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Chapter 8

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ABSTRACT

Assessing uncertainties in measurements must become a standard practice in the field of urban drainage and stormwater management. This chapter presents three standard methods to estimate uncertainties: the Type A method (repeated measurements), the Type B method (law of propagation of uncertainties) and the MC method (Monte Carlo method). Each method is described with its fundamental principles and equations, various examples are presented in detail and Matlab® codes are given to facilitate the calculations for routine applications. An advanced method to account for partial autocorrelation in time series is presented. Lastly, typical orders of magnitude of standard uncertainties for usual sensors used in urban drainage and stormwater management are given.

Keywords: Coverage interval, error, guide for uncertainty in measurements, law of propagation of uncertainties, Monte Carlo method, standard uncertainty.

SYMBOLS

a	low boundary of an interval or index of the iterative calculation of δ or numerical coefficient
A	matrix containing data related to constant quantities
b	high boundary of an interval or vector of coefficients b_i or numerical coefficient
b_i	regression coefficients
B	channel width (m)
B_c	notch width (m)
B_e	effective width (m)
c	integer value used to calculate δ
c_i	sensitivity coefficient related to the quantity x_i in the measurement function f
C_d	discharge coefficient (-)
COV	covariance matrix
d_{low}	distance between low boundaries of Type B and MCM coverage intervals
d_{high}	distance between high boundaries of Type B and MCM coverage intervals
D	pipe diameter (m)
e_{fs}	numerical factor to calculate V from V_{fs} (-)
e_{max}	numerical factor to calculate V from V_{max} (-)
f	function of quantities x_i representing the measurement process
fc	as index: full autocorrelation
F_y	numerical coefficient for velocity-area methods
F_z	numerical coefficient for velocity-area methods
g	gravity (m/s^2)
h	water level (m)
h_e	effective head (m)
h_p	crest height (m)
i	index
I	slope of a channel or a pipe (m/m)
I_{est}	estimated rainfall intensity (mm/h)
I_m	measured rainfall intensity (mm/h)
I_r	reference rainfall intensity (mm/h)
IC_{95min}	shortest 95% coverage interval calculated with the Monte Carlo method
j	index
J	smallest integer greater than or equal to $100/\alpha$
k	coverage factor
K	Manning-Strickler coefficient (m$^{1/3}$/s)
K_b	correction factor in the calculation of Q_{RW} (-)
K_h	correction factor in the calculation of Q_{RW} (-)
l	integer value used to calculate δ
L	distance upstream a weir where the water level is measured (m)
m	mean value of a normal distribution or number of quantities in the matrix Z
M	number of Monte Carlo simulations
MC	as index: refers to the Monte Carlo method
n	number of repeated measurements in the Type A method
nc	as index: no autocorrelation

n_{dig}	number of significant digits used to calculate δ
N	number of quantities x_i used in the function f
$N(m,s)$	normal (Gaussian) probability distribution with mean value m and standard deviation s
p	number of quantities in the matrix A
pc	as index: partial autocorrelation
q	integer used for estimating the narrowest coverage interval in Monte Carlo simulations
Q	discharge (m^3/s)
Q_c	discharge in a circular pipe (m^3/s)
Q_{MS}	discharge calculated with the Manning-Strickler formula (m^3/s)
Q_p	perimeter flow for velocity-area methods (m)
Q_{RW}	discharge over a rectangular weir (m^3/s)
r	coefficient of correlation
r	as index: index of Monte Carlo simulations
$r(x_i, x_j)$	coefficient of correlation of x_i and x_j
r_{ij}	coefficient of correlation of x_i and x_j
R_c	circular pipe radius (m)
R_h	hydraulic radius (m)
$s(y)$	standard deviation of y
S	wet cross section (m^2)
t	Student t value
TB	as index: refers to the Type B method
$Trap(a,b,\beta)$	trapezoidal probability distribution in the interval $[a,b]$ with the coefficient β
$Tri(a,b)$	triangular probability distribution in the interval $[a,b]$
$u(x_i, x_j)$	covariance of x_i and x_j
$u(Y)$	standard uncertainty of Y
$u^*(Y)$	relative standard uncertainty of Y
$U(a,b)$	uniform probability distribution in the interval $[a,b]$
$U(Y)$	enlarged uncertainty of Y
v	flow velocity at a given position within a wet cross section (m/s)
V	cross section mean flow velocity (m/s)
V_d	daily volume (m^3)
V_{fs}	free surface flow velocity (m/s)
V_{max}	maximum flow velocity (m/s)
x_i	quantities used in the measurement function f to calculate y
X_i	random variable corresponding to the quantity x_i
$\bar{y}$	mean value of y
Y	measured or calculated quantity
$Y_{\alpha,low}$	low boundary of a coverage interval for Y calculated by the Monte Carlo method for a level of confidence α
$Y_{\alpha,high}$	high boundary of a coverage interval for Y calculated by the Monte Carlo method for a level of confidence α
z	matrix containing data related to time varying quantities (time series)
α	level of confidence
β	numerical coefficient of a trapezoidal probability distribution
δ	tolerance

Δt	time step (s)
ε_i	finite difference used in the 2nd order approximation of c_i
ν	degree of freedom
ν_{eff}	effective degree of freedom

8.1 INTRODUCTION

Why is uncertainty assessment important and should be systematically done? In urban drainage and stormwater management (UDSM), like in numerous other professional fields and disciplines, information, knowledge, performance analysis, modelling, scenario analysis, planning and decision making are based on or use measurement results. However, measurements are never perfect and cannot be carried out without uncertainties. Consequently, 'when reporting the result of a measurement of a physical quantity, it is obligatory that some quantitative indication of the quality of the result be given so that those who use it can assess its reliability. Without such an indication, measurement results cannot be compared, either among themselves or with reference values given in a specification or standard. […] When all of the known or suspected components of error have been evaluated and the appropriate corrections have been applied, there still remains an uncertainty about the correctness of the stated result, that is, a doubt about how well the result of the measurement represents the value of the quantity being measured' (ISO, 2008b, p. vii).

Uncertainty assessment (UA) should thus become a standard professional practice in UDSM, aiming to comply with laws and regulations, quality control requirements, expected professional skills, basic and applied research needs, etc. This chapter aims to provide information, concepts, methods, tools, and detailed examples facilitating knowledge transfer and implementation of uncertainty assessment. However, as UA is not always obvious and requires some training, 'critical thinking, intellectual honesty and professional skills' (ISO, 2008b, p. 8) remain fundamental.

This chapter is organized in three main sections:

- Section 8.2 presents the methods and international standards for UA, with their principles, conditions of application, step by step explanations and basic examples of application.
- Section 8.3 provides some additional examples for various aspects of UDSM.
- Section 8.4 gives complements including *in situ* uncertainties and some reference values for typical sensors and measurement methods used in UDSM.

BOX 0: EXAMPLES WITH MATLAB®

Detailed examples of calculations with Matlab® are shown in dedicated boxes throughout this chapter: the instructions and codes are written with the Matlab® syntax and `courier new` font to distinguish them from the rest of the text. The instructions and code lines can be copied-pasted directly by the reader who would like to replicate them for training or to adapt them to his/her own needs.

Numerical results in boxes are usually given with 4 digits (`format short`). In the main text, numerical values are rounded to the number of significant digits. It is also important to note, for readers who would like to reproduce them, that all calculations have been run without rounding in the successive steps.

Matlab® codes and associated data csv files are available for download at https://doi.org/10.2166/9781789060102.

8.2 INTERNATIONAL STANDARDS AND METHODS FOR UNCERTAINTY ASSESSMENT

8.2.1 Introduction and common rules of application

The first internationally unified frame for UA in measurements was published in 1993 as an ISO (International Organization for Standardization) guide entitled GUM – Guide for Uncertainty in Measurements (ISO, 1993), re-published with revisions in 1995 and also as a European standard in 1999 (CEN, 1999). It was later on revised, adapted and completed as parts of a new Guide for Uncertainty in Measurement, abbreviated hereafter as the ISO Guide 98, elaborated at international level by the JCGM – Joint Committee for Guides in Metrology – convened by the Bureau International des Poids et Mesures (BIPM), the International Electrotechnical Commission (IEC), the International Organization for Standardization (ISO), and the International Organization of Legal Metrology (OIML). The Supplement 1 published in 2008 introduces the Monte Carlo method for uncertainty assessment.

The ISO Guide 98 is based on a statistical approach to estimate uncertainties in measurements, in agreement with definitions given in Chapter 12.

In this chapter, we refer to the following parts of the ISO Guide 98:

- As general introduction for all concepts and methods:
  ISO (2009a). *ISO/IEC Guide 98-1:2009(E) Uncertainty of measurement – Part 1: Introduction to the expression of the uncertainty in measurement*. Geneva (Switzerland): ISO, September 2009, 32 p.
- As Guide for uncertainty in measurements method (abbreviated as GUM):
  ISO (2008a). *ISO/IEC Guide 98-3:2008(E) Uncertainty of measurement – Part 3: Guide to the expression of uncertainty in measurement (GUM: 1995)*. Geneva (Switzerland): ISO, December 2008, 130 p.
- As Monte Carlo method (abbreviated as MCM):
  ISO (2008b). *ISO/IEC Guide 98-3/Suppl.1:2008(E) Uncertainty of measurement – Part 3: Guide to the expression of uncertainty in measurement (GUM: 1995) Supplement 1: Propagation of distributions using a Monte Carlo method*. Geneva (Switzerland): ISO, December 2008, 98 p. and
  ISO (2009b). *ISO/IEC Guide 98-3/S1/AC1:2009(E) Uncertainty of measurement – Part 3: Guide to the expression of uncertainty in measurement (GUM: 1995), Supplement 1: Propagation of distributions using a Monte Carlo method, Technical corrigendum 1*. Geneva (Switzerland): ISO, May 2009, 2 p.

GUM and MCM may also be referred to as the 'propagation of uncertainties' method and the 'propagation of distributions' method, respectively.

This chapter does not reproduce the full content of the above detailed standards. A brief introduction is presented below in Section 8.2, and additional examples are given in Section 8.3.

For any measured or calculated quantity Y, there are three steps in UA:

- (1) Estimation of the true value* of Y. **(Note: the symbol * indicates that the definition of the word or the expression is given in Chapter 12).**
- (2) Estimation of the standard uncertainty* of Y noted $u(Y)$.

(3)   Estimation of the coverage interval* of Y for a given level of probability α (typically 95%):

   $[Y - ku(Y), Y + ku(Y)]$ in the case of methods A and B (described respectively in Sections 8.2.2 and
         8.2.3), where k is the coverage factor*, or
   $[Y_{\alpha,low}, Y_{\alpha,high}]$ in the case of MCM (described in Section 8.2.4).

The third step is optional but is almost systematically applied in practice.

   Reporting UA should be done systematically, and the following information should be provided (ISO, 2008b, p. 25):

- A detailed and clear description of (i) the measurement process, and (ii) the methods used.
- A list of all uncertainty components that are accounted for and how they are evaluated.
- All values, constants, corrections used in the UA analysis process, so that it could independently repeated if necessary.

A test of the foregoing list is to ask oneself 'Have I provided enough information in a sufficiently clear manner that my result can be updated in the future if new information or data become available?' (ISO, 2008b, p. 25).

   An important precondition for UA is the absence of coarse errors and systematic deviations in measurements. This is ensured by the rigorous application of metrological best practices, including sensor calibration, and periodic maintenance and checking (see Chapter 7, especially Section 7.6).

8.2.2 Type A method for uncertainty assessment of repeated measurements

8.2.2.1 Principle

The Type A method assumes that the quantity of interest Y can be measured directly and repeatedly, according to repeatability conditions*. It is applicable to stationary quantities that do not change with time (at least at the timescale of measurements), and to dynamic processes provided they are repeatable. Examples are the diameter of a pipe, the width of a channel, the angle of a weir, the hydraulic conductivity of a soil, etc.

   One assumes that the measurement of the quantity Y is a random process, due to all possible sources of variabilities attached to the instruments used, the measurement conditions, the operator and to the quantity itself. Each measurement y_i is assumed to be an independent observation of Y. The best estimate of the true value of Y is given by the mean $\bar{y}$ of the $i = 1{:}n$ repeated measurements y_i:

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \tag{8.1}$$

The unbiased standard deviation $s(y)$ of the measured values y_i is calculated as follows:

$$s(y) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2} \tag{8.2}$$

The unbiased standard deviation $s(\bar{y})$ of the mean value $\bar{y}$ is given by:

$$s(\bar{y}) = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (y_i - \bar{y})^2} = \frac{s(y)}{\sqrt{n}} \tag{8.3}$$

The standard uncertainty* $u(\bar{y})$ is then assumed to be equal to the standard deviation $s(\bar{y})$.

The expanded uncertainty* $U(\bar{y})$ with a given level of probability α (typically 95%) is calculated with the coverage factor* k. The value of k depends on α. If n is lower than 30, the level of information about Y and the distribution of its measurements y_i is limited. One assumes that the distribution of the y_i values is a Student t distribution. In this case, which is frequent in practice as making more than 30 repeated measurements may be too long or too expensive, the value of k is given by:

$$k = t_{\frac{1+\alpha}{2}}(\nu) \tag{8.4}$$

where $t_{\frac{1+\alpha}{2}}(\nu)$ is the Student t value with $\alpha = 0.95$ for a symmetric probability level of 95% and $\nu = n - 1$ degrees of freedom. The Student t value is found in statistics tables and can be obtained from software tools like Excel®, Matlab® or Octave® (see Table 8.2).

The expanded uncertainty is:

$$U(\bar{y}) = ku(\bar{y}) \tag{8.5}$$

and the coverage interval with the probability level α is then calculated as follows:

$$[\bar{y} - U(\bar{y}), \bar{y} + U(\bar{y})] = [\bar{y} - ku(\bar{y}), \ \bar{y} + ku(\bar{y})] \tag{8.6}$$

If n is above 30, the distribution of the measurements y_i is usually assumed to be normal (i.e. Gaussian) and, in this case, $k = 1.96$ for $\alpha = 0.95$ (Table 8.3).

In practice, due to (i) the unavoidable approximations in the measurement process, (ii) the fact that the measured values are not necessarily exactly normally distributed and (iii) 'the impracticality of trying to distinguish between intervals having levels of confidence that differ by one or two percent', the ISO Guide 98 (ISO, 2008a, appendix G) indicates that it is also acceptable to approximate $k = 1.96$ by $k = 2$ (which corresponds to the exact value $\alpha = 0.9545$ in case of the normal distribution). The 95% coverage interval of $\bar{y}$ is then approximated by:

$$[\bar{y} - 2u(\bar{y}), \ \bar{y} + 2u(\bar{y})] \tag{8.7}$$

In this chapter, we use $k = 1.96$ to approximate 95% coverage intervals with the hypothesis of the normal distribution. It is recommended to systematically apply Equations (8.4) and (8.6) for any number of measurements n.

The coverage interval (Equation (8.6)) is usually interpreted, in a simplified way, as 'the true value of the mean $\bar{y}$ of the quantity Y has an approximately 95% probability to lie between $\bar{y} - ku(\bar{y})$ and $\bar{y} + ku(\bar{y})$'. This can be acceptable only if (i) there is no bias (systematic error) in the measurements, which is ensured only by proper calibration of the sensor used for measurements and careful checking of the complete measurement process, and (ii) the number of measurements is high enough to ensure that the mean of the measured values is reasonably close to the true value of Y.

As indicated by Equation (8.3), increasing n allows decreasing the uncertainty in $\bar{y}$ proportionally to the square root of n. Multiplying n by 4 and 10 leads to dividing the uncertainty respectively by 2 (i.e. $\sqrt{4}$) and 3.16 (i.e. $\sqrt{10}$).

8.2.2.2 Basic example with Matlab®

The diameter D of a 1 m circular sewer pipe has been measured four times with a 2 m long class II meter (i.e. true length of this meter is between 1.9993 and 2.0007 m, according to the class definition given in OJEU, 2014). The four measured values D_i with $i = 1:4$ are given in Table 8.1.

Table 8.1 Four measurements of the pipe diameter D.

D_i (mm)
1002
1000
997
1002

Note: Examples and codes written for Matlab® (https://fr.mathworks.com) can also be used without any modification with the free software tool Octave® (https://www.gnu.org/software/octave/). The compatibility has been checked by the authors with Matlab® 2017b and Octave 5.1.0.

As shown in Box 1 below, the best estimate of the pipe diameter is $\bar{D} = 1000.2$ mm and its 95% coverage interval is [996.5, 1004.0] mm, with only one meaningful digit. Box 2 shows how to apply the Type A method with the Matlab® code uTypeA.

BOX 1: STEP BY STEP APPLICATION OF THE TYPE A METHOD WITH MATLAB®

For the data given in Table 8.1, the Matlab® instructions are as follows.

Create the vertical vector D_i with the four measured values:
```
Di=[1002 1000 997 1002]'
```
Calculate the mean value $\bar{D}$:
```
Dbar=mean(Di)
```
One gets $\bar{D} = 1000.2500$ mm.
Calculate the standard uncertainty $u(\bar{D})$:
```
uDbar=std(Di)/sqrt(length(Di))
```
One gets $u(\bar{D}) = 1.1814$ mm.
Calculate k with $\alpha = 95\%$ and $\nu = n\text{-}1$ degrees of freedom:
```
alpha=0.95
k=tinv((1+alpha)/2, length(Di)-1)
```
One gets $k = 3.1824$. The expanded uncertainty $k \times u(\bar{D}) = 3.7599$ mm.
Calculate the coverage interval with the probability level α:
```
Dbar-k*uDbar
Dbar+k*uDbar
```
One gets respectively 996.4900 mm and 1004.0099 mm.

BOX 2: APPLICATION OF THE TYPE A METHOD WITH THE MATLAB® CODE `uTypeA`

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).

All calculations presented in Box 1 are automated in the Matlab® code `uTypeA`.

With the previous notations defined in Box 1, enter `uTypeA(yi,alpha)` with `yi` the vertical vector of n measured values y_i and `alpha` the level of probability. The `uTypeA` function provides respectively the mean value $\bar{y}$, the standard uncertainty $u(\bar{y})$, and the boundaries of the coverage interval $[\bar{y} - ku(\bar{y}), \; \bar{y} + ku(\bar{y})]$ with the level of probability α and the coverage factor k calculated with the Student t value.

For the Box 1 example, type

```
Dbar=uTypeA(Di,alpha)
```

One gets $\bar{D} = 1000.2500$ mm, $u(\bar{D}) = 1.1814$ mm, $a = 996.4900$ mm and $b = 1004.0099$ mm.

8.2.3 Type B method for uncertainty assessment by the law of propagation of uncertainties

8.2.3.1 Principle

The Type B method is applied in cases where the quantity of interest Y can be measured neither directly nor repeatedly, i.e. when the Type A method cannot be applied. Some examples: rainfall intensities calculated from measured tips of a rain gauge bucket, discharge calculated from the water level measured over a weir, discharge calculated from both measured water level and mean flow velocity, infiltration flow in a stormwater infiltration tank calculated from water level and mass balance, pollutant load calculated from measured water quality and discharge, etc. In many cases in urban hydrology, the quantities of interest vary with time: repeated measurements are not possible. Measured process data are usually recorded as time series.

It is assumed that Y is determined from N other quantities X_i by means of a function f representing the measurement process. All quantities are assumed to be random quantities. All measured, estimated or known values x_i of the quantities X_i and their standard uncertainties $u(x_i)$ shall be known from Type A repeated measurements, previous applications of the Type B method, sensor calibration, experiments, expertise, standards, scientific literature, textbooks, etc. Previous knowledge on uncertainties $u(x_i)$ and their distribution is a pre-requisite for the Type B method and is discussed in Section 8.2.3.3.

The estimate y of the quantity Y is given by:

$$y = f(x_1, x_2, \ldots x_i, \ldots x_N) \tag{8.8}$$

The combined standard uncertainty $u(y)$ is obtained using the following equation, also referred to as the Law of Propagation of Uncertainties (LPU):

$$u(y)^2 = \sum_{i=1}^{N} u(x_i)^2 \left(\frac{\partial f}{\partial x_i}\right)^2 + 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} u(x_i, x_j)\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) \tag{8.9}$$

where $u(x_i, x_j)$ is the covariance of x_i and x_j:

$$u(x_i, x_j) = r(x_i, x_j)u(x_i)u(x_j) \tag{8.10}$$

where $r(x_i, x_j)$ is the correlation coefficient of x_i and x_j.

The partial derivatives, also called sensitivity coefficients c_i, are evaluated at $X_i = x_i$ using:

$$c_i = \left(\frac{\partial f}{\partial X_i}\right) \tag{8.11}$$

In the case where f has a complicated expression, its derivatives may be difficult to establish analytically. They can be replaced by numerical second order approximations:

$$c_i = \left(\frac{\partial f}{\partial X_i}\right) \approx \frac{f(x_i + \varepsilon_i) - f(x_i - \varepsilon_i)}{2\varepsilon_i} \tag{8.12}$$

where ε_i is very small compared to x_i. Typically, one can use $\varepsilon_i = u(x_i)/1000$.

The expanded uncertainty $U(\bar{y})$ with a given level of probability α (typically 95%) is calculated with the coverage factor k:

$$U(y) = ku(y) \tag{8.13}$$

The value of k depends on α. Ideally, uncertainty estimates $u(x_i)$ are based upon reliable Type A and Type B evaluations with a sufficient number n of observations such that using the coverage factor of $k = 1.96$ will ensure a confidence level close to 95%.

