V.d.S.
Receptie na afloop.

ON THE ASYMPTOTIC SOLUTION OF WAVE PROPAGATION AND OSCILLATION PROBLEMS.

P R O E F S C H R I F T

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOGESCHOOL TE DELFT OP GEZAG VAN DE RECTOR MAGNIFICUS DR O. BOTTEMA, HOOGLERAAR IN DE AFDELING DER ALGEMENE WETENSCHAPPEN, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 26 JANUARI 1955 DES NAMIDDAGS TE 2 UUR

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CONTENTS.

Summary.

Boundary value problems with the wave equation for harmonic time dependence are transformed, by a one-sided Laplace transform, into hyperbolic problems with one more dimension. Using asymptotic properties of the transform, the solution can be found in the form of an asymptotic series. The method is applied to a pair of two-dimensional problems which are mathematically largely equivalent, viz. diffraction by a plane screen with slit, and the oscillating airfoil at high frequencies (or in near-sonic flow).

INTRODUCTION.

Recently Kline (1954) has given a method for the asymptotic solution of certain linear, second order hyperbolical problems, such as to provide the solution of wave problems in the form of an asymptotic series for high frequencies, with the approximation of "geometrical optics" as first term. This expansion contains only the zero and positive integral powers of the reciprocal frequency; however, it is known that solutions of diffraction problems contain also fractional powers of this quantity, and thus the concluding remark of Kline is that "the theory of asymptotic solution required to treat such problems is not at present adequate."

The aim of the present paper is to sketch, by a few examples , a method of solving problems of "diffraction" for high frequencies. The problems chosen for this purpose have two space dimensions, but the method may, in principle, be extended to more dimensions.

As far as previous rigorous results are concerned, only one diffraction problem has thus far been solved in closed form, viz. the problem of diffraction by a perfectly reflecting screen in the form of a half-plane, (with slight generalizations) which has been treated by Sommerfeld (1896). Further, in a limited number of plane screen configurations separation of the wave equation is possible, and rigorous results have been obtained in terms of series of special functions, e.g. by Sieger (1908) and Strutt (1931) for an infinite slit, and by Bouwkamp (1941) and Meixner and Andejewsky (1950) for a circular aperture. These solutions may be used for calculations in the low frequency range, but they are no longer serviceable at high frequencies, while also the series occurring here are rather unsurveyable.

It is therefore natural that in general diffraction problems are treated by methods of approximation. In the regime of high frequencies, the classical method is that of Kirchhoff (1891). It is, however, subject to the serious draw-back that only a first approximation is yielded, while no systematic method of obtaining subsequent approximations is available. Several modifications have been proposed for the Kirchhoff method, and also in the present paper the form used for comparison, is a variant which is specially applicable to plane screens, and which may be traced to Bouwkamp (1941). This modified solution, which is in the sequel called the Kirchhoff approximation, should more properly be termed a "Raleigh solution", after Bouwkamp (1954).

Among recent attempts at improving on the Kirchhoff theory, is the method of Franz (1950). It consists of a sequence of approximations, in which ad hoc adjustments are made to the trial solution, using the boundary conditions and the wave equation in alternation. Franz's theory is intended to be useful for all wavelengths, and it is not clear that this will yield something of the nature of an asymptotic series for high frequencies.

Also noteworthy in this respect, is the work of Braunbek (1950) on plane screen problems. He has made use of Sommerfeld's exact solution for the half-plane problem, in an attempt at giving a first estimate of the neglected term in the modified Kirchhoff solution. Numerical computations give good support to this estimate.

Specially for diffraction by a slit, an older recursion method is due to Schwarzschild (1902). His method has been shown by Baker and Copson (1950) to be equivalent to the solution by successive substitution of a pair of simultaneous integral equations, starting from Sommerfeld's solution for each of the two halves of the screen separately. Slow convergence of the process for real k seems to be its main draw-back.

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In the present paper, the two main problems treated are, firstly, that of diffraction of a wave train by an infinitely long strip or slit, and, secondly, that of a similar oscillating airfoil in subsonic compressible flow. The entire treatment of the diffraction problem is directly applicable to the aerodynamical problem, but an additional complication arises in the latter case, from the occurrence of a singularity in the acceleration potential at the leading edge of the airfoil.

The first chapter contains the formulation of the diffraction and oscillation problems concerned, and reference is made to the physically important quantities in the two cases, viz. the plane wave transmission cross section and the aerodynamic force and moment derivatives. The results obtained in two-dimensional scalar diffraction theory, have direct rigorous meaning not only in acoustical, but also in electromagnetic diffraction, though in three-dimensional electromagnetic problems a vector treatment becomes necessary. For comparison with the results of later chapters, this chapter is concluded by applying a (modified) Kirchhoff approximation to the calculation of the transmission cross section for normal incidence. Differing results, obtained by substitution in two different rigorous formulae, yield a preliminary indication of the order of approximation.

In chapter 2 an outline is given of the method of solution and its underlying theory. Following a suggestion of Prof. Dr. R. Timman, the given problem, which is of elliptical character, is transformed to one which is hyperbolical with one more dimension and of the nature of a transience problem. The transformation effecting this change, may be termed an inverse Laplace transform (with respect to the frequency parameter), since the relationship is best characterized by viewing it in the opposite sense, and stating that the original problem is a one-sided Laplace transform of the transformed problem. The ultimate reason for introducing this transform is that, by a type of Tauberian theorem, the required behaviour of the solution for high frequencies may be determined from the transformed solution for small values of the newly-introduced "time" variable.

The theory of characteristics is applied to the hyperbolical problem, and this problem proves to be mathematically equivalent to that of a plate in the form of a semi-infinite strip, in stationary supersonic flow, under a small angle of attack. The relations found are therefore in general either equivalent to known formulae in lifting surface theory, such as have been deduced by Evvard (1950) and Ward (1949), or generalizations of such formulae. The fundamental relation constitutes a direct integral representation of the solution in one part of space, while elsewhere it has the form of an integro-differential equation. From the continuity and symmetry properties of the solution, an additional set of integral equations is obtained, from which the derivatives occurring in the integro-differential equation may be solved by a recurrence process, with a finite number of steps for each finite region. The details of this whole process, as well as of the back-transformation to the original problem, are left over for discussion in the third chapter.

The transformed problem occurs also in the work done by Fox (1948) on the diffraction of a step-function pulse by a screen in the form of a strip, using the method of Friedlander (1946), which was developed for the problem of diffraction of a pulse by a half-plane.

Chapter 3, which primarily contains the application of the method of chapter 2 to the diffraction problem of a slit at high frequencies, starts with a check on the reliability of the method. This is afforded by an application to Sommerfeld's half-plane problem, and the known solution of this problem proves to be correctly reproduced. In the strip problem, the solution is found as an infinite series, which is shown to be an asymptotic series for high frequencies. This solution is applied to the calculation of the transmission cross section for normal incidence, retaining the terms of second and lower order in the reciprocal frequency. The result shows that the (modified) Kirchhoff approximation does possess a qualitative indication of the leading diffraction term, though quantitatively it is in error. Numerical values computed from the asymptotic formula, agree surprisingly well with exact results, even though the available values of Skavlem (1951) pertain to still rather low frequencies.

The final chapter contains the main features in the extension of the previous solution to the aerodynamical problem of the oscillating airfoil in subsonic compressible flow. This procedure of extension is necessary to ensure that the correct types of singularity occur at the leading and trailing edges of the airfoil, and comprises some rather awkward limiting processes performed on the Green's function of the diffraction problem. The physical significance of the frequency parameter in this case indicates that the solution holds not only for high frequencies, but also for near-sonic flight speeds, as far as the linearization and the two-dimensional character of the problem are physically valid in this case. A first approximation to the solution of this aerodynamical problem for high frequencies has been obtained by Timman (1951), by application of a Kirchhoff approximation.

Chapter 1.

THE BOUNDARY VALUE PROBLEM AND ITS PHYSICAL SIGNIFICANCE.

1.1. The two-dimensional problem of diffraction by a plane screen.

1.1.1. The differential equation.

The velocity field $\underline{v}$ in a frictionless fluid medium without external forces is related to the pressure and density in the medium, by Euler's equation of motion and the continuity equation, viz.:

$$\left.\begin{aligned} &\frac{\partial \underline{v}}{\partial t}+(\underline{v}\cdot\nabla)\,\underline{v}=-\frac{\nabla p}{\rho} \\ &\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\,\underline{v})=0 \end{aligned}\right\} \qquad (1).$$

Assume that the pressure is a function of the density only:

$$p=p(\rho)\,,$$

which is the case when all changes are isentropic; this implies that the density ρ_0 for the fluid at rest is a constant, independent of position and time. Further assume that the density differs only slightly from the rest density, so that it is possible to introduce a small parameter ε by

$$\rho-\rho_0=\varepsilon\rho_1\,.$$

We write accordingly

$$\underline{v}=\varepsilon\,\underline{v}_1+O(\varepsilon^2)$$

and

$$\begin{aligned}\nabla p&=p'(\rho)\,\nabla\rho\\ &=p'(\rho_0)\,\nabla\rho_1+O(\varepsilon)\end{aligned}$$

where O is the order symbol. Substituting these relations in (1), and retaining only first order terms in ε, and writing $\underline{v}$ instead of $\varepsilon\,\underline{v}_1$, etc., and c^2 instead of $p'(\rho_0)$, yields:

$$\begin{aligned}&\frac{\partial \underline{v}}{\partial t}=-\frac{c^2}{\rho_0}\nabla\rho\\ &\frac{\partial \rho}{\partial t}=-\rho_0\,\nabla\cdot\underline{v}\end{aligned}$$

If initially the motion is vortex-free, i.e. if there exists a potential ψ such that $\underline{v}_{t=0}=\nabla\psi$, then the equation of motion can be integrated to:

$$\underline{v} = \nabla \left(\psi - \frac{c^2}{\rho_0} \int_0^t \rho \, dt \right)$$

which means that the motion remains vortex-free and is characterized by a velocity potential

$$\chi = \psi - \frac{c^2}{\rho_0} \int_0^t \rho \, dt \, .$$

Thus, finally

$$\begin{aligned} \frac{\partial \chi}{\partial t} &= -\frac{c^2}{\rho_0} \rho \\ \frac{\partial \rho}{\partial t} &= -\rho_0 \Delta_3 \chi \end{aligned} \qquad (2)$$

where Δ_3 denotes the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \frac{\partial^2}{\partial z^2}$; and therefore, eliminating ρ :

$$\Delta_3 \chi - \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} = 0 \, ,$$

viz. the standard form of the wave equation.

The time dependence is assumed to be given by

$$\chi = \varphi e^{-ikct}$$

in which the potentials χ and φ are complex numbers, the real parts of which are to be taken eventually for physical application. Then φ satisfies Helmholtz's wave equation:

$$\left(\Delta_3 + k^2 \right) \varphi = 0 \qquad (3)$$

If, in analogy with damped mechanical systems, the case is considered where the equation for χ has a damping term, viz.

$$\Delta_3 \chi - \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} - \frac{\sigma}{c^2} \frac{\partial \chi}{\partial t} = 0 \, , \qquad \sigma \geq 0 \, ,$$

the equation for φ becomes

$$\left(\Delta_3 + k^2 + i \frac{\sigma k}{c} \right) \varphi = 0 \qquad (4)$$

We shall interest ourselves for the undamped case and, taking regard of the sign of the extra term in (4), interpret (3) as a limiting form for real k, reached through complex k with positive values of Im k. (It should be noted that it is essentially more complicated to regard the undamped case strictly as a limiting form for vanishing viscosity, since the Navier-Stokes equation, which applies in the viscous case, is of higher order than the Euler equation to which it tends).

In the sequel it will be assumed that the field is cylindrical, i.e. that it is the same in all planes perpendicular to the z-axis, so that only two space coordinates x and y are needed.

We are thus concerned with solutions of the wave equation

$$(\Delta+k^2)\,\varphi=0$$

in which Δ denotes the two dimensional Laplacian.

1.1.2. The boundary conditions.

If an obstacle is inserted into a given initial field, the field is modified, and we say that it is diffracted by the obstacle. The problem of diffraction is to determine the disturbed field from a knowledge of the initial or "incident" field (i.e. the field that would subsist if the obstacle were absent), and of the position and nature of the obstacle.

The obstacle will be assumed to be in the form of an infinitesimally thin screen in the (x,z) plane, with edges parallel to the z-axis, thus serving to preserve the two-dimensional character of the problem. The part of the x-axis occupied by the screen, will be denoted by S, and the rest of the x-axis by S'.

The boundary conditions are determined by the nature of the screen. For a perfectly rigid screen, the fluid velocity at the screen is tangential to the surface, so that the normal derivative of the total velocity potential φ_T vanishes there. For the other extreme case of a perfectly "soft" screen the fluid motion at the screen is normal to the surface, so that the tangential derivative of φ_T vanishes; this means that φ_T is constant on the screen, and we may choose this constant value to be zero. It will be convenient to consider the velocity potential as composed of the incident field plus a scattered field:

$$\varphi_T=\varphi_i-\varphi_s$$

and the two above cases are thus characterized by the boundary conditions:

$$\frac{\partial\varphi_s}{\partial y}=\frac{\partial\varphi_i}{\partial y} \quad \text{on } S \qquad \text{(rigid screen)},$$

$$\varphi_s=\varphi_i \quad \text{on } S \qquad \text{("soft" screen)},$$

in which the y-derivative is used because S is on the x-axis.

Together with the wave equation, these boundary conditions, being valid on part of the x-axis, and being the same when the screen is approached from different sides, impose on φ_s the condition that it be antisymmetrical in y for a rigid screen and symmetrical for a soft screen. For a rigid screen, this is illustrated by considering the incident wave

$$\varphi_i(x,y)+\varphi_i(x,-y) \tag{5}$$,

of which the y-derivative vanishes on the screen, so that the boundary conditions are automatically satisfied. This means that the screen causes no disturbance, or, since the problem is linear, and the differential equation is symmetrical in y,

$$\varphi_s(x,y)+\varphi_s(x,-y)=0$$

which is the relation of antisymmetry. The symmetry of φ_s for a soft screen is proved analogously by using the difference instead of the sum in (5). Since it is required that φ_s and $\frac{\partial\varphi_s}{\partial y}$ be continuous everywhere outside S, these symmetry relations imply the further boundary conditions:

$$\varphi_s=0 \qquad \text{on } S' \qquad \text{(rigid screen)},$$

$$\frac{\partial\varphi_s}{\partial y}=0 \qquad \text{on } S' \qquad \text{(soft screen)}.$$

Now first the case of a rigid strip for which S is the interval $|x|<1$, will be considered. The boundary conditions are:

$$\frac{\partial\varphi_s}{\partial y}=\frac{\partial\varphi_i}{\partial y} \quad \text{for} \quad y=0\pm\,,\ |x|<1,$$

$$\varphi_s=0 \quad \text{for} \quad y=0\,,\ |x|>1\,.$$

Next to these boundary conditions on the x-axis, conditions must be imposed at infinity in order to eliminate certain physically undesired solutions. Since S is bounded, it is natural to conclude that the scattered wave φ_s will behave at large distances from the origin like a cylindrical wave expanding from the origin, apart from a directional factor. We therefore digress for a moment to consider such waves due to a line source at the origin.

Cylindrical waves due to a line source at the origin, are given by a solution of the wave equation, singular at the origin, dependent only on the distance r_o from the origin, and representing an expanding wave motion. The first Hankel function of zero order and with argument kr_o, viz. $H_o^{(1)}(kr_o)$, possesses these properties, being a solution of Bessel's equation and behaving like

$$\frac{2i}{\pi}\log r_o$$

in the origin and like

$$\sqrt{\frac{2}{\pi i k r_o}}\; e^{ikr_o}$$

at infinity. Other solutions, differing from this one in their behaviour at infinity, are

$$Y_0(kr_0) \sim \sqrt{\frac{2}{\pi k r_0}} \sin\left(kr_0 - \frac{\pi}{4}\right)$$

which represents standing waves, and

$$H_0^{(2)}(kr_0) \sim \sqrt{\frac{2i}{\pi k r_0}}\, e^{-ikr_0}$$

which represents incoming waves. Allowing Im $k>o$ as mentioned in (1.1.1), $H_0^{(1)}(kr_0)$ is the only one of the above solutions which remains finite for large r_0.

Returning therefore to the wave φ_S scattered from the strip S, we shall, following Baker and Copson (1950), initially consider Im $k>o$ and impose the condition that φ_S, together with its first partial derivatives, be bounded, uniformly for all directions, as r_o tends to infinity. This is a simplified formulation of Sommerfeld's radiation condition.

Due to the sharp edges of the obstacle, it is found to be necessary to ensure uniqueness of the solution by imposing some condition of integrability, which may be derived from energy considerations. The sharp edge is a branch point of the solution, and, as noticed by Bouwkamp (1946, 1954), further solutions may be obtained by differentiation of a given solution, thus increasing the order of the singularity at the edge. Meixner (1949) has pointed out that the order of the singularity is restricted physically by the reasonable condition that the space energy should be finite in any finite region of space, including the vicinity of the edge, i.e. that the perturbation fluid velocity grad φ_S be quadratically integrable everywhere in space. This implies that φ_S must be finite everywhere outside the screen, since an unbounded term in the development of φ_S at a point would make $(\text{grad } \varphi_S)^2$ non-integrable. We shall call this the edge condition, since practically it is critical only at the edge of the screen.

No strict uniqueness proof seems to have been given for this two-dimensional problem for real k. However, the above conditions are generally believed to determine uniquely the solution of the problem, and we shall therefore adhere to this view in the sequel. (See also (2.1.1)).

Next consider the case of a soft screen complementary to the previous one, viz. an infinite plane screen on the x-axis, with a slit S' on $|x|<1$. It will be assumed that the incident field originates entirely in the negative half-space ($y<o$). Since the

screen is now infinite, the conditions at infinity for φ_s are more difficult to formulate than in the previous case. We therefore rather use a different decomposition of φ_T, suggested by comparing the problem with the case where the slit is absent. For a soft screen without slit the field, given by the method of images, is:

$$\varphi_0(x,y) = \begin{cases} \varphi_i(x,y) - \varphi_i(x,-y) & \text{for } y<0 \\ 0 & \text{for } y>0 \end{cases}$$

in accordance with the boundary condition $\varphi_0 = 0$ on the screen. The problem with slit will be considered as a perturbation of the one without it, thus

$$\varphi_i - \varphi_s = \varphi_T = \varphi_0 - \varphi_1$$

so that

$$\varphi_1(x,y) = \begin{cases} \varphi_s(x,y) - \varphi_i(x,-y) & \text{for } y<0, \\ \varphi_s(x,y) - \varphi_i(x,y) & \text{for } y>0, \end{cases}$$

for which

$$\frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_s}{\partial y} \pm \left.\frac{\partial\varphi_i}{\partial y}\right|_{y=|y|} \qquad \text{for } y \lessgtr 0.$$

Thus, the properties of φ_s of continuity in the slit S' and of symmetry apply also to φ_1.

Keeping in mind that the boundary conditions for φ_s are in this case:

$$\frac{\partial\varphi_s}{\partial y} = 0 \qquad \text{for } y=0,\ |x|<1$$
$$\varphi_s = \varphi_i \qquad \text{for } y=0\pm,\ |x|>1,$$

we obtain for φ_1 the boundary conditions:

$$\frac{\partial\varphi_1}{\partial y} = \pm\frac{\partial\varphi_i}{\partial y} \qquad \text{for } y=0\pm,\ |x|<1,$$
$$\varphi_1 = 0 \qquad \text{for } y=0\pm,\ |x|>1.$$

Since φ_1 is a perturbation due to a slit with projection on the x,y-plane finitely situated, the considerations on the behaviour at infinity for the function φ_s in the case of a strip, also hold here, i.e. φ_1 also satisfies the radiation condition. Further, of course, the edge condition also applies.

Evidently therefore, the two halves of the function φ_1, symmetrical in y, are, with adjustment of sign, identical with the antisymmetrical function φ_s of the previous problem. This statement is a special case of Babinet's principle, which states that two plane screen diffraction problems in which the screens are complementary, one being rigid and the other soft, may be described by the same functions.

In the sequel the boundary value problem connected with the two cases outlined above, viz. the cases of a rigid strip or of a complementary soft screen with slit, will be treated. An alternative pair, also equivalent by Babinet's principle, would be a soft strip and a complementary rigid screen with slit. Then the problem is the same as ours , but with the roles of the function and its y-derivative reversed.

Summing up, the problem before us is to find a function φ with the following properties:

1. $(\Delta+k^2)\,\varphi=0$ excepting on $y=0,\ |x|<1$
2. $\dfrac{\partial\varphi}{\partial y}=-\dfrac{1}{ik}f(x)$ for $y=0\pm,\ |x|<1$.
3. $\varphi=0$ for $y=0,\ |x|>1$.
4. φ satisfies the radiation condition
5. φ satisfies the edge condition.

The factor $-\frac{1}{ik}$ in condition 2 is written for later convenience. The conditions 4 and 5 imply that φ is bounded when both the real and imaginary parts of k are positive.

This problem can be treated analytically by introducing elliptic coordinates, which separates the wave equation, and developing the solution in terms of characteristic functions, in this case Mathieu functions. This method was followed by Sieger (1908) and Strutt (1931) and numerical results were computed by Morse and Rubinstein (1938). Later Skavlem (1951) independently solved the problem by a similar method, also presenting numerical values.

In the present treatise, however, interest will be focussed solely on the domain of large k (high frequencies), for which the characteristic function method is impracticable, due to slow convergence of the series and the difficult nature of the functions involved.

1.2. The transmission cross section.

In the diffraction problem of an infinite plane screen with slit, the energy transfer through the slit, more specially for the case of a plane incident wave, is of physical interest. The transmission cross section is defined as the ratio of the time mean of the power transmitted through a certain height of the slit, expressed as power per unit area, to that which falls on the slit per unit area normal to the direction of the incident wave.

The power dW transmitted at a given moment through an area df is

$$dW = (\mathrm{Re}\, p)\,(\mathrm{Re}\, v_n)\, df \tag{1}$$

in which v_n is the magnitude of the fluid velocity normal to df, and p should strictly be the total pressure, but may be taken to be the perturbation pressure, since the time mean of the transmitted power corresponding to a constant pressure, is zero.

By an obvious modification, eq. (2) of (1.1.1) may be written

$$-\frac{p}{\rho_0} = \frac{\partial \chi}{\partial t} \tag{2},$$

in which p is the perturbation pressure. Further

$$v_n = \frac{\partial \chi}{\partial n}$$

where $\frac{\partial}{\partial n}$ denotes differentiation normal to df. The real parts of p and v_n are needed in (1), thus:

$$(\mathrm{Re}\, p)(\mathrm{Re}\, v_n) = -\rho_0 \left(\mathrm{Re}\frac{\partial \chi}{\partial t}\right)\left(\mathrm{Re}\frac{\partial \chi}{\partial n}\right)$$

$$= -\frac{1}{4}\rho_0 \left\{\frac{\partial \chi}{\partial t}\cdot\frac{\partial \bar{\chi}}{\partial n} + \frac{\partial \bar{\chi}}{\partial t}\cdot\frac{\partial \chi}{\partial n} + 2\,\mathrm{Re}\left(\frac{\partial \chi}{\partial t}\cdot\frac{\partial \chi}{\partial n}\right)\right\}$$

$$= -\frac{1}{4}\rho_0 \left\{-ikc\,\varphi\frac{\partial \bar{\varphi}}{\partial n} + ikc\,\bar{\varphi}\frac{\partial \varphi}{\partial n} + 2\,\mathrm{Re}\left(ikc\,\varphi\frac{\partial \varphi}{\partial n}e^{-2ikct}\right)\right\}$$

in which a bar over a number denotes the complex conjugate value. The time mean of the last term is zero (taken over a period, or over a large interval). Thus, substitution in (1) and integration over the whole area considered, yields

$$W = \frac{1}{4} ikc\rho_0 \iint \left(\varphi\frac{\partial \bar{\varphi}}{\partial n} - \bar{\varphi}\frac{\partial \varphi}{\partial n}\right) df\,.$$

The power transmitted through the slit per unit area is therefore

$$W' = \frac{1}{8} ikc\,\rho_0 \int \left(\varphi\frac{\partial \bar{\varphi}}{\partial n} - \bar{\varphi}\frac{\partial \varphi}{\partial n}\right) ds \tag{3}$$

in which the path of integration lies in the shadow half-plane $y \geq 0$ and connects the two edges of the slit, or, more generally, connects the two halves of the screen, since $\varphi = 0$ on the screen itself. Application of Green's theorem makes evident that W is independent of the special path of integration which is chosen.

If we consider the plane incident wave with angle of incidence θ' :

$$\varphi_i = e^{ik(x \sin\theta' + y\cos\theta')}$$

and integrate over the position of the slit, $\frac{\partial}{\partial n}$ becomes $\frac{\partial}{\partial y}$, and

$$\frac{8}{ikc\rho_0} W_0' = \int_{-1}^{1} (-2ik\cos\theta')\,dx = -4ik\cos\theta'$$

Thus the transmission cross section is

$$\sigma(\theta') = \frac{W'}{W_0'\cos\theta'} = \frac{1}{4ik}\int\left(\bar{\varphi}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\bar{\varphi}}{\partial n}\right)ds \qquad (4)$$

for a slit of width 2 and a plane wave with direction of incidence θ'.

A more convenient expression for $\sigma(\theta')$ may be found by use of Green's function of the first kind for the half-space $y \geqslant 0$. To find this, the following lemma, which is Weber's two-dimensional analogue of the theorem of Helmholtz, will be used:

If φ satisfies the two-dimensional wave equation

$$(\Delta + k^2)\,\varphi = 0$$

and has continuous partial derivatives of first and second order within and on a sufficiently smooth closed curve Γ , then for every point (x,y) within Γ , holds

$$\varphi(x,y) = \frac{1}{4i}\int_{\Gamma}\left\{\varphi\frac{\partial}{\partial n}H_0^{(1)}(kr) - \frac{\partial\varphi}{\partial n}H_0^{(1)}(kr)\right\}ds \qquad (5),$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal to Γ , and r the distance from (x,y) to the element of integration. For a point outside Γ the integral is zero.

This can be proved for an internal point (x,y) by applying Green's theorem

$$\iint(\varphi\Delta\psi - \psi\Delta\varphi)\,dx\,dy = \int\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)ds$$

to the region bounded by Γ and a circle γ of radius ε around (x,y), and choosing

$$\psi(x,y) = H_0^{(1)}(kr)$$

The surface integral is then zero, so that

$$\int_{\Gamma}\left\{\varphi\frac{\partial}{\partial n}H_0^{(1)}(kr) - \frac{\partial\varphi}{\partial n}H_0^{(1)}(kr)\right\}ds = \int_{\gamma}\left\{\varphi\frac{\partial}{\partial r}H_0^{(1)}(kr) - \frac{\partial\varphi}{\partial r}H_0^{(1)}(kr)\right\}ds,$$

and since the left-hand member is independent of ε , we may let $\varepsilon \to 0$ in the right-hand side. Noting that the Hankel function and its derivative behave like $\frac{2i}{\pi}\log\varepsilon$ and $\frac{2i}{\pi\varepsilon}$ respectively at the point (x,y), this yields the value $4i\varphi(x,y)$, as required.

When (x,y) is an external point, Green's theorem may be applied at once to the whole domain inside Γ, since no singularities of the integrand occur in this domain. This yields the value zero for the line integral, as required, and the lemma is proved.

The above result can now be applied to obtain Green's functions for the half space $y \geq 0$, for the function φ with no singularities in $y \geq 0$, and satisfying the radiation condition at infinity.

Let Γ be the boundary of the semi-circle $\eta \geq 0$, $R \leq a$, in which ξ, η are running coordinates in the x,y plane, so that $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, and $\xi = R \sin\theta, \eta = R\cos\theta$. Then (5) gives

$$\varphi(x,y) = \frac{1}{4i}\int_{-a}^{a}\left\{-\varphi\frac{\partial}{\partial\eta}H_0^{(1)}(kr) + \frac{\partial\varphi}{\partial\eta}H_0^{(1)}(kr)\right\}_{\eta=0+} d\xi$$
$$+\frac{1}{4i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\varphi\frac{\partial}{\partial R}H_0^{(1)}(kr) - \frac{\partial\varphi}{\partial R}H_0^{(1)}(kr)\right\}_{R=a} a\,d\theta.$$

The contribution of the curved part vanishes when $a \to \infty$, since, by the radiation condition, φ and $\frac{\partial\varphi}{\partial R}$ are uniformly bounded for Im $k>0$, while both $H_0^{(1)}(kr)$ and $\frac{\partial}{\partial r}H_0^{(1)}(kr) = -kH_1^{(1)}(kr)$ are of exponentially vanishing order. Further,

$$\frac{\partial}{\partial\eta}H_0^{(1)}(kr) = -\frac{\partial}{\partial y}H_0^{(1)}(kr)$$

so that finally, making $a \to \infty$, we have

$$\varphi(x,y) = \frac{1}{4i}\int_{-\infty}^{\infty}\left\{\varphi\frac{\partial}{\partial y}H_0^{(1)}(kr) + \frac{\partial\varphi}{\partial\eta}H_0^{(1)}(kr)\right\}_{\eta=0+} d\xi \qquad (6).$$

Again, for $y<0$ the line integral is zero, so that replacing y by $-y$,

$$0 = \frac{1}{4i}\int_{-\infty}^{\infty}\left\{-\varphi\frac{\partial}{\partial y}H_0^{(1)}(kr) + \frac{\partial\varphi}{\partial\eta}H_0^{(1)}(kr)\right\}_{\eta=0+} d\xi$$

Subtraction and addition now yield the required results, viz.

$$\varphi(x,y) = \frac{1}{2i}\int_{-\infty}^{\infty}\varphi(\xi,0+)\frac{\partial}{\partial y}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)d\xi \qquad (7)$$

$$\varphi(x,y) = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{\partial\varphi(\xi,0+)}{\partial\eta}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)d\xi \qquad (8),$$

demonstrating Green's functions of the first and second kind for the half-space $y \geq 0$.

If φ satisfies condition 3 of (1.1.2), viz.

$$\varphi = 0 \quad \text{for} \quad y=0, \; |x|>1,$$

expression (7) becomes

$$\varphi(x,y) = \frac{1}{2i}\int_{-1}^{1} \varphi(\xi,0+)\,\frac{\partial}{\partial y} H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right) d\xi \qquad (9)$$

This equation will now be used to simplify the expression for the transmission cross section, and we begin by determining the behaviour of φ at great distances from the origin. With $x = r_0 \sin\theta$, $y = r_0\cos\theta$, (9) becomes:

$$\varphi(x,y) = -\frac{1}{2i}\int_{-1}^{1}\varphi(\xi,0+)\,H_1^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)\frac{ky\,d\xi}{\sqrt{(x-\xi)^2+y^2}}$$

$$= -\frac{kr_0\cos\theta}{2i}\int_{-1}^{1}\varphi(\xi,0+)\,H_1^{(1)}\left(kr_0\sqrt{1-\frac{2\xi}{r_0}\sin\theta+\frac{\xi^2}{r_0^2}}\right)\frac{d\xi}{r_0\sqrt{1-\frac{2\xi}{r_0}\sin\theta+\frac{\xi^2}{r_0^2}}}$$

$$\sim -\frac{k\cos\theta}{2i}\int_{-1}^{1}\varphi(\xi,0+)\,\frac{1}{i}\sqrt{\frac{2}{\pi i}}\,\frac{e^{ik(r_0-\xi\sin\theta)}}{\sqrt{kr_0}}\,d\xi$$

$$\sim \frac{e^{ikr_0}}{\sqrt{r_0}}\sqrt{\frac{k}{2\pi i}}\cos\theta.\int_{-1}^{1}\varphi(\xi,0+)\,e^{-ik\xi\sin\theta}\,d\xi \qquad (10),$$

for large r_0, by developing under the sign of integration.

Thus φ does in fact behave like an expanding cylindrical wave with a directional factor, as was anticipated in (1.1.2). Defining the amplitude $A(\theta)$ as the coefficient of $\frac{e^{ikr_0}}{\sqrt{r_0}}$ in the development of φ for large r_0, (10) yields

$$A(\theta) = \sqrt{\frac{k}{2\pi i}}\cos\theta.\int_{-1}^{1}\varphi(\xi,0+)\,e^{-ik\xi\sin\theta}\,d\xi \qquad (11).$$

Returning now to eq. (4), choosing the path of integration directly over the slit, and noting that φ satisfies there the boundary condition

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi_i}{\partial y} = ik\cos\theta'.\,e^{ikx\sin\theta'} \qquad \text{for } y = 0+,\ |x|<1,$$

we have

$$\sigma(\theta') = \frac{1}{4ik}\,ik\cos\theta'.\int_{-1}^{1}\left\{\bar\varphi(x,0+)\,e^{ikx\sin\theta'} + \varphi(x,0+)\,e^{-ikx\sin\theta'}\right\}dx$$

$$= \frac{1}{2}\cos\theta'.\ \mathrm{Re}\int_{-1}^{1}\varphi(x,0+)\,e^{-ikx\sin\theta'}\,dx \qquad (12),$$

so that, from (11),

$$\sigma(\theta') = \mathrm{Re}\sqrt{\frac{\pi i}{2k}}\,A(\theta') \qquad (13).$$

This result is due to Levine and Schwinger (1948).

