Critical Exponents in Long-Range Percolation

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Theory and Estimation

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Layman's summary

English version

One question that appears regularly in various fields is when many small disconnected networks merge into one large network. This can describe, for example, how information spreads on social networks, or how bonds form in a material. To study this question, we consider a model in which such networks are represented as collections of random connections between points, where short connections are more common than long ones. When the connection strength is small, many small disconnected networks exist. When this connection strength is increased, a large connected network eventually appears.

We study what this transition from many small networks to one large network looks like. We combine theoretical arguments with computer simulations to understand where the transition happens and what the network looks like near this point. We also study how the sizes of connected groups behave at the transition, which gives information about the structure of all the networks at this critical stage.

Nederlandse versie

Een vraag die in verschillende vakgebieden regelmatig aan bod komt, is wanneer precies veel kleine netwerken verbonden raken en samen een groter netwerk vormen. Dit kan bijvoorbeeld beschrijven hoe informatie zich verspreidt op sociale netwerken, of hoe bindingen ontstaan in een materiaal. Om deze vraag te beantwoorden, kijken we naar een model waarin we zulke netwerken voorstellen als een verzameling willekeurige verbindingen tussen punten, waarbij korte verbindingen veel vaker voorkomen dan lange verbindingen. Wanneer de verbindingssterkte klein is, bestaan er veel kleine losse netwerken. Wanneer we deze verbindingssterkte vergroten, vormen de kleine netwerken op een gegeven moment samen één groot netwerk.

Wij onderzoeken deze overgang van veel kleine netwerken naar één groot netwerk. Door theoretische argumenten en simulaties te combineren, kunnen we onderzoeken wanneer deze overgang plaatsvindt en hoe het netwerk er rond dit overgangspunt uitziet. Ook onderzoeken wij hoe groot de verbonden groepen zijn tijdens deze overgang. Dit geeft inzicht in de structuur van alle netwerk op dit punt.

Abstract

In this thesis we study the long-range percolation model on $\mathbb{Z}^d$, where each pair of vertices $x, y \in \mathbb{Z}^d$ form a connection with probability $1 - \exp(-\beta J(x, y))$, and $J(x, y)$ decays asymptotically with the form $\|x - y\|^{-d\alpha}$. The parameter $\alpha > 0$ is fixed, while β can be varied.

This model is an extension of the classical Bernoulli bond percolation model allowing the modeling of connection phenomena where long-distance connections play a crucial role. In this paper we study the critical value β_c, the percolation probability $\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty)$ and we investigate the critical exponent δ explaining the cluster decay at criticality.

We compile known bounds for the critical exponent δ and derive foundational results for β_c. In the numerical part we simulate long-range percolation on finite boxes, create new estimators for the percolation probability $\theta(\beta)$ and critical parameter β_c. Using this estimate for β_c we estimate the critical parameter δ using linear regression.

The estimators for $\theta(\beta)$ and β_c show consistent stable behaviour conforming to theory. The estimates for δ in contrast are sensitive to the finite size approximation, showcasing the limits of simulating critical parameters on a finite scale.

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Contents

Chapter 1

Introduction

Percolation is a mathematical model for studying the emergence of large-scale connectivity in random systems. The percolation model was originally developed to study fluid flow through porous media [14], but has since evolved into a central tool in probability theory and the natural sciences for modeling phenomena in which large-scale connectivity emerges from local interactions. Percolation plays a key role in Physics, as it is one of the simplest models exhibiting phase transitions [30].

In this thesis we study *long-range percolation* on $\mathbb{Z}^d$. Long-range percolation allows connections between any two arbitrary vertices where the probability of connection decreases as the distances increases. The formal model we study is outlined below in Definition 1.0.1.

Definition 1.0.1 (Long-range percolation model)**.** Let $d \geq 1$, $\alpha \geq 0$, and $\beta \geq 0$. Let $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ be a symmetric, translation-invariant kernel satisfying

$$J(x, y) = J(x - y, 0), \qquad J(x, y) = J(y, x), \qquad J(x, x) = 0.$$

Furthermore assume that the kernel behaves like a power law.

$$J(x, y) \asymp \|x - y\|^{-d\alpha},$$

where throughout this paper we maintain the notation of $\|\cdot\|$ being the Euclidean norm on $\mathbb{R}^d$.

Consider the edge set

$$\mathcal{E} = \{\{x, y\} : x, y \in \mathbb{Z}^d, x \neq y\}.$$

Each edge $e = \{x, y\} \in \mathcal{E}$ is declared open independently with probability

$$\mathbb{P}_\beta(x \sim y) = 1 - \exp(-\beta J(x, y)).$$

Where $x \sim y$ denotes that the edge $\{x, y\}$ is open. We denote by $\mathbb{P}_\beta$ the resulting product measure on $\{0, 1\}^{\mathcal{E}}$. We write $G = (\mathbb{Z}^d, E)$ for the resulting random graph, where $E \subseteq \mathcal{E}$ is the set of open edges.

Notation Throughout this thesis we use the parametrisation

$$J(x,y) \asymp \|x-y\|^{-d\alpha}.$$

This corresponds to the alternative convention $J(x,y) \asymp \|x-y\|^{-(d+\sigma)}$ through the change of variables $\sigma = d(\alpha - 1)$.

For the long-range percolation model, the parameter β controls the density of open edges. The parameter β can be thought of as the inverse temperature. Increasing β makes any edge more likely to be open, leading to a more interconnected graph. The parameter α controls the strength of the long range effect. A smaller value of α makes long edges more likely, whereas a larger value of α makes the model behave more locally by reducing the likelihood of long edges.

Many real-world systems contain interactions that are not purely local but also depend on long-range connections. Examples include social networks, transportation networks, and the spread of infectious diseases, where long-distance connections can significantly influence global connectivity [7]. Long-range percolation provides a mathematical model in which such long-distance interactions can be studied.

We would like to understand how the connected components of the long-range percolation random graph behave as the parameter β varies. The central object is the connected component, also called cluster, containing the origin. We define this cluster by

$$K_0 := \{x \in \mathbb{Z}^d : x \leftrightarrow 0\},$$

where $x \leftrightarrow y$ means that there exists a path of open edges between x and y.

Definition 1.0.2 (Theta function)**.** The theta function is defined by

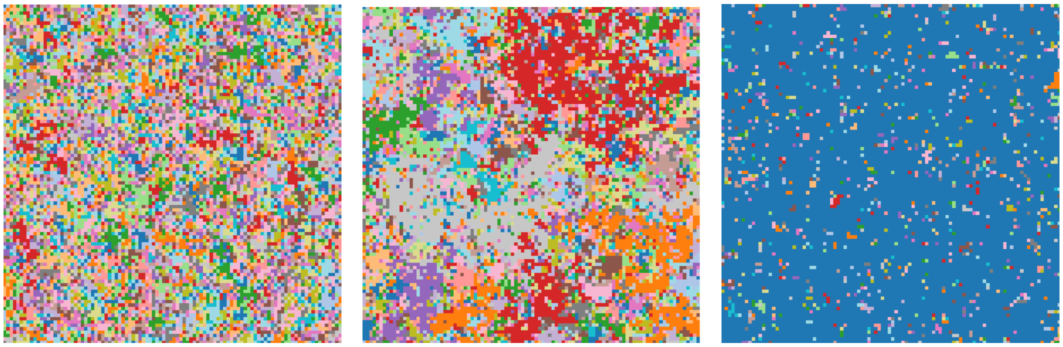
$$\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty).$$

Thus $\theta(\beta)$ is the probability that the origin belongs to an infinite cluster.

Definition 1.0.3 (Critical parameter)**.** The critical parameter is defined by

$$\beta_c = \inf\{\beta \geq 0 : \theta(\beta) > 0\}.$$

The value β_c separates the different macroscopic regimes of the model. For $\beta < \beta_c$, the model is called subcritical and the origin does not belong to an infinite cluster. For $\beta > \beta_c$, the model is called supercritical and the origin belongs to an infinite component with positive probability. The transition point $\beta = \beta_c$ is called the critical point. Figure 1.1 gives a finite-volume illustration of these three regimes.



(a) Subcritical phase (b) Critical phase (c) Supercritical phase

Figure 1.1: A simulation of long-range percolation on a finite 2 dimensional grid $\Lambda = \{0, \ldots, 99\}^2$ with $\alpha = 1.8$. For a) $\beta = 0.1$, for b) $\beta = 0.28$ and for c) $\beta = 0.4$. Each colour represents a connected component.