If the above assumption is not valid, the effective degree of freedom ν_{eff} needs to be estimated using the Welch-Satterthwaite formula:

$$\nu_{eff} = u(y)^4 \left(\sum_{i=1}^{N} \frac{[c_i u(x_i)]^4}{\nu_i}\right)^{-1} \tag{8.14}$$

If $u(x_i)$ is determined from a Type A estimation based on n repeated measurements, then $\nu_i = n-1$. If $u(x_i)$ is determined from a previous Type B estimation, and if the distribution of x_i is exactly known (i.e. the type and the boundaries of the distribution are known), which is frequent in practice, then $\nu_i \to \infty$. Otherwise, ν_i is estimated from the following equation:

$$\nu_i = \frac{1}{2}\frac{u(x_i)^2}{\sigma[u(x_i)]^2} \approx \frac{1}{2}\left[\frac{\Delta u(x_i)}{u(x_i)}\right]^{-2} \tag{8.15}$$

where $\sigma[u(x_i)]$ is the standard deviation of the standard uncertainty $u(x_i)$.

The quantity between large brackets in the last part of Equation (8.15) corresponds to the relative uncertainty of the standard uncertainty $u(x_i)$, i.e. how exactly the standard uncertainty $u(x_i)$ itself is known. This is usually based on scientific judgement and expertise.

In the case where the value of ν_{eff} obtained from Equation (8.14) is not an integer, then round ν_{eff} to the nearest lower integer.

The value of k is then calculated from the Student t distribution:

$$k = t_{\frac{1+\alpha}{2}}(\nu_{eff}) \tag{8.16}$$

and the coverage interval with the probability level α is calculated as follows:

$$[y - U(y), y + U(y)] = [y - ku(y), \ y + ku(y)] \tag{8.17}$$

Note that the coverage interval is, by definition, symmetric around the estimate y of the measurand Y.

8.2.3.2 Covariances in the type B method

The main equation of the LPU is Equation (8.9). The right-hand part contains two sums. The first one is always applied and corresponds to cases where all measured values x_i and their uncertainties $u(x_i)$ are fully independent of each other and are not correlated: all covariances $u(x_i, x_j) = 0$ or all coefficients of correlation $r(x_i, \ x_j) = 0$. This is the case when all x_i values are estimated with separate sensors or independent measurement processes and sources of information.

The second double sum has to be taken into account when measured values x_i or their uncertainties $u(x_i)$ are not independent of each other and are correlated: covariances $u(x_i, x_j) \neq 0$ or coefficients of correlation $r(x_i, \ x_j) \neq 0$ shall be (i) detected by means of a detailed analysis of the measurement process and (ii) quantified to be included in the calculation. Detection and quantification of covariances are not always obvious and should receive special attention. As covariances and coefficients of correlation may be positive or negative, they may contribute, sometimes in a very high proportion, to respectively increasing or decreasing the standard uncertainty $u(y)$. More details are given in Section 8.2.6.

It is frequent in practice that estimating the value y of the measurand Y includes intermediate quantities X_i in the function f which are themselves based on common measured quantities X_j. This shall be avoided as it generates covariance and complicates the estimation of the standard uncertainty $u(y)$. It is thus very important to avoid intermediate quantities as much as possible, and to write the function f in a way which may be less usual but reflects closely the measurement process with independent quantities.

Example 1: Calculation of flow by the Manning-Strickler equation

The discharge Q (m^3/s) in a rectangular channel is commonly calculated using the Manning-Strickler equation as:

$$Q = f(K, I, S, R_h) = KI^{\frac{1}{2}} S R_h^{\frac{2}{3}} \tag{8.18}$$

where K (m$^{1/3}$/s) is the Manning-Strickler coefficient, I (m/m) is the pipe invert slope, S (m^2) is the flow cross section and R_h (m) is the hydraulic radius.

The coefficient K is estimated either from field experiments or, more frequently, from tables or textbooks. The slope I is estimated from field measurements, maps or GIS data. Both the cross section S and the hydraulic radius R_h are calculated from the channel width B (m) and the measured water level h (m):

$$S = Bh \tag{8.19}$$

$$R_h = \frac{Bh}{B + 2\,h} \tag{8.20}$$

K, I, B and h are the truly independent quantities estimated by means of different and independent instruments and information. But S and R_h are obviously not independent quantities and are highly correlated as both depend on B and h. Clearly Equation (8.18) is not appropriate to be used in Equation (8.9) as covariance between S and R_h has been introduced. It is thus recommended to rewrite Equation (8.18) without correlated intermediate quantities and only with the truly independent quantities

in a way which reflects closely the real measurement process, as:

$$Q = f(K, I, B, h) = KI^{\frac{1}{2}}(Bh)\left(\frac{Bh}{B + 2\,h}\right)^{\frac{2}{3}} = KI^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2\,h)^{-\frac{2}{3}} \tag{8.21}$$

8.2.3.3 Estimation of standard uncertainties u(x_i) from prior information on distributions

As the Type B method requires all values $u(x_i)$ as inputs in Equation (8.9), and as these values are not necessarily obtained from Type A estimations (i.e. series of repeated observations), it is necessary to provide additional information on the probability distributions of the quantities X_i.

If the value x_i of the quantity X_i is known from repeated measurements and if the Type A method is applied to estimate $u(x_i)$, the degrees of freedom $v(x_i)$ are known and $u(x_i)$ can be used directly in Equation (8.9). In the case where x_i is not given with its standard uncertainty $u(x_i)$ but with (i) a coverage interval $[a, b]$ and (ii) a coverage factor k or a level of probability α, then $u(x_i)$ is calculated from Equation (8.6), by assuming the values of the quantity X_i are distributed according to a normal (i.e. Gaussian) distribution:

$$u(x_i) = \frac{(b - a)}{2k} \tag{8.22}$$

or

$$u(x_i) = \frac{(b - a)}{2k(\alpha)} \tag{8.23}$$

where $k(\alpha)$ is calculated either from the normal distribution or from the Student t distribution for an infinite number of degrees of freedom $v = +\infty$. The value $k(\alpha)$ is found in statistics tables and can be obtained from software tools like Excel®, Matlab® or Octave® (Table 8.2). Most typical values are given in Table 8.3.

Example 2: If $x_i = 100$ and $[a, b] = [99, 101]$ with $k = 2$ (i.e. $\alpha \approx 0.95$), then Equation (8.22) gives $u(x_i) = 0.5$. If $x_i = 100$ and $[a, b] = [99, 101]$ with $\alpha = 0.99$, then $k(\alpha) = 2.58$ and Equation (8.23) gives $u(x_i) = 0.39$.

Table 8.2 Excel® and Matlab®/Octave® functions to calculate the coverage factor k from the probability level α and reciprocally.

Normal distribution			Student distribution	
From α to k				
Excel®*	$k(\alpha) = \text{NORMINV}((1 + \alpha)/2, 0, 1)$	(8.24)	$k(\alpha) = \text{TINV}((1 - \alpha), 1\text{e}6)$	(8.25)
Matlab®	$k(\alpha) = \text{norminv}((1 + \alpha)/2)$	(8.26)	$k(\alpha) = \text{tinv}((1 + \alpha)/2, \text{inf})$	(8.27)
From k to α				
Excel®*	$\alpha(k) = (\text{NORM.DIST}(k, 0, 1, 1) * 2) - 1$	(8.28)	$\alpha(k) = (\text{T.DIST}(k, 1\text{e}6) * 2) - 1$	(8.29)
Matlab®	$\alpha(k) = (\text{normcdf}(k) * 2) - 1$	(8.30)	$\alpha(k) = (\text{tcdf}(k, \text{inf}) * 2) - 1$	(8.31)

* With Excel®, the infinite value is replaced by one million (1e6).

Table 8.3 Most typical values of probability level α and corresponding coverage factor k for the normal distribution. Note: as indicated in ISO (2008a) appendix G, it is acceptable to replace $k = 1.96$ by $k = 2$ to approximate 95% coverage intervals.

Probability level α	Coverage factor k
0.68	1.00
0.90	1.64
0.95	1.96
0.99	2.58

The probability distribution of the quantities X_i can take a variety of other forms. In addition to the normal distribution, three typical examples are the uniform (or rectangular), the triangular and the trapezoidal distributions (Figure 8.1).

The symmetric uniform distribution corresponds to cases with limited information about the value x_i of the quantity X_i. One knows for example that the value x_i lies in the interval $[a, b]$ with a probability close to 1 but without any additional information about the shape of the distribution. It is thus assumed that (i) any value within the interval $[a, b]$ has the same probability to be the most likely value of x_i, and (ii) any value outside this interval is almost unlikely. In this case, x_i and $u(x_i)$ are given respectively by:

$$x_i = \frac{a + b}{2} \tag{8.32}$$

and

$$u(x_i) = \frac{b - a}{2\sqrt{3}} \tag{8.33}$$

Example 3: If $[a, b] = [99, 101]$, then Equations (8.32) and (8.33) give respectively $x_i = 100$ and $u(x_i) = 0.58$.

The symmetric triangular distribution corresponds to cases with more information. One knows for example that the most likely value x_i is the central value of the interval $[a, b]$ and that the probability declines regularly towards the lower and upper bounds a and b, with the assumption that any value outside the interval is unlikely. In this case, x_i is given by Equation (8.32) and $u(x_i)$ is given by:

$$u(x_i) = \frac{b - a}{2\sqrt{6}} \tag{8.34}$$

Example 4: If $[a, b] = [99, 101]$, then Equations (8.32) and (8.34) give respectively $x_i = 100$ and $u(x_i) = 0.41$.

The symmetric trapezoidal distribution is used to account for the fact that in the uniform distribution, the abrupt probability step below a and above b is likely unphysical. Slopes on each side of the distribution are thus included to get a more realistic distribution (Figure 8.1). The trapezoidal distribution is characterized by both the bottom interval $[a, b]$ and the coefficient β which represents the width of the
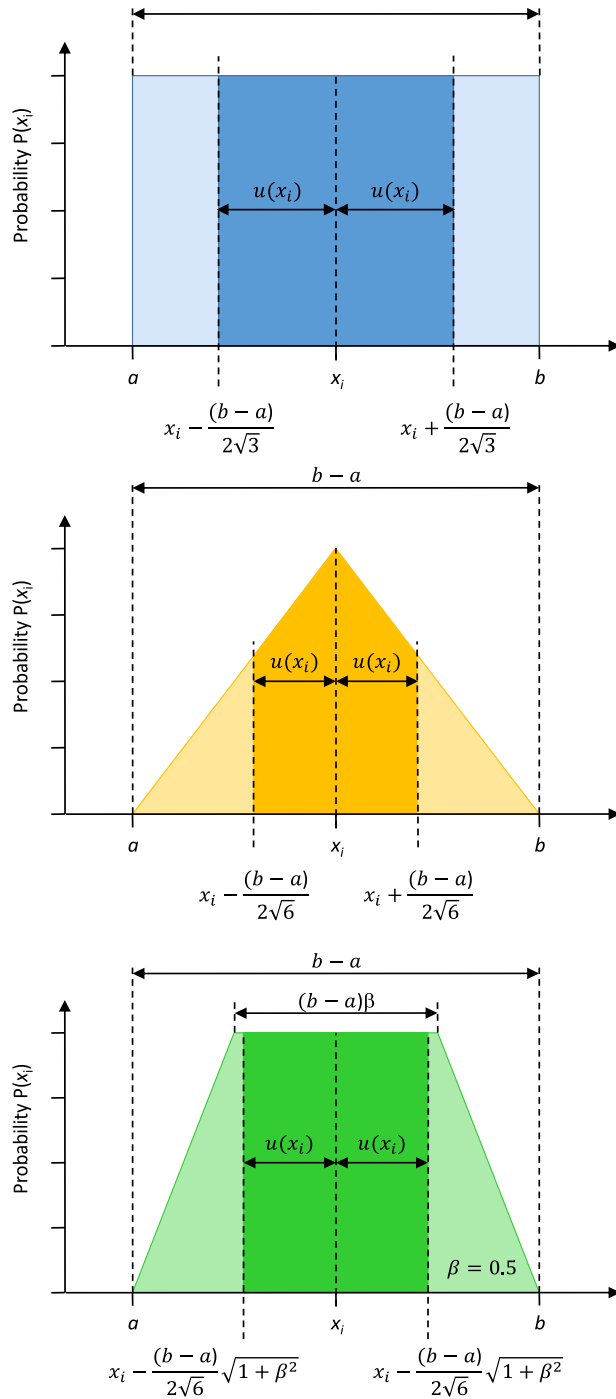
Figure 8.1 Uniform, triangular and trapezoidal distributions for the Type B method. *Source*: adapted from ISO (2008a) by Jean-Luc Bertrand-Krajewski (INSA Lyon).

Table 8.4 Choice of typical distributions for the Type B method.

Distribution	Examples/available information
Normal	• Calibration certificates, handbooks, material or sensor specifications, knowledge quoting either (i) a probability level α or a coverage factor k with the expanded uncertainty, (ii) a number of standard deviations, or (iii) a given probability level interval α. • Information from Type A estimations based on repeated measurements.
Uniform	• Maximum bounds within which all values of the quantity are assumed to lie with equal probability. • Maximum instrument drift between calibrations. • Error due to limited resolution of an instrument's display or digitizer. • Manufacturers' tolerance limits.
Triangular	• Maximum bounds within which all values of the quantity are assumed to lie with higher probability for the central value and decreasing probabilities towards the interval bounds.
Trapezoidal	• Maximum bounds within which all values of the quantity are assumed to lie with equal probability in the central part of the interval.

top of the distribution as a fraction of the interval $[a, b]$. In this case, x_i is given by Equation (8.32) and $u(x_i)$ is given by:

$$u(x_i) = \frac{b - a}{2\sqrt{6}} \sqrt{1 + \beta^2} \tag{8.35}$$

Example 5: If $[a, b] = [99, 101]$ and $\beta = 0.5$, then Equations (8.32) and (8.35) give respectively $x_i = 100$ and $u(x_i) = 0.46$.

It is worth noting that Equation (8.35) is similar to Equation (8.33) when $\beta = 1$ (uniform distribution) and to Equation (8.34) when $\beta = 0$ (triangular distribution).

Other distributions, including non-symmetrical ones, are described in ISO (2008a) and also in textbooks (e.g. Gentle, 2003; Thomopoulos, 2018). The choice of an appropriate distribution for each quantity X_i in running the Type B method is based on knowledge and experience. Some examples are given in Table 8.4. In practice, normal and uniform distributions are among the most frequently used ones.

8.2.3.4 Basic example with Matlab®

The discharge Q (m³/s) in an open rectangular channel is calculated by means of the Manning-Strickler formula, written as discussed in Section 8.2.3.2:

$$Q = f(K, I, B, h) = KI^{\frac{1}{2}}(Bh)\left(\frac{Bh}{B + 2\,h}\right)^{\frac{2}{3}} = KI^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2\,h)^{-\frac{2}{3}} \tag{8.36}$$

where K (m$^{1/3}$/s) is the Manning-Strickler coefficient, I (m/m) is the channel invert slope, B (m) is the channel width and h (m) is the water level in the channel.

The channel is made of smooth concrete, with no deposits, no biofilm, and no surface degradation. Textbooks (e.g. Lencastre, 1999) indicate that the value of K is usually between 70 and 80 m$^{1/3}$/s. Consequently, in the absence of *in situ* measurements, it is reasonable to assume that the value of K

lies with a symmetric uniform probability in the interval $[a, b] = [70, 80]$ m$^{1/3}$/s (uniform distribution $U(70, 80)$).

The mean slope I of the open channel has been measured by a land surveyor along a reach of 50 m with the direct levelling method. The result is $I = 0.0032$ m/m, with a standard uncertainty $u(I) = 6 \times 10^{-6}$ m/m (normal distribution $N(0.0032, 6 \times 10^{-6})$).

The channel width B has been measured four times with a class II meter, similarly as the pipe diameter in Section 8.2.2.2. The results are $B = 0.805$ m, $u(B) = 0.002$ m and $v_B = 3$ degrees of freedom.

The water level $h = 0.32$ m is measured by means of a calibrated ultrasound sensor. *In situ* sensor calibration accounting for uncertainties related to both the sensor itself and local measurement conditions results in a whole standard uncertainty $u(h) = 1.5$ mm (normal distribution $N(0.32, 0.0015)$).

With the above values, the discharge is $Q = 0.346$ m^3/s (detailed calculations are given in Box 3).

The next step consists of calculating the standard uncertainty $u(Q)$ by applying Equation (8.9). As the quantities K, I, B and h are measured with independent sensors, there is no covariance between them and only the first part of Equation (8.9) is applied. This leads to:

$$u(Q)^2 = \sum_{i=1}^{4} (u(x_i)^2 \left(\frac{\partial Q}{\partial x_i}\right)^2 = u(K)^2 \left(\frac{\partial Q}{\partial K}\right)^2 + u(I)^2 \left(\frac{\partial Q}{\partial I}\right)^2 + u(B)^2 \left(\frac{\partial Q}{\partial B}\right)^2 + u(h)^2 \left(\frac{\partial Q}{\partial h}\right)^2 \tag{8.37}$$

The partial derivatives of Q need to be estimated first. There are two possibilities: algebra derivation or second order numerical approximation. In this example, both approaches are used and compared.

The algebra derivation gives (with 6 digits for illustrative purpose):

$$\frac{\partial Q}{\partial K} = I^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2h)^{-\frac{2}{3}} = \frac{Q}{K} = 0.004615 \tag{8.38}$$

$$\frac{\partial Q}{\partial I} = \frac{1}{2}KI^{-\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2h)^{-\frac{2}{3}} = \frac{Q}{2I} = 54.090477 \tag{8.39}$$

$$\frac{\partial Q}{\partial B} = \frac{5}{3}hKI^{\frac{1}{2}}(Bh)^{\frac{2}{3}}(B + 2h)^{-\frac{2}{3}} - \frac{2}{3}KI^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2h)^{-\frac{5}{3}} = \frac{Q}{3}\left(\frac{5}{B} - \frac{2}{B + 2h}\right) = 0.557013 \tag{8.40}$$

$$\frac{\partial Q}{\partial h} = \frac{5}{3}BKI^{\frac{1}{2}}(Bh)^{\frac{2}{3}}(B + 2h)^{-\frac{2}{3}} - \frac{4}{3}KI^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2h)^{-\frac{5}{3}} = \frac{Q}{3}\left(\frac{5}{h} - \frac{4}{B + 2h}\right) = 1.483588 \tag{8.41}$$

The second order numerical approximation, calculated according to Equation (8.12), gives:

$$\frac{\partial Q}{\partial K} = \frac{Q(K + \varepsilon_K, I, B, h) - Q(K - \varepsilon_K, I, B, h)}{2\varepsilon_K} = 0.004615 \tag{8.42}$$

$$\frac{\partial Q}{\partial I} \approx \frac{Q(K, I + \varepsilon_I, B, h) - Q(K, I - \varepsilon_I, B, h)}{2\varepsilon_I} = 54.090477 \tag{8.43}$$

$$\frac{\partial Q}{\partial B} \approx \frac{Q(K, I, B + \varepsilon_B, h) - Q(K, I, B - \varepsilon_B, h)}{2\varepsilon_B} = 0.557013 \tag{8.44}$$

$$\frac{\partial Q}{\partial h} \approx \frac{Q(K, I, B, h + \varepsilon_h) - Q(K, I, B, h - \varepsilon_h)}{2\varepsilon_h} = 1.483588 \tag{8.45}$$

Both approaches provide results which, in this example, are identical to the 6th digit at least. The advantage of the numerical approximation is the possibility to run calculations automatically without algebra.

The resulting standard uncertainty is $u(Q) = 0.013$ m^3/s and the relative standard uncertainty is $u*(Q) = u(Q)/Q = 0.039$, i.e. 3.9%. All calculations with Matlab$^{\circledR}$ are given in Box 3.

BOX 3: STEP BY STEP CALCULATIONS OF *Q* AND *u*(*Q*) IN A RECTANGULAR CHANNEL WITH MATLAB®

Let us first set the values and the standard uncertainties for *K*, *I*, *B,* and *h*.

Define the interval for *K*:

```
intK=[70 80]
```

Calculate *K* and its standard uncertainty *u*(*K*):

```
K=mean(intK)
uK=diff(intK)/2/sqrt(3)
```

One gets $K = 75$ m$^{1/3}$/s and $u(K) = 2.8867$ m$^{1/3}$/s.

Then type

```
I=3.2e-3
uI=6e-6
B=0.805
uB=2e-3
h=0.32
uh=1.5e-3
```

Calculate the discharge *Q*(*K,I,B,h*) by Equation (8.36):

```
Q=K.*power(I,1/2).*power(B.*h,5/3).*power(B+2.*h,-2/3)
```

One gets $Q = 0.3462$ m^3/s.

Calculate the standard uncertainty *u*(*Q*) by Equation (8.37).