A more direct but less convenient simplification of (4) is obtained if we choose a semi-circle $R = a$, $\psi \geqslant 0$ as path of integration and make $a \to \infty$, using the asymptotic property of φ which is expressed by (10) and (11):

$$\varphi \sim A(\theta) \frac{e^{ikr_0}}{\sqrt{r_0}} \qquad \text{for large } r_0 \tag{14}$$

Thus (4) becomes

$$\sigma(\theta') = \frac{1}{4ik} \lim_{a \to \infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\bar{\varphi} \frac{\partial \varphi}{\partial R} - \varphi \frac{\partial \bar{\varphi}}{\partial R} \right)_{R=a} a\, d\theta$$

$$= \frac{1}{4ik} \lim_{a \to \infty} 2ik \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A(\theta)\, \overline{A(\theta)}\, d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |A(\theta)|^2\, d\theta \tag{15}$$

1.3. The aerodynamics of a vibrating airfoil in a subsonic, compressible flow.

It will now be demonstrated that the problem of finding the air forces on a vibrating airfoil in subsonic, compressible flow, is to a large extent equivalent to the above diffraction problem.

We consider a compressible fluid medium, with the coordinate system x_0, y_0, z_0, at rest with respect to the medium. A two-dimensional airfoil in the form of a flat plate situated "approximately" on part of the x_0-axis, with edges parallel to the z_0-axis, moves with constant velocity V in the direction of the negative x_0-axis, such that the Mach number $M = \frac{V}{c} < 1$. Further the airfoil executes small oscillations, independent of z_0, but otherwise arbitrarily prescribed. We thus again have a problem with only two space dimensions x_0 and y_0.

Both the velocity potential X and the acceleration potential x, defined by

$$x = \frac{\partial X}{\partial t_0} \tag{1}$$

where t_0 is the time, satisfy the wave equation, since the oscillations are assumed small. If either a Galileo transformation

$$x' = x_0 + Vt_0,\ y' = y_0,\ t' = t_0 \tag{2}$$

or a Lorentz transformation (with scale factor a)

$$a\beta x = x_0 + Vt_0,\ ay = y_0,\ a\beta t = \frac{V}{c^2} x_0 + t_0, \tag{3}$$

in which $\beta = \sqrt{1-M^2} = \frac{1}{c}\sqrt{c^2 - V^2}$, is applied, the forward motion of the airfoil is reduced to rest, while with the Lorentz transformation also the wave equation remains invariant.

Since the oscillations are small, the boundary values on the airfoil will be interpreted as given exactly on the x-axis. If the width of the airfoil is called 2ℓ, its position in the Lorentz coordinates may be taken as $y=0, |x|<\frac{\ell}{a\beta}$. Evidently it will be convenient to choose the scale factor a so as to have the airfoil, and therefore the boundary conditions, on the interval $|x|<1$, viz.

$$a=\frac{\ell}{\beta}\,,$$

so that (3) becomes

$$\ell x = x_0+Vt_0\,,\quad \frac{\ell}{\beta}y=y_0\,,\quad \ell t=\frac{V}{c^2}x_0+t_0 \tag{3a}$$

We assume the oscillation to be harmonic and write

$$\chi=\varphi' e^{-i\nu t'} \tag{4}$$

in which φ' is independent of t'. Solving (2) and (3a) in the form

$$x'=\ell x,\; y'=\frac{\ell}{\beta}y,\quad t'=-\frac{\ell V}{\beta^2c^2}x+\frac{\ell}{\beta^2}t \tag{5}$$

(4) may be written as

$$\chi=\varphi' e^{\frac{i\nu\ell Vx}{\beta^2c^2}-\frac{i\nu\ell t}{\beta^2}} \tag{6}$$

while φ' is seen to be independent not only of t', but also of t.
Writing

$$k=\frac{\ell\nu}{\beta^2c} \tag{7}$$

and

$$\varphi=\varphi' e^{\frac{i\nu\ell Vx}{\beta^2c^2}}$$

$$=\varphi' e^{ik\frac{V}{c}x} \tag{8}$$

(6) becomes

$$\chi=\varphi e^{-ikct} \tag{9}$$

with φ independent of t, so that φ satisfies Helmholtz's wave equation

$$(\Delta+k^2)\varphi=0 \tag{10}$$

which is condition 1 of section (1.1.2).

Next we consider the boundary values of φ, and investigate which other conditions of (1.1.2) apply. The boundary condition is furnished by noting that the fluid particles may not penetrate the airfoil, so that the prescribed oscillation determines the vertical velocity component of the adjacent fluid. Using for the velocity potential a similar notation as above for the acceleration potential, we thus have

$$\frac{\partial\phi'}{\partial y'}=w(x')\quad \text{for}\quad y=0^{\pm},\ |x'|<\ell\,,$$

where the oscillation of the airfoil is described by its (small) deviation in the y'-direction, viz.

$$w(x')\, e^{-i\nu t'} \quad . \qquad (11)$$

Therefore, as in (8), and using (5),

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial y} &= w(\ell x)\, e^{ik\frac{V}{c}x}, \quad \text{for } y=0^{\pm},\ |x|<1 \\ &= W(x) \end{aligned}\right\} \qquad (12),$$

by which W(x) is defined. Using (9), (1) and (3a), we obtain

$$\begin{aligned} \varphi e^{-ikct} = \chi &= \frac{\partial X}{\partial t_0} \\ &= \frac{V}{\ell}\frac{\partial X}{\partial x} + \frac{1}{\ell}\frac{\partial X}{\partial t} \\ &= \frac{1}{\ell}\left(V\frac{\partial\Phi}{\partial x} - ikc\,\Phi\right) e^{-ikct} \end{aligned} \qquad (13).$$

Therefore, from (12) the boundary condition for φ may be written as

$$\frac{\partial\varphi}{\partial y} = \frac{V}{\ell} W'(x) - \frac{ikc}{\ell} W(x), \quad \text{for } y=0^{\pm},\ |x|<1 \qquad (14),$$

which can be expressed in terms of $w(x')$ by use of (12) viz.

$$\begin{aligned} \frac{\partial\varphi}{\partial y} &= \left\{ V w'(\ell x) + \left(\frac{ikV^2}{\ell c} - \frac{ikc}{\ell}\right) w(\ell x) \right\} e^{ik\frac{V}{c}x} \\ &= \left\{ V w'(\ell x) - \frac{ikc\beta^2}{\ell} w(\ell x) \right\} e^{ik\frac{V}{c}x} \end{aligned} \qquad (15).$$

Again (14) may be written as

$$\frac{\partial\varphi}{\partial y} = -\frac{1}{ik} f(x), \quad \text{for } y=0^{\pm},\ |x|<1, \qquad (16),$$

as in condition 2 of (1.1.2).

By eq. (2) of (1.2) we have (in adjusted notation)

$$-\frac{p}{\rho_0} = \frac{\partial X}{\partial t_0} = \chi \qquad (17).$$

This means that, since the pressure is to be continuous everywhere outside the airfoil, the same applies to the acceleration potential. Since again φ is antisymmetrical in y, this means that φ also satisfies condition 3 of (1.1.2).

Obviously the radiation condition is also satisfied, so that the validity of only condition 5, the edge condition, is still to be investigated. Suffice to remark that at the trailing edge,*) viz. y=0, x=-1, in general a quadratically integrable singularity occurs. The occurrence of a singularity at this edge is made plausible by remarking that a point of stagnation with respect to the airfoil occurs there, so that the perturbations are no longer small, as required by linearized theory.

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*) viz. y=0, x=1, the Kutta condition may be imposed, which requires that p and therefore φ be finite; whereas at the leading edge

Thus φ satisfies conditions 1 to 4 of (1.1.2), together with a modification of 5, allowing a singularity at the leading edge. Thus φ may be decomposed into two parts:

$$\varphi = \varphi^r + \varphi^s \tag{18}$$

such that φ^r, the "regular" part, satisfies all the conditions 1 to 5 of (1.1.2), whereas φ^s, the "singular" part, satisfies these conditions with 2 replaced by

$$\frac{\partial \varphi^s}{\partial y} = 0 \qquad \text{for } y = 0^{\pm},\ |x|<1 ,$$

and 5 modified to allow the leading edge singularity.

When φ is known, the pressure distribution and therefore the air forces on the airfoil resulting from the vibration, may be calculated. The pressure jump across the airfoil (downward thrust positive) is, using (17), (4) and (8):

$$\begin{aligned} 2p(x',0+,t') &= -2\rho_0\, \chi(x',0+,t') \\ &= -2\rho_0\, \varphi'(x',0+)\, e^{-i\nu t'} \\ &= -2\rho_0\, \varphi(x,0+)\, e^{-iMkx-i\nu t'} . \end{aligned}$$

Therefore the total (complex) downward force F on unit length of the airfoil is

$$\begin{aligned} K &= -2\rho_0\, e^{-i\nu t'} \int_{-l}^{l} \varphi(x,0+)\, e^{-iMkx}\, dx' \\ &= -2\rho_0\, l\, e^{-i\nu t'} \int_{-1}^{1} \varphi(x,0+)\, e^{-iMkx}\, dx \end{aligned} \tag{19}$$

from (5), a result which is in direct analogy to the expression (12) of (1.2) for $\sigma(\theta')$. The moment about the midpoint (trailing edge downward) is

$$m = -2\rho_0\, l^2 e^{-i\nu t'} \int_{-1}^{1} x\, \varphi(x,0+)\, e^{-iMkx}\, dx \tag{20}$$

A rigid airfoil may execute translational and torsional oscillations, for which the function w(x') in (11) becomes

$$\left.\begin{aligned} w(x') &= A \quad \text{(translation)} \\ w(x') &= Bx' \quad \text{(torsion)} \end{aligned}\right\} \tag{21}$$

respectively. These two types of vibration may be characterized by dimensionless aerodynamic derivatives, introduced as follows:

$$\left.\begin{aligned} K &= \pi\rho_0\, l\, V^2 e^{-i\nu t'} \left[Ak_a + Bk_b \right] \\ m &= \pi\rho_0\, l^2 V^2 e^{-i\nu t'} \left[Am_a + Bm_b \right] \end{aligned}\right\} \tag{22}$$

The above aerodynamic problem has been solved analytically by Timman (1946), and a numerical computation of the aerodynamic derivatives was given by Timman, Van de Vooren and Greijdanus (1951 and 1954). In the domain of large k, Timman (1951) applied the Kirchhoff method to obtain a first approximation to the air forces, and the method of the present treatise may be used to approach the asymptotic behaviour more closely.

The method presented here yields an asymptotic solution for large k, and it is therefore of interest to take note of the physical meaning of k. By (7), k is proportional to the frequency $\frac{\nu}{2\pi}$. However, it is customary in this field rather to use the reduced frequency ω, which is a non-dimensional parameter defined by

$$\omega = \frac{\nu l}{V} \qquad (23),$$

and k may therefore be written, using (7) and (23), as

$$k = \frac{V\omega}{c\beta^2} = \frac{M\omega}{\beta^2} \qquad (24).$$

From this it is seen that k is large for high frequencies and also for small β, i.e. for near-sonic speeds, so that not only does the present theory yield results for oscillations at high reduced frequency, but it also gives results in linear theory for oscillations of moderate frequency at high subsonic (near-sonic) speeds, though, as is well known, the linearization becomes rather questionable in this region.

The method of obtaining the "regular" part φ^r of the solution will be given in the sequel. For the "singular" part φ^s, use is made of the Green's function obtained for the regular solution. The method, which is due to Timman (1954), will now be briefly sketched.

Let the regular solution be written in the form

$$\varphi^r(x,y) = \int_{-1}^{1} \frac{\partial\varphi(\xi,0)}{\partial\eta}\, G(x,y;\xi,0)\, d\xi \qquad (25).$$

Then the Green's function $G(x,y;\xi,\eta)$ with parameters ξ and η, satisfies:

$$\left.\begin{array}{lll} (\Delta+k^2)\, G = 0 & & \\ \frac{\partial G}{\partial y} = 0 & \text{for} & y = 0^{\pm}\,,\ |x|<1 \\ G = 0 & \text{for} & y = 0\,,\ |x|>1 \end{array}\right\} \qquad (26),$$

plus radiation condition and edge condition, while further G is symmetrical in (x,y) and (ξ, η), and possesses a logarithmic singularity where (x,y) = (ξ, η). Therefore the function G with (ξ,η) = (-1,0), satisfies all conditions imposed on φ^s, excepting that the singularity should be of the nature of a branch point. The nature of the singularity may be changed by differentiation with respect to ξ; G then still satisfies the differential equation and boundary values, but the singularity becomes of order minus one, which is still not what is required. To set this right, a new coordinate x_1 is introduced by

$$x = \begin{cases} -1+\frac{1}{4}x_1^2 & \text{for } x>-1 \\ -1-\frac{1}{4}x_1^2 & \text{for } x<-1 , \end{cases}$$

or, inversely

$$x_1 = \begin{cases} 2\sqrt{1+x} & \text{for } x>-1 \\ -2\sqrt{-1-x} & \text{for } x<-1 , \end{cases}$$

so that

$$\frac{\partial}{\partial x_1} = \sqrt{|1+x|}\,\frac{\partial}{\partial x}$$

and similarly ξ_1 is defined in terms of ξ, so that

$$\frac{\partial}{\partial \xi_1} = \sqrt{|1+\xi|}\,\frac{\partial}{\partial \xi} \tag{27}$$

If, now, G is differentiated with respect to ξ_1 instead of ξ, and (ξ, η) made equal to (-1,0), all conditions prove to be satisfied by the result, which may be written as

$$\lim_{\xi\to -1} \frac{\partial}{\partial \xi_1} G(x,y;\xi,0) \equiv \frac{\partial}{\partial \xi_1} G(x,y;-1,0) . \tag{28}$$

Since, however, this problem is homogeneous, the solution still contains an undetermined factor, which must be determined separately.

Therefore φ may be written, taking regard of (14), (18), (25) and (28), as

$$\varphi(x,y) = \int_{-1}^{1} \left[\frac{V}{\ell} W^1(\xi) - \frac{ikc}{\ell} W(\xi) \right] G(x,y;\xi,0)\,d\xi + \frac{\alpha_0 V}{\ell} \frac{\partial}{\partial \xi_1} G(x,y;-1,0) \tag{29}$$

in which α_0 is still unknown. This formula will first be simplified before attempting to determine α_0. We introduce the function

$$\Phi^*(x,y) = \int_{-1}^{1} W(\xi)\,G(x,y;\xi,0)\,d\xi \tag{30}$$

which, by (25), satisfies the boundary condition (12) imposed on Φ. With this notation, (29) may be written as

$$\varphi(x,y) = \frac{V}{\ell} \int_{-1}^{1} W^1(\xi) G(x,y;\xi,0)\,d\xi - \frac{ikc}{\ell} \Phi^*(x,y) + \frac{\alpha_0 V}{\ell} \frac{\partial}{\partial \xi_1} G(x,y;-1,0) \tag{31}$$

Further, arguing formally, it might be expected that the first term of the right-hand side is equal to $\frac{V}{\ell}\cdot\frac{\partial \Phi^*}{\partial x}$. This is, however, not the case, since differentiation increases the order of the singularities at the two edges, and in view of this and the properties of (28) it is natural to write

$$\frac{\partial \Phi^*}{\partial x} = \int_{-1}^{1} W^1(\xi) G(x,y;\xi,0)\,d\xi - \mu_1 \frac{\partial}{\partial \xi_1} G(x,y;-1,0) - \mu_2 \frac{\partial}{\partial \xi_2} G(x,y;1,0) \tag{32}$$

in which μ_1 and μ_2 are constants, and ξ_2 is defined in analogy with (27), so that

$$\frac{\partial}{\partial \xi_2} = \sqrt{|1-\xi|}\,\frac{\partial}{\partial \xi} \tag{33}$$

Evidently both sides of (32) satisfy the differential equation (10) and have y-derivative equal to $W'(x)$ on the airfoil. The relation (32) is in fact proved by Timman (1954) by making use of the known Green's function for k=0, and of the property of similar behaviour of solutions of the wave equation and Laplace's equation at singular points. The condition that the singularities at the two edge points should be the same on the left- and right-hand side of (32), determines the two coefficients, viz.

$$\left.\begin{aligned} \mu_1 &= -\pi \int_{-1}^{1} W(\xi) \frac{\partial}{\partial x_1} G(-1,0;\xi,0)\, d\xi \\ \mu_2 &= \pi \int_{-1}^{1} W(\xi) \frac{\partial}{\partial x_2} G(1,0;\xi,0)\, d\xi \end{aligned}\right\} \quad (34)$$

with $\frac{\partial}{\partial x_2}$ defined as in (33). Therefore (31) may now be written as

$$\varphi(x,y) = \frac{V}{l}\frac{\partial \Phi^*}{\partial x} - \frac{ikc}{l}\Phi^*(x,y) + \frac{V}{l}(\alpha_0+\mu_1)\frac{\partial}{\partial \xi_1}G(x,y;-1,0) + \mu_2 \frac{V}{l}\frac{\partial}{\partial \xi_2}G(x,y;1,0)\,. \quad (35)$$

The above formulae for φ are not sufficient to determine α_0, since the physical condition is a prescribed normal velocity on the airfoil, whereas φ, which is a (Lorentz) acceleration potential, defines the velocity only up to a constant of integration. Therefore we make use of the relation (13) between the Lorentz acceleration and velocity potentials, which, may be interpreted as an ordinary, linear, first order differential equation in Φ with solution

$$\Phi(x,y) = \frac{l}{V}\int_{-\infty}^{x} e^{\frac{ikc}{V}(x-x')}\varphi(x',y)\,dx' \quad (36),$$

in which Φ is assumed to be zero at $x=-\infty$. Substitution of (35) in (36), and partial integration of the first term, yields

$$\Phi(x,y) = \Phi^*(x,y) + \int_{-\infty}^{x} e^{\frac{ikc}{V}(x-x')}dx' \; (\alpha_0+\mu_1)\frac{\partial}{\partial \xi_1}G(x,y;-1,0) + \mu_2 \frac{\partial}{\partial \xi_2}G(x',y;1,0) \quad (37)$$

Since $\frac{\partial \Phi}{\partial y}$ and $\frac{\partial \Phi^*}{\partial y}$ both satisfy (12), the derivative with respect to y of the integral in (37) tends to zero for vanishing y, identically for all x with $|x|<1$, i.e.

$$0 = \lim_{y\to 0+}\frac{\partial}{\partial y}\int_{-\infty}^{-1} e^{-\frac{ikc}{V}x'}dx' \left\{(\alpha_0+\mu_1)\frac{\partial}{\partial \xi_1}G(x',y;-1,0) + \mu_2\frac{\partial}{\partial \xi_2}G(x',y;1,0)\right\} \quad (38)$$

in which -1 is substituted as upper limit of integration since the y-derivative of the integrand vanishes for vanishing y and $-1<x'<x$. Therefore finally

$$\alpha_0 = -\mu_1 - \mu_2\frac{R_2}{R_1} \quad (39)$$

in which

$$R_1 = \left[\frac{\partial}{\partial y}\int_{-\infty}^{-1} e^{-\frac{ikcx'}{V}}\frac{\partial}{\partial \xi_1}G(x',y;-1,0)\,dx'\right]_{y=0+} \quad (40)$$

$$R_2 = \left[\frac{\partial}{\partial y}\int_{-\infty}^{-1} e^{-\frac{ikcx'}{V}}\frac{\partial}{\partial \xi_2}G(x',y;1,0)\,dx'\right]_{y=0+} \quad (41).$$

Evidently the differentiation in (40) may not be performed under the sign of integration, since the singularity at the upper limit then becomes non-integrable; the evaluation can, however, be done by taking the "finite part" of the divergent integral thus formed, as defined by Hadamard (1932). We shall later return to this matter.

1.4. Relationship with electromagnetic diffraction.

Since the diffraction theory for high frequency electro-magnetic waves approaching optical frequencies, is assuming an increasing importance, it is of interest to point out that the scalar theory discussed here, is directly applicable to electromagnetic problems.

An electro-magnetic field in free space is described by Maxwell's equations for the electric and magnetic vectors $\underline{E}$ and $\underline{H}$, viz. (using "rational" Gaussian units):

$$\operatorname{curl} \underline{E} = -\frac{1}{c}\frac{\partial \underline{H}}{\partial t}, \quad \operatorname{div} \underline{H} = 0$$

$$\operatorname{curl} \underline{H} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{\sigma}{c}\underline{E}, \quad \operatorname{div} \underline{E} = 0,$$

where c is the velocity of light and σ the conductivity.

Let the time dependence be given by

$$\underline{E} = \underline{d}\, e^{-ikct}, \quad \underline{H} = \underline{h}\, e^{-ikct},$$

in which the vector components are regarded as given by complex numbers, the real parts of which are to be taken eventually for physical application. Then Maxwell's equations become

$$\operatorname{curl} \underline{d} = ik\, \underline{h} \qquad (1),$$

$$\operatorname{curl} \underline{h} = \left(-ik + \frac{\sigma}{c}\right)\underline{d} \qquad (2).$$

Both $\underline{d}$ and $\underline{h}$ satisfy the Helmholtz wave equation, as is seen by eliminating either from the above equations, e.g.

$$\begin{aligned} ik\left(-ik+\frac{\sigma}{c}\right)\underline{d} &= \operatorname{curl}\operatorname{curl}\underline{d} \\ &= \operatorname{grad}\operatorname{div}\underline{d} - \Delta_3 \underline{d} \\ &= -\Delta_3 \underline{d} \end{aligned}$$

$$\text{i.e.} \left(\Delta_3 + k^2 + \frac{ik\sigma}{c}\right)\underline{d} = 0.$$

Each cartesian component of $\underline{d}$ and $\underline{h}$ therefore satisfies Helmholtz's (scalar) wave equation.

In two-dimensional electromagnetic problems, the general difficulties of vector solutions do not appear, since a two-dimensional field may be written as the sum of two plane polarized disturbances:

$$\underline{d} = \underline{d}_1 + \underline{d}_2, \quad \underline{h} = \underline{h}_1 + \underline{h}_2,$$

where $\underline{d}_1 = d_x \underline{i} + d_y \underline{j}$, $\underline{h}_1 = h_x \underline{k}$
represents a wave polarized parallel to the z-axis, while
$\underline{d}_2 = d_x \underline{k}$, $\underline{h}_2 = h_x \underline{i} + h_y \underline{j}$,
is polarized perpendicular to the z-axis.

This separation is significant because the two component disturbances individually satisfy Maxwell's equations, due to the fact that all derivatives with respect to z vanish.

Further, from a knowledge of h_z, the whole of the first wave is determined by (2), and similarly, using (1), the second wave follows from a knowledge of d_z, so that the whole two-dimensional field is determined when d_z and h_z are given. All interest may thus be focussed on these two components, both of which satisfy Helmholtz's wave equation.

Again the case in which $\sigma \rightarrow 0$, may be interpreted as a limiting form of the equation $(\Delta + k^2)\varphi = 0$ for real k, reached through complex k with positive values of Im k. Evidently this passage to the limit is more natural here than in the acoustical case.

The same form of plane screen is considered as previously. The boundary conditions may be obtained by assuming that the screen is a perfect conductor, which implies that the electrical vector is normal to the surface, i.e. that its tangential components vanish.

For the wave $\underline{d}_1$, $\underline{h}_1$ polarized parallel to the edge of the screen, the condition on the screen is

$$d_x = 0$$

but, since h_z determines the whole field, we write this, using
$\mathrm{curl}\, \underline{h} = \left(ik + \frac{\sigma}{c}\right) \underline{d}$, as

$$\frac{\partial h_x}{\partial y} = 0$$

For the wave $\underline{d}_2$, $\underline{h}_2$, polarized perpendicular to the edge, the condition is simply

$$d_x = 0$$

on the screen.

Thus the same type of boundary conditions occur as previously, with softness or rigidity of the screen in the acoustical case corresponding respectively to polarization perpendicular or parallel to the edge in the electromagnetic case.

Also the radiation condition continues to hold for the perturbation quantities, and further an appropriate form of the edge condition must be applied. The space energy is half the sum of the squares of the electric and magnetic real field strengths, and the edge condition therefore requires both $\underline{d}$ and $\underline{h}$ to be quadratically integrable. Since from d_z or h_z the rest of the field is obtained by using (1) or (2), i.e. by differentiation, the condition for d_z or h_z takes the same form as for the acoustical velocity potential.

The general two-dimensional electromagnetic problem for a perfectly conducting screen is therefore fully equivalent to the scalar acoustical theories for rigid and soft screens.

Finally, it appears that also the electromagnetic transmission cross section can be calculated as in the acoustical case. The flow of energy per unit area per unit time is given by Poynting's vector

$$c \operatorname{Re} \underline{E} \times \operatorname{Re} \underline{H} ,$$

and this is equal to

$$c \operatorname{Re} \left(\underline{E} \times \underline{\bar{H}} + \underline{E} \times \underline{H}\right)$$
$$= c \operatorname{Re}\left(\underline{d} \times \underline{\bar{h}}\right) + c \operatorname{Re}\left(\underline{d} \times \underline{h}\, e^{-2ikct}\right) ,$$

of which the last term gives a zero time mean. Therefore, using Maxwell's equations (1) and (2), the energy vector is

$$c \operatorname{Re}\left(\underline{d} \times \underline{\bar{h}}\right) = \operatorname{Re}\left(-\frac{c}{ik}\, \underline{d} \times \operatorname{curl} \underline{\bar{d}}\right) \qquad (3),$$
$$= \operatorname{Re}\left(-\frac{c}{ik}\, \underline{h} \times \operatorname{curl} \underline{\bar{h}}\right) \qquad (4).$$

For polarization in the (x,y) plane, $\underline{d}$ has only a z-component, of magnitude d_z and (3) yields

$$\operatorname{Re}\left(-\frac{c}{ik}\, d_z \nabla \bar{d}_z\right)$$

in which ∇ is the two-dimensional gradient operator, and this can be written as

$$\frac{c}{2ik}\left(\bar{d}_z \nabla d_z - d_z \nabla \bar{d}_z\right)$$

The power transmitted through the slit per unit area is therefore

$$W' = \frac{c}{4ik} \int \left(\bar{d}_z \frac{\partial d_z}{\partial n} - d_z \frac{\partial \bar{d}_z}{\partial n}\right) ds ,$$

in analogy with eq. (3) of (1.2), and, as in (4) of (1.2),

$$\sigma(\theta') = \frac{1}{4ik} \int \left(\bar{d}_z \frac{\partial d_z}{\partial n} - d_z \frac{\partial \bar{d}_z}{\partial n}\right) ds .$$

Using (4) instead of (3), the same analysis applied to the case of polarization parallel to the edge, with h_z replacing d_z.

1.5. Kirchhoff's theory of diffraction.

The classical theory for the treatment of diffraction problems for the high frequencies of optics, is that of Kirchhoff (1891). Accounts of Kirchhoff's work are given by Baker and Copson (1949) and by Sommerfeld in Frank-Mises (1930), and only a brief sketch of the method and its deficiencies is given here.

Kirchhoff's treatment depends, for the two-dimensional case, on Weber's formula (5) of (1.2). In order to discuss our case of plane screen diffraction, the relevant formulae (6), (7) and (8) of (1.2), valid for $y>0$, are reproduced here for reference:

$$\varphi(x,y)=\frac{1}{4i}\int_{-\infty}^{\infty}\left\{\varphi\frac{\partial}{\partial y}H_0^{(1)}(kr)+\frac{\partial\varphi}{\partial\eta}H_0^{(1)}(kr)\right\}_{\eta=0+}d\xi \tag{1}$$

$$\varphi(x,y)=\frac{1}{2i}\int_{-\infty}^{\infty}\varphi(\xi,0+)\frac{\partial}{\partial y}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)d\xi \tag{2}$$

$$\varphi(x,y)=\frac{1}{2i}\int_{-\infty}^{\infty}\frac{\partial\varphi(\xi,0+)}{\partial\eta}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)d\xi \tag{3}$$

By (2) and (3), it is evident that φ is determined in a general point if either φ or $\frac{\partial\varphi}{\partial y}$ is given on the whole x-axis, so that they may not be prescribed independently in (1). Kirchhoff's method consists in prescribing plausible boundary values in (1) for both φ and $\frac{\partial\varphi}{\partial y}$, regardless of the inconsistency arising thus. Kirchhoff's theory was supposed to be valid for a "black" screen and he actually assumed that on the backside of the screen there was no excitation, while on the free part of the x-axis the incident wave was unaltered, an assumption which is suggested by geometrical optics. This means that both φ and $\frac{\partial\varphi}{\partial y}$ are taken to be zero on the screen, and equal to φ_i and $\frac{\partial\varphi_i}{\partial y}$ elsewhere on the x-axis, but of course the solution obtained by substituting these in (1) cannot be expected to reproduce these boundary values.

In the sequel, however, not this original form of the theory will be referred to, but a modified form obtained by using (3) instead of (1). In our problem $\frac{\partial\varphi}{\partial y}$ is given for $y=0, |x|<1$.
On the rest of the x-axis, φ is given to be zero, but if, instead, it is assumed that not φ but $\frac{\partial\varphi}{\partial y}$ is zero there, (3) becomes

$$\varphi_K(x,y)=\frac{1}{2i}\int_{-1}^{1}\frac{\partial\varphi(\xi,0+)}{\partial\eta}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+y^2}\right)d\xi \tag{4}$$

This function φ_K, which is explicitly determined and satisfies the given boundary condition for $\frac{\partial \varphi}{\partial y}$ but not for φ , will be called "the" Kirchhoff approximation to our problem.

The expression (4) will now be employed to obtain an approximation $\sigma_K(0)$ to the transmission cross section for a normally incident plane wave on a screen with slit, using (13) and (15) of (1.2), viz.

$$\sigma(0) = \mathrm{Re}\sqrt{\frac{\pi i}{2k}}\, A(0) \tag{5}$$

$$= \int_0^{\frac{\pi}{2}} |A(\theta)|^2 d\theta \tag{6}$$

The amplitude is obtained from the value of φ_K far from the origin:

$$\varphi_K \sim \frac{1}{2i} \int_{-1}^{1} \frac{\partial \varphi(\xi, 0+)}{\partial \eta} \sqrt{\frac{2}{\pi i k r_0}}\, e^{ik(r_0 - \xi \sin\theta)} d\xi$$

$$= \frac{e^{ikr_0}}{\sqrt{r_0}} \frac{1}{i\sqrt{2\pi i k}} \int_{-1}^{1} \frac{\partial \varphi(\xi, 0)}{\partial \eta} e^{-ik\xi\sin\theta} d\xi ,$$

and since $\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi_i}{\partial y} = ik$ for $y = 0$, $|x| < 1$,

the amplitude of φ_K is

$$A_K(\theta) = \frac{ik}{i\sqrt{2\pi i k}} \int_{-1}^{1} e^{-ik\xi \sin\theta} d\xi$$

$$= \sqrt{\frac{2k}{\pi i}} \frac{\sin(k\sin\theta)}{k\sin\theta} \tag{7}$$

Substitution in (5) yields

$$\sigma_K(0) = \mathrm{Re}\sqrt{\frac{\pi i}{2k}} \cdot \sqrt{\frac{2k}{\pi i}} = 1 \tag{8}$$

which is identical with the result of geometrical optics, and gives no evidence of diffraction effects.

For the sake of comparison with (8), it is interesting also to substitute (7) into (6) instead of into (5), obtaining

$$\sigma_K(0) = \frac{2}{\pi k} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\sin\theta)}{\sin^2\theta} d\theta = \frac{2}{\pi k} \int_0^1 \frac{\sin^2 ku}{u^2} \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{2}{\pi k}\left\{ \int_0^\infty \frac{\sin^2 ku}{u^2} du - \int_1^\infty \frac{\sin^2 ku}{u^2} du + \int_0^1 \frac{\sin^2 ku}{u^2}\left(\frac{1}{\sqrt{1-u^2}} - 1\right) du \right\}$$

$$= \frac{2}{\pi k}\left\{ k\int_0^\infty \frac{\sin 2ku}{u} du - \frac{1}{2}\int_1^\infty \frac{1-\cos 2ku}{u^2} du + \frac{1}{2}\int_0^1 \frac{1-\cos 2ku}{u^2}\left(\frac{1}{\sqrt{1-u^2}} - 1\right) du \right\}$$

$$= \frac{2}{\pi k}\left\{ \frac{\pi k}{2} - \frac{1}{2}\left(1 - \int_1^\infty \frac{\cos 2ku}{u^2} du\right) + \frac{1}{2}\left[1 - \int_0^1 \frac{\cos 2ku}{u^2}\left(\frac{1}{\sqrt{1-u^2}} - 1\right) du \right]\right\}$$

$$= 1 - \frac{1}{\pi k} \int_0^\infty \frac{\cos 2ku}{u^2} \left\{ \frac{U(1-u)}{\sqrt{1-u^2}} - 1 \right\} du$$

in which U(u) is the unit step function. By the principle of stationary phase, the integral contributes to the asymptotic behaviour for large k only near u=1, since also the lower limit does not contribute, so that

$$\sigma_k(0) \sim 1 - \frac{1}{\pi k} \int_{1-}^{1} \frac{\cos 2ku}{u^2} \cdot \frac{du}{\sqrt{1-u^2}}$$

$$\sim 1 - \frac{1}{2} \cdot \frac{\sin(2k + \frac{\pi}{4})}{\sqrt{\pi k^3}} + O\left(k^{-\frac{5}{2}}\right) \qquad (9).$$

Comparison of (9) with (8) shows that the error in the Kirchhoff approximation is at least of the order of $k^{-\frac{3}{2}}$, since both (5) and (6) are exact.

