchanging summation and integration, and inserting the definition,

$$V_k := \frac{2}{b-a} \int_a^b v(y, T) \cos(k\pi \frac{y-a}{b-a}) dy, \tag{3.2.6}$$

we obtain

$$v_1(x, t_0) = \frac{1}{2}(b-a) e^{-r\triangle t} \cdot \sum_{k=0}^{\infty}{}' A_k(x) V_k. \tag{3.2.7}$$

The pricing formula approximation then reads:

$$v(x, t_0) \approx e^{-r\triangle t} \sum_{k=0}^{N-1}{}' Re\{\phi(\frac{k\pi}{b-a}; x) \exp(-ik\pi \frac{a}{b-a})\} V_k, \tag{3.2.8}$$

which is the COS formula for general underlying processes. We will subsequently show that the terms V_k can be obtained analytically for plain vanilla options, and that many strike values can be handled simultaneously. In summary, the essence of the COS method is to have an appropriate characteristic function (Chf) in terms of the form (3.1.5) but the availability of a Chf strongly depends on model assumptions.

3.3 Black-Scholes model

Denote the log-asset prices by

$$x := \ln(S_0/K) \,\text{and}\, y := \ln(S_T/K), \tag{3.3.1}$$

with S_t the underlying price at time t and K the strike price.

For the Black-Scholes model, we have

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ \Longrightarrow d\ln(S_t) &= (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \\ \Longrightarrow \ln S_T &= \ln S_0 + (r - \frac{1}{2}\sigma^2)T + \sigma \int_0^T dW_t \\ \Longrightarrow y &= x + (r - \frac{1}{2}\sigma^2)T + \sigma W_T. \end{aligned} \quad (3.3.2)$$

Thus, $y \sim N(x + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$ and its characteristic function reads

$$\phi(\omega) = \exp\{i(x + (r - \frac{1}{2}\sigma^2)T)\omega - \frac{1}{2}\sigma^2 T\omega^2\}. \quad (3.3.3)$$

The payoff for European options, in terms of the log-asset price, reads

$$v(y,T) = [\alpha K(e^y - 1)]^+ \text{ with } \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases} \quad (3.3.4)$$

The corresponding cosine series coefficients read, respectively,

$$V_k^{Call} = \frac{2}{b-a}\alpha K(\chi_k(0,b) - \psi_k(0,b)), \quad (3.3.5)$$

and

$$V_k^{Put} = \frac{2}{b-a}\alpha K(-\chi_k(a,0) + \psi(a,0)), \quad (3.3.6)$$

where

$$\begin{aligned} \chi_k(c,d) = & \frac{1}{1+(\frac{k\pi}{b-a})^2}[\cos(k\pi\frac{d-a}{b-a})e^d - \cos(k\pi\frac{c-a}{b-a})e^c \\ & + \frac{k\pi}{b-a}\sin(k\pi\frac{d-a}{b-a})e^d - \frac{k\pi}{b-a}\sin(k\pi\frac{c-a}{b-a})e^c], \end{aligned}$$

and

$$\psi_k(c,d) = \begin{cases} [\sin(k\pi\frac{d-a}{c-a}) - \sin(k\pi\frac{c-a}{b-a})] & k \neq 0, \\ d - c & k = 0. \end{cases} \quad (3.3.7)$$

Notice that for plain vanilla options Equations (3.3.5) and (3.3.6) are independent of the model we use. Here, we have taken the Black-Scholes model, but we could also use other models, such as the Heston model or the Heston-Hull-White model.

3.4 Heston model

In the Heston model, the volatility, denoted by $\sqrt{\nu_t}$, is modeled by a stochastic differential equation,

$$\begin{aligned} dx_t &= (r - \frac{1}{2}\nu_t)dt + \sqrt{\nu_t}dW_{1t}, \quad (3.4.1) \\ d\nu_t &= \kappa(\overline{\upsilon} - \upsilon_t)dt + \eta\sqrt{\nu_t}dW_{2t}, \end{aligned}$$

where x_t denotes the log-asset price variable and υ_t the variance of the asset price process. The parameters $\kappa \geqslant 0, \overline{\upsilon} \geqslant 0$ and $\eta \geqslant 0$ are called the speed of mean reversion, the mean level of variance and the volatility of the volatility, respectively. Furthermore, the Brownian motions W_{1t} and W_{2t} are assumed to be correlated with coefficient ρ.

For this model, the COS pricing equation simplifies, since

$$\phi(\omega; x, \upsilon_0) = \varphi_{hes}(\omega; \upsilon_0) e^{i\omega x}, \tag{3.4.2}$$

with ν_0 the volatility of the underlying at the initial time and $\varphi_{hes}(\omega; \nu_0) = \phi(\omega; 0, \nu_0)$. We find

$$\upsilon(x, t_0, \upsilon_0) \approx e^{-r\triangle t} Re\{\sum_{k=0}^{N-1}{}' \varphi_{hes}(\frac{k\pi}{b-a}; \upsilon_0) e^{ik\pi\frac{x-a}{b-a}} V_k\}, \tag{3.4.3}$$

where the characteristic function of the log-asset price, $\varphi_{hes}(\omega; \upsilon_0)$, reads

$$\begin{aligned}\varphi_{hes}(\omega; \upsilon_0) = \exp(i\omega r\triangle t + \frac{\upsilon_0}{\eta^2}(\frac{1-e^{-D\triangle t}}{1-Ge^{-D\triangle t}})(\kappa - i\rho\eta\omega - D)) \\ \cdot \exp(\frac{\kappa\overline{\upsilon}}{\eta^2}(\triangle t(\kappa - i\rho\eta\omega - D) - 2\ln(\frac{1-Ge^{-D\triangle t}}{1-G}))),\end{aligned} \tag{3.4.4}$$

with

$$D = \sqrt{(\kappa - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2}, \tag{3.4.5}$$

$$G = \frac{\kappa - i\rho\eta\omega - D}{\kappa - i\rho\eta\omega + D}. \tag{3.4.6}$$

3.5 Numerical Results for Heston Model

3.5.1 COS Method versus Carr-Madan Method

In this subsection we carry out some numerical tests for the Heston model with pricing formula (3.4.4) as well as with Carr & Madan's FFT method. The Fourier transform based option pricing method introduced by Carr and Madan [CM99] is often used in the calibration of the Heston model. We aim to compare the COS pricing method with the Carr-Madan method. We follow the tests introduced by Fang and Oosterlee and choose similar parameters. In order to ensure that the Feller condition is satisfied, we choose a larger mean reversion parameter. For the reference value, we also use the Carr-Madan method with $N = 2^{17}$ points and the prescribed truncated Fourier domain is [0,1200]. The values are chosen as follows:

$$\begin{aligned} S &= 100; r = 0.07; T = 1; q = 0; K = 100;\ \text{Reference price} = 11.1299... \\ \kappa &= 5; \gamma = 0.5751; \overline{\upsilon} = 0.0398; \upsilon(0) = 0.0175; \rho = -0.5711. \end{aligned}$$

There is quite an extensive literature on option pricing based on the Heston model. It is well-known that the price of a European call option is given by

$$C = F - \frac{e^{\eta k}}{\pi} Re \int_0^{\infty} e^{-iuk} \frac{\phi(u - i(\eta + 1))}{(u - i(\eta + 1))(u - i\eta)} du \tag{3.5.1}$$

where F is the Forward price that can be obtained by $F = Se^{(r-q)T}$, k is the log-transform of the option strike and η is an arbitrary dampening parameter which we choose as 0.75. The Carr-Madan method is based on application of the Fast Fourier Transform for the above integral. The numerical results are as follows:

Absolute error	COS	Carr-Madan
N=32	6.35E-01	1.37E+06
N=64	7.50E-03	6.17E+07
N=128	7.52E-07	3.74E+07
N=256	3.57E-09	1.63E+07
N=512	3.57E-09	8.79E+04
N=1024	3.57E-09	1.64E+02
N=2048	3.57E-09	5.63E-01
N=4096	3.57E-09	1.07E-02
N=8192	3.57E-09	3.45E-06
N=16384	3.57E-09	2.61E-13
N=32768	3.57E-09	3.82E-13
Time (seconds)	COS	Carr-Madan
N=32	1.09E-03	4.39E-03
N=64	1.31E-03	6.10E-03
N=128	1.83E-03	9.10E-03
N=256	2.80E-03	2.23E-02
N=512	4.87E-03	2.71E-02
N=1024	8.88E-03	6.53E-02
N=2048	1.72E-02	1.23E-01
N=4096	4.92E-02	1.40E-01
N=8192	8.31E-02	2.74E-01
N=16384	9.12E-02	6.85E-01
N=32768	1.61E-01	2.08E+00

Table 3.1: COS v.s. Carr-Madan

It is clear that even with a significantly smaller value of N the COS pricing formula gives a better approximation than the plain Carr-Madan method. More importantly for our applications, the COS method may significantly improve the speed of calibration.

3.5.2 COS versus *quadl*

We do not need to use the Fourier method in the pricing formula. A direct numerical integration can also be used to compute the result of formula (3.5.1). In practice, we can use Matlab to numerically evaluate the integral by using the function $quadl(fun, a, b)$. This function can approximate the integral of function fun from a to b within an error of 10^{-6} , using the recursive adaptive Lobatto quadrature. However, the integral of formula (3.5.1) has an infinite upper bound, which has to be approximated. Experience tells us that 100 is a sufficiently large value to approximate infinity. Here we choose the same parameter values as in subsection 3.5.1 to see the impact of the upper bound of the integral by using the Matlab function *quadl* for pricing a call option. The reference value is 11.1299.

Upper Bound	1	10	100	1000	10000
Price error	6.6472	0.3183	3.1985E-9	3.7091E-9	4.6572E-9
Time (Seconds)	0.1165	0.1172	0.1136	0.1235	0.1244

Table 3.2: Impact of Upper bound

As seen before, the COS method has the same scale of error as the direct numerical integration but with significantly less computation time. In fact, the COS pricing method has another advantage if we look at the number of strikes. For practical use, as we will see in the chapters to follow, calibration is usually performed based on several strikes per maturity. In the numerical test shown above, when the number of strikes increases, the numerical integration method can become quite time-consuming. Since the function *quadl* is used for the integral of a single variable, many loops are needed for more than one strike. On the other hand, no loop is needed for both Fourier methods (COS and FFT). In order to confirm this issue, we can extend the above test to pricing options at more than one strike, and use the maximum absolute error to observe the final impact. The strikes chosen are $K = 50, 55, 60, 65, ..., 145, 150$.

Method	COS	*quadl*
Time (seconds)	0.0809	0.2599

Table 3.3: COS v.s. Numerical Integration

Table 3.3 gives the comparison of the use of the COS method with $N = 2^{10}$and numerical integration with upper bound set equal to 100. Both methods then result in an error of 10^{-9}. In terms of computation time, the COS pricing method is strongly preferred. Since both FFT and the direct numerical integration methods are significantly better than a Monte Carlo method for calibrating the Heston model, we can state that the COS method can save a large amount of time, without sacrificing accuracy, when we calibrate the Heston model. We will show in Chapter 4 this is also true for Heston-Hull-White model.

Chapter 4

Heston-Hull-White Model for Equity

Grzelak and Oosterlee [GO09] have studied the Heston-Hull-White model, which incorporates stochastic equity volatility and stochastic short rate. They obtained a closed-form formula for the characteristic function for an approximation of the combined model. In this chapter we will discuss this hybrid model and use the discounted ChF to price plain vanilla options by Fourier methods.

4.1 Combining two models

The full-scale Heston-Hull-White model is a combination of the Heston model for equity with stochastic volatility and the Hull-White model for a stochastic short rate process. Three factors are considered: the asset price $S(t)$, the short rate $r(t)$ and the volatility $\upsilon(t)$. The dynamics of this model can be presented as follows:

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\upsilon(t)}S(t)dW_x(t), & S(0) > 0, \\ dr(t) = \lambda(\theta(t) - r(t))dt + \eta dW_r(t), & r(0) > 0, \\ d\upsilon(t) = \kappa(\overline{\upsilon} - \upsilon(t))dt + \gamma\sqrt{\upsilon(t)}dW_\upsilon(t), & \upsilon(0) > 0, \end{cases} \tag{4.1.1}$$

where the correlations of the Brownian motions are given in the following way:

$$dW_x dW_r = \rho_{x,r}dt,\ dW_x dW_\upsilon = \rho_{x,\upsilon}dt,\ dW_\upsilon dW_r = \rho_{\upsilon,r}dt.$$

The parameters here are from the two individual models: λ, η and $\theta(t)$ are Hull-White parameters, where $\lambda > 0$ is the speed of mean reversion of the short rate, $\eta > 0$ represents the volatility of the interest rate and $\theta(t)$ is the term-structure. The variance process, which follows Cox-Ingersoll-Ross dynamics, includes the other three parameters, in which $\kappa > 0$ is also the speed of mean reversion, i.e. the speed of variance to its mean $\overline{\upsilon}$, and $\gamma > 0$ determines the volatility of the volatility.

System (4.1.1) does not fit in the class of affine diffusion processes (AD), as in [DPS00], not even when we make the log transform of the asset price. We cannot determine the characteristic function by standard procedures due to the non-affine form. Hence, accurate approximations are needed.

4.2 Approximation of Heston-Hull-White Model by an Affine Process

4.2.1 Model Reformulation

The full-scale HHW model is not affine because the symmetric instantaneous covariance matrix reads:

$$\sigma(X(t))\sigma(X(t))^T = \begin{bmatrix} \upsilon(t) & \rho_{x,\upsilon}\gamma\upsilon(t) & \rho_{x,r}\eta\sqrt{\upsilon(t)} \\ * & \gamma^2\upsilon(t) & \rho_{r,\upsilon}\eta\sqrt{\upsilon(t)} \\ * & * & \eta^2 \end{bmatrix}. \tag{4.2.1}$$

This matrix is of the affine form if we would set $\rho_{x,r}$ and $\rho_{r,\upsilon}$ to zero. However, for some interest rate sensitive products, it is important to keep $\rho_{x,r}$ non-zero.

In this section the following reformulated HHW model is considered:

$$dS(t)/S(t) = r(t)dt + \sqrt{\upsilon(t)}dW_x(t) + \Omega(t)dW_r(t) + \triangle\sqrt{\upsilon(t)}dW_\upsilon(t), \quad S(0) > 0, \tag{4.2.2}$$

with

$$\begin{cases} dr(t) = \lambda(\theta(t) - r(t))dt + \quad \eta dW_r(t), & r(0) > 0, \\ d\upsilon(t) = \kappa(\overline{\upsilon} - \upsilon(t))dt + \gamma\sqrt{\upsilon(t)}dW_\upsilon(t), & \upsilon(0) > 0. \end{cases} \tag{4.2.3}$$

Instead of a non-zero correlation between the asset price and interest rate, we assume

$$dW_x dW_r = 0,\ dW_x dW_\upsilon = \tilde{\rho}_{x,\upsilon}dt,\ dW_\upsilon dW_r = 0. \tag{4.2.4}$$

Grzelak and Oosterlee [GO09] have shown that (4.2.2) is equivalent to the full-scale model by setting $\Omega(t)$, $\triangle$ and $\tilde{\rho}_{x,\upsilon}$, as follows:

$$\Omega(t) = \rho_{x,r}\sqrt{\upsilon(t)},\ \tilde{\rho}^2_{x,\upsilon} = \rho^2_{x,\upsilon} + \rho^2_{x,r},\ \triangle = \rho_{x,\upsilon} - \tilde{\rho}_{x,\upsilon}. \tag{4.2.5}$$

The technique employed here is the Cholesky decomposition. The two systems (4.1.1) and (4.2.2) are presented in terms of their independent Brownian motions, respectively, and the three parameters introduced are obtained by matching the appropriate coefficients.

After the reformulation the Heston-Hull-White model is, of course, still not affine. We need some approximations. Now we take a look at the symmetric instantaneous covariance matrix for this reformulated model by log transforming and exchanging the order of state variables, to $X(t) = [r(t), \upsilon(t), \log S(t)]^T$:

$$\sum := \sigma(X(t))\sigma(X(t))^T = \begin{bmatrix} \eta^2 & 0 & \rho_{x,r}\eta\sqrt{\upsilon(t)} \\ * & \gamma^2\upsilon(t) & \rho_{x,\upsilon}\gamma\upsilon(t) \\ * & * & \upsilon(t) \end{bmatrix}. \tag{4.2.6}$$

It seems that only one term in this matrix is not of the affine term, $\sum_{(1,3)} = \rho_{x,r}\eta\sqrt{\upsilon(t)}$. By appropriate approximations of this term, $\sum_{(1,3)}$, an affine HHW model can be obtained. Grzelak and Oosterlee [GO09] discussed two approximations in their paper: A deterministic approach (called the H1-HW model) and a stochastic approach (called the H2-HW model).

In the following section, we will present the H1-HW model and determine the closed-form characteristic function of this model. As mentioned earlier, a characteristic function is essential for Fourier methods, such as the COS method.

4.2.2 The H1-HW Model

In the H1-HW model term $\sum_{(1,3)}$ is replaced by its expectation: $\sum_{(1,3)} \approx \rho_{x,r}\eta E(\sqrt{\upsilon(t)})$. By doing this, the Heston-Hull-White model turns into an affine model, since the stochastic variable has been approximated by a deterministic quantity. The closed-form expression for the expectation and the variance of $\sqrt{\upsilon(t)}$ can be found in [Du01]. Here, we only need its expectation:

$$E(\sqrt{\nu(t)}) = \sqrt{2c(t)}e^{-\lambda(t)/2}\sum_{k=0}^{\infty}\frac{1}{k!}(\lambda(t)/2)^k\frac{\Gamma(\frac{1+d}{2}+k)}{\Gamma(\frac{d}{2}+k)}, \tag{4.2.7}$$

where

$$c(t) = \frac{1}{4\kappa}\gamma^2(1-e^{-\kappa t}),\ d = \frac{4\kappa\overline{\upsilon}}{\gamma^2},\ \lambda(t) = \frac{4\kappa\upsilon(0)e^{-\kappa t}}{\gamma^2(1-e^{-\kappa t})}, \tag{4.2.8}$$

with $\Gamma(k)$ being the gamma function: $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$.

The analytical expression above is not sufficient to obtain the characteristic function of the H1-HW model, as many computations need to be performed to compute (4.2.7). A more efficient approximation is required.

One approximation mentioned in [GO09] is from the delta method, which gives

$$E(\sqrt{\nu(t)}) \approx \sqrt{c(t)(\lambda(t)-1) + c(t)d + \frac{c(t)d}{2(d+\lambda(t))}} =: \Lambda(t), \tag{4.2.9}$$

with the parameters from (4.2.8).