A central question in long-range percolation is how the model behaves at the critical point. Away from criticality, the main distinction is whether infinite-scale connectivity is present. At criticality, however, the focus shifts to the distribution of large finite clusters. In many percolation models this distribution is expected to decay polynomially, and this decay is described by critical exponents.

In this thesis we focus in particular on the critical exponent δ, which describes the decay of the critical cluster-size tail through the asymptotic relation

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-1/\delta + o(1)}.$$

Where $o(1)$ is some function tending to 0 as $n \to \infty$. When this exponent is well-defined, a larger value of δ corresponds to a heavier tail and therefore to a higher probability of observing large clusters at criticality. Recent work shows that critical behaviour in long-range percolation depends on both the dimension d and the long-range parameter α [2, 18].

The main goal of this thesis is to study the critical behaviour of long-range percolation by combining known theoretical results with finite-volume simulations. First, we prove that the phase transition is non-trivial in the parameter regimes considered, meaning that

$$0 < \beta_c < \infty.$$

We then discuss known theoretical bounds for the critical exponent δ. Finally, we use repeated finite-volume simulations to study the percolation probability, pseudo-critical estimates of β_c, and the numerical estimation of the critical exponent δ.

1.1 Thesis overview

In Chapter 2, we introduce the long-range percolation model and prove several fundamental results about its phase transition. In particular, we show that the critical parameter is non-trivial in the parameter regimes considered in this thesis. More precisely, for $d = 1$ and $\alpha \in (1, 2)$, and for $d \geq 2$ and $\alpha > 1$, the model has a nontrivial phase transition.

In Chapter 3, we study critical exponents. In particular, assuming that the critical exponent δ exists, we show that it satisfies the lower bound

$$\delta \geq \frac{d + \big(d(\alpha - 1) \wedge 1\big)}{d - \big(d(\alpha - 1) \wedge 1\big)}.$$

In Chapter 4, we introduce an efficient algorithm for sampling finite-volume long-range percolation graphs. This algorithm is then used in Chapter 5 to numerically study the model in one dimension. We consider finite-volume estimators for the percolation probability $\theta(\beta)$, the critical parameter β_c, and the critical exponent δ. The percolation probability is estimated using both a local estimator, based on the size of the cluster containing the origin, and a global estimator, based on the mass of large clusters. The critical parameter β_c is estimated using susceptibility-based and ratio-based methods, while δ is estimated from the decay of the cluster-size distribution near criticality.

Chapter 2

Foundational Results for Long-Range Percolation

In this thesis we desire to better understand how the long-rang percolation model behaves at criticality. However, prior to assessing this question we provide some foundational results of the long-range percolation model. First in Section 2.1 we present the necessary conditions on the parameters d and α for each vertex to have finite degree. In 2.2 we show that the critical parameter β_c is always positive $\beta_c > 0$. Finally, in section 2.3 we present the results that $\beta_c < \infty$.

2.1 Finite degree of vertices

For the behaviour of the model we are particularly interested in the number of vertices at a distance r:

$$S(r) = \left|\{x \in \mathbb{Z}^d : \|x\|_1 = r\}\right|$$

This quantity will be used to determine the expected number of edges, which is a keystone in many of these papers proofs.

Theorem 2.1.1 (Growth of a sphere on $\mathbb{Z}^d$)**.** Let $d \geq 1$. Let $S(r) = \left|\{x \in \mathbb{Z}^d : \|x\|_1 = r\}\right|$ denote the sphere of radius r on $\mathbb{Z}^d$. Then we have that $S(r) \asymp r^{d-1}$

The proof of Theorem 2.1.1 can be found in Appendix A.

We mostly want to concern ourselves with the emerging behaviour of the graph. To that end we are interested in the case where the expected degree of each vertex is finite, as to make sure that one single vertex does not produce an infinite component. One theorem we will make use of is Theorem 2.1.2.

Theorem 2.1.2 (Asymptotic equivalence of L^p norms on $\mathbb{R}^d$ [21].)**.** Let $p, q \in [1, \infty]$. Then for any two L^p norms $\|\cdot\|_p, \|\cdot\|_q$ on $\mathbb{R}^d$ we have that $\|\cdot\|_p \asymp \|\cdot\|_q$.

With both Theorem 2.1.1 and Theorem 2.1.2, we are now ready to prove Theorem 2.1.3.

Theorem 2.1.3 (Finite expected degree)**.** Let $d \geq 1$, $\alpha \geq 0$ and $\beta > 0$. Let $G = (\mathbb{Z}^d, E)$ be a long-range percolation graph as defined in Definition 1.0.1. Then we have that for any $x \in \mathbb{Z}^d$, $\mathbb{E}_\beta(\deg(x)) < \infty$ if and only if $\alpha > 1$.

Proof. Without loss of generality we may assume that $x = 0$, as the translation invariance of $J(x, y)$ guarantees that the expected degree of any vertex $x \in \mathbb{Z}^d$ is equal. We then have that:

$$\mathbb{E}_\beta(\deg(0)) = \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \mathbb{P}_\beta(0 \sim x) = \sum_{x\in\mathbb{Z}^d\setminus\{0\}} (1 - \exp(-\beta J(0, x))) \leq \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \beta J(0, x)$$

where in the last inequality we used that $1 - e^{-t} \leq t$ for $t \geq 0$.

Now, by the assumption that $J(x, 0) \asymp \|x\|^{-d\alpha}$, we can find a $C_1 > 0$ such that $J(x, 0) \leq C_1\|x\|^{-d\alpha}$. Additionally we have from Theorem 2.1.2 that $\|x\| \asymp \|x\|_1$, and so similarly there exists $C_2 > 0$ such that $\|x\| \geq C_2\|x\|_1$. Therefore it follows that

$$\sum_{x\in\mathbb{Z}^d\setminus\{0\}} \beta J(0, x) \leq C_1\beta \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \|x\|^{-d\alpha} \leq C_1 C_2^{-d\alpha}\beta \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \|x\|_1^{-d\alpha}.$$

We now can rearrange the sum by grouping terms $x \in \mathbb{Z}^d$ with equal L^1 norm:

$$C_1 C_2^{-d\alpha}\beta \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \|x\|_1^{-d\alpha} = C_1 C_2^{-d\alpha}\beta \sum_{r=1}^{\infty} S(r) r^{-d\alpha} \leq \tilde{C} \sum_{r=1}^{\infty} r^{d-1-d\alpha}.$$

Here in the last inequality we used Theorem 2.1.1 to bound $S(r)$ by $C_3 r^{d-1}$ for some $C_3 > 0$. The term on the last line is a p-series which converges if and only if $d - 1 - d\alpha < -1$. Some elementary computations then show that $-d\alpha < -d$, which implies $\alpha > 1$.

To see the reverse implication (divergence when $\alpha \leq 1$), note that as $\|x\| \to \infty$, $J(0, x) \to 0$. For y small enough we have that $1 - e^{-y} \geq \frac{1}{2}y$ and therefore $1 - \exp(-\beta J(0, x)) \geq \frac{1}{2}\beta J(0, x)$ for all sufficiently large x. By a similar bounding argument using the lower bounds of $J(0, x)$ and the norm equivalence, the sum is bounded below by a multiple of $\sum_{r=1}^{\infty} r^{d-1-d\alpha}$, which diverges if $d-1-d\alpha \geq -1$ (i.e., $\alpha \leq 1$). This completes the proof. □

From (2.1) we can see that having $\alpha > 1$ is a sufficient condition to have the expected degree of a vertex be finite, and so henceforth we will only consider the percolation model with $\alpha > 1$. A corollary which follows from the proof of Theorem 2.1.3 is Corollary 2.1.4.

Corollary 2.1.4. Let $d \geq 1$, $\alpha \geq 0$ and $\beta > 0$. Let $G = (\mathbb{Z}^d, E)$ be a long-range percolation graph as defined in 1.0.1. Then $\mathbb{E}_\beta(\deg(0)) < \infty$ if and only if $\sum_{x\in\mathbb{Z}^d} J(0, x) < \infty$.