Chapter 2.

THE METHOD OF SOLUTION.

2.1. Transformation to a "transient" problem.

2.1.1. The formal transformation.

In the present section we intend transforming the boundary value problem, formulated in chapter 1, to a "transient" problem. The original problem has the form

$$\left.\begin{aligned} &(\Delta+k^2)\varphi=0 && \\ &\frac{\partial\varphi}{\partial y}=-\frac{f(x)}{ik} && \text{for } y=0\pm,\ |x|<1 \\ &\varphi=0 && \text{for } y=0,\ |x|>1 \end{aligned}\right\} \qquad (1)$$

plus edge condition and radiation condition. Using the result (14) of (1.2), the radiation condition, may be expressed as

$$\varphi\sim\frac{e^{ikr_0}}{\sqrt{r_0}}A(\theta) \qquad \text{for large } r_0=\sqrt{x^2+y^2} \qquad (1a),$$

with

$$A(\theta)=\sqrt{\frac{k}{2\pi i}}\cos\theta\int_{-1}^{1}\varphi(\xi,0+)\,e^{-ik\xi\sin\theta}\,d\xi$$

The solution φ is a function of x,y and k, and may be regarded as analytically extended over the whole k-plane. From the radiation condition (1a) it is seen that the half planes of positive and negative Re k correspond respectively to incoming and outgoing waves, while Im $k>0$ implies positive damping for incoming and negative damping for outgoing waves. (Evidently, therefore, only the first quadrant is physically significant, though φ does possess hermitian symmetry about the imaginary k-axis viz. $\varphi(-\bar{k})=\overline{\varphi(k)}$).

Since (1a) implies that, at least for large r_0, φ vanishes exponentially for large positive Im k, we shall for convenience call the half plane Im $k>0$ the region of "damped" values of φ, although this designation is rather arbitrary in the light of the above physical interpretation.

As was remarked in (1.1.2), no uniqueness proof is available for the problem when k is real. Since, however, we are treating the problem as a limiting case for positive Im k tending to zero, it is of interest to know whether the solution is unique in the "damped" region and we shall now show that this is in fact the case.

If more than one solution exist, the difference function satisfies all conditions, but with $f(x) \equiv 0$ in (1), so that the problem for the difference function is homogeneous. We now investigate whether there are values of k (eigenvalues) with $\mathrm{Im}\, k > 0$, for which the homogeneous problem has a non-zero solution, i.e. for which the inhomogeneous problem is not uniquely soluble.

Suppose that k_1 and k_2 are two such values of k for which non-trivial solutions φ_1 and φ_2 of the homogeneous problem exist. Then, since both φ_1 and φ_2 satisfy the edge condition, Green's theorem may be applied to the semi-circular region $r_0 \leq R, y \geq 0$, and yields, with obvious notation,

$$\iint (\varphi_1 \Delta \varphi_2 - \varphi_2 \Delta \varphi_1)\, d\tau = \int \left(\varphi_1 \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1}{\partial n} \right) d\sigma \tag{2}$$

The line-integral vanishes for large R, viz. the part along the x-axis for all R by the boundary conditions, and the curved part for large R by the radiation condition. The surface integral is simplified by the differential equation in (1), so that (2) becomes

$$\left(k_1^2 - k_2^2\right) \iint \varphi_1 \varphi_2 \, d\tau = 0 \tag{3}$$

Now, if k is an eigenvalue, so is $-\bar{k}$, so that

$$k_2 = -\bar{k}_1 ,$$

and therefore
$$\varphi_2 = \bar{\varphi}_1 ,$$

so that (3) becomes

$$\left(k_1^2 - \bar{k}_1^2\right) \iint |\varphi_1|^2 \, d\tau = 0$$

i.e. $$k_1 = \pm \bar{k}_1 ,$$

which means that $\mathrm{Re}\; k_1 = 0$, since we are considering only the half plane $\mathrm{Im}\; k > 0$.

This proves that, unless possibly for purely imaginary k, the inhomogeneous problem is uniquely soluble in the damped region. It therefore only remains to show that the same is true on the positive imaginary k-axis. Let φ_1 again be the difference function belonging with the value k_1 of k. Then, applying a different form of Green's theorem to the same region as above, φ_1 satisfies

$$\iint \varphi_1 \Delta \varphi_1 \, d\tau = \int \varphi_1 \frac{\partial \varphi_1}{\partial n} \, d\sigma - \iint \left\{ \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 \right\} d\tau \tag{4}$$

Again the line integral disappears for large R, and (4) becomes

$$-k_1^2 \iint \varphi_1^2 \, d\tau + \iint \left\{ \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 \right\} d\tau = 0 ,$$

which means that φ_1 vanishes identically, since k_1 is here assumed to be purely imaginary, and therefore the uniqueness holds also in this case.

Since, therefore, our problem is uniquely soluble in the half plane Im $k>0$, φ possesses no singularities in this half plane. This is a property possessed by every function which is the one-sided Laplace transform of another function, and we are thus led to consider introducing a function ψ such that

$$\varphi(x,y;k) = \int_0^\infty e^{ikz} \, \psi(x,y,z) \, dz \tag{5}$$

in which Im k is positive. Since φ is unique, any significant solution obtained via the formally transformed problem corresponding to ψ , will be the correct one.

Working formally, ψ may be written by Fourier's inversion theorem in the form

$$\psi(x,y,z) = \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} e^{-ikz} \, \varphi(x,y;k) \, dk \tag{5a}$$

in which a is positive (so that the integration goes over "damped" values of φ), and in which

$$\psi(x,y,z) = 0 \text{ for } z<0 \tag{6}$$

(The z occurring here is not the third space variable of chapter 1). The differential equation of (1) is transformed by (5a) formally into

$$\Delta\psi - \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{7}$$

which is a wave equation of hyperbolic type, in which z plays the part of a time coordinate. This justifies the use of the term "transient" in connection with this problem to indicate the property(6). The boundary conditions in (1) are reproduced by the transformation (5) if ψ is subjected to the conditions

$$\frac{\partial \psi}{\partial y} = \begin{cases} f(x) & \text{for } y=0\pm, |x|<1, z>0, \quad (8) \\ 0 & \text{for } y=0\pm, |x|<1, z<0, \quad (8a) \end{cases}$$

$$\psi = 0 \qquad \text{for } y=0, |x|>1. \tag{9}$$

(It was for the sake of simplicity of (8) that the factor $-\frac{1}{ik}$ was introduced in (1) and in condition 2 of (1.1.2)).

The edge condition may for the moment be left out of consideration, since any solution obtained can afterwards be tested for this property. Reference to this point will, however, be made in (2.2). The radiation condition is imposed by the choice of the sign of the exponents in (5) and (5a).

2.1.2. The theory of characteristics, and application.

For the purposes of the further analysis, the notion of characteristic surfaces of a linear, hyperbolic, partial differential equation is needed, and we therefore briefly introduce the essential concepts and properties. Fuller discussions are found in Courant-Hilbert, vol. II (1937), and Sauer (1952).

A characteristic surface, or simply characteristic, of the linear, second order differential equation (7), is a surface $g(x,y,z)=0$, such that the second order normal derivative of a smooth function ψ satisfying the equation, is not determined by prescribing ψ and its first normal derivative on the surface. If new orthogonal coordinates $\lambda(x,y,z)$, $\mu(x,y,z)$ and $g(x,y,z)$ are introduced, the transformation being non-singular, then the second normal derivative is $\frac{\partial^2\psi}{\partial g^2}$, and its coefficient in the transformed differential equation is

$$N(g)=\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2-\left(\frac{\partial g}{\partial z}\right)^2 .$$

All other first and second order derivatives on the surface $g=0$ may be directly obtained from the prescribed values of ψ and $\frac{\partial\psi}{\partial g}$, since differentiation with respect to λ and μ are inner processes on this surface. Therefore, if $N(g)$ is non-zero, $\frac{\partial^2\psi}{\partial g^2}$ is uniquely determined by the differential equation, so that the condition for the surface $g(x,y,z)=0$ to be characteristic is that

$$\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2-\left(\frac{\partial g}{\partial z}\right)^2=0 \tag{10}$$

for all points on the surface. If $g(x,y,z)=0$ is solved in the form $z=z(x,y)$, the condition becomes

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2-1=0 \tag{10a}$$

which holds identically in x and y. This can also be written in the form

$$(\text{grad } z)^2=1$$

in which grad denotes the two-dimensional gradient with respect to x and y only. This means that the characteristics of (7) are ruled surfaces with generators making an angle $\frac{\pi}{4}$ with the z-axis, so that a characteristic surface passing through the point (x',y',z'), touches there the cone

$$z-z'=\pm\sqrt{(x-x')^2+(y-y')^2} . \tag{11}$$

This cone itself satisfies (10a) identically and is therefore a characteristic. We may thus say generally that cones of the form (11) and envelopes of such cones are characteristic surfaces of (7). Obviously the characteristics are invariant under transformation of coordinates.

The fact that the equation (7) possesses real characteristics is expressed by saying that the equation is hyperbolic, in contrast to the original equation in (1), which has only complex characteristics and is termed elliptic.

The characteristic condition may also be formulated in a different form, which will be useful in application. Equation (10a) may namely be interpreted as an orthogonality relation between two vectors, viz.

$$\left(\frac{\partial z}{\partial x}\underline{i}+\frac{\partial z}{\partial y}\underline{j}-\underline{k}\right)\cdot\left(\frac{\partial z}{\partial x}\underline{i}+\frac{\partial z}{\partial y}\underline{j}+\underline{k}\right)=0\,.$$

The first of these is directed along the normal $\underline{n}$ to the surface with direction cosines (n_1, n_2, n_3) and the second along the so-called conormal $\underline{\nu}$, with direction cosines $(n_1, n_2, -n_3)$. The characteristic condition therefore states that the conormal is tangential to the surface. This means that a differentiation in the conormal direction is an inner process on the surface, a property which we shall have occasion to use in the sequel.

The most interesting property of characteristic surfaces of an equation, is the fact that discontinuities of a special kind may occur only on such surfaces. In the first place, if it is required that, on passing through a given surface, a solution is continuous together with its first derivatives and all inner derivatives, but that its second normal derivative should be finitely discontinuous, then evidently, from the definition of a characteristic, this surface can only be a characteristic. A finite discontinuity in the first normal derivative may occur at any surface, since the two sides of the surface may be treated separately in determining higher derivatives, but even here characteristic surfaces play an exceptional role. If we namely require that a solution with such a discontinuity be stable in the sense that on and near the surface it is the uniform limit of a sequence of continuous solutions with continuous and uniformly bounded first derivatives, such that just off the surface also their first and second derivatives tend uniformly to those of the limit function, then it can easily be proved that the surface must be characteristic.

Discontinuities at characteristics have the important property that they do, in general, not die out along a generator (i.e. in the conormal direction) of the characteristic. This can be shown for instance for a discontinuous first normal derivative by writing the differential equation in coordinates g, λ and μ as above. Since the coefficient N(g) of the second normal derivative vanishes on the characteristic surface, only the solution ψ and its inner derivatives, together with the discontinuous first normal derivative $\frac{\partial\psi}{\partial g}$ and a first inner derivative, which proves to be directed along the conormal, occur in the equation. Subtraction of the two equations valid on the two sides of the surface, makes all continuous terms vanish and only terms with the jump in $\frac{\partial\psi}{\partial g}$ and with the conormal derivative of this jump, remain. The resulting linear, ordinary differential equation is therefore of first order and homogeneous, and, as is well-known, the solution of such an equation with real coefficients either vanishes identically, or else not at all in an ordinary point. (Singular points may arise through singular points of the characteristic, e.g. the vertex of a cone, and at infinity).

In accordance with the special role played by characteristic surfaces in the occurrence of discontinuities in the solutions of (7), it will be reasonable to consider solutions which are continuous everywhere (excepting of course on the strip where the boundary value of $\frac{\partial\psi}{\partial y}$ is prescribed) and with first and second derivatives in general continuous, but possibly containing finite discontinuities in the normal derivatives on passing through a characteristic surface. It may be noted in passing that this convention is further extended by the assumption of the transient character of our boundary value problem. In view of the above-mentioned property of persistence of discontinuities along generators of characteristics, it is seen that, if no disturbance is to enter the half-plane of negative z, the discontinuities which may occur in ψ are restricted to such characteristics as are formed by cones with the positive sign of the root in (11), thus

$$z-z' = +\sqrt{(x-x')^2+(y-y')^2}\,. \tag{11a}$$

We shall now sketch briefly that allowing discontinuous behaviour of the solution ψ, as is proposed above, is compatible with the requirement that the solution of the boundary value problem be unique. Assume that ψ and its normal derivative are prescribed on an initial surface z=h(x,y). If, in a given region, the solution is not unique, the difference of two solutions will also satisfy the differential equation and will vanish together with its normal derivative on $z=h(x,y)$. Therefore, if vanishing of a solution and its normal derivative on $z=h(x,y)$, implies vanishing of the solution in a given region, the solution of the above initial value problem is unique in this region. In fact, it will be proved that $\psi\ (x',y',z')$ is dependent only on the values of ψ and its normal derivative which are given on that part of the initial surface which lies inside the single characteristic cone with (x',y',z') as vertex:

$$z-z' = -\sqrt{(x-x')^2+(y-y')^2}\ , \tag{11b}$$

which will be called the forward characteristic cone from (x',y',z'). The proof depends on Gauss' divergence theorem, and the divergence expression

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}\right)\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial z}\ \frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\ \frac{\partial\psi}{\partial y}\right)-\frac{1}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right\} \tag{12}$$

is used for the purpose. First consider only solutions which are continuous together with their first and second derivatives. Then, if (12) is integrated over the region G bounded by the cone (11b), and the initial surface $z=h(x,y)$, the volume integral in Gauss' theorem vanishes by the differential equation, and the surface integral over the part of the initial surface involved, vanishes by virtue of the null initial conditions. Therefore

$$\iint_M \left[\frac{\partial\psi}{\partial x}\ \frac{\partial\psi}{\partial x}\ \frac{\partial x}{\partial n}+\frac{\partial\psi}{\partial z}\ \frac{\partial\psi}{\partial y}\ \frac{\partial y}{\partial n}-\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right\}\frac{\partial x}{\partial n}\right]d\tau = 0 \tag{13}$$

in which M denotes the part of the conic surface (11b) cut off by the initial surface, and the differentiation $\frac{\partial}{\partial n}$ is directed along the normal to M. Using the relation

$$\left(\frac{\partial x}{\partial n}\right)^2+\left(\frac{\partial y}{\partial n}\right)^2-\left(\frac{\partial z}{\partial n}\right)^2 = 0$$

which holds because M is characteristic, the integrand can be written as

$$\frac{1}{\frac{\partial z}{\partial n}}\left\{\left(\frac{\partial\psi}{\partial x}\ \frac{\partial z}{\partial n}-\frac{\partial\psi}{\partial z}\ \frac{\partial x}{\partial n}\right)^2+\left(\frac{\partial\psi}{\partial y}\ \frac{\partial z}{\partial n}-\frac{\partial\psi}{\partial x}\ \frac{\partial y}{\partial n}\right)^2\right\}\ , \tag{14}$$

which means that the two squared terms are separately equal to zero on M, since $\frac{\partial z}{\partial n}=\frac{1}{\sqrt{2}}$ from (11b). But these two terms are independent inner derivatives on M, so that this means that ψ is constant on M, and therefore zero, by continuity and the initial values, so that also at (x',y',z') it is zero, and, by using smaller conical regions, also in all other points of G. Thus the unique dependence of ψ in a point on initial values on only that part of the initial surface within the forward characteristic cone from the point, is proved for functionswith regular behaviour.

If, now, the possibility of discontinuity in the normal derivative at one characteristic surface is allowed, this surface divides the region G into two parts. In the part furthest from the vertex (x',y',z') of (11b), the above proof applies, yielding the value naught for ψ on the dividing surface. Integration over the second part again yields an integral of the form (13), since it was seen in (14) that the integrand contains only inner derivatives of ψ on a characteristic surface, which are therefore zero on the dividing surface. The same result as previously thus follows in the presence of one characteristic of discontinuity, and evidently, on applying the same procedure repeatedly, also in the presence of more such discontinuities.

Thus it is seen that ψ in any point is entirely determined by values prescribed inside its forward characteristic cone, which is therefore called its region of dependence. Evidently the points in whose region of dependence a given point (x',y',z') is situated, are contained in the so-called backward characteristic cone from this point, presented in (11a) and this cone will therefore be called its region of influence. It should further be stressed that the above considerations are no longer valid for discontinuities along arbitrary surfaces, so that for instance a region of dependence must not be allowed to include the strip on which the boundary value (8) is prescribed.

In all the above the consistency of the initial values were not considered. Evidently, however, if a point on the initial surface has other points of the surface in its region of influence, this fact imposes a relation between the initial values, and the function and its normal derivative may not be prescribed arbitrarily. This happens, as can easily be verified, when the initial surface $z=h(x,y)$ satisfies

$$\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2 \geqslant 1 .$$

This is the case with the plane $y=0$ on which the boundary value (8) of our problem is prescribed, and it is therefore reasonable that ψ and $\frac{\partial \psi}{\partial y}$ are prescribed alternately on different parts of the plane, and not together. We shall be content with this remark, and not prove an existence and uniqueness theorem for our problem, since the important point is that, after transformation, the conditions of the original problem for φ are satisfied.

The boundary condition (8a) has been imposed in order to represent the transient character of the problem, but it will be convenient to use the conditions

$$\psi = \frac{\partial\psi}{\partial x} = 0 \qquad \text{for} \qquad z = 0 \tag{15}$$

in its stead. These conditions may be imposed in keeping with the transience condition, because z=0 is no characteristic, so that both ψ and $\frac{\partial\psi}{\partial x}$ are continuous on it. In this way the region of dependence of a point can be considered as bounded by the forward characteristic cone from the point, and the two planes y=0 and z=0.

Evidently the characteristic surface of discontinuity approaching nearest to the initial plane z=0, is the envelope of backward characteristic cones from the points of the line segment z=0, y=0, $|x|<1$, where the prescribed boundary values are discontinuous. Let this surface be presented by $z=z_0(x,y)$. Then, from the uniqueness theorem for the initial value problem, ψ is naught, not only for $z<0$, but also for all $z<z_0(x,y)$.

2.1.3. The inverse transformation.

With the foregoing results on characteristic theory available, and under the imposed convention about the occurrence of discontinuities, we may now return to the formal transformation, and examine whether this convention is compatible with the requirement that (5) transforms the equation (7) into the original equation of (1).

Consider a line parallel to the z-axis. Since $\psi=0$ for negative z, the characteristic surfaces of discontinuous behaviour which cross this line are assumed to be

$$z = z_n(x,y) \qquad \text{with} \quad n = 0, 1, \ldots\ldots$$

Formal differentiation of (5) with respect to x now gives, remembering that $\psi=0$ for $z<z_0$,

$$\frac{\partial\varphi}{\partial x} = \int_{z_0}^{\infty} e^{ikz}\,\frac{\partial\psi}{\partial x}\,dz \,,$$

in which no extra terms occur, since ψ is continuous. In the next differentiation, however, the discontinuities in the gradient of x do contribute:

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\int_{z_n}^{z_{n+1}} e^{ikz}\,\frac{\partial\psi}{\partial x}\,dz$$

$$=\sum_{n=0}^{\infty}\left\{\frac{\partial z_{n+1}}{\partial x}\left(e^{ikz}\,\frac{\partial\psi}{\partial x}\right)_{z=z_{n+1}^-}-\frac{\partial z_n}{\partial x}\left(e^{ikz}\,\frac{\partial\psi}{\partial x}\right)_{z=z_n^+}+\int_{z_n}^{z_{n+1}} e^{ikz}\,\frac{\partial^2\psi}{\partial x^2}\,dz\right\}$$

$$=\int_0^{\infty} e^{ikz}\,\frac{\partial^2\psi}{\partial x^2}\,dz-\sum_{n=0}^{\infty} e^{ikz_n}\,\frac{\partial z_n}{\partial x}\left[\frac{\partial\psi}{\partial x}\right]_{z=z_n-}^{z=z_n+}\;.$$

The same treatment holds for the y-derivatives, while partial integration is needed in the case of the z-derivatives, viz.

$$\int_0^{\infty} e^{ikz}\,\frac{\partial^2\psi}{\partial z^2}\,dz=\sum_{n=0}^{\infty}\int_{z_n}^{z_{n+1}} e^{ikz}\,\frac{\partial^2\psi}{\partial z^2}\,dz$$

$$=\sum_{n=0}^{\infty}\left\{\left[e^{ikz}\,\frac{\partial\psi}{\partial z}\right]_{z_n^+}^{z_{n+1}^-}-ik\int_{z_n}^{z_{n+1}} e^{ikz}\,\frac{\partial\psi}{\partial z}\,dz\right\}$$

$$=-ik\int_0^{\infty} e^{ikz}\,\frac{\partial\psi}{\partial z}\,dz-\sum_{n=0}^{\infty} e^{ikz_n}\left[\frac{\partial\psi}{\partial z}\right]_{z=z_n^-}^{z=z_n^+}$$

$$=-k^2\int_0^{\infty} e^{ikz}\,\psi\,dz-\sum_{n=0}^{\infty} e^{ikz_n}\left[\frac{\partial\psi}{\partial z}\right]_{z=z_n^-}^{z=z_n^+}\;.$$

Therefore application of the transformation (5) to the differential equation (7), yields

$$0=\int_0^{\infty} e^{ikz}\left\{\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}\right\}dz$$

$$=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+k^2\varphi+\sum_{n=0}^{\infty} e^{ikz_n}\left[\frac{\partial z_n}{\partial x}\,\frac{\partial\psi}{\partial x}+\frac{\partial z_n}{\partial y}\,\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial z}\right]_{z=z_n^-}^{z=z_n^+}$$

$$=\left(\Delta+k^2\right)\varphi\,,$$

since the expression inside the square brackets is a derivative in the conormal direction of the characteristic surface $z=z_n(x,y)$, and therefore an inner derivative, which means that it is continuous. Thus it is seen that a continuous solution of the boundary value problem for ψ, may have discontinuous behaviour of the normal gradient at characteristic surfaces, and yet be formally equivalent, via the transform (5), to the solution of the problem for φ.

We are therefore now ready to attack the original problem by solving the boundary value problem (7), (8), (9), (15).

This problem appears to be equivalent to that of determining the linearized velocity potential in a supersonic stream flowing steadily in the positive z-direction at Mach number $\sqrt{2}$ past a thin plate of rectangular plan-form and with prescribed local angle of attack. What is, however, somewhat unusual from the aerodynamical point of view, is that the chord of this supposed plate has to be thought of as infinitely long; this means on the one hand that there is no wake behind the plate, but on the other hand that the pattern of interaction between the regions next to the plate, becomes rather complicated.

The linearized problem of steady supersonic lifting surface theory for general plan-forms has been treated by Evvard (1950) and Ward (1949). The derivation of the formulae of solution depend in principle on solving the initial value problem for the wave equation (7) with given values of the function and its normal derivative on a fixed initial surface. We shall therefore consider this problem, called Cauchy's problem, in the next section.

It should be noted here that Friedlander (1946) and Fox (1948) have used a similar method as in the present treatise to solve the problems of diffraction of a pulse at a half-plane and at a slit, but that no work seems to have been done along these lines to obtain an asymptotic solution for wave trains of high frequency.

2.2. The solution of Cauchy's problem.

The problem of determining a solution ψ of the wave equation

$$\mathcal{L}(\psi) \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} = 0 \tag{1}$$

which assumes prescribed values of ψ and its normal derivative on a given surface

$$g(x,y,z) = 0 \tag{2}$$,

has been solved by Volterra (1892) and by Hadamard (1932) by different methods. The method of Hadamard was refined by M. Riesz (1949) by replacing the process of picking out the "finite part" of a divergent integral, as defined by Hadamard, by the more elegant method of analytical continuation with respect to a parameter. A still further refinement of the method is offered by the theory of distributions due to L. Schwartz (1950, 1951), but since the gain seems to be mainly in elegance, the sketch given in the present treatise will follow Riesz in making use of analytical continuation.[1)]

The divergence theorem of Gauss is the starting point for the method of solution. In analogy with the derivation of Green's theorem in the case of the potential equation, use is made of the divergence expression:

1) R. Sauer (1954) refers to a forthcoming publication of Dorfner, in which Schwarz's theory will be consistently used.

$$\chi\mathcal{L}(\psi)-\psi\mathcal{L}(\chi)=\frac{\partial}{\partial x}\left(\chi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\chi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\chi\frac{\partial\psi}{\partial y}-\psi\frac{\partial\chi}{\partial y}\right)-\frac{\partial}{\partial z}\left(\chi\frac{\partial\psi}{\partial z}-\psi\frac{\partial\chi}{\partial z}\right)$$
$$=\operatorname{div}(\chi\,\operatorname{gradh}\psi-\psi\,\operatorname{gradh}\chi)$$
$$=\operatorname{div}\underline{b}\,, \qquad (3)$$

in which the symbol gradh is defined as the vector operator

$$\operatorname{gradh}=\underline{i}\frac{\partial}{\partial x}+\underline{j}\frac{\partial}{\partial y}-\underline{k}\frac{\partial}{\partial z}\,.$$

Gauss' theorem states that, for sufficiently regular functions and region,

$$\iiint \operatorname{div}\underline{b}\,dV=\iint \underline{b}\cdot\underline{n}\,dS$$

in which the surface integral is extended over the surface bounding the region of integration of the volume integral, and $\underline{n}=n_1\underline{i}+n_2\underline{j}+n_3\underline{k}$ is the unit outward normal to this surface. The normal component of gradh ψ is equal to

$$\underline{n}\cdot\operatorname{gradh}\psi=\left(n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}-n_3\frac{\partial}{\partial z}\right)\psi=\frac{\partial\psi}{\partial\nu}\,,$$

which is the derivative of ψ in the conormal direction $\underline{\nu}=n_1\underline{i}+n_2\underline{j}-n_3\underline{k}$, as defined in (2.1.2). Therefore the required analogy of Green's theorem can be written as

$$\iiint\left\{\chi\mathcal{L}(\psi)-\psi\mathcal{L}(\chi)\right\}dV=\iint\left(\chi\frac{\partial\psi}{\partial\nu}-\psi\frac{\partial\chi}{\partial\nu}\right)dS\,. \qquad (4)$$

In this result the bounding surface need not be everywhere smooth: it may consist of a finite number of smooth parts. The result is valid if ψ, χ, and their first derivatives are continuous everywhere in the region and integrable on the boundary, and if the second derivatives of ψ and χ are sectionally continuous. In our problem, everywhere outside the boundary value strip $(y=0, z>0, |x|<1)$, ψ is continuous, while on the surfaces of discontinuity of its gradient, $\frac{\partial\psi}{\partial\nu}$ is continuous. Therefore, if χ is well-behaved, the theorem may be applied as long as this strip does not cross the region of integration, and if $\frac{\partial\psi}{\partial\nu}$ remains integrable everywhere on the boundary. This condition of integrability on the boundary is applicable to the edges of the boundary value strip, and is the counterpart of the edge condition in the original problem.

Again in analogy with the treatment of the potential equation, χ may be taken to be an "elementary solution" of (1), viz.

$$\frac{1}{\rho}=\left\{(z-\zeta)^2-(x-\xi)^2-(y-\eta)^2\right\}^{-\frac{1}{2}}\,, \qquad (5)$$

which is singular on the characteristic cone

$$(z-\zeta)^2 = (x-\xi)^2 + (y-\eta)^2$$

from the point P : (x,y,z) and imaginary outside it. Here ξ , η , ζ denote the running coordinates. Following our considerations on regions of dependence, the obvious region of integration to consider, is the region G bounded by the forward characteristic cone

$$\zeta - z = \sqrt{(\xi-x)^2 + (\eta-y)^2} \tag{6}$$

from P, and the initial surface (2). Evidently, however, (4) then contains divergent integrals, due to the singularity of (5) on the surface of the cone (6), so that a more elaborate analysis is needed to obtain the solution of the initial value problem.

Following Riesz, we therefore introduce into (5) a complex parameter α , thus

$$\chi = \rho^{\alpha-1} \tag{7}$$

and evidently, when Re α is large enough, (4) contains no divergent integrals, and is analytical in α . Our task is therefore to obtain an analytical continuation up to $\alpha = 0$ of (4) with G as region of integration, ψ satisfying (1), and χ given by (7).

Of course $\mathcal{L}(\chi)$ does not now satisfy (1) for a general α. In fact

$$\begin{aligned}
\frac{\partial}{\partial\xi}\rho^{\alpha-1} &= (\alpha-1)\rho^{\alpha-3}(x-\xi) \\
\frac{\partial^2}{\partial\xi^2}\rho^{\alpha-1} &= -(\alpha-1)\rho^{\alpha-3} + (\alpha-1)(\alpha-3)\rho^{\alpha-5}(x-\xi)^2 \\
&= -(\alpha-1)\rho^{\alpha-5}\left\{\rho^2 - (\alpha-3)(x-\xi)^2\right\} \\
\frac{\partial^2}{\partial\eta^2}\rho^{\alpha-1} &= -(\alpha-1)\rho^{\alpha-5}\left\{\rho^2 - (\alpha-3)(x-\eta)^2\right\} \\
\frac{\partial^2}{\partial\zeta^2}\rho^{\alpha-1} &= (\alpha-1)\rho^{\alpha-5}\left\{\rho^2 + (\alpha-3)(z-\zeta)^2\right\}
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{L}(\chi) &= -(\alpha-1)\rho^{\alpha-5}\left\{3\rho^2 + (\alpha-3)\rho^2\right\} \\
&= -\alpha(\alpha-1)\rho^{\alpha-3}
\end{aligned}$$

Therefore (4) becomes, for Re $\alpha > 3$,

$$\alpha(\alpha-1)\iiint_G \psi\rho^{\alpha-3}\,dV = \iint_{S_0}\left(\rho^{\alpha-1}\frac{\partial\psi}{\partial\nu} - \psi\frac{\partial}{\partial\nu}\rho^{\alpha-1}\right)dS, \tag{8}$$

in which it is assumed that the cone (6) is truncated by the part S_0 of the initial surface (2). No integral over the surface of the cone appears, since the integrand of (4) vanishes there for $\mathrm{Re}\,\alpha > 3$.

