A consistent first approximation in the general theory of thin elastic shells

Part 1, Foundations and linear theory

by

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PREFACE.

This report was intended for presentation at the I.U.T.A.M. Symposium on the Theory of Thin Elastic Shells (Delft, 24th to 28th August, 1959). However, when the report was completed in its present form - omitting an extensive discussion of nonlinear theory which was originally also envisaged to be included - it turned out to be too long for presentation at the symposium, and more in particular far too long for inclusion in the Proceedings. The author therefore decided to prepare, under the same title, a separate paper for presentation at the symposium, containing only the most important results of the analysis and the basic assumptions and arguments in their derivation. Since the symposium paper frequently refers to the present report for substantiation, this report is also issued to participants in the symposium. It is hoped to publish the present detailed report at some future date, if possible in a more complete form by addition of a full discussion of nonlinear theory.

Delft, 5th August 1959.

1. INTRODUCTION.

The general equations of the theory of thin elastic shells have often been discussed in recent years. The amount of literature on this subject is now nearly overwhelming, and the list of references at the end of this report quotes only some, in our opinion, more important papers; a comprehensive bibliography on shell theory has been compiled by NASH [33].

Most authors derive their equations on the basis of the well-known LOVE-KIRCHHOFF assumptions:

(a) points which lie on one and the same normal to the undeformed middle surface also lie on one and the same normal to the deformed middle surface;

(b) the effect of the normal stress on surfaces parallel to the middle surface may be neglected in the stress-strain relations;

(c) the displacements in the direction of the normal to the middle surface are approximately equal for all points on the same normal.

The wide variety of resulting equations, to be found in the literature, is due to variations in rigor and to different approximations in the subsequent analysis. Some authors claim a higher accuracy for their equations, as compared to those of other writers, on the grounds of a more rigorous derivation from the basic assumptions. In view of the approximative character of the LOVE-KIRCHHOFF assumptions these claims are open to some doubt. It may well happen, and this is even actually so in most cases, that the refinements made are of the same order of magnitude as the errors which remain on account of the basic assumptions, and such refinements are of course meaningless in a general theory.

It is the primary purpose of the present paper to give the foundations for shell theory in as simple a form as is consistent with the basic assumptions; although large deflections are permitted, it will be assumed that the strains are small everywhere. Our purpose is achieved by a strain-energy approach (section 2). It is shown that LOVE's so-called first approximation for the strain energy, as the sum of stretching or extensional energy and bending or flexural energy, is a consistent first approximation, and that no refinement of this first approximation is justified, in general, if the basic LOVE-KIRCHHOFF assumptions (or equivalent assumptions) are retained. This fact is proved by showing that the errors of LOVE's first approximation, as compared to a rigorous elaboration of the LOVE-KIRCHHOFF assumptions, are of the same order of magnitude as the strain energy due to transverse normal and shear stresses, which are inherently neglected if the LOVE-KIRCHHOFF assumptions are employed. Moreover, it is shown that it is always permissible, in LOVE's first approximation, to add terms of type ε/R (where ε is any physical middle surface strain component and R any radius of curvature or torsion of the middle surface) to expressions for the physical changes of curvature. In other words, expressions for the changes of curvature, which differ only by terms of type ε/R, are equivalent in LOVE's consistent first approximation.

The foundations of section 2 are applied to the linear theory of shells in section 3. The expressions for the extensional strains and rotations are well-known. Our expressions for the changes of curvature, in tensor form initially defined as the covariant derivative of the strain tensor in surfaces parallel to the middle surface with respect to the coordinate normal to the middle surface, appear to be new and more graphic than existing expressions. The stress-strain relations have a particularly simple form, and the equations of equilibrium and boundary conditions, obtained by the variational method, agree with the well-known rigorous equations. A comparison with existing theories shows that most writers have obtained results which, although different in appearance, are equivalent from the point of view of the first approximation in shell theory. On the other hand, some expressions for the changes of curvature given in the literature are shown to be inadequate for general application.

The advantages of tensor analysis in the theory of shells are now widely recognized [e.g. 5,12,14,26,34, 36,42,48], and no apology is needed for its free use in the present paper. An explanation of the notations employed, mostly taken from [12], is given in the appendix, together with some more involved geometric derivations. The main results have also been given in the more usual notation, in order to facilitate comparison with existing literature; the translation rules from general tensor notation to the more conventional notation for general orthogonal parametric curves on the middle surface are also summarized in the appendix.

2. FOUNDATIONS: LOVE's APPROXIMATE STRAIN-ENERGY EXPRESSION.

2.1 Basic assumptions.

A complete theory of thin elastic shells in a consistent first approximation, and valid for deflections of any magnitude, may be based on LOVE's approximate strain-energy expression. This strain-energy expression will be derived on the basis of three assumptions:

(a) the shell is thin, i.e. $h/R \ll 1$, where h is the shell thickness, and R is the smallest principal radius of curvature of the middle surface;

(b) the strains are small everywhere, although large deflections are admitted, and the strain energy per unit volume of the undeformed body is represented by the quadratic function of the strain components for an isotropic solid (HOOKE's law);

(c) the state of stress is approximately plane, i.e. the effect of transverse shear stresses and of the transverse normal stress may be neglected.

It will be observed that we have replaced the usual LOVE-KIRCHHOFF assumptions ((a) to (c) in section 1) by the single assumption of plane stress (c) above, which is of course effectively equivalent to the usual assumptions.

2.2 Derivation of strain-energy expression.

On the basis of assumptions (b) and (c) of par. 2.1 the strain energy per unit volume of the undeformed body is given by [12, p.384]

$$\Phi = \tfrac{1}{2}E^{\alpha\beta\mu\nu}\gamma_{\alpha\beta}\gamma_{\mu\nu} \,, \tag{2.1}$$

where $\gamma_{\alpha\beta}$ is the covariant strain tensor in surfaces parallel to the middle surface, $E^{\alpha\beta\mu\nu}$ is the contravariant tensor of elastic moduli, defined by

$$E^{\alpha\beta\mu\nu} = 2G\left[g^{\alpha\mu}g^{\beta\nu} + \frac{\nu}{1-\nu}\, g^{\alpha\beta}g^{\mu\nu}\right], \tag{2.2}$$

$g^{\alpha\beta}$ is the contravariant metric tensor, G is the shear modulus, and ν is POISSON's ratio *). In our approximation of plane stress the transverse shear strains are zero

$$\gamma_{\alpha 3} = 0 \,, \tag{2.3}$$

and the tranverse normal strain is given by

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*) The tensor notation employed here is fully explained in [12] and in the appendix.

$$\gamma_{33} = -\frac{\nu}{1-\nu} g^{\alpha\beta}\gamma_{\alpha\beta} \text{ .} \tag{2.4}$$

Multiplying (2.1) by (cf. par. A1)

$$\sqrt{g/a} = 1 - 2Hz + Kz^2 \text{ ,} \tag{2.5}$$

where H and K are the mean and GAUSSIAN curvatures of the middle surface, and z is the distance to the middle surface, the strain energy per unit area of the undeformed middle surface is obtained by integrating with respect to z

$$V = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \left[1 - 2Hz + Kz^2\right] \Phi dz \text{ .} \tag{2.6}$$

The TAYLOR-expansion of the energy density Φ with respect to the coordinate $x^3 = z$ may be written in the form (cf. par. A2)

$$\begin{aligned} \Phi(x^\lambda, z) &= \Phi(x^\lambda, 0) + z\Phi_{,3}(x^\lambda, 0) + \tfrac{1}{2}z^2\Phi_{,33}(x^\lambda, 0) + \ldots \\ &= \Phi(x^\lambda, 0) + z\Phi_{\|3}(x^\lambda, 0) + \tfrac{1}{2}z^2\Phi_{\|33}(x^\lambda, 0) + \ldots \end{aligned} \tag{2.7}$$

The covariant derivatives of the tensor of elastic moduli with respect to x^3 are all zero, and (2.7) is reduced to

$$\begin{aligned} \Phi(x^\lambda, z) = \tfrac{1}{2}\overset{o}{E}{}^{\alpha\beta\mu\nu} &\left[\overset{o}{\gamma}_{\alpha\beta} + z\overset{o}{\gamma}_{\alpha\beta\|3} + \tfrac{1}{2}z^2\overset{o}{\gamma}_{\alpha\beta\|33} + \ldots\right] \cdot \\ \cdot &\left[\overset{o}{\gamma}_{\mu\nu} + z\overset{o}{\gamma}_{\mu\nu\|3} + \tfrac{1}{2}z^2\overset{o}{\gamma}_{\mu\nu\|33} + \ldots\right], \end{aligned} \tag{2.8}$$

where subscripts and superscripts o indicate values at the middle surface. Substituting (2.8) into (2.6), performing the integration, and retaining only two terms *), we obtain the equivalent in tensor notation of LOVE's

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*) It will be shown in par. 2.3 that these two terms indeed provide a first approximation.

approximate strain-energy expression

$$V = \frac{1}{2} h \overset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta} \overset{\circ}{\gamma}_{\mu\nu} + \frac{1}{24} h^3 \overset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta\|3} \overset{\circ}{\gamma}_{\mu\nu\|3} . \tag{2.9}$$