Proof. Note that for $t \in [0, 1]$,

$$\frac{1}{2}t \leq 1 - e^{-t} \leq t.$$

Thus, as $J(0, x) \asymp \|x\|^{-d\alpha}$, we have that $J(0, x) \to 0$ as $\|x\| \to \infty$. Therefore

$$J(0, x) \asymp 1 - \exp(-\beta J(0, x))$$

Consequently,

$$\sum_{x\in\mathbb{Z}^d} J(0, x) \asymp \sum_{x\in Z^d} 1 - \exp(-\beta J(0, x)) = \mathbb{E}_\beta(\deg(0)).$$

It then follows that $\sum_{x\in\mathbb{Z}^d} J(0, x)$ converges if and only if $\mathbb{E}_\beta(\deg(0))$ is finite. □

Having established basic properties of the long-range percolation model, we now turn to one of its central questions: the existence of a phase transition. Recall from Definition 1.0.2 that

$$\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty)$$

denotes the probability that the origin belongs to an infinite cluster. The critical parameter

$$\beta_c = \inf\{\beta \geq 0 : \theta(\beta) > 0\}$$

marks the transition between the subcritical and supercritical regimes. A natural question is whether this phase transition is nontrivial, i.e. whether

$$0 < \beta_c < \infty.$$

We investigate conditions under which this holds. In section 2.2 we prove that if the expected degree of a vertex is finite, then the critical parameter β_c is strictly positive. In section 2.3 we present a proof that that the critical parameter β_c is finite if $d = 1$ and $\alpha \in (1, 2)$ or $d \geq 2$ and $\alpha > 1$.

2.2 Positivity of the critical parameter

The aim of this section is to prove that $\beta_c > 0$ whenever $\alpha > 1$. The key idea is to dominate the cluster exploration process by a subcritical Galton–Watson branching process.

Theorem 2.2.1 (Positive critical parameter)**.** Let $G = (\mathbb{Z}^d, E)$ be a long-range percolation model with $\alpha > 1$, $d \geq 1$. Then there exists $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$, the cluster containing the origin $K(0)$ is finite almost surely. In particular, $\theta(\beta) = 0$ and hence $\beta_c > 0$.

We will prove (2.2.1) by proving that there is a dominating model that has finite components almost surely. For this we need to introduce the idea of stochastic domination.

Definition 2.2.1 (Stochastic domination [31])**.** Let X and Y be two real valued random variables. We say that Y *stochastically dominates* X if there exists a joint construction (a coupling) of X and Y on a shared probability space such that

$$X \leq Y \qquad \text{almost surely.}$$

Lemma 2.2.2 (Branching process domination)**.** *Let $\beta \geq 0$, $\alpha > 1$. Let $G = (V, E)$ be a long percolation model as defined in Definition 1.0.1. Denote the connected component that includes the origin 0 by $K(0)$. Define the exploration process as follows:*

$$K_0 = \{0\}, \qquad K_n = \{x \in \mathbb{Z}^d : d(0, x) \leq n\}.$$

Where $d(x, y)$, the graph distance, denotes number of edges on the shortest path of open edges from $\mathbf{x}$ to $\mathbf{y}$, and if no such path exists then $d(x, y) = \infty$. Then exists a coupling of the exploration process $(K_n)_{n\geq 0}$ and a Galton-Watson process $(X_n)_{n\geq 0}$ such that

$$|K_n| \leq \sum_{k=0}^{n} X_k \qquad \textit{almost surely } \forall n \geq 0.$$

Proof. Define a branching process $(X_n)_{n\geq 0}$ by

$$X_0 = 1, \qquad X_{n+1} = \sum_{k=1}^{X_n} \xi_k,$$

where (ξ_k) are i.i.d. copies of

$$\xi = \sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\{0\sim x\}}.$$

Construct the exploration process and the branching process on the same probability space as follows. Starting from 0, reveal all open edges incident to each newly discovered vertex independently according to the percolation law. The exploration process only keeps vertices that have not been discovered previously.

For the branching process, however, every revealed open edge gives rise to a new offspring, even if its endpoint has already appeared earlier in the exploration. Consequently, the branching process may count the same vertex multiple times, while the exploration process counts each vertex at most once.

We thus have that the number of vertices counted by our exploration process K_n will always be strictly less than the size of the Galton-watson tree up to generation n. In other words

$$|K_n| \leq \sum_{k=0}^{n} X_k, \qquad n \geq 0,$$

almost surely. □

Now that we have bounded K_n, the number of vertices at a graph distance n from 0, we need to show that the Galton-Watson tree dies out almost surely to prove boundedness.

Lemma 2.2.3 (Expected offspring bound for the Galton Watson-process [15])**.** *Let* $(X_n)_{n\geq 0}$ *be a branching process with offspring random variable*

$$\xi = \sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\{0\sim x\}}.$$

The offspring random variable ξ *satisfies*

$$\mathbb{E}_\beta[\xi] \leq \beta \sum_{x\in\mathbb{Z}^d} J(0,x).$$

Proof. By linearity of expectation,

$$\mathbb{E}_\beta[\xi] = \sum_{x\in\mathbb{Z}^d} \mathbb{P}_\beta(0\sim x) = \sum_{x\in\mathbb{Z}^d} \Big(1-\exp\big(-\beta J(0,x)\big)\Big).$$

Using the inequality $1-e^{-t}\leq t$ for $t\geq 0$, we obtain

$$\mathbb{E}_\beta[\xi] \leq \sum_{x\in\mathbb{Z}^d} \beta J(0,x).$$

□

For a branching process the following lemma is a well known fact:

Lemma 2.2.4 (Extinction of the branching process [15])**.** *Let $(X_n)_{n\geq 0}$ be a branching process with the offspring random variable ξ. If $\mathbb{E}[\xi] < 1$, then*

$$\sum_{n=0}^{\infty} X_n < \infty \quad \textit{almost surely.}$$

Now we have all the pieces needed to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Let G=(V,E) be a random percolation model as defined in Definition 1.0.1 with parameters $\alpha > 1$, $d \geq 1$ and $0 < \beta < \left(\sum_{x\in\mathbb{Z}^d} J(0,x)\right)^{-1}$. Note that we can choose β in such a way as by Theorem 2.1.3 we have that

$$\alpha > 1 \iff \mathbb{E}_\beta(\deg(0)) < \infty \iff \sum_{x\in\mathbb{Z}^d} J(0,x) < \infty$$

Let $(X_n)_{n\geq 0}$ be the branching process as in 2.2.2. Define a coupling of $|K_n|$ and $\sum_{k=0}^{n} X_n$ as in 2.2.2. By Lemma 2.2.3 we have that for this coupled branching process $(X_n)_{n\geq 0}$, its offspring ξ satisfies

$$\mathbb{E}_\beta[\xi] < 1.$$

Therefore by Lemma 2.2.4:

$$\sum_{n=0}^{\infty} X_n < \infty \quad \text{almost surely.}$$

Finally, using lemma 2.2.2, together with the fact that $K_n \uparrow K(0)$ as $n \to \infty$, we obtain that:

$$|K(0)| = \lim_{n\to\infty} |K_n| \leq \sum_{n=0}^{\infty} X_n < \infty \quad \text{almost surely.}$$

Therefore $\theta(\beta) = 0$, and we conclude that $\beta_c > 0$. □

2.3 The finiteness of the critical parameter

We now wish to show that $\beta_c < \infty$. However, first it is necessary to simplify our asymptotical calculations. In Lemma 2.3.1 we show that it is sufficient to just consider the kernels in the form $J(x,y) = \|x-y\|^{-d\alpha}$.

Lemma 2.3.1 (Reduction to the standard power-law kernel)**.** *Let $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ be a symmetric, translation-invariant kernel satisfying*

$$J(x,y) \asymp \|x-y\|^{-d\alpha}.$$

Then there exists a constant $c_J > 0$ such that, for every $\beta \geq 0$, the long-range percolation model with kernel J and parameter β stochastically dominates the long-range percolation model with kernel

$$\widetilde{J}(x,y) = \|x-y\|^{-d\alpha}$$

and parameter $\widetilde{\beta} = \beta c_J$.

Proof. Since $J(x,y) \asymp \|x-y\|^{-d\alpha}$, there exists $c_J > 0$ such that

$$J(x,y) \geq c_J \|x-y\|^{-d\alpha}$$

for all distinct $x, y \in \mathbb{Z}^d$. Hence

$$1 - \exp(-\beta J(x,y)) \geq 1 - \exp(-\beta c_J \|x-y\|^{-d\alpha}).$$

The right-hand side is precisely the probability that an edge is open in the long-range percolation model with parameter $\widetilde{\beta} = \beta c_J$.