Firstly the left-hand side must be written in a form which is significant also for $\alpha = 0$. Since G is partly bounded by the cone (6), it is natural to introduce conical coordinates r, μ, θ by

$$\begin{aligned} z - \zeta &= r \\ x - \xi &= r(1-\mu)\cos\theta \\ y - \eta &= r(1-\mu)\sin\theta \end{aligned}$$

with the Jacobian

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(r,\mu,\theta)} = \begin{vmatrix} -(1-\mu)\cos\theta & -(1-\mu)\sin\theta & -1 \\ r(1-\mu)\sin\theta & -r(1-\mu)\cos\theta & 0 \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix}$$
$$= r^2(1-\mu)$$

Then
$$\rho = r\sqrt{1-(1-\mu)^2} = r\sqrt{\mu(2-\mu)} \quad ,$$
and if we write

$$\psi(\xi,\eta,\zeta) = \bar{\psi}(r,\mu,\theta)$$

and assume the initial surface (2) to be represented by

$$r = r_0(\mu,\theta) ,$$

the left-hand side of (8) becomes

$$\alpha(\alpha-1)\int_0^{2\pi} d\theta \int_0^1 \mu^{\frac{\alpha-3}{2}} (2-\mu)^{\frac{\alpha-3}{2}} (1-\mu)\, d\mu \int_0^{r_0(\mu,\theta)} \bar{\psi}\, r^{\alpha-1}\, dr$$
$$= \alpha(\alpha-1)\int_0^{2\pi} d\theta \left[\frac{2\mu^{\frac{\alpha-1}{2}}}{\alpha-1} (2-\mu)^{\frac{\alpha-3}{2}} (1-\mu) \int_0^{r_0(\mu,\theta)} \bar{\psi}\, r^{\alpha-1}\, dr \right]_0^1$$
$$-2\alpha \int_0^{2\pi} d\theta \int_0^1 \mu^{\frac{\alpha-1}{2}} \frac{\partial}{\partial\mu} \left\{ (2-\mu)^{\frac{\alpha-3}{2}} (1-\mu) \int_0^{r_0(\mu,\theta)} \bar{\psi}\, r^{\alpha-1}\, dr \right\} d\mu \qquad (9),$$

by partial integration with respect to μ. The first term vanishes, but the second contains the expression

$$\alpha \int_0^{r_0} \bar{\psi}\, r^{\alpha-1}\, dr ,$$

and this integral diverges when α vanishes. Partial integration yields

$$\alpha \int_0^{r_0} \bar{\psi} r^{\alpha-1} dr = \left[\bar{\psi} r^{\alpha} \right]_0^{r_0} - \int_0^{r_0} \frac{\partial \bar{\psi}}{\partial r} r^{\alpha} dr$$

$$\rightarrow \bar{\psi}(0,\mu,\theta), \text{ when } \alpha \rightarrow 0 \quad , \tag{10}$$

and this, substituted in the second member of (9), makes that integral convergent for $\alpha = 0$. The derivative occurring there, when multiplied by α, becomes for $\alpha \rightarrow 0$

$$\frac{\partial}{\partial \mu} \left\{ (2-\mu)^{-\frac{3}{2}} (1-\mu) \bar{\psi}(0,\mu,\theta) \right\}$$

$$= (2-\mu)^{-\frac{5}{2}} \left\{ \frac{3}{2}(1-\mu) - (2-\mu) \right\} \bar{\psi}(0,\mu,\theta) + (2-\mu)^{-\frac{3}{2}} (1-\mu) \frac{\partial \bar{\psi}(0,\mu,\theta)}{\partial \mu}$$

$$= -\frac{1}{2}(2-\mu)^{-\frac{5}{2}} (1+\mu) \bar{\psi}(0,\mu,\theta) + (2-\mu)^{-\frac{3}{2}} (1-\mu) \frac{\partial \bar{\psi}(0,\mu,\theta)}{\partial \mu} \quad . \tag{11}$$

This expression can be simplified, firstly by noting that r=0 corresponds to $\xi = \eta = \zeta = 0$, so that

$$\bar{\psi}(0,\mu,\theta) = \psi(x,y,z) \quad , \tag{12}$$

and secondly by using the relation

$$\frac{\partial}{\partial \mu} = r \cos\theta \frac{\partial}{\partial \xi} + r \sin\theta \frac{\partial}{\partial \eta} \quad ,$$

which implies $\dfrac{\partial \bar{\psi}(0,\mu,\theta)}{\partial \mu} = 0$. (13)

Substitution of (11), (12) and (13) in (9) gives, for $\alpha \rightarrow 0$,

$$\psi(x,y,z) \int_0^{2\pi} d\theta \int_0^1 \mu^{-\frac{1}{2}} (2-\mu)^{-\frac{5}{2}} (1+\mu)\, d\mu$$

$$= 2\pi\, \psi(x,y,z) \left[\mu^{\frac{1}{2}} (2-\mu)^{-\frac{3}{2}} \right]_0^1$$

$$= 2\pi\, \psi(x,y,z) \quad ,$$

which is the analytical continuation of the left-hand side of (8) for $\alpha = 0$. We may thus write (8) in the form

$$\psi(x,y,z) = \frac{1}{2\pi} \iint_{S_0} \frac{\partial \psi}{\partial \nu} \rho^{-1} dS - \frac{1}{2\pi} (C_\alpha) \iint_{S_0} \psi \frac{\partial}{\partial \nu} \rho^{\alpha-1} dS \quad , \tag{14}$$

where the symbol (C_α) before the second integral indicates that the analytical continuation for $\alpha = 0$ is meant.

This result holds under the supposition already mentioned, that S_0 totally truncates the forward characteristic cone from the point P: (x,y,z). In the case where the volume G which S_0 cuts out of this cone, does not include P, this volume may be regarded as the difference of two volumes like the above, which do contain P, and then of course the left-hand side of (14) vanishes.

It has thus been shown that, if a solution of the wave equation is continuous and has a continuous gradient, excepting for possible finite jumps in the normal gradient across characteristic surfaces, and if the conormal derivative is integrable on the boundary, then

$$\frac{1}{2\pi}\left\{\iint_{S_0} \frac{\partial\psi}{\partial\nu}\rho^{-1}\,dS-(C_\alpha)\iint_{S_0}\psi\frac{\partial}{\partial\nu}\rho^{\alpha-1}\,dS\right\}=\begin{cases}\psi(x,y,z) & (14)\\ 0\ , & (15)\end{cases}$$

in which (14) refers to the case where the sectionally smooth surface S_0 truncates the forward characteristic cone from (x,y,z), while (15) applies when S_0 cuts off from the surface of this cone a closed area not including the vertex (x,y,z).

Evidently (14) renders, in principle, the solution of the Cauchy problem. In the deduction of this solution the notation of distributions of Schwartz (1950, 1951) is extremely elegant, since his introduction of so-called distributions, which is an extension of the notion of measurable functions, allows the systematic use of Dirac's delta "function" δ and its derivatives. This makes it possible to include boundary value terms in the differential equation itself, and to write for instance Green's theorem without explicit occurrence of the surface integrals. For the sake of interest a few of his notations may be given here. The basic property of the delta distribution is represented by

$$\delta * T = T\ ,$$

in which T is any distribution, and the star represents the product of composition. The differential equation may be written as

$$A*T=B\ ,$$

in which T represents the required solution, A the differential operator, and B the boundary conditions. Then the left-hand side of (14) may be written as

$$E*B\ ,$$

in which E is the "elementary solution"

$$E=\frac{1}{2\pi\sqrt{z^2-x^2-y^2}}\ ,$$

which satisfies

$$A*E=\delta\ .$$

Green's theorem becomes, for this case,

$$(E*A)*T-E*(A*T)=0\ ,$$

so that the validity of (14) may be illustrated by

$$
\begin{aligned}
E*B &= E*(A*T)\\
&=(E*A)*T\\
&=\delta*T\\
&=T.
\end{aligned}
$$

2.3. Application to the boundary value problem.

We now return to the boundary value problem as expressed in (3), (5), (6) and (15) of (2.1):

$$\Delta\psi-\frac{\partial^2\psi}{\partial z^2}=0 \tag{1}$$

$$\frac{\partial\psi}{\partial y}=f(x) \quad \text{for} \quad y=0\pm,\ |x|<1,\ z>0 \tag{2}$$

$$\psi=0 \quad \text{for} \quad y=0,\ |x|>1\ , \tag{3}$$

$$\frac{\partial\psi}{\partial y}=\psi=0 \quad \text{for} \quad z=0 \tag{4}$$

together with the conditions of continuity formulated in (2.1.2) and the condition of integrability formulated in connection with the use of formula (4) of (2.2).

Firstly consider a point P:(x,y,z), for which both y and z are positive, and such that the forward characteristic cone from P,

$$\zeta=z-\sqrt{(x-\xi)^2+(y-\eta)^2} \tag{5}$$

is truncated by the half-planes $\zeta=0$, $\eta\geqslant 0$ and $\eta=0, \zeta\geqslant 0$, in which again ξ, η , ζ are running coordinates in the x,y,z directions. The truncating surface thus consists of two parts: the first part, is the major segment of a circle on the plane $\zeta=0$; the second, which we call D_+ , is part of the hyperbola

$$\zeta\leqslant z-\sqrt{(x-\xi)^2+y^2} \tag{6}$$

$$\eta=0\ .$$

On the negative side of the plane $\eta=0$, a further bounded region is cut out of the cone (5) by the planes $\zeta=0$ and $\eta=0$, and in this case the bounding surface consists firstly of the minor segment of the same circle as above in the plane $\zeta=0$, and secondly of (6), which will in this connection be called D_- . The difference of notation in D_+ and D_- is intended to convey that surface integrals over (6) are concerned with limiting values of the integrand, approaching from the side of positive and negative η respectively.

Formulae (14) and (15) of (2.2) may now be applied to the two bounded segments referred to above, which have the cone (5) and parts of the planes $\zeta=0$ and $\eta=0$ as bounding surfaces. Evidently the boundary condition (4) makes the contributions on the plane $\zeta=0$ vanish, while on the plane $\eta=0$ the conormal coincides with the (outward) normal, i.e.

$$\frac{\partial}{\partial\nu}=\begin{cases}-\dfrac{\partial}{\partial\eta} & \text{on } D_+\\[2mm] \dfrac{\partial}{\partial\eta} & \text{on } D_-\end{cases},$$

so that, finally, we get

$$\psi(x,y,z)=-\frac{1}{2\pi}\left\{\iint_{D_+}\frac{\partial\psi}{\partial\eta}\,\frac{1}{\rho}\,d\xi\,d\zeta-(c_\alpha)\iint_{D_+}\psi\frac{\partial}{\partial\eta}\rho^{\alpha-1}\,d\xi\,d\zeta\right\} \tag{7}$$

$$0=\frac{1}{2\pi}\left\{\iint_{D_-}\frac{\partial\psi}{\partial\eta}\,\frac{1}{\rho}\,d\xi\,d\zeta-(c_\alpha)\iint_{D_-}\psi\frac{\partial}{\partial\eta}\rho^{\alpha-1}\,d\xi\,d\zeta\right\} \tag{8}$$

Since, on the same arguments as for φ in (1.1.2), ψ is antisymmetrical in y or η, we may write

$$\psi(\xi,0+,\zeta)+\psi(\xi,0-,\zeta)=0$$

$$\frac{\partial\psi(\xi,0+,\zeta)}{\partial\eta}+\frac{\partial\psi(\xi,0-,\zeta)}{\partial\eta}=2\,\frac{\partial\psi(\xi,0+,\zeta)}{\partial\eta}.$$

Therefore subtraction of (7) and (8), and writing D for either D_+ or D_-, gives

$$\psi(x,y,z)=-\frac{1}{\pi}\iint_D\left[\frac{\partial\psi(\xi,\eta,\zeta)}{\partial\eta}\,\frac{1}{\rho}\right]_{\eta=0+}d\xi\,d\zeta$$

$$=-\frac{1}{\pi}\iint_D\frac{\dfrac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\,d\xi\,d\zeta}{\sqrt{(z-\zeta)^2-(x-\xi)^2-y^2}}\,, \tag{9}$$

which is valid for $y>0$ and is in accord with the formula found by Evvard (1950) for the velocity potential on a lifting surface in steady, supersonic flow. The value for negative y follows from the property of antisymmetry

$$\psi(x,y,z)=-\psi(x,-y,z)\,.$$

Formula (9) is not yet the solution of the problem, because the value of $\dfrac{\partial\psi}{\partial\eta}$ occurring in the integrand is prescribed in (2) only for $|\xi|<1$, whereas the hyperbola sector D does certainly extend beyond this if, for instance, z is large enough. The area D, situated on the plane $\eta=0$, (compare (6)), may be written,

$$D:\ \left.\begin{aligned}&z-\zeta=\sqrt{(x-\xi)^2+y^2}\equiv r(x,y;\xi)\equiv r(\xi)\\&\zeta\geqslant 0\end{aligned}\right\} \tag{10}$$

which serves as definition for the expression $r(\xi)$. This area is therefore to be divided into parts namely the part D' for which $|\xi|<1$, and the rest, called D", which may or may not exist, projecting beyond this strip for a fixed point P:(x,y,z). The formula (9) then becomes

$$\psi(x,y,z) = -\frac{1}{\pi}\iint_{D'}\frac{f(\xi)\,d\xi\,d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} - \frac{1}{\pi}\iint_{D''}\frac{\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\,d\xi\,d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}\,, \qquad (11)$$

which is a solution of the problem for positive values of z smaller than both r(-1) and r(+1), (since then D" vanishes), but which is an integro-differential equation in ψ for more general points.

In order to determine the unknown value of $\frac{\partial\psi}{\partial\eta}$ in the second integral, use is made of the boundary value (3), which, when substituted in (11), yields

$$0 = \iint_{D'}\frac{f(\xi)\,d\xi\,d\zeta}{\sqrt{(x-\zeta)^2-(x-\xi)^2}} + \iint_{D''}\frac{\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\,d\xi\,d\zeta}{\sqrt{(x-\zeta)^2-(x-\xi)^2}} \qquad (12),$$

which is valid for $|x|>1$. The domain of integration D which is in general a hyperbola sector, is now degenerated into a triangle. On condition that the entire domain D" in (12) lies on one and the same side of the strip $|\xi|<1$, (which is the case when $0<x-1<z<x+1$ or $0<-x-1<z<-x+1$), then (12) can be simply solved analytically for $\frac{\partial\psi}{\partial\eta}$. These values can be used in the second term of (11), thus increasing the range of points P for which (11) gives the value of ψ; but the same values of $\frac{\partial\psi}{\partial\eta}$ also increase the range of points for which $\frac{\partial\psi}{\partial\eta}$ may be obtained from (12). Thus, repeated application of (12) yields, in succession, the required values of $\frac{\partial\psi}{\partial\eta}$ over a larger and larger domain of the plane $\eta=0$, thus continually extending the range of usefulness of the formula (11). The process becomes rather laborious after a few steps, but in principle the solution of ψ over the whole space can thus be found in the form of an integral recursion formula.

On applying the Laplace transform (4) of (2.1.1), it appears that domains for which ψ is successively found by the process sketched above, yield contributions to the value of φ which are of decreasing order for large values of the frequency-parameter k. It is in this circumstance that the value of the present method lies, since only a finite number of steps of the recursion process are required to yield a result which is asymptotically valid to a specified order in k.

The details of the process are deferred till the next chapter, where they will become plain in the course of the application to two examples.

Chapter 3.

APPLICATION OF THE METHOD TO DIFFRACTION PROBLEMS.

3.1. Sommerfeld's half-plane diffraction problem.

3.1.1. Solution by the Laplace transform method.

As a preliminary illustration of the application of the method, the dase is treated where the screen, which is assumed to be rigid (or perfectly reflecting or conducting), takes the form of a half plane, instead of a strip as was described in chapter 1. We shall again require that only the two space dimensions x,y occur, so that the screen may be assumed to be situated on the positive half of the x-axis, thereby implying that the edge coincides with the axis of the third space variable.

There are certain advantages in starting with this problem. Firstly, the results obtained in this relatively simple case are directly applicable to the first stage of the more elaborate problem of a strip, and, secondly, it is an example of the only type of diffraction problem that has been solved in closed form, namely by Sommerfeld (1896), which thus afford a valuable check on the correctness of the method.

The boundary value problem for the perturbation velocity potential may be formulated as

$$(\Delta+k^2)\,\varphi=0 \qquad (1)$$

$$\frac{\partial\varphi}{\partial y}=-\frac{\partial\varphi_i}{\partial y}=-\frac{f(x)}{ik}\,, \qquad \text{for } y=0\pm\,,\ x>0 \qquad (2)$$

$$\varphi=0 \qquad \text{for } y=0\ ,\ x<0 \qquad (3),$$

together with the edge condition and the conditions of continuity of φ and its first and second derivatives, excepting on the positive x-axis. We impose also the radiation condition, but the fact that the region of (2) stretches to infinity, implies that this condition is only satisfied if the incident wave may be regarded as originating in a bounded domain of the half plane $y<0$, which means that no plane waves occur. In case plane waves do occur, the solution may be regarded as a limiting case in which the source region tends to infinity, so that even plane wave behaviour may be obtained from a treatment which imposes the radiation condition.

Evidently, the problem thus formulated is directly analogous to our problem for a strip, and the transformed problem takes the form

$$\Delta\psi - \frac{\partial^2\psi}{\partial z^2} = 0 \tag{4}$$

$$\frac{\partial\psi}{\partial y} = f(x), \quad \text{for } y = 0^{\pm}, x>0, z>0 \tag{5}$$

$$\psi = 0 \quad \text{for } y = 0, x<0 \tag{6}$$

$$\psi = \frac{\partial\psi}{\partial y} = 0 \quad \text{for } z<0$$

with the same conventions regarding continuity and integrability as in (2.3).

The fundamental relation for obtaining the solution is formula (9) of (2.3), which may be written as

$$\psi(x,y,z) = -\frac{1}{\pi}\iint_D \frac{\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\, d\xi\, d\zeta}{\sqrt{(z-\zeta)^2 - r^2(\xi)}}, \quad \text{for } y>0 \tag{7}$$

with

$$r(\xi) = \sqrt{(x-\xi)^2 + y^2}$$

and the surface of integration D given by the hyperbola sector,

$$D:\ 0 \leq \zeta \leq z - r(\xi) \tag{8}$$

Again the values for negative y follow by antisymmetry, and we therefore henceforth need consider only the case of positive y.

The region in which ψ is non-zero, (cf. end of (2.1.2)), is the wedge $z>|y|$ in the half space $x>0$, supplemented by the half of the backward cone from the origin which lies in the other half space, viz. $z>r(0)$, $x<0$. In analogy to the division represented by (11) of (2.3), this region of non-zero ψ is divided into two parts, viz. firstly, the part for whose points the surface of integration D in (7) is such that only values of $\frac{\partial\psi}{\partial\eta}$ given in (5) occur, so that ψ is directly determined, and secondly, the part for which some of the values of $\frac{\partial\psi}{\partial\eta}$ occurring in (7), are not given by (5).

Evidently the first region consists of points (x,y,z) with $z>|y|$ and $x>0$ and for which the hyperbola sector (8) lies entirely in the half plane of positive ξ. This is the case if $z<r(0)$ and of course $z>|y|$ and $x>0$, which indicates the region for positive x which is situated between the wedge surface $z=|y|$ and the backward cone from the origin $z=r(0)\equiv\sqrt{x^2+y^2}$. We may thus write, using (7) and (5),

$$\psi(x,y,z) = -\frac{1}{\pi}\int_{x-\sqrt{z^2-y^2}}^{x+\sqrt{z^2-y^2}} f(\xi)\,d\xi \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2 - r^2(\xi)}} \tag{9}$$

valid in the region

$$0<y\leqslant x\leqslant r(0)\quad,\quad x>0 \tag{10}$$

The second part consists of the rest of the region of non-zero ψ, viz. the inside of the backward cone from the origin, $z>r(0)$. In this case (7) and (5) yield

$$\psi(x,y,z)=-\frac{1}{\pi}\int_0^{x+\sqrt{z^2-y^2}} f(\xi)\,d\xi\int_0^{z-r(\xi)}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}-\frac{1}{\pi}\iint_{D''}\frac{\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\,d\xi\,d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} \tag{11}$$

in which D" is the part of (8) for which ξ is negative.

Now the values of $\frac{\partial\psi}{\partial\eta}$ over the region D" may be determined by using condition (6), as was indicated in equation (12) of (2.3). We therefore apply (7) to a point for which y=0 and $x<0$, so that the region (8) of integration degenerates into the triangle

$$0\leqslant\zeta\leqslant z-|x-\xi|\,.$$

Since ψ is identically zero for $z<r(0)$ and $x<0$, i.e. for $z<-x$, we further choose $z>-x$, and the triangle reduces to the trapezium

$$\left.\begin{array}{ll} 0\leqslant\zeta\leqslant z+x-\xi & \text{for } \xi\geqslant 0\\ -\xi\leqslant\zeta\leqslant z-|x-\xi| & \text{for } \xi<0\,. \end{array}\right\} \tag{8a}$$

Introducing new coordinates by

$$\left.\begin{array}{l} \sigma=\zeta+\xi\\ \tau=\zeta-\xi \end{array}\right\} \tag{12}$$

and writing

$$\sigma_1=z+x$$
$$\tau_1=z-x$$
$$\psi_\eta(\sigma,\tau)=\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}$$
$$f(\sigma,\tau)=f(\xi)$$

(7) and (6) yield

$$\begin{aligned} 0&=\int_0^{\sigma_1}\frac{d\sigma}{\sqrt{\sigma_1-\sigma}}\int_{-\sigma}^{\tau_1}\frac{\psi_\eta\,d\tau}{\sqrt{\tau_1-\tau}}\\ &=\int_0^{\sigma_1}\frac{d\sigma}{\sqrt{\sigma_1-\sigma}}\left\{\int_{-\sigma}^{\sigma}\frac{f\,d\tau}{\sqrt{\tau_1-\tau}}+\int_{\sigma}^{\tau_1}\frac{\psi_\eta\,d\tau}{\sqrt{\tau_1-\tau}}\right\} \end{aligned} \tag{13}$$

valid in the region y=0, $x<0$ and $z+x>0$, that is in the region $\tau_1>\sigma_1$ and $\sigma_1>0$. This is the equivalent of equation (12) of (2.3). Now since (13) holds identically in σ_1 in the range $0<\sigma_1<\tau_1$, the expression in brackets must be identically zero, thus yielding the simple integral equation

$$\int_{-\sigma}^{\sigma}\frac{f\,d\tau}{\sqrt{\tau_1-\tau}}=-\int_{\sigma}^{\tau_1}\frac{\psi_\eta\,d\tau}{\sqrt{\tau_1-\tau}} \tag{14}$$

valid for $0<\sigma<\tau$. It now proves to be possible to eliminate $\frac{\partial\psi}{\partial\eta}$ from the second term of (11) without even solving (14). Remembering that $\psi=0$ for $x<-x$, the domain D" reduces to

$$D'': 0\leqslant-\xi\leqslant\zeta\leqslant z-r(\xi),$$

in which the boundaries are

$$\xi+\zeta=0,\ \xi=0\ ,\ \text{and}\ z-\zeta=r(\xi)\equiv\sqrt{(x-\xi)^2+y^2}$$

or, in the coordinates σ and τ of (12),

$$\sigma=0,\ \sigma=\tau\ \ \text{and}\ (\sigma_1-\sigma)(\tau_1-\tau)-y^2=0\ .$$

Therefore the second term of (11) becomes

$$-\frac{1}{2\pi}\int_0^{z-r(0)}d\sigma\int_\sigma^{\tau_1-\frac{y^2}{\sigma_1-\sigma}}\frac{\psi_\eta\,d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}}=-\frac{1}{2\pi}\int_0^{z-r(0)}\frac{d\sigma}{\sqrt{\sigma_1-\sigma}}\int_\sigma^{\tau_1-\frac{y^2}{\sigma_1-\sigma}}\frac{\psi_\eta\,d\tau}{\sqrt{\left(\tau_1-\frac{y^2}{\sigma_1-\sigma}\right)-\tau}}$$

$$=\frac{1}{2\pi}\int_0^{z-r(0)}\frac{d\sigma}{\sqrt{\sigma_1-\sigma}}\int_{-\sigma}^{\sigma}\frac{f\,d\tau}{\sqrt{\left(\tau_1-\frac{y^2}{\sigma_1-\sigma}\right)-\tau}}\qquad\text{from (14),}$$

$$=\frac{1}{\pi}\int_0^{z-r(0)}f(\xi)\,d\xi\int_0^{z-r(0)-\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}\qquad(15),$$

which is a statement of equivalence between two integrals of $\frac{\partial\psi}{\partial\eta}$ with a fixed weight function, over different areas. This is an extension of Evvard's method of equivalent areas.

Substitution of (15) in (11) yields

$$\psi(x,y,z)=-\frac{1}{\pi}\int_0^{x+\sqrt{z^2-y^2}}f(\xi)\,d\xi\int_0^{z-r(\xi)}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}+\frac{1}{\pi}\int_0^{z-r(0)}f(\xi)\,d\xi\int_0^{z-r(0)-\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}\qquad(16),$$

valid for $z>r(0)$ and $y>0$. The problem of determining ψ is therefore completely solved by (9) and (16). This is possible because $\frac{\partial\psi}{\partial y}$ is given by (5) for a whole quadrant of the plane y=0, so that only in one triangular area $0<-x<z$ of this plane extra values must be determined for the application of (7). In our more general problem in which $\frac{\partial\psi}{\partial y}$ is given on a strip, two such triangular areas occur, and interaction between them prevents the solution to be obtained in closed form as above.

The required solution φ of the problem (1), (2), (3) is now obtained from ψ by application of the Laplace transform (5) of (2.1.1) to (9) and (16),

$$\varphi(x,y) = \int_0^\infty e^{ikz}\ \psi(x,y,z)\, dz$$

$$= -\frac{1}{\pi}\int_y^{r(0)} e^{ikz} dz \int_{x-\sqrt{z^2-y^2}}^{x+\sqrt{z^2-y^2}} f(\xi)d\xi \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$-\frac{1}{\pi}\int_{r(0)}^\infty e^{ikz} dz \left\{ \int_0^{x+\sqrt{z^2-y^2}} f(\xi)d\xi \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} - \int_0^{z-r(0)} f(\xi)d\xi \int_0^{z-r(0)-\xi} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} \right\}$$

$$= -\frac{1}{\pi}\int_0^\infty f(\xi)d\xi \left\{ \int_{r(\xi)}^\infty e^{ikz} dz \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} - \int_{r(0)+\xi}^\infty e^{ikz} dz \int_0^{z-r(0)-\xi} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} \right\}$$

$$= \frac{1}{ik\pi}\int_0^\infty f(\xi)d\xi \left\{ \int_{r(\xi)}^\infty e^{ikz} \frac{dz}{\sqrt{z^2-r^2(\xi)}} - \int_{r(0)+\xi}^\infty e^{ikz} \frac{dz}{\sqrt{z^2-r^2(\xi)}} \right\} \qquad (17),$$

by partial integration with respect to z, since

$$\frac{\partial}{\partial z}\int_0^{z-c} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} = \frac{\partial}{\partial z}\int_c^{z} \frac{d\zeta}{\sqrt{\zeta^2-r^2(\xi)}}$$

$$= \frac{1}{\sqrt{z^2-r^2(\xi)}}$$

for c independent of z. Finally (17) becomes, by using (2),

$$\varphi(x,y) = -\frac{1}{\pi}\int_0^\infty \frac{\partial\varphi(\xi,0)}{\partial\eta}\, d\xi \int_{r(\xi)}^{r(0)+\xi} e^{ikz} \frac{dz}{\sqrt{z^2-r^2(\xi)}} \qquad (18),$$

or, remembering that (cf. § 6.13 of Watson (1944)),

$$H_0^{(1)}(kr) = \frac{2}{\pi i}\int_1^\infty e^{ikrz} \frac{dz}{\sqrt{z^2-1}}$$

$$= \frac{2}{\pi i}\int_r^\infty e^{ikz} \frac{dz}{\sqrt{z^2-r^2}} \quad , \qquad (18a)$$

(17) may be written as

$$\varphi(x,y) = \frac{1}{2i}\int_0^\infty \frac{\partial\varphi(\xi,0)}{\partial\eta}\, d\xi \left\{ H_0^{(1)}\left[kr(\xi)\right] + \frac{2i}{\pi}\int_{r(0)+\xi}^\infty e^{ikz} \frac{dz}{\sqrt{z^2-r^2(\xi)}} \right\} \qquad (19).$$

This affords an interesting comparison with the result that would be obtained by applying the Kirchhoff method, (cf. formula (4) of (1.5), in which $|\xi| \leq l$ instead of $0 \leq \xi \leq \infty$). The correct Green's function for the problem proves to be an "incomplete" Hankel function, instead of the Hankel function occurring in Kirchhoff's formula, and the error in the Kirchhoff formula is given by the second term of (19).

3.1.2. Equivalence to Sommerfeld's solution.

Sommerfeld's solution was obtained for the case of a plane incident wave, and we shall here illustrate its equivalence to the solution (18) for the special case of the normally incident plane wave

$$\varphi_i = e^{iky} \quad .$$

The boundary condition (2) in this case takes the form

$$\frac{\partial \varphi}{\partial y} = -ik \quad \text{for} \quad y = 0^{\pm}, \; x > 0$$

so that (18) becomes (for $y > 0$),

$$\varphi(x,y) = \frac{ik}{\pi} \int_0^\infty d\xi \int_{r(\xi)}^{r(0)+\xi} e^{ikz} \frac{dz}{\sqrt{z^2 - r^2(\xi)}} \qquad (20).$$

Our aim is now to prove that this is equal to Sommerfeld's result

$$\varphi(x,y) = -\frac{e^{iky}}{\sqrt{\pi i}} \int_{T_1}^\infty e^{i\lambda^2} d\lambda + \frac{e^{-iky}}{\sqrt{\pi i}} \int_{T_2}^\infty e^{i\lambda^2} d\lambda \qquad (21),$$

in which T_1 and T_2 are most conveniently written in terms of polar coordinates r_o and θ with

$$x = r_o \sin\theta$$
$$y = r_o \cos\theta ,$$

viz.

$$T_1 = -\sqrt{2kr_o} \; \sin\frac{\theta}{2} = \pm\sqrt{k(r_o - y)}$$
$$T_2 = \sqrt{2kr_o} \; \cos\frac{\theta}{2} = \pm\sqrt{k(r_o + y)} \quad .$$

This result may be found for instance in Baker and Copson (1949), in a somewhat different notation.

We now simplify (20) in an obvious way, always keeping both x and y positive, and noting that $r(o) = r_o$:

$$\varphi(x,y) = \frac{ik}{\pi} \int_y^{r_o} e^{ikz} dz \int_{z-\sqrt{z^2-y^2}}^{z+\sqrt{z^2-y^2}} \frac{d\xi}{\sqrt{z^2 - y^2 - (z-\xi)^2}} + \frac{ik}{\pi} \int_{r_o}^\infty e^{ikz} dz \int_{z-r_o}^{z+\sqrt{z^2-y^2}} \frac{d\xi}{\sqrt{z^2 - y^2 - (z-\xi)^2}}$$

$$= ik \int_y^{r_o} e^{ikz} dz + \frac{ik}{\pi} \int_{r_o}^\infty e^{ikz} \left\{ \frac{\pi}{2} - \sin^{-1}\frac{z + r_o - z}{\sqrt{z^2 - y^2}} \right\} dz$$

$$= \frac{1}{2} e^{ikr_o} - e^{iky} - \frac{1}{\pi}\left[e^{ikz} \sin^{-1}\frac{z - r_o - z}{\sqrt{z^2 - y^2}} \right]_{r_o}^\infty + \frac{1}{\pi} \int_{r_o}^\infty e^{ikz} \frac{z(r_o + z) - y^2}{(z^2 - y^2)\sqrt{2(z + r_o)(z - r_o)}} dz$$

$$= -e^{iky} + \frac{1}{\pi}\sqrt{\frac{r_o + x}{2}} \int_{r_o}^\infty e^{ikz} \frac{z - (r_o - x)}{(z^2 - y^2)\sqrt{z - r_o}} dz \quad , \qquad (22),$$

since $y^2 = r_o^2 - x^2$. Using the relations

$$\frac{z - (r_o - x)}{z^2 - y^2} = \frac{y + (r_o - x)}{2y(z + y)} + \frac{y - (r_o - x)}{2y(z - y)} \quad ,$$

$$\frac{y+(r_0-x)}{y}\sqrt{r_0+x}=\frac{\sqrt{r_0-x}\left(\sqrt{r_0+x}+\sqrt{r_0-x}\right)}{\sqrt{(r_0-x)(r_0+x)}}\sqrt{r_0+x}\quad , \qquad \text{for } y>0$$
$$=\sqrt{r_0+x}+\sqrt{r_0-x}$$
$$=\sqrt{(\sqrt{r_0+x}+\sqrt{r_0-x})^2}$$
$$=\sqrt{2(r_0+y)}\quad ,$$

and

$$\frac{y-(r_0-x)}{y}\sqrt{r_0+x}=\sqrt{r_0+x}-\sqrt{r_0-x}$$
$$=\mathrm{sg}(x)\sqrt{(\sqrt{r_0+x}-\sqrt{r_0-x})^2}$$
$$=\mathrm{sg}(x)\sqrt{2(r_0-y)}$$

changes (22) into

$$\varphi(x,y)=-e^{iky}+\frac{\sqrt{r_0+y}}{2\pi}\int_{r_0}^{\infty}e^{ikx}\frac{dx}{(x+y)\sqrt{x-r_0}}+\frac{\sqrt{r_0-y}}{2\pi}\int_{r_0}^{\infty}e^{ikx}\frac{dx}{(x-y)\sqrt{x-r_0}} \qquad (23).$$

Now, in order to simplify this, let I(k,y) be defined by

$$I(k,y)=e^{iky}\int_{r_0}^{\infty}e^{ikx}\frac{dx}{(x+y)\sqrt{x-r_0}} \quad .$$

Differentiation with respect to k gives

$$\frac{\partial I}{\partial k}=ie^{iky}\int_{r_0}^{\infty}e^{ikx}\frac{dx}{\sqrt{x-r_0}}=-\sqrt{\frac{\pi}{ik}}\,e^{ik(r_0+y)} \quad ,$$

which, on integration, yields

$$I(k,y)=\sqrt{\frac{\pi}{i}}\int_{k}^{\infty}e^{i(r_0+y)\mu}\frac{d\mu}{\sqrt{\mu}}$$
$$=2\sqrt{\frac{\pi}{i(r_0+y)}}\int_{\sqrt{k(r_0+y)}}^{\infty}e^{i\lambda^2}d\lambda$$

in which the upper boundary is determined by the condition

$$I(0,y)=\int_{r_0}^{\infty}\frac{dx}{(x+y)\sqrt{x-r_0}}=\frac{\pi}{\sqrt{r_0+y}} \quad .$$

Substituting in (23) yields

$$\varphi(x,y)=-e^{iky}+\frac{e^{-iky}}{\sqrt{\pi i}}\int_{\sqrt{k(r_0+y)}}^{\infty}e^{i\lambda^2}d\lambda+\frac{e^{iky}}{\sqrt{\pi i}}\int_{\sqrt{k(r_0-y)}}^{\infty}e^{i\lambda^2}d\lambda \,,$$

or, making use of

$$\frac{1}{\sqrt{\pi i}}\int_{-\infty}^{\infty}e^{i\lambda^2}d\lambda=1 \quad ,$$

this becomes

$$\varphi(x,y)=\frac{e^{-iky}}{\sqrt{\pi i}}\int_{\sqrt{k(r_0+y)}}^{\infty}e^{i\lambda^2}d\lambda-\frac{e^{ik\lambda}}{\sqrt{\pi i}}\int_{-\sqrt{k(r_0-y)}}^{\infty}e^{i\lambda^2}d\lambda,$$

valid for both x and y positive, and equal to (21) for this region.