The first term in (2.9) represents the extensional strain energy V_ε , i.e. the energy due to the middle surface strains or extensional strains $\overset{\circ}{\gamma}_{\alpha\beta}$. The second term represents the flexural strain energy V_κ, i.e. the energy due to the flexural or bending strains, corresponding to the changes of curvature $\rho_{\alpha\beta}$ defined by *)

$$\rho_{\alpha\beta} = -\overset{\circ}{\gamma}_{\alpha\beta\|3} . \tag{2.10}$$

LOVE's approximate result may therefore be described by the statement that the energy per unit area of the middle surface is the sum of extensional and flexural strain energies

$$V = V_\varepsilon + V_\kappa ; \tag{2.11}$$

$$V_\varepsilon = \frac{1}{2} h \overset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta} \overset{\circ}{\gamma}_{\mu\nu} , \qquad V_\kappa = \frac{1}{24} h^3 \overset{\circ}{E}^{\alpha\beta\mu\nu} \rho_{\alpha\beta} \rho_{\mu\nu} . \tag{2.12}$$

Our result in tensor notation is easily translated into the more usual notation for orthogonal parametric curves on the middle surface by means of the translation scheme given in par. A5 of the appendix. Let ε_1, ε_2 and ψ denote the extensions along the parametric curves and the shear strain between these curves, and let κ_1, κ_2 and τ denote the physical components of the changes of curvature and torsion; we have

$$\overset{\circ}{\gamma}_{11} = A^2 \varepsilon_1 , \quad \overset{\circ}{\gamma}_{22} = B^2 \varepsilon_2 , \quad \overset{\circ}{\gamma}_{12} = \overset{\circ}{\gamma}_{21} = \frac{1}{2} AB\psi ; \tag{2.13}$$

$$\rho_{11} = A^2 \kappa_1 , \quad \rho_{22} = B^2 \kappa_2 , \quad \rho_{12} = \rho_{21} = AB\tau . \tag{2.14}$$

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*) The minus sign is introduced for the sake of convenience.

Expressions (2.12) may now be written in LOVE's form

$$V_\varepsilon = \frac{1}{2}C\left[(\varepsilon_1+\varepsilon_2)^2-2(1-\nu)(\varepsilon_1\varepsilon_2-\frac{1}{4}\psi^2)\right], \tag{2.15}$$

$$V_\kappa = \frac{1}{2}D\left[(\kappa_1+\kappa_2)^2-2(1-\nu)(\kappa_1\kappa_2-\tau^2)\right], \tag{2.16}$$

where

$$C=\frac{Eh}{1-\nu^2}, \qquad D=\frac{Eh^3}{12(1-\nu^2)}, \tag{2.17}$$

and $E=2G(1+\nu)$ is YOUNG's modulus.

2.3 Justification of LOVE's expression as a consistent first approximation.

In order to justify LOVE's approximation *) (2.9) or (2.11) it will be shown that the order of magnitude of the terms which have been omitted after integrating (2.6) is indeed negligible compared with (2.9) or (2.11). All neglected terms, which might conceivably be critical in this respect, are listed below

$$\frac{1}{24}h^3 K\overset{\circ}{E}{}^{\alpha\beta\mu\nu}\overset{\circ}{\gamma}_{\alpha\beta}\overset{\circ}{\gamma}_{\mu\nu} \; ; \tag{2.18}$$

$$\frac{1}{160}h^5 K\overset{\circ}{E}{}^{\alpha\beta\mu\nu}\overset{\circ}{\gamma}_{\alpha\beta\|3}\overset{\circ}{\gamma}_{\mu\nu\|3} \; ; \tag{2.19}$$

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*) LOVE's derivation of his approximate expression (2.11) was restricted to infinitesimal deflections, and his justification was based on more or less qualitative, although entirely sound, physical arguments. The following argument is valid for deflections of any magnitude, provided that the strains remain small. Reference should also be made to [20], where a justification is given, starting from the usual LOVE-KIRCHHOFF assumptions.

$$\frac{1}{6}h^3 H \underset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta} \overset{\circ}{\gamma}_{\mu\nu\|3} \quad ; \tag{2.20}$$

$$\frac{1}{24}h^3 \underset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta} \overset{\circ}{\gamma}_{\mu\nu\|33} \quad ; \tag{2.21}$$

$$\frac{1}{80}h^5 H \underset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta\|3} \overset{\circ}{\gamma}_{\mu\nu\|33} \quad ; \tag{2.22}$$

$$\frac{1}{640}h^5 \underset{\circ}{E}^{\alpha\beta\mu\nu} \overset{\circ}{\gamma}_{\alpha\beta\|33} \overset{\circ}{\gamma}_{\mu\nu\|33} \quad . \tag{2.23}$$

It is almost obvious that the first two neglected terms are indeed negligible in a thin shell. In fact, if R denotes the smallest principal radius of curvature of the middle surface, we have the estimates

$$\left|(2.18)\right| \leqq \frac{h^2}{12R^2} V_\varepsilon \ , \qquad \left|(2.19)\right| \leqq \frac{3h^2}{20R^2} V_\kappa \ . \tag{2.24}$$

By means of SCHWARZ's inequality, holding for the essentially positive strain-energy density (2.1), we obtain an estimate for the third neglected term

$$\left|(2.20)\right| \leqq \frac{2h}{\sqrt{3}\ R} (V_\varepsilon V_\kappa)^{1/2} \quad . \tag{2.25}$$

In order to assess the order of magnitude of the last three neglected terms (2.21) to (2.23), it is convenient to determine first the order of magnitude of the second covariant derivatives of the strain tensor with respect to the coordinate normal to the middle surface $\overset{\circ}{\gamma}_{\alpha\beta\|33}$. This object is achieved by appropriate use of the compatibility conditions; the somewhat laborious analysis is given in par. A4 of the appendix. Let ε denote the (in absolute value) largest extension in the middle surface, κ the largest change of curvature, R the smallest radius of curvature of the undeformed middle surface, and L the smallest "wave length" of the deformation pattern on the middle surface, defined by

$$\left|\frac{d\varepsilon_1}{ds}\right|, \quad \left|\frac{d\varepsilon_2}{ds}\right|, \quad \left|\frac{d\psi}{ds}\right| \quad = \quad \mathcal{O}\left(\frac{\varepsilon}{L}\right) , \tag{2.26}$$

$$\left|\frac{d\kappa_1}{ds}\right|, \quad \left|\frac{d\kappa_2}{ds}\right|, \quad \left|\frac{d\tau}{ds}\right| \quad = \quad \mathcal{O}\left(\frac{\kappa}{L}\right) , \tag{2.27}$$

where ds is any arc element along the middle surface. The order of magnitude of $a^{-1/2} \, \overset{\circ}{\gamma}_{\alpha\beta\|33}$ now does not exceed the order of magnitude of

$$\frac{\varepsilon}{L^2}, \quad \frac{\varepsilon}{R^2}, \quad \frac{\kappa}{R} \quad \text{or} \quad \kappa^2 , \tag{2.28}$$

whichever of these may be critical (cf. par. A4). It is now easily seen that the relative error in the neglection of (2.21) to (2.23) compared with (2.9) or (2.11) does not exceed the orders of magnitude

$$\frac{h^2}{L^2} , \quad \frac{h}{R} \quad \text{or} \quad h\kappa , \tag{2.29}$$

whichever of these may be critical; our assumption of small strains implies of course that the last order given in (2.29) is always negligible.

Combining (2.24), (2.25) and the orders of magnitude (2.29) for the neglected terms (2.21) to (2.23), we obtain our final result that the relative error in neglecting all terms (2.18) to (2.23) does not exceed the orders of magnitude h^2/L^2 or h/R, whichever of these may be critical. Hence LOVE's strain-energy expression (2.11) is indeed justified as a consistent first approximation on the basic assumption of plane stress, and the relative error in this approximation does not exceed h^2/L^2 or h/R, whichever of these may be critical.

2.4 Inconsistency of higher approximations.

Many writers [e.g. 13,23,24,25,36] retain in their energy expressions, derived on the basis of the single assumption of plane stress or its combination with the KIRCHHOFF hypothesis, additional terms, equivalent to some or all of the terms (2.18) to (2.23) neglected in LOVE's expression. Such a supposedly higher approximation is also implied in the analysis of other writers [e.g. 4,6,10,11,19,48], who develop the theory without direct reference to the strain energy. It has been pointed out repeatedly by several writers [e.g. 12, 20, 21,22,30,40] that such a refinement may be meaningless if the assumption of plane stress and/or the KIRCHHOFF hypothesis (which assumptions are of course only approximately valid) are retained.

In fact, the transverse shear stresses, obtained from equilibrium conditions, are in general of order h/L times the bending stresses, and neglection of the corresponding strain energy therefore already implies a relative error of order h^2/L^2. Moreover, the transverse normal stress is, in general, of order h^2/L^2 or h/R times the bending or direct stresses, and its neglection in the strain-energy density (2.1) involves relative errors of the same orders. Hence a refinement of LOVE's approximation (2.11) is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time *).

This general conclusion is confirmed by a detailed analysis by JOHNSON and REISSNER [17] of a semi-infinite cylindrical shell under axisymmetric radial loads and bending moments at its end cross-section. The three-dimensional stress distribution is expanded with respect to the small parameter $\lambda=h/2a$, where a is the mean radius.

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*) Several writers have developed theories in which the effects of transverse stresses are taken into account [e.g. 1,15,31,32,40,41]; a discussion of these theories falls outside the scope of the present paper.

The terms of order λ^{o} represent the classical shell solution, and in order to obtain all terms of order λ^{1} it is necessary to take into account both the transverse shear stresses and the transverse normal stress.

Our general argument on the order of magnitude of the errors, introduced by the assumption of plane stress, gives of course an estimate for the general case. In some problems it may happen that the errors of the plane stress assumption are of smaller order than h/R. In such problems retention of some or all of the additional terms (2.18) through (2.23) may conceivably result in a better accuracy. However, such an improvement is by no means certain, since in such cases the neglected terms in LOVE's expression may also be of a smaller order of magnitude. E.g., if the extensional strain energy A_{ε} is of order h^2/R^2 times the flexural strain energy A_{κ} (or vice versa), and if L is of the same order as R, the error in LOVE's approximation through neglection of (2.18) through (2.23) is of order h^2/R^2 instead of order h/R. An example of such higher accuracy of LOVE's expression occurs in the case of the helicoidal shell under normal loading [6,7], where retention of additional terms in the energy expression is again meaningless unless the effect of transverse shear stresses is also taken into account at the same time.