Let us first define the quantities `cK`, `cI`, `cB` and `ch` respectively equal to the numerical values of the partial derivatives of *Q* (Equations 8.42 to 8.45):

```
epsK=uK/1000
epsI=uI/1000
epsB=uB/1000
epsh=uh/1000
cK=((K+epsK).*power(I,1/2).*power(B.*h,5/3).*power(B+2*h,-2/3)-
(K-epsK).*power(I,1/2).*power(B.*h,5/3).*power(B+2*h,-2/3))/epsK/2
cI=(K.*power(I+epsI,1/2).*power(B.*h,5/3).*power(B+2*h,-2/3)-
K.*power(I-epsI,1/2).*power(B.*h,5/3).*power(B+2*h,-2/3))/epsI/2
cB=(K.*power(I,1/2).*power((B+epsB).*h,5/3).*power(B+epsB+2*h,-2/3)-
K.*power(I,1/2).*power((B-epsB).*h,5/3).*power(B-epsB+2*h,-2/3))/epsB/2
ch=(K.*power(I,1/2).*power(B.*(h+epsh),5/3).*power(B+2*(h+epsh),-2/3)-
K.*power(I,1/2).*power(B.*(h-epsh),5/3).*power(B+2*(h-epsh),-2/3))/
epsh/2
```

Then create two intermediate vectors `Vu` and `Vc` and apply Equation (8.37):

```
Vu=[uK uI uB uh]'
Vc=[cK cI cB ch]'
uQ=sqrt(sum(power(Vu.*Vc,2)))
```

One gets

$u(Q) = 0.0136$ m^3/s

The relative standard uncertainty is calculated by

```
uQ/Q
```

One gets $u*(Q) = 0.0392$, i.e. 3.9%.

The 95% coverage interval is given by

```
[ Q-1.96*uQ, Q+1.96*uQ]
```

One gets [0.3196, 0.3728] m^3/s.

Let us now calculate the 95% coverage interval. In the first and most simple way, one may assume that the coverage factor $k = 1.96$ for the probability level $\alpha = 0.95$. In this case, the 95% coverage interval for Q is determined from Equation (8.17):

$$[Q - 1.96 \times u(Q),\ Q + 1.96 \times u(Q)] = [0.320,\ 0.373]\ \text{m}^3/\text{s} \tag{8.46}$$

A more detailed approach is based on the estimation of the effective degree of freedom from the Welch-Satterthwaite formula (Equations 8.14 to 8.16). Detailed Matlab® calculations are given in Box 4. The effective degrees of freedom v_i are estimated as follows.

As $u(K)$ can be considered to be known itself with a relative uncertainty of 20% according e.g. to hydraulic textbooks, then, according to Equation (8.15):

$$v(K) = \frac{1}{2}[0.20]^{-2} = 12.49 \tag{8.47}$$

to be approximated to the nearest lower integer, i.e. $v(K) = 12$.

$u(I)$ is given by the land surveyor, based on repeated calibrations of the sensors. One assumes here that $v(I) = \infty$.

$u(B)$ is calculated from a Type A estimation based on $n = 4$ repeated measurements: $v(B) = n - 1 = 3$.

$u(h)$ is estimated from a sensor calibration based on $n = 60$ measurements (12 repeated measurements for 5 values along the sensor measurement range). Thus $v(h) = 60 - 1 = 59$. For comparison purposes, one may also assume that $v(h) = \infty$.

Applying Equation (8.14) gives $v_{eff} = 12$, with both $v(h) = 59$ or $v(h) = \infty$.

BOX 4: STEP BY STEP CALCULATIONS OF THE COVERAGE INTERVAL OF *Q* WITH THE EFFECTIVE DEGREE OF FREEDOM

Set the respective values of the degrees of freedom for *K*, *I*, *B*, and *h*:
```
nuK=floor(0.5*power(0.20, -2))
```
One gets $v(K) = 12$.
Type
```
nuI=Inf
nuB=3
nuh=59 (or nuh = Inf)
```
Define `Vnu` an intermediate vertical vector:
```
Vnu=[nuK nuI nuB nuh]'
```
The effective degree of freedom v_{eff} is calculated by Equation (8.14), with `Vu` and `Vc` defined in Box 3:
```
nueff=floor(power(uQ,4)/sum(power(Vu.*Vc,4)./Vnu))
```
One gets $v_{eff} = 12$, with both $v(h) = 59$ or $v(h) = \infty$.
Then type
```
alpha=0.95
k=tinv((1+alpha)/2, nueff)
```
One gets $k = 2.1788$.
The coverage interval is then calculated by:
```
[Q-k*uQ,Q+k*uQ]
```
One gets $[0.3166, 0.3757]\ \text{m}^3/\text{s}$.

The coverage factor k corresponding to $v_{eff} = 12$ is equal to 2.18 (Equation (8.16)), which is a little bit higher than the default value $k = 1.96$ used previously.

Lastly, applying Equation (8.17), the coverage interval for the discharge Q is:

$$[Q - k \times u(Q), Q + k \times u(Q)] = [0.317, \quad 0.376] \text{ m}^3/\text{s}.$$

Looking at the details of the calculations reveals that the strongest contribution to the combined uncertainty is due to the uncertainty in the Manning-Strickler coefficient K. If one assumes that the relative uncertainty of $u(K)$ is 10% instead of 20%, then $v(K)$ increases from 12 to 49. Consequently, v_{eff} increases from 12 to 52, and $k = 2.006$, which is now equivalent to the default value. This emphasizes the importance of a reliable assessment of all components contributing to the estimation of the coverage interval, including the degrees of freedom.

All calculations presented in this example can be run automatically with the Matlab® code `uTypeB` presented in Box 5.

BOX 5: APPLICATION OF THE TYPE B METHOD WITH THE MATLAB® CODE `uTypeB`

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
The calculations of the discharge Q and its standard uncertainty $u(Q)$ shown in Box 3 and Box 4 are automated in the Matlab® code `Y = uTypeB(Z,A,chaine,alpha,MatCor,NuZ,NuA)`, where the quantities X_i of the measurement process are divided into two groups Z and A, where:
`Z` is the matrix containing data related to time varying quantities (time series). Its structure is as follows: Z has as many lines as time steps in the time series. Each line contains, in successive columns from left to right, m pairs of data for each time varying quantity Z: Z_1, $u(Z_1)$, Z_2, $u(Z_2)$,…Z_i, $u(Z_i)$, …Z_m, $u(Z_m)$.
`A` is the matrix containing data related to constant quantities. The single line of A contains, in the successive columns from left to right, p pairs of data for each constant quantity A: A_1, $u(A_1)$, A_2, $u(A_2)$,…A_j, $u(A_j)$,…A_p, $u(A_p)$.
The sum $m + p$ is equal to N, the number of quantities X_i in Equation (8.9).
`chaine` is the equation string describing the measurement process of Y using Z_i and A_j quantities.
`alpha` is the level of probability.
`MatCor` is the matrix of correlation between Z_i and A_j quantities.
`NuZ` is the vector of the degrees of freedom $v(Z_i)$ of each quantity Z_i, in the same order as in the matrix Z.
`NuA` is the vector of the degrees of freedom $v(A_i)$ of each quantity A_j, in the same order as in the matrix A.
`NuZ` and `NuA` are optional inputs in the function `uTypeB`: if they are not used, the coverage interval is estimated solely with the default values of k given in Tables 8.2 and 8.3 for an infinite degree of freedom.
The `uTypeB` function provides respectively the following results in successive columns of the output matrix Y, with one line per time step: the value y of Y, its standard uncertainty $u(Y)$, the boundaries of the coverage interval calculated with an infinite degree of freedom and the boundaries with the effective degree of freedom v_{eff}.
For the above example, there is one time-varying quantity: the water level h, and three constant quantities: respectively the Manning-Strickler coefficient K, the channel slope I and the channel width B. Consequently, type

---

BOX 5: (Continued)

```
Z=[h uh]
```
with only one line as there is only one single value of *h*.
```
A=[K uK I uI B uB]
```
The previous measurement process equation (see Box 3):
```
Q=K.*power(I,1/2).*power(B.*h,5/3).*power(B+2.*h,-2/3)
```
is then re-written with notations indicating the quantities with their rank in matrices *Z* and *A*.

h is the first quantity in the matrix *Z*: h is replaced by Z(:,1) in the above expression of *Q*. Similarly, K, I and B are replaced, respectively, by A(:,1), A(:,2) and A(:,3) as they are, respectively, the first, second and third quantities in the matrix *A*. It is important to note that the indices refer to the rank of the quantities in matrices *Z* and *A*, and not to the rank of the columns.

Consequently, type
```
chaine='A(:,1).*power(A(:,2),0.5).*power(A(:,3).*Z(:,1),5/3).*power
      (A(:,3)+2.*Z(:,1),-2/3)'
alpha=0.95
```
In this example, all four quantities in *Z* and *A* are independent as there is no correlation or covariance between them. Consequently, type
```
MatCor=eye(4)
```
which gives

$$MatCor = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the columns from left to right and the lines from top to bottom refer successively to the Z_i and A_j quantities.

In addition, type
```
NuZ=[59]
NuA=[12 inf 3]
```
Lastly type
```
Q=uTypeB(Z,A,chaine,alpha,MatCor,NuZ,NuA)
```
One gets $Q = 0.3462$ m^3/s, $u(Q) = 0.0136$ m^3/s, and

coverage interval with an infinite degree of freedom: [0.3196, 0.3728]

coverage interval with the effective degree of freedom: [0.3166, 0.3757].

8.2.4 Monte Carlo method for uncertainty assessment

8.2.4.1 Principle

The Monte Carlo method (MCM) is a generic simulation method, which can be applied to estimate uncertainties under various conditions and in particular when the conditions of the Type B method are not satisfied (non-linearity, non-symmetric distributions, significance of second order terms in the derivation of the LPU) or are very difficult to apply. It is considered as the reference method.

The MCM consists basically of repeatedly simulating the measurement process f calculating the measurand Y from the quantities X_i as described by Equation (8.8) in Section 8.2.3.1:

$$y = f(x_1, x_2, \ldots x_i, \ldots x_N) \tag{8.48}$$

Samples of size M are built for all quantities X_i involved in the measurement process, according to their distributions and with appropriate correlation coefficients. Then the samples are used to calculate M times the value y of the measurand Y and the M values of y allows calculation of the mean value $\bar{y}$ of the measurand Y, the standard uncertainty $u(y)$ and the coverage interval corresponding to the defined probability level α.

As for the Type B method, it is of particular importance:

- To write the measurement process $Y = f(X_1, X_2, \ldots X_N)$ in a way which closely reflects the measurement process with independent quantities X_i.
- To define the probability distributions for all quantities X_i.
- To carefully analyse the possible correlations between quantities and to quantify them when they exist.

The principle of the MCM is illustrated in Figure 8.2. The sample of each quantity X_i for $i = 1{:}N$ in the measurement process f is represented by a vector of M values with $r = 1{:}M$: $x_{i,1}, x_{i,2}, \ldots x_{i,r}, \ldots x_{i,M}$. Within each sample, the successive values $x_{i,r}$ are sorted neither in ascending nor in descending order but are randomly listed.

The samples are built according to two requirements: (i) each sample distribution should represent the information and knowledge about each quantity X_i, as in the Type B method (see Section 8.2.4.2), and (ii) the correlation between samples should represent the correlation as described in Equation (8.10) between the quantities X_i in the measurement process f (see Section 8.2.4.3). Creating large samples cannot be done manually and software tools are required.

Once all samples of the quantities X_i are built appropriately, M values of y are calculated as follows:

$$y_r = f(x_{1,r}, x_{2,r}, \ldots x_{i,r}, \ldots x_{N,r}) \tag{8.49}$$

with $r = 1{:}M$.

Sample for X_1	Sample for X_2	$\cdots$	Sample for X_i	$\cdots$	Sample for X_N	$f(X_1, X_2, \ldots X_i, \ldots X_N)$	Sample for Y
$x_{1,1}$	$x_{2,1}$		$x_{i,1}$		$x_{N,1}$	$\longrightarrow$	y_1
$x_{1,2}$	$x_{2,2}$		$x_{i,2}$		$x_{N,2}$	$\longrightarrow$	y_2
$x_{1,3}$	$x_{2,3}$		$x_{i,3}$		$x_{N,3}$	$\longrightarrow$	y_3
.	.		.		.		.
.	.		.		.		.
.	.		.		.		.
$x_{1,r}$	$x_{2,r}$		$x_{i,r}$		$x_{N,r}$	$\longrightarrow$	y_r
.	.		.		.		.
.	.		.		.		.
.	.		.		.		.
$x_{1,M}$	$x_{2,M}$		$x_{i,M}$		$x_{N,M}$	$\longrightarrow$	y_M

Figure 8.2 Principle of the Monte Carlo method. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

The estimate of the measurand Y is given by the mean $\bar{y}$ of the y_r values:

$$\bar{y} = \frac{1}{M} \sum_{r=1}^{M} y_r \tag{8.50}$$

The estimate of the standard uncertainty $u(y)$ is given by the standard deviation $s(y)$ of the values y_r:

$$u(y) = \sqrt{\frac{1}{M-1} \sum_{r=1}^{M} (y_r - \bar{y})^2} \tag{8.51}$$

One of the advantages of the MCM, contrarily to the Type B method, is its ability to deal with non-symmetric distributions. As a consequence, the distribution of the y_r values may be also non-symmetric, and thus the coverage interval is not necessarily symmetric around the mean value $\bar{y}$, as this is the case in Equation (8.17) with the Type B method.

With the MCM, the coverage interval $[y_{\alpha,low}, y_{\alpha,high}]$ for the given level of probability α corresponds to the narrowest interval containing the fraction α of the values y_r. Its estimation requires some preliminary steps for processing of the values y_r:

(1) Sort all values y_r in ascending and non-decreasing order (in case of possible equalities among values y_r). The sorted values are then noted $y_{(r)}$ with $r = 1{:}M$.
(2) Define an integer $q = \alpha M$ if αM is an integer. Otherwise, define q as the integer part of $(\alpha M + 1/2)$.
(3) Determine the confidence interval $[y_{\alpha,low\ r}, y_{\alpha,high\ r}]$ for Y where, for any $r = 1{:}M\text{-}q$, $y_{\alpha,low\ r} = y_{(r)}$ and $y_{\alpha,high\ r} = y_{(r+q)}$.
(4) The shortest coverage interval with probability level α is obtained with r^* such that, for $r = 1{:}M\text{-}q$,

$$y_{(r*+q)} - y_{(r*)} \leq y_{(r+q)} - y_{(r)} \tag{8.52}$$

8.2.4.2 Creating non-correlated samples

Creating samples of the quantities X_i according to a given probability density function (pdf) requires a random number generator. Basic Matlab® functions to create uniform, normal, triangular and trapezoidal samples are given in Box 6. Similar functions exist with other software tools. They are parameterized and used to create samples following predefined pdfs.

Other pdfs can be used with the MCM: Student t pdf, exponential pdf for values which cannot be higher or lower than a threshold (typically non-negative values), gamma pdf, empirical pdfs based on experiments, truncated normal pdf, lognormal pdf, etc.

Additional information to generate samples according to these pdfs can be found in ISO (2008b) and in textbooks (e.g. Gentle 2003; Press *et al.*, 2007; Robert & Casella, 2005).

Non-correlated samples can be created individually one after another or simultaneously by using appropriate software functions. It is however important to ensure that they are really not correlated, usually by controlling the parameters of the random number generator.

8.2.4.3 Creating correlated samples

In the case where two quantities X_i and X_j with $i \neq j$ are correlated in the measurement process f with a correlation coefficient $r(x_i, x_j)$ as given in Equation (8.10), their corresponding samples shall be built in such a way that (i) each quantity is distributed according to its own pdf and (ii) the correlation of the two samples is equal to $r(x_i, x_j)$. Generating correlated samples may be complex and presenting the details of

BOX 6: MATLAB® FUNCTIONS AND CODES TO CREATE NON-CORRELATED SAMPLES WITH PRE-DEFINED PDFS

Two basic Matlab® functions are available: `rand` and `randn`.

`rand(M,1)`

generates a vector of random numbers x uniformly distributed between 0 and 1, noted $U(0, 1)$ with M lines and 1 column. An example of results is given in Figure 8.3 with $M = 10,000$.

`randn(M,1)`

generates a vector of random numbers x normally distributed with mean value $m = 0$ and standard deviation $s = 1$, noted $N(0,1)$. An example of results is given in Figure 8.4 with $M = 10,000$.

These basic functions can be parameterized and used to create samples following predefined pdfs.

Uniform (rectangular) pdf

To create a sample of M values x uniformly distributed in the interval [a, b] (see Figure 8.1), noted $U(a, b)$:

`x=a+(b-a).*rand(M,1)`

or

`x=unifrnd(a,b,M,1)`

Normal (Gaussian) pdf

To create M values x normally distributed with mean value m and standard deviation s, noted $N(m, s)$:

`x=m+s.*randn(M,1)`

Triangular pdf

To create a sample of M values x distributed according to a triangular pdf in the interval [a, b] (see Figure 8.1), noted $Tri(a, b)$:

0.223770404697041
0.373563807642645
0.087500349576586
0.640116548246715
0.180616887753108
0.045051107473574
0.723173479183095
0.347437645581790
0.660616824502904
0.383868601071971
0.627346502443467
0.021649814630306
0.910569988523029
0.800558656278811
0.745847484342721
0.813112813610761
0.383306318625529
0.617279232316449
0.575494859702814
0.530051704765016
0.275069755821935
0.248628959661970
0.451638770451973
0.227712826026548
0.804449583613070

Figure 8.3 First 25 values (left) and histogram (right) of the 10,000 values x created by `rand (10000,1)`. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

BOX 6: (Continued)

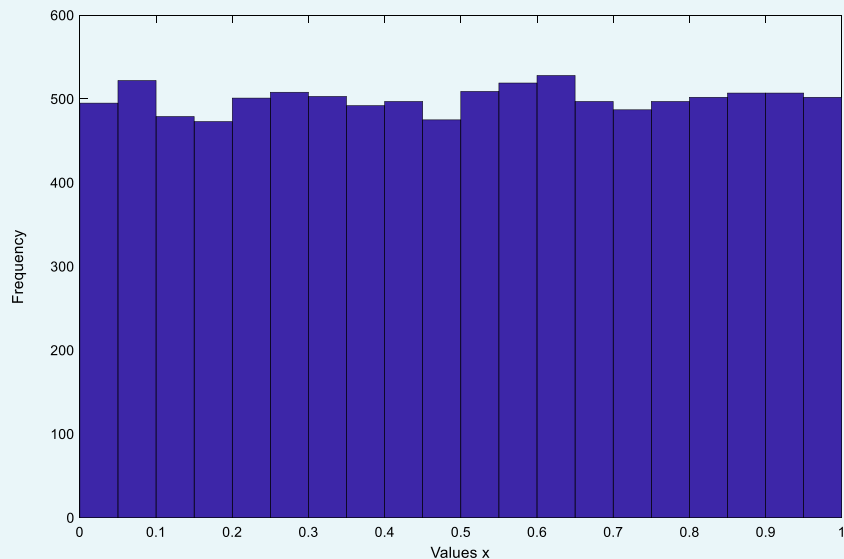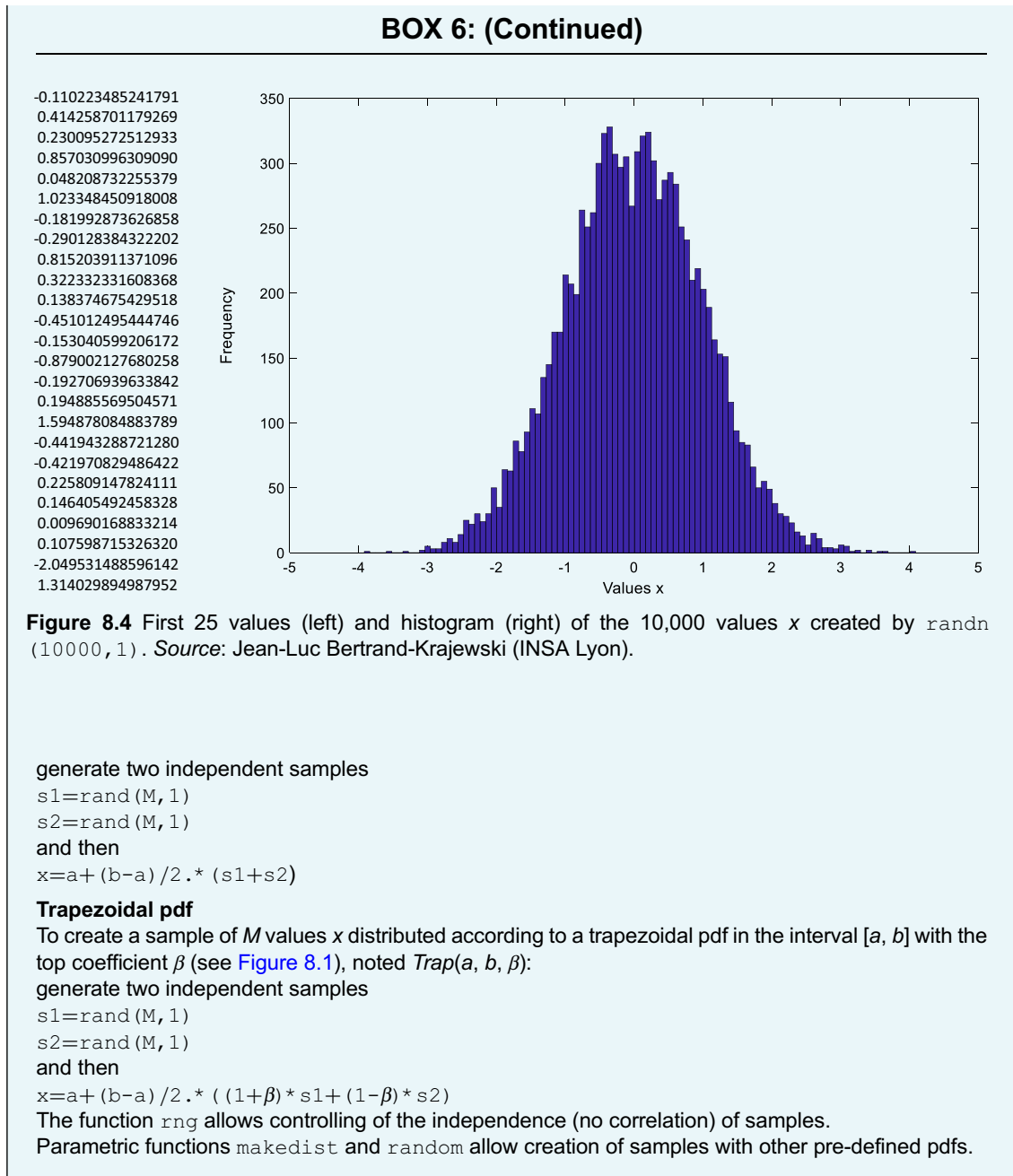| |
|---|
| -0.110223485241791 |
| 0.414258701179269 |
| 0.230095272512933 |
| 0.857030996309090 |
| 0.048208732255379 |
| 1.023348450918008 |
| -0.181992873626858 |
| -0.290128384322202 |
| 0.815203911371096 |
| 0.322332331608368 |
| 0.138374675429518 |
| -0.451012495444746 |
| -0.153040599206172 |
| -0.879002127680258 |
| -0.192706939633842 |
| 0.194485569504571 |
| 1.594878084883789 |
| -0.441943288721280 |
| -0.421970829486422 |
| 0.225809147824111 |
| 0.146405492458328 |
| 0.009690168833214 |
| 0.107598715326320 |
| -2.049531488596142 |
| 1.314029894987952 |

Figure 8.4 First 25 values (left) and histogram (right) of the 10,000 values *x* created by `randn (10000,1)`. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

generate two independent samples
```
s1=rand(M,1)
s2=rand(M,1)
```
and then
```
x=a+(b-a)/2.*(s1+s2)
```
Trapezoidal pdf
To create a sample of *M* values *x* distributed according to a trapezoidal pdf in the interval [*a*, *b*] with the top coefficient β (see Figure 8.1), noted *Trap*(*a*, *b*, β):
generate two independent samples
```
s1=rand(M,1)
s2=rand(M,1)
```
and then
```
x=a+(b-a)/2.*((1+β)*s1+(1-β)*s2)
```
The function `rng` allows controlling of the independence (no correlation) of samples.
Parametric functions `makedist` and `random` allow creation of samples with other pre-defined pdfs.

available methods (e.g. copulas) is beyond the scope of this chapter (Gentle, 2003; Nelsen, 2010). Creating samples with various pdfs and predefined correlation coefficients requires specific methods. Among them, the copula functions are very convenient (Nelsen, 2010) and can generate bi- and multivariate samples with correlated marginal distributions. Generation of correlated uniform and normal samples with Matlab® is described in Box 7. Other examples are given below in Section 8.3.