Consider now the case where y is still positive, but x negative. Then (20) becomes simply

$$\varphi(x,y)=\frac{ik}{\pi}\int_{r_0}^{\infty} e^{ikz}dz \int_{z-r_0}^{x+\sqrt{z^2-y^2}} \frac{d\xi}{\sqrt{z^2-y^2-(x-\xi)^2}}$$

$$=\frac{1}{\pi}\sqrt{\frac{r_0+x}{2}}\int_{r_0}^{\infty} e^{ikz}\frac{z-(r_0-x)}{(z^2-y^2)\sqrt{z-r_0}}\,dz$$

$$=\frac{\sqrt{r_0+y}}{2\pi}\int_{r_0}^{\infty} e^{ikz}\frac{dz}{(z+y)\sqrt{z-r_0}}-\frac{\sqrt{r_0-y}}{2\pi}\int_{r_0}^{\infty} e^{ikz}\frac{dz}{(z-y)\sqrt{z-r_0}}$$

$$=\frac{e^{-iky}}{\sqrt{\pi i}}\int_{\sqrt{k(r_0+y)}}^{\infty} e^{i\lambda^2}d\lambda-\frac{e^{iky}}{\sqrt{\pi i}}\int_{\sqrt{k(r_0-y)}}^{\infty} e^{i\lambda^2}d\lambda\,,$$

again equal to (21) for negative x and positive y.

Thus it has been proved that (20) is equivalent to Sommerfeld's solution (21) for positive y and all x, and therefore for all x and y, since both solutions are antisymmetrical in y.

3.2. Diffraction by a strip.

3.2.1. The initial stage.

We now come to the solution of the diffraction problem formulated in chapter 1. The two-dimensional elliptical boundary value problem

$$\left.\begin{aligned} &(\Delta+k^2)\,\varphi=0 \\ &\frac{\partial\varphi}{\partial y}=-\frac{f(x)}{ik} \quad \text{for } y=0\pm,\ |x|<1 \\ &\varphi=0 \quad \text{for } y=0,\ |x|>1 \end{aligned}\right\} \qquad (1)$$

with continuity, radiation and edge conditions, is replaced by the three-dimensional hyperbolical problem

$$\Delta\psi-\frac{\partial^2\psi}{\partial z^2}=0 \qquad (2)$$

$$\frac{\partial\psi}{\partial y}=f(x) \quad \text{for } y=0\pm,\ |x|<1,\ z>0 \qquad (3)$$

$$\psi=0 \quad \text{for } y=0,\ |x|>1 \qquad (4)$$

$$\psi=\frac{\partial\psi}{\partial y}=0 \quad \text{for } z=0 \qquad (5),$$

with continuity and integrability conditions, as was described in chapter 2. Again formula (9) of (2.3) is the fundamental relation to be used in solving the problem, viz.

$$\psi(x,y,z) = -\frac{1}{\pi} \iint_D \frac{\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta}\, d\xi\, d\zeta}{\sqrt{(z-\zeta)^2 - r^2(\xi)}} \qquad \text{for } y>0 \qquad (6),$$

with

$$r(\xi) = \sqrt{(x-\xi)^2 + y^2} \qquad (7),$$

and the surface of integration D given by the hyperbola sector

$$D:\ 0 \leqslant \zeta \leqslant z - r(\xi) \qquad (8).$$

We also again need consider only positive values of y, due to the antisymmetry in y.

Before proceeding to obtain the solution, we consider briefly in which fashion the subdivision in different regions takes place. This division is done on the basis of which part of the plane $\eta=0$ is covered by the domain D of integration in (6).

Firstly, if D is situated entirely on the boundary value strip $|\xi|<1$, the solution is directly determined by (3) and (6). This is the case when the point

$$P:\ (x,y,z)$$

lies outside both of the backward cones from the two corners of the boundary value strip, viz. the cones z=r(-1) and z=r(1), but, of course, still inside the region of non-trivial values; thus P satisfies

$$z<r(-1) \quad \text{and} \quad z<r(1)$$

and also $\quad |x|<1 \quad$ and $\quad z>|y|$.

This may (for y > 0) be written in the form

$$\left.\begin{array}{ll} 0<y<z<r(-1) & \text{for } -1<x\leqslant 0 \\ 0<y<z<r(1) & \text{for } 0\leqslant x<1 \end{array}\right\} \qquad (9).$$

Next, if P does lie inside either one of these cones but still outside the other, that is, if either

$$r(-1)<z<r(1) \quad \text{which implies } x<0 \qquad (10a)$$

or

$$r(1)<z<r(-1) \quad \text{which implies } x>0 \qquad (10b)$$

then D protrudes beyond the strip $|\xi|<1$ on only one side, and the situation is identical with that which occurred inside the cone z > r(0) in the half-plane problem of the preceding section. The solution obtained there therefore applies directly, with the necessary adjustment of notation. However, the region of applicability of the results of the half-plane problem is larger than is suggested by these considerations. The criterion is namely in what region an equivalence relation of the form of (15) of (3.1.1) is valid. The region of integration of its right-hand side is a triangle with hypotenuse starting at the point where the hyperbola of D leaves the boundary value strip (or quadrant, previously), and with gradient 45°. Therefore, as soon as the region D includes either of the points (-1,0,2) or (+1,0,2), this triangle protrudes beyond the boundary value strip, and the validity of the equivalence relation, of the form of (15) of (3.1.1), ceases. Evidently this means that the point P is situated inside the region of influence of either of these two points, i.e. inside either of the backward cones z=r(-1)+2 and z=r(1)+2. Thus the results of the half-plane problem are directly applicable also in certain cases where the region D of integration protrudes on both sides of the boundary value strip $|\xi|<1$, viz. in the region

$$\left.\begin{array}{ll} r(1)<x<r(-1)+2\,, & x<0 \\ r(-1)<x<r(1)+2\,, & x>0 \end{array}\right\} \qquad (11).$$

To summarize, it is thus seen that the solution outside the cones z=r(-1) and z=r(1) is either trivial (i.e. zero) or else is found directly by substituting the boundary values (3) in the relation (6). Then comes a region bounded on one side by the previous one, and on the other by the two cones z=r(-1)+2 and z=r(1)+2, in which the solution is not fully determined by substitution, because part of the values of $\frac{\partial\psi}{\partial\eta}$ occurring in (6) are not contained in the prescribed boundary values, but in which the solution can be completed by direct elimination of these values.

The above is indicative of what happens when the solution is sought in the rest of space: the situation changes every time a member of either of the two sets of coaxial cones, z=r(-1)+2n and z=r(1)+2n, is crossed. However, before passing on to consider this in more detail, we shall sketch here the deduction of that part of the solution which corresponds to the result for the half-plane problem.

Firstly, then, in the region (9), direct substitution of (3) in (6) gives

$$\psi(x,y,z)=-\frac{1}{\pi}\int_{x-\sqrt{z^2-y^2}}^{x+\sqrt{z^2-y^2}} f(\xi)\,d\xi \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)-r^2(\xi)}} \qquad (12),$$

as in the half-plane problem.

In order to apply the rest of the results of (3.1.1), notably (15) and (16) of that section, we first introduce a new coordinate x' (with "running" coordinate ξ'), viz.

$$x'=1+x; \qquad \xi'=1+\xi \qquad (13).$$

In accordance with this, primed coordinates are introduced also in the (σ,τ) system (cf. (12) of (3.1.1)) by

$$\left.\begin{aligned}\sigma_1' &= \sigma_1+1\ (=x+x') \quad ; \quad \sigma' = \sigma+1 \quad (=\zeta+\xi') \\ \tau_1' &= \tau_1-1\ (=z-x') \quad ; \quad \tau' = \tau-1 \quad (=\zeta-\xi')\end{aligned}\right\} \tag{14}$$

Then the situation inside the cone z=r(-1) and outside z=r(1) is the same with respect to the primed coordinates as it was in (3.1.1) inside the cone z=r(0) in unprimed coordinates. Therefore, writing

$$\psi_{1,\eta}(\sigma',\tau') = \frac{\partial\psi(\xi,0,\zeta)}{\partial\eta} \qquad \text{and} \qquad f_1(\sigma',\tau') = f(\xi),$$

we obtain, as in (13) of (3.1.1),

$$0 = \int_0^{\sigma_1'} \frac{d\sigma'}{\sqrt{\sigma_1'-\sigma'}} \left[\int_{-\sigma'}^{\sigma'} \frac{f_1(\sigma',\tau')d\tau'}{\sqrt{\tau_1'-\tau'}} + \int_{\sigma'}^{\tau_1'} \frac{\psi_{1,\eta}(\sigma',\tau')\,d\tau'}{\sqrt{\tau_1'-\tau'}}\right] \tag{15}$$

for y=0, $x'<0$ and $2>x+x'>0$, that is for $\tau_1'-\sigma_1'>0$ and $2>\sigma_1'>0$, identically in σ_1'. (The only extra feature, due to the strip-character of the boundary values in the present case, is the condition $2>\sigma_1'$). As in (14) of (3.1.1) we obtain the integral equation

$$\int_{-\sigma'}^{\sigma'} \frac{f_1(\sigma',\tau')\,d\tau'}{\sqrt{\tau_1'-\tau'}} = -\int_{\sigma'}^{\tau_1'} \frac{\psi_{1,\eta}(\sigma',\tau')\,d\tau'}{\sqrt{\tau_1'-\tau'}} \tag{15a}$$

valid this time in $\tau_1'-\sigma'>0$ and $2>\sigma'>0$. For the sake of later convenience τ_1' may be replaced by τ' and the integration variable called t', thus

$$\int_{-\sigma'}^{\sigma'} \frac{f_1(\sigma',t')\,dt'}{\sqrt{\tau_1'-\tau'}} = -\int_{\sigma'}^{\tau'} \frac{\psi_{1,\eta}(\sigma',t')\,dt'}{\sqrt{\tau'-t'}},$$

valid in $\tau'-\sigma'>0$ and $2>\sigma'>0$, that is in $\tau'>\sigma'>0$ and $2>\sigma'$. In unprimed coordinates this becomes

$$\begin{aligned} 0 &= \int_{-\sigma-1}^{\sigma+1} \frac{f_1(\sigma+1,t')\,dt'}{\sqrt{\tau-1-t'}} + \int_{\sigma+1}^{\tau-1} \frac{\psi_{1,\eta}(\sigma+1,t')\,dt'}{\sqrt{\tau-1-t'}} \\ &= \int_{-\sigma}^{\sigma+2} \frac{f_1(\sigma+1,t-1)\,dt}{\sqrt{\tau-t}} + \int_{\sigma+2}^{\tau} \frac{\psi_{1,\eta}(\sigma+1,t-1)\,dt}{\sqrt{\tau-t}} \\ &= \int_{-\sigma}^{\sigma+2} \frac{f(\sigma,t)\,dt}{\sqrt{\tau-t}} + \int_{\sigma+2}^{\tau} \frac{\psi_\eta(\sigma,t)\,dt}{\sqrt{\tau-t}} \end{aligned} \tag{16}$$

with region of validity $\tau-1>\sigma+1>0$ and $1>\sigma$. Returning to the analogy with the previous section, and noting that, by (7),

$$r(\xi) = \sqrt{(x-\xi)^2+y^2} = \sqrt{(x'-\xi')+y^2}$$

$$r(\xi)\big|_{\xi'=0} = r(-1),$$

relation (15) of (3.1.1) corresponds to

$$-\frac{1}{2\pi}\int_0^{x-r(-1)} d\sigma' \int_{\sigma'}^{\tau_1'-\frac{y^2}{\sigma_1'-\sigma'}} \frac{\psi_{1,\eta}(\sigma',\tau')\,d\tau'}{\sqrt{(\sigma_1'-\sigma')(\tau_1'-\tau')-y^2}} = \frac{1}{2\pi}\int_0^{x-r(-1)} d\sigma' \int_{-\sigma'}^{\sigma'} \frac{f_1(\sigma',\tau')\,d\tau'}{\sqrt{(\sigma_1'-\sigma')(\tau_1'-\tau')-y^2}}$$

$$= \frac{1}{\pi}\int_0^{x-r(-1)} f(\xi'-1)\,d\xi' \int_0^{x-r(-1)-\xi'} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}}$$

$$= \frac{1}{\pi} \int_{-1}^{x-r(-1)-1} f(\xi)d\xi \int_{0}^{x-r(-1)-1-\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} \qquad (17).$$

This is valid in the region for which the integration with respect to ξ concerns only values given by (3), viz. the region

$$r(-1) < x < r(-1)+2 \qquad (18),$$

which is situated between two coaxial cones.

Similarly, introducing double-primed coordinates

$$\left.\begin{array}{ll} x'' = 1-x & ;\ \xi'' = 1-\xi \\ \sigma_1'' = \tau_1 + 1 \quad (= z + x'') & ;\ \sigma'' = \tau + 1 \quad (= \zeta + \xi'') \\ \tau_1'' = \sigma_1 - 1 \quad (= z - x'') & ;\ \tau'' = \sigma - 1 \quad (= \zeta - \xi'') \end{array}\right\} \qquad (19),$$

the above argument may be repeated, with the notation adjusted by

$$\psi_{2,\eta}(\sigma'', \tau'') = \frac{\partial \psi(\xi, 0, \zeta)}{\partial \eta} \quad \text{and} \quad f_2(\sigma'', \tau'') = f(\xi),$$

thus yielding

$$-\frac{1}{2\pi} \int_{0}^{x-r(1)} d\sigma'' \int_{\sigma''}^{\tau_1'' - \frac{y^2}{\sigma_1'' - \sigma''}} \frac{\psi_{2,\eta}(\sigma'', \tau'')\, d\tau''}{\sqrt{(\sigma_1'' - \sigma'')(\tau_1'' - \tau'') - y^2}} = \frac{1}{\pi} \int_{0}^{x-r(1)} f(1-\xi'')d\xi'' \int_{0}^{x-r(1)-\xi''} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}}$$

$$= \frac{1}{\pi} \int_{-x+r(1)+1}^{1} f(\xi)d\xi \int_{0}^{x-r(1)-1+\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} \qquad (20),$$

again valid in a region between two coaxial cones, viz.

$$r(1) < x < r(1) + 2 \qquad (21).$$

The solution in the three regions (10a), (10b) and (11) may now be written down, in analogy to (16) of (3.1.1). The first follows directly by use of the primed variables x' and ξ', viz.

$$\psi(x,y,z) = -\frac{1}{\pi} \int_{0}^{x'+\sqrt{z^2-y^2}} f(\xi'-1)d\xi' \int_{0}^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} + \frac{1}{\pi} \int_{0}^{x-r(-1)} f(\xi'-1)d\xi' \int_{0}^{x-r(-1)-\xi'} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}}$$

$$= -\frac{1}{\pi} \int_{-1}^{x+\sqrt{z^2-y^2}} f(\xi)d\xi \int_{0}^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} + \frac{1}{\pi} \int_{-1}^{x-r(-1)-1} f(\xi)d\xi \int_{0}^{x-r(-1)-1-\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} \qquad (22),$$

in which the second integral is significant only in (18), while the first requires that also $z < r(1)$, thus yielding an over-all significance in region (10a), to which, of course, is to be added the condition $y > 0$. (The expression represents $-\psi$ for $y < 0$). Similarly, using double primed variables, we obtain

$$\psi(x,y,z) = -\frac{1}{\pi} \int_{x-\sqrt{z^2-y^2}}^{1} f(\xi)d\xi \int_{0}^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} + \frac{1}{\pi} \int_{-x+r(1)+1}^{1} f(\xi)d\xi \int_{0}^{x-r(1)-1+\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2 - r^2(\xi)}} \qquad (23),$$

for positive y in the region (10b). In the case where P lies in (11), the domain of integration (8) protrudes on both sides beyond the strip $|\xi| < 1$, but since the conditions for application of both (17) and (20) are satisfied, we may write at once

$$\psi(x,y,z)=-\frac{1}{\pi}\int_{-1}^{1}f(\xi)d\xi\int_{0}^{z-r(\xi)}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}+\frac{1}{\pi}\int_{-1}^{x-r(-1)-1}f(\xi)d\xi\int_{0}^{z-r(-1)-1-\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$+\frac{1}{\pi}\int_{-x+r(1)+1}^{1}f(\xi)d\xi\int_{0}^{z-r(1)-1+\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} \qquad (24)$$

for positive y in the region (11). The formulae (12), (22), (23) and (24) represent the solution in the whole region outside the two cones z=r(-1)+2 and z=r(1)+2, and, as has been remarked, inside either of these cones the integral equation corresponding to (16) has to be actually solved to ensure further progress.

At this stage it will be convenient to introduce a simplification of notation. The function f(x), which, by (3) is prescribed in the interval $|x|<1$, may be continued beyond this interval by requiring

$$f(x)=0 \quad \text{for} \quad |x|>1 \qquad (25).$$

This convention allows integrals over $f(\xi)$ to be interpreted for all real values of the limits of integration; for instance the integral in (12), which as it stands becomes meaningless as soon as $z>r(-1)$ or $z>r(1)$, may now be interpreted in the whole region $z>|y|$. Moreover, the respective first terms of (22), (23) and (24) may all be represented by the same formula as in (12), viz.

$$-\frac{1}{\pi}\int_{x-\sqrt{z^2-y^2}}^{x+\sqrt{z^2-y^2}}f(\xi)d\xi\int_{0}^{z-r(\xi)}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}} \qquad (26).$$

Now, what is needed ultimately, is the Laplace transform of ψ, which means that we have to integrate with respect to z while keeping x and y fixed. In this process, the values of ψ which are successively encountered with increasing z, are given, for instance for both x and y positive, by the formulae (22), (24) and (25), followed by more extensive formulae which still have to be determined. Written in the unifying notation suggested here, each formula is identical with the previous one, but with an extra term added. In this way (26) may be regarded as a first approximation to the solution in the whole region of non-zero ψ, and the extra terms which are added for the successive regions, as so many correction terms. Our task in the sequel will be to determine what order of approximation to the value of the required solution φ of the original problem is represented by each of these correction terms; in fact, it will be shown that they are of descending order for large k, and therefore suited to serve the purpose of obtaining an asymptotic expansion for large k.

We shall now show what contribution to the solution φ is rendered by the part of the solution ψ which has already been determined above. Since φ is given by

$$\varphi(x,y) = \int_0^\infty e^{ikz}\,\psi(x,y,z)\,dz \qquad (27),$$

the contribution of the first term (26) is

$$-\frac{1}{\pi}\int_y^\infty e^{ikz}dz \int_{x-\sqrt{z^2-y^2}}^{x+\sqrt{z^2-y^2}} f(\xi)d\xi \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{r(\xi)}^{\infty} e^{ikz}\,dz \int_0^{z-r(\xi)} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$=\frac{1}{ik\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{r(\xi)}^{\infty} e^{ikz}\,\frac{dz}{\sqrt{z^2-r^2(\xi)}}$$

by partial integration, as in (17) of (3.1.1). Using (18a) of (3.1.1), and also (1) and (25) above, this becomes

$$\frac{1}{2i}\int_{-1}^{1} \frac{\partial\varphi(\xi,0)}{\partial\eta}\,H_0^{(1)}\left\{kr(\xi)\right\}d\xi \qquad (28),$$

which is identical with the Kirchhoff approximation φ_K given in (4) of (1.5). This means that all further contributions are correction terms to the Kirchhoff approximation.

The next contribution is obtained by applying (27) to (23) and (24) for positive x and y, and excluding the first term, thus

$$\frac{1}{\pi}\int_{r(1)}^{\infty} e^{ikz}dz \int_{-z+r(1)+1}^{1} f(\xi)d\xi \int_0^{z-r(1)-1+\xi} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$+\frac{1}{\pi}\int_{r(-1)}^{\infty} e^{ikz}dz \int_{-1}^{z-r(-1)-1} f(\xi)d\xi \int_0^{z-r(-1)-1-\xi} \frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}$$

$$=\frac{1}{\pi}\int_{-1}^{1} f(\xi)d\xi\left\{\int_{r(1)+1-\xi}^{\infty} e^{ikz}dz\int_0^{z-r(1)-1+\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}+\int_{r(-1)+1+\xi}^{\infty} e^{ikz}dz\int_0^{z-r(-1)-1-\xi}\frac{d\zeta}{\sqrt{(z-\zeta)^2-r^2(\xi)}}\right\},$$

using (25). Integrating partially with respect to z as in (17) of (3.1.1), and substituting for f(ξ) from (1), this becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\partial\varphi(\xi,0)}{\partial\eta}d\xi\left\{\int_{r(1)+1-\xi}^{\infty} e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}-\int_{r(-1)+1+\xi}^{\infty} e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}\right\} \qquad (29).$$

This result has been deduced for positive x (and y), but if (23) is replaced by (22), the same result is obtained, so that (29) is valid for all x (and positive y).

3.2.2. The integral equations and their solution.

The next stage is to generalize the integral equation

(16), and to solve this (generalized) equation in the form of recurrence relations, for the sake of applying the solution to find φ in a general point.

The integral equations concern those values of $\frac{\partial\varphi}{\partial y}$ needed in (6) to find φ , namely the values pertaining to the two lateral quadrants formed by removing the boundary value strip $|x|<|$ from the half-plane $z>0$, $y=0$. Since φ is zero for $z<r(-1)$ if $x<-1$, and for $z<r(1)$ if $x>1$, the two quadrants reduce to sectors viz.

$$z>-x-1 \qquad \text{for} \quad x<-1$$

and

$$z>x-1 \qquad \text{for} \quad x>1,$$

which may be written in the (σ,τ) system, using the running coordinates, as

$$\sigma>-1 \qquad \text{for} \quad \tau-\sigma>2$$

$$\tau>-1 \qquad \text{for} \quad \sigma-\tau>2,$$

or rather

$$\tau-1>\sigma+1>0 \tag{30}$$

$$\sigma-1>\tau+1>0 \tag{31}.$$

(Written in primed coordinates these become

$$\tau'>\sigma'>0$$

$$\tau''>\sigma''>0).$$

Either of these two regions may be respectively referred to as the "left" or "right" lateral region, and the fact that the roles of σ and τ are commuted in the two cases, will allow a simple transfer of results from one to the other.

Now, before proceeding to the more general case, we shall first solve (16), which renders $\frac{\partial\varphi}{\partial\eta}$ near the "front" edge of the left lateral region, i.e. in that part of (30) for which $\sigma<1$. The integral equation (16) is of Abel's type, and may be simply solved analytically. For the sake of later application (16) is written in the slightly more general form,

$$0=\int_{\alpha}^{\beta}\frac{f(\sigma,t)\,dt}{\sqrt{\tau-t}}+\int_{\gamma}^{\tau}\frac{\varphi_{\eta}(\sigma,t)\,dt}{\sqrt{\tau-t}} \tag{32},$$

where in this case evidently $\alpha=-\sigma$ and $\beta=\gamma=\sigma+2$. The solution of this Abel equation in φ_{η} is obtained by multiplying by the factor $\frac{1}{\sqrt{T-\tau}}$ (in which T is for the moment still undetermined) and integrating with respect to τ from γ to T, thus

$$0=\int_{\gamma}^{T}\frac{d\tau}{\sqrt{T-t}}\left\{\int_{\alpha}^{\beta}\frac{f(\sigma,t)\,dt}{\sqrt{\tau-t}}+\int_{\gamma}^{\tau}\frac{\varphi_{\eta}(\sigma,t)\,dt}{\sqrt{\tau-t}}\right\}$$

$$=\int_{\alpha}^{\beta} f(\sigma,t)\,dt \int_{\gamma}^{T} \frac{d\tau}{\sqrt{(T-\tau)(\tau-t)}} + \int_{\gamma}^{T} \varphi_\eta(\sigma,t)\,dt \int_{t}^{T} \frac{d\tau}{\sqrt{(T-\tau)(\tau-t)}}$$

$$=2\int_{\alpha}^{\beta} f(\sigma,t)\arctan\sqrt{\frac{T-\gamma}{\gamma-t}}\,dt + \pi\int_{\gamma}^{T} \varphi_\eta(\sigma,t)\,dt.$$

Differentiation with respect to T gives

$$0=\frac{1}{\sqrt{T-\gamma}}\int_{\alpha}^{\beta} f(\sigma,t)\frac{\sqrt{\gamma-t}}{T-t}\,dt+\pi\varphi_\eta(\sigma,T) \qquad (33).$$

Substituting in (33) for α, β and γ, and choosing T to be equal to τ, yields

$$\varphi_\eta(\sigma,t)=-\frac{1}{\pi\sqrt{\tau-\sigma-2}}\int_{-\sigma}^{\sigma+2} f(\sigma,t)\frac{\sqrt{\sigma+2-t}}{\tau-t}\,dt \qquad (34),$$

which is therefore the solution of (16), valid for $\tau-1>\sigma+1>0$ and $\sigma<1$. This may be simplified by noting that f may be written in the form

$$f(\sigma,\tau)=f(\xi)=f\left(\frac{\sigma-\tau}{2}\right) \qquad (35).$$

This means that

$$f(\sigma,\sigma-2\xi)=f(\xi),$$

thus suggesting the substitution

$$t=\sigma-2u\ , \qquad (36)$$

in (34), which yields

$$\varphi_\eta(\sigma,t)=-\frac{2\sqrt{2}}{\pi\sqrt{\tau-\sigma-2}}\int_{-1}^{\sigma} f(u)\frac{\sqrt{1+u}}{\tau-\sigma+2u}\,du \qquad (37),$$

a result which is valid in the region

$$\tau-1>\sigma+1>0 \qquad \text{and } \sigma<1$$

The above considerations leading to (34) also apply with the roles of σ and τ commuted, and we get

$$\varphi_\eta(\sigma,\tau)=-\frac{1}{\pi\sqrt{\sigma-\tau-2}}\int_{-\tau}^{\tau+2} f(s,\tau)\frac{\sqrt{\tau+2-s}}{\sigma-s}\,ds\ ,$$

while in this case, since the exchange of σ and τ changes the sign of ξ in (35), the substitution (36) is replaced by

$$s=\tau+2u,$$

yielding

$$\varphi_\eta(\sigma,\tau)=-\frac{2\sqrt{2}}{\pi\sqrt{\sigma-\tau-2}}\int_{-\tau}^{1} f(u)\frac{\sqrt{1-u}}{\tau-\sigma-2u}\,du \qquad (38),$$

and this is valid if

$$\sigma-1>\tau+1>0 \qquad \text{and } \tau<1 \qquad (39).$$

The identities (37) and (38), together with the boundary condition (3), explicitly determine $\frac{\partial\varphi}{\partial\eta}$ in a connex region consisting of three strips, viz. two strips along the front edges of the lateral regions, and the boundary value strip. Now the procedure leading to (16) is no longer limited by the condition that the domain of integration of (15) should not protrude beyond the boundary value strip, but by the less stringent condition that it should not enter the region of unknown $\frac{\partial\varphi}{\partial\eta}$, which lies between the boundary value strip and (39). This condition is fulfilled if the point (σ_1', τ_1') in (15) lies outside the domain of influence of the point where these two strips meet, viz. the point where

$$\xi = 1 \quad \text{and} \quad \tau = 1 ,$$

which becomes, in the (σ , τ) notation,

$$\sigma - \tau = 2 \quad \text{and} \quad \tau = 1$$

i.e. $\sigma = 3, \quad \tau = 1$

The domain of influence of this point in the plane $\eta = 0$ is bounded by the lines $\sigma = 3$ and $\tau = 1$. Therefore evidently the integral equation (16) may be extended to include on its left-hand side values of $\varphi_\eta(\sigma,\tau)$ known from (38), so as to cover a region of points (σ , τ) such that

$$\tau - 1 > \sigma + 1 > 0$$

and $\quad 3 > \sigma > 1$.

This means that if the integral equation thus found is solved, the domain of known values of $\frac{\partial\varphi}{\partial\eta}$ is extended by widening of the front strip in the left lateral region. The same reasoning holds with the roles of primed and double primed coordinates inverted, so that also the strip of known $\frac{\partial\varphi}{\partial\eta}$ in the right lateral region is widened, symmetrically with the left-hand side.

The above evidently points to a recursion process, in which strips of the left and right lateral regions, respectively of the form $2n+1 > \sigma > 2n-1$ and $2n+1 > \tau > 2n-1$, in which n is a non-negative integer, are successively added to the domain of explicitly known values of $\frac{\partial\varphi}{\partial\eta}$. Since the formulae obtained for $\frac{\partial\varphi}{\partial\eta}$ may be expected to differ in the different regions, it will be convenient to use a different notation for the function $\frac{\partial\varphi}{\partial\eta}$ in each region. For this purpose the regions will be numbered as follows:

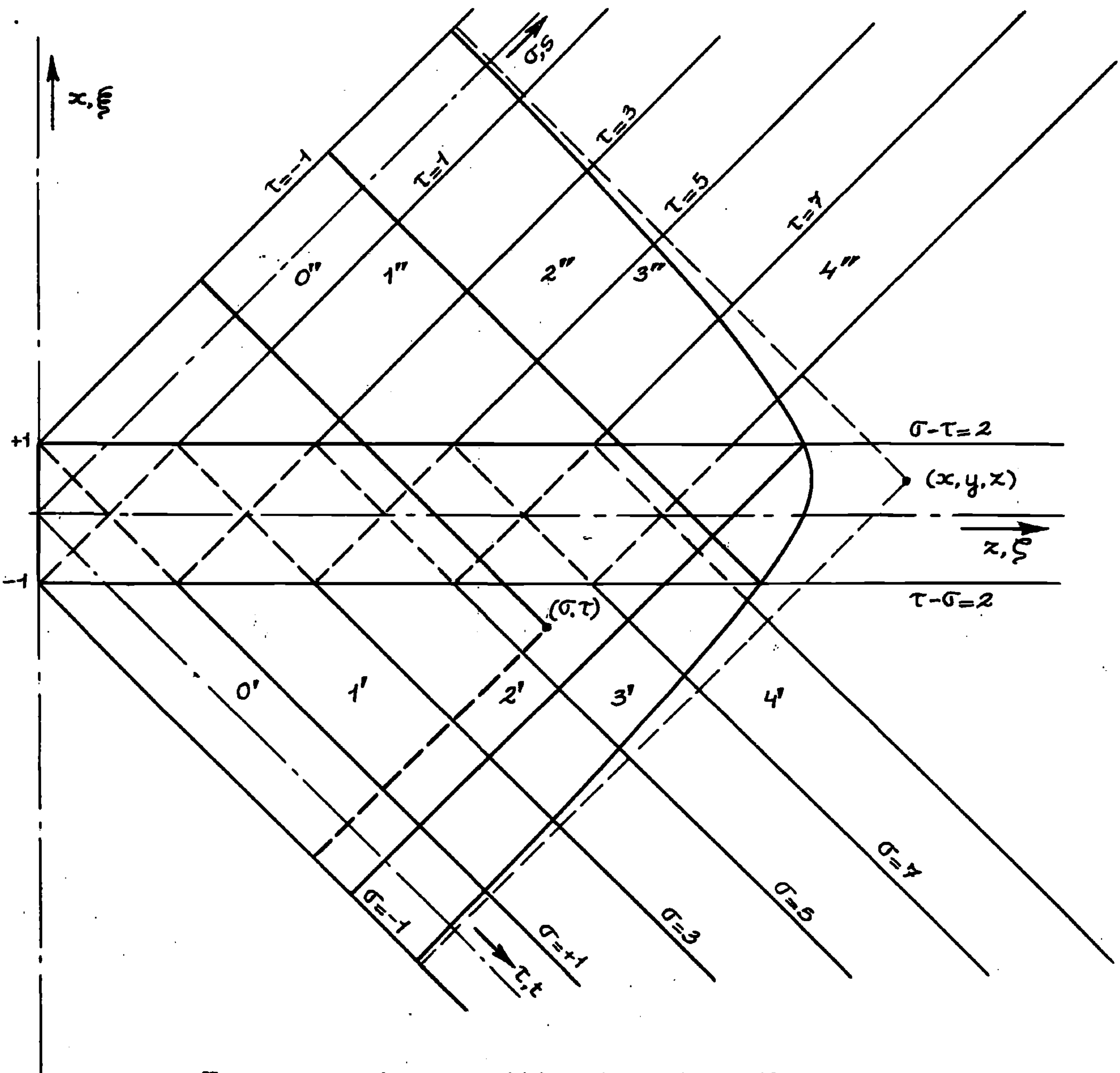
$$\left.\begin{array}{ll} \text{region } n' & \begin{cases} \tau' - 1 > \sigma + 1 > 0 \\ 2n+1 > \sigma > 2n-1 \quad (n = 0, 1, \ldots\ldots) \end{cases} \\ & \\ \text{region } n'' & \begin{cases} \sigma - 1 > \tau + 1 > 0 \\ 2n+1 > \tau > 2n-1 \quad (n = 0, 1, \ldots\ldots) \end{cases} \end{array}\right\} \quad (40)$$

as is indicated in the figure on the next page.