2.5 Discussion of results.

The fact that LOVE's expression (2.11) has inherent errors of the type of the neglected terms, has important consequences for practical applications of the theory. These inherent errors imply that the accuracy of LOVE's expression is not affected by the addition or subtraction of certain terms of the same type. In particular, it is therefore permissible (cf. (2.20)) to add to the expressions for the physical components of the changes of curvature and torsion (κ_1, κ_2 and τ) terms of type ε/R (where ε is any of the physical middle surface strains

ε_1, ε_2 or ψ, and R is any radius of curvature or torsion of the middle surface R_1, R_2 or T), multiplied by a numerical factor, provided this factor is not large compared to unity.

This freedom is of considerable importance for two reasons. Firstly, it allows to distinguish between significant and non-essential differences in the wide variety of expressions for the changes of curvature in linear shell theory, obtained by various writers. Differences of type ε/R may be regarded as unimportant, whereas differences which are not reducible to the form ε/R should be regarded as essential differences which may imply a significant loss in accuracy for at least one of the corresponding strain-energy expressions; a detailed comparison of various expressions is discussed in par. 3.5 and summarized in table 1. On the other hand, the expressions for the changes of curvature can often be simplified by addition or subtraction of suitable terms of type ε/R; this freedom is of particular value in selecting the appropriate simplest expressions for the changes of curvature for applications to specific problems [cf. 20,21].

3. LINEAR THEORY (INFINITESIMAL DEFLECTIONS).

3.1 Extensional strains and rotations.

In linear theory the general expression for the strain tensor in surfaces parallel to the middle surface (cf. par. A3) is simplified into

$$2\gamma_{\alpha\beta} = u_{\alpha||\beta} + u_{\beta||\alpha} . \qquad (3.1)$$

In the middle surface itself the spatial covariant derivatives may be expressed in surface derivatives (cf. par. A2). The resulting linear expression for the middle surface strain tensor is

$$2\mathring{\gamma}_{\alpha\beta} = u_{\alpha|\beta} + u_{\beta|\alpha} - 2wb_{\alpha\beta} \,. \qquad (3.2)$$

The translation into the usual **non-tensorial** notation is obtained by means of (2.13) and par. A5; the resulting well-known expressions for the extensions ε_1, ε_2 along, and the shear strain ψ between the orthogonal parametric curves (α,β) are

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{v}{AB}\frac{\partial A}{\partial \beta} - \frac{w}{R_1} \,, \qquad (3.3)$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{u}{AB}\frac{\partial B}{\partial \alpha} - \frac{w}{R_2} \qquad (3.4)$$

$$\psi = \frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha} - \frac{2w}{T} \,, \qquad (3.5)$$

where u, v and w are the physical displacement components in the directions of the parametric curves α,β and of the normal.

The rotation in the middle surface around the normal is described by the antisymmetric surface tensor $\omega_{\alpha\beta}$ (cf. par. A3)

$$2\omega_{\alpha\beta} = u_{\beta\|\alpha} - u_{\alpha\|\beta} = u_{\beta|\alpha} - u_{\alpha|\beta} = u_{\beta,\alpha} - u_{\alpha,\beta} \,. \qquad (3.6)$$

The rotation of the normal at the middle surface is described by the surface vector φ_α , given by

$$\varphi_\alpha = \omega_{\alpha 3} = \tfrac{1}{2}\left[u_{3\|\alpha} - u_{\alpha\|3}\right] = u_{3\|\alpha} \,, \qquad (3.7)$$

where the last step is a consequence of (2.3)

$$2\gamma_{\alpha 3} = u_{\alpha\|3} + u_{3\|\alpha} = 0 \,. \qquad (3.8)$$

Expressing the spatial derivative in a surface derivative by means of par. A2 we have

$$\varphi_\alpha = w_{|\alpha} + u_\lambda b^\lambda_\alpha = w_{,\alpha} + u_\lambda b^\lambda_\alpha \,. \qquad (3.9)$$

Translating into the usual non-tensorial notation the physical rotation around the normal, Ω, is given by

$$2\Omega = \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{u}{AB}\frac{\partial A}{\partial \beta} + \frac{v}{AB}\frac{\partial B}{\partial \alpha} , \tag{3.10}$$

and the physical rotation components of the normal ϕ_1, ϕ_2 are given by

$$\phi_1 = \frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1} + \frac{v}{T} , \tag{3.11}$$

$$\phi_2 = \frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2} + \frac{u}{T} . \tag{3.12}$$

3.2 Changes of curvature.

In order to obtain the tensor of changes of curvature we differentiate (3.1) covariantly with respect to x^3, and interchange the order of covariant differentiation, which is admissible in EUCLIDEAN space

$$2\gamma_{\alpha\beta||3} = u_{\alpha||\beta 3} + u_{\beta||\alpha 3} = u_{\alpha||3\beta} + u_{\beta||3\alpha} . \tag{3.13}$$

Applying the basic formulae of covariant differentiation, and using (3.8), we obtain

$$2\gamma_{\alpha\beta||3} = -(u_{3||\alpha})_{,\beta} - (u_{3||\beta})_{,\alpha} + 2\Gamma^{\lambda}_{\alpha\beta}u_{3||\lambda} + \\ - 2\Gamma^{3}_{\alpha\beta}u_{3||3} - \Gamma^{\lambda}_{3\beta}u_{\alpha||\lambda} - \Gamma^{\lambda}_{3\alpha}u_{\beta||\lambda} . \tag{3.14}$$

From (2.4) we have

$$u_{3||3} = - \frac{\nu}{1-\nu} g^{\lambda\mu}\gamma_{\lambda\mu} , \tag{3.15}$$

and since at the middle surface (cf. par. A2)

$$\Gamma^{3}_{\alpha\beta} = b_{\alpha\beta} \tag{3.16}$$

we may neglect the term involving $u_{3||3}$ in the evaluation of (3.14) at the middle surface on account of our discussion in par. 2.5. Similarly, we may write

$$\Gamma^{\lambda}_{3\beta} u_{\alpha\|\lambda} = \Gamma^{\lambda}_{3\beta} \gamma_{\alpha\lambda} - \Gamma^{\lambda}_{3\beta} \omega_{\alpha\lambda} \tag{3.17}$$

where the first term may again be neglected at the middle surface on account of par. 2.5. Rewriting (3.14) at the middle surface in terms of surface derivatives, introducing (3.7), and remembering that at the middle surface (cf. par. A2)

$$\Gamma^{\lambda}_{3\beta} = -b^{\lambda}_{\beta} \, , \tag{3.18}$$

we obtain finally

$$2\rho_{\alpha\beta} = -2\mathring{\gamma}_{\alpha\beta\|3} = \varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^{\lambda}_{\beta}\omega_{\alpha\lambda} + b^{\lambda}_{\alpha}\omega_{\beta\lambda} \, . \tag{3.19}$$

This result is again easily transformed into the usual non-tensorial notation for orthogonal parametric curves, by means of (2.14) and the translation rules of par. A5; we obtain

$$\kappa_1 = \frac{1}{A}\frac{\partial\phi_1}{\partial\alpha} + \frac{\phi_2}{AB}\frac{\partial A}{\partial\beta} + \frac{\Omega}{T} \, , \tag{3.20}$$

$$\kappa_2 = \frac{1}{B}\frac{\partial\phi_2}{\partial\beta} + \frac{\phi_1}{AB}\frac{\partial B}{\partial\alpha} - \frac{\Omega}{T} \, , \tag{3.21}$$

$$\tau = \frac{1}{2A}\frac{\partial\phi_2}{\partial\alpha} + \frac{1}{2B}\frac{\partial\phi_1}{\partial\beta} - \frac{\phi_1}{2AB}\frac{\partial A}{\partial\beta} - \frac{\phi_2}{2AB}\frac{\partial B}{\partial\alpha} - \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\Omega . \tag{3.22}$$

The result in this form (as in the tensor form (3.19)) is more graphic than the usual expressions, since it clearly demonstrates the dependence of the changes of curvature on the rotations ϕ_1, ϕ_2, Ω and their derivatives.

3.3 Stress-strain relations.

Stress resultants and stress couples may be introduced in various manners. In our energy approach the most obvious manner in a definition of stress resultants and stress couples as partial derivatives of the strain energy per unit area of the middle surface with respect to the middle surface strains and the changes of curvature. In this way we obtain the symmetric contravariant tensors of stress resultants and stress couples and the corresponding stress-strain relations

$$n^{\alpha\beta} = \frac{\partial V}{\partial \mathring{\gamma}_{\alpha\beta}} = h\underset{\circ}{E}^{\alpha\beta\mu\nu}\mathring{\gamma}_{\mu\nu} , \tag{3.23}$$

$$m^{\alpha\beta} = \frac{\partial V}{\partial \rho_{\alpha\beta}} = \frac{1}{12} h^3\underset{\circ}{E}^{\alpha\beta\mu\nu}\rho_{\mu\nu} . \tag{3.24}$$

For the purpose of deriving the equations of equilibrium by variational methods, it is more convenient to introduce a modified asymmetric tensor of stress resultants, defined by

$$n_*^{\alpha\beta} = \frac{\partial V}{\partial u_{\beta|\alpha}} = n^{\mu\nu}\frac{\partial \mathring{\gamma}_{\mu\nu}}{\partial u_{\beta|\alpha}} + m^{\mu\nu}\frac{\partial \rho_{\mu\nu}}{\partial u_{\beta|\alpha}} . \tag{3.25}$$

By means of (3.2), (3.6) and (3.19) we have now

$$n_*^{\alpha\beta} = n^{\alpha\beta} + \frac{1}{2}b^{\beta}_{\lambda}m^{\alpha\lambda} - \frac{1}{2}b^{\alpha}_{\lambda}m^{\beta\lambda} . \tag{3.26}$$

A similarly modified definition of the tensor of stress couples

$$m_*^{\alpha\beta} = \frac{\partial V}{\partial \varphi_{\beta|\alpha}} = m^{\mu\nu}\frac{\partial \rho_{\mu\nu}}{\partial \varphi_{\beta|\alpha}} = m^{\alpha\beta} . \tag{3.27}$$

does not result in a modification of this tensor.