BOX 7: MATLAB® FUNCTIONS AND CODES TO CREATE UNIFORM AND NORMAL CORRELATED SAMPLES

Bivariate normal (Gaussian) pdf

To create two correlated samples of size M for two quantities X_i and X_j normally distributed, with their own respective means m_i and m_j, and standard deviations s_i and s_j, noted $N(m_i, s_i)$ and $N(m_j, s_j)$, with a defined coefficient of correlation $r_{ij} = r(x_i, x_j) = r_{ji} = r(x_j, x_i)$ (knowing that $r_{ii} = r_{jj} = 1$):

create the vector of means $m = [m_i, m_j]$

create the covariance matrix:

$$COV = \begin{bmatrix} cov(x_i, x_i) & cov(x_i, x_j) \\ cov(x_j, x_i) & cov(x_j, x_j) \end{bmatrix} = \begin{bmatrix} r_{ii}s_i s_i & r_{ij}s_i s_j \\ r_{ji}s_j s_i & r_{jj}s_j s_j \end{bmatrix} = \begin{bmatrix} s_i^2 & r_{ij}s_i s_j \\ r_{ji}s_j s_i & s_j^2 \end{bmatrix}$$

and then type

```
X=mvnrnd(m,COV,M)
```

The output matrix X contains 2 columns, respectively with the values of X_i and X_j.

Example 1: two samples X_1 and X_2 normally distributed with $M = 10,000$ values, with $m_1 = 10$, $s_1 = 25$, $m_2 = 25$, $s_2 = 1.2$ and $r_{12} = 0.45$.

First define

```
m=[10 25]
s1=0.5
s2=1.2
r12=0.45
COV=[s1^2 r12*s1*s2; r12*s2*s1 s2^2]
M=10000
```

and then

```
X=mvnrnd(m,COV,M)
```

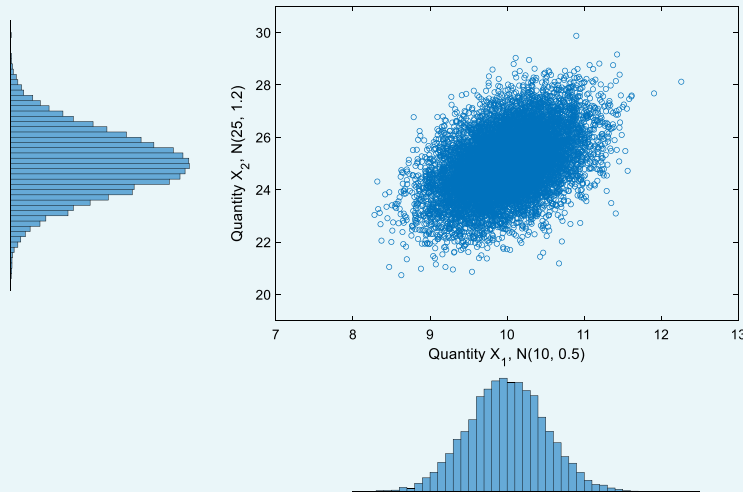The results are illustrated in Figure 8.5.

Figure 8.5 Scatter plot and marginal histograms of two correlated normally distributed samples with 10,000 values x_1 ($m_1 = 10$, $s_1 = 0.5$) and x_2 ($m_2 = 25$, $s_2 = 1.2$) and $r_{12} = 0.45$ created by mvnrnd. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

BOX 7: (Continued)

Bivariate uniform (rectangular) pdf

The easiest way to generate two correlated uniformly distributed samples X_i and X_j is to start with two correlated standard normally distributed samples $N(0, 1)$ and then to use the normal cumulative probability function (cdf) usually noted Φ. The two new standard uniform samples $U(0, 1)$ are then re-scaled as $U(a_i, b_i)$ and $U(a_j, b_j)$ on their respective intervals.

Example 2: two samples X_1 and X_2 uniformly distributed with $M = 10,000$ values, with $[a_1, b_1] = [5, 15]$, $[a_2, b_2] = [18, 32]$ and $r_{12} = 0.45$.
First create two normally distributed samples $N(0, 1)$ in the matrix Z_1 :

```
m=[0 0]
COV=[1 0.45; 0.45 1]
M=10000
Z1=mvnrnd(m,COV,M)
```

Then generate the two uniformly distributed samples $U(0, 1)$ in the matrix Z_2 by means of the function Φ, noted `normcdf` in Matlab®, which transforms a standard normal random sample $N(0, 1)$ into a random sample that is uniform $U(0, 1)$:

```
Z2=normcdf(Z1)
```

Lastly, create the two samples $U(5, 15)$ and $U(18, 32)$ from the two samples in Z_2 :

```
X1=5+(15-5).*Z2(:,1)
X2=18+(32-18).*Z2(:,2)
```
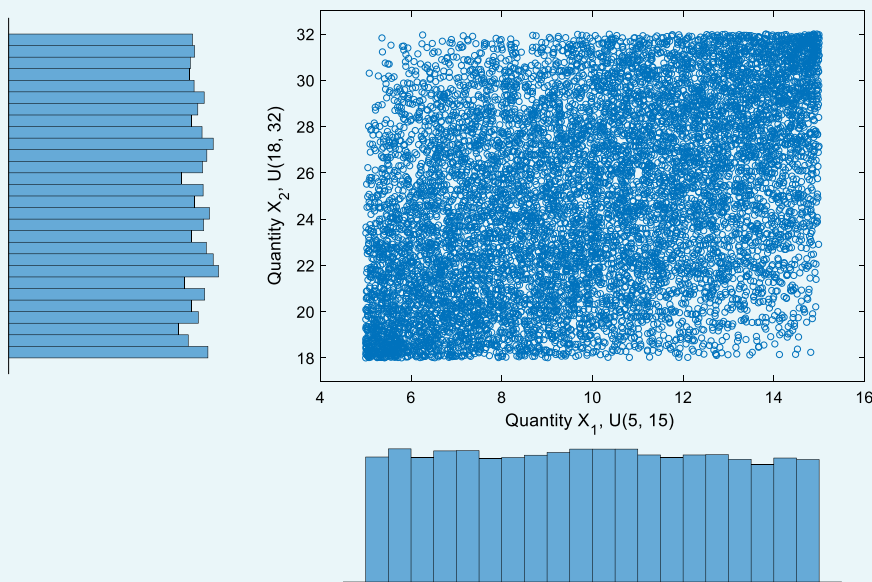
The results are illustrated in Figure 8.6.

Figure 8.6 Scatter plot and marginal histograms of two correlated uniformly distributed samples with 10,000 values x_1 ($a_1 = 5$, $b_1 = 15$) and x_2 ($a_2 = 18$, $b_2 = 32$) and $r_{12} = 0.45$ created by `mvnrnd` and `normcdf`. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).
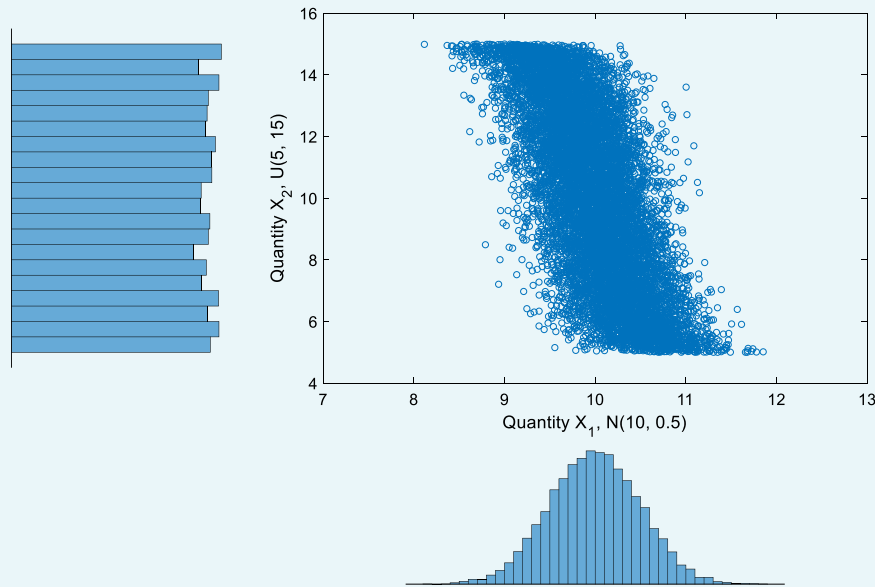
BOX 7: (Continued)

Figure 8.7 Scatter plot and marginal histograms of two correlated samples with $M = 10{,}000$ values and $r_{12} = -0.7$: x_1 as $N(10, 0.5)$ and x_2 as $U(5, 15)$ created by `copularnd`. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Other bi- and multivariate pdfs

Example 3: create two samples X_1 normally distributed with mean $m = 10$ and standard deviation $s = 0.5$, and X_2 uniformly distributed in the interval $[a, b] = [5, 15]$, with $M = 10{,}000$ values and a correlation coefficient $r_{12} = -0.7$.
First create the copula Z:
```
M=10000
r12=-0.7
Z=copularnd('gaussian',r12,M)
```
Then create the two samples X_1 and X_2 in the matrix X:
```
X=[norminv(Z(:,1),10,0.5) 5+(15-5).*Z(:,2)]
```
The results are illustrated in Figure 8.7.
The correlation coefficient $r(X_1, X_2) = -0.69$, which is close to the target value $r_{12} = -0.7$.

8.2.4.4 Size of samples

As the MCM is stochastic in nature, each run of the MCM giving a sample of M estimates y_r of Y from the random samples of the quantities X_i will generate slightly different outputs for $\bar{y}$, $u(y)$ and $[y_{\alpha,low}, y_{\alpha,high}]$. If M is too small (some tens or hundreds), the results may be significantly different for each run, especially for the interval $[y_{\alpha,low}, y_{\alpha,high}]$ which strongly depends on the representativeness and exhaustiveness of the possible combinations of the values $x_{i,r}$ involved in the calculation of the values y_r. Large values of M ensure that successive runs of the MCM deliver successive output estimates which are stable and do not differ from each other more than a required numerical tolerance to be defined by the operator.

ISO (2008b) indicates in its Section 7.2.2 that 'the choice of a value of M that is large compared with $1/(1-\alpha)$, e.g. M at least 10^4 times greater than $1/(1-\alpha)$, should be made'. For example, with $\alpha = 0.95$, a minimum value of $M = 2 \times 10^5$ should be used.

In practice, in many cases, 'a value of $M = 10^6$ can often be expected to deliver a 95% coverage interval for the output quantity such that its length is correct to one or two significant decimal digits' (note in Section 7.2.1 of ISO, 2008b). However, it is always recommended to check that this prior choice of M is appropriate.

A procedure that selects M adaptively by iterations can be used (Section 7.9 of ISO, 2008b). It is based on the number of significant digits n_{dig} considered as meaningful in a numerical value z by the operator. The numerical tolerance δ is then defined as follows:

Express z in the form $c \times 10^{\,l}$ where c is a n_{dig} decimal digit integer and l is an integer.

$$\text{Set } \delta = \frac{1}{2} 10^l \tag{8.53}$$

Example: tolerance

If a discharge $Q = 0.346 \text{ m}^3/\text{s}$ with a standard uncertainty $u(Q) = 0.013 \text{ m}^3/\text{s}$ (taken from Section 8.2.3.4), and if the last two digits are significant, then $n_{dig} = 2$ and $u(Q)$ can be written as 13×10^{-3}, therefore $c = 13$ and $l = -3$. Consequently, the tolerance $\delta = \frac{1}{2} \times 10^{-3} = 0.0005 = 5 \times 10^{-4} \text{ m}^3/\text{s}$. If only one digit is significant, then $Q = 0.35 \text{ m}^3/\text{s}$, $u(Q) = 0.01 \text{ m}^3/\text{s}$, $u(Q)$ can be written as 1×10^{-2}, therefore $c = 1$ and $l = -2$. Consequently, the tolerance $\delta = \frac{1}{2} \times 10^{-2} = 0.005 = 5 \times 10^{-3} \text{ m}^3/\text{s}$.

The objective of the MCM adaptive procedure is to estimate $\bar{y}$, $u(y)$ and $[y_{\alpha,low}, y_{\alpha,high}]$ in a way that ensures they meet the numerical tolerance required. The adaptive procedure includes the following steps (ISO, 2008b):

(1) Set n_{dig}, usually 1 or 2.
(2) Set $M = \max(J, 10^4)$ where J is the smallest integer greater than or equal to $100/\alpha$.
(3) Set $a = 1$, first application of the MCM in the sequence.
(4) Carry out the M Monte Carlo simulations.
(5) From the M outputs, calculate $\bar{y}^{(a)}$, $u(y)^{(a)}$, $y_{\alpha,low}^{(a)}$ and $y_{\alpha,high}^{(a)}$ respectively, for the ath member of the sequence.
(6) If $a = 1$, increase a by one and return to step 4.
(7) Calculate the standard deviation s_y associated with the average estimates $y^{(1)}$, $y^{(2)}$, …$y^{(a)}$ by:

$$s_y^2 = \frac{1}{a(a-1)} \sum_{r=1}^{a} (y^{(r)} - y_m)^2 \tag{8.54}$$

where

$$y_m = \frac{1}{a} \sum_{r=1}^{a} y^{(r)} \tag{8.55}$$

(8) Calculate the counterpart of this statistic also for $u(y)^{(a)}$, $y_{\alpha,low}^{(a)}$ and $y_{\alpha,high}^{(a)}$ respectively.
(9) Use all $a \times M$ values available so far to calculate $\bar{y}$, $u(y)$, $y_{\alpha,low}$ and $y_{\alpha,high}$.
(10) Calculate the numerical tolerance δ according to Equation (8.53) respectively for $\bar{y}$, $u(y)$, $y_{\alpha,low}$ and $y_{\alpha,high}$.
(11) If any of $2s_y$, $2s_{u(y)}$, $2s_{y\alpha,low}$ and $2s_{y\alpha,high}$ exceeds its respective tolerance δ, increase a by one and return to step 4. Otherwise, go to step 12.
(12) As all outputs of the MCM are now stabilized, use all $a \times M$ values available to calculate $\bar{y}$, $u(y)$, and $[y_{\alpha,low}, y_{\alpha,high}]$.

The above adaptive method is appropriate to minimize the total number $a \times M$ of simulations to be run to achieve a given tolerance. However, if the choice of M in step 2 is too low, it will be necessary to increase the number a of applications, and the global time required for simulations will increase. So, in practice, it is suggested to use preferably $M = 10^5$ rather than 10^4 in step 2. In addition, for repetitive calculations (e.g. calculations of uncertainties in discharge in a sewer with measurements every 2 or 5 minutes), the adaptive procedure does not need to be carried out for each time step, which is counterproductive. A first estimation can be done to estimate the minimum necessary total number $a \times M$ of simulations to reach the required tolerance, and then run this total number of Monte Carlo simulations, possibly multiplied by a safety factor, to save computation times. For example, if preliminary assessments indicate that a total number $a \times M = 4 \times 10^5$ simulations is necessary, the routine number can be increased to 10^6 for repetitive simulations. This is only a case-by-case decision.

8.2.4.5 Basic example with Matlab®

For this example, let us take the same case as for the Type B example in Section 8.2.3.4. The four measured quantities K, I, B and h involved in the measurement process given by Equation (8.21) are independent and not correlated, so independent samples will be created. Detailed Matlab® calculations are given in Box 8.

BOX 8: STEP BY STEP APPLICATION OF THE MCM WITH MATLAB®

Define $M = 10^6$:
```
M=1e6
```
For K, create a sample $U(70, 80)$:
```
K=unifrnd(70,80,M,1)
```
One gets
```
mean(K)=74.99
std(K)=2.8854
```
For I, create a sample $N(0.0032, 6 \times 10^{-6})$:
```
I=normrnd(0.0032, 6e-6,M,1)
```
One gets
```
mean(I)=0.003200
std(I)=6.003e-06
```
For B, create a sample $N(0.8005, 0.002)$:
```
B=normrnd(0.805, 0.002,M,1)
```
One gets
```
mean(B)=0.80500
std(B)=0.00200
```
For h, create a sample $N(0.32, 0.0015)$ (as with a degree of freedom $\nu = 59$, the Student t distribution is equivalent to a normal distribution):
```
h=normrnd(0.32,0.0015,M,1)
```
One gets
```
mean(h)=0.3199
std(h)=0.001499
```
The histograms of the four samples are shown in Figure 8.8.

BOX 8: (Continued)
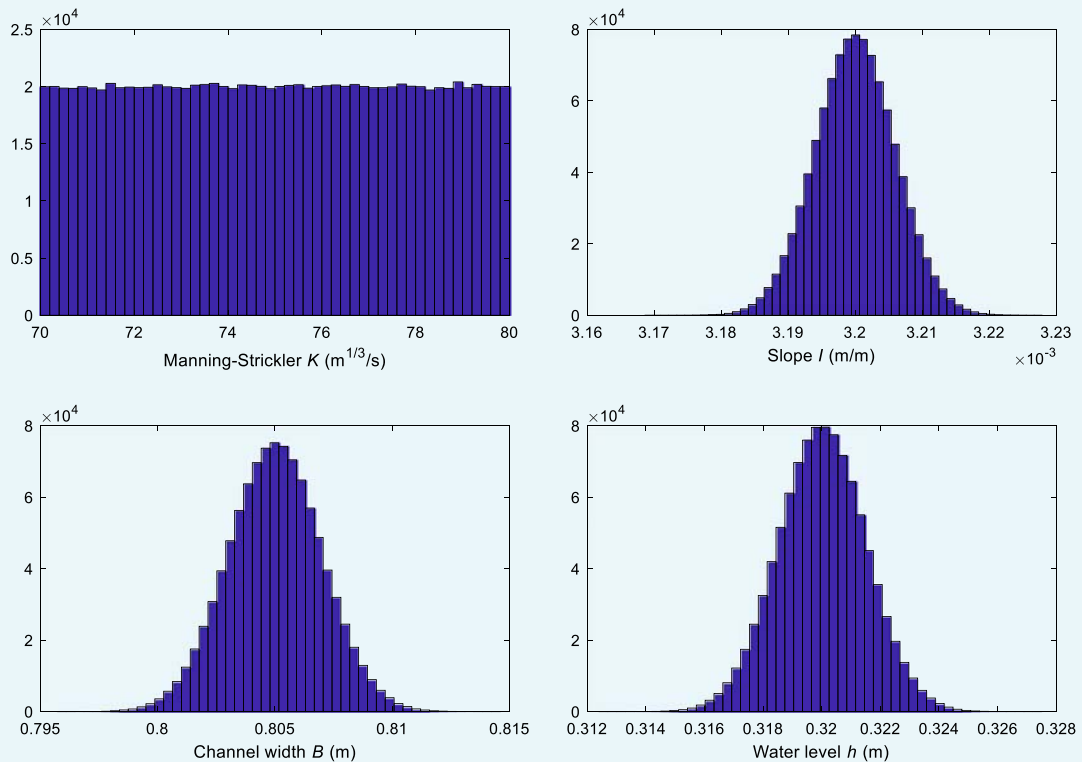
Figure 8.8 Histograms of K, I, B and h used in the MCM with $M = 1{,}000{,}000$ values. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Then calculate the vector Q with M values:
```
Q=K.*power(I,1/2).*power(B.*h,5/3).*power(B+2.*h,-2/3)
```
(Note the importance of the dot in front of the $*$ operators in the above expression to ensure sample vectors are multiplied term by term according to the Matlab® syntax).