We then denote $\frac{\partial\varphi}{\partial\eta}$ as follows:

$$\frac{\partial\varphi(\xi,0,\zeta)}{\partial\eta} = \left.\begin{cases} \varphi_\eta(\sigma,\tau) = \begin{cases} w_n'(\sigma,\tau) & \text{in region } n' \\ w_n''(\tau,\sigma) & \text{in region } n'' \end{cases} \\ f(\xi) = w'(\sigma,\tau) = w''(\tau,\sigma), \text{ for } |\xi| < 1, \eta = 0, \zeta > 0 \end{cases}\right\} \quad (41)$$

The different order in which the arguments σ and τ are written, as well as the different notations for $f(\xi)$ are introduced to attain that all formulae pertaining to either the left or the right lateral region may be applied to the other simply by commuting primes with double primes and σ with τ .



Now we are in a position to set up the general integral equations for determining the different $w_{n'}$'s and $w_{n''}$'s . For this purpose we return to (16), and note that in the general case it is necessary to replace $f(\sigma,t)$ by a set of different symbols for $\frac{\partial\psi}{\partial\eta}$ along the path of integration in accordance with the notation introduced above in (41). We note first that $\frac{\partial\psi}{\partial y}=0$ for $x>1$ and

$x<r(1)$, and therefore more specially for the part $z<x-1$ of the right lateral region, i.e. for the region $\tau_1<-1$. This means that for (σ,τ) in that part of the left lateral region where $\sigma>1$ (i.e. just the part which is excluded in (16)), the lower limit of integration of the first integral is -1 instead of $-\sigma$. We may therefore write the following generalization of (16) for $n \geqslant 1$:

$$-\int_{\sigma+2}^{\tau} \frac{w_{n'}(\sigma,t)\,dt}{\sqrt{\tau-t}}$$

$$=\sum_{m=0}^{n-2}\int_{2m-1}^{2m+1} \frac{w_{m''}(t,\sigma)\,dt}{\sqrt{\tau-t}}+\int_{2n-3}^{\sigma-2} \frac{w_{(n-1)''}(t,\sigma)\,dt}{\sqrt{\tau-t}}+\int_{\sigma-2}^{\sigma+2} \frac{w''(t,\sigma)\,dt}{\sqrt{\tau-t}} \tag{42}$$

in which the paths of integration may be vizualized as lying along the line $s=\sigma$, as indicated in the figure. Equation (42), as it stands, is valid for (σ,t) in region n'. However, we may suppose that each of the formulae for the different functions is interpreted in the whole region backwards of the region in which it represents $\frac{\partial\psi}{\partial\eta}$, as was done previously in (25) and (26). Then equation (42) is also significant if (σ,τ) lies in the regions p' with $p \geqslant n$, though of course the solution $w_{n'}(\sigma,\tau)$ would not there represent the true value of $\frac{\partial\psi}{\partial\eta}$, since $\frac{\partial\psi}{\partial\eta}=w_{p'}(\sigma,t)$ in region p'. This extension allows (42) to be compared with the corresponding equation for $w_{(n-1)'}(\sigma,\tau)$, viz.

$$-\int_{\sigma+2}^{\tau} \frac{w_{(n-1)'}(\sigma,t)\,dt}{\sqrt{\tau-t}}$$

$$=\sum_{m=0}^{n-3}\int_{2m-1}^{2m+1} \frac{w_{m''}(t,\sigma)\,dt}{\sqrt{\tau-t}}+\int_{2n-5}^{\sigma-2} \frac{w_{(n-2)''}(t,\sigma)\,dt}{\sqrt{\tau-t}}+\int_{\sigma-2}^{\sigma+2} \frac{w''(t,\sigma)\,dt}{\sqrt{\tau-t}}\ .$$

Subtracting this from (42) then yields

$$-\int_{\sigma+2}^{\tau} \frac{w_{n'}(\sigma,t)-w_{(n-1)'}(\sigma,t)}{\sqrt{\tau-t}}\,dt$$

$$=\int_{2n-5}^{2n-3} \frac{w_{(n-2)''}(t,\sigma)\,dt}{\sqrt{\tau-t}}+\int_{2n-3}^{\sigma-2} \frac{w_{(n-1)''}(t,\sigma)\,dt}{\sqrt{\tau-t}}-\int_{2n-5}^{\sigma-2} \frac{w_{(n-2)''}(t,\sigma)\,dt}{\sqrt{\tau-t}}$$

$$=\int_{2n-3}^{\sigma-2} \frac{w_{(n-1)''}(t,\sigma)-w_{(n-2)''}(t,\sigma)}{\sqrt{\tau-t}}\,dt\ .$$

Introducing the notation

$$\left.\begin{aligned} \delta w_{n'}(\sigma,\tau) &= \begin{cases} w_{n'}(\sigma,\tau) & \text{for } n=0 \\ w_{n'}(\sigma,\tau)-w_{(n-1)'}(\sigma,\tau), & \text{for } n\geqslant 1 \end{cases} \\ \delta w_{n''}(\tau,\sigma) &= \begin{cases} w_{n''}(\tau,\sigma) & \text{for } n=0 \\ w_{n''}(\tau,\sigma)-w_{(n-1)''}(\tau,\sigma), & \text{for } n\geqslant 1 \end{cases} \end{aligned}\right\} \quad (43),$$

this becomes

$$0 = \int_{2n-3}^{\sigma-2} \frac{\Delta w_{(n-1)''}(t,\sigma)}{\sqrt{\tau-t}}\,dt + \int_{\sigma+2}^{\tau} \frac{\Delta w_{n'}(\sigma,t)}{\sqrt{\tau-t}}\,dt \qquad (44)$$

in the regions p' with $p \geqslant n$, i.e. in

$$\tau > \sigma+2 > 2n+1, \quad \text{where } n \geqslant 1 \qquad (45).$$

Now if $\Delta w_{(n-1)''}$ is assumed to be known, (44) is an integral equation in $\Delta w_{n'}$, of the form given in (32), and with solution of the form (33). Therefore, substituting for α, β, γ and τ, and replacing the functions f and φ_η occurring in (33) by the difference functions in (44), we obtain

$$\Delta w_{n'}(\sigma,\tau) = -\frac{1}{\pi\sqrt{\tau-\sigma-2}} \int_{2n-3}^{\sigma-2} \Delta w_{(n-1)''}(t,\sigma) \frac{\sqrt{\sigma+2-t}}{\tau-t}\,dt \qquad (46),$$

valid in the region (45). The corresponding result for the right lateral region is

$$\Delta w_{n''}(\tau,\sigma) = -\frac{1}{\pi\sqrt{\sigma-\tau-2}} \int_{2n-3}^{\tau-2} \Delta w_{(n-1)'}(s,\tau) \frac{\sqrt{\tau+2-s}}{\sigma-s}\,ds \qquad (47),$$

which is valid in

$$\sigma > \tau+2 > 2n+1, \text{ where } n \geqslant 1 .$$

In the notation of (41) and (43), the solutions (37) and (38) obtained for $\frac{\partial\psi}{\partial\eta}$ in the regions 0' and 0", are

$$\Delta w_{0'}(\sigma,\tau) = -\frac{2\sqrt{2}}{\pi\sqrt{\tau-\sigma-2}} \int_{-1}^{\sigma} f(u) \frac{\sqrt{1+u}}{\tau-\sigma+2u}\,du \qquad (48)$$

$$\Delta w_{0''}(\tau,\sigma) = -\frac{2\sqrt{2}}{\pi\sqrt{\sigma-\tau-2}} \int_{\tau}^{1} f(u) \frac{\sqrt{1-u}}{\sigma-\tau-2u}\,du \qquad (49),$$

and, using (25), these remain significant for all regions p' or p" respectively, with $p \geqslant 0$, i.e. in the whole left or right lateral region. The set of relations (46) to (49) above, form an integral recursion system which completely determines the difference functions $\Delta w_{n'}$ and $\Delta w_{n''}$ for all positive n. This is in fact the solution of the problem of obtaining $\frac{\partial\psi}{\partial\eta}$ in the plane $\eta=0$, (as is required for the application of (6) to obtain ψ everywhere in space), since evidently, by (41) and (43), $\frac{\partial\psi}{\partial\eta}$ may be written as

$$\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta} = \varphi_\eta(\sigma,\tau) = \sum_{m=0}^{n} \Delta w_{m'}(\sigma,\tau) \qquad (50)$$

if (σ,τ) lies in region n', and as

$$\frac{\partial\psi(\xi,0,\zeta)}{\partial\eta} = \sum_{m=0}^{n} \Delta w_{m''}(\tau,\sigma) \qquad (51),$$

for (σ,τ) in the region n".

3.2.3. Completion of the solution.

The above results may now be used to complete the solution of the problem of determining ψ and therefore eventually φ. We have already obtained the first two approximations to the final solution φ, viz. firstly the function (28), which is identical with the Kirchhoff approximation, and secondly the correction terms (29), which were obtained by elimination from (6) of what is evidently the two functions that are now called $\delta w_{0'}$ and $\delta w_{0''}$, as defined in (41) and (43). The values from which these first terms in the solution of φ were obtained by applying the transformation (27), represent the correct values of ψ only if the point P:(x,y,z) lies outside both of the cones z=r(-1)+2 and z=r(1)+2, which is the region for which the only values of $\frac{\partial\psi}{\partial\eta}$ contained in (6) are f, $w_{0'}$ and $w_{0''}$. If P lies further backwards, (6) contains also values of $\frac{\partial\psi}{\partial\eta}$ which are composed, by (50) and (51), of terms $\delta w_{m'}$ and $\delta w_{m''}$ with $m \geqslant 0$.

We therefore consider a general point P in this further region, and note that it satisfies the conditions

$$r(-1)+2n_1 < x < r(-1)+2(n_1+1)$$

$$r(1)+2n_2 < x < r(1)+2(n_2+1)$$

in which now both n_1 and n_2 are at least equal to 1 (while their difference is at most 1). Then ψ may, by using (6), be written in the form

$$2\pi\psi(x,y,z) = -2\int_{x-\sqrt{x^2-y^2}}^{x+\sqrt{x^2-y^2}} f(\xi)\,d\xi \int_0^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}}$$

$$-\int_{-1}^{x-r(-1)-1} d\sigma \int_{\sigma+2}^{\tau_1-\frac{y^2}{\sigma_1-\sigma}} \frac{\psi_\eta(\sigma,\tau)\,d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} - \int_{-1}^{x-r(1)-1} d\tau \int_{\tau+2}^{\sigma_1-\frac{y^2}{\tau_1-\tau}} \frac{\psi_\eta(\sigma,\tau)\,d\sigma}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}},$$

which, by (50) and (51), becomes

$$2\pi\psi(x,y,z) = -2\int_{x-\sqrt{x^2-y^2}}^{x+\sqrt{x^2-y^2}} f(\xi)\,d\xi \int_0^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}}$$

$$-\int_{-1}^{x-r(-1)-1} d\sigma \int_{\sigma+2}^{\tau_1-\frac{y^2}{\sigma_1-\sigma}} \frac{\delta w_{0'}(\sigma,\tau)\,d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} - \int_{-1}^{x-r(1)-1} d\tau \int_{\tau+2}^{\sigma_1-\frac{y^2}{\tau_1-\tau}} \frac{\delta w_{0''}(\tau,\sigma)\,d\sigma}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}}$$

$$-\sum_{m=1}^{n_1}\int_{2m-1}^{x-r(-1)-1} d\sigma \int_{\sigma+2}^{\tau_1-\frac{y^2}{\sigma_1-\sigma}} \frac{\delta w_{m'}(\sigma,\tau)\,d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} - \sum_{m=1}^{n_2}\int_{2m-1}^{x-r(1)-1} d\tau \int_{\tau+2}^{\sigma_1-\frac{y^2}{\tau_1-\tau}} \frac{\delta w_{m''}(\tau,\sigma)\,d\sigma}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} \quad (52).$$

The first three terms have been treated in (3.2.1), yielding the results (17) and (20) for the second and third respectively. The rest of the terms may be similarly treated by means of the integral equations(44) and its counterpart for (σ,τ) in the right lateral region. It is sufficient to do so for only one of the two sums, since the other follows by commuting σ with τ and primes with double primes. The general term of the first sum in (52) then yields:

$$-\int_{2m-1}^{x-r(-1)-1} d\sigma \int_{\sigma+2}^{\tau_1-\frac{y^2}{\sigma_1-\sigma}} \frac{\delta w_{m'}(\sigma,\tau)d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} = -\int_{2m-1}^{x-r(-1)-1} \frac{d\sigma}{\sqrt{\sigma_1-\sigma}} \int_{\sigma+2}^{\tau_1-\frac{y^2}{\sigma_1-\sigma}} \frac{\delta w_{m'}(\sigma,\tau)d\tau}{\sqrt{\left(\tau_1-\frac{y^2}{\sigma_1-\sigma}\right)-\tau}}$$

$$= +\int_{2m-1}^{x-r(-1)-1} \frac{d\sigma}{\sqrt{\sigma_1-\sigma}} \int_{2m-3}^{\sigma-2} \frac{\delta w_{(m-1)''}(\tau,\sigma)d\tau}{\sqrt{\left(\tau_1-\frac{y^2}{\sigma_1-\sigma}\right)-\tau}}$$

$$= \int_{2m-1}^{x-r(-1)-1} d\sigma \int_{2m-3}^{\sigma-2} \frac{\delta w_{(m-1)''}(\tau,\sigma)d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} ,$$

by application of (44). Therefore finally the identity (52) becomes

$$2\pi\psi(x,y,z) = -2\int_{x-\sqrt{x^2-y^2}}^{x+\sqrt{x^2-y^2}} f(\xi)d\xi \int_0^{x-r(\xi)} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}}$$

$$+2\int_{-1}^{x-r(-1)-1} f(\xi)d\xi \int_0^{x-r(-1)-1-\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}} +2\int_{-x+r(1)+1}^{1} f(\xi)d\xi \int_0^{x-r(1)-1+\xi} \frac{d\zeta}{\sqrt{(x-\zeta)^2-r^2(\xi)}}$$

$$+\sum_{m=0}^{n_1-1} \int_{2m+1}^{x-r(-1)-1} d\sigma \int_{2m-1}^{\sigma-2} \frac{\delta w_{m''}(\tau,\sigma)d\tau}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} + \sum_{m=0}^{n_2-1} \int_{2m+1}^{x-r(1)-1} d\tau \int_{2m-1}^{\tau-2} \frac{\delta w_{m'}(\sigma,\tau)d\sigma}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} \qquad (53).$$

In this expression the values bordering the hyperbola $\zeta = x - r(\xi)$ have been eliminated everywhere excepting in the first term, (so that also the highest value among the index numbers m occurring in the functions $dw_{m'}$ and $dw_{m''}$, has been decreased by 1), and z occurs linearly in the limits of integration of the remaining parts. The above process is also best vizualized by referring to the figure.

In (53) the point P lies outside the cones $z=r(-1)+2(n_1+1)$ and $z=r(1)+2(n_2+1)$, which conditions are reflected in the occurrence of functions $\delta w_{m'}$ and $\delta w_{m''}$ for only such values of m which are at most equal to n_2-1 or n_1-1 respectively. That this should be the case, is evident, because the domain of influence of for instance the region n' is easily seen to be the inside of the cone $z=r(-1)+2n$.

We shall now analyse the general term in the second sum of (53), viz.

$$\int_{2m+1}^{x-r(1)-1} d\tau \int_{2m-1}^{\tau-2} \frac{\delta w_{m'}(\sigma,\tau)d\sigma}{\sqrt{(\sigma_1-\sigma)(\tau_1-\tau)-y^2}} \qquad (54),$$

in which $\delta w_{m'}$ is given by the recursion system (46) to (49), which may be rewritten in slightly different notation as

$$\left.\begin{aligned}
\delta w_{0'}(s_0,s_1) &= -\frac{2\sqrt{2}}{\pi\sqrt{s_1-s_0-2}} \int_{-1}^{s_0} f(u)\,\frac{\sqrt{1+u}}{s_1-s_0+2u}\,du \\
\delta w_{0''}(s_0,s_1) &= -\frac{2\sqrt{2}}{\pi\sqrt{s_1-s_0-2}} \int_{-1}^{s_0} f(-u)\,\frac{\sqrt{1+u}}{s_1-s_0+2u}\,du \\
\delta w_{n'}(s_n,s_{n+1}) &= -\frac{1}{\pi\sqrt{s_{n+1}-s_n-2}} \int_{2n-3}^{s_n-2} \delta w_{(n-1)''}(s_{n-1},s_n)\,\frac{\sqrt{s_n-s_{n-1}+2}}{s_{n+1}-s_{n-1}}\,ds_{n-1} \\
\delta w_{n''}(s_n,s_{n+1}) &= -\frac{1}{\pi\sqrt{s_{n+1}-s_n-2}} \int_{2n-3}^{s_n-2} \delta w_{(n-1)'}(s_{n-1},s_n)\,\frac{\sqrt{s_n-s_{n-1}+2}}{s_{n+1}-s_{n-1}}\,ds_{n-1}
\end{aligned}\right\}(55)$$

In order to demonstrate explicitly the influence of the boundary values f(x), the function δw_m, may be written, for $m \geqslant 1$, in extended form, thus

$$\delta w_m(s_m, s_{m+1}) = \frac{2\sqrt{2}}{(-\pi)^{m+1}\sqrt{s_{m+1}-s_m-2}} \int_{2m-3}^{s_m-2} \frac{ds_{m-1}}{s_{m+1}-s_{m-1}} \sqrt{\frac{s_m-s_{m-1}+2}{s_m-s_{m-1}-2}} \int_{2m-1}^{s_{m-1}-2} \cdots\cdots$$

$$\cdots\cdots \int_1^{s_1-2} \frac{ds_0}{s_2-s_0} \sqrt{\frac{s_1-s_0+2}{s_1-s_0-2}} \int_{-1}^{s_0} f\left\{(-)^m u\right\} \frac{\sqrt{1+u}}{s_1-s_0+2u}\, du \, .$$

Therefore the expression (54) becomes, with σ and τ replaced by s_m and s_{m+1} respectively,

$$\frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{2m+1}^{x-r(1)-1} ds_{m+1} \int_{2m-1}^{s_{m+1}-2} \frac{ds_m}{\sqrt{s_{m+1}-s_m-2}\,\sqrt{(z+x-s_m)(z-x-s_{m+1})-y^2}} \int_{2m-3}^{s_m-2} \frac{ds_{m-1}}{s_{m+1}-s_{m-1}} \sqrt{\frac{s_m-s_{m-1}+2}{s_m-s_{m-1}-2}} \int \cdots$$

$$\cdots \int_1^{s_1-2} \frac{ds_0}{s_2-s_0} \sqrt{\frac{s_1-s_0+2}{s_1-s_0-2}} \int_{-1}^{s_0} f\left\{(-)^m u\right\} \frac{\sqrt{1+u}}{s_1-s_0+2u}\, du$$

$$= \frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{-1}^{x-r(1)-2m-3} f\left\{(-)^m u\right\} \sqrt{1+u}\, du \int_u^{x-r(1)-2m-3} ds_0 \int_{s_0+2}^{x-r(1)-2m-1} \frac{ds_1}{s_1-s_0+2u} \sqrt{\frac{s_1-s_0+2}{s_1-s_0-2}} \int_{s_1+2}^{x-r(1)-2m+1} \frac{ds_2}{s_2-s_0} \sqrt{\frac{s_2-s_1+2}{s_2-s_1-2}} \int \cdots$$

$$\cdots \int_{s_{m-1}+2}^{x-r(1)-3} \frac{ds_m}{s_m-s_{m-2}} \sqrt{\frac{s_m-s_{m-1}+2}{s_m-s_{m-1}-2}} \int_{s_m+2}^{x-r(1)-1} \frac{ds_{m+1}}{(s_{m+1}-s_{m-1})\sqrt{s_{m+1}-s_m-2}\,\sqrt{(z+x-s_m)(z-x-s_{m+1})-y^2}}$$

by changing the order of integration. Making the substitutions

$$s_p = \left\{x-r(1)-2(m-p)-3\right\} - t_{m+1-p}$$

this becomes

$$\frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{-1}^{x-r(1)-2m-3} f\left\{(-)^m u\right\} \sqrt{1+u}\, du \int_0^{x-r(1)-2m-3-u} dt_{m+1} \int_0^{t_{m+1}} \frac{dt_m}{t_{m+1}-t_m+2+2u} \sqrt{\frac{t_{m+1}-t_m+4}{t_{m+1}-t_m}} \int_0^{t_m} \frac{dt_{m-1}}{t_{m+1}-t_{m-1}+4} \sqrt{\frac{t_m-t_{m-1}+4}{t_m-t_{m-1}}} \int \cdots\cdots$$

$$\cdots\cdots \int_0^{t_2} \frac{dt_1}{t_3-t_1+4} \sqrt{\frac{t_2-t_1+4}{t_2-t_1}} \int_0^{t_1} \frac{dt_0}{(t_2-t_0+4)\sqrt{t_1-t_0}\sqrt{(r(1)+x+3+t_1)(r(1)-x+1+t_0)-y^2}} \qquad (56).$$

The recursion system (55) may therefore be replaced by a simpler one which is independent of f(x), viz.

$$\left.\begin{aligned} D_0(t_0, t_1; x, y) &= \frac{1}{\sqrt{t_1-t_0}\sqrt{(r(1)+x+3+t_1)(r(1)-x+1+t_0)-y^2}} \\ D_n(t_n, t_{n+1}; x, y) &= \sqrt{\frac{t_{n+1}-t_n+4}{t_{n+1}-t_n}} \int_0^{t_n} D_{n-1}(t_{n-1}, t_n; x, y) \frac{dt_{n-1}}{t_{n+1}-t_{n-1}+4} \end{aligned}\right\} \quad (57).$$

In this notation (56) becomes

$$\frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{-1}^{x-r(1)-2m-3} f\left\{(-)^m u\right\} \sqrt{1+u}\, du \int_0^{x-r(1)-2m-3-u} dt_{m+1} \int_0^{t_{m+1}} \frac{D_m(t_m, t_{m+1}; x, y)}{t_{m+1}-t_m+2+2u}\, dt_m \qquad (58),$$

which is therefore the form which the general term (54) assumes when the influence of the boundary values is made explicit. This is valid for all $m \geqslant 0$.

In order to obtain the contribution of (58) to the final solution φ, it has to be Laplace-transformed (cf. (27)). Since this term does not occur unless $x > r(1)+2(m+1)$, the lower limit of integration is $r(1)+2(m+1)$, and the transform is therefore

$$\frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{r(1)+2m+2}^{\infty} e^{ikx}\,dx \int_{-1}^{x-r(1)-2m-3} f\{(-)^m u\}\sqrt{1+u}\,du \int_{0}^{x-r(1)-2m-3-u} dt_{m+1} \int_{0}^{t_{m+1}} \frac{D_m(t_m,t_{m+1};x,y)}{t_{m+1}-t_m+2+2u}\,dt_m$$

$$= \frac{2\sqrt{2}}{(-\pi)^{m+1}} \int_{-1}^{\infty} f\{(-)^m u\}\sqrt{1+u}\,du \int_{r(1)+2m+3+u}^{\infty} e^{ikx}\,dx \int_{0}^{x-r(1)-2m-3-u} dt_{m+1} \int_{0}^{t_{m+1}} \frac{D_m(t_m,t_{m+1};x,y)}{t_{m+1}-t_m+2+2u}\,dt_m$$

$$= \frac{2\sqrt{2}}{(-\pi)^{m+1}} e^{ik\{r(1)+2m+3\}} \int_{-1}^{1} f\{(-)^m u\} e^{iku}\sqrt{1+u}\,du \int_{0}^{\infty} e^{ik\mu}\,d\mu \int_{0}^{\mu} dt_{m+1} \int_{0}^{t_{m+1}} \frac{D_m(t_m,t_{m+1};x,y)}{t_{m+1}-t_m+2+2u}\,dt_m \quad (59),$$

by introduction of the variable

$$\mu = x - \{r(1)+2m+3+u\}$$

and in which the upper limit of integration with respect to u is 1, by (25). The expression (59) may be further simplified by partial integration with respect to μ. Doing this, and substituting

$$(-)^m u = \xi$$

and $t_m = \lambda$,

yields

$$-\frac{2\sqrt{2}}{(-\pi)^{m+1}ik} e^{ik\{r(1)+2m+3\}} \int_{-1}^{1} f(\xi) e^{\pm ik\xi}\sqrt{1\pm\xi}\,d\xi \int_{0}^{\infty} e^{ik\mu}\,d\mu \int_{0}^{\mu} \frac{D_m(\lambda,\mu;x,y)}{\mu-\lambda+2(1\pm\xi)}\,d\lambda \quad (60),$$

in which $\pm\xi$ is written to indicate $(-)^m\xi$.

The corresponding result for the general term of the first sum of (53) is

$$-\frac{2\sqrt{2}}{(-\pi)^{m+1}ik} e^{ik\{r(1)+2m+3\}} \int_{-1}^{1} f(\xi) e^{\mp ik\xi}\sqrt{1\mp\xi}\,d\xi \int_{0}^{\infty} e^{ik\mu}\,d\mu \int_{0}^{\mu} \frac{D_m(\lambda,\mu;-x,y)}{\mu-\lambda+2(1\mp\xi)}\,d\lambda \quad (61),$$

in which $\mp\xi$ denotes $(-)^{m+1}\xi$.

We can now write down the formal solution of the boundary value problem (1) for φ. The identity (53) contains the general solution of ψ, from which φ follows by applying the Laplace transform (27). The transforms of the first three terms of (53) have been obtained in (3.2.1), and are given in (28) and (29), while the contributions of the general terms of (53) have been obtained above, and are represented in (60) and (61). Therefore finally we may write, on substituting for $f(\xi)$ from (1), an expression of the form

$$\varphi(x,y) = \int_{-1}^{1} \frac{\partial\varphi(\xi,0)}{\partial\eta}\, G(x,y;\xi,0)\,d\xi \quad (62),$$

in which G is the Green's function for the problem, viz.

$$G(x,y;\xi,0) \doteq \frac{1}{2i} H_0^{(1)}\left\{kr(\xi)\right\} + \frac{1}{\pi}\left\{\int_{r(1)+1-\xi}^{\infty} e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}} + \int_{r(-1)+1+\xi}^{\infty} e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}\right\}$$

$$-\frac{\sqrt{2}}{\pi^2} e^{3ik}\sum_{m=0}^{\infty}\frac{e^{2ikm}}{(-\pi)^m}\left\{e^{ik\{r(1)+(-)^m\xi\}}\sqrt{1+(-)^m\xi}\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{D_m(\lambda,\mu;x,y)\,d\lambda}{\mu-\lambda+2\{1+(-)^m\xi\}}\right.$$

$$\left.+e^{ik\{r(-1)+(-)^{m+1}\xi\}}\sqrt{1+(-)^{m+1}\xi}\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{D_m(\lambda,\mu;-x,y)\,d\lambda}{\mu-\lambda+2\{1+(-)^{m+1}\xi\}}\right\} \qquad (63)$$

in which the functions D_m are given by (57). Here the infinite sum is for the moment written purely formally, and the task of the next section will be to indicate in which way the formal series occurring here should be interpreted. The problem is therefore formally solved in terms of a Green's function consisting of a sum of terms, of which Kirchhoff's kernel is the first.

3.2.4. The asymptotic character of the solution.

Since what we have set out to find, is the solution φ for large values of the (eventually real) parameter k, the formal series occurring in (63) need not converge to be useful. The property which is needed, is that successive terms should be of decreasing order in k, in other words that the series should be asymptotic in the sense of Poincaré. We shall now justify the work of the preceding sections by showing that this is in fact the case.

A glance at the series in (63) shows that the general term has a factor e^{2ikm}, which is due to the fact that the regions of action of the functions $\delta w_{n'}$ and $\delta w_{n''}$ (cf. (41) and (43)), lie progressively further away from the plane z=0. It should be noted that for complex k this factor plays a dominating role, but that in the case in which we are eventually interested, viz. the case where k is real, this factor becomes purely oscillatory, so that a closer analysis is needed to determine the order of each term for large k.

Evidently the asymptotic behaviour of the general term of the solution φ is sufficiently represented by considering the slightly simplified expression

$$\int_{-1}^{1}\frac{\partial\varphi(\xi,0)}{\partial\eta}e^{ik\xi}\sqrt{1+\xi}\,d\xi\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{D_m(\lambda,\mu;x,y)\,d\lambda}{\mu-\lambda+2(1+\xi)} \qquad (64).$$

We shall allow for the possibility that $\frac{\partial\varphi(\xi,0)}{\partial\eta}$ contains a factor $e^{ik(a-1)\xi}$, in which a is a constant, which is, for illustration, chosen to be positive, and write

$$-\frac{\partial\varphi(\xi,0)}{\partial\eta} = h(\xi)e^{ik(a-1)\xi} \quad \text{for } |\xi|<1 \tag{65}$$

where $h(\xi)$ is assumed to be a polynomial in ξ and k, of degree q in k. This choice is in accordance with a common type of boundary value occurring in both diffraction and aerodynamic problems, and is sufficient for our present purpose. In actual fact, however, much wider classes of functions may be allowed. Substitution of (65) in (64) yields

$$\int_{-1}^{1} e^{ika\xi} h(\xi)\sqrt{1+\xi}\,d\xi \int_0^\infty e^{ik\mu}d\mu \int_0^\mu \frac{D_m(\lambda,\mu;x,y)\,d\lambda}{\mu-\lambda+2(1+\xi)} \tag{66}.$$

Following a suggestion of J. Berghuis, the substitutions

$$1+\xi = 2\mu X_1 \quad \text{and} \quad \lambda = \mu\lambda_1 \tag{67}$$

transform this into

$$2\sqrt{2}\,e^{ika}\int_0^\infty dX_1 \int^{\frac{1}{X_1}} d\mu \int_0^1 d\lambda_1 e^{ik\mu(1+2aX_1)} h(\xi) X_1^{\frac{1}{2}} \mu^{\frac{3}{2}} \frac{D_m(\lambda,\mu;x,y)}{1-\lambda_1+4X_1}$$

which, by the further substitution

$$\mu(1+2a\,X_1) = \mu_1 \tag{68}$$

becomes

$$2\sqrt{2}\,e^{-ika}\int_0^\infty dX_1 \int_0^{2a+\frac{1}{X_1}} d\mu_1 \int_0^1 d\lambda_1 e^{ik\mu_1} h(\xi) X_1^{\frac{1}{2}} \frac{\mu_1^{\frac{3}{2}}}{(1+2aX_1)^{\frac{5}{2}}} \cdot \frac{D_m(\lambda,\mu;x,y)}{1-\lambda_1+4X_1} \tag{69}$$

in which only the single oscillatory factor $e^{ik\mu_1}$ occurs. For the sake of simplicity of analysis we still suppose that Re k is positive, and use

$$\int_0^{2a+\frac{1}{X_1}} = \int_0^\infty - \int_{2a+\frac{1}{X_1}}^\infty .$$

Then (69) becomes, by inverting the order of integration,

$$2\sqrt{2}\,e^{-ika}\Bigg\{\int_0^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1 \int_0^\infty h(\xi)\frac{X_1^{\frac{1}{2}}dX_1}{(1+2aX_1)^{\frac{5}{2}}}\int_0^1 \frac{D_m(\lambda,\mu;x,y)\,d\lambda_1}{1-\lambda_1+4X_1}$$
$$-\int_{2a}^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1 \int_{\frac{1}{\mu_1-2a}}^\infty h(\xi)\frac{X_1^{\frac{1}{2}}dX_1}{(1+2aX_1)^{\frac{5}{2}}}\int_0^1 \frac{D_m(\lambda,\mu;x,y)\,d\lambda_1}{1-\lambda_1+4X_1}\Bigg\}$$
$$=2\sqrt{2}\,e^{-ika}\int_0^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1 \int_0^\infty h(\xi)\frac{X_1^{\frac{1}{2}}dX_1}{(1+2aX_1)^{\frac{5}{2}}}\int_0^1 \frac{D_m(\lambda,\mu;x,\lambda)\,d\lambda_1}{1-\lambda_1+4X_1}$$
$$-2\sqrt{2}\,e^{ika}\int_0^\infty e^{ik\mu_2}(2a+\mu_2)^{\frac{3}{2}}\mu_2^{\frac{1}{2}}d\mu_2 \int_0^1 h(\xi)\frac{X_2\,dX_2}{(\mu_2X_2+2a)^{\frac{5}{2}}}\int_0^1 \frac{D_m(\lambda,\mu;x,y)\,d\lambda_1}{(1-\lambda)\mu_2X_2+4} \tag{70}$$

where the substitutions

$$\mu_1 = 2a+\mu_2 \quad \text{and} \quad X_1 = \frac{1}{\mu_2 X_2} \tag{71}$$

are used in the second term.

Now, the integrals occurring in (70), considered as integrals over μ_1 and μ_2 respectively, have the well-known property that their asymptotic behaviour for large k is entirely determined by the behaviour of the integrand near the origin. This is a statement in what is generally called the domain of Tauberian theorems. The prototype of the above situation is formulated by Doetsch (1937) for complex k (theorem 1 of his § 12.3), but when k becomes real, the position is more difficult. The general ideas occurring here are at the basis of the method of stationary phase of Kelvin (1887), and rigorous theorems in this field have been proved by Watson (1918) and Van der Corput (1934, 1936).