The symmetrical physical stress resultants N_1, N_2, S and stress couples M_1, M_2, W for orthogonal parametric curves, corresponding to (3.23) and (3.24), and the modified asymmetrical physical stress resultants N_1^*, N_2^*, S_{12}^*, S_{21}^*, corresponding to (3.26), may be obtained from the translation scheme in par. A5. Alternatively, these quantities may be found directly by partial differentiation of the strain-energy expressions (2.11), (2.15) and (2.16). The results are

$$N_1 = \frac{\partial V}{\partial \varepsilon_1} = C(\varepsilon_1 + \nu\varepsilon_2), \tag{3.28}$$

$$N_2 = \frac{\partial V}{\partial \varepsilon_2} = C(\varepsilon_2 + \nu\varepsilon_1), \tag{3.29}$$

$$S = \frac{\partial V}{\partial \psi} = \tfrac{1}{2}(1-\nu)C\psi; \tag{3.30}$$

$$M_1 = \frac{\partial V}{\partial \kappa_1} = D(\kappa_1 + \nu\kappa_2), \tag{3.31}$$

$$M_2 = \frac{\partial V}{\partial \kappa_2} = D(\kappa_2 + \nu\kappa_1), \tag{3.32}$$

$$2W = \frac{\partial V}{\partial \tau} = 2(1-\nu)D\tau; \tag{3.33}$$

$$N_1^* = \frac{\partial V}{\partial\left(\frac{1}{A}\frac{\partial u}{\partial \alpha}\right)} = N_1, \tag{3.34}$$

$$N_2^* = \frac{\partial V}{\partial\left(\frac{1}{B}\frac{\partial v}{\partial \beta}\right)} = N_2, \tag{3.35}$$

$$S_{12}^* = \frac{\partial V}{\partial\left(\frac{1}{A}\frac{\partial v}{\partial \alpha}\right)} = S + \frac{M_1 - M_2}{2T} - \frac{1}{2}W\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{3.36}$$

$$S_{21}^* = \frac{\partial V}{\partial\left(\frac{1}{B}\frac{\partial u}{\partial \beta}\right)} = S - \frac{M_1 - M_2}{2T} + \frac{1}{2}W\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{3.37}$$

3.4 Equations of equilibrium and boundary conditions.

The equations of equilibrium and boundary conditions are obtained by the well-known variational method. The first variation of the total elastic energy U, i.e. the integral of V (2.11) over the entire middle surface of the shell, is given by

$$\delta U=\iint\left[n_*^{\alpha\beta}\delta u_{\beta|\alpha} - n^{\alpha\beta}b_{\alpha\beta}\delta w + m^{\alpha\beta}\delta\varphi_{\beta|\alpha}\right]\sqrt{a}\ dx_1 dx_2. \qquad (3.38)$$

On account of the symmetry of $b_{\alpha\beta}$ we may replace $n^{\alpha\beta}$ in the second term by the asymmetric tensor $n_*^{\alpha\beta}$. Applying GREEN's theorem (cf. par. A2), and defining the contravariant vector of transverse shear forces q^β by

$$q^\beta = -m^{\alpha\beta}_{\ \ |\alpha}\ , \qquad (3.39)$$

the first variation (3.38) is reduced to

$$\delta U=\int\left[n_*^{\alpha\beta}\delta u_\beta + m^{\alpha\beta}\delta\varphi_\beta\right]\varepsilon_{\alpha\lambda}dx^\lambda +$$
$$-\iint\left[n_{*|\alpha}^{\alpha\beta}\ \delta u_\beta - q^\beta\delta\varphi_\beta + n_*^{\alpha\beta}b_{\alpha\beta}\delta w\right]\sqrt{a}\ dx^1dx^2\ , \qquad (3.40)$$

where the first integral is taken along the boundary of the shell's middle surface. Remembering (3.9), the derivatives $\delta w_{|\beta}= \delta w_{,\beta}$ in the surface integral are removed by a second application of GREEN's theorem. In order to remove the same derivatives, as far as possible, from the line integral, it is convenient to choose a special coordinate system in which the boundary of the middle surface is a coordinate line, say an x^2-line (i.e. a line x^1=constant). Remembering the definition of the antisymmetric ε-tensor (cf. par. A1), we have by integration by parts *)

$$\int m^{\alpha\beta}\delta w_{,\beta}\varepsilon_{\alpha 2}dx^2 = \int\left[m^{11}\sqrt{a}\delta w_{,1} - (m^{12}\sqrt{a})_{,2}\delta w\right]dx^2. \qquad (3.41)$$

Our final result for the first variation of the elastic energy is

$$\delta U = \int\left[(n_*^{1\beta}+m^{1\mu}b_\mu^\beta)\sqrt{a}\,\delta u_\beta + m^{11}\sqrt{a}\delta w_{,1} + \left\{q^1\sqrt{a}-(m^{12}\sqrt{a})_{,2}\right\}\delta w\right]dx^2 +$$
$$-\iint\left[(n^{\alpha\beta}_{*|\alpha} - q^\alpha b_\alpha^\beta)\delta u_\beta + (n_*^{\alpha\beta}b_{\alpha\beta}+q^\beta_{|\beta})\delta w\right]\sqrt{a}\;dx^1dx^2. \qquad (3.42)$$

Let p^α, p^3 denote the contravariant vector of external surface loads per unit area of the middle surface, and let $\bar{n}^\beta$, $\bar{q}$, and $\bar{m}^1$ denote the contravariant vector of edge forces, and the edge bending moment, both per unit length of the edge. The work $\delta\bar{U}$, performed by the surface loads and edge loads in any variation of the displacements is then given by

$$\delta\bar{U} = \int\left[\bar{n}^\beta\delta u_\beta + \bar{q}\delta w + \bar{m}^1\delta\varphi_1\right]\sqrt{a_{22}}\;dx^2 + \iint\left[p^\beta\delta u_\beta + p^3\delta w\right]\sqrt{a}\;dx^1dx^2. \qquad (3.43)$$

The conditions of equilibrium are now obtained from the principle of virtual work, i.e. $\delta U = \delta\bar{U}$ for all kinematically admissible variations of the displacements. The fundamental lemma of the calculus of variations now results in the equations of equilibrium

$$n^{\alpha\beta}_{*|\alpha} - q^\alpha b_\alpha^\beta + p^\beta = 0, \qquad (3.44)$$

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*) It has been assumed here that the entire boundary is a smooth x^2-line. If the boundary has corners, stock terms arise in these corners as a result of the integration by parts; the coefficients of δw in these stock terms represent concentrated forces normal to the shell middle surface, which are well known to occur in corners of plates and shells, but we shall not pursue this complication here.

$$n_{*}^{\alpha\beta} b_{\alpha\beta} + q^{\beta}_{|\beta} + p^3 = 0 , \qquad (3.45)$$

holding in the interior of the middle surface. Moreover, unless the boundary displacements u_β, w and the rotation φ_1 are specified (in which case their variations are zero along the boundary), the boundary conditions

$$(n_{*}^{1\beta} + m^{1\mu} b^{\beta}_{\mu})\sqrt{a} = (\bar{n}^{\beta} + \bar{m}^{1} b^{\beta}_{1}) \sqrt{a_{22}} , \qquad (3.46)$$

$$q^1\sqrt{a} - (m^{12}\sqrt{a})_{,2} = \bar{q}\sqrt{a_{22}} , \qquad (3.47)$$

$$m^{11}\sqrt{a} = \bar{m}^1\sqrt{a_{22}} \qquad (3.48)$$

must hold along the edge of the middle surface.

It should be noted that our equations of equilibrium (3.44), (3.45), and the definition of the surface vector of transverse shear forces agree completely with the rigorous equations of equilibrium, as derived by GREEN and ZERNA [12] from the general threedimensional equations*). Likewise, GREEN's and ZERNA's equation for equilibrium of moments around the normal is equivalent to our equation (3.26). Hence, in spite of the approximative character of our theory, based on LOVE's approximate strain-energy expression, our equations of equilibrium are completely rigorous.