This approximation is faster computed than (4.2.7), but it is still non-trivial. When it comes to the standard ODE routines to determine a characteristic function, $E(\sqrt{\nu(t)})$ is always inside integrands to be computed. So, the form (4.2.9) may cause difficulties during numerical integration, which is why we use another approximation for $E(\sqrt{\nu(t)})$ as:

$$E(\sqrt{\nu(t)}) \approx a + be^{-ct}. \tag{4.2.10}$$

The values a, b and c can be approximated by:

$$a = \sqrt{\overline{\upsilon} - \frac{\gamma^2}{8\kappa}},\ b = \sqrt{\upsilon(0)} - a,\ c = -\ln(b^{-1}(\Lambda(1) - a)), \tag{4.2.11}$$

where $\Lambda(t)$ is given by (4.2.9).

The attractive approximation given by (4.2.10) can significantly improve the speed of calculating the characteristic function. As proved in [GO09], it is a good approximation. Unfortunately, we can not use this approximation all the time. This approximation is well-defined only when the Feller condition is satisfied, which is $\overline{\upsilon} > \gamma^2/2\kappa$. When the Feller condition is violated, we have to use the exact formula (4.2.7).

With the final approximation (4.2.10), we have obtained the affine H1-HW model. In the next section the standard methods from [DPS00] can be used to obtain the discounted ChF of this model. The ChF forms the basis for many Fourier option pricing methods. We will use this ChF to value Variable Annuities.

4.3 Characteristic function for H1-HW

With the affine form of Heston-Hull-White model (H1-HW) obtained, the discounted characteristic function can be derived by the method mentioned in [DPS00]. In order to simplify the problem we assume that the term-structure for interest rate $\theta(t)$ is constant θ. Also, the Feller condition is assumed to be satisfied. For situations without these two assumptions numerical integration has to be used.

According to Duffie, Pan and Singleton [DPS00], the discounted Chf is of the following form:

$$\phi_{H1-HW}(\mu, X(t), \tau) = \exp(A(\mu,\tau) + B(\mu,\tau)x(t) + C(\mu,\tau)r(t) + D(\mu,\tau)\upsilon(t)), \tag{4.3.1}$$

with boundary conditions $A(\mu, 0) = 0, B(\mu, 0) = i\mu, C(\mu, 0) = 0$ and $D(\mu, 0) = 0$, and also $\tau := T - t$.

The following ordinary differential equations (ODEs), related to the H1-HW model, will help us derive the ChF:

$$\begin{cases} B'(\tau) = 0, & B(\mu,0) = i\mu, \\ C'(\tau) = -1 - \lambda C(\tau) + B(\tau) & C(\mu,0) = 0, \\ D'(\tau) = B(\tau)(B(\tau)-1)/2 + (\gamma\rho_{x,\upsilon}B(\tau) - \kappa)D(\tau) + \gamma^2 D^2(\tau)/2 & D(\mu,0) = 0, \\ A'(\tau) = \lambda\theta C(\tau) + \kappa\overline{\upsilon}D(\tau) + \eta^2 C^2(\tau)/2 + \eta\rho_{x,r}E(\sqrt{\upsilon(t)})B(\tau)C(\tau) & A(\mu,0) = 0, \end{cases} \tag{4.3.2}$$

with all the parameters as shown before.

The solution is given by:

$$\begin{cases} B(\mu,\tau) = i\tau, \\ C(\mu,\tau) = (i\mu - 1)\lambda^{-1}(1 - e^{-\lambda\tau}), \\ D(\mu,\tau) = \frac{1-e^{-D_1\tau}}{\gamma^2(1-ge^{-D_1\tau})}(\kappa - \gamma\rho_{x,\upsilon}i\mu - D_1), \\ A(\mu,\tau) = \lambda\theta I_1(\tau) + \kappa\overline{\upsilon}I_2(\tau) + \frac{1}{2}\eta^2 I_3(\tau) + \eta\rho_{x,r}I_4(\tau). \end{cases} \tag{4.3.3}$$

where $D_1 = \sqrt{(\gamma\rho_{x,\upsilon} i\mu - \kappa)^2 - \gamma^2 i\mu(i\mu - 1)}$, $g = \frac{\kappa - \gamma\rho_{x,\upsilon} i\mu - D_1}{\kappa - \gamma\rho_{x,\upsilon} i\mu + D_1}$ and the four integrals $I_1(\tau)$, $I_2(\tau)$, $I_3(\tau)$, $I_4(\tau)$ can be solved analytically and semi-analytically:

$$\begin{aligned}
I_1(\tau) &= \frac{1}{\lambda}(i\mu - 1)(\tau + \frac{1}{\lambda}(e^{-\lambda\tau} - 1)), \\
I_2(\tau) &= \frac{\tau}{\gamma^2}(\kappa - \gamma\rho_{x,\upsilon} i\mu - D_1) - \frac{2}{\gamma^2}\ln(\frac{1 - ge^{-D_1\tau}}{1 - g}), \\
I_3(\tau) &= \frac{1}{2\lambda^3}(i + \mu)^2(3 + e^{-2\lambda\tau} - 4e^{-\lambda\tau} - 2\lambda\tau), \\
I_4(\tau) &= i\mu \int_0^\tau E(\sqrt{\upsilon(T-s)})C(\mu, s)ds \\
&= -\frac{1}{\lambda}(i\mu + \mu^2)\int_0^\tau E(\sqrt{\upsilon(T-s)})(1 - e^{-\lambda s})ds \\
&\approx -\frac{1}{\lambda}(i\mu + \mu^2)\int_0^\tau (a + be^{-c(T-s)})(1 - e^{-\lambda s})ds \\
&= -\frac{1}{\lambda}(i\mu + \mu^2)[\frac{b}{c}(e^{-ct} - e^{-cT}) + a\tau + \frac{a}{\tau}(e^{-\lambda\tau} - 1) + \frac{b}{c - \lambda}e^{-cT}(1 - e^{-\tau(\lambda - c)})].
\end{aligned} \tag{4.3.4}$$

The two assumptions we mentioned are used in deriving $A(\mu, \tau)$. In particular, by taking the approximation of the last section $E(\sqrt{\upsilon(T-s)}) \approx a + be^{-c(T-s)}$, i.e when the Feller condition is satisfied, we can obtain a closed-form expression for $I_4(\tau)$.

Given the discounted ChF of the H1-HW model, in which we use the deterministic approach to approximate the non-affine term, we are ready to proceed with numerical experiments.

4.4 Numerical Result

In this section, we present some numerical results. The parameters we use are as follows:

$$\begin{aligned}
S(0) &= 100, r(0) = \theta = 0.07, \lambda = 0.05, \eta = 0.005, \rho_{S,r} = 0.2, \\
\kappa &= 1.5768, \gamma = 0.0571, \overline{\nu} = 0.0398, \nu(0) = 0.0175, \rho_{S,\nu} = -0.5711,
\end{aligned}$$

and the maturity is $T = 1$ year and $T = 10$ years.

In Tables 4.1 and 4.2, we benchmark our approximation formula against the full-scale HHW model using the Monte Carlo method with 10000 simulations. We can see that the approximation error is small, especially for short maturities. A more convincing proof can be found in Figure 4.4.1. As the COS method is significantly faster than the Monte Carlo method, the use of the COS method for the approximate H1-HW model is strongly recommended during the calibration process.

Strike	COS Price	MC Price	Price Error	COS Vol	M C Vol	Vol Error
50	53.3802	53.2918	0.0884	0.6581	0.6524	0.0058
55	48.7188	48.6306	0.0881	0.6053	0.6002	0.0051
60	44.0595	43.9715	0.0879	0.5549	0.5504	0.0045
65	39.4077	39.3205	0.0873	0.5068	0.5028	0.0040
70	34.7775	34.6918	0.0857	0.4609	0.4573	0.0036
75	30.1981	30.1134	0.0847	0.4175	0.4143	0.0032
80	25.7204	25.6351	0.0853	0.3774	0.3745	0.0029
85	21.4190	21.3309	0.0881	0.3415	0.3388	0.0027
90	17.3863	17.2966	0.0897	0.3106	0.3081	0.0025
95	13.7190	13.6280	0.0910	0.2848	0.2824	0.0024
100	10.5001	10.4121	0.0880	0.2640	0.2617	0.0022
105	7.7827	7.6983	0.0844	0.2473	0.2452	0.0021
110	5.5809	5.5070	0.0740	0.2341	0.2322	0.0019
115	3.8703	3.8080	0.0623	0.2236	0.2219	0.0018
120	2.5958	2.5471	0.0487	0.2152	0.2136	0.0016
125	1.6846	1.6505	0.0341	0.2083	0.2069	0.0014
130	1.0587	1.0355	0.0232	0.2026	0.2014	0.0012
135	0.6450	0.6289	0.0161	0.1978	0.1967	0.0011
140	0.3813	0.3715	0.0099	0.1937	0.1927	0.0010
145	0.2191	0.2133	0.0058	0.1902	0.1893	0.0008
150	0.1225	0.1201	0.0024	0.1871	0.1866	0.0005

Table 4.1: H1-HW v.s. the full-scale HHW model, 1 year.

Strike	COS Price	MC Price	Price Error	COS Vol	M C Vol	Vol Error
50	75.2837	75.0646	0.2191	0.5762	0.5723	0.0040
55	72.8956	72.6748	0.2208	0.5603	0.5566	0.0037
60	70.5405	70.3175	0.2230	0.5456	0.5421	0.0035
65	68.2222	67.9982	0.2241	0.5320	0.5288	0.0032
70	65.9459	65.7207	0.2252	0.5194	0.5164	0.0031
75	63.7143	63.4882	0.2261	0.5078	0.5049	0.0029
80	61.5303	61.3038	0.2266	0.4969	0.4942	0.0028
85	59.3969	59.1701	0.2268	0.4868	0.4842	0.0026
90	57.3157	57.0894	0.2264	0.4773	0.4748	0.0025
95	55.2881	55.0617	0.2264	0.4685	0.4661	0.0024
100	53.3159	53.0901	0.2258	0.4602	0.4579	0.0023
105	51.4001	51.1745	0.2256	0.4524	0.4502	0.0023
110	49.5397	49.3135	0.2262	0.4451	0.4429	0.0022
115	47.7361	47.5089	0.2273	0.4382	0.4361	0.0021
120	45.9896	45.7617	0.2278	0.4317	0.4296	0.0021
125	44.2992	44.0700	0.2292	0.4256	0.4236	0.0021
130	42.6632	42.4320	0.2312	0.4199	0.4178	0.0020
135	41.0849	40.8490	0.2360	0.4144	0.4124	0.0020
140	39.5580	39.3208	0.2372	0.4092	0.4072	0.0020
145	38.0851	37.8465	0.2385	0.4043	0.4023	0.0020
150	36.6641	36.4242	0.2399	0.3997	0.3977	0.0020

Table 4.2: H1-HW v.s. the full-scale HHW model, 10 years.

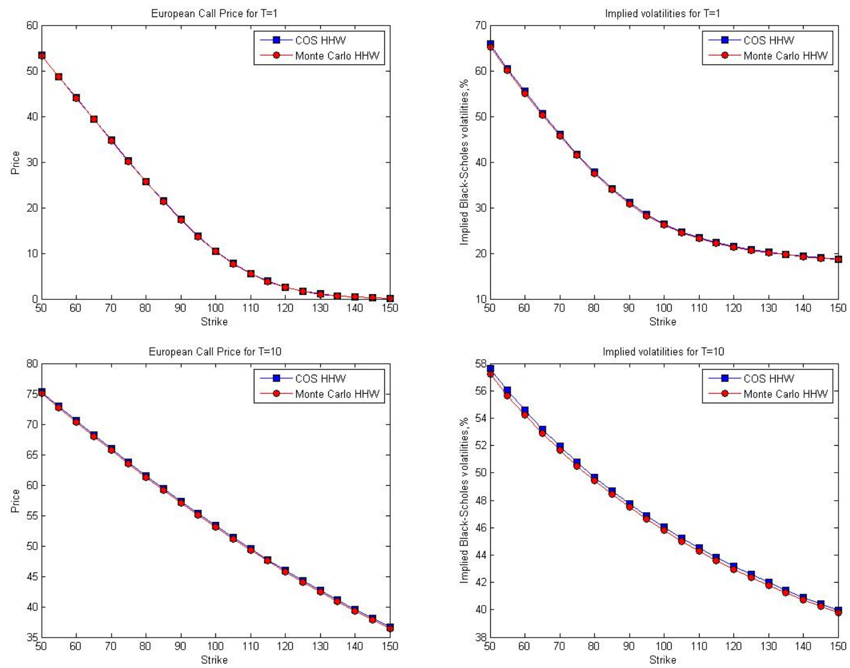

Figure 4.4.1: Equity-COS v.s. Monte Carlo

4.5 Heston-Hull-White Model under the Forward Measure

In this section, we want to move to the forward measure which would be preferred for our model and payoffs. We will also find an expression for the approximate characteristic function of the Heston-Hull-White model under the forward measure. Reducing the pricing complexity is one of the great advantages of the forward measure approach.

4.5.1 Forward HHW full-scale model

The H1-HW model is under the spot measure, which is generated by the money market account $M(t)$. The Euro money market account is a security that is worth 1 Euro at time zero and earns the risk-free rate r at time t. The spot measure is consistent with the risk neutral valuation result we presented before. However, for the valuation of bond options, interest rate caps and swap options, the forward measure is preferred. The numéraire of the forward measure is the zero-coupon bond, $P(t,T)$ -the price at time t that pays 1 Euro at maturity time T. Under the forward measure, the forward asset price is defined as:

$$F(t) = \frac{S(t)}{P(t,T)}. \tag{4.5.1}$$

Here we refer to [BG09] to obtain the full-scale HHW model under the forward measure.

First of all, the zero-coupon bond is analytically expressed as follows based on the Hull-White model:

$$P(t,T) = \exp[A_r(t,T) - B_r(t,T)r(t)], \tag{4.5.2}$$

where

$$\begin{aligned} A_r(t,T) &= \ln\left(\frac{P(0,T)}{P(0,t)}\right) + B_r(t,T)f(0,t) - \\ &\quad \frac{\eta^2}{4\lambda}(1 - e^{-2\lambda t}B_r(t,T)^2), & (4.5.3) \\ B_r(t,T) &= \frac{1-e^{-\lambda(T-t)}}{\lambda}, & (4.5.4) \end{aligned}$$

and $f(0,t)$ is the forward rate, λ and η are Hull-White parameters. The dynamics for the zero-coupon bond under the spot measure $\mathbb{Q}$ are

$$dP(t,T)/P(t,T) = r(t)dt - \eta B_r(t,T)dW_r(t). \tag{4.5.5}$$

We can use this to derive the dynamics of the forward price by applying Ito's lemma to (4.5.1):

$$dF(t) = (\eta^2 B_r^2(t,T) + \rho_{s,r}\sqrt{\nu(t)}\eta B_r(t,T)F(t)dt + \sqrt{\nu(t)}F(t)dW_s(t) + \eta B_r(t,T)F(t)dW_r(t). \tag{4.5.6}$$

Under the T-forward measure the forward asset price $F(t)$ is a martingale, i.e. the coefficients of the drift should be zero. An appropriate transform of the Brownian motions can achieve this.

Under the forward measure $\mathbb{Q}^{\mathbb{T}}$, the SDE system is given by:

$$\begin{cases} dF(t) = \sqrt{\nu(t)}F(t)dW_S^T(t) + \eta B_r(t,T)F(t)dW_r^T(t), \\ d\nu(t) = \kappa(\overline{\nu} - \nu(t))dt + \gamma\sqrt{\nu(t)}dW_\nu^T(t). \end{cases} \tag{4.5.7}$$

Taking the log-transform of the forward price by $x(t) = \ln F(t)$, we can get the forward full-scale HHW model with only two processes in the SDE system.

$$\begin{cases} dx(t) = -\frac{1}{2}(\nu(t) + 2\rho_{s,r}\sqrt{\nu(t)}\eta B_r(t,T) + \eta^2 B_r^2(t,T))dt + \sqrt{(\nu(t)}dW_S^T(t), \\ d\nu(t) = \kappa(\overline{\nu} - \nu(t))dt + \gamma\sqrt{\nu(t)}dW_\nu^T(t). \end{cases} \tag{4.5.8}$$

4.5.2 Approximation of the Forward Characteristic Function

We also wish to use the deterministic approximation from the H1-HW model to obtain the closed form of the Chf of system (4.5.8). According to Benhamou and Gauthier [BG09], the characteristic function is of the following form:

$$\phi_{Forward}(\mu, X(t), \tau) = \exp(A(\mu,\tau) + B(\mu,\tau)\nu(t) + i\mu x(t)), \tag{4.5.9}$$

where

$$\begin{cases} A(\mu,\tau) = \kappa\overline{\nu} I_2(\tau) - \frac{1}{2}\mu(i+\mu)V(\tau) - I(f), \\ B(\mu.\tau) = \frac{1-e^{-D_1\tau}}{\gamma^2(1-ge^{-D_1\tau})}(\kappa - \gamma\rho_{x,\nu}i\mu - D_1), \end{cases}$$

with $D_1 = \sqrt{(\gamma\rho_{x,\nu}i\mu - \kappa)^2 - \gamma^2 i\mu(i\mu - 1)}$, $g = \frac{k-\gamma\rho_{x,\nu}i\mu - D_1}{k-\gamma\rho_{x,\nu}i\mu + D_1}$,$I_2(\tau)$ in (4.3.4), and

$$V(\tau) = \frac{\eta^2}{\lambda^2}(\tau + \frac{2}{\lambda}e^{-\lambda\tau} - \frac{1}{2\lambda}e^{-2\lambda} - \frac{3}{2\lambda}), \tag{4.5.10}$$

$$I(f) = \rho_{S,r}\mu(i+\mu)\eta\int_t^T \sqrt{\nu(s)}B_r(s,T)ds. \tag{4.5.11}$$

Here we also assume that $\rho_{r,\nu} = 0$.