To make the domination explicit, assign to each edge $e = \{x, y\}$ an independent random variable $U_e \sim \text{Uniform}(0,1)$. Declare e open in the standard model if

$$U_e \leq 1 - \exp(-\beta c_J \|x-y\|^{-d\alpha}),$$

and declare e open in the model with kernel J if

$$U_e \leq 1 - \exp(-\beta J(x,y)).$$

By the inequality above, every edge that is open in the standard model is also open in the model with kernel J. Therefore the model with kernel J stochastically dominates the standard power-law model. □

By Lemma 2.3.1, it is enough to prove finiteness of β_c for the standard kernel

$$J(x,y) = \|x-y\|^{-d\alpha}.$$

Indeed, if the standard model percolates for some finite parameter $\widetilde{\beta}$, then the original model percolates for $\beta = \widetilde{\beta}/c_J$. Hence the original model also has finite critical parameter.

2.3.1 The one dimensional case

The main result for the 1-dimensional case is Theorem 2.3.2. This theorem was originally proven by C. M. Newman and L. S. Schulman [28]. We present a slightly different proof than the one they use, but base our proof on the same core ideas present in their paper.

Theorem 2.3.2 (The finiteness of the critical parameter in one dimension [28])**.** Let $G = (\mathbb{Z}^d, E)$ be a random percolation model as defined in Definition 1.0.1 with parameters α, β, d. Then for $d = 1$, $\alpha \in (1,2)$, there exists a β such that $\theta(\beta) > 0$ and thus $\beta_c < \infty$

The main idea is to construct a sequence of growing intervals that contain large connected components with probabilities bounded away from zero. We first introduce some terminology. Consider the set $\{0, \dots, L-1\} \subset \mathbb{Z}$, and denote the largest cluster fully connected within this set by $B(L)$. Denote the number of vertices inside this cluster by $|B(L)|$.

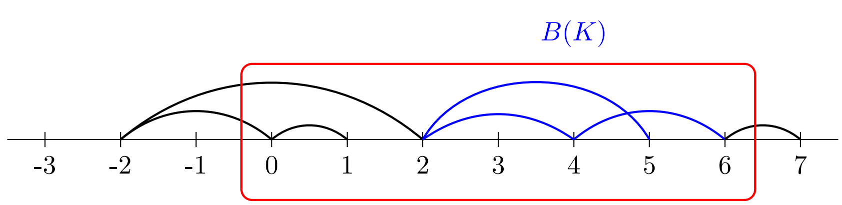


Figure 2.1: Example of the largest component B(K) in blue for K = 7

We then have the following result:

Lemma 2.3.3. *Consider the random percolation model $G = (\mathbb{Z}, E)$ defined in Definition 1.0.1 with parameters $\alpha > 1, \beta \geq 0$, and dimension $d = 1$. Suppose there exist sequences $\psi_n, \lambda_n \in [0,1]$ and $L_n > 0$ satisfying:*

1. $\liminf_{n\to\infty} \lambda_n \psi_n > 0$,
2. $\lim_{n\to\infty} L_n = \infty$,
3. $\mathbb{P}_\beta\big(|B(L_n)| \geq L_n\psi_n\big) \geq \lambda_n$ *for all* $n \geq 0$.

Then, $\theta(\beta) > 0$.

Proof. Let $K(y)$ be the cluster containing y, i.e.

$$K(y) = \{x \in \mathbb{Z} : x \leftrightarrow y\}.$$

Consider the random variable $|K(0) \cap \{0, \ldots, L-1\}|$. By the translation invariance of the probability kernel J, we have that this random variable has the same distribution as $|K(i) \cap \{i, \ldots, L-1+i\}|$. Hence for every i,

$$\begin{aligned}\mathbb{P}_\beta\left(\frac{|K(0)\cap\{0,\ldots,L-1\}|}{L} \geq \frac{\psi}{2}\right) &= \mathbb{P}_\beta\left(\frac{|K(i)\cap\{i,\ldots,L+i-1\}|}{L} \geq \frac{\psi}{2}\right)\\ &\geq \mathbb{P}_\beta\left(\frac{|K(i)\cap\{i,\ldots,L-1\}|}{L} \geq \frac{\psi}{2},\ i \in B(L)\right). \quad (2.1)\end{aligned}$$

Here we used that

$$K(i) \cap \{i, \ldots, L-1\} \subset K(i) \cap \{i, \ldots, L+i-1\}.$$

Moreover, on the event $\{i \in B(L)\}$ we have $B(L) \subset K(i)$. Hence

$$\mathbb{P}_\beta\left(\frac{|K(i)\cap\{i,\ldots,L-1\}|}{L} \geq \frac{\psi}{2},\ i \in B(L)\right) \geq \mathbb{P}_\beta\left(\frac{|B(L)\cap\{i,\ldots,L-1\}|}{L} \geq \frac{\psi}{2},\ i \in B(L)\right).$$

Now, define the variable W as:

$$W = \sum_{i=0}^{L-1} \mathbf{1}_{\{B(L)\cap\{i,\ldots,L-1\}| \geq \frac{1}{2}L\psi\} \text{ and } i \in B(L)\}}$$

Then summing over $i \in \{0, \ldots, L-1\}$ in the inequality

$$\mathbb{P}_\beta\left(\frac{|K(0)\cap\{0,\ldots,L-1\}|}{L} \geq \frac{\psi}{2}\right) \geq \mathbb{P}_\beta\left(\frac{|B(L)\cap\{i,\ldots,L-1\}|}{L} \geq \frac{\psi}{2},\ i \in B(L)\right)$$

yields:

$$L\mathbb{P}_\beta\left(\frac{|K(0)\cap\{0,\ldots,L-1\}|}{L} \geq \frac{\psi}{2}\right) \geq \sum_{i=0}^{L-1} \mathbb{P}_\beta\left(\frac{|B(L)\cap\{i,\ldots L-1\}|}{L} \geq \frac{\psi}{2}, i \in B(L)\right) = \mathbb{E}(W)$$

Now, if we have $|B(L)| \geq L\psi$, $|B(L) \cap \{i, \ldots L-1\}| \geq \frac{1}{2}L\psi$ for at least $\frac{1}{2}L\psi$ values of $i \in B(L)$. This holds, as if we consider the elements of $B(L)$: $i_1 < \ldots < i_m$, where $m \geq \lceil L\psi \rceil$. For any i_j in $B(L)$, $B(L) \cap \{i_j, \ldots, L-1\}$ contains exactly the elements $\{i_j, \ldots, i_m\}$. In this case $|\{i_j, \ldots, i_m\}| = m - j + 1$. Now, to satisfy $|B(L) \cap \{i, \ldots L-1\}| \geq \frac{1}{2}L\psi$ we need that:

$$m - j + 1 \geq \frac{1}{2}L\psi \iff j \leq m - \frac{1}{2}L\psi + 1$$

Using that $m \geq L\psi$ this results in:

$$j \leq \frac{1}{2}L\psi + 1 \implies j \leq m - \frac{1}{2}L\psi + 1$$

This inequality on the left side is satisfied for $j \in \{1, \ldots, \lceil \frac{1}{2}L\psi \rceil\}$. Hence there are at least $\frac{1}{2}L\psi$ values of j such that $|B(L) \cap \{i_j, \ldots, L-1\}| \geq \frac{1}{2}L\psi$. Consequently, on the event $\{|B(L)| \geq L\psi\}$, we have

$$W \geq \frac{1}{2}L\psi.$$

Therefore

$$\mathbb{E}[W] \geq \frac{1}{2}L\psi\, \mathbb{P}_\beta(|B(L)| \geq L\psi).$$

Combining all inequalities yields that

$$\mathbb{P}_\beta(\frac{|C(0) \cap \{0, \ldots, L-1\}|}{L} \geq \frac{1}{2}\psi) \geq L^{-1}\mathbb{E}(W) \geq \frac{1}{2}\psi\mathbb{P}(|B(L)| \geq L\psi). \tag{2.2}$$