The translation of our results into the usual non-tensorial notation for orthogonal parametric curves is again easily obtained by means of par. A5 of the appendix. The non-tensorial formulation may of course also be achieved by a direct application of the calculus of variations to the energy expression in non-tensorial form. Let Q_1 and Q_2 denote the transverse shear forces

*) Due allowance should of course be made for the difference in sign between our definition of $m^{\alpha\beta}$ and the definition in [12].

per unit length; they are expressed in the bending and torsional moments by the translation of (3.39)

$$Q_1 = -\frac{1}{A}\frac{\partial M_1}{\partial \alpha} - \frac{1}{B}\frac{\partial W}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}(M_1 - M_2) - \frac{2}{AB}\frac{\partial A}{\partial \beta} W , \qquad (3.49)$$

$$Q_2 = -\frac{1}{B}\frac{\partial M_2}{\partial \beta} - \frac{1}{A}\frac{\partial W}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}(M_2 - M_1) - \frac{2}{AB}\frac{\partial B}{\partial \alpha} W . \qquad (3.50)$$

Let p_I, p_{II} and $p_{III} = p^3$ denote the physical external loads per unit area of the middle surface. The equations of equilibrium (3.44) are now translated into

$$\frac{1}{A}\frac{\partial N_1^*}{\partial \alpha} + \frac{1}{B}\frac{\partial S_{21}^*}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}(N_1^* - N_2^*) + \frac{1}{AB}\frac{\partial A}{\partial \beta}(S_{12}^* + S_{21}^*) + \\ - \frac{Q_1}{R_1} - \frac{Q_2}{T} + p_I = 0 , \qquad (3.51)$$

$$\frac{1}{B}\frac{\partial N_2^*}{\partial \beta} + \frac{1}{A}\frac{\partial S_{12}^*}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}(N_2^* - N_1^*) + \frac{1}{AB}\frac{\partial B}{\partial \alpha}(S_{12}^* + S_{21}^*) + \\ - \frac{Q_2}{R_2} - \frac{Q_1}{T} + p_{II} = 0 , \qquad (3.52)$$

and equation (3.45) now reads

$$\frac{N_1^*}{R_1} + \frac{N_2^*}{R_2} + \frac{S_{12}^* + S_{21}^*}{T} + \frac{1}{A}\frac{\partial Q_1}{\partial \alpha} + \frac{1}{B}\frac{\partial Q_2}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} Q_1 + \frac{1}{AB}\frac{\partial A}{\partial \beta} Q_2 + p_{III} = \\ = 0 . \qquad (3.53)$$

Equations (3.49) to (3.53) and (3.26) agree completely with the six equations of equilibrium in their well-known form for arbitrary orthogonal parametric curves [e.g. 6,7, 11,18,19].*)

*) Here again, allowance must of course be made for differences in sign in the definitions employed here and those in the cited references; moreover, no distinction is made in our analysis between the torsional moments W_{12} and W_{21}, which are both equal to W in our approximation.

The geometric boundary conditions along a boundary coinciding with a ß-line (α=constant) are in the usual notation simply that prescribed values of u, v, w and $\frac{\partial w}{\partial \alpha}$ are specified. The dynamic boundary conditions are obtained by a translation of (3.46) to (3.48). Let $\bar{N}_1$, $\bar{N}_2$, $\bar{Q}$ and $\bar{M}$ denote the specified forces and bending moment per unit length of the boundary; we have

$$\bar{N}_1 = A\bar{n}^1, \quad \bar{N}_2 = B\bar{n}^2, \quad \bar{Q} = q, \quad \bar{M} = A\bar{m}^1, \tag{3.54}$$

and the dynamic boundary conditions read, if (3.48) is used in order to simplify (3.46)

$$N_1^* + \frac{W}{T} = \bar{N}_1 \ , \tag{3.55}$$

$$S_{12}^* + \frac{W}{R_2} = \bar{N}_2 \ , \tag{3.56}$$

$$Q_1 - \frac{1}{B}\frac{\partial W}{\partial \beta} = \bar{Q} \ , \tag{3.57}$$

$$M_1 = \bar{M} \ . \tag{3.58}$$

Dynamic boundary conditions have been formulated explicitly by only a few writers [6,7,22,27], and they agree with the present conditions, if again due allowance is made for differences in sign in the definitions.

3.5 Discussion and comparison with earlier writers.

All authors are in agreement on the formulae for the middle surface strains, (3.2) in tensor notation, and (3.3) to (3.5) in the more conventional notation. Likewise, full agreement exists on five equations of equilibrium, (3.39) and (3.44), (3.45) in tensor notation, and (3.49) to (3.53) in the usual notation. Considerable differences occur, however, in the expressions for the changes of curvature, in the stress-strain relations, and in the importance attached to the equation of equilibrium of moments around the normal, expressed by

(3,26) in tensor notation, and by (3.36), (3.37) in the usual notation.

3.5.1 Changes of curvature. The expressions given in 13 papers and books are compared with our results in table 1; the differences have been reduced to their simplest form by appropriate use of the MAINARDI-CODAZZI equations (cf. par. A5). It appears that not less than 10 different expressions have been proposed, although all of them are based on the same basic LOVE-KIRCHHOFF assumptions, or, as in our theory, the equivalent assumption of plane stress. Fortunately, all differences of type ε/R are immaterial from the point of view of a first approximation (cf. par. 2.5). Some authors present their results as "second" or "better" approximations; in the absence of a complete analysis in their papers of the effect of transverse normal and shear stresses, these claims lack adequate substantion. On the other hand, differences listed in table 1 which are not of type ε/R are essential discrepancies. Since our expressions have been proved to provide a consistent first approximation if used in conjunction with LOVE's strain-energy expression (2.11), it follows that the expressions in the cited reference may be seriously in error. These doubtful expressions are marked by an asterisk in table 1.

The questionable term in COHEN's change of twist [6, eq. (1.6.21)] is obviously

$$\frac{w}{T}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) , \qquad (3.59)$$

and in a subsequent check COHEN discovered an arithmetical error in his original analysis; the term (3.59) should be deleted from his equation (1.6.21). *)

*) It will be observed that COHEN's expressions for the change of twist in [6] and [7] do not agree after deletion of (3.59) from eq. (1.6.21) in [6]. The remaining difference (non-essential in the first approximation) is due to a slight difference in definition of the change of twist between [6] and [7].

In papers by REISSNER [38] and by KNOWLES and REISSNER [18] the questionable terms are those in which the (physical) rotation around the normal, Ω, occurs, divided by a radius of curvature or torsion. These discrepancies are again unimportant, if the rotation is small, i.e. of the same order of magnitude as the middle surface strains. This is actually so in the majority of interesting specific problems, and the expressions of [18] and [38] are then essentially equivalent to our expressions. The exception arises e.g. if the shell geometry and boundary conditions permit inextensional deformations of the middle surface. In such cases Ω may be large, even infinite, compared with the middle surface strains, and the expressions in [18] and [38], used in conjunction with LOVE's strain-energy expression (2.11), are inadequate as a first approximation.

Whenever the rotation Ω is small, of the same order of magnitude as ε_1, ε_2, ψ, we may simplify (3.19) into

$$2\rho_{\alpha\beta} = \varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} \, . \tag{3.60}$$

Remembering (3.9), we may now write

$$2\rho_{\alpha\beta} = 2w_{|\alpha\beta} + b^{\lambda}_{\beta}u_{\lambda|\alpha} + b^{\lambda}_{\alpha}u_{\lambda|\beta} + 2u_{\lambda}b^{\lambda}_{\alpha|\beta} \, , \tag{3.61}$$

where the MAINARDI-CODAZZI equations (par. A2) have been used. From (3.2) and (3.6) we have in the present approximation

$$b^{\lambda}_{\beta}u_{\lambda|\alpha} = b^{\lambda}_{\beta}(\mathring{\gamma}_{\lambda\alpha} - \omega_{\lambda\alpha} + wb_{\lambda\alpha}) = wc_{\alpha\beta} \, , \tag{3.62}$$

and hence

$$\rho_{\alpha\beta} = w_{|\alpha\beta} + wc_{\alpha\beta} + u_{\lambda}b^{\lambda}_{\alpha|\beta} \, . \tag{3.63}$$

GREEN and ZERNA [12] introduce a further approximation by retention of only the first term in (3.63). This approximation is equivalent to DONNELL's [9] in the case of cylindrical shells. It has been noted [16,29] that DONNELL's approximation is sometimes inaccurate, even though the rotation Ω may remain small of the same order as the strains, and in such cases the second term in (3.63) should evidently not be omitted; the last term is of course always zero in cylindrical or spherical shells.

LOVE's expressions for the changes of curvature have often been criticized in the literature [e.g. 23, 35,47] for the lack of symmetry in his change of twist and for supposed other defects. Hence it may be worthwhile to reconfirm by means of the comparison in table 1 that LOVE's result actually is consistent and adequate within the framework of the first approximation in shell theory [cf. also 20,21]; it is entirely equivalent to other consistent results.

3.5.2 <u>Stress-strain relations and equilibrium of moments around the normal.</u> Consistent stress-strain relations for stress resultants and stress couples to be used in the equations of equilibrium, are given by (3.26) and (3.27) in tensor form, and by (3.31) to (3.37) in the more conventional notation. In many investigations the asymmetric shear stress resultants are replaced by their symmetric part (3.30) *). The condition of equilibrium of moments around the normal is of course violated in this simplification. In our energy approach, neglection of the asymmetric terms in the shear stress resultants

*) The corresponding stress-strain relations (3.28) to (3.33) are often referred to in literature as "LOVE's first approximation". We shall not use this terminology in order to avoid confusion with our definition of LOVE's first approximation, i.e. the strain-energy expression (2.11) with any appropriate consistent expressions for the extensional strains and changes of curvature.

(3.33) and (3.34) is obviously equivalent to omitting the terms involving the rotation Ω in our expressions for the changes of curvature (3.20) to (3.22). The simplification is therefore justified, if the rotation Ω is small, of the same order of magnitude as the middle surface strains ε_1, ε_2, ψ. On the other hand, considerable errors may be introduced if a symmetric shear stress resultant S is assumed in problems where the rotation Ω may be large compared with the extensional strains. This conclusion is borne out by COHEN's detailed analysis of a helicoidal shell [6,7].

Numerous more complicated stress-strain relations have been derived in the literature, and it is often believed that the additional terms would imply a higher approximation. However, this object can actually not be achieved without due account of the effect of transverse normal and shear stresses (cf. par. 2.4), and it is meaningless to use more "refined" stress-strain relations than (3.31) to (3.37) if the LOVE-KIRCHHOFF assumptions are retained in their derivation. This view is again confirmed by COHEN's analysis in the case of the helicoidal shell [7]. On the basis of the stress-strain relations (3.31) to (3.37) the same result is obtained as from the more elaborate stress-strain relations in [6].