We therefore need the asymptotic development near $\mu_1 = 0$ or $\mu_2 = 0$, of $D_m(\lambda,\mu;x,y)$, which may be written, by (67), (68) and (71), as

$$\left.\begin{aligned} D_m(\lambda,\mu;x,y) &= D_m\left(\frac{\mu_1\lambda_1}{1+2a\,x_1},\ \frac{\mu_1}{1+2a\,x_1};x,y\right)\\ &= D_m\left(\frac{\mu_2 x_2\lambda_1(2a+\mu_2)}{\mu_2 x_2+2a},\ \frac{\mu_2 x_2(2a+\mu_2)}{\mu_2 x_2+2a};x,y\right)\end{aligned}\right\} \qquad (72),$$

so that what is needed is the development of D_m near the origin in both arguments μ and λ. It is convenient to simplify the function D_o of (57) by introducing the variables

$$\left.\begin{aligned}\alpha_1 &= r(1)+(1-x)\\ \beta_1 &= r(1)-(1-x)\end{aligned}\right\} \qquad (73)$$

so that D_o becomes

$$D_o(t_o t_1;x,y) = \frac{1}{\sqrt{t_1-t_o'}\sqrt{4\alpha_1+\alpha_1 t_1+(4+\beta_1)t_o+t_1 t_o'}} .$$

We now perform in the recursion relations (57) the substitutions

$$\left.\begin{aligned} t_{m+1} &= \mu\\ t_n &= \mu t_n' \qquad \text{for} \quad 0 \leq n \leq m\end{aligned}\right\} \qquad (74).$$

The formulae (57) therefore become

$$D_o(t_o,\mu;x,y) = \frac{1}{\sqrt{\mu(1-t_o')}\sqrt{4\alpha_1+\mu\{\alpha_1+(4+\beta_1)t_o'+\mu t_o'\}}} \qquad \text{for } m=0 \qquad (75)$$

$$D_o(t_o,t_1;x,y) = \frac{1}{\sqrt{\mu(t_1'-t_o')}\sqrt{4\alpha_1+\mu\{\alpha_1 t_1'+(4+\beta_1)t_o'+\mu t_1' t_o'\}}} \qquad \text{for } m \geq 1 \qquad (75a)$$

and

$$\begin{aligned} D_n(t_n,t_{n+1};x,y) &= \sqrt{\frac{\mu(t_{n+1}'-t_n')+4}{\mu(t_{n+1}'-t_n')}}\int_0^{t_n'} D_{n-1}(t_{n-1},t_n;x,y)\frac{\mu\,dt_{n-1}'}{\mu(t_{n+1}'-t_{n-1}')+4}\\ &= \frac{1}{2}\sqrt{\mu}\sqrt{\frac{1+\frac{1}{4}\mu(t_{n+1}'-t_n')}{t_{n+1}'-t_n'}}\int_0^{t_n'}\frac{D_{n-1}(t_{n-1},t_n;x,y)\,dt_{n-1}'}{1+\frac{1}{4}\mu(t_{n+1}'-t_{n-1}')}\\ &= \frac{1}{2}\sqrt{\frac{\mu}{t_{n+1}'-t_n'}}\int_0^{t_n'} D_{n-1}(t_{n-1},t_n;x,y)\,dt_{n-1}' + O(\mu)\end{aligned} \qquad (76)$$

for $0<n<m$, while the formula for $0<n=m$, written in extended form, becomes

$$
\left.\begin{aligned}
&D_m(t_m,\mu;x,y)\\
&=\left(\frac{\sqrt{\mu}}{2}\right)^m \sqrt{\frac{1+\frac{1}{4}\mu(1-t'_m)}{1-t'_m}} \int_0^{t'_m} \frac{dt'_{m-1}}{1+\frac{1}{4}\mu(1-t'_{m-1})} \sqrt{\frac{1+\frac{1}{4}\mu(t'_m-t'_{m-1})}{t'_m-t'_{m-1}}} \int_0^{t'_{n-1}} \cdots\cdots\\
&\cdots\int_0^{t'_2} \frac{dt'_1}{1+\frac{1}{4}\mu(t'_2-t'_0)} \sqrt{\frac{1+\frac{1}{4}\mu(t'_2-t'_1)}{t'_2-t'_1}} \int_0^{t'_1} \frac{dt'_0}{\left\{1+\frac{1}{4}\mu(t'_2-t'_0)\right\}\sqrt{t'_1-t'_0}\,\sqrt{4\alpha_1+\mu\left\{\alpha_1 t'_1+(4+\beta_1)t'_0+\mu t'_1 t'_0\right\}}}\\
&=\frac{\mu^{\frac{m}{2}}}{2^m\sqrt{1-t'_m}} \int_0^{t'_m} \frac{dt'_{m-1}}{\sqrt{t'_m-t'_{m-1}}} \int_0^{t'_{m-1}} \cdots \int_0^{t'_2} \frac{dt'_1}{\sqrt{t'_2-t'_1}} \int_0^{t'_1} \frac{dt'_0}{\sqrt{t'_1-t'_0}\,\sqrt{4\alpha_1+\mu\left\{\alpha_1 t'_1+(4+\beta_1)t'_0\right\}}}+A
\end{aligned}\right\} \quad (77)
$$

In this, A is $\theta(\mu^{\frac{m}{2}+1})$ if $\alpha \neq 0$, and $\theta(\mu^{\frac{m+1}{2}})$ if $\alpha = 0$, for small values of μ . Substituting

$$
\mu = \frac{\mu_1}{1+2aX_1}
$$

$$
t_m = \mu t'_m = \mu\lambda' = \lambda ,
$$

in accordance with (72) and (74), it is seen that

$$
D_m(\lambda,\mu;x,y) = \theta(\mu_1^{\frac{m}{2}})
$$

for small μ_1 , unless $\alpha_1 = 0$, in which case it is $\theta(\mu_1^{\frac{m-1}{2}})$. Therefore the factor of the exponential in the first term of (70) is $\theta(\mu_1^{\frac{m+3}{2}})$ or $\theta(\mu_1^{\frac{m}{2}+1})$, so that, taking (66) into account, the first term of (70) is $\theta(k^{q-\frac{m+5}{2}})$ for $\alpha_1 \neq 0$, and $\theta(k^{q-\frac{m+4}{2}})$ for $\alpha_1 = 0$. Similarly, using the substitutions

$$
\mu = \frac{\mu_2 x_2 (2a+\mu_2)}{\mu_2 x_2 + 2a}
$$

$$
\lambda = t_m ,
$$

it is seen that the second term is of order $k^{-\frac{1}{2}}$ times that of the first.

Therefore the series occurring in (63) is an asymptotic series for large k, not necessarily convergent, but such that a finite number of terms give an approximation to the solution φ , becoming increasingly more accurate for large k.

3.2.5. The transmission cross section.

As an example of the more detailed application of the results of the preceding section, the transmission cross section for a normally incident plane wave $\varphi_i e^{-ikct} = e^{ik(x-ct)}$ on a soft screen with slit, will now be calculated. The result may then be compared with the Kirchhoff approximations obtained respectively in (8) and (9) of (1.5), viz.

$$
\sigma_k(0) = \begin{cases} 1 & (78a) \\ 1-\frac{1}{2}\,\frac{\sin(2k+\frac{\pi}{4})}{\sqrt{\pi k^3}}+\theta(k^{-\frac{5}{2}}) & (78b). \end{cases}
$$

Using formula (13) of (1.2), we may write

$$\sigma(0)=\mathrm{Re}\sqrt{\frac{\pi i}{2k}}\,A(0) \qquad (79),$$

in which A(0) is the amplitude in the direction of the positive y-axis, being the coefficient of $\frac{e^{ikr_0}}{\sqrt{r_0}}$ in the asymptotic expansion of φ for large $r_0=\sqrt{x^2+y^2}$.

The boundary values are given by

$$\frac{\partial\varphi(\xi,0)}{\partial\eta}=\frac{\partial\varphi_i(\xi,0)}{\partial\eta}=ik, \qquad \text{for } |x|<1.$$

With this substituted in (62) and (63), the first term, (being the Kirchhoff approximation), yields, by using (79), the value 1 for large k, without any powers of k in the development (cf. (78a) above).

Next we consider the second term of (62) and (63), viz.

$$\frac{ik}{\pi}\int_{-1}^{1}d\xi\left\{\int_{r(1)+1-\xi}^{\infty}e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}+\int_{r(-1)+1+\xi}^{\infty}e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}\right\} \qquad (80).$$

Again it is sufficient to treat only the first of these two integrals, since the second follows by replacing x and ξ by -x and $-\xi$ respectively. Substituting, therefore,

$$z=\mu+\{r(1)+1-\xi\}$$

in the first integral of (80), we obtain for the root in the integrand

$$\sqrt{(\mu+r(1)+1-\xi)^2-(x-\xi)^2-y^2}=\sqrt{\mu^2+2\mu(r(1)+1-\xi)+2(1-\xi)(r(1)+1-x)}$$

$$=\sqrt{\mu^2+\mu(\alpha_1+\beta_1+2-2\xi)+2\alpha_1(1-\xi)}$$

in the notation of (73), while the integral itself becomes

$$e^{ik\{r(1)+1\}}\int_{-1}^{1}e^{-ik\xi}d\xi\int_0^{\infty}e^{ik\mu}\frac{d\mu}{\sqrt{\mu^2+\mu(\alpha_1+\beta_1+2-2\xi)+2\alpha_1(1-\xi)}}$$

$$=2e^{ikr(1)}\int_0^1 e^{2ikX}dX\int_0^{\infty}e^{ik\mu}\frac{d\mu}{\sqrt{\mu^2+\mu(\alpha_1+\beta_1+4X)+4\alpha_1X}}$$

by the substitution

$$1-\xi=2X$$

Putting now first $X=\mu X'$ and then $\mu(1+2X')=\mu'$, we obtain, (again assuming, for convenience, that Im k is positive),

$$2e^{ikr(1)}\int_0^{\infty}dX'\int_0^{\frac{1}{X'}}e^{ik\mu(1+2X')}\frac{\mu\,d\mu}{\sqrt{\mu^2+\mu(\alpha_1+\beta_1+4\mu X')+4\alpha_1\mu X'}}$$

$$=2e^{ikr(1)}\int_0^{\infty}\frac{dX'}{1+2X'}\int_0^{2+\frac{1}{X'}}e^{ik\mu'}\frac{\sqrt{\mu'}\,d\mu'}{\sqrt{\mu'(1+4X')+(\alpha_1+\beta_1+4\alpha_1X')(1+2X')}}$$

$$=2e^{ikr(1)}\left\{\int_0^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'\int_0^\infty\frac{dx'}{(1+2X')\sqrt{\mu'(1+4X')+(\alpha_1+\beta_1+4\alpha_1X')(1+2X')}}\right.$$

$$\left.-\int_2^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'\int_{\frac{1}{\mu'-2}}^\infty\frac{dx'}{(1+2X')\sqrt{\mu'(1+4X')+(\alpha_1+\beta_1+4\alpha_1X')(1+2X')}}\right\}$$

$$=2e^{ikr(1)}\int_0^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'\int_0^\infty\frac{dx'}{(1+2X')\sqrt{\mu'(1+4X')+(\alpha_1+\beta_1+4\alpha_1X')(1+2X')}}$$

$$-2e^{ik\{r(1)+2\}}\int_0^\infty e^{ik\mu''}\mu''\sqrt{2+\mu''}\,d\mu''\int_0^1\frac{dx''}{(\mu X''+2)\sqrt{\mu''X''(2+\mu'')(\mu''X''+4)+\{(\alpha_1+\beta_1)\mu''X''+4\alpha_1\}(\mu''X''+2)}}\quad(81),$$

where in the second integral

$$\mu''=\mu'-2\quad\text{and}\quad X''=\frac{1}{\mu''X'}$$

The amplitude A(0), which is needed in (79), applies to values with x=0, i.e. with $\alpha_1-\beta_1=2$, from (73). We therefore put x=0 in (81) and obtain, eliminating β_1,

$$2e^{ik(\alpha_1-1)}\int_0^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'\int_0^\infty\frac{dx'}{(1+2X')\sqrt{2\alpha_1(1+2X')^2+(1+4X')\mu'-2(1+2X')}}$$

$$-2e^{ik(\alpha_1+1)}\int_0^\infty e^{ik\mu''}\mu''\sqrt{2+\mu''}\,d\mu''\int_0^1\frac{dx''}{(\mu''X''+2)\sqrt{2\alpha_1(\mu''X''+2)^2+\mu''X''\{(2+\mu'')(\mu''X''+4)-2(\mu''X''+2)\}}}$$

$$\backsim\sqrt{2}\,\frac{e^{ik(\alpha_1-1)}}{\sqrt{\alpha_1}}\left\{\int_0^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'\int_0^\infty\frac{dx'}{(1+2x')^2}-e^{2ik}\int_0^\infty e^{ik\mu''}\mu''\sqrt{2+\mu''}\,d\mu''\int_0^1\frac{dx''}{(\mu''X''+2)^2}\right\}$$

for large $\alpha_1=1+y\sqrt{1+\frac{1}{y^2}}$, i.e. for large y. Evaluating the inner integrals, this yields

$$\sqrt{2}\,\frac{e^{iky}}{\sqrt{y}}\left\{\frac{1}{2}\int_0^\infty e^{ik\mu'}\sqrt{\mu'}\,d\mu'-\frac{1}{2}e^{2ik}\int_0^\infty e^{ik\mu''}\frac{\mu''}{\sqrt{2+\mu''}}\,d\mu''\right\}$$

$$=\sqrt{2}\,\frac{e^{iky}}{\sqrt{y}}\left\{\frac{e^{\frac{3\pi i}{4}}}{2k^{\frac{3}{2}}}\int_0^\infty e^{-u}u^{\frac{1}{2}}du+\frac{e^{2ik}}{2\sqrt{2}\,k^2}\int_0^\infty e^{-u}u\left(1-\frac{u}{2ik}\right)^{-\frac{1}{2}}du\right\}$$

$$=\frac{e^{iky}}{\sqrt{y}}\left\{\frac{\sqrt{\pi}\,e^{\frac{3\pi i}{4}}}{2\sqrt{2}\,k^{\frac{3}{2}}}+\frac{e^{2ik}}{2k^2}+O(k^{-3})\right\}$$

for large k. Therefore, the two integrals of (80) being identical for x=0, the contribution of the expression (80), viz. of the second term of (63), to the value of A(0) is

$$\frac{2ik}{\pi}\left\{\frac{\sqrt{\pi}\,e^{\frac{3\pi i}{4}}}{2\sqrt{2}\,k^{3/2}}+\frac{e^{2ik}}{2k^2}\right\}+O(k^{-2})$$

$$=-\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi k}}+\frac{e^{i(2k+\frac{\pi}{2})}}{\pi k}+O(k^{-2})$$

Substitution of this in (79) gives the contribution to $\sigma(0)$, viz.

$$Re\left[\frac{\sqrt{\pi}}{2k}e^{i\frac{\pi}{4}}\left\{-\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi k}}+\frac{e^{i(2k+\frac{\pi}{2})}}{\pi k}\right\}\right]+O(k^{-\frac{5}{2}})$$

$$=Re\left[-\frac{i}{2k}+\frac{e^{i(2k+\frac{3\pi}{4})}}{\sqrt{2\pi k^3}}\right]+O(k^{-\frac{5}{2}})$$

$$=-\frac{\sin(2k+\frac{\pi}{4})}{\sqrt{2\pi k^3}}+O(k^{-\frac{5}{2}}) \qquad (82).$$

Now the contribution of the terms under the summation sign in (63) to the transmission cross section $\sigma(0)$ may be determined. Again the parts constituting each term of the sum are identical for x=0 i.e. for $\alpha_1-\beta_1=2$, and the entire first term becomes

$$-\frac{2\sqrt{2}\,ik}{\pi^2}e^{ik(\alpha_1+2)}\int_{-1}^{1}e^{ik\xi}\sqrt{1+\xi}\,d\xi\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{d\lambda}{\{\mu-\lambda+2(1+\xi)\}\sqrt{\mu-\lambda}\sqrt{4\alpha_1+\alpha_1\mu+(2+\alpha_1)\lambda+\mu\lambda}}$$

which corresponds to (66) with $a=h(\xi)=1$ and $m=0$, so that, using (70) and (75), we get

$$-\frac{8ik}{\pi^2}e^{ik(\alpha_1+1)}\int_0^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1\int_0^\infty\frac{X_1^{\frac{1}{2}}dX_1}{(1+2X_1)^{\frac{5}{2}}}\int_0^1\frac{d\lambda_1}{(1-\lambda_1+4X_1)\sqrt{\mu(1-\lambda_1)}\sqrt{4\alpha_1+\mu\{\alpha_1+(2+\alpha_1)\lambda_1+\mu\lambda_1\}}}$$

$$+\frac{8ik}{\pi^2}e^{ik(\alpha_1+3)}\int_0^\infty e^{ik\mu_2}(2+\mu_2)^{\frac{3}{2}}\mu_2^2d\mu_2\int_0^1\frac{X_2dX_2}{(2+\mu_2X_2)^{\frac{5}{2}}}\int_0^1\frac{d\lambda_1}{\{(1-\lambda_1)\mu_2X_2+4\}\sqrt{\mu(1-\lambda_1)}\sqrt{4\alpha_1+\mu\{\alpha_1+(2+\alpha_1)\lambda_1+\mu\lambda_1\}}} \qquad (83)$$

in which is to be put

$$\mu=\frac{\mu_1}{1+2X_1}=\frac{\mu_2X_2(2+\mu_2)}{2+\mu_2X_2}$$

in accordance with (72). For large α_1 we have

$$\left[4\alpha_1+\mu\{\alpha_1+(2+\alpha_1)\lambda_1+\mu\lambda_1\}\right]^{-\frac{1}{2}}\sim\alpha_1^{-\frac{1}{2}}\{4+\mu(1+\lambda_1)\}^{-\frac{1}{2}}\left\{1-\frac{\mu\lambda_1(2+\mu)}{2\alpha_1\{4+\mu(1+\lambda_1)\}}+\ldots\ldots\right\}$$

so that the contribution of (83) to the amplitude A(0) is

$$-\frac{8ik}{\pi^2}e^{2ik}\int_0^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1\int_0^\infty\frac{X_1^{\frac{1}{2}}dX_1}{(1+2X_1)^{\frac{5}{2}}}\int_0^1\frac{d\lambda_1}{(1-\lambda_1+4X_1)\sqrt{\frac{\mu_1(1-\lambda_1)}{1+2X_1}}\sqrt{4+\frac{\mu_1(1+\lambda_1)}{1+2X_1}}}$$

$$+\frac{8ik}{\pi^2}e^{4ik}\int_0^\infty e^{ik\mu_2}(2+\mu_2)^{\frac{3}{2}}\mu_2^2d\mu_2\int_0^1\frac{X_2dX_2}{(2+\mu_2X_2)^{\frac{5}{2}}}\int_0^1\frac{d\lambda_1}{\{4+\mu_2X_2(1-\lambda_1)\}\sqrt{\frac{\mu_2X_2(2+\mu_2)}{2+\mu_2X_2}(1-\lambda_1)}\sqrt{4+\frac{\mu_2X_2(2+\mu_2)}{2+\mu_2X_2}(1+\lambda_1)}} \qquad (84).$$

Considering the first term of this expression, it is seen that the behaviour of the non-exponential part of the integrand near $\mu_1=0$ is represented by

$$\frac{1}{2}\mu_1\int_0^\infty\frac{\sqrt{X_1}dX_1}{(1+2X_1)^2}\int_0^1\frac{d\lambda_1}{(1-\lambda_1+4X_1)\sqrt{1-\lambda_1}}+O(\mu^2) \qquad (85).$$

Evaluating this integral, we get

$$\int_0^\infty\frac{\sqrt{X_1}dX_1}{(1+2X_1)^2}\int_0^1\frac{d\lambda_1}{(1-\lambda_1+4X_1)\sqrt{1-\lambda_1}}=\int_0^\infty\frac{\arctan\frac{1}{2\sqrt{X_1}}}{(1+2X_1)^2}dX_1$$

$$=\left[-\frac{\arctan\frac{1}{2\sqrt{X_1}}}{2(1+2X_1)}\right]_0^\infty-\frac{1}{2}\int_0^\infty\frac{dX_1}{(1+2X_1)(1+4X_1)\sqrt{X_1}}$$

$$= \frac{\pi}{4} - \int_0^\infty \frac{ds}{(1+2s^2)(1+4s^2)}$$

$$= \frac{\pi}{4} + \int_0^\infty \frac{ds}{1+2s^2} - 2\int_0^\infty \frac{ds}{1+4s^2}$$

$$= \frac{\pi}{4} + \frac{\pi}{2\sqrt{2}} - \frac{\pi}{2}$$

$$= \frac{\pi}{4}(\sqrt{2}-1),$$

so that the first term of (84) behaves, for large k, like

$$-\frac{8ik}{\pi^2} e^{2ik} \frac{\pi}{8}(\sqrt{2}-1)\frac{1}{-k^2}\int_0^\infty e^{-u} u\,du + O(k^{-2})$$

$$= -\frac{e^{2ik}}{\pi i k}(\sqrt{2}-1) + O(k^{-2})$$

The non-exponential part of the second term of (84) is, near $\mu_2 = 0$, equal to

$$\left(\frac{\mu_2}{2}\right)^{\frac{3}{2}} \int_0^1 \frac{\sqrt{X_2}\,dX_2}{4} \int_0^1 \frac{d\lambda_1}{8\sqrt{2(1-\lambda_1)}} + O\left(\mu^{+\frac{5}{2}}\right)$$

$$= \frac{1}{8}\mu_2^{\frac{3}{2}} \int_0^1 \sqrt{X_2}\,dX_2 + O\left(\mu^{\frac{5}{2}}\right)$$

$$= \frac{1}{12}\mu_2^{\frac{3}{2}} + O\left(\mu^{\frac{5}{2}}\right),$$

so that the term itself becomes, for large k,

$$\frac{8ik}{\pi^2} e^{4ik} \frac{1}{12} \frac{e^{i\frac{5\pi}{4}}}{k^{\frac{5}{2}}} \int_0^\infty e^{-u} u^{\frac{3}{2}} du + O(k^{-\frac{5}{2}})$$

$$= \frac{2e^{i(4k-\frac{\pi}{4})}}{3\pi^2 k^{3/2}} \quad \frac{3\sqrt{\pi}}{4} + O(k^{-\frac{5}{2}})$$

$$= \frac{e^{i(4k-\frac{\pi}{4})}}{2(\pi k)^{3/2}} + O\left(k^{-\frac{5}{2}}\right).$$

Therefore, by application of (79), the contribution of the term with m=0 in (63) to the transmission cross section is

$$\mathrm{Re}\sqrt{\frac{\pi}{2k}}\, e^{i\frac{\pi}{4}}\left\{-\frac{e^{2ik}}{\pi i k}(\sqrt{2}-1) + \frac{e^{i(4k-\frac{\pi}{4})}}{2(\pi k)^{3/2}}\right\} + O(k^{-3})$$

$$= \mathrm{Re}\left\{-(\sqrt{2}-1)\frac{e^{i(2k-\frac{\pi}{4})}}{\sqrt{2\pi k^3}} + \frac{e^{4ik}}{2\sqrt{2}\pi k^2}\right\} + O(k^{-3})$$

$$= -(\sqrt{2}-1)\frac{\sin(2k+\frac{\pi}{4})}{\sqrt{2\pi k^3}} + \frac{\cos 4k}{2\sqrt{2}\pi k^2} + O(k^{-3}) \qquad (86).$$

The next step is to consider the term with m=1 in (62) and (63). Evidently this is

$$\frac{2\sqrt{2}\,ik}{\pi^3}e^{ik(\alpha_1+4)}\int_{-1}^{1}e^{ik\xi}\sqrt{1+\xi}\,d\xi\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{d\lambda}{\mu-\lambda+2(1+\xi)}\sqrt{\frac{\mu-\lambda+4}{\mu-\lambda}}\;.$$

$$.\int_0^\lambda\frac{dt_0}{(\mu-t_0+4)\sqrt{\lambda-t_0}\sqrt{4\alpha_1+\alpha_1\lambda+(2+\alpha_1)t_0+\lambda t_0}}$$

with contribution to A(0) equal to

$$\frac{2\sqrt{2}\,ik}{\pi^3}e^{5ik}\int_{-1}^{1}e^{ik\xi}\sqrt{1+\xi}\,d\xi\int_0^\infty e^{ik\mu}d\mu\int_0^\mu\frac{d\lambda}{\mu-\lambda+2(1+\xi)}\sqrt{\frac{\mu-\lambda+4}{\mu-\lambda}}\int_0^\lambda\frac{dt_0}{(\mu-t_0+4)\sqrt{\lambda-t_0}\sqrt{4+\lambda+t_0}}$$

$$=\frac{8ik}{\pi^3}e^{4ik}\int_0^\infty e^{ik\mu_1}\mu_1^{\frac{3}{2}}d\mu_1\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{\frac{5}{2}}}\int_0^1\frac{d\lambda_1}{1-\lambda_1+4x_1}\sqrt{\frac{4+\frac{\mu_1}{1+2x_1}(1-\lambda_1)}{\frac{\mu_1}{1+2x_1}(1-\lambda_1)}}\;.$$

$$.\int_0^1\frac{\frac{\mu_1\lambda_1}{1+2x_1}\,dt_0}{\left\{4+\frac{\mu_1}{1+2x_1}(1-\lambda_1t_0')\right\}\sqrt{\frac{\mu_1\lambda_1}{1+2x_1}(1-t_0')}\sqrt{4+\frac{\mu_1\lambda_1}{1+2x_1}(1+t_0')}}+O(k^{-2})\qquad(87),$$

in which the order symbol refers to the second term of (70) for m=1. Again the behaviour of non-exponential part of the integrand near $\mu_1=0$ is required, and this is given by

$$\mu_1^{\frac{3}{2}}\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{\frac{5}{2}}}\int_0^1\frac{2\sqrt{\lambda_1}\,d\lambda_1}{(1-\lambda_1+4x_1)\sqrt{1-\lambda_1}}\int_0^1\frac{dt_0'}{8\sqrt{1-t_0'}}+O(\mu^{\frac{5}{2}})$$

$$=\frac{1}{2}\mu_1^{\frac{3}{2}}\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{\frac{5}{2}}}\int_0^1\frac{d\lambda}{1-\lambda_1+4x_1}\sqrt{\frac{\lambda_1}{1-\lambda_1}}+O(\mu^{\frac{5}{2}})$$

$$=\mu_1^{\frac{3}{2}}\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{5/2}}\int_0^\infty\frac{s^2ds}{(1+s^2)(1+4x_1+4x_1s^2)}+O(\mu^{\frac{5}{2}})$$

$$=\mu_1^{\frac{3}{2}}\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{5/2}}\int_0^\infty\left\{-\frac{1}{1+s^2}+\frac{1+4x_1}{1+4x_1+4x_1s^2}\right\}ds+O(\mu^{\frac{5}{2}})$$

$$=\mu_1^{\frac{3}{2}}\int_0^\infty\frac{\sqrt{x_1}\,dx_1}{(1+2x_1)^{5/2}}\left\{-\frac{\pi}{2}+\frac{\pi}{4}\sqrt{\frac{1+4x_1}{x_1}}\right\}+O(\mu^{\frac{5}{2}})$$

$$=\mu_1^{\frac{3}{2}}\left\{-\frac{\pi}{2}\left[\frac{2}{3}\left(\frac{x_1}{1+2x_1}\right)^{\frac{3}{2}}\right]_0^\infty+\frac{\pi}{4}\left[\frac{1}{3}\left(\frac{1+4x_1}{1+2x_1}\right)^{\frac{3}{2}}\right]_0^\infty\right\}+O(\mu^{\frac{5}{2}})$$

$$=\mu_1^{\frac{3}{2}}\,\frac{\pi}{12}(\sqrt{2}-1)+O(\mu^{\frac{5}{2}})\;.$$

Therefore (87) becomes, for large k,

$$\frac{8ik}{\pi^3}e^{4ik}\,\frac{\pi}{12}(\sqrt{2}-1)\,\frac{e^{i\frac{5\pi}{4}}}{k^{\frac{5}{2}}}\int_0^\infty e^{-u}u^{\frac{3}{2}}du+O(k^{-2})$$

$$=\frac{\sqrt{2}-1}{2(\pi k)^{3/2}}e^{i(4k-\frac{\pi}{4})}+O(k^{-2})$$

which contributes to the transmission cross section an amount

$$\mathrm{Re}\sqrt{\frac{\pi}{2k}}\,e^{i\frac{\pi}{4}}\,\frac{\sqrt{2}-1}{2(\pi k)^{3/2}}\,e^{i(4k-\frac{\pi}{4})}+O(k^{-\frac{5}{2}})$$

$$=\mathrm{Re}\,\frac{\sqrt{2}-1}{2\sqrt{2}\,\pi k^2}\,e^{4ik}+O(k^{-\frac{5}{2}})$$

$$=\left(1-\frac{1}{\sqrt{2}}\right)\frac{\cos 4k}{2\pi k^2}+O(k^{-\frac{5}{2}}) \qquad (88).$$

Finally adding the contributions (78a), (82), (86) and (88), we get

$$\sigma(0)=1-\frac{\sin(2k+\frac{\pi}{4})}{\sqrt{\pi k^3}}+\frac{\cos 4k}{2\pi k^2}+O(k^{-\frac{5}{2}}) \qquad (89).$$

This affords an interesting comparison with the second form of the Kirchhoff approximation, viz. (78b), showing that the form of the term in k is correct, but that its coefficient is only half of the actual value. Thus the conclusion of (1.5) that this term was in error, is confirmed, but, on the other hand, the marked resemblance with the correct value is in accord with the success of Kirchhoff's method in physical optics.

The result (89) is valid asymptotically for large k, and the question naturally arises how small k may be taken to still obtain a reasonable approximation. In this respect the exact results of Skavlem (1951) afford an interesting comparison. In the table below, σ (exact) denotes Skavlem's values, and σ (asymptotic) the values computed from (89):

k	σ (exact)	σ (asymptotic)
1	0.5454	0.6992
2	1.1843	1.1932
3	0.9720	0.9627
4	0.9424	0.9484
5	1.0499	1.0519
6	0.9956	0.9935
7	0.9717	0.9726
8	1.0233	1.0240
9	1.0020	1.0011
10	0.9822	0.9823

Thus, evidently, the range of significance of (89) reaches down to values of k below 10.

Chapter 4.

THE AERODYNAMIC PROBLEM AND THE "SINGULAR" SOLUTION.

4.1. Recapitulation.

The results of the preceding chapter may now be applied to solve the problem of the two-dimensional airfoil with chord ℓ, oscillating in subsonic compressible flow. This problem has been discussed in (1.3) and we shall first briefly recapitulate the main features.

The Lorentz acceleration potential φ was shown to satisfy the differential equation

$$(\Delta+k^2)\varphi=0 \tag{1}$$

with boundary values

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial y}&=\frac{V}{\ell}W'(x)-\frac{ikc}{\ell}W(x) \qquad &&\text{for } y=0\pm,\ |x|<1 \\ \varphi&=0 &&\text{for } y=0,\ |x|>1 \end{aligned}\right\} \tag{2},$$

in which W(x) is defined by

$$W(x)=w(\ell x)e^{ikMx} \qquad \text{for } |x|<1 \tag{3},$$

where the normal velocity of the airfoil due to its prescribed oscillation is, in Galileo-coordinates,

$$w(x')e^{-i\nu t'} \tag{4}.$$

The parameter k was shown to be

$$k=\frac{\ell\nu}{\beta^2 c} \tag{5},$$

which means that it is large for high frequencies and for near-sonic flight speeds.

Further, in (1.3) φ was split into two parts φ^r and φ^s. Of these the "regular" part φ^r satisfies the differential equation (1) and the boundary values (2), together with the edge and radiation conditions, exactly as was the case with φ in the diffraction problem of (3.2). Therefore, as was written in (25) of (1.3),

$$\begin{aligned} \varphi^r(x,y)&=\int_{-1}^{1}\frac{\partial\varphi(\xi,0)}{\partial\eta}G(x,y;\xi,0)\,d\xi \\ &=\int_{-1}^{1}\left\{\frac{V}{\ell}W'(\xi)-\frac{ikc}{\ell}W(\xi)\right\}G(x,y;\xi,0)\,d\xi \end{aligned} \tag{6},$$

by substitution of (2). In this the Green's function G is given for large k by (62) and (63) of (3.1.3). On the other hand, the "singular" part φ^s satisfies (1), while in (2) both conditions are homogeneous, and also φ^s possesses an integrably infinite singularity at the leading edge x=-1, y=0. Thus φ^s was shown (cf (29) of (1.3)) to be the function

$$\varphi^s(x,y)=\frac{\alpha_0 V}{\ell}\frac{\partial}{\partial\xi_1}G(x,y;-1,0) \tag{7}$$

in which

$$\frac{\partial}{\partial\xi_1}=\sqrt{|1+\xi|}\frac{\partial}{\partial\xi} \tag{8}$$

while α_0 is given by (39), (34), (40) and (41) of (1.3), i.e.

$$\alpha_0=-\mu_1-\mu_2\frac{R_2}{R_1} \tag{9}$$

with

$$\left.\begin{aligned}\mu_1&=-\pi\int_{-1}^{1}W(\xi)\frac{\partial}{\partial x_1}G(1,0;\xi,0)\,d\xi\\ \mu_2&=\pi\int_{-1}^{1}W(\xi)\frac{\partial}{\partial x_2}G(1,0;\xi,0)\,d\xi\end{aligned}\right\} \tag{10}$$

$$R_1=\left[\frac{\partial}{\partial y}\int_{\infty}^{1}e^{-\frac{ikx'}{M}}\frac{\partial}{\partial\xi_1}G(x;y;-1,0)\,dx'\right]_{y=0+} \tag{11}$$

$$R_2=\left[\frac{\partial}{\partial y}\int_{\infty}^{1}e^{-\frac{ikx'}{M}}\frac{\partial}{\partial\xi_2}G(x;y;1,0)\,dx'\right]_{y=0+} \tag{12}$$

Here the coordinate ξ_2 is given by

$$\frac{\partial}{\partial\xi_2}=\sqrt{|1-\xi|}\frac{\partial}{\partial\xi_2} \tag{13}$$

while x_1 and x_2 are the fixed coordinates corresponding to ξ_1 and ξ_2 respectively.