Therefore

$$\theta(\beta) \geq \liminf_{n\to\infty} \mathbb{P}_\beta(|C(0) \cap \{0, \ldots, L_n - 1\}| \geq \frac{1}{2}L_n\psi_n) \tag{2.3}$$

$$\geq \liminf_{n\to\infty} \frac{1}{2}\psi_n\mathbb{P}(|B(L_n)| \geq L_n\psi_n) \tag{2.4}$$

$$\geq \liminf_{n\to\infty} \frac{1}{2}\psi_n\lambda_n > 0. \tag{2.5}$$

To justify 2.3 we have that as $\liminf_{n\to\infty} \psi_n > 0$ (which follows from $\liminf_{n\to\infty} \psi_n\lambda_n > 0$ and $\lambda_n \in [0,1]$) and from $L_n \to \infty$ there exists an n_0 such that for all $n \geq n_0$ we have that $L_n\psi_n \geq M$ for some $M \in \mathbb{R}$. Thus

$$\{|K(0) \cap \{0, \ldots, L_n - 1| \geq L_n\psi_n\} \subseteq \{|K(0)| \geq M\}.$$

Taking the probability and $\liminf_{n\to\infty}$ on both sides and then $M \to \infty$ we obtain that by the continuity of increasing events that

$$\theta(\beta) = \mathbb{P}(|K(0)| = \infty) \geq \liminf_{n\to\infty} \mathbb{P}_\beta(|K(0) \cap \{0, \ldots, L_n\} \geq L_n\psi_n).$$

In 2.4 we used the inequality 2.2 and in 2.5 we used the assumption that $\liminf_{n\to\infty} \psi_n\lambda_n > 0$ Therefore

$$\theta(\beta) > 0.$$

□

In Lemma 2.3.3 we have found sufficient conditions for $\theta(\beta) > 0$ for $\beta > 0$, which means that $\beta_c < \infty$. Now we construct sequences that satisfy the sufficient conditions

1. $\liminf_{n\to\infty} \lambda_n \psi_n > 0$,
2. $\lim_{n\to\infty} L_n = \infty$,
3. $\mathbb{P}_\beta\big(|B(L_n)| \geq L_n\psi_n\big) \geq \lambda_n$ for all $n \geq 0$.

The outline of the proof is the following: At each scale n, the interval $\{0, \dots, L_n - 1\}$ is partitioned into smaller disjoint subblocks of length L_{n-1}. A subblock is called *good* if it contains a connected component contained entirely inside of it of size at least $\psi_n L_n$. We then show:

1. with high probability, sufficiently many subblocks are good;
2. with high probability, these good subblocks are mutually connected;
3. these two estimates yield a recursive lower bound for

$$\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n).$$

Finally, satisfying the conditions of Lemma 2.3.3.

Lemma 2.3.4 (Recursive bound)**.** *Consider the random percolation model $G = (\mathbb{Z}, E)$ defined in Definition* (1.0.1) *with parameters $\alpha > 1, \beta \geq 0$, and dimension $d = 1$. Let L_n be a sequence of integer block lengths satisfying $L_{n+1} = M_n L_n$ for some integers $M_n \geq 1$, and let $W_n := |B(L_n)|$ denote the size of the largest connected component contained in the interval $\{0, \dots, L_n - 1\}$.*

Given $\psi_n \in [0, 1]$ and $\varepsilon_n \in (0, 1)$, define $\psi_{n+1} = (1 - \varepsilon_n)\psi_n$ and $m_n = (1 - \varepsilon_n)M_n$. Partition the interval $\{0, \dots, L_{n+1} - 1\}$ into M_n consecutive subblocks of length L_n:

$$I_{n,j} := \{jL_n, \dots, (j + 1)L_n - 1\}, \qquad 0 \leq j \leq M_n - 1.$$

We call a subblock $I_{n,j}$ good *if its largest connected component $B(I_{n,j})$ satisfies $|B(I_{n,j})| \geq \psi_n L_n$. Let X_n denote the number of good scale-n subblocks inside $\{0, \dots, L_{n+1} - 1\}$. Let Y_n denote the number good subblocks that are connected, where we say that two distinct subblocks $I_{n,j}$ and $I_{n,k}$ are connected if there exists an open edge $\{x, y\}$ with $x \in B(I_{n,j})$ and $y \in B(I_{n,k})$. Then for all $n \geq 0$,*

$$\mathbb{P}_\beta(W_{n+1} \geq \psi_{n+1} L_{n+1}) \geq \mathbb{P}_\beta(X_n \geq m_n)\mathbb{P}_\beta(Y_n \geq m_n \mid X_n \geq m_n).$$

Proof. We note that for $W_{n+1} = |B(L_{n+1})| = |B(I_{n+1,j})|$, a sufficient condition for a block $I_{n+1,j}$ to be good is that

1. at least m_n of its scale-n subblocks are good;
2. all of these good subblocks are connected.

Indeed, if both conditions hold, then $I_{n+1,j}$ contains a connected component of size at least

$$m_n \psi_n L_n.$$

Since

$$L_{n+1} = M_n L_n,$$

it follows that

$$m_n \psi_n L_n = (1-\varepsilon_n)\psi_n L_{n+1} = \psi_{n+1} L_{n+1}.$$

Hence,

$$W_{n+1} \geq \psi_{n+1} L_{n+1}.$$

Therefore

$$\begin{aligned}\mathbb{P}_\beta(W_{n+1} \geq \psi_{n+1}L_{n+1}) &\geq \mathbb{P}_\beta(X_n \geq m_n, Y_n \geq m_n) \\ &= \mathbb{P}_\beta(X_n \geq m_n)\mathbb{P}_\beta(Y_n \geq m_n \mid X_n \geq m_n).\end{aligned}$$

□

We now work on bounding both $\mathbb{P}_\beta(X_n \geq m_n)$ and $\mathbb{P}_\beta(Y_n \geq m_n \mid X_n \geq m_n)$ as presented in Lemma 2.3.4.

Lemma 2.3.5 (Lower bound on the number of good boxes)**.** *Consider the random percolation model $G = (\mathbb{Z}, E)$ defined in Definition 1.0.1 with parameters $\alpha > 1, \beta \geq 0$, and dimension $d = 1$. Let L_n be a sequence of integer block lengths satisfying $L_{n+1} = M_n L_n$ for some integers $M_n \geq 1$, and let $W_n := |B(L_n)|$ denote the size of the largest connected component contained in the interval $\{0, \dots, L_n - 1\}$. Let $\varepsilon_n \in (0,1)$, and $m_n = (1-\varepsilon_n)M_n$. Partition the interval $\{0, \dots, L_{n+1} - 1\}$ into M_n consecutive subblocks of length L_n:*

$$I_{n,j} := \{jL_n, \dots, (j+1)L_n - 1\}, \qquad 0 \leq j \leq M_n - 1.$$

We call a subblock $I_{n,j}$ good *if its largest connected component $B(I_{n,j})$ satisfies $|B(I_{n,j})| \geq \psi_n L_n$. Let X_n denote the number of good scale-n subblocks inside $\{0, \dots, L_{n+1} - 1\}$. Then we have that*

$$\mathbb{P}_\beta(X_n \geq m_n) \geq 1 - \exp(\frac{-\eta_n^2 M_n \lambda_n}{2})$$

given that $\eta_n = 1 - \frac{1-\varepsilon_n}{\lambda_n} > 0$

Proof. In total there are M_n subblocks $I_{n,j}$. Let Z_i the indicator if box i is good. Then we have that

$$X = \sum_{i=1}^{M_n} Z_i \qquad \text{and} \qquad \mathbb{E}_\beta(X) \geq \lambda_n M_n.$$

Consequently

$$\mathbb{P}(X < m_n) = \mathbb{P}(X < (1-\varepsilon_n)M_n) = \mathbb{P}(X < (1-\eta_n)M_n\lambda_n) \leq \mathbb{P}(X \leq (1-\eta_n)\mathbb{E}(X)).$$

As $0 < \eta_n < 1$, from the Chernoff bound [27], it follows that

$$\mathbb{P}(X \leq (1-\eta_n)\mathbb{E}(X)) \leq \exp(\frac{-\eta_n^2 \mathbb{E}(X)}{2}) \leq \exp(\frac{-\eta_n^2 M_n \lambda_n}{2}).$$

So it can be concluded that

$$\mathbb{P}(X \geq m) = 1 - \mathbb{P}(X < m) \geq 1 - \exp(\frac{-\eta_n^2 M_n \lambda_n}{2}).$$

□

For the term probability of all the good boxes connecting we have Lemma 2.3.6

Lemma 2.3.6 (Lower bound on all good subblocks connecting)**.** *Consider the random percolation model $G = (\mathbb{Z}, E)$ defined in Definition 1.0.1 with parameters $\alpha > 1, \beta \geq 0$, and dimension $d = 1$. Let L_n be a sequence of integer block lengths satisfying $L_{n+1} = M_n L_n$ for some integers $M_n \geq 1$, and let $W_n := |B(L_n)|$ denote the size of the largest connected component contained in the interval $\{0, \ldots, L_n - 1\}$.*