Table 1

Comparis of various expressions for the physical changes of curvature.

N.B. The entrances in his table indicate the corrections $\Delta\kappa_1$, $\Delta\kappa_2$, $\Delta\tau$ which must be added to our expressions r κ_1, κ_2, τ in order to obtain the expressions in the cited references. Where necessary, justments for sign and/or a numerical factor 2 have been made to achieve conformity with r notation.
Essential differ ces in the sense of paras. 2.5 and 3.5 are marked by an asterisk.
References emplo ng the lines of curvature as parametric curves are marked by a small circle.

AUTHORS and REFERENCES	$\Delta\kappa_1$	$\Delta\kappa_2$	$\Delta\tau$
LOVE, 1888 [27] o)	—	—	$\frac{1}{4}\psi(\frac{1}{R_2}-\frac{3}{R_1})$
LAMB, 1891 [22] o) REISSNER, 1942 [39] o) WLASSOW, 1949 [47] o) OSGOOD and JOSEPH, 1950 5] o) HAYWOOD and WILSON, 1958 13] o)	$-\frac{\varepsilon_1}{R_1}$	$-\frac{\varepsilon_2}{R_2}$	$-\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})$
REISSNER, 1941 [38] o)	—	—	$\frac{1}{2}\Omega(\frac{1}{R_1}-\frac{1}{R_2})$ (*)
KOITER, 1945 [20] o)	$\frac{2\varepsilon_1+\varepsilon_2}{R_1}$	$\frac{\varepsilon_1+2\varepsilon_2}{R_2}$	$\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})$
GOLDENWEISER, 1953 [11], .54	—	—	$\frac{\varepsilon_1+\varepsilon_2}{T}+\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})$
COHEN, 1955 [6]	$-\frac{\varepsilon_1}{R_1}+\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}+\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}+\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})+\frac{w}{T}(\frac{1}{R_1}+\frac{1}{R_2})$ ( )
COHEN, 1959 [7]	$-\frac{\varepsilon_1}{R_1}+\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}+\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}-\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})$
KNOWLES and REISSNER, 195 [18]	$-\frac{\Omega}{T}$ (*)	$+\frac{\Omega}{T}$ (*)	$\frac{1}{2}\Omega(\frac{1}{R_1}-\frac{1}{R_2})$ (*)
KNOWLES and REISSNER, 195 [19]	$-\frac{\varepsilon_1}{R_1}-\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}-\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}-\frac{1}{4}\psi(\frac{1}{R_1}+\frac{1}{R_2})$

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APPENDIX: TENSOR NOTATION AND GEOMETRICAL RELATIONS [cf. also 8, 12, 43]

A1. Geometry of shells.

A point on the middle surface of a shell is described by its surface coordinates x^{α} $(\alpha=1,2)$. Greek subscripts and superscripts will always be used to refer to the pair of surface coordinates. A point of the shell outside the middle surface is described by its coordinates x^{i} $(i=1,2,3)$, where x^{1}, x^{2} are the surface coordinates of its projection on the middle surface, and $x^{3}=z$ $(-\frac{1}{2}h \leq z \leq \frac{1}{2}h)$ is its distance to the middle surface. Roman subscripts and superscripts will always be used to refer to this triple of space coordinates. The thickness of the shell is expressed by the assumption that the shell thickness h is small compared with the smallest principal radius of curvature R of the middle surface, i.e. $h/R \ll 1$.

The (symmetric) covariant metric tensor on the undeformed middle surface is denoted by $a_{\alpha\beta}$, its determinant by

$$a = a_{11}a_{22} - (a_{12})^2 , \tag{A1.1}$$

and its associated (symmetric) contravariant tensor by $a^{\alpha\beta}$. The (symmetric) covariant second and third fundamental tensors of the undeformed middle surface are denoted by $b_{\alpha\beta}$ and $c_{\alpha\beta}$; the determinant of $b_{\alpha\beta}$ is denoted by

$$b = b_{11}b_{22} - (b_{12})^2 . \tag{A1.2}$$

The three fundamental tensors are connected by

$$c_{\alpha\beta} = a^{\lambda\mu}b_{\alpha\lambda}b_{\beta\mu} = b_{\alpha\lambda}b^{\lambda}_{\beta} , \tag{A1.3}$$

where the summation convention has been employed for repeated indices, once as a subscript and once as a [illegible] superscript. The metric tensors $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are always used to lower and raise indices in surface tensors. A special surface tensor of some importance in the analysis is the antisymmetric ε-tensor, defined by $\varepsilon_{11}=\varepsilon_{22}=0$, $\varepsilon_{12}=-\varepsilon_{21}=\sqrt{a}$.

The (symmetric) covariant metric tensor in space in the undeformed shell is denoted by g_{ij}, its determinant by g, and its associated (symmetric) contravariant tensor by g^{ij}. These metric tensors will always be used to raise and lower indices in space tensors. Since the x^3-direction is normal to the middle surface, we have

$$g_{\alpha\beta}=a_{\alpha\beta}-2zb_{\alpha\beta}+z^2c_{\alpha\beta}\,,\qquad g_{\alpha3}=0\,,\qquad g_{33}=1\,; \tag{A1.4}$$

$$\left.\begin{array}{lll} gg^{11}=g_{22}\,, & gg^{22}=g_{11}\,, & gg^{12}=-g_{12}\,, \\ g^{\alpha3}=0\,, & g^{33}=1\,. & \end{array}\right\} \quad (A1,5)$$

Defining the mean curvature and the GAUSSIAN curvature of the undeformed middle surface by the invariants

$$H=\frac{1}{2}b^{\alpha}_{\alpha}\,,\qquad K=\frac{b}{a}=b^1_1b^2_2-b^1_2b^2_1\,, \tag{A1.6}$$

the volume-element of the undeformed shell is described by

$$dv=\sqrt{g}\;dx^1dx^2dx^3=\sqrt{a}\left[1-2Hz+Kz^2\right]dx^1dx^2dz\,. \tag{A1.7}$$

A2. <u>Covariant differentiation.</u>

Partial differentiation of a scalar Φ, a vector u^h or u_h, and a tensor t^{hi} or t_{hi} with respect to a coordinate x_j will be indicated by an additional subscript j preceded by a comma. Likewise, covariant differentiation in space will be indicated by an additional subscript preceded by a double vertical line. The rules of covariant differentiation are given by

$$\left.\begin{aligned}
\Phi_{\|j} &= \Phi_{,j} \\
u^h_{\|j} &= u^h_{,j} + \Gamma^h_{kj}u^k \\
u_{h\|j} &= u_{h,j} - \Gamma^k_{hj}u_k \\
t^{hi}_{\|j} &= t^{hi}_{,j} + \Gamma^h_{kj}t^{ki} + \Gamma^i_{kj}t^{hk} \\
t_{hi\|j} &= t_{hi,j} - \Gamma^k_{hj}t_{ki} - \Gamma^k_{ij}t_{hk} \quad ,
\end{aligned}\right\} \tag{A2.1}$$

where the CHRISTOFFEL-symbols of the second kind $\Gamma^h_{ij} = \Gamma^h_{ji}$ are defined by

$$\Gamma^h_{ij} = g^{hk}\Gamma_{ijk} = \tfrac{1}{2}g^{hk}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \ . \tag{A2.2}$$

The covariant derivatives of the metric tensor are zero. Moreover, in EUCLIDEAN space the order of covariant differentiation of any tensor is immaterial.

In our special coordinate system with metric tensor (A1.4) we have for the CHRISTOFFEL-symbols of the second kind

$$\left.\begin{aligned}
&\Gamma^3_{33} = \Gamma^\alpha_{33} = \Gamma^3_{3\alpha} = \Gamma^3_{\alpha 3} = 0, \\
&\Gamma^3_{\alpha\beta} = \Gamma^3_{\beta\alpha} = b_{\alpha\beta} - zc_{\alpha\beta}, \\
&\Gamma^\alpha_{3\beta} = \Gamma^\alpha_{\beta 3} = -b^\alpha_\beta + zc^\alpha_\beta \ .
\end{aligned}\right\} \tag{A2.3}$$

It should be noted that in view of these results the n-th covariant derivative of a scalar Φ with respect to x^3 is equal to the n-th partial derivative.