There is only one stochastic term in the characteristic function, i.e. in $I(f)$. We also use the same technique in [GO09] as we substitute $\sqrt{\nu(t)}$ by its expectation $E(\sqrt{\nu(t)})$. Details can be found in subsection (4.2.2). Thus:

$$\begin{aligned} I(f) &\approx \rho_{S,r}\mu(i+\mu)\eta/\lambda \cdot \int_0^\tau E(\sqrt{\upsilon(T-s)})(1-e^{-\lambda s})ds \\ &\approx \rho_{S,r}\mu(i+\mu)\eta/\lambda \cdot \int_0^\tau (a + be^{-c(T-s)})(1-e^{-\lambda s})ds \\ &= \rho_{S,r}\mu(i+\mu)\eta/\lambda \cdot [\frac{b}{c}(e^{-ct} - e^{-cT}) + a\tau + \frac{a}{\tau}(e^{-\lambda\tau} - 1) + \frac{b}{c-\lambda}e^{-cT}(1 - e^{-\tau(\lambda - c)})]. \end{aligned}$$

Chapter 5

Foreign Exchange Model of Heston and Hull-White

In the previous chapter, we gave a detailed discussion on how to use the Heston-Hull-White model in the equity market. Now we move to the foreign exchange (FX) market - a worldwide decentralized over-the-counter financial market for the trading of currencies. A foreign currency is analogous to a stock paying a known dividend yield, which is the foreign risk-free rate of interest r_f .The value of interest paid in a currency depends on the value of the FX. For a domestic investor, this value of interest can be regarded as an income equal to r_f of the value of the foreign currency per time period. To sum up, it is an asset that provides a yield of r_f per period. We define S_0 as the spot FX price and F_0 as the forward price with maturity T . If we make the assumptions that the domestic risk-free interest rate r_d and the foreign interest rate r_f are both constant, then the well-known interest rate parity relationship is

$$F_0 = S_0 e^{(r_d - r_f)T}. \tag{5.0.1}$$

5.1 Valuation of FX options: The Garman-Kohlhagen model

A foreign exchange (FX) option is a foreign currency derivative, where the owner has the right but not the obligation to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date. Due to the existence of FX exposure in a VA product, better understanding of this FX options is required. Garman and Kohlhagen [GK83] extended the Black-Scholes model to cope with the presence of two interest rates (one for each currency). It is a standard model to price simple FX options. By replacing the dividend yield q by r_f in the Black-Scholes formula on stocks paying known dividend yields, the domestic currency value of a call option into the foreign currency is

$$c = S_0 \exp(-r_f T)N(d_1) - K \exp(-r_d T)N(d_2). \quad (5.1.1)$$

The value of the put option is

$$p = K \exp(-r_d T)N(-d_2) - S_0 \exp(-r_f T)N(-d_1), \quad (5.1.2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad (5.1.3)$$

$$d_2 = d_1 - \sigma\sqrt{T}. \quad (5.1.4)$$

K is the strike price and σ is the volatility of FX. $N(\cdot)$ is the cumulative density function of the normal distribution.

5.2 FX model with stochastic interest rate and volatility

The Garman-Kohlhagen model is a basic extension of the Black-Scholes model, which should not be used for pricing and hedging long-term FX contracts. Hence the construction of multi-currency models with stochastic volatility and stochastic interest rates (both domestic and foreign) is required. In this section we want to extend the framework of the last chapter of the HHW model. The difference between the application in two markets is the existence of two kinds of stochastic interest rates in one currency. Here we assume that both interest rates are driven by the Hull-White one factor model:

$$dr_d(t) = \lambda_d(\theta_d(t) - r_d(t))dt + \eta_d dW_d^{\mathbb{Q}}(t), \quad (5.2.1)$$

$$dr_f(t) = \lambda_f(\theta_f(t) - r_f(t))dt + \eta_f dW_f^{\mathbb{Z}}(t), \quad (5.2.2)$$

where $r_d(t)$ and $r_f(t)$ are domestic and foreign interest rates. The dynamics (5.2.1) and (5.2.2) are mean-reverting processes from Hull and White, in which the two Brownian motions $W_d^{\mathbb{Q}}(t)$ and $W_f^{\mathbb{Z}}(t)$ are under different measures: the $\mathbb{Q}$-domestic spot risk-neutral measure and the $\mathbb{Z}$-foreign spot risk-neutral measure. Now we wish to define the hybrid FX Heston-Hull-White model with all Brownian motions under the same measure. The process of the FX spot price $FX(t)$ can be easily obtained under the $\mathbb{Q}$ measure:

$$dFX(t)/FX(t) = (r_d(t) - r_f(t))dt + \sqrt{\sigma(t)}dW_{FX}^{\mathbb{Q}}(t), \quad (5.2.3)$$

which is just by substituting the constant parameters into time-dependent ones in the Garman-Kohlhagen model. The variance process $\sigma(t)$ is also derived by the Heston model, which is also under the $\mathbb{Q}$ measure.

Now let's change the underlying measure from the foreign-spot to the domestic-spot measure. According to [Shr04], the following prices should be martingales under the domestic-spot measure. ($\chi_1(t)$ is the foreign money account in a local currency and $\chi_2(t)$ is the foreign zero-coupon bond in a local currency.)

$$\chi_1(t) := FX(t)\frac{B_f(t)}{B_d(t)}, \tag{5.2.4}$$

$$\chi_2(t) := FX(t)\frac{P_f(t,T)}{B_d(t)}, \tag{5.2.5}$$

where $B_f(t)$ and $B_d(t)$ are domestic and foreign saving accounts and $P_f(t,T)$ is the foreign zero coupon bond price. According to Hull and White, the dynamics of the zero-coupon bond are

$$dP_f(t,T)/P_f(t,T) = r_f(t)dt + \frac{\eta_f}{\lambda_f}(e^{-\lambda_f(T-t)} - 1)dW_f^{\mathbb{Z}}(t). \tag{5.2.6}$$

Applying the Ito's rule to the processes $\chi_1(t)$ and $\chi_2(t)$, we get

$$\begin{aligned} d\chi_1(t) &= \sqrt{\sigma(t)}\chi_1(t)dW_{FX}^{\mathbb{Q}}(t) \\ d\chi_2(t) &= \sqrt{\sigma(t)}\chi_2(t)dW_{FX}^{\mathbb{Q}}(t) + \frac{\eta_f}{\lambda_f}(e^{-\lambda_f(T-t)} - 1)dW_f^{\mathbb{Z}}(t) + \\ &\quad \rho_{FX,f}\frac{\eta_f}{\lambda_f}(e^{-\lambda_f(T-t)} - 1)\sqrt{\sigma(t)}\chi_2(t)dt, \end{aligned} \tag{5.2.7}$$

where $\rho_{FX,f}$ is the correlation coefficient in $dW_{FX}^{\mathbb{Q}}(t)dW_f^{\mathbb{Z}}(t) = \rho_{FX,f}dt$. Now we change the measure in the following way to make sure that$\chi_1(t)$ and $\chi_2(t)$ are martingales under the domestic spot measure:

$$dW_f^{\mathbb{Z}}(t) = dW_f^{\mathbb{Q}}(t) - \rho_{FX,f}\sqrt{\sigma(t)}dt. \tag{5.2.8}$$

To sum up, the full-scale FX-HHW model under the domestic risk-neutral measure, $\mathbb{Q}$ reads

$$\begin{cases} dFX(t)/FX(t) = (r_d(t) - r_f(t))dt + \sqrt{\sigma(t)}dW_{FX}^{\mathbb{Q}}(t), & FX(0) > 0, \\ d\sigma(t) = \kappa(\overline{\sigma} - \sigma(t))dt + \gamma\sqrt{\sigma(t)}dW_{\sigma}^{\mathbb{Q}}(t), & \sigma(0) > 0, \\ dr_d(t) = \lambda_d(\theta_d(t) - r_d(t))dt + \eta_d dW_d^{\mathbb{Q}}(t), & r_d(0) > 0, \\ dr_f(t) = (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{FX,f}\sqrt{\sigma(t)})dt + \eta_f dW_f^{\mathbb{Q}}(t), & r_f(0) > 0, \end{cases} \tag{5.2.9}$$

The parameters come from the Heston part and the Hull-White part, which are the same as we defined for the full-scale HHW model for equity. Unlike the equity HHW model, there are four SDEs in system (5.2.9), which is therefore more difficult to calibrate. Some assumptions have to be made to simplify the system. First of all, we assume the correlation matrix between the Brownian motions is as follows:

$$\sum := dW(t)(dW(t))^T = \begin{bmatrix} 1 & \rho_{FX,\sigma} & \rho_{FX,d} & \rho_{FX,f} \\ \rho_{FX,\sigma} & 1 & \rho_{\sigma,d} & \rho_{\sigma,f} \\ \rho_{FX,d} & \rho_{\sigma,d} & 1 & \rho_{d,f} \\ \rho_{FX,f} & \rho_{\sigma,f} & \rho_{d,f} & 1 \end{bmatrix} dti, \tag{5.2.10}$$

where $W(t) := [W^{\mathbb{Q}}_{FX}(t), W^{\mathbb{Q}}_{\sigma}(t), W^{\mathbb{Q}}_{d}(t), W^{\mathbb{Q}}_{f}(t)]^T$.

This full-scale FX-HHW model is not affine for the same reasons as discussed in the previous chapter. In order to get an accurate approximation for the characteristic function of this model, we again assume that the correlation between interest rate and volatility is zero, i.e.$\rho_{\sigma,d} = \rho_{\sigma,f} = 0$.

5.3 Forward Domestic Measure

System (5.2.9) is still too complex for the calibration process. An appropriate approximation should be performed so that we can reduce the dimension of the system. For this reason we move to the forward domestic measure. The measure change procedure can be done similarly as in section (4.5.1). More specifically, we choose as the numéraire the domestic zero-coupon bond $P_d(t,T)$, whose dynamics can be easily obtained by the Hull-White model. According to (5.1.1) the forward FX price is given by

$$FX^{Forward}(t) = FX(t)\frac{P_f(t,T)}{P_d(t,T)} \tag{5.3.1}$$

It has been discussed earlier that the forward FX price $FX^{Forward}(t)$ should be a martingale under the domestic forward measure $\mathbb{Q}^{\mathbb{T}}$. If we use Ito's lemma to determine the dynamics of $FX^{Forward}(t)$, we will see that

$$\begin{aligned}
dFX^{Forward}(t) &= \frac{P_f(t,T)}{P_d(t,T)}dFX(t) + \frac{FX(t)}{P_d(t,T)}dP_f(t,T) - FX(t)\frac{P_f(t,T)}{P_d^2(t,T)}dP_d(t,T) \\
&\quad + FX(t)\frac{P_f(t,T)}{P_d^3(t,T)}(dP_d(t,T))^2 + \frac{1}{P_d(t,T)}(dFX(t)dP_f(t,T)) \\
&\quad - \frac{P_f(t,T)}{P_d^2(t,T)}(dP_d(t,T)dFX(t)) - \frac{FX(t)}{P_d^2(t,T)}dP_d(t,T)dP_f(t,T).
\end{aligned}$$

We insert the dynamics of the FX spot price $FX(t)$ (5.2.3) and the SDEs for the zero-coupon bond prices, both domestic and foreign:

$$\begin{aligned}
dP_f(t,T)/P_f(t,T) &= (r_f(t) - \rho_{FX,f}\frac{\eta_f}{\lambda_f}(e^{-\lambda_f(T-t)} - 1)\sqrt{\sigma(t)})dt \\
&\quad + \frac{\eta_f}{\lambda_f}(e^{-\lambda_f(T-t)} - 1)dW^{\mathbb{Q}}_f(t) \\
dP_d(t,T)/P_d(t,T) &= r_d(t) + \frac{\eta_d}{\lambda_d}(e^{-\lambda_d(T-t)} - 1)dW^{\mathbb{Q}}_d(t).
\end{aligned}$$

For convenience, we define $B_{rd}(t,T) = \frac{1}{\lambda_d}(\exp(-\lambda_d(T-t)) - 1)$, $B_{rf}(t,T) = \frac{1}{\lambda_f}(\exp(-\lambda_f(T-t)) - 1)$. Then, we finally get

$$\begin{aligned}
\frac{dFX^{Forward}(t)}{FX^{Forward}(t)} &= \eta_d B_{rd}(\eta_d B_{rd} - \rho_{FX,d}\sqrt{\sigma(t)} - \rho_{d,f}\eta_f B_{rf})dt \\
&\quad + \sqrt{\sigma(t)}dW^{\mathbb{Q}}_{FX}(t) - \eta_d B_{rd} dW^{\mathbb{Q}}_d(t) + \eta_f B_{rf} dW^{\mathbb{Q}}_f(t).
\end{aligned}$$

Now, the appropriate Brownian motions under the T- forward measure $dW^T_{FX}(t)$, $dW^T_\sigma(t)$ and $dW^T_d(t)$, $dW^T_f(t)$ can be determined. More details of this transformation can be found in [GO10]. Here we just refer the lemma 2.2 in the paper [GO10] which presents the derivation of the full-scale FX-HHW model under the forward measure

$$\begin{cases} \frac{dFX^{Forward}(t)}{FX^{Forward}(t)} = \sqrt{\sigma(t)}dW^{\mathbb{T}}_{FX}(t) - \eta_d B_{rd} dW^T_d(t) + \eta_f B_{rf} dW^T_f(t), \\ d\sigma(t) = \kappa(\overline{\sigma} - \sigma(t))dt + \gamma\sqrt{\sigma(t)}dW^{\mathbb{T}}_\sigma(t), \\ dr_d(t) = (\lambda_d(\theta_d(t) - r_d(t)) + \eta_d^2 B_{rd})dt + \eta_d dW^T_d(t), \\ dr_f(t) = (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f \rho_{FX,f}\sqrt{\sigma(t)} + \eta_d\eta_f\rho_{d,f}B_{rd})dt + \eta_f dW^{\mathbb{T}}_f(t). \end{cases} \tag{5.3.2}$$

Note that here we still assume $\rho_{\sigma,d} = \rho_{\sigma,f} = 0$. Even though there are still four SDEs in this new system, the first two dynamics are sufficient to approximate the forward characteristic function, as will be explained in the next section in detail.

5.4 Forward characteristic function

For the full-scale FX system above, an approximation of the characteristic function is obtained by using the techniques as in section (4.3). From (5.3.2), we see that there is no drift term for the first SDE of the forward FX price, which means that we have already reduced the complexity significantly.

First, we take the log-transform of the $FX^{Forward}(t)$. The dynamics of this term $x^T(t) := \ln(FX^{Forward}(t))$ are

$$\begin{aligned} dx^T(t) : &= ((\rho_{x,d}\eta_d B_{rd} - \rho_{x,f}\eta_f B_{rf})\sqrt{\sigma(t)} + \rho_{d,f}\eta_d\eta_f B_{rd}B_{rf} \\ &- \frac{1}{2}(\eta_d^2 B_{rd}^2 + \eta_f^2 B_{rf}^2) - \frac{1}{2}\sigma(t))dt + \sqrt{\sigma(t)}dW^{\mathbb{T}}_{FX}(t) - \eta_d B_{rd} dW^T_d(t) \\ &+ \eta_f B_{rf} dW^T_f(t), \end{aligned}$$

with the variance process

$$d\sigma(t) = \kappa(\overline{\sigma} - \sigma(t))dt + \gamma\sqrt{\sigma(t)}dW^{\mathbb{T}}_\sigma(t).$$

Before we use the standard approach in [DPS00], we approximate the non-affine term as in the previous chapter. That is, we replace the square root of the variance by its expectation:

$$E(\sqrt{\sigma(t)}) = \sqrt{2c(t)}e^{-\lambda(t)/2}\sum_{k=0}^{\infty}\frac{1}{k!}(\lambda(t)/2)^k\frac{\Gamma(\frac{1+d}{2}+k)}{\Gamma(\frac{d}{2}+k)}, \tag{5.4.1}$$

where

$$c(t) = \frac{1}{4\kappa}\gamma^2(1-e^{-\kappa t}),\ d = \frac{4\kappa\overline{\sigma}}{\gamma^2},\ \lambda(t) = \frac{4\kappa\sigma(0)e^{-\kappa t}}{\gamma^2(1-e^{-\kappa t})}. \tag{5.4.2}$$

As Grzelak and Oosterlee mentioned in [GO09], another approximation for $E(\sqrt{\sigma(t)})$ may be helpful, i.e.

$$E(\sqrt{\sigma(t)}) \approx a + be^{-ct}. \tag{5.4.3}$$

The values a, b and c can be approximated by:

$$a = \sqrt{\overline{\sigma} - \frac{\gamma^2}{8\kappa}},\ b = \sqrt{\sigma(0)} - a,\ c = -\ln(b^{-1}(\Lambda(1) - a)), \tag{5.4.4}$$

where $\Lambda(t)$ is given by (4.2.9). The details have been presented in section (4.2).

After this preparation step, the corresponding forward characteristic function $\phi^T(\mu, X(t), \tau) = \exp(A(\mu,\tau)+B(\mu,\tau)x^T(t)+C(\mu,\tau)\sigma(t))$, where $\tau := T-t$ can be analytically derived by solving the following ODEs with respect to τ:

$$\begin{aligned}
B'(\tau) &= 0,\\
C'(\tau) &= -\kappa C(\tau) + (B^2(\tau) - B(\tau))/2 + \rho_{x,\sigma}\gamma B(\tau)C(\tau) + \gamma^2 C^2(\tau)/2,\\
A'(\tau) &= \kappa\overline{\sigma}C(\tau) - ((\rho_{x,d}\eta_d B_{rd} - \rho_{x,f}\eta_f B_{rf})\sqrt{\sigma(t)} + \rho_{d,f}\eta_d\eta_f B_{rd}B_{rf}\\
&\quad -\frac{1}{2}(\eta_d^2 B_{rd}^2 + \eta_f^2 B_{rf}^2))((B^2(\tau) - B(\tau)).
\end{aligned}$$

As we already solved this type of ODEs in section (4.3), we need not repeat the derivation and we just show the result:

$$\begin{cases}
B(\mu,\tau) = i\tau,\\
C(\mu,\tau) = \frac{1-e^{-D_1\tau}}{\gamma^2(1-ge^{-D_1\tau})}(\kappa - \gamma\rho_{x,\sigma}i\mu - D_1),\\
A(\mu,\tau) = \kappa\overline{\sigma}I_1(\tau) + (\mu^2 + i\mu)I_2(\tau).
\end{cases}$$

where $D_1 = \sqrt{(\gamma\rho_{x,\sigma}i\mu - \kappa)^2 - \gamma^2 i\mu(i\mu - 1)}$, $g = \frac{\kappa-\gamma\rho_{x,\sigma}i\mu-D_1}{\kappa-\gamma\rho_{x,\sigma}i\mu+D_1}$ and

$$\begin{aligned}
I_1(\tau) &= \frac{\tau}{\gamma^2}(\kappa - \gamma\rho_{x,v}i\mu - D_1) - \frac{2}{\gamma^2}\ln(\frac{1-ge^{-D_1\tau}}{1-g}),\\
I_2(\tau) &= \int_0^\tau \{(\rho_{x,d}\eta_d B_{rd} - \rho_{x,f}\eta_f B_{rf})\sqrt{\sigma(s)} + \rho_{d,f}\eta_d\eta_f B_{rd}B_{rf}\\
&\quad -\frac{1}{2}(\eta_d^2 B_{rd}^2 + \eta_f^2 B_{rf}^2)\}ds.
\end{aligned}$$

The closed-form solution for $A(\mu,\tau)$ is available similarly to section (4.3), which involves the integration for the expectation of the square root of the variance. Now, we are ready to price European options with this analytic expression of the forward characteristic function. The pricing method we will use is again the COS method.