Given $\psi_n \in [0, 1]$ and $\varepsilon_n \in (0, 1)$, define $\psi_{n+1} = (1 - \varepsilon_n)\psi_n$ and $m_n = (1 - \varepsilon_n)M_n$. Partition the interval $\{0, \ldots, L_{n+1} - 1\}$ into M_n consecutive subblocks of length L_n:

$$I_{n,j} := \{jL_n, \ldots, (j + 1)L_n - 1\}, \qquad 0 \leq j \leq M_n - 1.$$

We call a subblock $I_{n,j}$ good *if its largest connected component $B(I_{n,j})$ satisfies $|B(I_{n,j})| \geq \psi_n L_n$. Let X_n denote the number of good scale-n subblocks inside $\{0, \ldots, L_{n+1} - 1\}$. Let Y_n denote the number good subblocks that are connected, where we say that two distinct subblocks $I_{n,j}$ and $I_{n,k}$ are connected if there exists an open edge $\{x, y\}$ with $x \in B(I_{n,j})$ and $y \in B(I_{n,k})$. Then*

$$\mathbb{P}_\beta(Y_n \geq m_n | X \geq m_n) \geq 1 - 2^{M_n} \exp(-\beta \psi_n^2 L_n^{2-\alpha} M_n^{-\alpha})$$

Proof. Denote the set of good blocks by

$$\mathcal{I}_n = \{I_{n,i} : 0 \leq i < M_n, |B(I_{n,i})| \geq \psi_n L_n\}.$$

Consider 2 good blocks $I_{n,i}$ and $I_{n,j}$. We write $B_i = B(I_{n,i})$ for the largest component within a block $I_{n,i}$. Then the largest components B_i and B_j connect with probability:

$$p_{ij} = 1 - \prod_{x \in B_i} \prod_{y \in B_j} \exp\big(-\beta |x - y|^{-\alpha}\big) = 1 - \exp\left(\sum_{x \in B_i} \sum_{y \in B_j} -\beta |x - y|^{-\alpha}\right)$$

Now, note that the maximum distance between any 2 points on $\{0, \ldots, M_n L_n - 1\}$ is $M_n L_n - 1$. Furthermore for good subblocks $I_{n,i}$ we have that $|B_i| \geq \psi_n L_n$. It follows that:

$$\begin{aligned} 1 - \exp\left(\sum_{x \in B_i} \sum_{y \in B_j} -\beta |x - y|^{-\alpha}\right) &\geq 1 - \exp\left(\sum_{x \in B_i} \sum_{y \in B_j} -\beta (M_n L_n)^{-\alpha}\right) \\ &\geq 1 - \exp\left(-\beta (\psi_n L_n)^2 (M_n L_n)^{-\alpha}\right) \\ &= 1 - \exp\left(-\beta \psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}\right) \end{aligned} \tag{2.6}$$

Now, consider looking at only $l = \lceil m_n \rceil$ of these good blocks. Then clearly if these blocks connect, $Y_n \geq l \geq m_n$. Thus it suffices to bound the probability of l good blocks being connected.

These blocks are not fully connected if we can partition the l blocks into at least 2 disjoint non-empty components on $\{0, \ldots M_n L_n - 1\}$. Then the largest component will contain $< m_n$ of these blocks.

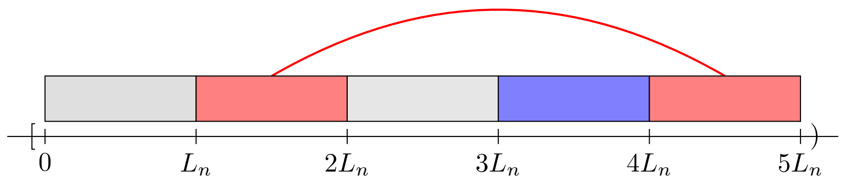


Figure 2.2: Renormalised block structure with number of blocks $M_n = 5$ on $\{0, \dots, M_n L_n - 1\}$ with blocks $[cL_n, (c+1)L_n)$. Gray blocks are not good. The good blocks have been partitioned into two disjoint sets, where there is no edge from the red to the blue partition.

Consider a partition of the l good blocks into disjoint sets C and D. C contains k good blocks, D $l-k$ good blocks. The probability that they connect p_{CD} is:

$$p_{CD} \geq 1 - \exp(-k(l-k)\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) \tag{2.7}$$
$$\geq 1 - \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) \tag{2.8}$$

By a similar argument as for 2 single clusters. There are $2^{l-1} - 1 \leq 2^{M_n}$ different ways to partition a group of l elements into 2 non-empty sets where the sets are unlabeled. We now consider the probability of one of these partitions not being connected, where by union bound:

$$\begin{aligned}\mathbb{P}(\text{one of the partitions A and B do not connect}) &\leq (2^{l-1} - 1) \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) \\ &\leq 2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha})\end{aligned}$$

Therefore all of them connect with probability at least:

$$1 - 2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha})$$

And as shown before we have that $Y_n \geq m_n$ if all blocks connect, and therefore

$$\mathbb{P}(Y_n \geq m_n | X_n \geq m_n) \geq 1 - 2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}).$$

□

We now have bounds on both of the probabilities $\mathbb{P}_\beta(X_n \geq m_n)$ and $\mathbb{P}_\beta(Y_n \geq m_n | X_n \geq m_n)$ mentioned in Theorem 2.3.4. We want our bounds to be sufficiently sharp. To this end we will show that by choosing β sufficiently large the bounds (2.3.5) and (2.3.6) are "sharp enough". We will define what is sharp enough later.

Lemma 2.3.7 (Connection bounds)**.** *Let*

$$M_0 = 2^6, M_n = M_0 2^{3(n+1)}.$$

Define

$$L_n = \prod_{j=0}^{n-1} M_j = M_0^n 2^{3n(n+1)/2}.$$

Define $\varepsilon_n = 2^{-(n+1)}$, $\psi_0 = 1, \psi_{n+1} = \psi_n(1 - \varepsilon_n)$. *Let*

$$A_n = \ln(2) M_n - \beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}.$$

Then for β sufficiently large

$$A_n \leq -\ln(2)(n+4) \qquad \forall n \geq 0$$

Proof. Consider the negative term of A_n

$$\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha} = \beta\psi_n^2 M_0^{n(2-\alpha)-\alpha} 2^{3n(n+1)\frac{1}{2}(2-\alpha)-3\alpha n}.$$

Because $\alpha \in (1,2)$, we have $2-\alpha > 0$. This implies that the exponent of base 2 grows quadratically with n (specifically, $\frac{3}{2}(2-\alpha)n^2$).

On the other hand, the positive term in A_n is $\ln(2)M_n = \ln(2)M_0 2^{3(n+1)}$, which only grows linearly in the exponent. Furthermore, ψ_n is bounded below by a positive constant $\psi_\infty > 0$.

Therefore, the quadratic exponent in the subtracted term strictly dominates for large n. Thus, $A_n \to -\infty$ exponentially. This means there exists some N such that $A_n \leq -\ln(2)(n+4)$ for all $n \geq N$ regardless of $\beta > 0$. Since there are only finitely many terms $1 \leq n < N$, we can choose β sufficiently large to ensure the inequality holds for those initial terms as well. □

Lemma 2.3.8 (Growth bounds)**.** *Define M_0,M_n,ε_n as in* (2.3.7)*. Let $1 - \frac{1}{2}\varepsilon_n \leq \lambda_n \leq 1$, and define $\eta_n = 1 - \frac{1-\varepsilon_n}{\lambda_n}$. Let*

$$B_n = -\frac{1}{2}\eta_n^2 M_n \lambda_n.$$

Then $B_n \leq -\ln(2)(n+4)$ for all $n \geq 1$.