A space vector u^h or u_h at a point on the middle surface may alternatively be represented by the surface vector $u^\alpha_{.}$ or u_α together with the surface invariant $w(=u^3=u_3)$. The covariant surface derivatives of a surface invariant, a surface vector or a surface tensor will be indicated by an additional (Greek) subscript, preceded by a single vertical line, e.g.

$$u_{\alpha|\beta} = u_{\alpha,\beta} - A^{\lambda}_{\alpha\beta} u_{\lambda} \text{ , etc.,} \tag{A2.4}$$

where $A^{\lambda}_{\alpha\beta}$ is the CHRISTOFFEL-symbol of the second kind for the twodimensional middle surface geometry

$$A^{\lambda}_{\alpha\beta} = a^{\lambda\mu} \frac{1}{2}(a_{\mu\alpha,\beta} + a_{\mu\beta,\alpha} - a_{\alpha\beta,\mu}) = \Gamma^{\lambda}_{\alpha\beta}(x^{\nu},0) \tag{A2.5}$$

It should be noted that the covariant surface derivative of a surface invariant is again identical to the partial derivative (cf. (A2.1)). Moreover, although the surface geometry is non-EUCLIDEAN, and the order of repeated covariant differentiation of a surface vector or tensor is therefore not interchangeable, this order of repeated differentiation is again immaterial for a surface invariant, e.g.

$$w_{|\alpha\beta} = w_{|\beta\alpha} = w_{,\alpha\beta} - A^{\lambda}_{\alpha\beta} w_{,\lambda} \text{ .} \tag{A2.6}$$

The GAUSS equation for the middle surface is expressed by

$$b_{\alpha\lambda} b_{\beta\mu} - b_{\alpha\mu} b_{\beta\lambda} = K \varepsilon_{\alpha\beta} \varepsilon_{\lambda\mu} \text{ ,} \tag{A2.7}$$

where K is the GAUSSIAN curvature (A1.6) and $\varepsilon_{\alpha\beta}$ is the ε-tensor on the middle surface. The MAINARDI--CODAZZI equations are expressed by

$$b_{\alpha\beta|\lambda} = b_{\alpha\lambda|\beta} \text{ .} \tag{A2.8}$$

The spatial derivative of a space vector u^h or u_h at a point of the middle surface with respect to a surface coordinate x^{β} may also be expressed by means of surface derivatives

$$\left.\begin{aligned} u^{\alpha}_{\|\beta} &= u^{\alpha}_{|\beta} + \Gamma^{\alpha}_{3\beta}u^{3} = u^{\alpha}_{|\beta} - wb^{\alpha}_{\beta} , \\ u_{\alpha\|\beta} &= u_{\alpha|\beta} - \Gamma^{3}_{\alpha\beta}u_{3} = u_{\alpha|\beta} - wb_{\alpha\beta} , \\ u^{3}_{\|\beta} &= u^{3}_{,\beta} + \Gamma^{3}_{\lambda\beta}u^{\lambda} + \Gamma^{3}_{3\beta}u^{3} = w_{,\beta} + b_{\lambda\beta}u^{\lambda} , \\ u_{3\|\beta} &= u_{3,\beta} - \Gamma^{\lambda}_{3\beta}u_{\lambda} - \Gamma^{3}_{3\beta}u_{3} = w_{,\beta} + b^{\lambda}_{\beta}u_{\lambda} . \end{aligned}\right\} \qquad \text{(A2.9)}$$

Finally, we note GREEN's theorem on the middle surface for the conversion of surface integrals into line integrals and vice versa. If u^{α} is a contravariant surface vector and $\varepsilon_{\alpha\beta}$ the ε-tensor on the middle surface, GREEN's theorem is expressed by

$$\iint_S u^{\alpha}_{|\alpha} \sqrt{a}\, dx^1 dx^2 = \int_C \varepsilon_{\alpha\beta} u^{\alpha} dx^{\beta} . \qquad \text{(A2.10)}$$

The integral in the left-hand member is taken over the area S of the middle surface bounded by the contour C, and the sense of the contour integral in the right-hand member is defined by the sense in which a curvilinear quadrangle is described by the sequence of points with surface coordinates 0,0; 1,0; 1,1; 0,1.

A3. The strain tensor.

The LAGRANGIAN (symmetric) covariant strain tensor in space γ_{ij}, produced by a field of finite displacements with covariant components u_i, is given by

$$2\gamma_{ij} = u_{i\|j} + u_{j\|i} + g^{mn} u_{m\|i} u_{n\|j} . \qquad \text{(A3.1)}$$

The strain components with respect to the surface direction x^{α}, x^{β}, i.e. the strain components in surfaces parallel to the middle surface, are therefore given by

$$2\gamma_{\alpha\beta} = u_{\alpha\|\beta} + u_{\beta\|\alpha} + g^{\lambda\mu}u_{\lambda\|\alpha}u_{\mu\|\beta} + u_{3\|\alpha}u_{3\|\beta} \; . \qquad (A3.2)$$

It is also convenient to introduce the (anti-symmetric) covariant rotation tensor, defined by *)

$$2\omega_{ij} = u_{j\|i} - u_{i\|j} = u_{j,i} - u_{i,j} \; . \qquad (A3.3)$$

Its components with respect to the surface directions x^{α}, x^{β} are given by

$$2\omega_{\alpha\beta} = u_{\beta\|\alpha} - u_{\alpha\|\beta} = u_{\beta,\alpha} - u_{\alpha,\beta} \; , \qquad (A3.4)$$

and its components with respect to the directions x^{α} and x^{3} by

$$2\omega_{\alpha 3} = u_{3\|\alpha} - u_{\alpha\|3} \; . \qquad (A3.5)$$

A4. Compatibility conditions.

The six independent equations of compatibility for the strain tensor are contained in the set of equations**)

$$\gamma_{hi\|jk} + \gamma_{jk\|hi} - \gamma_{hk\|ij} - \gamma_{ij\|hk} +$$

$$+ G^{mn}\Big[(\gamma_{mh\|i} + \gamma_{mi\|h} - \gamma_{hi\|m})(\gamma_{nj\|k} + \gamma_{nk\|j} - \gamma_{jk\|n}) +$$

$$- (\gamma_{mh\|k} + \gamma_{mk\|h} - \gamma_{hk\|m})(\gamma_{ni\|j} + \gamma_{nj\|i} - \gamma_{ij\|n})\Big] = 0 \; , \qquad (A4.1)$$

---

*) Actually this tensor describes the rotation completely only in the case of infinitesimal displacements; however, it is convenient to call this tensor the rotation tensor generally.

**) GREEN and ZERNA [12, p. 62] give (A4.1) in a somewhat more complicated form, which is equivalent to our form in CARTESIAN coordinates. SOKOLNIKOFF [43, p.300] gives the equivalent equation for the EULERIAN strain tensor.

where G^{mn} is the contravariant metric tensor in the deformed body, defined by the equations

$$G^{mn}(g_{nr} + 2\gamma_{nr}) = \delta^m_r \, , \qquad \text{(A4.2)}$$

and δ^m_r represents the KRONECKER delta. If the strains are small everywhere in the body (although the displacements need not be small), the tensor G^{mn} in (A4.1) may be replaced by the metric tensor g^{mn} of the undeformed body, and a considerable simplification results which is of particular importance in practical applications.*) These simplified compatibility conditions will be used in order to evaluate the second derivatives of the strains in surfaces parallel to the middle surface with respect to the normal direction x^3 **)

$$\begin{aligned}\gamma_{\alpha\beta\|33} = & - \gamma_{33\|\alpha\beta} + \gamma_{\alpha3\|3\beta} + \gamma_{\beta3\|3\alpha} + \\ & + g^{\lambda\mu}\Big[(\gamma_{\lambda\alpha\|\beta}+\gamma_{\lambda\beta\|\alpha}-\gamma_{\alpha\beta\|\lambda})(2\gamma_{\mu3\|3}-\gamma_{33\|\mu}) + \\ & - (\gamma_{\lambda\alpha\|3}+\gamma_{\lambda3\|\alpha}-\gamma_{\alpha3\|\lambda})(\gamma_{\mu\beta\|3}+\gamma_{\mu3\|\beta}-\gamma_{\beta3\|\mu})\Big] + \\ & + (\gamma_{3\alpha\|\beta}+\gamma_{3\beta\|\alpha}-\gamma_{\alpha\beta\|3})\gamma_{33\|3} - \gamma_{33\|\alpha}\,\gamma_{33\|\beta} \, . \end{aligned} \qquad \text{(A4.3)}$$

In order to estimate the order of magnitude of the right-hand member of (A4.3), the basic approximate assumption of shell theory, i.e. the assumption

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*) This simplification in the case of small strains and finite displacements seems to have been overlooked by most writers.

**) This intrinsic approach, without direct reference to the displacements, seems to be due to CHIEN [5].

of plane stress will be used, i.e. (cf. (2.3),(2.4))

$$\gamma_{\alpha 3} = 0 \,, \qquad \gamma_{33} = -\frac{\nu}{1-\nu} g^{\alpha\beta}\gamma_{\alpha\beta} \,. \tag{A4.4}$$

We obtain now from this assumption

$$\begin{aligned}\gamma_{3\alpha\|\beta} &= \gamma_{3\alpha,\beta} - \Gamma^{\lambda}_{3\beta}\gamma_{\lambda\alpha} - \Gamma^{3}_{3\beta}\gamma_{3\alpha} - \Gamma^{\lambda}_{\alpha\beta}\gamma_{3\lambda} - \Gamma^{3}_{\alpha\beta}\gamma_{33} = \\ &= -\Gamma^{\lambda}_{3\beta}\gamma_{\lambda\alpha} + \frac{\nu}{1-\nu}\Gamma^{3}_{\alpha\beta} g^{\lambda\mu}\gamma_{\lambda\mu} \,,\end{aligned} \tag{A4.5}$$

and hence on the middle surface

$$\overset{\circ}{\gamma}_{3\alpha\|\beta} = b^{\lambda}_{\beta}\overset{\circ}{\gamma}_{\lambda\alpha} + \frac{\nu}{1-\nu} b_{\alpha\beta} a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu} \,. \tag{A4.6}$$