The histogram of the M values of Q is shown in Figure 8.9.

Calculate the mean value:
```
mean(Q)
```
One gets $Q = 0.3462$ m³/s.

Calculate the standard deviation:
```
std(Q)
```
One gets $u(Q) = 0.0136$ m³/s.

Calculate the shortest 95% coverage interval:
```
alpha=0.95
Qsort=sort(Q)
q=round(alpha*M)
```
which is an integer.

One gets $q = 950{,}000$.

BOX 8: (Continued)
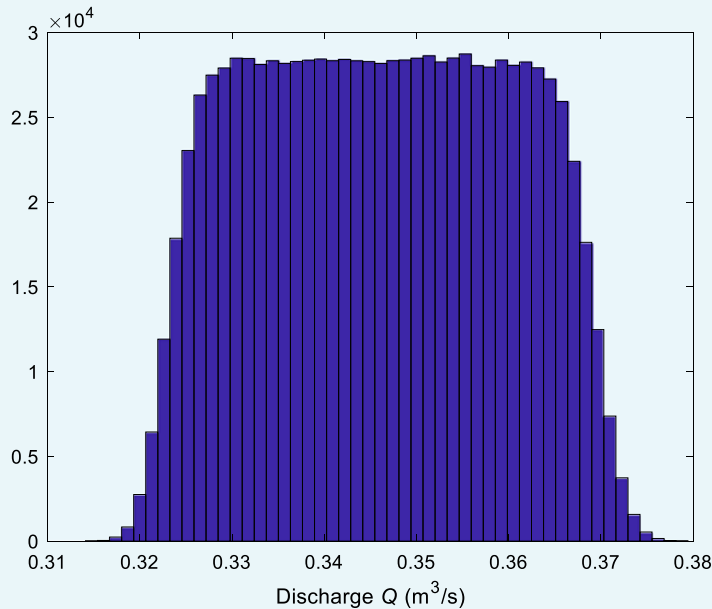
Figure 8.9 Histogram of the discharge Q calculated with the MCM with $M = 1,000,000$ values. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Create the matrix *IC95* with three columns (respectively low boundary, high boundary and width of the interval) as follows:

```
for r=1:M-q
IC95(r,1:3)=[Qsort(r) Qsort(r+q) Qsort(r+q)-Qsort(r)];
end
```

Find the boundaries (in columns 1 and 2) of the shortest interval (in column 3) in *IC95*:

```
IC95(find(IC95(:,3)==min(IC95(:,3))),1:2)
```

The result is $[Q_{95,\,low},\ Q_{95,\,high}] = [0.3233, 0.3689]$ m^3/s.

Let us create non-correlated samples with size $M = 10^6$ for the four quantities:

- K with a uniform distribution between 70 and 80 m$^{1/3}$/s.
- I, B and h with normal distributions given, respectively, by $N_I(0.0032, 6 \times 10^{-6})$, $N_B(0.8005, 0.002)$ and $N_h(0.32, 0.0015)$.

The histograms of the four samples are shown in Figure 8.8. The M values of the discharge Q are then calculated by Equation (8.21). The resulting histogram of the Q values is shown in Figure 8.9.

The mean value is $\bar{Q} = 0.346$ m^3/s, the standard uncertainty is $u(Q) = 0.014$ m^3/s and the shortest 95% coverage interval is $[Q_{95,low}, Q_{95,high}] = [0.323, 0.369]$ m^3/s. This interval is shorter than the interval [0.320, 0.373] obtained with the Type B method in Section 8.2.3.4. This difference is discussed in Section 8.2.5. Box 9 shows how to apply the Monte Carlo method with the Matlab$^®$ code uMCM.

BOX 9: APPLICATION OF THE MONTE CARLO METHOD WITH THE MATLAB® CODE uMCM

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102). The calculations of the discharge Q, its standard uncertainty $u(Q)$ and its shortest coverage interval shown in Box 8 are automated in the Matlab® code `Y = uMCM(Z,A,chaine,alpha,MatCor,Mmc, distrib)`, where the quantities X_i of the measurement process are divided into two groups Z and A, where:

`Z` is the matrix containing data related to time varying quantities (time series). Its structure is as follows: Z has as many lines as time steps in the time series. Each line contains, in successive columns from left to right, m pairs of data for each time varying quantity Z: Z_1, $u(Z_1)$, Z_2, $u(Z_2)$,…Z_i, $u(Z_i)$, …Z_m, $u(Z_m)$.

`A` is the matrix containing data related to constant quantities. The single line of A contains, in the successive columns from left to right, p pairs of data for each constant quantity A: A_1, $u(A_1)$, A_2, $u(A_2)$,…A_j, $u(A_j)$,…A_p, $u(A_p)$.

`chaine` is the equation string describing the measurement process of Y using Z_i and A_j quantities.

`alpha` is the level of probability.

`MatCor` is the $N{\times}N$ matrix of correlation between Z_i and A_j quantities, with $N = m + p$.

`Mmc` is the length of the samples (number of Monte Carlo simulations).

`distrib` is the vertical vector containing the codes of the pdfs for all quantities Z_i and A_j: 0 for no pdf, 1 for normal pdf, 2 for uniform pdf, 3 for triangular pdf.

The uMCM function provides, respectively, the following results in successive columns of the output matrix Y, with one line per time step: the value y of Y, its standard uncertainty $u(Y)$, and the boundaries of the shortest coverage interval.

For the above example, there is one time-varying quantity: the water level h, and three constant quantities: respectively, the Manning-Strickler coefficient K, the channel slope I and the channel width B. Consequently, type

`Z=[h uh]` with only one line as there is only one single value of h.

`A=[K uK I uI B uB]`

The measurement process equation:

`Q=K.*power(I,1/2).*power(B.*h,5/3).*power(B+2.*h,-2/3)`

is then re-written with notations indicating the quantities with their rank in matrices Z and A.

h is the first quantity in the matrix Z: h is replaced by `Z(:,1)` in the above expression of Q. Similarly, `K`, `I` and `B` are replaced, respectively, by `A(:,1)`, `A(:,2)` and `A(:,3)` as they are, respectively, the first, second and third quantities in the matrix A. It is important to note that the indices refer to the rank of the quantities in matrices Z and A, and not to the rank of the columns.

Consequently, type

`chaine='A(:,1).*power(A(:,2),0.5).*power(A(:,3).*Z(:,1),5/3).*power(A (:,3)+2.*Z(:,1),-2/3)'`

and

`alpha=0.95`

In this example, all four quantities in Z and A are independent as there is no correlation or covariance between them. Consequently, type

`MatCor=eye(4)`

BOX 9: (Continued)

---

which gives

$$MatCor = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Type
```
Mmc=1e6
```
Distribution of quantities in Z and A are, respectively, normal (for h), uniform (for K), normal (for l and B).
Consequently, type
```
distrib=[ 1 2 1 1]'
```
Lastly type
```
Q=uMCM(Z,A,chaine,alpha,MatCor,Mmc,distrib)
```
One gets $Q = 0.346\,\mathrm{m^3/s}$, $u(Q) = 0.0136\,\mathrm{m^3/s}$, and the 95% coverage interval is [0.3233, 0.3688] $\mathrm{m^3/s}$.
The values are slightly different from those given in Box 8 as each MC run generates new samples.

8.2.5 Comparison of uncertainties estimated with Type B and Monte Carlo methods

The Type B method is less demanding than the MCM in term of calculations: once the literal expression of the Law of Propagation of Uncertainties (Equation (8.9)) for the quantity Y of interest is established, it can be applied instantaneously to any value of Y. On the contrary, the MCM imposes that the simulations are run systematically for each value of Y, which requires more computation time. However, the MCM is much more flexible and does not require the conditions of application of the Type B method. This is why MCM is the reference method.

In practice, it is frequent that the operator compares the Type B method and the MCM: if both methods give equivalent results, then the Type B method can be applied, which may be more convenient, especially with repetitive calculations. If the results are not equivalent, then the MCM should be used.

The comparison of both methods, based on their coverage intervals, is made with the following steps:

(1) Apply the Type B method to calculate y and the interval $[y - U(y), y + U(y)]$ (Equation (8.13)) for the given level of probability α.
(2) Apply the MCM to calculate $u(y)$ and the interval $[y_{\alpha,low}, y_{\alpha,high}]$ for the same level of probability α.
(3) Set the required tolerance δ for the coverage interval according to Equation (8.53).
(4) Compare the above coverage intervals by determining:

$$d_{low} = |y - U(y) - y_{\alpha,low}| \tag{8.56}$$

$$d_{high} = |y + U(y) - y_{\alpha,high}| \tag{8.57}$$

(5) If both d_{low} and d_{high} are not larger than the tolerance δ for $u(y)$, then the comparison is positive, and the Type B method can be applied. Otherwise, the MCM should be applied. It is important to note that the comparison applies only for the specified probability level α.

Example 1:

The comparison is made with the results obtained in Sections 8.2.3.4 and 8.2.4.5. The probability level α is 0.95.

Coverage interval obtained with the Type B method: [0.320, 0.373] m^3/s
Coverage interval obtained with the MCM: [0.323, 0.369] m^3/s
Standard uncertainty $u(y) = 0.013$ m^3/s.
If $n_{dig} = 2$, then $u(y) = 13 \times 10^{-3}$, $l = -3$ and $\delta = 0.5 \times 10^{-3}$.
$d_{low} = |0.320–0.323| = 3 \times 10^{-3} > \delta$
$d_{high} = |0.373–0.369| = 4 \times 10^{-3} > \delta$

Consequently, the Type B method and the MCM give non-equivalent results and the MCM shall be applied.

The same approach can be used to compare the coverage intervals obtained in Section 8.2.3.4 with both an infinite degree of freedom and the effective degree of freedom. With $n_{dig} = 2$ and $\delta = 0.5 \times 10^{-3}$, the conclusion is that the results are different.

Example 2:

Let us modify the above example by assuming now that all quantities follow normal distributions, for both Type B and MCM calculations, with

$K = 75$ m$^{1/3}$/s, $u(K) = 2.88$ m$^{1/3}$/s
$I = 3.2 \times 10^{-3}$ m/m, $u(I) = 6 \times 10^{-6}$ m/m
$B = 0.805$ m, $u(B) = 2 \times 10^{-3}$ m
$h = 0.32$ m, $u(h) = 1.5 \times 10^{-3}$ m

One gets (with 4 digits for illustrative purpose):

Coverage interval with the Type B method: $Q_{TB} = [0.3196, 0.3728]$
Standard uncertainty with the Type B method: $u(y)_{TB} = 0.0136$ m^3/s
Coverage interval with the MCM: $Q_{MC} = [0.3195, 0.3726]$ m^3/s
Standard uncertainty with the MCM: $u(y)_{MC} = 0.0136$ m^3/s
If $n_{dig} = 2$, then $u(y) = 13 \times 10^{-3}$, $l = -3$ and $\delta = 0.5 \times 10^{-3}$.
$d_{low} = abs(Q_{TB,low} - Q_{MC,low}) = 1.5287 \times 10^{-4} < \delta$
$d_{high} = abs(Q_{TB,high} - Q_{MC,high}) = 1.0596 \times 10^{-4} < \delta$

In this case, the Type B method and the MCM give equivalent results and can be both used indifferently.

8.2.6 Correlation between quantities

Correlation (or covariance) between quantities involved in a measurement process may have a significant effect on the estimation of the resulting uncertainty. Therefore, special attention should be devoted to checking and estimating correlations when they exist. The best practice in uncertainty assessment thus requires systematic checking and accounting for the possible correlation and estimating it. Only if its influence is proved to be negligible over the entire range of expected values of all quantities, may it be neglected in further calculations. It is worth mentioning that detecting and quantifying correlation is not always obvious and may require expertise.

Correlation can occur:

- Between different input quantities X_i used to calculate an output variable Y.
- As autocorrelation in time series used to calculate aggregated values.

8.2.6.1 Correlation between input quantities

According to ISO (2008a, Sections 5.2.4 and 5.2.5, p. 22–23), 'there may be significant correlation between two input quantities if the same measuring instrument, physical measurement standard, or reference datum having a significant standard uncertainty is used in their determination. […] Correlations between input quantities cannot be ignored if present and significant. The associated covariances should be evaluated experimentally if feasible by varying the correlated input quantities, or by using the pool of available information on the correlated variability of the quantities in question.'

As explained in Section 8.2.3.2, it is recommended to write the function f

$$Y = f(X_1, X_2, \ldots X_i, \ldots X_N) \tag{8.58}$$

with the quantities X_i corresponding to separate and thus uncorrelated measurement instruments.

Other frequent cases where covariance should be accounted for occur when quantities X_i used in Equation (8.58) are obtained from previous regression functions like e.g. calibration functions, rating curves, etc.

If covariance remains and cannot be avoided between some quantities X_i, it should then be quantified. As covariance may be positive or negative, it may increase or decrease the resulting uncertainty in Y. Some examples are given in Section 8.2.6.3.

8.2.6.2 Autocorrelation in time series

Time series data of flow, water level or water quality are not randomly distributed as they result from dynamic processes where each value can be considered as equal to the previous one with a small increase or decrease depending on the evolution of the quantity. In such cases, the successive values of a given quantity may be considered as partly autocorrelated. Analogous to geostatistics, the chronostatistics approach (Gy, 1988, 2012) may be applied in such cases, using the variograph of the data to estimate the time horizon over which the autocorrelation between successive values is meaningful. Details of the variograph approach are given in Bertrand-Krajewski & Bardin (2001).

The uncertainty in the sum of a time series can be calculated with the Type B method under three different assumptions:

(1) No autocorrelation ($r = 0$) between successive values of the time series.
(2) Full autocorrelation ($r = 1$) between successive values of the time series.
(3) Partial autocorrelation estimated between successive values of the time series calculated from the variograph.

The third assumption is the recommended approach. The two first ones are given for comparison only. The Matlab® code `uTypeBsum` given in Box 10 allows calculation of the uncertainty in the sum of a time series with the three above assumptions.

8.2.6.3 Examples of correlation

8.2.6.3.1 Covariance in measurements

The previous example of a discharge measurement in a rectangular open channel (Section 8.2.3.4) is revisited, with the following equations:

$$Q = f(K, I, S, R_h) = KI^{\frac{1}{2}}SR_h^{\frac{2}{3}} \tag{8.59}$$

$$S = Bh \tag{8.60}$$

BOX 10: CALCULATING THE UNCERTAINTY IN THE SUM OF A TIME SERIES WITH THE MATLAB® CODE `uTypeBsum`

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).

Let us consider a times series of volume (m^3) measured during 24 hours with a two-minute time step. The three columns of the `vol1.csv` file (separator ;) contain successively the date and time, the volume *Vol* (m^3) and the standard uncertainty *u(Vol)* (m^3).

One should first import the data of the `vol1.csv` file in a matrix, without the column of date and time:

`data=dlmread('vol1.csv',';',1,1)`

Calculate the daily volume V_d and its standard uncertainty $u(V_d)$ with the three assumptions about correlation (i.e. no autocorrelation – nc, full autocorrelation – fc, and partial autocorrelation – pc):

`Vd=uTypeBsum(data)`

One gets, respectively, $V_d = 328.0588$ m^3, $u(V_d)_{nc} = 0.4495$ m^3, $u(V_d)_{fc} = 12.0623$ m^3 and $u(V_d)_{pc} = 4.6842$ m^3.

$$R_h = \frac{Bh}{B + 2\,h} \tag{8.61}$$

$$Q = f(K, I, B, h) = KI^{\frac{1}{2}}(Bh)\left(\frac{Bh}{B + 2\,h}\right)^{\frac{2}{3}} = KI^{\frac{1}{2}}(Bh)^{\frac{5}{3}}(B + 2\,h)^{-\frac{2}{3}} \tag{8.62}$$

To ensure no covariance between the quantities *K*, *I*, *B* and *h*, Equation (8.62) was used for the previous uncertainty calculations with the Type B method and the MCM (see Sections 8.2.3.4 and 8.2.4.5). The minimum 95% coverage interval of *Q* given by the MCM was [0.323, 0.369] m^3/s. It can be considered as the reference value.

The most usual writing of Equation (8.59) uses the wet section *S* and the hydraulic radius R_h. Both values are calculated by Equations (8.60) and (8.61), respectively, from the same uncorrelated measured quantities *B* and *h*, and are thus correlated.

With the previous values

$B = 0.805$ m, $u(B) = 2 \times 10^{-3}$ m
$h = 0.32$ m, $u(h) = 1.5 \times 10^{-3}$ m

one gets with MCM:

$S = 0.2576$ m^2, $u(S) = 0.0014$ m^2, $R_h = 0.1783$ m and $u(R_h.) = 5 \times 10^{-4}$ m.

One can assume that *S* and R_h are positively fully correlated, with $r(S, R_h) = +1$. This assumption can be tested by means of a basic Monte Carlo simulation with Matlab® (see Box 11).

If this correlation is not accounted for, applying the MCM gives $Q_1 = 0.3462$ m^3/s and $u(Q_1) = 0.0134$ m^3/s. If this correlation is accounted for, the MCM gives $Q_2 = 0.3462$ m^3/s and $u(Q_2) = 0.0135$ m^3/s (see detailed calculations in Box 12).

The covariance very slightly increases *u(Q)* (from 0.0134 to 0.0135 m^3/s) and the width of the 95% coverage interval (from [0.3199, 0.3726] to [0.3198, 0.3728] m^3/s). In this example, the increase may be considered as negligible for operational use but reflects the influence of the writing of the equation *f* used in uncertainty assessment for representing the measurement process.

BOX 11: ESTIMATING THE CORRELATION BETWEEN S AND R_h WITH THE MATLAB® CODE uMCM

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
Type
```
B=0.805
uB=2e-3
h=0.32
uh=1.5e-3
M=1e6
Bmc=normrnd(B, uB, M, 1)
hmc=normrnd(h, uh, M, 1)
Smc=Bmc.*hmc
Rhmc=Bmc.*hmc./(Bmc+2*hmc)
```
Calculate the coefficient of correlation between S and R_h:
```
corr(Smc, Rhmc)
```
One gets $r(S, R_h) = 0.9961$, which is equivalent to one in practice.

BOX 12: IGNORING OR ACCOUNTING FOR THE CORRELATION BETWEEN S AND R_h WITH THE MATLAB® CODE uMCM

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
In addition to the variables created in Box 11, type
```
S=mean(Smc)
uS=std(Smc)
Rh=mean(Rhmc)
uRh=std(Rhmc)
K=75
uK=2.88
I=3.2e-3
uI=6e-6
Z=[S uS Rh uRh]
A=[K uK I uI]
distrib=[1 2 1 1]'
```
If the correlation $r(S, R_h)$ between S and R_h is ignored, the correlation matrix is
```
MatCor1=eye(4)
```

$$MatCor1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Type
```
chaine='A(:,1).*power(A(:,2),0.5).*Z(:,1).*power(Z(:,2),2/3)'
```

BOX 12: (Continued)

```
Mmc=1e6
alpha=0.95
```
and lastly
```
Q1=uMCM(Z,A,chaine,alpha,MatCor1,Mmc,distrib)
```
One gets $Q_1 = 0.3462$ m^3/s, $u(Q_1) = 0.0134$ m^3/s and $IC95\ min_1 = [0.3199, 0.3726]$ m^3/s.
If the correlation $r(S, R_h) = +1$ between S and R_h is accounted for, applying the MCM requires modification of the correlation matrix $MatCor$:
```
MatCor2=MatCor1
MatCor2(1,2)=1
MatCor2(2,1)=1
```

$$MatCor2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then type
```
Q2=uMCM(Z,A,chaine,alpha,MatCor2,Mmc,distrib)
```
One gets $Q_2 = 0.3462$ m^3/s, $u(Q_2) = 0.0135$ m^3/s and $IC95\ min_2 = [0.3198, 0.3728]$ m^3/s.

The initial minimum 95% coverage interval of Q obtained with the MCM and the four separate and independent quantities (Section 8.2.4.5) was [0.323, 0.369] m^3/s. It can be considered as the reference value as it does not include any correlation between the input quantities.

8.2.6.3.2 Covariance resulting from a regression function

Let us consider the discharge in a 1.8 m high A180 egg-shape sewer measured by means of both a water level sensor and a flow velocity sensor. *In situ* measurements by a land surveyor have been carried out to collect a series of 21 pairs of points (h_i, S_i) with h_i (m) the water level and S_i (m^2) the corresponding wet section (see Table 8.5, Figure 8.10, and the file eggshape1.csv). They are used to establish a function $S(h)$ defined as a third order polynomial function:

$$S = b_1\ h + b_2\ h^2 + b_3\ h^3 \tag{8.63}$$

The regression (detailed calculations are given in Box 13) gives the following results:

- Values of the parameters b_i: $[b_1, b_2, b_3] = [0.7825, 0.3601, -0.1473]$.
- Standard uncertainties $u(b_i)$: $[u(b_1), u(b_2), u(b_3)] = [0.0173, 0.0287, 0.0114]$.