Proof. Consider the term $\eta_n^2\lambda_n$. Then we have that

$$\eta_n^2\lambda_n = (\frac{\lambda_n - 1 + \varepsilon_n}{\lambda_n})^2\lambda_n \geq (\lambda_n - 1 + \varepsilon_n)^2 \geq (\frac{1}{2}\varepsilon_n)^2 = \frac{1}{4}\varepsilon_n^2.$$

Where we used the fact that $\lambda_n \leq 1$. Utilizing the definition of $\varepsilon_n = 2^{-(n+1)}$, we have that:

$$\frac{\eta_n^2 M_n \lambda_n}{2} \geq \frac{M_n}{2}\frac{1}{4}(2^{-n-1})^2$$

Thus:

$$\begin{aligned} \frac{M_n}{2}\frac{1}{4}(2^{-n-1})^2 \geq \ln(2)(n+4) &\iff \\ M_0 2^{3n+3} \geq 8\ln(2)(n+4)2^{2n+2} &\iff \\ M_0 \geq 8\ln(2)(n+4)2^{-n-1} & \end{aligned}$$

As $(n+4)2^{-n-1}$ is a decreasing sequence, it obtains its maximum at $n=1$. With $M_0 = 2^6$ we get that that the first entry of $(n+4)2^{-n-1}$ is smaller than M_0:

$$M_0 = 2^6 \geq 8\ln(2)(1+4)2^{-1-1}$$

Therefore:

$$B_n \leq -\frac{M_n}{2}\frac{1}{4}(2^{-n-1})^2 \leq -\ln(2)(n+4) \qquad \forall n \geq 1$$

□

Lemma 2.3.9 (Diminishing failure)**.** *Let X_n be the number of good blocks and Y_n the size of the largest connected component of good blocks as in Lemma 2.3.4. Let μ_{n+1} be an upper bound for failure, defined as*

$$\mu_{n+1} = \mathbb{P}_\beta(X < m_n) + P_\beta(Y < m_n | X \geq m_n)$$

Then if β sufficiently large

$$\mu_n \leq \frac{1}{2}\varepsilon_n$$

Proof. we prove the bounds by an induction proof.
Define

$$\lambda_n = 1 - \mu_n.$$

Base case $n = 0$:
We have that $\mathbb{P}_\beta(X_0 < m_0) = 0$ as we have M_0 1 point blocks. These are good as they are trivially fully connected. Thus the only point of failure is the connection probability. We have that

$$\mathbb{P}_\beta(Y < m_0 | X \geq m_0) \leq 2^{M_0} \exp(-\beta\psi_n^2 L_n^{2-\alpha}) = \exp(A_0) \leq \left(\frac{1}{2}\right)^4 = \frac{1}{4}\varepsilon_1$$

Thus $\mu_1 \leq \frac{1}{2}\varepsilon_1$ holds. Holds.
Case $n \geq 1$:
Now assume for $n \geq 1$ we have that: $\mu_n \leq \frac{1}{2}\varepsilon_n$. That is $\lambda_n \geq 1 - \frac{1}{2}\varepsilon_n$. Then for μ_{n+1} we have that by lemma

$$\mu_{n+1} \leq 2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) + \exp(\frac{-\delta_n^2 M_n \lambda_n}{2})$$

We will now bound each term on the right separately by $\frac{1}{4}\varepsilon_{n+1}$. Let

$$A_n = \ln(2) M_n - \beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}.$$

By Lemma 2.3.7 we have that:

$$2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) = \exp(A_n) \leq \left(\frac{1}{2}\right)^{(n+4)} = \frac{1}{4}\varepsilon_{n+1}$$

Now, let

$$B_n = -\frac{1}{2}\eta_n^2 M_n \lambda_n$$

As $\mu_n \leq \frac{1}{2}\varepsilon_n$, we have that $\lambda_n = 1 - \mu_n \geq 1 - \frac{1}{2}\varepsilon_n$. Thus we can use Lemma 2.3.8, yielding us:

$$\exp(-\frac{1}{2}\eta_n^2 M_n \lambda_n) = \exp(B_n) \leq \left(\frac{1}{2}\right)^{(n+4)} = \frac{1}{4}\varepsilon_{n+1}$$

Therefore we have that:

$$\begin{aligned}\mu_{n+1} &\leq 2^{M_n} \exp(-\beta\psi_n^2 L_n^{2-\alpha} M_n^{-\alpha}) + \exp(\frac{-\delta_n^2 M_n \lambda_n}{2}) \\ &\leq \frac{1}{4}\varepsilon_{n+1} + \frac{1}{4}\varepsilon_{n+1} = \frac{1}{2}\varepsilon_{n+1}\end{aligned}$$

Thus $\mu_{n+1} \leq \varepsilon_{n+1}$ holds, concluding the proof by induction. □

Lemma 2.3.10 (Increasing successes). *Let* $M_0 = 2^6$, $M_n = M_0 2^{3(n+1)}$, $L_n = \prod_{j=0}^{n-1} M_j$. *Let* $\varepsilon_n = 2^{-(n+1)}$, $\psi_0 = 1, \psi_{n+1} = \psi_n(1-\varepsilon_n)$. *Define* $m_n = M_n(1-\varepsilon_n)$. *Let* X_n *be the number of good blocks at scale* n, *and* Y_n *the number of good connected blocks as defined in* (2.3.4). *Define for* $n \geq 1$

$$\mu_n = \inf\{\mu : \mathbb{P}_\beta(X_{n-1} < m_n) + \mathbb{P}_\beta(Y_n < m_n | X_{n-1} \geq m_n) \leq \mu\}.$$

Define

$$\lambda_n = 1 - \mu_n$$

Then if β *sufficiently large*

$$\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n) \geq \lambda_n.$$

Furthermore $\lambda_n \to 1$ *as* $n \to \infty$.

Proof. From Lemma 2.3.4 we have that

$$\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n) \geq \mathbb{P}_\beta(X_{n-1} \geq m_n)\mathbb{P}_\beta(Y_n \geq m_n | X_{n-1} \geq m_n).$$

Using some simple probability theory have then have that

$$\begin{aligned}\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n) &\geq \mathbb{P}_\beta(X_{n-1} \geq m_n)\mathbb{P}_\beta(Y_n \geq m_n | X_{n-1} \geq m_n) \\ &\geq 1 - \mathbb{P}_\beta(X_{n-1} < m_n) - \mathbb{P}_\beta(Y_n < m_n | X_{n-1} \geq m_n) \\ &\geq 1 - \mu_n = \lambda_n\end{aligned}$$

From (2.3.9) we have that if β is sufficiently large, then $\mu_n \leq \frac{1}{2}\varepsilon_n$. Hence $\mu_n \to 0$ and consequently $\lambda_n \to 1$. □

Proof of the Main Theorem

We are now in a state to prove Theorem 2.3.2.

Proof of Theorem 2.3.2. Fix $\alpha \in (1,2)$. Choose the sequences

$$M_0 = 2^6, \qquad M_n = M_0\, 2^{3(n+1)},$$

and define recursively

$$L_{n+1} = M_n L_n, \qquad L_0 = 1,$$

together with

$$\varepsilon_n = 2^{-(n+1)}, \qquad \psi_{n+1} = (1-\varepsilon_n)\psi_n, \qquad \psi_0 = 1.$$

Since

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty,$$

the infinite product

$$\prod_{n=0}^{\infty}(1-\varepsilon_n)$$

converges to a strictly positive limit [9]. Hence

$$\psi_n \to \psi_\infty > 0.$$

Moreover,

$$L_n \to \infty.$$

Choose $\beta > 0$ sufficiently large so that the conclusion of Lemma 2.3.10 holds:

$$\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n) \geq \lambda_n$$

and $\lambda_n \to 1$. Then the sequences $(L_n)_{n\geq 0}$, $(\psi_n)_{n\geq 0}$ and $(\lambda_n)_{n\geq 1}$ satisfy

1. $\liminf_{n\to\infty} \lambda_n \psi_n > 0$,
2. $L_n \to \infty$,
3. $\mathbb{P}_\beta(|B(L_n)| \geq \psi_n L_n) \geq \lambda_n$ for all n.

Indeed,

$$\liminf_{n\to\infty} \lambda_n \psi_n = \psi_\infty \liminf_{n\to\infty} \lambda_n = \psi_\infty > 0,$$

since $\lambda_n \to 1$. Therefore all assumptions of Lemma 2.3.3 are satisfied. Thus

$$\theta(\beta) > 0.$$

Hence there exists a finite value of β for which percolation occurs, and consequently

$$\beta_c < \infty.$$

□

2.3.2 The multidimensional case

In the previous section 2.3.1 we saw that the one dimensional case required an extensive framework utilizing a renormalization approach to prove that a phase transition exists. For higher dimensional cases where $d \geq 2$, the critical value β_c at which a transition occurs can be shown to be finite by comparing it to a simpler embedded model. Similar to the proof used to show that $\beta_c > 0$ in section 2.2, we prove that $\beta_c < \infty$ for $d \geq 2$ and $\alpha > 1$ by stochastic domination. We compare the long-range percolation model to the Bernoulli nearest percolation model, where we define Bernoulli nearest percolation as:

Definition 2.3.1 (Nearest-neighbour bond percolation)**.** Let $d \geq 1$ and $p \in [0, 1]$. Consider the lattice $\mathbb{Z}^d$ with edge set

$$\mathcal{E} = \{\{x, y\} : x, y \in \mathbb{Z}^d, \|x - y\|_1 = 1\}.$$

Nearest-neighbour (bond) percolation is defined by declaring each edge $e \in \mathcal{E}$ to be *open* independently with probability p and *closed* otherwise.