By a similar analysis we obtain

$$\overset{\circ}{\gamma}_{3\alpha\|3} = 0 \,, \tag{A4.7}$$

$$\overset{\circ}{\gamma}_{33\|\alpha} = -\frac{\nu}{1-\nu} a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu|\alpha} \,, \tag{A4.8}$$

$$\overset{\circ}{\gamma}_{33\|3} = -\frac{\nu}{1-\nu} a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu\|3} \,, \tag{A4.9}$$

where the superscript $\circ$ again refers to the middle surface. The second derivatives in (A4.3) are reduced in a similar way: we have

$$\begin{aligned}\gamma_{33\|\alpha\beta} &= \left(\gamma_{33\|\alpha}\right)_{,\beta} - \Gamma^{\lambda}_{3\beta}\gamma_{\lambda 3\|\alpha} - \Gamma^{3}_{3\beta}\gamma_{33\|\alpha} - \Gamma^{\lambda}_{3\beta}\gamma_{3\lambda\|\alpha} + \\ &\quad - \Gamma^{3}_{3\beta}\gamma_{33\|\alpha} - \Gamma^{\lambda}_{\alpha\beta}\gamma_{33\|\lambda} - \Gamma^{3}_{\alpha\beta}\gamma_{33\|3} = \\ &= \left(\gamma_{33\|\alpha}\right)_{,\beta} - 2\Gamma^{\lambda}_{3\beta}\gamma_{\lambda 3\|\alpha} - \Gamma^{\lambda}_{\alpha\beta}\gamma_{33\|\lambda} - \Gamma^{3}_{\alpha\beta}\gamma_{33\|3} \,.\end{aligned} \tag{A4.10}$$

By appropriate use of (A4.6), (A4.8) and (A4.9), and of the rules for covariant space and surface derivatives, we obtain finally on the middle surface

$$\overset{\circ}{\gamma}_{33\|\alpha\beta} = -\frac{\nu}{1-\nu}\left(a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu}\right)_{|\alpha\beta} + \frac{\nu}{1-\nu}\, b_{\alpha\beta}a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu\|3} +$$

$$+ 2b^{\lambda}_{\alpha}b^{\mu}_{\beta}\overset{\circ}{\gamma}_{\lambda\mu} + \frac{2\nu}{1-\nu}\, c_{\alpha\beta}a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu} \ . \qquad \text{(A4.11)}$$

By an entirely similar analysis we obtain *)

$$\overset{\circ}{\gamma}_{\alpha 3\|3\beta} = b^{\lambda}_{\beta}\overset{\circ}{\gamma}_{\alpha\lambda\|3} + \frac{\nu}{1-\nu}\, b_{\alpha\beta}a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu\|3} +$$

$$+ c^{\lambda}_{\beta}\overset{\circ}{\gamma}_{\alpha\lambda} + \frac{\nu}{1-\nu}\, c_{\alpha\beta}a^{\lambda\mu}\overset{\circ}{\gamma}_{\lambda\mu} \ . \qquad \text{(A4.12)}$$

The order of magnitude of the second derivatives $\overset{\circ}{\gamma}_{\alpha\beta\|33}$ at the middle surface is now conveniently assessed by assuming a suitable system of surface coordinates in which the components of the metric tensor $a_{\alpha\beta}$ and its determinant are of order unity. The tensorial strain components in the middle surface $\overset{\circ}{\gamma}_{\alpha\beta}$ are then of the same order of magnitude as the physical middle surface strain components ε (ε_1, ε_2, ψ) (cf. par. A5), the derivatives $\gamma_{\alpha\beta\|\lambda} = \gamma_{\alpha\beta|\lambda}$ have the order $\frac{d\varepsilon}{ds}$, where ds is the arc element along any curve on the middle surface, and the normal covariant derivatives $\overset{\circ}{\gamma}_{\alpha\beta\|3}$ have the same order as the physical changes pf curvature κ (κ_1, κ_2, τ) (cf. par. A5). The quadratic terms in the right-hand member of (A4.3) are now easily seen to be of order

$$\left(\frac{d\varepsilon}{ds}\right)^2 , \quad \frac{\varepsilon^2}{R^2} , \quad \frac{\varepsilon}{R}\frac{d\varepsilon}{ds} , \quad \kappa^2 \ \text{ or } \ \kappa\frac{\varepsilon}{R} \ , \qquad \text{(A4.13)}$$

---

*) The fact that the order of covariant differentiation in space is immaterial is easily seen to be confirmed by the symmetry of (A4.11) with respect to α and ß. Likewise, it may be verified, be it after a more laborious analysis, that interchanging the order of differentiation with respect to x^3 and x^{β} does not affect the result (A4.12).

where R is the smallest principal radius of curvature of the middle surface. Likewise, the order of magnitude of the linear terms in the right-hand member of (A4.3) is given by

$$\frac{d^2\varepsilon}{ds^2}, \quad \frac{\varepsilon}{R^2} \quad \text{or} \quad \frac{\kappa}{R}. \tag{A4.14}$$

If L denotes the "wave length" of the deformation pattern on the middle surface, defined by $\frac{d\varepsilon}{ds} = \mathcal{O}(\varepsilon/L)$, the order of magnitude of $\overset{\circ}{\gamma}_{\alpha\beta\|33}$ is evidently given by

$$\frac{\varepsilon}{L^2}, \quad \frac{\varepsilon}{R^2}, \quad \frac{\kappa}{R} \quad \text{or} \quad \kappa^2, \tag{A4.15}$$

whichever of these may be critical. The same estimate holds of course for $a^{-1/2}\,\overset{\circ}{\gamma}_{\alpha\beta\|33}$ in any coordinate system on the middle surface if a_{11}, a_{22} and $a^{1/2}$ are of the same order of magnitude (not necessarily of order unity).

A5. Translation into non-tensorial notation.

Let α and β denote any set of orthogonal parametric curves on the middle surface, $A d\alpha$ and $B d\beta$ the line elements along these curves, and R_1, R_2, T the radii of curvature and the radius of torsion of the middle surface along the parametric curves. If we now identify x^1 with α and x^2 with β, we have for the components of the fundamental tensors of the middle surface

$$
\left.
\begin{aligned}
& a_{11} = A^2 , \quad a_{12} = a_{21} = 0 , \quad a_{22} = B^2 ; \\
& a^{11} = \frac{1}{A^2} , \quad a^{12} = a^{21} = 0 , \quad a^{22} = \frac{1}{B^2} ; \\
& b_{11} = \frac{A^2}{R_1} , \quad b_{12} = b_{21} = \frac{AB}{T} , \quad b_{22} = \frac{B^2}{R_2} ; \\
& b_1^1 = \frac{1}{R_1} , \quad b_1^2 = \frac{A}{BT} , \quad b_2^1 = \frac{B}{AT} , \quad b_2^2 = \frac{1}{R_2} ; \\
& c_{11} = A^2 \left[\left(\frac{1}{R_1} \right)^2 + \left(\frac{1}{T} \right)^2 \right] , \quad c_{12} = c_{21} = \frac{AB}{T} \left[\frac{1}{R_1} + \frac{1}{R_2} \right] , \\
& c_{22} = B^2 \left[\left(\frac{1}{R_2} \right)^2 + \left(\frac{1}{T} \right)^2 \right] .
\end{aligned}
\right\} \quad \text{(A5.1)}
$$

The mean and GAUSSIAN curvatures are given by

$$
2H = \frac{1}{R_1} + \frac{1}{R_2} , \qquad K = \frac{1}{R_1 R_2} - \left(\frac{1}{T} \right)^2 , \qquad \text{(A5.2)}
$$

and the CHRISTOFFEL-symbols of the second kind for the middle surface (A2.5) are given by

$$
\left.
\begin{aligned}
& A_{11}^1 = \frac{1}{A} \frac{\partial A}{\partial \alpha} , \qquad A_{22}^2 = \frac{1}{B} \frac{\partial B}{\partial \beta} , \\
& A_{12}^1 = A_{21}^1 = \frac{1}{A} \frac{\partial A}{\partial \beta} , \quad A_{12}^2 = A_{21}^2 = \frac{1}{B} \frac{\partial B}{\partial \alpha} , \\
& A_{22}^1 = - \frac{B}{A^2} \frac{\partial B}{\partial \alpha} , \quad A_{11}^2 = - \frac{A}{B^2} \frac{\partial A}{\partial \beta} .
\end{aligned}
\right\} \quad \text{(A5.3)}
$$

The equations of GAUSS and MAINARDI-CODAZZI now take the well-known form

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -ABK \; ; \tag{A5.4}$$

$$\left.\begin{aligned} \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{T}\right) &= \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{R_1}\right) - \frac{2}{ABT}\frac{\partial B}{\partial\alpha} + \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{AB}\frac{\partial A}{\partial\beta} \, , \\ \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{T}\right) &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{R_2}\right) - \frac{2}{ABT}\frac{\partial A}{\partial\beta} - \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{AB}\frac{\partial B}{\partial\alpha} \, . \end{aligned}\right\} \tag{A5.5}$$

The physical components of any surface vector or tensor are defined as the components in a (locally) CARTESIAN coordinate system with axes along the (orthogonal) α- and β-curves. For the displacement vector with physical components u and v in the α- and β-directions we have

$$u_1 = Au \, , \quad u^1 = \frac{u}{A} \, , \quad u_2 = Bv \, , \quad u^2 = \frac{v}{B} \, , \tag{A5.6}$$

with similar translation rules for other surface vectors. The translation formulae for the symmetric covariant tensors of extensional strains and of changes of curvature have already been given in (2.13) and (2.14). The antisymmetric covariant tensor of the rotation $\omega_{\alpha\beta}$ around the normal (3.6) is given in terms of the physical rotation Ω (3.10) by

$$\omega_{12} = -\omega_{21} = AB\,\Omega \, . \tag{A5.7}$$

Likewise, the translation rules for the asymmetric contravariant tensor of stress resultants $n_*^{\alpha\beta}$ (3.26) read in terms of the physical components (3.34) to (3.37)

$$n_*^{11} = \frac{1}{A^2} N_1^* \, , \qquad n_*^{22} = \frac{1}{B^2} N_2^* \, ,$$

$$n_*^{12} = \frac{1}{AB} S_{12}^* \, , \qquad n_*^{21} = \frac{1}{AB} S_{21}^* \, ,$$

with similar translation rules for the contravariant tensor of stress couples (3.27) in terms of its physical components (3.31) to (3.33).