5.5 Numerical Experiment

In this section, we mimic the situation of chapter 4 when comparing the COS method with the Monte Carlo method for the Heston-Hull-White model. As we have already shown, the expression of the forward characteristic function of the FX-HHW model can be obtained semi-analytically. Then we can use a set-up from [Pit04] to start the experiment. The original experiment is organized as follows: we first pick up the appropriate parameters for the Heston part and for the Hull-White part independently

$$\begin{array}{rll} \text{Heston} & \kappa = 0.5, & \gamma = 0.3, \overline{\sigma} = 0.1, \rho_{FX,\sigma} = -0.4, \sigma(0) = 0.1, \\ \text{Hull-White} & \lambda_d = 0.01 & , \eta_d = 0.007, \lambda_f = 0.05, \eta_f = 0.012. \end{array}$$

Then, we choose the zero-coupon bond prices from both the domestic and foreign markets that can be used to get the forward FX price for a certain maturity by $P_d(0,T) = \exp(-0.02T), P_f(0,T) = \exp(-0.05T)$. Next, the correlation matrix is determined for which is it assured that it is symmetric positive definite.

$$\begin{bmatrix} 1 & \rho_{FX,\sigma} & \rho_{FX,d} & \rho_{FX,f} \\ \rho_{FX,\sigma} & 1 & \rho_{\sigma,d} & \rho_{\sigma,f} \\ \rho_{FX,d} & \rho_{\sigma,d} & 1 & \rho_{d,f} \\ \rho_{FX,f} & \rho_{\sigma,f} & \rho_{d,f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.4 & -0.15 & -0.15 \\ -0.4 & 1 & 0 & 0 \\ -0.15 & 0 & 1 & 0.25 \\ -0.15 & 0 & 0.25 & 1 \end{bmatrix}.$$

Here we use a different matrix than [Pit04] because we use the assumption that the correlation between the interest rate and the FX volatility is equal to zero. The spot FX price is set to 1.35. The strikes have to be chosen conveniently to coincide with the FX market data. We will explain this in the following chapter. Here we use the strikes set as follows:

$$\begin{array}{rcl} K(T) & = & FX^{Forward}(0)\exp(0.1\sqrt{T}\delta), \\ \delta & = & \{-2.5, -2, -1.5, -1, -0.5, 0\,, 0.5\,, 1\,, 1.5\,, 2\,, 2.5\}. \end{array}$$

To simplify, we compare two maturities $T = 1$ and $T = 10$.

We use the COS pricing method with $N = 50000$ to calculate the FX call option price and corresponding implied volatility based on the forward FX-HHW characteristic function. We compare them with the full-scale model simulation by a Monte Carlo method with 10000 simulations and 1000 time steps.

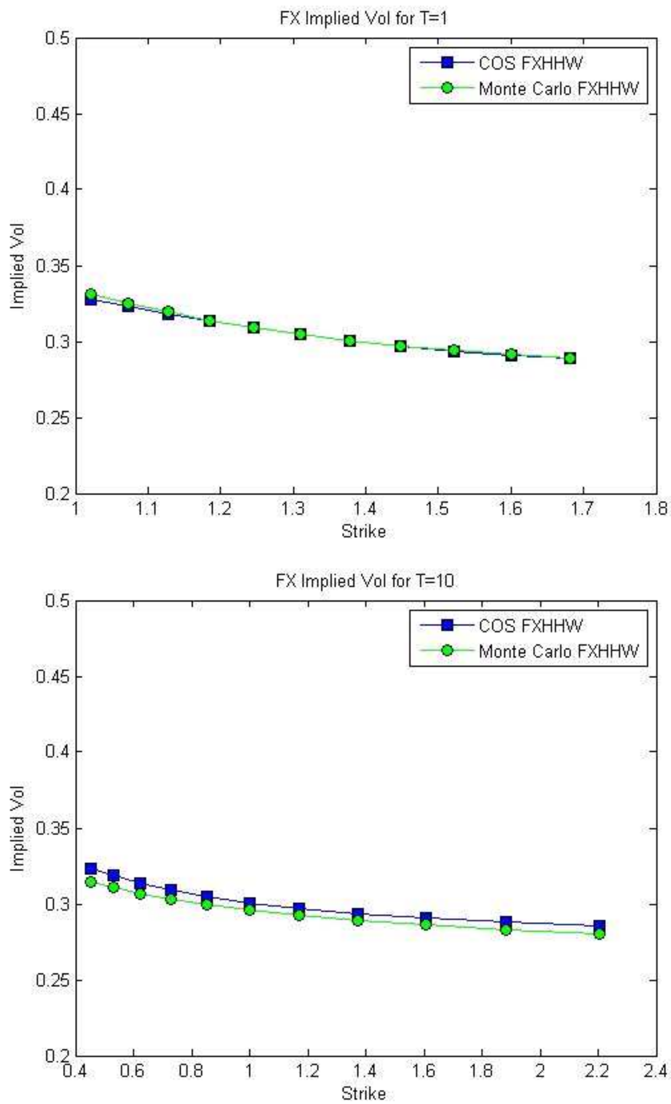

Figure 5.5.1: FX-COS v.s. Monte Carlo

The approximate forward FX-HHW model explained above seems a good approximation of the full-scale FX-HHW model, as can be seen from the above figures. Even with a long maturity, like 10 years, the distance between two resulting implied volatilities is small. Based on these two experiments we presume that the approximation can be used for pricing and hedging FX related contracts. This kind of characteristic function is most-of-all needed during the calibration process in the next chapter.

Chapter 6

Calibration of the HHW model

Calibration implies finding appropriate values for the open parameters in the pricing model, so that model prices for traded instruments match market prices as good as possible. Due to the complexity of the Heston-Hull-White model, several parameters need to be fitted to equity and FX data simultaneously. This is clearly an optimization problem in which the distance between market and theoretical prices should be minimized.

As we pointed out, the HHW model is a combination of the Heston and Hull-White models. There is a large body of work on the calibration of pure Heston models and pure Hull-White models. In this chapter, we will first give a review of the calibration approach of these two popular models. Then a realistic method based on two steps is proposed to calibrate the Heston-Hull-White model. The pricing formulas for plain vanilla options have been given in Chapters 4 & 5.

6.1 Equity and FX market

A mentioned, a Variable Annuity is similar to a long-maturity basket put option. There are domestic stock indices as well as foreign stock indices in a VA basket. In the Heston-Hull-White world, each interest rate and volatility is assumed to be stochastic. Hence, the pricing of VAs benefits from the application of the HHW model to both the equity and the FX markets. A volatility smile can be observed in both markets as option markets for both equity and FX are liquid. The realized volatility of an asset is a measure of how the asset price fluctuated over a specific period of time. It is also called "historical volatility", because it reflects the past. The "implied volatility" - a volatility that can be extracted from the prices of liquid traded instruments, is representative for what the market is implying in terms of volatility for future dates. Volatility smile is the phrase used to describe how the implied volatility of options varies with the strike price. A smile means that out-of-the-money puts and out-of-the-money calls both have higher implied volatilities than the at-the-money options. In the financial industry traders deal with implied volatility data every day. This volatility is the result of extracting the option price by the Black-Scholes formula. We can usually observe an implied volatility surface directly from the two markets. An implied volatility surface is a 3-D plot that combines the volatility smile and the term-structure into a consolidated view of all options for an underlier. Here, "surface" means the implied volatility for all strikes and all maturities. The purpose of calibration is to fit this surface as closely as possible.

In the equity markets, when implied volatility is plotted against the strike price, the resulting graph is typically downward sloping. Usually, we use the term 'volatility skew' for equity options referring to the downward sloping plot. For FX options, we prefer to use 'volatility smile' to describe the situation in which the graph turns up at either end. A stock market index is composed of a basket of stocks and provides a way to measure a specific sector's performance. They can give an overall idea about the whole economy. These indices are the most regularly quoted and are composed of large-cap stocks of a specific stock exchange, such as the American S&P 500, the British FTSE 100 and the EuroStoxx 50 (SX5E). Let's take the SX5E as an example example, we can easily see the skew of the implied volatility. As we can see from Figure 6.1.1, the strike is always chosen around the spot price of the index at a certain time and the maturity can vary from 1 week to 10 years. The implied volatility surface of SX5E is especially skewed for short maturities which are usually hard to fit with Heston-type models.

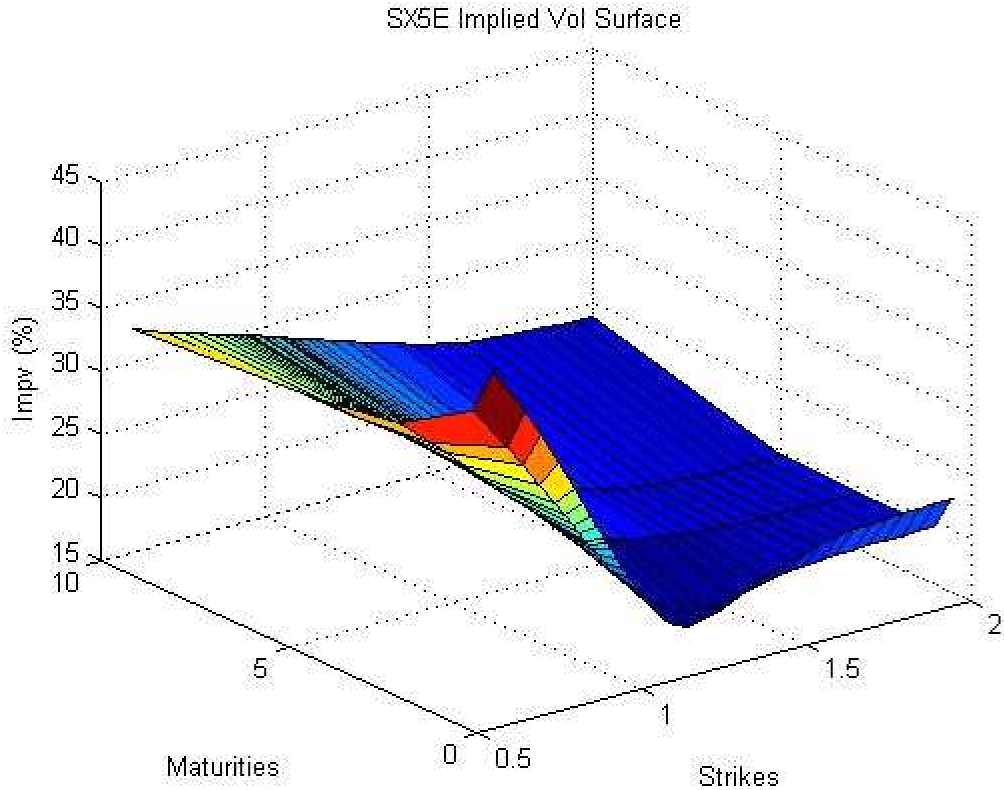

Figure 6.1.1: SX5E implied volatility surface

Unlike the equity market in which the implied volatility is function of strike and maturity, the FX implied volatility is quoted by Delta and maturity as shown in Table 6.1. Delta is the first derivative of the option price with respect to the underlying FX spot rate. As seen in section 5.1, the fair value of a call FX vanilla option in the Black-Scholes model is calculated as follows:

$$c = S_0 \exp(-r_f T) N(d_1) - K \exp(-r_d T) N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

The Delta can be mathematically obtained as:

$$\triangle = \frac{\partial C}{\partial S} = \omega \exp(-r_f T) N(\omega d_1), \tag{6.1.1}$$

where $\omega = 1$ for a call type and $\omega = -1$ for a put. Delta represents the amount of base currency units, expressed as a percentage of the notional, that is equivalent to the position in the option. If a trader wants to be hedged against the movements of the underlying FX rate, he has to trade in the market a spot contract with equal amount and opposite sign to this Delta.

The definition of Delta may help us understand the structure of the FX option market in a better way. The first type of Delta is the ATM straddle, which means the sum of a call and a put struck at the at-the-money level. Different types of these quotes exist in the market. ATM spot means that the strike of the option is equal to the FX spot rate and ATM forward indicates that this strike is set equal to the forward price of the underlying pair for the same expiry of the option. The last kind is the so-called "0-Delta-STDL", where the strike is chosen so that the call and the put have the same Delta but with different signs. This one is most often used in the FX option market for at-the-money implied volatility (obtained from inverting the Black-Scholes formula when Delta is equal to 50). Adjustment to this value is undertaken by incorporating the values of Risk Reversal (RR) and Vega-Weighted Butterfly (VWB). The RR is a structure set up when one buys a call and sells a put both featured with the same level of Delta. On the other hand, the VWB is the structure referring to buy a call and a put and sell an ATM STDL with the same Delta level. The percentages 25% and 10% are usually available in the markets, which stand for two Delta levels. We can get the value of Delta quoted implied volatility for different levels, based on the following:

$$\begin{aligned}\sigma(xD\,Call) &= \sigma(ATM) + 0.5 * \sigma(xD\,RR) + \sigma(xD\,VWB),\\ \sigma(xD\,Put) &= \sigma(ATM) - 0.5 * \sigma(xD\,RR) + \sigma(xD\,VWB),\end{aligned}$$

where x means 25 or 10. As we can see from Table 6.1, five quotes for each maturity exist in the market. We can also see the shape of smile from Figure 6.1.2. However, it is not a conventional plot since the implied volatility is presented against the Delta and not versus the strike. The implied strike can be recovered from the Black-Scholes formula when we calculate the Delta. If we invert Formula (6.1.1), the strike is calculated as follows:

$$K = S_0 \exp[(r_d - r_f)T] \cdot \exp\{-\omega\sigma\sqrt{T}N^{-1}[|\triangle| \exp(r_f T)] + 0.5\sigma^2 T\}.$$

Table 6.2 gives the strike level, when we choose the spot price of GBPEUR as 1.1986.

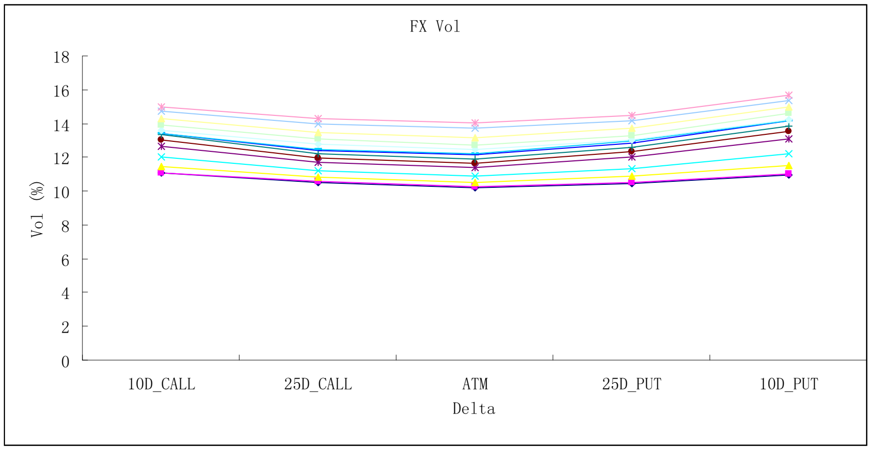

Figure 6.1.2: GBPEUR implied volatility (%)

GBP versus EUR	10D_CALL	25D_CALL	ATM	25D_PUT	10D_PUT
1W	11.083	10.536	10.205	10.479	10.948
1M	11.063	10.548	10.260	10.533	11.008
2M	11.463	10.826	10.528	10.889	11.543
3M	12.000	11.206	10.885	11.344	12.200
6M	12.643	11.680	11.394	11.995	13.113
9M	13.019	11.973	11.670	12.308	13.531
1Y	13.313	12.206	11.886	12.579	13.848
18M	13.424	12.428	12.124	12.838	14.136
2Y	13.423	12.478	12.24	12.948	14.183
3Y	13.623	12.783	12.453	13.038	14.253
4Y	13.895	13.061	12.732	13.299	14.570
5Y	14.285	13.465	13.159	13.695	15.000
7Y	14.710	13.954	13.715	14.144	15.382
10Y	15.003	14.302	14.005	14.465	15.673

Table 6.1: Implied volatility (%)

GBPEUR	10D_CALL	25D_CALL	ATM	25D_PUT	10D_PUT
1W	1.2226	1.2106	1.1987	1.1870	1.1756
1M	1.2495	1.2242	1.1993	1.1750	1.1515
2M	1.2744	1.2364	1.2000	1.1648	1.1300
3M	1.2973	1.2472	1.2008	1.1561	1.1111
6M	1.3505	1.2725	1.2031	1.1376	1.0703
9M	1.3954	1.2937	1.2055	1.1245	1.0409
1Y	1.4363	1.3127	1.2078	1.1136	1.0165
18M	1.5013	1.3443	1.2109	1.0968	0.9789
2Y	1.5572	1.3707	1.2136	1.0851	0.9519
3Y	1.6602	1.4169	1.2126	1.0680	0.9091
4Y	1.7550	1.4536	1.2045	1.0555	0.8723
5Y	1.8509	1.4870	1.1901	1.0478	0.8407
7Y	1.9964	1.5190	1.1325	1.0458	0.7976
10Y	2.1303	1.5001	0.9870	1.0743	0.7623

Table 6.2: Strike

In this MSc project, we will use the implied volatility surfaces of SX5E(EUR), SP500(USD), AEX(EUR), FTSE(GBP), IBEX(EUR) and TOPIX(JPY) index options. In this case, we should use the three currencies from the FX market: USDEUR, GBPEUR and JPYEUR.