Thus it is seen that the function φ, and with it the pressure distribution and airforces on the airfoil, are fully determined from a knowledge of G. Therefore what is to be done in the present chapter is simply an application of the previous work, notably of the formulae (62) and (63) of (3.1.3), from which we may write

$$\begin{aligned}G(x,y;\xi,0)\sim&\frac{1}{2i}H_0^{(1)}\{kr(\xi)\}+\frac{1}{\pi}\left\{\int_{r(1)+1-\xi}^{\infty}e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}+\int_{r(-1)+1+\xi}^{\infty}e^{ikz}\frac{dz}{\sqrt{z^2-r^2(\xi)}}\right\}\\ &-\frac{\sqrt{2}}{\pi^2}e^{2ik}\sum_{m=0}^{\infty}\frac{e^{2ikm}}{(-\pi)^m}\left[e^{ik\{r(1)+(-)^m\xi\}}\sqrt{1+(-1)^m\xi}\int_0^{\infty}e^{ik\mu}\,d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;x,y)\,d\lambda}{\mu-\lambda+2\{1+(-)^m\xi\}}\right.\\ &\left.+e^{ik\{r(-1)+(-)^{m+1}\xi\}}\sqrt{1+(-)^{m+1}\xi}\int_0^{\infty}e^{ik\mu}\,d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;-x,y)\,d\lambda}{\mu-\lambda+2\{1+(-)^{m+1}\xi\}}\right]\end{aligned} \tag{14}$$

in which $r(\xi)=\sqrt{(x-\xi)^2+y^2}$.

Since the determination of φ^r presents no new aspects, we may at once proceed to a few considerations on obtaining φ^s.

4.2. The singular solution.

The main step in determining φ^s is to evaluate

$$\frac{\partial}{\partial\xi_1}G(x,y;-1,0)\equiv\lim_{\xi\to-1}\sqrt{|1+\xi|}\frac{\partial}{\partial\xi}G(x,y;\xi,0)$$

and we shall now sketch this calculation, and show that in the resulting series the asymptotic property is preserved. Since G is defined primarily for $|\xi|<1$, we shall in the sequel always suppose ξ to satisfy $|\xi|<1$, so that the above expression becomes

$$\frac{\partial}{\partial \xi_1} G(x,y;-1,0) \equiv \lim_{\xi \to -1} \sqrt{1+\xi}\, \frac{\partial}{\partial \xi} G(x,y;\xi,0) \qquad (15).$$

Starting with the first term of (14), i.e. with the Kirchhoff kernel, we may write

$$\frac{\partial}{\partial \xi} H_0^{(1)}\{kr(\xi)\} = -H_1^{(1)}\{kr(\xi)\}\, k\, \frac{\partial r(\xi)}{\partial \xi}$$

$$= \frac{x-\xi}{r(\xi)} H_1^{(1)}\{kr(\xi)\}$$

$$\to \frac{x+1}{r(-1)} H_1^{(1)}\{kr(-1)\} \quad \text{when } \xi \to -1,$$

and therefore

$$\lim_{\xi \to -1} \frac{1}{2i} \sqrt{1+\xi}\, \frac{\partial}{\partial \xi} H_0^{(1)}\{kr(\xi)\} = 0$$

which means that from Kirchhoff's approximation no contribution to φ^s is obtained.

We proceed, therefore, to the next term of (14), and get, by partial integration of either of its two integrals,

$$\int_{r(\pm 1)+1\mp\xi}^{\infty} e^{ikx} \frac{dx}{\sqrt{x^2-r^2(\xi)}} = -e^{ik\{r(\pm 1)+1\mp\xi\}} \cosh^{-1}\frac{r(\pm 1)+1\mp\xi}{r(\xi)} - ik \int_{r(\pm 1)+1\mp\xi}^{\infty} e^{ikx} \cosh^{-1}\frac{x}{r(\xi)}\, dx$$

in which again, for convenience, Im k has been assumed to be positive. Therefore, considering the first integral,

$$\frac{\partial}{\partial \xi} \int_{r(1)+1-\xi}^{\infty} e^{ikx} \frac{dx}{\sqrt{x^2-r^2(\xi)}}$$

$$= e^{ik\{r(1)+1-\xi\}} \left[ik \cosh^{-1}\frac{r(1)+1-\xi}{r(\xi)} - \frac{(x-\xi)\{r(1)+1-\xi\} - r^2(\xi)}{r^2(\xi)\sqrt{\{r(1)+1-\xi\}^2 - r^2(\xi)}} \right]$$

$$-ike^{ik\{r(1)+1-\xi\}} \cosh^{-1}\frac{r(1)+1-\xi}{r(\xi)} - \frac{ik(x-\xi)}{r^2(\xi)} \int_{r(1)+1-\xi}^{\infty} e^{ikx} \frac{x\,dx}{\sqrt{x^2-r^2(\xi)}}$$

$$\to \frac{(x+1)\{r(1)+2\} - r^2(-1)}{r^2(-1)\sqrt{\{r(1)+2\}^2 - r^2(-1)}} - \frac{ik(x+1)}{r^2(-1)} \int_{r(1)+2}^{\infty} e^{ikx} \frac{x\,dx}{\sqrt{x^2-r^2(-1)}}$$

if $\xi \to -1$, and finally

$$\lim_{\xi \to -1} \frac{1}{\pi} \sqrt{1+\xi}\, \frac{\partial}{\partial \xi} \int_{r(1)+1-\xi}^{\infty} e^{ikx} \frac{dx}{\sqrt{x^2-r^2(\xi)}} = 0,$$

so that also this integral does not contribute to (15) and therefore to φ^s. Its counterpart, (the second integral of the second term of (14)), does, however, contribute. Differentiation in this case yields

$$\frac{\partial}{\partial\xi}\int_{r(-1)+1+\xi}^{\infty} e^{ikx}\frac{dx}{\sqrt{x^2-r^2(\xi)}}$$

$$=-e^{ik\{r(-1)+1+\xi\}}\frac{(x-\xi)\{r(-1)+1+\xi\}+r^2(\xi)}{r^2(\xi)\sqrt{\{r(-1)+1+\xi\}^2-r^2(\xi)}}-\frac{ik(x-\xi)}{r^2(\xi)}\int_{r(-1)+1+\xi}^{\infty} e^{ikx}\frac{x\,dx}{\sqrt{x^2-r^2(\xi)}} ,$$

and if $\xi \to -1$ the first term of this expression becomes infinite, while the second remains finite. The contribution to (15) is therefore

$$-\lim_{\xi\to -1}\frac{1}{\pi}\sqrt{1+\xi}\; e^{ik\{r(-1)+1+\xi\}}\frac{(x-\xi)\{r(-1)+1+\xi\}+r^2(\xi)}{r^2(\xi)\sqrt{\{r(-1)+1+\xi\}^2-r^2(\xi)}} \qquad (16).$$

In this, we may write

$$\{r(-1)+1+\xi\}^2-r^2(\xi)=2(1+\xi)\{r(-1)+x+1\}$$

so that (16) becomes

$$-\lim_{\xi\to -1}\frac{1}{\pi}\sqrt{1+\xi}\; e^{ikr(-1)}\frac{r(-1)+x+1}{r(-1)\sqrt{2(1+\xi)\{r(-1)+x+1\}}}$$

$$=-\frac{1}{\sqrt{2}\,\pi}\frac{\sqrt{r(-1)+x+1}}{r(-1)}\,e^{ikr(-1)} \qquad (17).$$

As in (73) of (3.2.4) we introduce the notation

$$\begin{aligned} r(1)+(1-x) &= \alpha_1\\ r(1)-(1-x) &= \beta_1\\ r(-1)+(1+x) &= \alpha_2\\ r(-1)-(1+x) &= \beta_2 \end{aligned} \qquad (18)$$

so that (17) may be written as

$$-\frac{\sqrt{2}}{\pi}\frac{\sqrt{\alpha_2}}{\alpha_2+\beta_2}\,e^{ikr(-1)} \qquad (17a).$$

Turning now to the general term in (14), we have to evaluate expressions of the form

$$\lim_{\xi\to -1}\sqrt{1\pm\xi}\,\frac{\partial}{\partial\xi}\left\{e^{\pm ik\xi}\sqrt{1\pm\xi}\int_0^{\infty}e^{ik\mu}d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+2(1\pm\xi)}d\lambda\right\} ,$$

but those in which ξ occurs with the negative sign are evidently zero. The terms which are to be investigated therefore contain expressions of the form

$$\lim_{\xi\to -1}\sqrt{1+\xi}\,\frac{\partial}{\partial\xi}\left\{e^{ik\xi}\sqrt{1+\xi}\int_0^{\infty}e^{ik\mu}d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+2(1+\xi)}d\lambda\right\}$$

$$=\lim_{\xi\to -1}e^{ik\xi}\sqrt{1+\xi}\left(ik+\frac{\partial}{\partial\xi}\right)\left[\sqrt{1+\xi}\int_0^{\infty}e^{ik\mu}d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+2(1+\xi)}d\lambda\right]$$

$$=\frac{1}{2}e^{-ik}\lim_{\xi_1\to 0}\left(\frac{1}{2}ik\xi_1+\frac{\partial}{\partial\xi_1}\right)\left[\xi_1\int_0^{\infty}e^{ik\mu}d\mu\int_0^{\mu}\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+\frac{1}{2}\xi_1^2}d\lambda\right] \qquad (19),$$

using the coordinate ξ_1 of (8). The passage to the limit in (19) may not be performed under the inner sign of integration, since the denominator in the integrand causes the integral

$$\int_0^{\mu} \frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda}\, d\lambda \tag{20}$$

to diverge at its upper limit, $\lambda=\mu$. Moreover, the function occurring in the numerator has a factor $(\mu-\lambda)^{-\frac{1}{2}}$, as is apparent from the recursion relations (57) of (3.2.3) defining D_m, viz.

$$\left.\begin{aligned}
D_0(t_0,t_1;x,y) &= \frac{1}{\sqrt{t_1-t_0}\sqrt{4\alpha_1+\alpha_1 t_1+(\beta_1+4)t_0+t_1 t_0}}\\
D_0(t_0,t_1;-x,y) &= \frac{1}{\sqrt{t_1-t_0}\sqrt{4\alpha_2+\alpha_2 t_1+(\beta_2+4)t_0+t_1 t_0}}\\
D_n(t_n,t_{n+1};\pm x,y) &= \sqrt{\frac{t_{n+1}-t_n+4}{t_{n+1}-t_n}}\int_0^{t_n} \frac{D_{n-1}(t_{n-1},t_n;\pm x,y)}{t_{n+1}-t_{n-1}+4}\, dt_{n-1}
\end{aligned}\right\} \tag{21}$$

We have to determine the behaviour of the functions $D_m(\lambda,\mu;\pm x,y)$ when λ tends to μ , and therefore put

$$\lambda=\mu\,(1-s) \tag{22}.$$

Writing $D_m(\lambda,\mu)$ to indicate either $D_m(\lambda,\mu;+x,y)$ or $D_m(\lambda,\mu;-x,y)$, and α , β to denote either α_1 , β_1 or α_2 , β_2 , and supposing α to be non-zero, we obtain

$$\begin{aligned}
D_0\{\mu(1-s),\mu\} &= \frac{1}{\sqrt{\mu s}\sqrt{4\alpha+\mu\{\alpha+(\beta+4)(1-s)+\mu(1-s)\}}}\\
&= \frac{1}{\sqrt{\mu s}\sqrt{4\alpha+(\alpha+\beta+4)\mu+\mu^2-\mu s\,(\beta+4+\mu)}}\\
&= \frac{1}{\sqrt{\mu s}}\left\{\frac{1}{\sqrt{4\alpha+(\alpha+\beta+4)\mu+\mu^2}}+O(\mu s)\right\}
\end{aligned} \tag{23}$$

for small s or μ , and

$$\begin{aligned}
D_1\{\mu(1-s),\mu\} &= \sqrt{\frac{4+\mu s}{\mu s}}\int_0^{\mu(1-s)} \frac{D_0\{t_0,\mu(1-s)\}}{\mu-t_0+4}\, dt_0\\
&= \sqrt{\frac{4+\mu s}{\mu s}}\int_0^{\mu(1-s)} \frac{dt_0}{(\mu-t_0+4)\sqrt{\mu(1-s)-t_0}\sqrt{4\alpha+\alpha\mu(1-s)+(\beta+4)t_0+t_0\mu(1-s)}}
\end{aligned}$$

Substituting in this

$$t_0=\mu\,(1-s)\,t_0'$$

yields

$$\begin{aligned}
D_1\{\mu(1-s),\mu\} &= \sqrt{\frac{4+\mu s}{\mu s}}\int_0^1 \frac{\mu(1-s)\,dt_0'}{\{4+\mu-\mu(1-s)t_0'\}\sqrt{\mu(1-s)(1-t_0)}\sqrt{4\alpha+\mu(1-s)\{\alpha+(\beta+4)t_0'+\mu(1-s)t_0'\}}}\\
&= \sqrt{\frac{(4+\mu s)(1-s)}{s}}\int_0^1 \frac{dt_0'}{\{4+\mu(1-t_0')+\mu s t_0'\}\sqrt{1-t_0'}\sqrt{4\alpha+\mu\{\alpha+(\beta+4)t_0'+\mu t_0'\}-\mu s\{\alpha+(\beta+4)t_0'+\mu(2-s)t_0'\}}}\\
&= 2\sqrt{\frac{1-s}{s}}\left[\int_0^1 \frac{dt_0'}{\{4+\mu(1-t_0')\}\sqrt{1-t_0'}\sqrt{4\alpha+\mu\{\alpha+(\beta+4)t_0'+\mu t_0'\}}}+O(\mu s)\right]
\end{aligned} \tag{24}.$$

Generally, for m > 1, we obtain

$$D_m\{\mu(1-s),\mu\} = \sqrt{\frac{4+\mu s}{\mu s}} \int_0^{\mu(1-s)} \frac{dt_{m-1}}{4+\mu-t_{m-1}} \; \frac{4+\mu(1-s)-t_{m-1}}{\mu(1-s)-t_{m-1}} \int_0^{t_{m-1}} \frac{D_{m-2}(t_{m-2},t_{m-1})}{4+\mu(1-s)-t_{m-2}} dt_{m-2}$$

which, by the substitutions

$$t_n = \mu(1-s)\,t'_n \tag{25}$$

becomes, when written in extended form,

$$D_m\{\mu(1-s),\mu\} = \sqrt{\frac{4+\mu s}{\mu s}}\, \mu^{\frac{m}{2}} (1-s)^{\frac{m}{2}} \int_0^1 \frac{dt'_{m-1}}{4+\mu-\mu(1-s)t'_{m-1}} \sqrt{\frac{4+\mu(1-s)(1-t'_{m-1})}{1-t'_{m-1}}}$$

$$\cdot \int_0^{t'_{m-1}} \frac{dt'_{m-2}}{4+\mu(1-s)(1-t'_{m-2})} \sqrt{\frac{4+\mu(1-s)(t'_{m-1}-t'_{m-2})}{t'_{m-1}-t'_{m-2}}} \int_0^{t'_{m-2}}$$

$$\cdots\cdots\cdots \int_0^{t'_2} \frac{dt'_1}{4+\mu(1-s)(t'_3-t'_1)} \sqrt{\frac{4+\mu(1-s)(t'_2-t'_1)}{t'_2-t'_1}}$$

$$\cdot \int_0^{t'_1} \frac{dt'_0}{\{4+\mu(1-s)(t'_2-t'_0)\}\sqrt{t'_1-t'_0}\sqrt{4\alpha+\mu(1-s)\{\alpha t'_1+(\beta+4)t'_0+\mu(1-s)t'_0 t'_1\}}}$$

$$= \frac{2\mu^{\frac{m-1}{2}}(1-s)^{\frac{m}{2}}}{\sqrt{s}} \Bigg\{ \int_0^1 \frac{dt'_{m-1}}{4+\mu(1-t'_{m-1})} \sqrt{\frac{4+\mu(1-t'_{m-1})}{1-t'_{m-1}}} \int_0^{t_{m-1}} \cdots \int_0^{t'_2} \frac{dt'_1}{4+\mu(t'_3-t'_1)} \sqrt{\frac{4+\mu(t'_2-t'_1)}{t'_2-t'_1}}$$

$$\int_0^{t'_1} \frac{dt'_0}{\{4+\mu(t'_2-t'_1)\}\sqrt{t'_1-t'_0}\sqrt{4\alpha+\mu\{\alpha t'_1+(\beta+4)t'_0+\mu t'_0 t'_1\}}} + O(\mu s) \Bigg] \tag{26}$$

The results (23), (24) and (26) may all be written in the form

$$D_m\{\mu(1-s),\mu\} = s^{-\frac{1}{2}} \mu^{\frac{m-1}{2}} F_m\{\mu(1-s),\mu\} \tag{27}$$

with

$$F_m\{\mu(1-s),\mu\} = f_0(\mu) + s f_1(\mu) + s^2 f_2(s,\mu) \tag{28}$$

where f_0, f_1 and f_2 are $O(1)$ if either s or μ tends to zero, excepting for m=0, in which case f_1 and f_2 are $O(\mu)$ for small μ.

Returning now to the expression (19), the inner integral takes the form

$$\int_0^{\mu} \frac{D_m(\lambda,\mu)\,d\lambda}{\mu-\lambda+\frac{1}{2}\xi_1^2} = \mu \int_0^1 \frac{D_m\{\mu(1-s),\mu\}}{(\mu s+\frac{1}{2}\xi_1^2)^2} ds$$

$$= \mu^{\frac{m+1}{2}} \left[f_0 \int_0^1 \frac{ds}{(\mu s+\frac{1}{2}\xi_1^2)\sqrt{s}} + f_1 \int_0^1 \frac{\sqrt{s}\,ds}{\mu s+\frac{1}{2}\xi_1^2} + \int_0^1 \frac{s^{3/2} f_2\,ds}{\mu s+\frac{1}{2}\xi_1^2} \right]$$

$$= \mu^{\frac{m+1}{2}} \left[f_0 \frac{2\sqrt{2}}{\xi_1\sqrt{\mu}} \arctan\frac{\sqrt{2\mu}}{\xi_1} + f_1 \left\{ \frac{2}{\mu} - \frac{\sqrt{2}\,\xi_1}{\mu^{3/2}} \arctan\frac{\sqrt{2\mu}}{\xi_1} \right\} + \int_0^1 \frac{s^{\frac{1}{2}} f_2\,ds}{\mu s+\frac{1}{2}\xi_1^2} \right]$$

$$= \frac{2\sqrt{2}\,\mu^{\frac{m}{2}} f_0}{\xi_1} \arctan\frac{\sqrt{2\mu}}{\xi_1} + \mu^{\frac{m-1}{2}} \int_0^1 \frac{f_1+s f_2}{\sqrt{s}} ds + \mu^{\frac{m}{2}-1} O(\xi_1) \tag{29}$$

for small values of ξ_1 , for $m \geqslant 1$, while for m=0 the factor of the order-term is 1, and not μ^{-1} . The ξ_1 - derivative is of the form

$$\frac{\partial}{\partial \xi_1}\int_0^{\mu} \frac{D_m(\lambda,\mu)d\lambda}{\mu-\lambda+\frac{1}{2}\xi_1^2} = -\xi_1\mu\int_0^1 \frac{D_m\{\mu(1-s),\mu\}}{(\mu s+\frac{1}{2}\xi_1^2)^2}ds$$

$$= -\xi_1\mu^{\frac{m+1}{2}}\left[f_0\int_0^1\frac{ds}{(\mu s+\frac{1}{2}\xi_1)^2\sqrt{s}} + f_1\int_0^1\frac{\sqrt{s}\,ds}{(\mu s+\frac{1}{2}\xi_1^2)^2} + \int_0^1\frac{s^{3/2}f_2\,ds}{(\mu s+\frac{1}{2}\xi_1^2)^2}\right]$$

$$= -\xi_1\mu^{\frac{m+1}{2}}\left[f_0\left\{\frac{2\sqrt{2}}{\xi_1^3\sqrt{\mu}}\arctan\frac{\sqrt{2\mu}}{\xi_1} + \frac{2}{\xi_1^2(\mu+\frac{1}{2}\xi_1^2)}\right\}\right.$$

$$\left. + f_1\left\{-\frac{\sqrt{2}}{\xi_1\mu^{3/2}}\arctan\frac{\sqrt{2\mu}}{\xi_1} + \frac{1}{\mu(\mu+\frac{1}{2}\xi_1^2)}\right\} + \int_0^1\frac{s^{3/2}f_2\,ds}{(\mu s+\frac{1}{2}\xi_1^2)^2}\right]$$

$$= -f_0\left\{\frac{2\sqrt{2}\mu^{\frac{m}{2}}}{\xi_1^2}\arctan\frac{\sqrt{2\mu}}{\xi_1} + \frac{2\mu^{\frac{m-1}{2}}}{\xi_1}\right\} + \mu^{\frac{m}{2}-1}.O(1) \qquad (30),$$

for small values of ξ_1 , if $m \geqslant 1$, while again for m=0 the factor of the order-term is 1.

Therefore we may write

$$\left(\frac{1}{2}ik\xi_1 + \frac{\partial}{\partial\xi_1}\right)\left[\xi_1\int_0^{\mu}\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+\frac{1}{2}\xi_1^2}d\lambda\right]$$

$$= \frac{\partial}{\partial\xi_1}\left[\xi_1\int_0^1\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+\frac{1}{2}\xi_1^2}d\lambda\right] + \mu^{\frac{m}{2}}\,O(\xi_1)$$

$$= \int_0^1\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+\frac{1}{2}\xi_1^2}d\lambda + \xi_1\frac{\partial}{\partial\xi_1}\int_0^1\frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda+\frac{1}{2}\xi_1^2}d\lambda + \mu^{\frac{m}{2}}\,O(\xi_1),$$

which, by using (29) and (30), becomes

$$\left\{\frac{2\sqrt{2}\mu^{\frac{m}{2}}f_0}{\xi_1}\arctan\frac{\sqrt{2\mu}}{\xi_1} + \mu^{\frac{m-1}{2}}\int_0^1\frac{f_1+sf_2}{\sqrt{s}}ds\right\}$$

$$-\left\{\frac{2\sqrt{2}\mu^{\frac{m}{2}}f_2}{\xi_1}\arctan\frac{\sqrt{2\mu}}{\xi} + 2\mu^{\frac{m-1}{2}}f_0\right\} + \mu^{\frac{m}{2}-1}\,O(\xi_1)$$

$$= \mu^{\frac{m-1}{2}}\left\{-2f_0 + \int_0^1\frac{f_1+sf_2}{\sqrt{s}}ds\right\} + \mu^{\frac{m}{2}-1}\,O(\xi_1) \qquad (31).$$

The same qualification about the order-term as previously, continues to hold here and in the sequel. By using (28) and (22), this expression may be written as

$$\mu^{\frac{m-1}{2}}\left[-2F_m(\mu,\mu) + \int_0^1\frac{ds}{s^{3/2}}\left\{F_m\{\mu(1-s),\mu\} - F_m(\mu,\mu)\right\}\right] + \mu^{\frac{m}{2}-1}\,O(\xi_1)$$

$$= \mu^{\frac{m-1}{2}}\left[-2F_m(\mu,\mu) + \sqrt{\mu}\int_0^{\mu}\frac{d\lambda}{(\mu-\lambda)^{3/2}}\left\{F_m(\lambda,\mu) - F_m(\mu,\mu)\right\}\right] + \mu^{\frac{m}{2}-1}\,O(\xi_1) \qquad (32),$$

which, by using (27) in the form

$$D_m(\lambda,\mu) = \frac{\mu^{\frac{m}{2}}}{\sqrt{\mu-\lambda}}\, F_m(\lambda,\mu) \;,$$

becomes

$$\lim_{\lambda_1\to\mu}\left[-2\sqrt{\frac{\mu-\lambda_1}{\mu}}\, D_m(\lambda_1,\mu) + \int_0^{\mu} \frac{D_m(\lambda,\mu) - \sqrt{\frac{\mu-\lambda_1}{\mu-\lambda}}\, D_m(\lambda_1,\mu)}{\mu-\lambda}\, d\lambda\right] + \mu^{\frac{m}{2}-1}\, \theta(\xi_1) \qquad (33).$$

The integral occurring here is identical with Hadamard's "finite part" of the divergent integral (20), and we may for convenience write (33) symbolically as

$$-\frac{2}{\sqrt{\mu}}\left[\sqrt{\mu-\lambda}\, D_m(\lambda,\mu)\right]_{\lambda=\mu} + \not\int_0^{\mu} \frac{D_m(\lambda,\mu)}{\mu-\lambda}\, d\lambda + \mu^{\frac{m}{2}-1}\, \theta(\xi_1) \qquad (34).$$

Remembering that this expression represents

$$\left(\frac{1}{2} ik\xi_1 + \frac{\partial}{\partial \xi_1}\right)\left[\xi_1 \int_0^{\mu} \frac{D_m(\lambda,\mu)}{\mu-\lambda+\frac{1}{2}\xi_1^2}\, d\lambda\right] \;,$$

and substituting in (19), we get

$$\frac{1}{2} e^{-ik} \int_0^{\infty} e^{ik\mu}\, d\mu \left\{-\frac{2}{\sqrt{\mu}}\left[\sqrt{\mu-\lambda}\, D_m(\lambda,\mu)\right]_{\lambda=\mu} + \not\int_0^{\mu} \frac{D_m(\lambda,\mu)}{\mu-\lambda}\, d\lambda\right\} \;,$$

so that the general term of (14) contributes to the function

$$\frac{\partial}{\partial \xi_1} G(x,y;-1,0)$$

an amount

$$-\frac{\sqrt{2}}{\pi^2} e^{3ik} \frac{e^{2ikm}}{(-\pi)^m} e^{ikr(\pm 1)} \frac{1}{2} e^{ik} \int_0^{\infty} e^{ik\mu}\, d\mu \left\{-\frac{2}{\sqrt{\mu}}\left[\sqrt{\mu-\lambda}\, D_m(\lambda,\mu;\pm x,y)\right]_{\lambda=\mu} + \not\int_0^{\mu} \frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda}\, d\lambda\right\}$$

$$= -\frac{1}{\sqrt{2}\,(-\pi)^{m+2}} e^{ik\{r(\pm 1)+2(m+1)\}} \int_0^{\infty} e^{ik\mu}\, d\mu \left\{-\frac{2}{\mu}\left[\sqrt{\mu-\lambda}\, D_m(\lambda,\mu;\pm x,y)\right]_{\lambda=\mu} + \not\int_0^{\mu} \frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda}\, d\lambda\right\} \qquad (35),$$

in which $\pm$ is to be interpreted as $(-)^m$. We may therefore write formally, using (17a) and (35),

$$\frac{\partial}{\partial \xi_1} G(x,y;-1,0) \doteq -\frac{\sqrt{2\alpha_2}}{\pi(\alpha_2+\beta_2)} e^{ikr(-1)} - \frac{1}{\sqrt{2}\,\pi^2} e^{2ik} \sum_{m=0}^{\infty} e^{ik\{r(\pm 1)+2m\}} \;.$$

$$\cdot \int_0^{\infty} e^{ik\mu}\, d\mu \left\{-\frac{2}{\sqrt{\mu}}\left[\sqrt{\mu-\lambda}\, D_m(\lambda,\mu;\pm x,y)\right]_{\lambda=\mu} + \not\int_0^{\mu} \frac{D_m(\lambda,\mu;\pm x,y)}{\mu-\lambda}\, d\lambda\right\} \qquad (36).$$

The asymptotic character of the series thus formed, follows as for the "regular" solution, by noting from (31) that the non-exponential part of the integrand in (35) is $\theta(\mu^{\frac{m-1}{2}})$ for small μ, so that the whole expression (35) is $\theta(k^{-\frac{m+1}{2}})$ for large k.

We shall now calculate explicitly the term for m=0. From (21) we may write

$$D_0(\lambda,\mu;x,y) = \frac{1}{\sqrt{\mu-\lambda}\sqrt{4\alpha_1+\alpha_1\mu+(\beta_1+4)\lambda+\mu\lambda}} \;,$$

so that (33) becomes

$$- \frac{2}{\sqrt{\mu}\sqrt{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2}}$$

$$+ \int_0^\mu \frac{d\lambda}{(\mu-\lambda)^{3/2}} \left\{ \frac{1}{\sqrt{4\alpha_1+\alpha_1\mu+(\beta_1+4)\lambda+\mu\lambda}} - \frac{1}{\sqrt{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2}} \right\} \qquad (37).$$

Evaluation of the indefinite integrals gives

$$\int \frac{d\lambda}{(\mu-\lambda)^{3/2}\sqrt{\alpha_1(4+\mu)+(\beta_1+4+\mu)\lambda}} = \frac{2}{\alpha_1(4+\mu)+(\beta_1+4+\mu)\mu}\sqrt{\frac{\alpha_1(4+\mu)+(\beta_1+4+\mu)\lambda}{\mu-\lambda}}$$

and

$$\int \frac{d\lambda}{(\mu-\lambda)^{3/2}} = \frac{2}{\sqrt{\mu-\lambda}} ,$$

so that the definite integral in (37) becomes

$$\frac{2}{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2}\left[\frac{1}{\sqrt{\mu-\lambda}}\left\{\sqrt{\alpha_1(4+\mu)+(\beta_1+4+\mu)\lambda} - \sqrt{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2}\right\}\right]_0^\mu$$

$$= - \frac{2}{\sqrt{\mu}\{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2\}}\left\{\sqrt{\alpha_1(4+\mu)} - \sqrt{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2}\right\} .$$

Therefore (37) is equal to

$$- \frac{2\sqrt{\alpha_1(4+\mu)}}{\sqrt{\mu}\{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2\}} ,$$

so that (35) becomes

$$\frac{\sqrt{2\alpha_1}}{\pi^2} e^{ik\{r(1)+2\}} \int_0^\infty e^{ik\mu} \frac{\sqrt{4+\mu}\, d\mu}{\sqrt{\mu}\{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2\}} \qquad (38).$$

We may therefore write (36) as

$$\frac{\partial}{\partial \xi_1} G(x,y;-1,0) \sim - \frac{\sqrt{2\alpha_2}}{\pi(\alpha_2+\beta_2)} e^{ikr(-1)} + \frac{\sqrt{2\alpha_1}}{\pi^2} e^{ik\{r(1)+2\}} \int_0^\infty e^{ik\mu} \frac{\sqrt{4+\mu}\, d\mu}{\sqrt{\mu}\{4\alpha_1+(\alpha_1+\beta_1+4)\mu+\mu^2\}}$$

$$+ O(k^{-1}) \qquad (39).$$

This function may now also be used to determine α_0 in (7), thus completing the "singular" solution. From (9) it is evident that for this purpose the function

$$\frac{\partial}{\partial \xi_2} G(x,y;1,0) \qquad (40)$$

occurring in (12), as well as

$$\frac{\partial}{\partial x_1} G(-1,0;\xi,0) \quad \text{and} \quad \frac{\partial}{\partial x_2} G(1,0;\xi,0) \qquad (41)$$

in (10), still have to be determined.

Evidently the function (40) is the symmetrical counterpart with respect to x of the function just determined, thus

$$\frac{\partial}{\partial \xi_2} G(x,y;1,0) \sim - \frac{\sqrt{2\alpha_1}}{\pi(\alpha_1+\beta_1)} e^{ikr(1)} + \frac{\sqrt{2\alpha_2}}{\pi^2} e^{ik\{r(-1)+2\}} \int_0^\infty e^{ik\mu} \frac{\sqrt{4+\mu}\, d\mu}{\sqrt{\mu}\{4\alpha_2+(\alpha_2+\beta_2+4)\mu+\mu^2\}} + \dots \qquad (42).$$

Special care should, however, be taken here, since the above results have been deduced for $\alpha \neq 0$, while in (11) and (12) the values are needed for y=0 and x< -1, which means, from (18), that

$$\alpha_2 = \beta_1 = 0 .$$

Further, the functions (41) may be directly found from the previous results, by making use of the symmetry of the Green's function in the points (x,y) and (ξ, η). Thus, these two functions are found respectively from

$$\frac{\partial}{\partial \xi_1} G(x,0;-1,0) \quad \text{and} \quad \frac{\partial}{\partial \xi_2} G(x,0;1,0)$$

by replacing x by ξ . In this case the integration is over values for which $\eta = 0$ and $|\xi| < 1$, which corresponds to

$$\beta_1 = \beta_2 = 0$$

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