- Correlation matrix of the parameters b: $MatCor(b) = \begin{bmatrix} 1 & -0.9656 & 0.9098 \\ -0.9656 & 1 & -0.9848 \\ 0.9098 & -0.9848 & 1 \end{bmatrix}$.

The values of the coefficients of correlation in *MatCor(b)* show that the parameters b_i are strongly correlated, either positively (b_1 with b_3) or negatively (b_1 with b_2, b_2 with b_3).

Let us now calculate the discharge Q and $u(Q)$ when the measured water level is $h = 0.45$ m and the mean flow velocity $V = 0.42$ m/s (detailed calculations are given in Box 14). Their respective standard

Table 8.5 Couples of points (h_i, S_i) for the A180 egg-shape sewer (file `eggshape1.csv`).

h_i (m)	S_i (m^2)
0.00	0.00
0.10	0.07
0.11	0.08
0.12	0.09
0.20	0.17
0.30	0.27
0.40	0.37
0.50	0.47
0.60	0.57
0.70	0.78
0.80	0.68
0.90	0.88
1.00	0.99
1.10	1.09
1.20	1.20
1.30	1.30
1.40	1.40
1.50	1.49
1.60	1.58
1.70	1.66
1.83	1.72

uncertainties are $u(h) = 3 \times 10^{-3}$ m (from sensor calibration and *in situ* conditions) and $u(V) = 0.05$ m/s (from expertise).

Q is given by the following equation:

$$Q = S(h)V = (b_1 h + b_2 h^2 + b_3 h^3)V \tag{8.64}$$

One gets $Q = 0.1729$ m^3/s.

The quantities h and V are measured independently and are not correlated. They are also not correlated with the parameters b_i which have been established from the land surveyor's data. However, *MatCor(b)* shows that the parameters b_i are correlated with each other.

$u(Q)$ and the 95% coverage intervals are calculated with the MCM for two cases: case 1 accounting for the correlation of the parameters b_i, and case 2 neglecting the correlation. The results given in Table 8.6
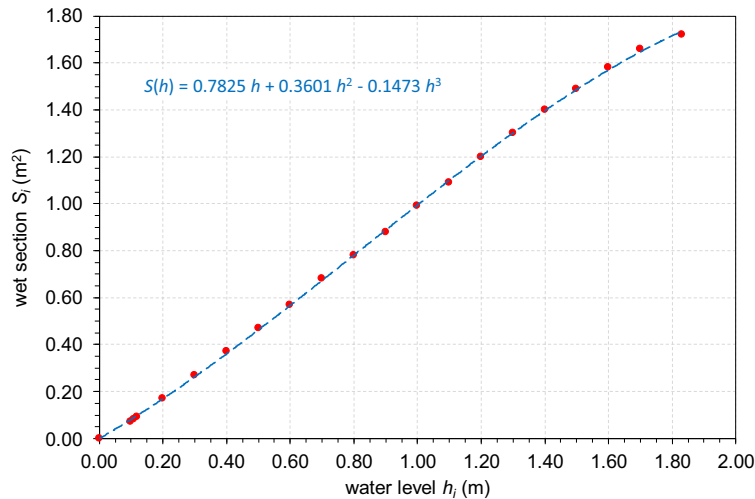
Figure 8.10 Plot of measured points (h_i, S_i) for the A180 egg-shape sewer (red dots) and fitted 3rd order polynomial regression function $S(h)$ (dashed blue line). *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

BOX 13: REGRESSION OF THE $S(h)$ FUNCTION FOR AN A180 EGG-SHAPE SEWER WITH MATLAB®

With `hi` and `Si` the vertical vectors containing respectively the 21 values of h_i and S_i in Table 8.5, one can estimate the values of the parameters b, their standard uncertainties $u(b)$, their covariance matrix $COV(b)$ and the mean squared error of the regression *mse*.
Type
`[b ub mse COVb] = lscov([hi hi.^2 hi.^3], Si)`
One gets
$[b_1, b_2, b_3] = [0.7825, 0.3601, -0.1473]$
$[u(b_1), u(b_2), u(b_3)] = [0.0173, 0.0287, 0.0114]$
$mse = 7.04 \times 10^{-5}$
and

$$COV(b) = 10^{-3} \times \begin{bmatrix} 0.2982 & -0.4789 & 0.1795 \\ -0.4789 & 0.8251 & -0.3232 \\ 0.1795 & -0.3232 & 0.1306 \end{bmatrix}$$

By applying Equation (8.10), the correlation matrix of b, noted *MatCor(b)*, is calculated by
`MatCorb=COVb./(ub*ub')`
which gives

$$MatCor(b) = \begin{bmatrix} 1 & -0.9656 & 0.9098 \\ -0.9656 & 1 & -0.9848 \\ 0.9098 & -0.9848 & 1 \end{bmatrix}$$

Table 8.6 Comparison of results of uncertainty in discharge Q in the A180 egg-shape sewer, by accounting for the correlation between the parameters b (case 1) or by neglecting it (case 2).

	Case 1 With correlation	Case 2 Without correlation
Q (m^3/s)	0.1729	0.1729
$u(Q)$ (m^3/s)	0.0207	0.0211
IC95 min (m^3/s)	[0.1322, 0.2131]	[0.1316, 0.2141]
Relative standard uncertainty (%)	11.9%	12.2%

BOX 14: CALCULATION OF $u(Q)$ AND 95% COVERAGE INTERVALS WITH THE MATLAB® CODE uMCM

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
In addition to the variables defined in Box 13, type
```
h=0.45
V=0.42
Q=sum(b.*[h h.^2 h.^3]')*V
```
One gets $Q = 0.1729$ m^3/s.
To calculate the standard uncertainty $u(Q)$ and the 95% coverage interval with the MCM, type
```
uh=3e-3
uV=0.05
Z=[h uh V uV]
A=[b(1) ub(1) b(2) ub(2) b(3) ub(3)]
chaineQ='(A(:,1).*Z(:,1)+A(:,2).*Z(:,1).^2+A(:,3).*Z(:,1).^3).*Z(:,2)'
alpha=0.95
Mmc=1e6
distrib=[1 1 1 1 1]'
```
For case 1, type:

$$MatCor1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.9656 & 0.9098 \\ 0 & 0 & -0.9656 & 1 & -0.9848 \\ 0 & 0 & 0.9098 & -0.9848 & 1 \end{bmatrix}$$

For case 2, type:
```
MatCor2=eye(5)
```

$$MatCor2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

<div style="border:1px solid">

BOX 14: (Continued)

Lastly type
```
Q1=uMCM(Z,A,chaineQ,alpha,MatCor1,Mmc,distrib)
```
and
```
Q2=uMCM(Z,A,chaineQ,alpha,MatCor2,Mmc,distrib)
```
Results are given in Table 8.6.

</div>

indicate that, in this case, accounting for the correlation of the parameters b_i slightly reduces the resulting uncertainty in the discharge Q. Similar results are obtained with the Type B method.

8.2.6.3.3 Covariance in rain gauge calibration function

Tipping bucket rain gauges are affected by systematic underestimation when rainfall intensities increase (see e.g. Bertrand-Krajewski *et al.*, 2000). Therefore, in addition to the bucket static calibration, it is recommended to establish a dynamic calibration function to account for and then correct the underestimation error (see Chapter 7).

An experiment is carried out with known and controlled rainfall intensities I_r (mm/h) over a range of measurement. The response of the rain gauge, i.e. the measured rainfall intensity I_m (mm/h) for each applied intensity I_r, is recorded (the protocol is described in Bertrand-Krajewski *et al.*, 2000 and in Chapter 2). An example of data for a rain gauge in the Mediterranean region where high intensities are expected is given in the file `raingauge1.csv` and shown in Table 8.7.

The typical calibration function is a power function

$$I_r = b_1 I_m^{b_2} \tag{8.65}$$

Let us estimate the parameters b_1 and b_2, their standard uncertainties, and their co-variance and correlation (detailed calculations are given in Box 15). One gets:

$$[b_1, b_2] = [2.0826, 0.8401]$$
$$[u(b_1), u(b_2)] = [0.4440, 0.0398]$$
$$\text{and } MatCor = \begin{bmatrix} 1.0000 & -0.9969 \\ -0.9969 & 1.0000 \end{bmatrix}$$

Table 8.7 Rain gauge dynamic calibration data set (file `raingauge1.csv`).

I_r (mm/h)	I_m (mm/h)
0	0
30	30
60	60
110	109
160	155
220	199
290	237

BOX 15: REGRESSION OF A RAIN GAUGE CALIBRATION
FUNCTION WITH MATLAB®

This non-linear regression can be done by means of the Matlab® function `nlinfit`.
Type
```
Ir=[0 30 60 110 160 220 290]'
Im=[0 30 60 109 155 199 237]'
fun=@(b,Ir) b(1).*power(Ir,(b(2)))
b0=ones(1,2)
```
and lastly
```
[b,R,J,CovB] =nlinfit(Ir,Im,fun,b0)
```
One gets
$b = [b_1, b_2] = [2.0826, 0.8401]$
Then
```
ub=sqrt(diag(CovB))
```
gives
$u(b) = [u(b_1), u(b_2)] = [0.4440, 0.0398]$
The covariance matrix is

$$CovB = \begin{bmatrix} 0.1971 & -0.0176 \\ -0.0176 & 0.0016 \end{bmatrix}$$

The correlation matrix is given by
```
MatCor=CovB./(ub*ub')
```

$$MatCor = \begin{bmatrix} 1.0000 & -0.9969 \\ -0.9969 & 1.0000 \end{bmatrix}$$

The parameters b_i are strongly negatively correlated.

From these results, one can now estimate the most likely true rainfall intensity I_{est} from a measured value I_{mes} and its standard uncertainty $u(I_{est})$ by using the Type B method and the inverse function

$$I_{est} = \left(\frac{I_{mes}}{b_1}\right)^{\frac{1}{b_2}} \tag{8.66}$$

Let us consider $I_{mes} = 152$ mm/h with $u(I_{mes}) = 5\%$.

The standard uncertainty $u(I_{est})$ is calculated with the Type B method for two cases: case 1 accounting for the correlation of the parameters b_i, and case 2 neglecting the correlation (detailed calculations are given in Box 16). One gets, respectively:

$I_{est1} = 165$ mm/h and $u(I_{est1}) = 10$ mm/h.
$I_{est2} = 165$ mm/h and $u(I_{est2}) = 59$ mm/h.

In this example, not accounting for the correlation between quantities leads to a standard uncertainty multiplied by a factor close to 6.

BOX 16: CALCULATION OF $u(I_{est})$ WITH THE MATLAB® CODE `uTypeB`

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
In addition to the variables defined in Box 15, type

```
Imes=152
uImes=0.05*Imes
b1=b(1)
ub1=ub(1)
b2=b(2)
ub2=ub(2)
Z=[Imes uImes]
A=[b1 ub1 b2 ub2]
alpha=0.95
```

Case 1: correlation is accounted for.
Type

```
MatCor1=eye(3)
MatCor1(2:3,2:3)=MatCor
```

$$MatCor1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.9969 \\ 0 & -0.9969 & 1 \end{bmatrix}$$

```
chaineIest='(Z(:,1)./A(:,1))^(1/A(:,2)))'
```

and then

```
Iest1=uTypeB(Z,A,chaineIest,alpha,MatCor1)
```

One gets $I_{est1} = 165$ mm/h and $u(I_{est1}) = 10$ mm/h.
Case 2: correlation is not accounted for.
Type

```
MatCor2=eye(3)
```

$$MatCor2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
Iest2=uTypeB(Z,A,chaineIest,alpha,MatCor2)
```

One gets $I_{est2} = 165$ mm/h and $u(I_{est2}) = 59$ mm/h.

8.3 EXAMPLES OF APPLICATIONS

All data and the following examples, along with the Matlab® codes, are available at https://doi.org/10.2166/9781789060102, so that the readers can re-do the examples themselves and also use the software codes for their own needs and applications.

8.3.1 Uncertainty in discharge calculation with a thin plate rectangular weir formula

Let us consider a thin plate rectangular weir with lateral contraction and the following characteristics (Figure 8.11):

- Channel width $B = 0.8$ m.
- Notch width $B_c = 0.48$ m.
- Crest height $h_p = 0.25$ m.
- Water head $h = 0.15$ m, measured at the distance L upstream of the weir.

The Kindsvater-Carter formula and the other parameters to estimate the discharge Q_{RW} over the rectangular weir are given in hydraulics textbooks and in the international standard ISO (2017):

$$Q_{RW} = \frac{2}{3} C_d \sqrt{2\,g} B_e h_e^{\frac{3}{2}} \tag{8.67}$$

where C_d (-) is the discharge coefficient, B_e (m) is the effective width and h_e (m) is the effective head, which themselves depend on other parameters related to the geometry of the weir:

$$C_d = f\left(\frac{B_c}{B}, \frac{h}{h_p}\right) \tag{8.68}$$

$$B_e = B + K_b \tag{8.69}$$

$$h_e = h + K_h \tag{8.70}$$

where K_b and K_h are quantities compensating for combined effects of viscosity and surface tension of water.

***Step 1*: Check of the applicability of the formula**

Before applying the Kindsvater-Carter formula (Equation (8.67)), the operator shall check that the numerical values of the variables respect the five conditions given in ISO (2017), as listed and checked in Table 8.8. In the present case, all conditions are satisfied, and the formula can be applied.
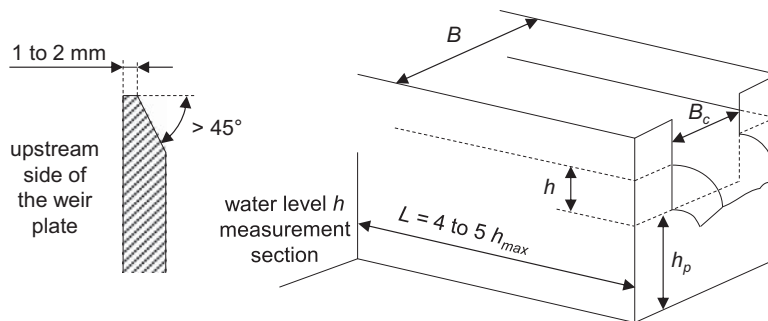
Figure 8.11 Rectangular weir with lateral contraction. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Table 8.8 Checking of the conditions of applicability of the Kindsvater-Carter formula.

Condition required by ISO 1438 (2017)	Example data	Check
h/h_p shall be less than 2.5	$h/h_p = 0.6 < 2.5$	☑
h shall be higher than 0.03 m	$h = 0.15 > 0.03$ m	☑
B_c shall be larger than 0.15 m	$B_c = 0.48 > 0.15$ m	☑
h_p shall be higher than 0.10 m	$h_p = 0.25 > 0.10$ m	☑
$(B-Bc)/2$ shall be larger than 0.10	$(B-Bc)/2 = 0.16 > 0.10$	☑

***Step 2*: Calculation of the discharge Q_{RW}**

For $B_c/B = 0.6$, and $h/h_p = 0.6$, the ISO standard gives (ISO 2017, pp. 11–13):

$$C_d = 0.593 + 0.018 \, \frac{h}{h_p} \tag{8.71}$$

$K_b = 3.6 \times 10^{-3}$ m
$K_h = 10^{-3}$ m

Accordingly, Equation (8.67) can be rewritten as

$$Q_{RW} = \frac{2}{3}\left(a + b\,\frac{h}{h_p}\right)\sqrt{2\,g}(B_c + K_b)(h + K_h)^{\frac{3}{2}} \tag{8.72}$$

where $a = 0.593$ and $b = 0.018$ (see Equation (8.71)).

The discharge over the rectangular weir is $Q_{RW} = 0.506$ m^3/s (detailed calculations are given in Box 17).

***Step 3*: Calculation of the 95% coverage interval of the discharge Q_{RW}**

This step includes the estimation of the standard uncertainties of all quantities in Equation (8.72). Water level sensor calibration, *in situ* repeated measurements of weir geometry and the ISO standard give, respectively:

$u(h) = 1$ mm
$u(h_p) = 1$ mm
$u(B_c) = 0.5$ mm
$u(a) = 0.75\%$
$u(b) = 0.75\%$
$u(K_b) = 0.15$ mm
$u(K_h) = 0.15$ mm.

One assumes that the uncertainty in the gravity $g = 9.81$ m/s is negligible.

Applying the MCM (detailed calculations are given in Box 17) gives the following results:

$u(Q_{RW}) = 0.006$ m^3/s
$IC95 \; min(Q_{RW}) = [0.049, 0.052]$ m^3/s.

<div style="border:1px solid">

BOX 17: CALCULATION OF Q_{RW} AND OF ITS STANDARD UNCERTAINTY WITH THE MATLAB® CODE uMCM

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
Type

```
h=0.15
uh=1e-3
hp=0.25
uhp=1e-3
Bc=0.48
uBc=0.5e-3
a=0.593
ua=0.75e-2*a
b=0.018
ub=0.75e-2*b
Kb=3.6e-3
uKb=0.15e-3
Kh=0.001
uKh=0.15e-3
g=9.81
Z=[h uh]
A=[hp uhp Bc uBc a ua b ub Kb uKb Kh uKh]
distrib=ones(7,1)
Nmc=1e6
alpha=0.95
MatCor=eye(7)
chaineRW='2/3*sqrt(2*9.81).*(A(:,3)+A(:,4).*Z(:,1)./A(:,1)).*
(A(:,2)+A(:,5)).*(Z(:,1)+A(:,6)).^(3/2)'
```

and lastly

```
QRW=uMCM(Z,A,chaineRW,alpha,MatCor,Nmc,distrib)
```

One gets $Q_{RW} = 0.5059$ m³/s, $u(Q_{RW}) = 6.3699 \times 10^{-4}$ m³/s and *IC95 min*$(Q_{RW}) = [0.0493,$ 0.0518] m³/s.

</div>

8.3.2 Uncertainty in discharge calculation with both water level and flow velocity measurements

Let us consider a circular sewer with radius R_c (Figure 8.12) equipped with a sensor measuring the water depth h (m) and another sensor measuring the mean flow velocity V (m/s).

The discharge Q_c (m³/s) is calculated by the following equation:

$$Q_c = S(h)V = R_c^2\left[\arccos\left(1 - \frac{h}{R_c}\right) - \left(1 - \frac{h}{R_c}\right)\sin\left(\arccos\left(1 - \frac{h}{R_c}\right)\right)\right]V \tag{8.73}$$

According to four repeated measurements (Type A method), the radius R_c is equal to 0.6 m and its standard uncertainty $u(R_c) = 0.002$ m. The standard uncertainties $u(h)$ and $u(V)$ in the file `hV1.csv` are calculated from sensors calibration and *in situ* experiments.
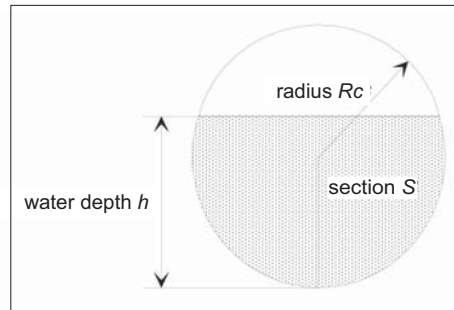
Figure 8.12 Scheme of the circular pipe. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Both the water level h and the mean flow velocity V are measured every 2 minutes during 24 hours. The file hV1.csv (separator ;) contains 720 values in the successive order of columns: h, $u(h)$, V and $u(V)$. The first 15 lines of hV1.csv are shown in Table 8.9.

This example shows successively

- The calculation of the discharge Q_c and of its uncertainty $u(Q_c)$ for each time step.
- The calculation of the daily volume V_d, its standard uncertainty $u(V_d)$ and its 95% coverage interval.

Step 1: Calculation of the discharge Q_c and of its uncertainty $u(Q_c)$ for each time step

One assumes that the quantities h, V and R_c are independent and normally distributed. Detailed calculations with both the MCM and the Type B method are given in Box 18. The results are shown graphically in Figure 8.13 and the first 15 lines of the numerical values are given in Table 8.10.

Table 8.9 First 15 lines of the file hV1.csv.

Date	*h* (m)	*u(h)* (m)	*V* (m/s)	*u(V)* (m/s)
01/01/2017 00:00	0.368	0.008	0.634	0.05
01/01/2017 00:02	0.368	0.008	0.632	0.05
01/01/2017 00:04	0.356	0.008	0.642	0.05
01/01/2017 00:06	0.356	0.008	0.642	0.05
01/01/2017 00:08	0.356	0.008	0.628	0.05
01/01/2017 00:10	0.349	0.008	0.634	0.05
01/01/2017 00:12	0.349	0.008	0.638	0.05
01/01/2017 00:14	0.349	0.008	0.628	0.05
01/01/2017 00:16	0.336	0.008	0.627	0.05
01/01/2017 00:18	0.336	0.008	0.634	0.05
01/01/2017 00:20	0.336	0.008	0.634	0.05
01/01/2017 00:22	0.349	0.008	0.629	0.05
01/01/2017 00:24	0.349	0.008	0.614	0.05
01/01/2017 00:26	0.356	0.008	0.614	0.05
01/01/2017 00:28	0.349	0.008	0.613	0.05
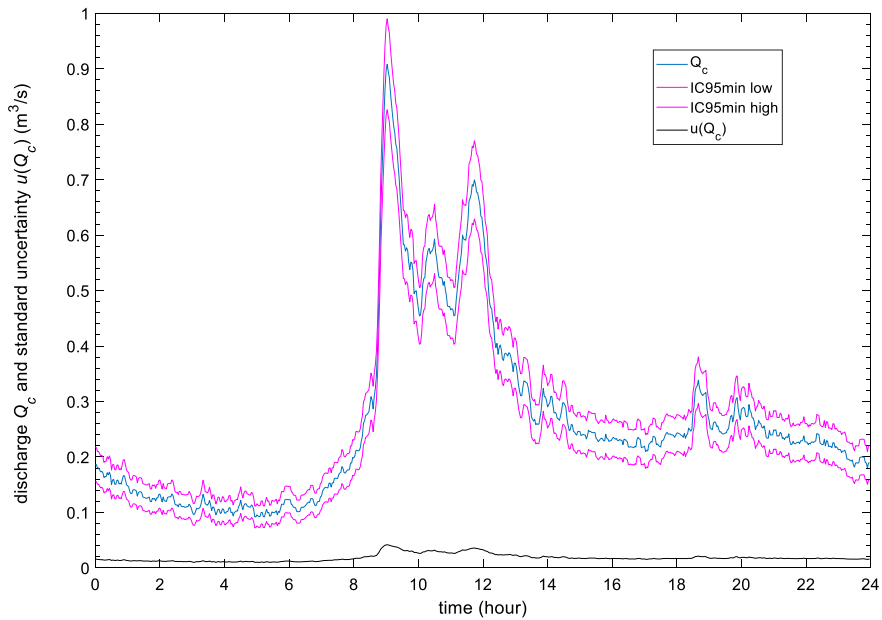
Figure 8.13 Plot of discharge Q_c over 24 hours, standard uncertainty $u(Q_c)$ and the 95% coverage interval. *Source*: Jean-Luc Bertrand-Krajewski (INSA Lyon).

Table 8.10 First 15 lines of the matrix QcMC (Q_c, $u(Q_c)$, *IC95min low* and *IC95min high*).

Q (m³/s)	*u(Q)* (m³/s)	*IC95min low* (m³/s)	*IC95min high* (m³/s)
0.1866	0.0157	0.1557	0.2173
0.1860	0.0157	0.1552	0.2166
0.1803	0.0151	0.1508	0.2099
0.1803	0.0151	0.1507	0.2099
0.1764	0.0150	0.1469	0.2058
0.1730	0.0147	0.1443	0.2019
0.1741	0.0147	0.1454	0.2029
0.1714	0.0146	0.1428	0.2002
0.1628	0.0140	0.1355	0.1903
0.1646	0.0140	0.1373	0.1922
0.1646	0.0140	0.1371	0.1921
0.1717	0.0147	0.1431	0.2006
0.1676	0.0146	0.1392	0.1965
0.1726	0.0150	0.1433	0.2021
0.1674	0.0146	0.1391	0.1964

BOX 18: CALCULATION OF THE DISCHARGE Q_C AND OF ITS UNCERTAINTY $u(Q_C)$ FOR EACH TIME STEP

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
Import the data of the `hV1.csv` file:

```
data=dlmread('hV1.csv',';',1,1)
```

Then type

```
Z=data
A=[0.6 0.002]
alpha=0.95
Nmc=1e6
distrib=ones(3,1)
MatCor=eye(3)
chaineQc='(A(:,1).^2).*(acos(1-Z(:,1)./A(:,1))-(1-Z(:,1)./A(:,1))
.*sin(acos(1-Z(:,1)./A(:,1)))).*Z(:,2)'
```

and lastly, apply the MCM method for the entire time series of 720 time steps in one single instruction:

```
QcMC=uMCM(Z,A,chaineQc,alpha,MatCor,Nmc,distrib)
```

The results are shown graphically in Figure 8.13 and the first 15 lines of the matrix `Qcmc` are given in Table 8.10.
Apply now the Type B method for the entire time series of 720 values in one single instruction:

```
QcTB=uTypeB(Z,A,chaineQc,alpha,MatCor)
```

The Type B method runs much faster in such a case, also for the entire time series of 720 values. The maximum relative difference of the standard uncertainties $(u(Q_{cTB}) - u(Q_{cMC}))/u(Q_{cMC})$ is less than 0.25%: both methods deliver similar results.

Step 2: Calculation of daily volume V_d, standard uncertainty $u(V_d)$ and 95% coverage interval

With the time step $\Delta t = 120$ s, the daily volume $V_d = 22{,}763$ m^3. Applying the variograph method (see Section 8.2.6.2), the standard uncertainty is $u(V_d) = 620$ m^3 and the 95% coverage interval is [21547, 23979] m^3 (detailed calculations are given in Box 19).

8.3.3 Uncertainty in discharge calculation with the Manning-Strickler formula

It is frequent that the Manning-Strickler formula, due to its simplicity, is used to evaluate discharges in sewers where there is no backwater effect and where, as a first approximation, the discharge can be considered as locally uniform and permanent during each time step.

The Manning-Strickler formula (Equation (8.74)), in addition to the water level h (m) used to estimate the wet section S (m^2) and the hydraulic radius R_h (m), requires the values of both the roughness coefficient K (m$^{1/3}$/s) and the pipe invert slope I (m/m).

$$Q_{MS} = K\sqrt{I}SR_h^{\frac{2}{3}}$$

$$(8.74)$$

BOX 19: CALCULATION OF DAILY VOLUME V_d, STANDARD UNCERTAINTY $u(V_d)$ AND 95% COVERAGE INTERVAL

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
Type
```
Dt=120
```
Calculate the daily volume V_d from the 720 values of the discharge Q_c given in the first column of QcMC calculated in Box 18:
```
Vd=sum(QcMC(:,1).*Dt)
```
which gives 22,763 m³.
To use the Matlab® code uTypeBsum (see Box 10), select the first and second columns of QcMC containing, respectively, the values of the discharge Q_c and their standard uncertainties $u(Q_c)$, and multiply them by the time step Δt to get the corresponding values of the volume. In this example, one assumes that the uncertainty in the time step Δt is negligible.
Type
```
Vd=uTypeBsum(QcMC(:,1:2).*Dt)
```
The four components of the vector Vd correspond respectively to

- $V_d = 22{,}763$ m³
- standard uncertainty with no autocorrelation $u(V_d)_{nc} = 62$ m³, i.e. relative standard uncertainty of 0.3%.
- standard uncertainty with full autocorrelation $u(V_d)_{fc} = 1563$ m³, i.e. relative standard uncertainty of 6.9%.
- standard uncertainty with partial autocorrelation $u(V_d)_{pc} = 620$ m³, i.e. relative standard uncertainty of 2.7%.

Calculate the 95% coverage interval for the case of partial autocorrelation:
```
[Vd(1)-1.96*Vd(4), Vd(1)+1.96*Vd(4)]
```
One gets [21547, 23979] m³.

Roughness and slope are critical quantities as the discharge Q_{MS} is directly proportional to K and to the square root of I. Instead of using (i) approximate values of K found in textbooks for the material of the sewer pipe, and (ii) values of I based on non-verified maps or GIS data, it may be better to estimate the value of the quantity $K\sqrt{I}$ from temporary field measurements.

Let us consider the case of a circular pipe with a radius $R_c = 0.8$ m, equipped with a water level sensor to estimate discharges during dry weather and most frequent storm events generating water levels lower than 0.5 m. The water sensor is installed permanently. For one week, a flow velocity sensor has been added temporarily and verified with tracing experiments (for tracing experiments in sewers, see e.g. Lepot *et al.*, 2014).

Two days of data recorded with a time step of 2 minutes are available in the file manning1.csv. The first 15 lines are shown in Table 8.11. The ranges of water level h and mean flow velocity V are, respectively, 0.009 to 0.51 m and 0.16 to 2.93 m/s. Standard uncertainties in h, V and R_c are, respectively, $u(h) = 0.003$ m, $u(V) = 0.03$ m/s and $u(R_c) = 0.002$ m.

Table 8.11 First 15 lines of the file `manning1.csv`.

Date UT	h (m)	V (m/s)
13/08/2018 00:00	0.104	1.50
13/08/2018 00:02	0.103	1.53
13/08/2018 00:04	0.103	1.53
13/08/2018 00:06	0.101	1.52
13/08/2018 00:08	0.098	1.51
13/08/2018 00:10	0.096	1.48
13/08/2018 00:12	0.093	1.46
13/08/2018 00:14	0.090	1.44
13/08/2018 00:16	0.088	1.42
13/08/2018 00:18	0.085	1.37
13/08/2018 00:20	0.083	1.36
13/08/2018 00:22	0.082	1.34
13/08/2018 00:24	0.079	1.30
13/08/2018 00:26	0.079	1.30
13/08/2018 00:28	0.079	1.30

Let us determine the values of the quantity $K\sqrt{I}$ and of its standard uncertainty. Using the independent quantities h, V and R_c, Equation (8.74) can be rewritten as

$$Q_{MS} = K\sqrt{I}\left[\arccos\left(1 - \frac{h}{R_c}\right) - \left(1 - \frac{h}{R_c}\right)\sin\left(\arccos\left(1 - \frac{h}{R_c}\right)\right)\right]^{\frac{5}{3}}\left[2R_c \arccos\left(1 - \frac{h}{R_c}\right)\right]^{-\frac{2}{3}}$$

(8.75)

One calculates directly the values of the quantity $K\sqrt{I}$ by combining Equations (8.75) and (8.73):

$$K\sqrt{I} = R_c^{-\frac{2}{3}}\left[\arccos\left(1 - \frac{h}{R_c}\right) - \left(1 - \frac{h}{R_c}\right)\sin\left(\arccos\left(1 - \frac{h}{R_c}\right)\right)\right]^{-\frac{2}{3}}\left[2 \arccos\left(1 - \frac{h}{R_c}\right)\right]^{\frac{2}{3}}V$$

(8.76)

The set of 1440 values of the quantity $K\sqrt{I}$ calculated from the recorded data can be considered as a sample for the Type A method.

Calculations (detailed in Box 20) give $\overline{K\sqrt{I}} = 7.562$ m$^{1/3}$/s and $u\left(\overline{K\sqrt{I}}\right) = 0.034$ m$^{1/3}$/s.

Based on these results, future values of the discharge in the circular pipe can be calculated by means of Equation (8.75) by using $\overline{K\sqrt{I}}$ as an independent quantity. For example, let us calculate the discharge and its uncertainty for a water level $h = 0.3$ m (detailed calculations are given in Box 21). One gets $Q = 1.33$ m^3/s, $u(Q) = 0.03$ m^3/s, and the 95% coverage interval is [1.28, 1.39] m^3/s.

This example does not account for the uncertainties in the 1440 values of $K\sqrt{I}$ due to uncertainties in the quantities h, V and R_c, which would add some additional uncertainty in the estimation of $\overline{K\sqrt{I}}$.

BOX 20: CALCULATION OF $K\sqrt{I}$ AND OF ITS STANDARD UNCERTAINTY

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
Import the data of the `manning1.csv` file:
```
data=dlmread('manning1.csv',';',1,1)
```
Type
```
Rc=0.8
```
For convenience, type
```
h=data(:,1)
V=data(:,2)
```
According to Equation (8.76), type
```
KVI=Rc^(-2/3).*(acos(1-h/Rc)-(1-h/Rc).*sin(acos(1-h/Rc))).^(-2/3)
.*(2*acos(1-h/Rc)).^(2/3).*V
```
Then apply the Type A method with a 95% level of probability:
```
KVIbar=uTypeA(KVI, 0.95)
```
One gets $\overline{K\sqrt{I}} = 7.5621$ m$^{1/3}$/s and $u\left(\overline{K\sqrt{I}}\right)$ is equal to 0.0338 m$^{1/3}$/s.

BOX 21: CALCULATION OF Q, $u(Q)$ AND 95% COVERAGE INTERVAL

(Matlab® codes and csv files available for download at https://doi.org/10.2166/9781789060102).
In addition to the variables defined in Box 20, type
```
h=0.3
uh=0.003
KVI=KVIbar(1)
uKVI=KVIbar(2)
Rc=0.8
uRc=0.002
Z=[h uh]
A=[KVI uKVI Rc uRc]
alpha=0.95
distrib=ones(3,1)
MatCor=eye(3)
Nmc=1e6
chaineQh='A(:,1).*(acos(1-Z(:,1)./A(:,2))-(1-Z(:,1)./A(:,2)).*sin(acos
(1-Z(:,1)./A(:,2)))).^(5/3).*(2.*A(:,2).*acos(1-Z(:,1)./A(:,2))).^(-2/3)'
```
and lastly
```
Q=uMCM(Z,A,chaineQh,alpha,MatCor,Nmc,distrib)
```
One gets $Q = 1.3342$ m^3/s, $u(Q) = 0.0292$ m^3/s and the 95% coverage interval is [1.2771, 1.3916] m^3/s.

This could be done by means of Monte Carlo simulations. In addition, this example implicitly assumes that K is constant, whereas it may happen that it depends on the water level h. A relation $K = f(h)$ can be investigated along with, if significant, another relation $u(K) = f(h)$.

8.3.4 Uncertainty in velocity-area methods

The velocity-area method is one of the most common non-continuous methods to determine flow data in open channels. The example introduces the calculation of flow uncertainty of velocity-area methods using the approach given in ISO (2020). Furthermore, it demonstrates the sensitivity of flow uncertainty on simplifications and inaccuracies in the measurement process.

8.3.4.1 Case study

In a rectangular channel (width $B = 0.8$ m, flow depth $h = 0.5$ m), flow velocities $v_x(y_i, z_j)$ (m/s) are measured at 135 points (y_i, z_j) of a grid with $\Delta y = \Delta z = 0.05$ m during uniform steady state (Figure 8.14).

8.3.4.2 Uncertainty of flow and mean velocity

Flow Q (m^3/s) and mean flow velocity V (m/s) can be calculated from a sufficiently dense velocity field $v_x = f(y,z,t)$ in a representative flow cross section S (m^2). For uniform steady state flow conditions, v_x values do not depend on time t and the following applies:

$$Q = \int_S v_x(y, z)dS \qquad (8.77)$$

For $m \times n$ elementary sections $S_{ij} = \Delta y_i \times \Delta z_j$ and point velocities $v_x(y_i, z_j)$ in the middle of each elementary section S_{ij}, the flow Q can be calculated as

$$Q = F_x F_z \sum_{i=1}^{m} \sum_{j=1}^{n} v_x(y_i, z_j)\Delta z_j \Delta y_i + Q_p \qquad (8.78)$$

where $m = 15$ is the number of points per row, $n = 9$ is the number of points per vertical (Figure 8.14), F_y
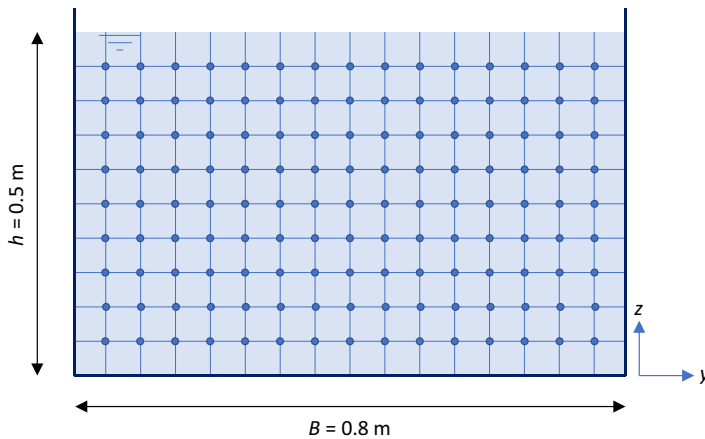
Figure 8.14 Scheme of the rectangular cross section with 135 points (y_i, z_j) where longitudinal flow velocity $v_x(y_i, z_j)$ is measured. *Source*: Mathias Uhl (Muenster University of Applied Sciences).

and F_z are factors relating the sums, respectively, in the y- and z-direction to an ideal integration of the real velocity profile, and Q_p (m^3/s) is the perimeter flow.

In the case of velocity grid measurements, F_y and F_z can be set to unity. The perimeter flow Q_p represents a part of the flow near the boundary of the cross section where measurements are impossible or very uncertain because of turbulence, velocity gradients or influences on the measurement sensor or just remaining areas between the real cross section and the measurement grid.

According to ISO (2020), the uncertainty of the flow $u(Q)$ is

$$
\begin{aligned}
u(Q)^2 &= \left[\sum_i^m \sum_j^n v_x(y_i z_j)\Delta y_i \Delta z_j \right] [u^*(F_y)^2 + u^*(F_z)^2] \\
&+ \sum_i^m \sum_j^n [v_x(y_i z_l)\Delta y_i \Delta z_j]^2 u^*|[v_x(y_i z_j)]^2 \\
&+ \sum_i^m \left[\Delta y_i^2 \cdot \left\{ \sum_j^n v_x(y_i z_j)\Delta z_j \right\}^2 \cdot u^*(\Delta y_i)^2 \right] \\
&+ \sum_j^n \left[\Delta z_i^2 \cdot \left\{ \sum_i^m v_x(y_i z_j)\Delta y_j \right\}^2 \cdot u^*(\Delta z_i)^2 \right] + u(Q_p)^2
\end{aligned}
\tag{8.79}
$$

with the mean velocity $V = Q/S$, its uncertainty $u(V)$ is

$$
u(V)^2 = \left(\frac{1}{S^2} \right) u^2(Q) + \frac{Q^2}{S^4} u^2(S)
\tag{8.80}
$$

8.3.4.3 Flow cross section uncertainty

The assumed range of uncertainty of the channel width $B = 0.80$ m is $0.002 \leq u(B) \leq 0.005$ m and as well $0.002 \leq u(h) \leq 0.005$ m for the water level $h = 0.5$ m.

8.3.4.4 Measurement segment uncertainty Δy, Δz

The uncertainty of Δy and Δz was estimated in the range of $0.002 \leq u(\Delta y) = u(\Delta z) \leq 0.005$ m.

8.3.4.5 Flow velocity uncertainty $u(v_x)$

The uncertainty of the flow measurements consists of (i) the relative uncertainty u_k^* of the measurement device as specified in the calibration certificate and (ii) a random relative component u_r^* resulting from the unsteady nature of flow in channels. The uncertainty depends on the sensor type and the local velocity and turbulence. The uncertainty of the local velocity $u(v_x)$ is

$$
u^*[v_x(y_i z_j)]^2 = u_k^*[v_x(y_i z_j)]^2 + u_r^*[v_x(y_i z_j)]^2
\tag{8.81}
$$

This example uses $0.001 \leq u_k \leq 0.01$ m/s with $u_k^* = u_k/v_x$ and $0.01 \leq u_r^* \leq 0.02$ as a Type B estimate.

8.3.4.6 Measurement grid relative uncertainties $u^*(F_y)$ and $u^*(F_z)$

The density of the measurement grid influences the number and the representativeness of the measurements. ISO (2020) provides recommended default values in its Annex D: Table D.4 for the number of measurement points in a vertical and Table D.6 for the number of verticals in a cross section. The respective values in Table 8.13 are taken from these tables.

8.3.4.7 Perimeter flow uncertainty $u(Q_p)$

Near to the walls of the cross section, measurements of the flow velocity are either impossible or have large errors or uncertainties due to boundary effects, large gradients of the flow velocities or disturbed measurement signals. In those perimeter sections, the flow has to be estimated by extrapolated flow velocities. A simple approach had been used in this study. The free surface velocity $v_x(y, z = h)$ was set equal to the first measured value $v_x(y, z = h - \Delta z)$. At walls and bottom, flow velocities were assumed to be 50% of the nearest measured values, i.e. $v_x(y = 0, z) = 0.5 \times v_x(y = 0 + \Delta y, z)$, $v_x(y = B, z) = 0.5 \times v_x(y = B - \Delta y, z)$ and $v_x(y, z = 0) = 0.5 \times v_x(y, z = 0 + \Delta z)$. The flow in the perimeter strip of 0.025 m width was calculated with a velocity at 0.0125 m derived from linear interpolation between the wall and bottom velocities and the next corresponding measured velocities.

It is very complicated or even impossible to accurately determine the perimeter flow uncertainty. Therefore ISO (2020) recommends a flat estimate for relative uncertainties of the perimeter flow in the range of $0.2 \leq u^*(Q_p) \leq 0.4$.

8.3.4.8 Scenarios using different measurement quality

Scenario 1 assumes the best practice in velocity measurement in a close measurement grid with a very accurate velocity sensor and careful positioning. The measurement grid consists of 135 points with $\Delta y = \Delta z = 0.05$ m. High sensor accuracy and precise handling is assumed and expressed by the values for uncertainty given in Table 8.13. Table 8.12 shows a screenshot of the Excel®-based calculations. The yellow marked perimeter section consists of a 0.025 m layer along the bottom and walls and at the surface. The measurement result of scenario 1 is $Q = 0.2458$ m³/s with the relative uncertainty $u^*(Q) = 0.0388$, and $V = 0.6144$ m/s with the relative uncertainty $u^*(V) = 0.0391$.

Scenario 2 assumes less accurate measurements in the same close measurement grid as scenario 1. Scenarios 3 and 4 use a less dense standard measurement grid with five measurements in eight verticals with measurement accuracy being high in scenario 3 and lower in scenario 4. Details on the parameters for all scenarios are given in Table 8.13.

Results in Table 8.14 show relative uncertainties for flow ranging in $0.0355 \leq u^*(Q) \leq 0.0653$ and with $0.0355 \leq u^*(V) \leq 0.0653$ slightly higher for the mean velocity. The low uncertainties confirm the state of velocity grid measurements as a standard for calibration of other measurement devices. The results of scenarios 2 and 3 show that accurate measurements can compensate for the uncertainties being introduced by larger measurement grids.

8.3.4.9 Scenarios using symmetry properties of the velocity field

Velocity grid measurement is undoubtedly one of the most accurate methods to determine flow and mean velocity. The high metrological effort can be reduced by using theoretically founded symmetry properties of fully developed flows. Necessary and often sufficient prerequisites are long and straight prismatic channels with low roughness and constant gradient without lateral junctions and installations. In these cases, the following symmetries of the velocity field can be observed and assumed:

- Prismatic channels with a free water surface: symmetry to the vertical centre axis.
- Prismatic channels with pressure flow: symmetry to the vertical and horizontal central axis.
- Circular cross sections with pressure flow: rotational symmetry.

Scenario 5 shows how the use of axial symmetry of velocity in the given rectangular cross section affects the uncertainty of Q and V. It is based on scenario 1 and assumes that velocity data are available only for the left side including the middle axis at $y = 0.5 \times B = 0.40$ m of the rectangular section. The cross section was split

scenario 1: best practice and close measurement grid

v(y,z) [m/s]

z [m]	0.303	0.378	0.605	0.671	0.746	0.711	0.795	0.797	0.801	0.762	0.771	0.793	0.775	0.728	0.692	0.676	0.633	0.396	0.317	Δz [m]
0.50	0.303	0.378	0.605	0.671	0.746	0.711	0.795	0.797	0.801	0.762	0.771	0.793	0.775	0.728	0.692	0.676	0.633	0.396	0.317	0.025
0.45	0.303	0.378	0.605	0.671	0.746	0.711	0.795	0.797	0.801	0.762	0.771	0.793	0.775	0.728	0.692	0.676	0.633	0.396	0.317	0.05