6.2 Hull-White calibration

The generalized Hull-White one-factor model has the following form for the short interest rate:

$$dr(t) = (\overline{\theta}(t) - a(t)r(t))dt + \sigma(t)dW_r(t), \tag{6.2.1}$$

where $a(t)$ is the mean reversion speed, $\sigma(t)$ is the volatility and $\overline{\theta}(t)$ is used to replicate the current term-structure. In this project, we use the classical Hull-White model, where the mean reversion speed and volatility are positive constants. Hence the dynamics for this interest rate are as presented in chapter 4.

$$dr(t) = \lambda(\theta(t) - r(t))dt + \eta dW_r(t).$$

Note that $\overline{\theta}(t) = \lambda\theta(t)$. The formula for short rate $r(t)$ is

$$r(t) = r(s)e^{-\lambda(t-s)} + \int_s^t \overline{\theta}(u)e^{-\lambda(t-u)}du + \eta\int e^{-\lambda(t-u)}dW_r(u) \tag{6.2.2}$$

Then we can also get the analytic formulas for $A_r(t,T)$ and $B_r(t,T)$ that we used in section 4.5.1:

$$P(t,T) = \exp[A_r(t,T) - B_r(t,T)r(t)], \tag{6.2.3}$$

where $P(t,T)$ is the zero-coupon bond and

$$\begin{aligned} A_r(t,T) &= \ln(\frac{P(0,T)}{P(0,t)}) + B_r(t,T)f(0,t) - \\ &\quad \frac{\eta^2}{4\lambda}(1-e^{-2\lambda t}B_r(t,T)^2) \qquad (6.2.4) \\ B_r(t,T) &= \frac{1-e^{-\lambda(T-t)}}{\lambda}. \qquad (6.2.5) \end{aligned}$$

Now, the analytic formula for $A_r(t,T)$ reads:

$$\ln A_r(t,T) = \frac{\eta^2}{2}\int_t^T B_r^2(u,T)du - \int_t^T \overline{\theta}(u)B_r(u,T)du. \qquad (6.2.6)$$

6.2.1 Fitting the term-structure

Here, we explain how to use the analytic formulas to calibrate the Hull-White model. For calibration purposes, the analytic formula is important to fit to market implied values highly efficiently. Let's first have a look at the current term-structure $\overline{\theta}(t)$. In the Hull-White model the current instaneous forward rates $f(0,T)$ can be analytically obtained, since $f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T}$. So:

$$\begin{aligned} f(0,T) &= \frac{\partial}{\partial T}\{B_r(0,T)r(0)\} - \frac{\partial}{\partial T}\ln A_r(0,T) \\ &= r(0)e^{-\lambda T} - \frac{\eta^2}{2\lambda^2}(1-e^{-\lambda T})^2 + e^{-\lambda T}\int_0^T \overline{\theta}(u)e^{\lambda u}du. \qquad (6.2.7) \end{aligned}$$

The term-structure can be fitted, if we invert the formula (6.2.7)

$$\overline{\theta}(t) = \frac{\partial}{\partial t}f(0,t) + \lambda f(0,t) + \frac{\eta^2}{2\lambda}(1-e^{-2\lambda t}) \qquad (6.2.8)$$

In practice, $\overline{\theta}(t)$ is fitted on a daily basis. The term-structure parameter of the short interest rate is essential for a Monte Carlo simulation process under the spot measure. In Figure 6.2.1, we use the Euro zero rates (interest rates of the zero coupon bond prices) data from August 2nd, 2010 to compute the instaneous forward rates $f(0,T)$. Then we use (6.2.7) to get the daily term-structure of Euro interest rate.

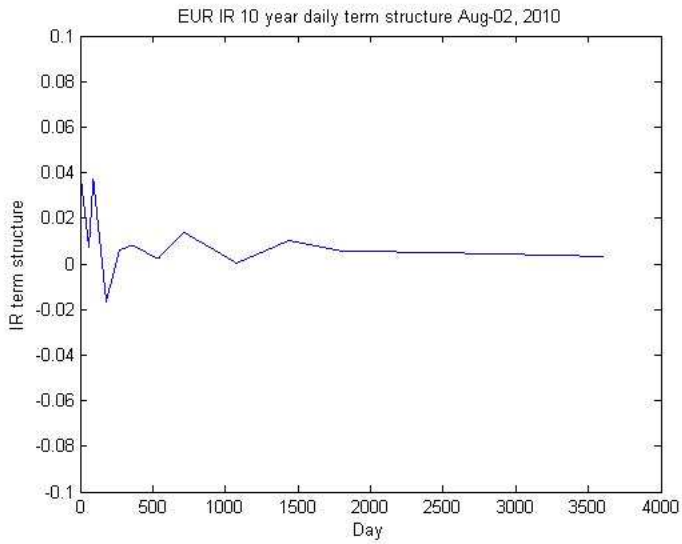

Figure 6.2.1: 10 year daily term-structure for Euro swap rate

6.2.2 Calibration by swaptions

A swap contract is a financial derivative, where counter-parties exchange cash-flows based on the return of reference indices. One type of contract is called the interest rate swap: it is the exchange of a fixed rate loan by a floating rate loan. The plain vanilla interest rate swap is the most popular swap in the over-the-counter market. A swaption is also a financial derivative. It can be seen as an option on a swap. It grants the owner the right, but not the obligation, to enter into the underlying swap. In the Hull-White case, we focus here on the interest rate swap only. In Brigo and Mercurio 's book, we can find the analytic formula for this swaption price, based on the Hull-White one-factor model. Let's first take a look at a European option on zero coupon bonds. The payoff for a European put option with maturity T_{i-1} on the T_i-maturity zero coupon bond should be as follows: (with X being the strike)

$$\max(X - P(T_{i-1}, T_i), 0)$$

Hence, the price of this option at time t reads:

$$ZBP(t, T_{i-1}, T_i, X) = E^{\mathbb{Q}}[e^{-\int_t^{T_{i-1}} r(s)ds} \cdot \max(X - P(T_{i-1}, T_i), 0)|\mathcal{F}_t].$$

German, EI Karoui and Rochet have given an analytic expression for this put option, based on the Hull-White one-factor model:

$$ZBP(t, T_{i-1}, T_i, X) = XP(t, T_{i-1})N(-h + \sigma_p) - P(t, T_i)N(-h), \qquad (6.2.9)$$

where

$$\begin{aligned}\sigma_P &= \eta\sqrt{\frac{1-e^{-2\lambda(T_{i-1}-t)}}{2\lambda}}B_r(T_{i-1},T_i),\\ h &= \frac{1}{\sigma_P}\ln\frac{P(t,T_i)}{P(t,T_{i-1})X}+\frac{\sigma_P}{2}.\end{aligned}$$

We still use the λ,η and $B_r(t,T)$ as in the Hull-White model.

We consider an interest rate swap contract where the tenor is $\tau_m = \{T_0, ..., T_m\}$ and the nominal value is N, the fixed rate K is paid and a floating rate (LIBOR) is received. We also refer to the book by Brigo and Mercurio. By defining $c_i = \tau_i K$ for $i = 1, 2..., m-1$ and $c_m = (1+\tau_m K)$ the value at time T_0 of the interest rate swap is $N(1-\sum_{i=1}^m c_i P(T_0, T_i))$. Since the payer swaption is an option on a payer swap, and we assume the option expires at time T_0, the payoff of this swaption reads $N \cdot \max(1-1-\sum_{i=1}^m c_i P(T_0, T_i), 0)$, Now we can get the swaption price from the earlier result:

$$\text{Swaption}(t, \tau_m, N, X) = N\sum_{i=1}^{m} c_i ZBP(t, T_0, T_i, P(T_0, T_i)). \tag{6.2.10}$$

This is the analytic formula, which can be applied within the calibration of the Hull-White model. More precisely, the mean-reversion parameter and the interest volatility can be obtained in a stable way, due to the existence of the swaption price in the market. Alternatively, we can calibrate via caplets, for which there is also an analytic formula.

In practice, the Hull-White parameters can be calibrated on a monthly basis based on either swaptions or caplets. Now, we take a look at swaption price data from the European market. These benchmark prices can be used to fit the Hull-White mean reversion and interest volatility parameters. The results are shown in Table 6.3.

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Date	mean reversion (%)	IR vol (%)
2010-01	2.45	0.85
2010-02	2.13	0.82
2010-03	1.96	0.80
2010-04	1.73	0.83
2010-05	1.70	0.86
2010-06	1.89	0.86
2010-07	1.93	0.82
2010-08	1.93	0.88
2010-09	2.19	0.91
2010-10	2.24	0.91
2010-11	2.12	0.99
2010-12	3.01	0.98

Table 6.3: EUR IR Hull-White monthly calibration results

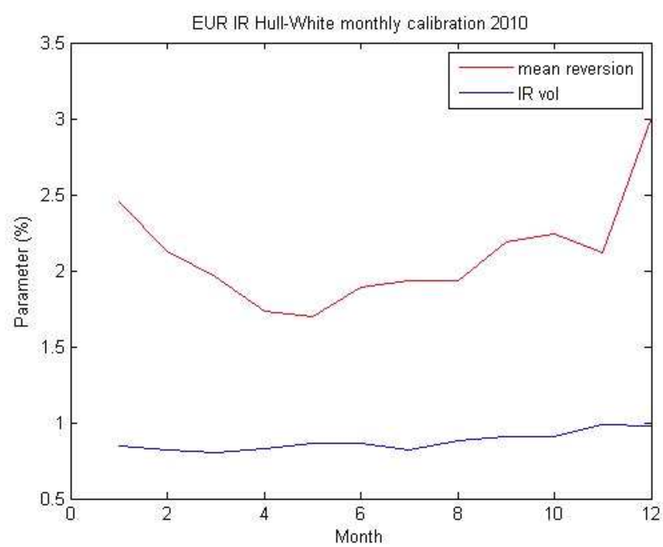

Figure 6.2.2: HW parameters v.s. time

Summarizing, the term-structure can be fitted from the instantaneous forward rate and the other two parameters can be calibrated from swaptions prices. The only parameter left for the pure Black-Scholes Hull-White model is the correlation between the interest rate and the stock. In reality, historical data is frequently used to calibrate this correlation. In this MSc project, for the six stock indices and four short rates in the Variable Annuities, we prefer to use the historical correlation between the 10-year swap rate (EUR, USD, GBP, and JPY) and each stock index to estimate. These can used for long maturities.

6.3 Heston calibration

The pure Heston model is well-known for its ability to capture the implied volatility smile in equity (or FX) and its tractability. As discussed in Chapter 3, the characteristic function for the Heston model reads

$$\varphi_{hes}(\omega; \upsilon_0) = \exp(i\omega r\triangle t + \frac{\upsilon_0}{\eta^2}(\frac{1-e^{-D\triangle t}}{1-Ge^{-D\triangle t}})(\kappa - i\rho\eta\omega - D)) \cdot \exp(\frac{\kappa\overline{\upsilon}}{\eta^2}(\triangle t(\kappa - i\rho\eta\omega - D) - 2\ln(\frac{1-Ge^{-D\triangle t}}{1-G}))), \quad (6.3.1)$$

with

$$D = \sqrt{(\kappa - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2}, \quad (6.3.2)$$

$$G = \frac{\kappa - i\rho\eta\omega - D}{\kappa - i\rho\eta\omega + D}. \quad (6.3.3)$$

Many scholars did fine research on how to calibrate the Heston model. The most common technique to calibrate the Heston model is by means of the Fast Fourier Transform (FFT) or the direct numerical integration. As we have seen, the COS pricing method is faster than the other methods. The COS formula for a plain vanilla option is

$$v(x, t_0) \approx e^{-r\triangle t} \sum_{k=0}^{N-1}{}' Re\{\phi_{hes}(\frac{k\pi}{b-a}; x) \cdot \exp(-ik\pi\frac{a}{b-a})\}V_k$$

where V_k is the pay-off coefficient for call or put options, see section 3.2 for their expressions.

Here, we should choose an appropriate calibration norm and method. The choice has a significant impact on the final result and on the speed of calibration as well. More importantly, the norm and method for calibrating the pure Heston model can be regarded as benchmark when we proceed to the calibration of the Heston-Hull-White model.

The most popular technique is to minimize the error between model and market prices, and evaluate the parameters in one bounded set by solving the following:

$$\min \sum w_i\{C_i^{Model}(K_i,T_i) - C_i^{Market}(K_i,T_i)\}^{Norm}, \quad (6.3.4)$$

where $C_i^{Model}(K_i,T_i)$ and $C_i^{Market}(K_i,T_i)$ are the ith option prices from the model and market, respectively, with strike K_i and maturity T_i. Parameter w_i represents the weight of each option among the total N options. $Norm$ is the specific norm that we can use to minimize the value. The most common norms are the quadratic norm and the relative norm:

$$\begin{cases} \text{quadratic } norm: & \min \sqrt{\sum_{i=1}^N \{C_i^{Model}(K_i,T_i) - C_i^{Market}(K_i,T_i)\}^2} \\ \text{relative } norm: & \min \sum_{i=1}^N \frac{|C_i^{Model}(K_i,T_i) - C_i^{Market}(K_i,T_i)|}{C_i^{Market}(K_i,T_i)}. \end{cases}$$

There is a trade-off between these two norms since a quadratic norm assigns more weight to long-maturity options and In-The-Money options. On the other hand, by using the relative norm we favor short-term and extremely skewed options. As Bin Chen [Bin07] already showed this in his MSc research, the Heston model has some problems when fitting skews to short maturities. That is why here we still use the quadratic norm but not an equally weighted version. We want to give weight to the At-The-Money options, so that we will choose the Black-Scholes vegas as the weights. i.e. $\omega_i = \frac{\partial C}{\partial \sigma} = S\sqrt{T}\exp(-qT)N'(d_1)$.

Choosing an accurate calibration method (with an appropriate optimization technique) is crucial, since calibration may take a long time. Even though we can significantly reduce the pricing time by the COS pricing formula, time is also needed to choose the appropriate parameters. For this reason, many of the complex models have limited power as several parameters need to be evaluated. For the pure Heston model, there are basically five parameters to be fitted: mean reversion κ , long term variance $\bar{\upsilon}$, volatility of variance γ, correlation between stock and variance ρ and initial variance $\upsilon(0)$.

Generally, there are two kinds of optimization schemes: global and local optimizers. Within the local algorithms, one has to choose a good initial value for the parameters. The algorithm then determines the optimal direction and may end up with good quality local minimization result. So it is essential to find a good initial guess. Global algorithms, on the other hand, aim to search everywhere in the constrained set determining the direction randomly. It is obvious that even though global schemes can find a better result than local schemes, they will cost much more time. In practice, we should perform the calibration by using the two different schemes. The aim is to use fewer time than global calibration and perhaps somewhat more time than the local calibration. In the financial industry, we may use global calibration at the very beginning (of a month) and then use this result as the starting point for local calibrate during the whole month.

In this project, we use Matlab as our computing software since it is strong in mathematical calculations and easy to use. As the local algorithm, we will use the Matlab function 'fminsearch'. This function uses the Nelder-Mead Simplex search method. For the global algorithms we will use adaptive simulated annealing. Simulation Annealing is a probability-based, non-linear, stochastic optimizer. Adaptive Simulated Annealing (ASA) is similar to this but more powerful. Ingber has already shown that ASA is a global optimizer and this optimizer can be implemented in Matlab by downloading the function asamin, written by S. Sakata. For the calibration part, we will use these two functions in Matlab to calibrate indices and currencies based on the two models. We will first give the results for the Heston model and then show the outcome of the Heston-Hull-White model in the next section.

6.3.1 Equity Heston Calibration

For equity index options, we use here the SX5E as an example. The dividend rate can be deduced from the forward price of the SX5E. The calibration results of Heston model (daily basis) are shown in Table 6.4. This calibration is based on the whole volatility surface with strikes between 60% and 200% of the spot index value. The maturities vary from 0.5 years to 10 years. The five variance-related parameters are stable throughout the month. Another important issue to notice is that the mean reversion level and the volatility-of-volatility parameter move in the same direction. This is easy to understand since it takes a long time for the initial variance to reach the long time variance, either when the mean reversion is high or when the vol-of-vol parameter is high.

date(Aug)	mean reversion	vol of variance	long-run variance	correlation	initial variance
2-Aug	0,3814	0,5168	0,1799	-0,9216	0,0648
3-Aug	0,7109	0,9079	0,1602	-0,8074	0,0753
4-Aug	0,6708	0,8737	0,1617	-0,8064	0,0729
5-Aug	0,6930	0,9017	0,1597	-0,8159	0,0758
6-Aug	0,3437	0,5539	0,1937	-0,8709	0,0729
9-Aug	0,6713	0,8904	0,1478	-0,7939	0,0812
10-Aug	0,7761	0,8529	0,1439	-0,8199	0,0843
11-Aug	0,6886	0,8543	0,1509	-0,8115	0,0988
12-Aug	0,8642	0,8053	0,1301	-0,8450	0,1020
13-Aug	0,7896	0,6170	0,1302	-0,9048	0,0942
16-Aug	0,9472	0,6898	0,1326	-0,9207	0,0969
17-Aug	0,8695	0,8173	0,1470	-0,8571	0,0921
18-Aug	0,7915	0,8613	0,1598	-0,8422	0,0915
19-Aug	0,8679	0,8261	0,1529	-0,8680	0,1008
20-Aug	0,8787	0,8193	0,1520	-0,8637	0,1048
23-Aug	0,8763	0,8144	0,1526	-0,8605	0,0967
24-Aug	0,8862	0,6340	0,1408	-0,9458	0,0989
25-Aug	0,8817	0,8180	0,1545	-0,8651	0,1109
26-Aug	0,8861	0,7855	0,1517	-0,8726	0,1063
27-Aug	0,8800	0,7034	0,1454	-0,8935	0,0972
30-Aug	0,8841	0,7854	0,1514	-0,8665	0,1024
31-Aug	0,8634	0,6911	0,1499	-0,9214	0,0977

Table 6.4: SX5E Heston monthly calibration

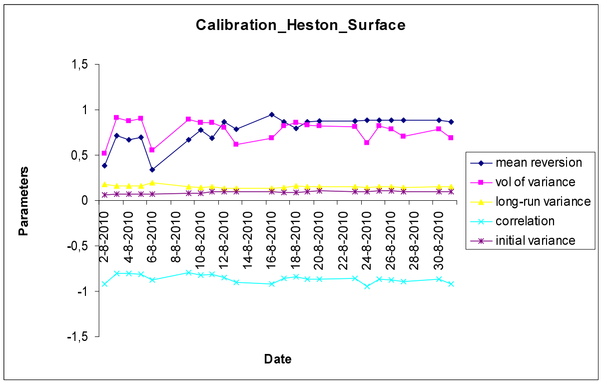

Figure 6.3.1: SX5E Heston monthly calibration

6.3.2 FX Heston Calibration

The calibration of the FX Heston model is almost the same as the equity case except that the dividend rate is now the foreign interest rate. Take the GBPEUR, for example, we will use the implied volatility data in Table 6.1. The corresponding strikes can be found in Table 6.2. The results should make the error as small as possible. We give the results from the ASA global minimization methods in Table 6.5. It is clear that the Heston model can fit the market better for long maturities. Now we move to the next part on calibrating Heston-Hull-White and we can compare the two results together and examine the impact.

T\Delta	10D_CALL	25D_CALL	ATM	25D_PUT	10D_PUT
0.5Y-Impv(%)	12.6430	11.6800	11.3940	11.9950	13.1130
0.5Y-HesImpv(%)	12.1103	11.5972	11.5362	12.0107	12.8314
Error	-0.0053	-0.0008	0.0014	0.0002	-0.0028
0.75Y-Impv(%)	13.0191	11.9730	11.6700	12.3080	13.5310
0.75Y-HesImpv(%)	12.4317	11.7271	11.5894	12.1663	13.1903
Error	-0.0059	-0.0025	-0.0008	-0.0014	-0.0034
1Y-Impv(%)	13.3130	12.2060	11.8860	12.5790	13.8480
1Y-HesImpv(%)	12.7104	11.8677	11.6706	12.3114	13.4753
Error	-0.0060	-0.0034	-0.0022	-0.0027	-0.0037
1.5Y-Impv(%)	13.4240	12.4280	12.1240	12.8381	14.1361
1.5Y-HesImpv(%)	13.1067	12.1370	11.8748	12.5628	13.8765
Error	-0.0032	-0.0029	-0.0025	-0.0028	-0.0026
2Y-Impv(%)	13.4230	12.4779	12.2400	12.9480	14.1830
2Y-HesImpv(%)	13.3961	12.3776	12.0972	12.7798	14.1416
Error	-0.0003	-0.0010	-0.0014	-0.0017	-0.0004
3Y-Impv(%)	13.6230	12.7830	12.4530	13.0379	14.2530
3Y-HesImpv(%)	13.8555	12.8062	12.5259	13.1401	14.5077
Error	0.0023	0.0002	0.0007	0.0010	0.0025
4Y-Impv(%)	13.8950	13.0610	12.7320	13.2990	14.5700
4Y-HesImpv(%)	14.1993	13.1499	12.9038	13.4328	14.7868
Error	0.0030	0.0009	0.0017	0.0013	0.0022
5Y-Impv(%)	14.2850	13.4649	13.1590	13.6950	15.0000
5Y-HesImpv(%)	14.4817	13.4350	13.2304	13.6668	14.9986
Error	0.0020	-0.0003	0.0007	-0.0003	0.0000
7Y-Impv(%)	14.7100	13.9540	13.7150	14.1440	15.3820
7Y-HesImpv(%)	14.8148	13.8377	13.7759	13.9896	15.2028
Error	0.0010	-0.0012	0.0006	-0.0015	-0.0018
10Y-Impv(%)	15.0030	14.3019	14.0050	14.4650	15.6730
10Y-HesImpv(%)	15.0359	14.2102	14.4648	14.2873	15.2920
Error	0.0003	-0.0009	0.0046	-0.0018	-0.0038

Table 6.5: GBPEUR Heston fitting result

6.4 Two steps of calibrating Heston-Hull-White

The calibration based on the pure Heston model and the Black-Scholes-Hull-White model have been independently introduced in the previous section. Now, we are fully prepared to calibrate the complete Heston-Hull-White model. Several options regarding computational methods exist. The first one is the direct method due to the semi-analytic pricing formula obtained based on the equity and FX Heston-Hull-White model. However, the direct market approach will take a long time and may give unstable results, since there are in total 10 parameters for the equity part and 16 parameters for the exchange rate part. This method is therefore not practical for a financial institution. An alternative is to first calibrate all parameters independently for each model without the correlations with interest rate. After that we calibrate the correlation $corr(EQ, IR)$ and $corr(FX, IR)$ by using the semi-analytic formula. (the approximation is based on the assumption that $corr(IR, Vol) = 0$.) The obvious advantage of doing this is to save computation time. If the calibrated correlation coincides with the market, this method of calibration will be acceptable. In reality, however, this does not happen all the time. Another implicit disadvantage is from the following formula (for equities only):

$$\sigma_{total}^2 = \sigma_{EQ}^2 + \sigma_{IR}^2 + 2\rho_{EQ,IR}\sigma_{EQ}\sigma_{IR},$$

which means that the volatility of interest rate, in the case of positive correlation between equity and interest, rate will reduce the parameters from the Heston part. (in the pure Heston model, it is obvious that $\sigma_{total}^2 = \sigma_{EQ}^2$).

In this project, we will use this technique to calibrate the Heston-Hull-White model: The calibration of the interest rate part is based on the swaption data by the Hull-White formula as introduced in section 6.2. Then, we include these results into our semi-analytic formula for pricing plain vanilla call and put options to calibrate the parameters from the Heston part. The second step is very similar as the calibration method of the pure Heston model in section 6.3, which may give us a good comparison. As we discussed earlier, the forward measure is used to reduce the complexity. Some results are as follows:

global			
	mean reversion	vol of variance	long-run variance
Heston	0.3814	0.5167	0.1798
HHW	0.6251	0.7191	0.1423
	correlation	initial variance	SSE
Heston	-0.9215	0.0648	0.4081
HHW	-0.8679	0.0752	0.1525
local			
	mean reversion	vol of variance	long-run variance
Heston	0.3814	0.5167	0.1798
HHW	0.3751	0.5139	0.1599
	correlation	initial variance	SSE
Heston	-0.9215	0.0648	0.4081
HHW	-0.924	0.0694	0.1593

Table 6.6: HHW v.s. Heston

In this Table 6.6, we calibrated the SX5E to the same surface for both models. The global minimization methodology shows that the HHW model is better than the pure Heston model. (SSE stands for Sum of Square Errors. It is is smaller for HHW.) We carried out another test to compare the two models. The initial value is crucial for the local minimization method in the calibration process. The Heston results are used as the starting points for the calibration of the HHW, which proves that the long-run variance is strongly reduced with other parameters changing little. Even though the local minimized results are not always ideal answers, as can be seen from the SSE, it it strongly advocated when the calibration is too time-consuming.

6.4.1 Equity-HHW Calibration

Here the calibration is based on the global method for the SX5E on a daily basis. Figure 6.4.1 gives the calibration results from August 23rd, 2010. It can be seen that both models work well when fitting the implied volatility smile. The monthly calibration results of the HHW model in August 2010 is shown in Table 6.7. Some characteristics can be observed when comparing with Table 6.4. Both calibrations have the same features connecting the mean reversion and the vol-of-vol parameter. Moreover, it can be seen again that the long-term variance is reduced by a certain amount for the HHW model.

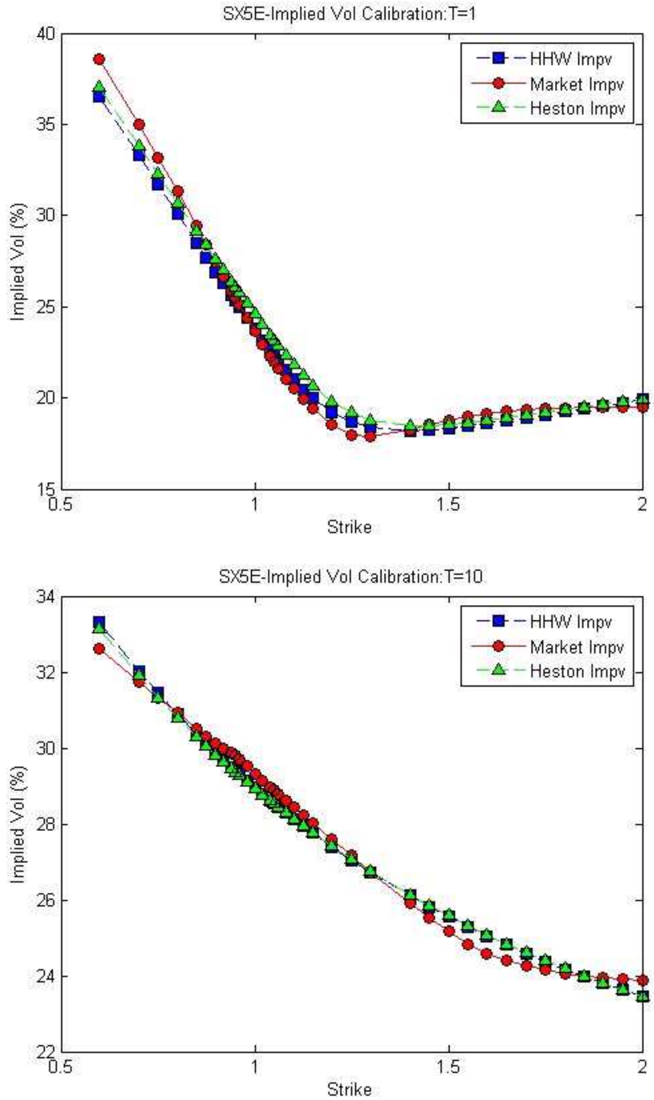

Figure 6.4.1: Fitting result of two models

Date	mean reversion	vol of variance	long-run variance	correlation	initial variance
2-Aug	0.3690	0.5153	0.1635	-0.9059	0.0691
3-Aug	0.4986	0.6100	0.1509	-0.8853	0.0712
4-Aug	0.7274	0.8303	0.1422	-0.8447	0.0754
5-Aug	0.3202	0.4902	0.1762	-0.9448	0.0677
6-Aug	0.4169	0.5677	0.1601	-0.8926	0.0772
9-Aug	0.2350	0.4345	0.1796	-0.9298	0.0690
10-Aug	0.7966	0.8512	0.1291	-0.8102	0.0898
11-Aug	0.7394	0.8012	0.1312	-0.8448	0.1020
12-Aug	0.8642	0.8123	0.1189	-0.8418	0.1092
13-Aug	0.7886	0.6109	0.1178	-0.9114	0.0999
16-Aug	0.9469	0.7006	0.1216	-0.9192	0.1047
17-Aug	0.8734	0.8199	0.1328	-0.8421	0.0989
18-Aug	0.8071	0.8554	0.1430	-0.8272	0.0975
19-Aug	0.8739	0.8310	0.1385	-0.8557	0.1083
20-Aug	0.8778	0.8226	0.1386	-0.8601	0.1129
23-Aug	0.9040	0.7733	0.1334	-0.8493	0.1013
24-Aug	0.8844	0.6377	0.1282	-0.9450	0.1065
25-Aug	0.7149	0.5413	0.1302	-0.9284	0.1039
26-Aug	0.6828	0.5604	0.1358	-0.9466	0.1037
27-Aug	0.8770	0.7113	0.1332	-0.8932	0.1057
30-Aug	0.8759	0.6554	0.1288	-0.8886	0.1030
31-Aug	0.7656	0.5563	0.1303	-0.9345	0.0981

Table 6.7: SX5E HHW monthly calibration

6.4.2 FX-HHW Calibration

We also perform the calibration based on the GBPEUR volatility surface data shown before. Here, the global minimization is also used. The results shown in Table 6.8 show that the fitting results are not better than the Heston model. The assumption of zero correlation between interest rate and the FX volatility is probably not acceptable. However the calibration error is still relatively small.

T\Delta	10D_CALL	25D_CALL	ATM	25D_PUT	10D_PUT
0.5Y-Impv(%)	12.6430	11.6800	11.3940	11.9950	13.1130
0.5Y-HHWImpv(%)	10.5711	10.4385	10.4477	10.5938	10.8748
Error	-0.0207	-0.0124	-0.0095	-0.0140	-0.0224
0.75Y-Impv(%)	13.0191	11.9730	11.6700	12.3080	13.5310
0.75Y-HHWImpv(%)	11.1750	11.0157	11.0214	11.1820	11.5010
Error	-0.0184	-0.0096	-0.0065	-0.0113	-0.0203
1Y-Impv(%)	13.3130	12.2060	11.8860	12.5790	13.8480
1Y-HHWImpv(%)	11.6680	11.4947	11.4994	11.6670	12.0055
Error	-0.0165	-0.0071	-0.0039	-0.0091	-0.0184
1.5Y-Impv(%)	13.4240	12.4280	12.1240	12.8381	14.1361
1.5Y-HHWImpv(%)	12.4035	12.2316	12.2390	12.4075	12.7556
Error	-0.0102	-0.0020	0.0011	-0.0043	-0.0138
2Y-Impv(%)	13.4230	12.4779	12.2400	12.9480	14.1830
2Y-HHWImpv(%)	12.9190	12.7570	12.7697	12.9327	13.2728
Error	-0.0050	0.0028	0.0053	-0.0002	-0.0091
3Y-Impv(%)	13.6230	12.7830	12.4530	13.0379	14.2530
3Y-HHWImpv(%)	13.5594	13.4120	13.4345	13.5805	13.9026
Error	-0.0006	0.0063	0.0098	0.0054	-0.0035
4Y-Impv(%)	13.8950	13.0610	12.7320	13.2990	14.5700
4Y-HHWImpv(%)	13.8868	13.7492	13.7822	13.9103	14.2234
Error	-0.0001	0.0069	0.0105	0.0061	-0.0035
5Y-Impv(%)	14.2850	13.4649	13.1590	13.6950	15.0000
5Y-HHWImpv(%)	14.0406	13.9074	13.9511	14.0594	14.3677
Error	-0.0024	0.0044	0.0079	0.0036	-0.0063
7Y-Impv(%)	14.7100	13.9540	13.7150	14.1440	15.3820
7Y-HHWImpv(%)	14.0623	13.9438	14.0167	14.0728	14.3625
Error	-0.0065	-0.0001	0.0030	-0.0007	-0.0102
10Y-Impv(%)	15.0030	14.3019	14.0050	14.4650	15.6730
10Y-HHWImpv(%)	14.0824	13.4732	13.8199	14.0014	14.0549
Error	-0.0092	-0.0083	-0.0019	-0.0046	-0.0162

Table 6.8: GBPEUR HHW fitting result

Calibration can give us insight into the model that we will use for pricing and hedging. In fact, it is difficult to say which model can fit the market better or which model is more realistic in the market. Calibration alone is not the standard to judge a model. We will see this in the next chapter when we aim to price long-term options.

Chapter 7

Multi-Asset Monte Carlo Pricing

From the calibration in the previous chapter, stable values for parameters for both the equity and FX Heston-Hull-White models have been obtained. The next step is the pricing of the Variable Annuities. As we already pointed out in Chapter 2, valuation should be based on Monte Carlo simulation since it is difficult to get a pricing formula for basket options. However, a basic Monte Carlo Euler scheme without any variance reduction technique may give rise to a significant bias. To make the result more accurate, we prefer to use other techniques and schemes. The structure of this chapter is as follows: Section 7.1 explains the antithetic sampling technique as well as the Milstein scheme. We will find that these two techniques can significantly improve the final result. For the Heston model, the Feller condition is crucial for pricing options. Section 7.2 discusses the pricing of the simple basket put options with one domestic stock and one foreign stock. We will compare the Heston-Hull-White model and the pure Heston model to observe the impact of a stochastic interest rate for long maturities. We will use the calibration results from the previous chapter to generate all the scenarios for six indices and three currencies for ten years in section 7.3. In section 7.4, some results for the GMWB prices based under Heston-Hull-White model and the basic Black-Scholes model are shown to see the impact on the final reserves.

7.1 Monte Carlo test for single asset

Monte Carlo simulation is a powerful pricing tool in the area of quantitative finance. It is a way of solving stochastic problems by evaluating many scenarios numerically and by calculating statistical properties. In quantitative finance, it is used to simulate the future behavior of assets. It can be used to determine the value of financial derivatives by exploiting the theoretical relation between option values and the expected payoff under a risk-neutral random walk. In the risk-neutral world, the fair value of an option is the present value of the expected payoff under the risk-neutral measure. With the Black-Scholes model for example, the dynamics of one market variable follows the process:

$$dS = rSdt + \sigma SdW,$$

where dW is a Wiener process, r is the risk-neutral interest rate and σ is the volatility. To simulate the path of S for maturity T, we can then subdivide T into M intervals, where $\Delta t = T/M$. Then we have

$$S(t + \Delta t) = S(t) + rS(t)\Delta t + \sigma S(t)\xi\sqrt{\Delta t},$$

where ξ is a random sample from a standard normal distribution. This equation can be used to generate future prices of S- i.e. $S(T)$ by using large numbers of paths. The value can be obtained and the derivatives be governed by a nonstandard payoff at time T. Monte Carlo methods have an essential advantage that if the payoff depends on the path. Nonetheless, it is computationally time-consuming to achieve a reasonable accuracy. This is the reason why we propose the use of variance reduction techniques.

7.1.1 Antithetic sampling

Antithetic sampling is a variance reduction technique. It is the easiest way of saving time by reducing the variance. The principle behind this is the computation of two values for the derivatives in one simulation trial. This technique uses the property that if $\xi \sim N(0,1)$, then $-\xi \sim N(0,1)$. Suppose we obtain the value of the derivative V_1 in the usual way; the second value V_2 is then calculated by changing the sign of all random samples from the standard normal distribution. Now, the sample value V is set to be the average of those two simulated values.

$$V = \frac{1}{2}(V_1 + V_2).$$

The final value is the total average of all the V variables. Suppose we use N simulations, and the standard deviation of each V is given by σ, then the standard error of the estimate is $\sigma/\sqrt{N}$. It is obvious that this error is significantly smaller than the one we obtain by using $2N$ simulations.

We can analyze this technique for the Heston model for a single asset. The reason why we choose the single Heston model is that this model is affine for a single asset. Hence, the analytic European call and put option prices can be obtained by the COS pricing method. For the Heston model we choose the parameters as:

$$\begin{aligned} T &= 1, r = 0.07, S = 100, K = 100, \\ \kappa &= 5, \quad \gamma = 0.5751, \overline{\upsilon} = 0.0398, \upsilon(0) = 0.0175, \rho = -0.5711. \end{aligned}$$

The analytic put option price obtained by the COS method with parameter $N = 5000$ (the number of Fourier cosine terms) is 4.3694. We implement the

Monte Carlo simulation with 1000 time steps to determine the accuracy and time spent with respect to the number of simulations

Absolute Error		N=10	N=1000	N=10000
	Antithetic	0.0802	0.0049	1.74E-04
	Standard	0.6870	0.0991	0.0066
Time(Seconds)		N=10	N=1000	N=10000
	Antithetic	0.2428	1.7240	15.8500
	Standard	0.2310	1.5125	13.6527

Table 7.1: Convergence with antithetic sampling

Table 7.1 shows that the antithetic sampling technique converges faster than the standard MC method.

7.1.2 Milstein scheme

In the Heston model, the basic Euler discretization of the variance process is as follows::

$$\upsilon(t+\Delta t) = \upsilon(t) + \kappa(\overline{\upsilon} - \nu(t))\Delta t + \gamma\sqrt{\upsilon(t)}\xi\sqrt{\Delta t},$$

where $\xi \sim N(0,1)$. This process may give however rise to a negative variance, which does not make any sense in practice. Generally, the method is combined with truncation in two ways. A negative variance is either set equal to zero or the absolute value. Both ways will give rise to some bias in the final result after a large number of simulations. However, another discretization method exists which can be used to alleviate the negative variance problem to some extent.