0.40	0.330	0.413	0.660	0.717	0.736	0.727	0.778	0.778	0.764	0.829	0.804	0.842	0.802	0.709	0.727	0.689	0.676	0.423	0.338	0.05
0.35	0.309	0.386	0.618	0.665	0.747	0.717	0.769	0.784	0.745	0.756	0.768	0.775	0.729	0.686	0.730	0.706	0.649	0.406	0.324	0.05
0.30	0.307	0.384	0.615	0.679	0.676	0.677	0.778	0.781	0.788	0.788	0.784	0.803	0.761	0.683	0.692	0.671	0.639	0.399	0.319	0.05
0.25	0.298	0.373	0.597	0.690	0.681	0.680	0.738	0.764	0.769	0.783	0.724	0.727	0.688	0.651	0.644	0.648	0.637	0.398	0.319	0.05
0.20	0.303	0.378	0.605	0.650	0.627	0.637	0.681	0.690	0.736	0.717	0.686	0.730	0.678	0.635	0.646	0.624	0.571	0.357	0.286	0.05
0.15	0.262	0.327	0.523	0.568	0.600	0.569	0.610	0.652	0.653	0.651	0.620	0.650	0.607	0.559	0.587	0.535	0.539	0.337	0.270	0.05
0.10	0.227	0.284	0.454	0.495	0.523	0.493	0.564	0.584	0.548	0.590	0.578	0.591	0.549	0.511	0.509	0.469	0.455	0.284	0.227	0.05
0.05	0.189	0.236	0.378	0.406	0.416	0.409	0.458	0.451	0.443	0.451	0.459	0.438	0.435	0.406	0.423	0.386	0.385	0.240	0.192	0.05
0.025	0.118	0.148	0.236	0.254	0.260	0.255	0.286	0.282	0.277	0.282	0.287	0.274	0.272	0.254	0.264	0.241	0.240	0.186	0.120	0.025
0.00	0.094	0.118	0.189	0.203	0.208	0.204	0.229	0.225	0.222	0.225	0.230	0.219	0.218	0.203	0.211	0.193	0.192	0.120	0.096	0.025
y [m]	0.00	0.0125	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.7875	0.80	
Δy [m]	0.00	0.025	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.025	0.025	

ΔQ(y,z) u*, u* Δy,Δz,

Parameters (perimeter / measured / sum)

	Δz	Δy	u*(Fy)	u*(Fz)	u*(Δy)	u*(Δz)	ur*	uk	u(h)	u(B)	u*(Qp)	Q	u(Q)	u*(Q)	S	u(S)	u*(S)	V	u(V)	u*(V)
unit	m	m	[m³/s]	[m³/s]	-	-	-	m/s	m	m	-	[m³/s]	[m³/s]	-	[m²]	[m²]	-	[m/s]	[m/s]	-
value	0.050	0.050	0.030	0.005	0.040	0.040	0.010	0.001	0.002	0.002	0.200	0.2458	0.0095	0.0388	0.4000	0.0019	0.0047	0.6144	0.0240	0.0391
			0.004	0.004									0.013	0.014		0.014	0.015		0.014	0.004

ΣQ [m³/s]

perimeter	0.02734
measured	0.21843
sum	0.24577

results

Q [m³/s]	u(Q) [m³/s]	u*(Q) [-]	S [m²]	u(S) [m²]	u*(S) [-]	V [m/s]	u(V) [m/s]	u*(V) [-]
0.2458	0.0095	0.0388	0.4000	0.0019	0.0047	0.6144	0.0240	0.0391

Table 8.13 Grid size and uncertainty parameters of scenarios 1–4.

Scenario	Δy m	Δz m	$u^*(F_y)$ –	$u^*(F_z)$ –	$u^*(\Delta y)$ –	$u^*(\Delta z)$ –	u_r^* –	u_k m/s	$u(h)$ m	$u(B)$ m	$u^*(Q_p)$ –
1	0.050	0.050	0.030	0.005	0.040	0.040	0.010	0.001	0.002	0.002	0.200
2	0.050	0.050	0.030	0.005	0.100	0.100	0.020	0.010	0.005	0.005	0.400
3	0.100	0.100	0.030	0.005	0.040	0.040	0.010	0.001	0.002	0.002	0.200
4	0.100	0.100	0.057	0.005	0.050	0.050	0.020	0.010	0.005	0.005	0.400

Table 8.14 Flow, flow cross section, and mean velocity with uncertainties for scenarios 1–4.

Scenario	Q m³/s	$u(Q)$ m³/s	$u^*(Q)$ –	S m²	$u(S)$ m²	$u^*(S)$ –	V m/s	$u(V)$ m/s	$u^*(V)$ –
1	0.2458	0.0095	0.0388	0.4000	0.0019	0.0047	0.6144	0.0240	0.0391
2	0.2458	0.0164	0.0667	0.4000	0.0047	0.0118	0.6144	0.0416	0.0677
3	0.2512	0.0103	0.0411	0.4000	0.0019	0.0047	0.6279	0.0260	0.0414
4	0.2512	0.0186	0.0741	0.4000	0.0047	0.0118	0.6279	0.0471	0.0751

into the left section 5l ($0 \le y < 0.5 \times B - 0.5 \times \Delta y$), the middle section 5m ($0.5 \times B - 0.5 \times \Delta y \le y \le 0.5 \times B + 0.5 \times \Delta y$) and right section 5r ($0.5 \times B + 0.5 \times \Delta y < y \le B$).

The flow velocities on the right side v_r ($y > 0.5 \times B, z$) are assumed to be equal to the axially mirrored data v_l ($y < 0.5 \times B, z$) on the left side. In this study case, data v ($y > 0.5 \times B, z$) are available. Therefore, the uncertainty of the mirrored data v_r can be derived from the differences $\Delta v_r = v_r - v_l$. They range from $\Delta v_{r,min} = -0.055$ to $\Delta v_{r,max} = 0.065$ (mean -0.005, standard deviation -0.025, skewness 0.341). The uncertainty $u(v_r)$ was determined by a Type B estimate using a uniform density function (Section 8.2.3.3, Figure 8.1) which might slightly overestimate the uncertainty. It amounts to $u(v_r) = 0.0344$ m/s.

The parameters of the uncertainty calculation (Table 8.15) were set equal to scenario 1 (Table 8.13) for the sections 5l and 5m. For section 5r the uncertainty of the velocity is higher due to (i) the lower number of measurement verticals ($u^*(F_y) = 0.063$ according to ISO (2020)) and (ii) the uncertainty of the estimated velocities on the right side which can be considered by $u_k = u(v_r) = 0.0344$ m/s.

Table 8.16 shows equal results for Q, S and V in sections 5l and 5r but higher relative uncertainties $u^*(Q)$ and $u^*(V)$ in section 5r due to the higher uncertainties u_k. In section 5m, $u^*(Q)$ and $u^*(V)$ are higher than in the left section due to the higher influence of $u^*(\Delta y)$ and $u^*(\Delta z)$ on S which results in a high relative uncertainty $u^*(S)$. This small segment introduces additional uncertainty which in the end decreases

Table 8.15 Grid size and uncertainty parameters of scenario 5.

Section	Δy m	Δz m	$u^*(F_y)$ –	$u^*(F_z)$ –	$u^*(\Delta y)$ –	$u^*(\Delta z)$ –	u_r^* –	u_k m/s	$u(h)$ m	$u(B)$ m	$u^*(Q_p)$ –
5l, 5m	0.05	0.05	0.030	0.005	0.04	0.04	0.010	0.0010	0.002	0.002	0.2
5r	0.05	0.05	0.063	0.005	0.04	0.04	0.010	0.0344	0.002	0.002	0.2

Table 8.16 Flow, flow cross section, and mean velocity with uncertainties for scenario 5.

Section	Q m³/s	u(Q) m³/s	u*(Q) –	S m²	u(S) m²	u*(S) –	V m/s	u(V) m/s	u*(V) –
5l	0.1146	0.0043	0.0380	0.1875	0.0013	0.0067	0.6111	0.0236	0.0385
5m	0.0171	0.0009	0.0550	0.0250	0.0010	0.0402	0.6822	0.0465	0.0681
5r	0.1146	0.0077	0.0676	0.1875	0.0013	0.0067	0.6111	0.0415	0.0679
5	0.2462	0.0089	0.0363	0.4000	0.0020	0.0051	0.6155	0.0666	0.1082
5l+m	0.1316	0.0049	0.0373	0.2125	0.0013	0.0062	0.6194	0.0234	0.0378
5r	0.1146	0.0077	0.0676	0.1875	0.0013	0.0067	0.6111	0.0415	0.0679
5	0.2462	0.0092	0.0372	0.4000	0.0018	0.0045	0.6155	0.0476	0.0774

slightly when summing up Q and S. A joint calculation for sections 5l and 5m, noted 5l+m, decreases $u^*(S)$ but increases $u^*(Q)$ due to the lower number of summands, and also slightly decreases $u^*(V)$.

Using symmetry properties of the velocity field in scenario 5 showed neglectable systematic deviations of Q, S and V compared to scenario 1 (Table 8.13) but higher uncertainties in V.

8.3.4.10 Scenarios using mean velocities estimates

Grid measurements of flow velocity fields are complex and unsuitable for continuous measurements of $v_x(y, z, t)$ in practice. Continuously-measuring devices capture a more or less well-defined part of the velocity field, and (i) extrapolate the non-measured part of the field with empirical or hydromechanical based algorithms or (ii) calculate the mean flow velocity V from the measured values using empirical factors or algorithms (see Chapter 3).

Exemplarily, two simple approaches are investigated with regard to their uncertainties by using the data set of scenario 1 in Table 8.12.

Scenario 6 uses the flow velocity V_{fs} in an area of the free water surface to calculate $V = e_{fs} \times V_{fs}$. The free surface velocity V_{fs} is calculated from $v(y = 0.25$ m, $z = 0.45$ m) to $v(y = 0.55$ m, $z = 0.45$ m) with an arithmetic mean 0.785 m/s, a median 0.793 m/s, standard deviation 0.0151 m/s, skewness -0.513, minimum 0.762 m/s and maximum 0.801 m/s. The Type B estimate using a uniform density distribution results in the uncertainty $u(V_{fs})$ and the uncertainty $u(e_{fs})$ for $e_{fs} = V/V_{fs}$ can be calculated by Equation (8.9) with results given in Table 8.17.

Scenario 7 is based on scenario 6 but uses the maximum flow velocity V_{max} to calculate the mean flow velocity $V = e_{max} \times V_{max}$. V_{max} is taken from the area of maximum velocities $v(y = 0.35$ m, $z = 0.45$ m), to $v(y = 0.45$ m, $z = 0.35$ m) with an arithmetic mean 0.7779 m/s, a median 0.7683 m/s, standard deviation 0.0274 m/s, skewness $+0.8795$, minimum 0.7451 m/s and maximum 0.8294 m/s. Results with uncertainties are given in Table 8.18.

Table 8.17 Mean velocity, mean surface velocity, and their ratio with uncertainties for scenario 6.

V m/s	u(V) m/s	V_fs m/s	V_fs,min m/s	V_fs,max m/s	u(V_fs) m/s	e_fs –	u(e_fs) –	u*(e_fs) –
0.6144	0.0240	0.7849	0.7625	0.8008	0.0111	0.7828	0.0325	0.0415

Table 8.18 Mean velocity, maximum velocity, and their ratio with uncertainties for scenario 7.

V m/s	u(V) m/s	V_{max} m/s	$V_{max,min}$ m/s	$V_{max,max}$ m/s	$u(V_{max})$ m/s	e_{max} –	$u(e_{max})$ –	$u^*(e_{max})$ –
0.6144	0.0240	0.7779	0.7451	0.8294	0.0244	0.7898	0.0395	0.0501

Table 8.19 Coverage intervals of e_{fs} and e_{max}.

	Scenario 6				Scenario 7			
k	e_{fs}	$k \times u(e_{fs})$	$e_{fs} - k \times u(e_{fs})$	$e_{fs} + k \times u(e_{fs})$	e_{max}	$k \times u(e_{max})$	$e_{max} - k \times u(e_{max})$	$e_{max} + k \times u(e_{max})$
1	0.7828	0.0325	0.7503	0.8153	0.7898	0.0395	0.7503	0.8294
1.96	0.7828	0.0637	0.7191	0.8465	0.7898	0.0775	0.7123	0.8673
2.58	0.7828	0.0839	0.6989	0.8667	0.7898	0.1020	0.6878	0.8918

The results of both scenarios 6 and 7 are rather similar and might indicate a reasonable accuracy when looking at $u^*(e_{fs})$ and $u^*(e_{max})$ only. But note that the values of e_{fs} and e_{max} are one of the potential realizations for these particular cases. Repeated tests of the same configuration would give other results. The range for e to be expected can be estimated by the coverage interval according to Equation (8.17) as $[e - k \times u(e), e + k \times u(e)]$ with k from Table 8.3. Table 8.19 shows the coverage intervals for e_{fs} and e_{max}.

The spreads between the low and high limits are 1.18 for e_{fs} and 1.22 for e_{max} at the 95% confidence level ($k = 1.96$), and 1.24 for e_{fs} and 1.30 for e_{max} at the 99% confidence level ($k = 2.58$). Note that these results were derived from one clearly defined steady state flow experiment. The rather wide ranges should be reason enough for equipment suppliers to carefully derive the values for e_{fs} or e_{max} for a broad variety of flow conditions and quantify their dependencies on influencing factors.

8.4 SENSOR UNCERTAINTY AND *IN SITU* MEASUREMENT UNCERTAINTY
8.4.1 Definitions and explanations

In uncertainty assessment, it is of crucial importance to account for two main and independent sources of uncertainty in measured data: (i) sensor uncertainty and (ii) *in situ* measurement uncertainty.

8.4.1.1 Sensor uncertainty

All sensors used in UDSM should be calibrated according to rigorous calibration procedures and protocols (see Section 7.6 on sensor calibration). It is very important to note that the calibration should involve the entire chain of sensors and instruments, from the transducer to the final data storage in data loggers, SCADA systems or databases, i.e. to the final state of the data as they are later used by operators, researchers, regulators, etc.

Calibration of the transducer or the sensor only (uncertainties reported by manufacturers are frequently given only for such conditions) will lead to systematic underestimation of uncertainties. Indeed, other

sources of uncertainties are due to data transmission, signal amplification, analogue-digital conversions, format conversion, re-scaling, display rounding, etc.

In addition, uncertainties reported by sensor manufacturers are obtained under specific conditions which may differ significantly for the *in situ* conditions. Therefore, it is the responsibility of the user to systematically calibrate all sensors under their real conditions of use, whatever manufacturers indicate.

8.4.1.2 In situ measurement uncertainty

This source of uncertainty may be summarized as 'the uncertainty due to the *in situ* conditions of measurement for a given sensor'. Its evaluation is mainly based on detailed *in situ* observations and expertise.

Let us consider water level measurement in a sewer system. In addition to the sensor standard uncertainty (e.g. 1 mm for an ultrasonic sensor determined from a calibration under static controlled conditions), there are other sources of *in situ* uncertainty which affect the measured value of the water level:

- The uncertainty in the position of the sensor in the pipe.
- The waves and oscillations of the free surface.
- The horizontality of the free surface through the width of the sewer.
- Local flow depending on changes of water level due to hydraulic effects.
- Etc.

Visual observations during dry weather in a man entry sewer showed that the free surface wave and oscillations had an amplitude up to 15 mm, which may be equivalent to a standard uncertainty of 7.5 mm. In the same sewer under wet weather conditions, the amplitude may reach 30 mm and even more. The resulting total uncertainty $u(h)$ in water level measurement h may be estimated from both the sensor uncertainty $u_s(h)$ and the *in situ* measurement uncertainty $u_i(h)$ by the law of propagation of uncertainty:

$$u(h) = \sqrt{u_s(h)^2 + u_i(h)^2} = \sqrt{1^2 + 7.5^2} = 7.6 \, \text{mm} \tag{8.82}$$

In this case, the *in situ* uncertainty is clearly the dominant contribution. It may be very different if the same sensor is used e.g. in a Venturi flume with an upstream channel of sufficient length to stabilize the free surface waves and oscillations.

In situ uncertainties should always be investigated case by case to evaluate their significance. Experience has revealed that they are too frequently ignored. Their importance may be great, especially for velocity sensors (Lepot *et al.*, 2014) and some quality sensors like e.g. turbidity sensors as suspended solids are not always evenly distributed through the entire cross section.

8.4.2 Examples/orders of magnitude for some common sensors and methods

Uncertainty assessment (UA) must always be done on a case-by-case basis, accounting for local conditions: sensors used, associated calibration and verification protocols and data, operating conditions including maintenance, and operators. However, to provide orders of magnitude for beginners in UA and to facilitate comparisons, Table 8.20 indicates typical relative standard uncertainties $u*(x) = u(x)/x$ for common sensors and methods used to measure water level, flow velocity and discharge in sewer systems. These relative standard uncertainties correspond to *in situ* best practice usual operational conditions.

Table 8.20 Order of magnitude of typical relative standard uncertainties $u^*(x)$ (in %) for most common sensors used to measure water level, velocity and discharge in sewer systems (adapted from DWA, 2011).

Percentage of the full measurement range	6	10	20	25	30	50	75	100
Water level *								
Ultra-sound sensor 0–2 m	1.398	0.893	0.388	0.311	0.311	0.194	0.116	0.116
Pressure sensor 0–2 m	2.407	1.398	0.699	0.582	0.505	0.311	0.194	0.116
Pressure sensor 0–6 m	1.902	1.203	0.582	0.505	0.388	0.194	0.194	0.116
Discharge								
Venturi flume	4.309	3.998	3.610	3.610	3.494	3.416	3.300	3.300
Electromagnetic sensor - full pipe	2.019	1.009	1.009	1.009	1.009	1.009	1.009	1.009
Electromagnetic sensor - partly filled pipe	2.989	2.989	2.019	2.019	2.019	2.019	2.989 **	2.989 **
Rectangular weir	2.484	2.096	1.786	1.708	1.708	1.514	1.514	1.281
Triangular weir	2.484	2.407	2.213	2.096	2.096	2.019	1.902	1.902
Free orifice outflow	na	na	na	1.281	1.203	1.203	1.087	1.087
Combined measurements of water level h and maximum velocity $v\,max$ in partly filled pipe, diameter from 600 to 1400 mm, $h \geq 8$ cm, and	6.599	6.599	6.483	6.483	6.483	6.289	6.095	6.483 **
Combined measurements of water level h and velocity $v\,(h)$ in partly filled circular pipe, diameter from 600 to 1400 mm, $h \geq 8$ cm, and	3.998	3.998	2.989	2.989	2.989	2.989	2.989	3.998 **
Tracing experiment with continuous injection	3.300	3.300	3.300	3.300	3.300	3.300	3.300	3.300

*Without accounting for uncertainty components from height measurement and surface waves (see *in situ* uncertainty in Section 8.4.1.2).
**In the upper part of the measurement range, uncertainty increases due to hydrodynamic and measurement effects.

Example: for a water level ultra-sound sensor with a measurement range of 0–2 m measuring a water level $h = 1$ m: this corresponds to 50% of the measurement range. Table 8.20 indicates that the relative standard uncertainty is $u^*(h) = u(h)/h = 0.194\%$. Consequently, the standard uncertainty is $u(h) = 1.94$ mm and the corresponding 95% coverage interval for $h = 1$ m is $[h - 1.96 \times u(h), h + 1.96 \times u(h)] = [0.996, 1.004]$ m.

8.5 SUMMARY AND TRANSITION

This chapter explained in detail and illustrated with various examples the three methods one can use to estimate uncertainties: (i) the Type A method (repeated measurements), (ii) the Type B method (law of propagation of uncertainties), and (iii) the Monte Carlo method (propagation of distributions). It is essential that uncertainty assessment becomes a routine practice in urban drainage and stormwater management. Maintenance, periodic calibration and verification of sensors (Chapter 7), and uncertainty assessment (this chapter) are key elements for the next step in data processing: data analysis, quality assessment and validation (Chapter 9).

ACKNOWLEDGEMENTS

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