The Milstein scheme provides an improved discretization method in terms of convergence and handles the negativity problem. By deriving the higher order derivatives in the Taylor expansion of $\upsilon(t+\Delta t)$, we obtain

$$\upsilon(t+\Delta t) = \upsilon(t) + \kappa(\overline{\upsilon} - \nu(t))\Delta t + \gamma\sqrt{\upsilon(t)}\xi\sqrt{\Delta t} + \frac{1}{4}\gamma^2\Delta t(\xi^2 - 1),$$

which can be rewritten as

$$\upsilon(t+\Delta t) = (\sqrt{\upsilon(t)} + \frac{\gamma}{2}\sqrt{\Delta t}\xi)^2 + \kappa(\overline{\upsilon} - \nu(t))\Delta t - \frac{1}{4}\gamma^2\Delta t.$$

This formula shows us that if we define $\upsilon(t) = 0$, and choose parameters that satisfy $\kappa\overline{\upsilon} - \frac{1}{4}\gamma^2 > 0$, we will have $\upsilon(t+\Delta t) > 0$. This will reduce the frequency of occurrence of negative variances. It can be easily determined that the Milstein and Euler discretizations are computationally equally expensive. In practice, when the specific condition ($\kappa\overline{\upsilon} - \frac{1}{4}\gamma^2 > 0$) is satisfied, experience has shown us that the Milstein discretization performs better than the Euler scheme. In order to make the Monte Carlo pricing engine more stable and less biased, we prefer the use of the Milstein scheme.

We employ the parameters from earlier experiments to see the impact of the discretization method. The Monte Carlo test is based on 10000 simulations with antithetic sampling to reduce the variance. The benchmark price is obtained by the COS pricing method. As expected, the Milstein scheme gives smaller errors compared to the Euler scheme.

Absolute Error	M=64	M=128	M=256	M=512
Euler	0.028	0.0124	0.0066	0.0048
Milstein	0.0189	0.0044	0.0023	0.0013
Time(Seconds)	M=64	M=128	M=256	M=512
Euler	1.0223	2.0373	4.0643	8.1816
Milstein	1.0530	2.0778	4.1218	8.2321

Table 7.2: Milstein convergence

7.1.3 Feller condition for Milstein scheme

The condition for the Milstein scheme $\kappa\overline{\upsilon}-\frac{1}{4}\gamma^2>0$ is very similar to the popular Feller condition $\kappa\overline{\upsilon}-\frac{1}{2}\gamma^2>0$ regarding the positivity of the variance process. The Feller condition is crucial when simulating the variance process with long maturities. The frequency of negative variance values can be significantly reduced when the Feller condition is satisfied. More specifically, according to the implicit scheme proposed by Alfonsi (2005),

$$\begin{aligned}\upsilon(t+\Delta t) &= \upsilon(t)+\kappa(\overline{\upsilon}-\nu(t))\Delta t+\gamma\sqrt{\upsilon(t)}\xi\sqrt{\Delta t}\\ &= \upsilon(t)+\kappa(\overline{\upsilon}-\nu(t+\Delta t))\Delta t+\gamma\sqrt{\upsilon(t+\Delta t)}\xi\sqrt{\Delta t}\\ &\quad -\gamma(\sqrt{\upsilon(t+\Delta t)}-\sqrt{\upsilon(t)})\xi\sqrt{\Delta t}+\mathit{higher\ order\ term},\end{aligned}$$

it can be found that

$$\sqrt{\upsilon(t+\Delta t)}-\sqrt{\upsilon(t)})=\xi\sqrt{\Delta t}/2+\mathit{higher\ order\ term}.$$

Further substitution gives the implicit discretization

$$\begin{aligned}\upsilon(t+\Delta t) &= \upsilon(t)+\kappa(\overline{\upsilon}-\nu(t+\Delta t))\Delta t+\gamma\sqrt{\upsilon(t+\Delta t)}\xi\sqrt{\Delta t}\\ &\quad -\xi\sqrt{\Delta t}/2.\end{aligned}$$

Hence,

$$\sqrt{\upsilon(t+\Delta t)}=\frac{\sqrt{4\upsilon(t)+\triangle t[(\kappa\overline{\upsilon}-0.5\gamma^2)(1+\kappa\triangle t)+\gamma^2\xi^2]}+\gamma\xi\sqrt{\Delta t}}{2(1+\kappa\triangle t)}.$$

If $2\kappa\overline{\upsilon}>\gamma^2$, we are guaranteed to have a real-valued root of the expression, which implies that the variance process is positive at all times.

This Feller condition is critical for the parameters in the Heston model. In practice, the condition can be violated. As a result, the Euler discretization may lead to a significant Monte Carlo bias. When we apply the Milstein discretization, the situation improves and is less critical. It is easy to see the connection between the two parameter sets:

$$\{(\kappa, \overline{v}, \gamma) \in A | 2\kappa\overline{v} > \gamma^2\} \subsetneqq \{(\kappa, \overline{v}, \gamma) \in A | 4\kappa\overline{v} > \gamma^2\} \subsetneqq A,$$

where A represents the usual parameter sets. We therefore calibrate the Heston model based on the larger constrained subset. For the calibration here, it is essential to restrict the parameters so that the Feller condition is satisfied to reduce the error of our Monte Carlo method. That's another reason why we employ the Milstein scheme especially for its relatively more moderate condition $4\kappa\overline{v} > \gamma^2$ compared to the Feller condition.

7.2 Basket put option

A basket option is an exotic option whose underlying is a weighted sum or average of different assets. Here we study particularly the index options. As introduced in Chapter 2, the Variable Annuity is similar to the European basket put option. In the basic case, the payoff at maturity of the contract is the maximum of the account value and the guaranteed level. If we choose the guaranteed level to be constant, this contract is a basket put option since the option payoff will be positive only when the account value is less than the guaranteed level. The payoff at maturity reads

$$\max(Guarante, Account) = \max(G - A_T, 0) + A_T,$$

where $A_T = \sum_{i=1}^{Number} w_i S_{iT}$ is the Account value at maturity, S_{iT} is the ith index value in the basket at T and w_i represents the weight. And G is the constant guarantee. According to the no arbitrage pricing theory, the fair value of this put option is the discounted payoff with respect to the risk-free interest rate. To make this somewhat more complicated, the index can be chosen globally. Since the guarantee is in a local currency, the foreign index in the domestic currency becomes $S_{iT}FX_i$.

It can be shown from the calibrated results of the Heston model that this model can fit the implied smile very well, especially for long maturities. However, a stochastic volatility model alone seems not enough for interest rate sensitive products. The impact on the final reserves of a stochastic interest rate can be found from simple testing based on two models: the pure Heston and the Heston-Hull-White model. In this basket, we choose one domestic stock index and one foreign stock index, which means that only one currency exists. In order to compare the two models, we first choose the parameters for the Heston-Hull-White model. Then, we will use the Hull-White formula to obtain the zero rate of the zero-coupon bond, as the constant interest rate for the Heston model.

For the simulation with the Monte Carlo method, we usually have one system with many Stochastic Differential Equations (SDEs) and these SDEs should be consistent with the probability measure. In this project, the domestic spot risk neutral measure is preferred since the only transformation which then needs to be done is with the dynamics of the foreign stock and its volatility. Generally, the dynamics under each measure are as follows:

$$\begin{cases} \frac{dS_d(t)}{S_d(t)} = r_d(t)dt + \sqrt{\upsilon_d(t)}dW^{\mathbb{Q}}_{S_d}(t), \\ dr_d(t) = \lambda_d(\theta_d(t) - r_d(t))dt + \eta dW^{\mathbb{Q}}_{r_d}(t), \\ d\upsilon_d(t) = \kappa_d(\overline{\upsilon_d} - \upsilon_d(t))dt + \gamma_d\sqrt{\upsilon_d(t)}dW^{\mathbb{Q}}_{\upsilon_d}(t), \\ \frac{dFX(t)}{FX(t)} = (r_d(t) - r_f(t))dt + \sqrt{\sigma_{FX}(t)}dW^{\mathbb{Q}}_{FX}(t), \\ d\sigma_{FX}(t) = \kappa_{FX}(\overline{\sigma_{FX}} - \sigma_{FX}(t))dt + \gamma_{FX}\sqrt{\sigma_{FX}(t)}dW^{\mathbb{Q}}_{\sigma_{FX}}(t), \\ \frac{dS_f(t)}{S_f(t)} = r_f(t)dt + \sqrt{\upsilon_f(t)}dW^{\mathbb{Z}}_{S_f}(t), \\ dr_f(t) = \lambda_f(\theta_f(t) - r_f(t))dt + \eta dW^{\mathbb{Z}}_{r_f}(t), \\ d\upsilon_f(t) = \kappa_f(\overline{\upsilon_f} - \upsilon_f(t))dt + \gamma_f\sqrt{\upsilon_f(t)}dW^{\mathbb{Z}}_{\upsilon_f}(t). \end{cases} \tag{7.2.1}$$

Here, S_i, r_i, $\upsilon_i(i = d, f)$ indicate the domestic and foreign stock and its own risk-neutral interest rate and the variance followed by the Heston-Hull-White process. $\mathbb{Q}$ and $\mathbb{Z}$ are the risk-neutral spot measures for the individual stocks. The currency part is also following the Heston-Hull-White dynamics, where FX is the spot price and σ_{FX} is its variance. All the correlations are non-zero except the ones between the interest rate and volatility. More details on this can be found in Chapter 5.

Unlike our approach in Chapter 5, here we want to use the spot measure to reduce the complexity. We will follow Grzelak and Oosterlee's work to make a uniform measure. We will focus on the domestic-spot risk neutral measure-$\mathbb{Q}$. With the closed-form dynamics of the foreign risk-free interest rate r_f obtained in Chapter 5, we will refer to the result from section 2.4 of [GO10] to get the dynamics of the foreign stock and its variance as follows:

$$\begin{cases} \frac{dS_f(t)}{S_f(t)} = (r_f(t) - \rho_{FX,S_f}\sqrt{\upsilon_f(t)}\sqrt{\sigma_{FX}(t)})dt + \sqrt{\upsilon_f(t)}dW^{\mathbb{Q}}_{S_f}(t), \\ dr_f(t) = (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{FX,f}\sqrt{\sigma(t)})dt + \eta_f dW^{\mathbb{Q}}_f(t), \\ d\upsilon_f(t) = [\kappa_f(\overline{\upsilon_f} - \upsilon_f(t)) - \rho_{S_f,\upsilon_f}\rho_{FX,S_f}\gamma_f\sqrt{\upsilon_f(t)}\sqrt{\sigma_{FX}(t)}]dt + \gamma_f\sqrt{\upsilon_f(t)}dW^{\mathbb{Q}}_{\upsilon_f}(t). \end{cases}$$

Now we can start the Monte Carlo simulation to obtain the following values for simple basket put options with maturity T :

$$E^{\mathbb{Q}}[\exp(-\int_0^T r_{d_s}ds) \cdot \max\{G - w_d S_d(T) - w_f S_f(T)FX(T), 0\}].$$

In the world of Heston, the dynamics are almost the same except for the deterministic domestic and foreign risk-free interest rate. The pure Heston model with the interest rate including the term-structure is sufficient to capture the implied volatility smile for long maturities. This will shown in the section 6.3. However, the structure of Variable Annuities indicates that the impact of a stochastic interest rate can not be neglected. In order to check that, we propose some numerical tests to compare the Heston-Hull-White and Heston models to see the impact. The parameters of both domestic stock and foreign stock are in Table 7.3.

	IR		Vol		EQ	
Domestic Stock	mean reversion	0.05	kappa	5	sd	100
	vol of IR	0.005	gamma	0.5751	corr(sd,FX)	0.4
	corr(sd,rd)	20%	mv	0.0398	corr(sd,sf)	0.1
	rd(0)	0.01	v(0)	0.0175		
	term-structure	0.01	correlation	-0.5711		
Foreign Stock	mean reversion	0.02	kappa	5	sf	100
	vol of IR	0.005	gamma	0.7	corr(sf,FX)	0.3
	corr(sf,rf)	40%	mv	0.08	corr(sf,sd)	0.1
	rf(0)	0.05	v(0)	0.02		
	term-structure	0.05	correlation	-0.79		
Exchange rate	corr(rd,FX)	0.5	kappa	5	FX(0)	0.7
	corr(rf,FX)	0.3	gamma	0.4		
	corr(rd,rf)	0.25	mv	0.01		
			v(0)	0.01		
			correlation	0.211		

Table 7.3: HHW parameters

The weights are chosen as: $\omega_d = 0.5, \omega_f = 0.5$. In order to compare with pure Heston model, the parameters from the interest rate can be used to compute the zero rates, that can serve as the constant risk-free interest rate for the Heston model.

In the Hull-White world, the zero-coupon bond price is:

$$P(t,T) = \exp[A_r(t,T) - B_r(t,T)r(t)],$$

where

$$\begin{aligned} B_r(t,T) &= \frac{1-e^{-\lambda(T-t)}}{\lambda}, \\ A_r(t,T) &= \frac{\sigma^2}{2} \int_t^T B_r^2(s,T)ds - \int_t^T \overline{\theta}(s)B_r(s,T)ds. \end{aligned}$$

So, the instantaneous short rate at T is the log-transform of the zero-coupon bond price divided by T. It can be analytically obtained from the Hull-White parameters, mean reversion, volatility of the IR and term-structure. The basket put option price from the two models are shown in Table 7.4. It is obtained by 50 Monte Carlo simulations with 10000 paths and 1000 time steps. We also apply the technique of antithetic sampling and use the Milstein scheme to make sure the Monte Carlo price is sufficiently accurate. When the maturity varies from 5 years to 50 years, the absolute difference from two models grows, as expected. The Heston-Hull-White price is higher because the total variance is a combination of two types of variance in the case of positive correlations which we already showed in section 6.4.

$$\sigma_{total}^2 = \sigma_{EQ}^2 + \sigma_{IR}^2 + 2\rho_{EQ,IR}\sigma_{EQ}\sigma_{IR}.$$

T	HHW	std dev	Heston	std dev	HHW-Heston
5	15.0531	0.1964	13.6067	0.1932	1.4464
10	20.0155	0.253	18.2624	0.2196	1.7531
15	23.1994	0.2565	21.1493	0.2303	2.0501
20	25.461	0.3172	23.3356	0.2244	2.1254
25	27.4779	0.2581	25.0297	0.2648	2.4482
30	29.0263	0.3894	26.4792	0.2634	2.5471
35	30.2525	0.3596	27.6074	0.2469	2.6451
40	31.3738	0.4037	28.6135	0.2463	2.7603
45	32.35	0.3968	29.4096	0.2798	2.9404
50	33.1641	0.3999	30.1143	0.2147	3.0498

Table 7.4: Impact of stochastic interest rate

We also show the impact of incorporating the stochastic interest rate in Figure 7.2.1.

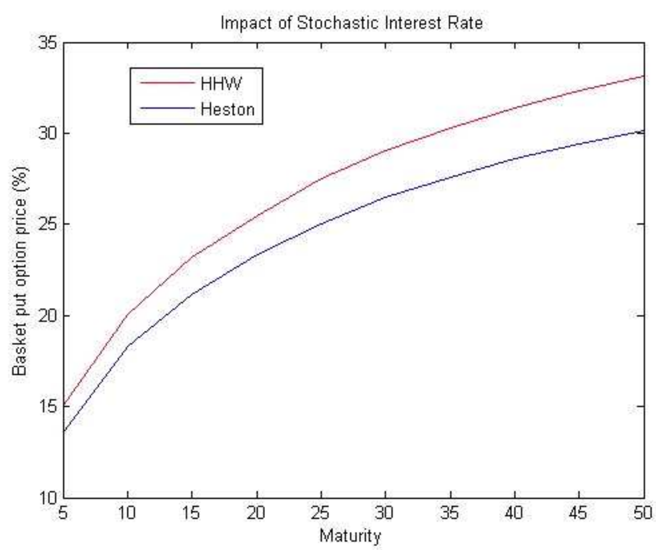

Figure 7.2.1: Impact of stochastic interest rate

As we can see from the formula, the total variance is an increasing function of interest rate volatility when the correlation between the interest rate and equity is positive. For some more about detail, we choose the weight in the basket as follows $\omega_d = 1, \omega_f = 0$. This is to make sure that there is only one index in the basket. In this case it is much easier to see the impact of the interest volatility. In Figure 7.2.2, we can easily see the results of the increasing interest volatility from 0% to 1%. Here, it is again proved that the impact of a stochastic interest rate can not be neglected especially for long maturities. As a result, for long-term options such as Variable Annuities, it is strongly preferred to use a stochastic interest rate in combination with a stochastic volatility.

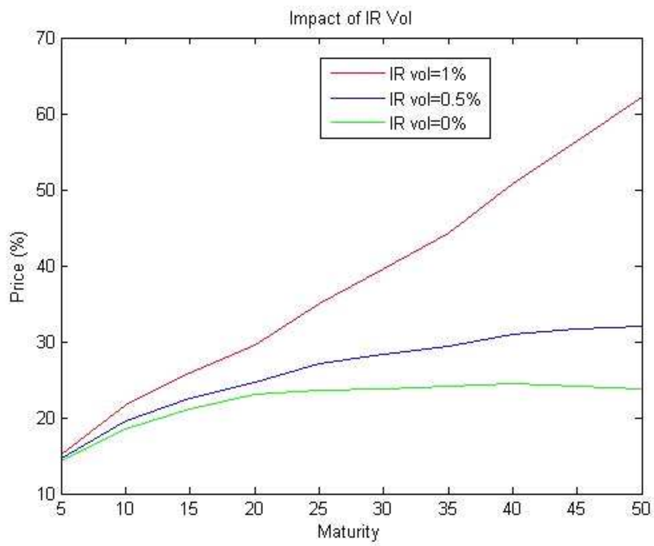

Figure 7.2.2: Impact of IR volatility

7.3 Variable Annuity scenario generation

For the use of risk management, millions of scenarios need to be simulated to evaluate some statistical properties for particular assets' behavior. It can be seen that the dimension of the system (7.2.1) for two assets is 8, and the pricing accuracy will strongly depend on the power of the numerical method and the computer. In this project, our aim is to generate the scenarios for six stock indices in their own currencies.

Consider the situation when the local currency is the Euro. Three of the stock indices are Euro indices. The other three are foreign indices which means that we need to simulate three currencies in total. A basic model like the Black-Scholes model with term-structure will significantly improve the speed of scenario generation. In fact, this model is still most frequently used when pricing Variable Annuities in the insurance sector. Now that we introduced the Heston-Hull-White model in our project, we need to quantify the impact on the final reserves when changing from the Black-Scholes to the Heston-Hull-White model. Before that, another interesting topic arising is to determine an accurate correlation matrix.

For finance institutioner's use, correlation must be accurate but also the matrix should obey certain properties. More precisely, the correlation matrix should be Symmetric-Positive-Definite (SPD). However, in reality the correlations among all the factors are determined from historical data, which cannot guarantee that the final resulting matrix is SPD. In this project, we use a simple technique to deal with this topic. The general idea is to replace a potential negative eigenvalue from the original correlation matrix by a small positive value and then to recover the matrix by a simple transformation. The correlation matrix is organized as follows: the correlations between indices, currencies and interest rates are computed by using the historical data. The correlation between interest rate and volatility is set to zero. The only correlations needed from the calibration are the ones between the indices and their volatilities and the ones between currencies and their volatilities. After that, we make a transformation to make it SPD. Some historical correlation results on November 30th, 2010 are shown in the appendix.

In order to satisfy the Milstein Feller condition, the scenarios are generated with one extra constrain. The calibration results on November 30th, 2010 are shown in Table (7.5) and (7.6).

	mean reversion	vol of vol	long variance	correlation	initial variance
SX5E	0.5403	0.3478	0.1026	-0.9996	0.0922
SP500	1.8119	0.2298	0.0607	-0.9882	0.0569
AEX	1.4772	0.5026	0.0855	-0.9619	0.0624
FTSE	1.0611	0.4400	0.0912	-0.9490	0.0659
IBEX	0.6196	0.4158	0.1394	-0.8919	0.1110
TOPIX	1.9369	0.6686	0.0744	-0.8255	0.0545
GBPEUR	0.0964	0.0278	0.0559	-0.1135	0.0136
JPYEUR	0.1056	0.2388	0.1735	-0.4636	0.0294
USDEUR	0.3319	0.0985	0.0077	0.2940	0.0294

Table 7.5: Calibration of variance

	vol	mean reversion
EURSWAP	0.99%	2.12%
GBPSWAP	1.03%	4.90%
JPYSWAP	0.59%	0.01%
USDSWAP	1.25%	3.42%

Table 7.6: Calibration of interest rate

With all the inputs ready, the scenarios can be generated by the Monte Carlo simulations for 10 years with 10000 paths and monthly time steps. These scenarios are crucial for pricing the GMxB products discussed in Chapter 2.

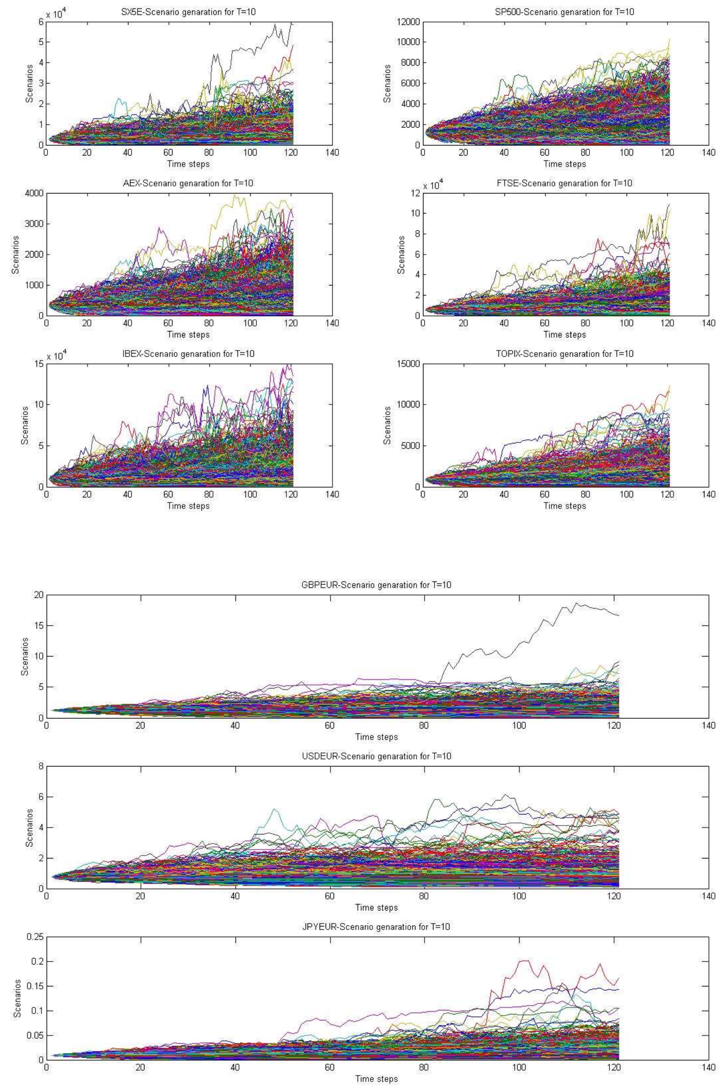

Figure 7.3.1: Scenarios

7.4 Valuation of GMWB

Scenarios are key to pricing and hedging Guaranteed Minimum Benefits products. Here we focus on pricing GMWBs. The GMWB promises a periodic payment (coupon) - regardless of the performance of the underlying policy. The investor can participate in market gains, but still has a guaranteed cash flow in the case of market losses. Even if the policy-holder's account value drops to zero as the annuitant makes withdrawals during the pay-out phase, the annuitant will be able to continue making withdrawals for the duration of the specified period or until the total amount of withdrawals adds up to a specified maximum lifetime amount. The GMWB is often used for retirement income protection. In 2007, 43% of all variable annuities sold in the US included a GMWB type option (including a lifetime benefit). As mentioned, this type of option is similar to a basket put option with long maturities but has notable differences in the combination of financial risk and insurance risk. A policy-holder's behavior may dramatically affect the cost of GMWBs in the real world.

Tables 7.7 & 7.8 provide a simple numerical example of the payoff for a GMWB rider, assuming a particular sequence of yearly investment returns for a typical Variable Annuity policy. Suppose the investor pays 100 Euros to an insurance company, which is invested in a risky asset. The contract will run for 10 years and the guaranteed withdrawal rate is 10 Euros per year. This is to make sure that the investor can at least have all the initial investment back at the end of the contract. The example assumes scenarios of good and bad returns of the market independently. It can be seen if the market does very well for these 10 years, the total withdrawal amount can go up to 455.3781 Euros. Even when the market goes down, the GMWB promises at least 100 Euros to the investor.

Year	Return (%)	Balance	Withdrawal
1	27.02	127.0200	10
2	19.48	139.8155	10
3	64.08	213.0013	10
4	5.42	214.0039	10
5	8.79	221.9359	10
6	11.02	235.2912	10
7	34.14	302.2056	10
8	-22.42	226.6931	10
9	24.66	270.1297	10
10	40.46	365.3781	10

Table 7.7: GMWB of good returns

Year	Return (%)	Balance	Withdrawal
1	-5.68	94.3200	10
2	-14.51	72.0852	10
3	-35.48	40.0574	10
4	32.32	39.7719	10
5	-1.44	29.3432	10
6	-22.52	14.9871	10
7	15.48	5.7591	10
8	-31.07	0	10
9	-17.47	0	10
10	-1.14	0	10

Table 7.8: GMWB protection of bad returns

The example shows us the protection effect of a GMWB. Typically, the GMWB carries an annual fee of 40-60 basis points as a percentage of the sub-account value. In this case, the withdrawal rate is 10% annually as a fixed percentage of the premium. This is called the static withdrawal policy. In a more complicated construction, the policy-holder can withdraw at a stochastic rate usually not greater than 7%. Other important issues arise due to the policy-holder's behavior, which may include death and lapse risk. These kinds of external features need to be incorporated in the valuation process of GMWBs, which will make the price cheaper.

The valuation of a GMWB is based on non-arbitrage pricing with an initial amount of money invested in a basket of assets. As described in Chapter 2, the dynamics of the asset without any underlying GMWB protection would be:

$$dS_t = rS_t dt + \sigma S_t dB_t^{\mathbb{Q}}.$$

The sub-account values should incorporate two additional effects, the proportional insurance fees q and withdrawal rate G. Therefore, the sub-account value of a GMWB would be in the following form.

$$dW_t = (r - q)W_t dt - Gdt + \sigma W_t dB_t^{\mathbb{Q}}.$$

If W_t ever reaches zero, it will remain zero to the maturity time. Consequently, the payoff of a GMWB is a collection of residual sub-account values at maturity and guaranteed withdrawals, i.e. the value of the GMWB reads:

$$GMWB = inforce * \{P(0,T)E^{\mathbb{Q}}[W_T|\mathcal{F}_0] + \int_0^T P(0,t)Gdt\}.$$

The $inforce$ is the combination rate of survival and lapse.

In this case, one basket of underlying assets is focused upon. The scenarios of all the assets in the basket are generated based on the stochastic model as shown before with multi-asset Monte Carlo simulations. The returns of the account value are simulated by the stochastic scenarios. We make the assumption that the annual fair fee is 50 basis points, which is deducted from the account value at the end of each year. For comparison purposes, we assume that the annual withdrawal rate is constant at 5%. The fair value of the embedded GMWB put option is calculated based on a maturity of T years, varying here from 5 to 20. In order to take the insurance risk into account, we will use some data of a life table in a particular area and the historical lapse rate. The impact of the stochastic model can be determined when comparing it with the basic Black-Scholes model. The Black-Scholes model performs well under particular assumptions.

We explore the pricing behavior of the GMWB with respect to two models. The result presented in Table 7.10 is the GMWB options value versus time. Suppose the age of the clients is 40 and the scenarios are based on a basket with two assets, the SX5E and FTSE. This choice is to make sure that the correlation between the European interest rate and the assets of the basket ($SX5E + GBPEUR * FTSE$) is positive. Under this assumption, the Heston-Hull-White gives a higher price than Black-Scholes model. We also compare the two corresponding prices graphically in Figure 7.4.1. It is shown in this figure that the impact of stochastic models can be significant when the maturities of the contract increase from 5 to 20 years.

T	Black-Scholes	Heston-Hull-White
5	0.0011	0.0387
6	0.0122	0.1060
7	0.0814	0.2551
8	0.2023	0.4921
9	0.4943	0.9067
10	0.9471	1.4055
11	1.5512	2.0765
12	2.2544	2.8359
13	3.2107	3.7238
14	4.1250	4.5461
15	5.2274	5.7371
16	6.3965	6.8187
17	7.6463	8.0800
18	8.8430	9.2782
19	9.9618	10.4493
20	11.1124	11.7096

Table 7.9: GMWB option price

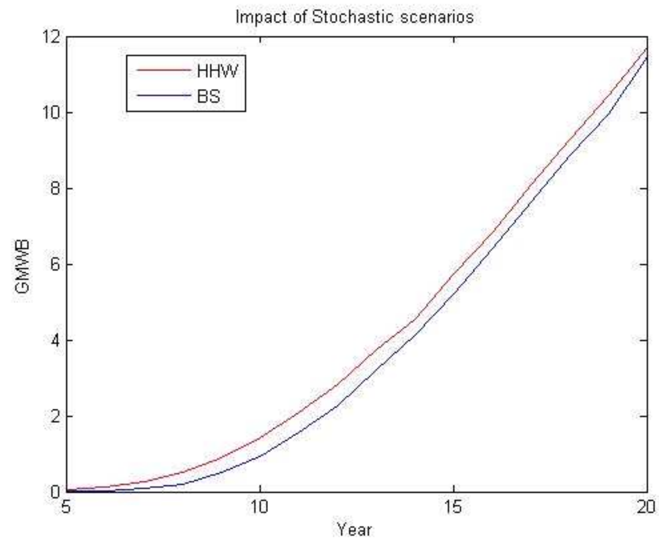

Figure 7.4.1: Impact of stochastic scenarios

This example is to show the impact of the use of a stochastic model under the assumption of a constant annual withdrawal rate of 5% for all the maturities. The GMWB option price increases with time, as expected. As seen in this plot, the use of the Black-Scholes model, which can not reproduce the overall volatility of the Heston model, will tend to under-price the contract, especially for longer maturities. This observation strongly motivates us to use Heston-Hull-White model for the modeling framework. Although these remarks are based on a GMWB product, they are expected to be valid for other GMxB products.

Chapter 8

Conclusion

In this thesis, we have studied the valuation of Variable Annuities based on the combined Heston and Hull-White model. This stochastic hybrid model has a significant impact on the embedded option price of a VA. For long-term options, such as VAs, the assumptions of stochastic volatility and interest rate are more in line with the market practice and can better capture the interest rate and equity risk. The key to realizing the application of Heston-Hull-White models for Variable Annuities is the power of the COS pricing method in calibration. It is one of the state-of-the-art numerical integration methods based on the Fourier technique, discussed in Chapter 3. The characteristic function of an affine model, such as the Black-Scholes or Heston model is analytically obtained, and subsequently used for pricing European options. We performed some tests to compare the COS pricing method with an FFT-based method and direct numerical integration. These two methods have already proved in the literature to be much better than Monte Carlo simulation for calibration. In terms of time consumed and accurate pricing, the COS method is strongly preferred. Additionally, the COS pricing method is easy to implement.

For a basket put option containing domestic stock and foreign stock, stochastic modeling refers to equity and FX, respectively. There is a notable difference between the equity-HHW model and the FX-HHW model. For the exchange rate, both domestic and foreign stochastic interest rates are considered, which means that the dimension of the FX-HHW hybrid model is four. We gave a full discussion in Chapters 4 and 5 about how to determine the characteristic function in approximate version of the two models. The same techniques are applied however for the two models. We used the expectation of the volatility as an approximation of a stochastic term to make both models affine. For affine models, the standard techniques can be employed to get the final expressions for the characteristic functions. The quality of the approximations can be observed from the numerical examples in Chapters 4 and 5. We presented the Monte Carlo price based on full-scale models as the benchmark price for the approximate hybrid model prices obtained by the COS method. This way we tested our approximate characteristic function to see the impact of the approximations made. The tables and figures confirm that this approximation can give us highly satisfactory answers. Since the COS method is much less time-consuming, the approximate affine hybrid models for equity and FX can be implemented within the calibration process.

Calibration is involved, especially for more complex models, such as the Heston and Hull-White model. Even though many scholars propose new calibration methods, this process may still take substantial time. It is even more difficult when we wish to have stable parameters for a certain period of time. The market implied approach strongly depends on analytic or semi-analytical pricing formulas. However, we have derived those in Chapter 4 and 5. We tried to fit the implied volatility smile for both equity and FX, as is observed in the market. The Heston and Heston-Hull-White models perform fine when fitting this smile, as confirmed by the calibration results in Chapter 6. It is hard to say which is model is better when fitting data. As long as the calibration can make the distance between the theoretical model and real market data as small as possible, a model can be regarded as robust. Nevertheless, the main motivation to use the Hull-White one-factor short rate process is the long maturity of VA contracts. The Hull-White model can be calibrated independently by using the interest rate curve and swaption data. Its calibrated results are stable and reasonable as shown in Chapter 6. After this robust calibration, the parameters for both models have been obtained. With the parameters fixed, we have started the multi-asset Monte Carlo engine to price the real VA contracts.

The Monte Carlo simulation costs a significant amount of CPU time and the variance can be an increasing function of the maturity. Variance reduction techniques, such as antithetic sampling, have been discussed in section 1 of Chapter 7. By changing the sign of random samples, we can reduce the variance significantly with the same number of simulations. The numerical results also show us that the Milstein scheme will improve the convergence compared to the Euler scheme. We paid particular attention to the Feller condition for the variance process, and we proposed a specifically tailored calibration method to reduce the calibration error and, at the same time, to satisfy the Milstein Feller condition. We saw some preliminary pricing results for basket put option in section 2 of chapter 7. The financial portfolio was composed of one domestic stock and one foreign stock. The impact of a stochastic interest rate for long maturities was clearly observed during the numerical test, comparing the pure Heston and the Heston-Hull-White models. Last but not least, we generated all the scenarios for Variable Annuities composed of 6 stock indices and 3 currencies on November 30th, 2010. Then we performed a final comparison between the Heston-Hull-White and Black-Scholes models during the valuation of Guaranteed Minimum Withdrawal Benefits. The reason for choosing a GMWB contract is because this option is somewhat different from the basic basket put option, for which we already shown that the impact of stochastic models can not be neglected. The last tables and figures of this thesis convinced us to use the hybrid stochastic model for long-term exotic options.

Future research

The performance of Monte Carlo can be improved when the Feller condition is not satisfied. In this thesis, we use the results from the conditioned calibration to give a guarantee that the variance process never reach negative. The further investigation should aim to deal with the situation of normal calibration process and improved Monte Carlo simulations. One of the possible solutions is to apply the discretization schemes other than Euler and Milstein during the Monte Carlo method.

The problem of how to specify a correlation matrix occurs in several important areas of finance and of risk management. To make this matrix symmetric-positive-definite without big bias from the original correlation data, better methods are still needed. Another interesting topic about the future work is the application of GPU (graphic processing unit) in the process of scenario generation, which is usually computational time-consuming in practice. GPU is a specialized circuit designed to rapidly manipulate and alter memory in certain ways. It can provide additional assistance for the combined system of calibration and scenario generation.

Appendix: market data

	SX5E	SP500	AEX	FTSE	IBEX	TOPIX	GBPEUR	USDEUR	JPYEUR
SX5E	1.00	0.82	0.94	0.90	0.91	0.46	0.06	-0.36	-0.59
SP500	0.82	1.00	0.80	0.84	0.73	0.37	-0.02	-0.42	-0.57
AEX	0.94	0.80	1.00	0.91	0.82	0.49	0.11	-0.30	-0.55
FTSE	0.90	0.84	0.91	1.00	0.80	0.41	-0.07	-0.34	-0.55
IBEX	0.91	0.73	0.82	0.80	1.00	0.45	0.00	-0.41	-0.56
TOPIX	0.46	0.37	0.49	0.41	0.45	1.00	0.01	-0.22	-0.41
GBPEUR	0.06	-0.02	0.11	-0.07	0.00	0.01	1.00	0.30	0.00
USDEUR	-0.36	-0.42	-0.30	-0.34	-0.41	-0.22	0.30	1.00	0.65
JPYEUR	-0.59	-0.57	-0.55	-0.55	-0.56	-0.41	0.00	0.65	1.00

Table 1: Correlation between Equities and FX

	SX5E	SP500	AEX	FTSE	IBEX	TOPIX	GBPEUR	USDEUR	JPYEUR
EURSWAP	0.41	0.31	0.38	0.31	0.38	0.32	0.10	-0.23	-0.50
GBPSWAP	0.35	0.31	0.38	0.25	0.32	0.31	0.36	-0.14	-0.49
JPYSWAP	0.33	0.26	0.33	0.29	0.29	0.40	0.17	-0.01	-0.26
USDSWAP	0.32	0.19	0.32	0.24	0.24	0.18	0.31	0.14	-0.34

Table 2: Correlations between Equities, FX and Interest Rates

	EURSWAP	GBPSWAP	JPYSWAP	USDSWAP
EURSWAP	1.00	0.77	0.52	0.61
GBPSWAP	0.77	1.00	0.47	0.64
JPYSWAP	0.52	0.47	1.00	0.5
USDSWAP	0.61	0.64	0.5	1.00

Table 3: Correlation between Interest Rates

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