We denote by $\mathbb{P}_p$ the corresponding product measure on $\{0, 1\}^{\mathcal{E}}$, and write $G = (\mathbb{Z}^d, E)$ for the resulting random subgraph, where $E \subseteq \mathcal{E}$ is the set of open edges.

Similar to the long-range percolation model we can define a critical parameter p_c signalling a phase transition:

Definition 2.3.2 (The critical parameter in nearest neighbour percolation)**.** Choose $d \geq 1$. Let $G = (V, E)$ be the random graph as defined in Definition 1.0.1. Then we define the critical parameter p_c as

$$p_c = \inf\{p : \mathbb{P}_p(|K_0| = \infty) > 0\}$$

where $K_0 = \{x \in V : x \leftrightarrow 0\}$ is the size of the open cluster containing 0.

A landmark result in Bernoulli nearest-neighbour percolation is the theorem of Harry Kesten (1980), which states that the critical probability on the two-dimensional square lattice is exactly $p_c = \frac{1}{2}$. [23]

Theorem 2.3.11 (Kesten (1980))**.** Let d=2, and consider the Bernoulli nearest neighbour percolation as defined in $G = (V, E)$ be the random graph as defined as in Definition (2.3.1). Let p_c be the critical parameter as in Definition 2.3.2. Then the critical parameter satisfies

$$p_c = \frac{1}{2}$$

Corollary 2.3.12 (finiteness of the critical parameter in long-range percolation)**.** Consider a long-range percolation model as in definition 1.0.1, with $d \geq 2$ and $\alpha > 0$. Then the critical parameter β_c satisfies

$$\beta_c < \infty$$

Proof. Let $c = \min_{\|x-y\|_1=1} J(x, y) > 0$. Choose $\beta > 0$ large enough such that

$$1 - \exp(-\beta c) > \frac{1}{2}.$$

Define $H := \mathbb{Z}^2 \times \{0\}^{d-2} \subseteq \mathbb{Z}^d$. We construct a coupling between Bernoulli nearest-neighbour percolation on H with parameter $p = 1 - \exp(-\beta c)$ and the long-range percolation model on $\mathbb{Z}^d$ with parameter β as follows.

Let

$$\mathcal{E} = \{\{x, y\} : x, y \in \mathbb{Z}^d, x \neq y\}.$$

For each edge $e = \{x, y\} \in \mathcal{E}$, let $U_e \sim \text{Uniform}(0, 1)$ independently. Declare e open in the Bernoulli model whenever $x, y \in H$, $\|x - y\|_1 = 1$ and $U_e \leq p$. Declare and edge open in the long-range model whenever

$$U_e \leq 1 - e^{-\beta J(x,y)}.$$

Now, since

$$1 - e^{-\beta J(x,y)} \geq 1 - e^{-\beta c} = p,$$

every edge that is open in the Bernoulli model is also open in the long-range model. Hence the long-range model stochastically dominates Bernoulli nearest-neighbour percolation on H with parameter p.

Since $p > \frac{1}{2} = p_c$, by Kesten's theorem Bernoulli nearest-neighbour percolation on H contains an infinite open cluster K_0 containing the origin with positive probability. By stochastic domination, the same is true for the long-range percolation model.

Therefore

$$\theta(\beta) > 0,$$

and consequently

$$\beta_c < \infty.$$

□

Chapter 3

Behaviour of the LRP model at criticality

Having established the existence of a phase transition for long-range percolation when either $d = 1$ and $\alpha \in (1, 2)$ or $d \geq 2$ and $\alpha > 1$, we now turn our attention to the critical regime. While the subcritical and supercritical phases have been studied extensively, the behaviour of the model at the critical parameter β_c remains an active area of research [2, 16]. One quantity of central interest is the distribution of cluster sizes. It is conjectured that at criticality the cluster containing the origin satisfies the power-law asymptotic

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-1/\delta + o(1)},$$

where δ is the critical cluster-size exponent [24, Chapter 9].

3.1 Bounds on the critical exponent

We wish to know how the critical exponent δ behaves. Whenever the critical exponent

$$\delta = \lim_{n \to \infty} -\frac{\log(n)}{\log \mathbb{P}_{\beta_c}(|K_0| \geq n)}$$

is well-defined, we can bound the possible values δ can take. In a recent result, Baumler and Berger have proven lower bounds on δ [2].

Theorem 3.1.1 (Bäumler and Berger (2022))**.** Consider the long-range percolation model. Suppose $\alpha \in (1, 2)$ and $d = 1$ or $\alpha > 1$ and $d \geq 2$. Then

$$\delta \geq \frac{d + (d(\alpha - 1) \wedge 1)}{d - (d(\alpha - 1) \wedge 1)}$$

where we write $b \wedge c = \min(b, c)$

Similarly an upper bound exists, proven by Hutchcroft [16].

Theorem 3.1.2 (Hutchcroft (2021))**.** Consider the long-range percolation model with parameter β. If $\alpha \in (1, 2)$, then:

$$\delta \leq \frac{\alpha + 1}{2 - \alpha}$$

Contrary to the general case, the existence of δ, and its exact value are known for the long-range percolation model with parameters $d \in \{1, 2\}$ and $\alpha \in (\frac{4}{3}, 1 + \frac{1}{d})$. See Theorem 3.1.3 below.[17]

Theorem 3.1.3. Consider the long-range percolation model with parameters $d \in \{1, 2\}$ and $\alpha \in (\frac{4}{3}, 1 + \frac{1}{d})$. Then the critical exponent δ is well defined and satisfies

$$\delta = \frac{\alpha}{2 - \alpha}.$$

See Figure 3.1 for a visualization of the known bounds for δ.

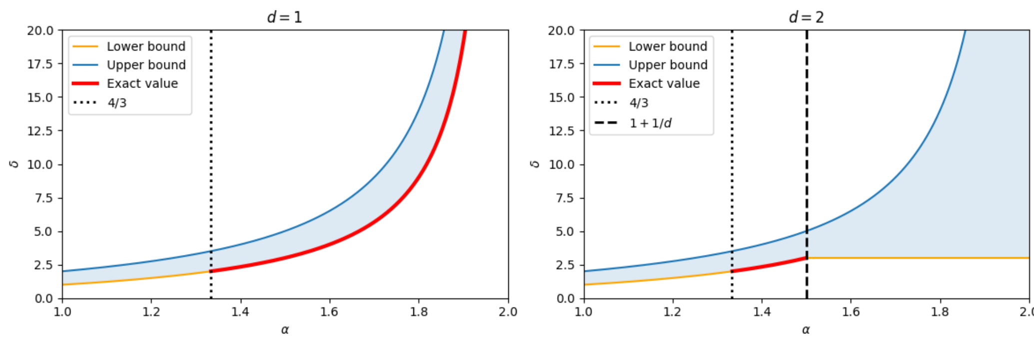


Figure 3.1: Illustrations for known bounds on the critical exponent δ as a function of α, assuming that δ is well-defined, for $d = 1$ on the left and $d = 2$ on the right. The shaded region represents the values of δ that are not excluded by the available lower and upper bounds. The red curve indicates the regime in which the value of δ is well-defined and known exactly. These curves are based on the bounds by Johannes Bäumler and Tom Hutcroft [2, 16, 17]

3.2 Lemmas for proof of the lower bound

We now present the proof for Theorem 3.1.1. First off, we write

$$\tau = \frac{1}{\delta}.$$

Let $\Lambda_n = \{-n, \dots, n\}^d$ be a finite d-dimensional box. Our main goal will be to prove the inquality chain

$$n^{d\frac{1-\tau}{1+\tau}} \gtrsim \mathbb{E}_\beta[|K_0(\Lambda_n)|] \gtrsim n^{(d(\alpha-1)\wedge 1)+o(1)},$$

where we write $a \lesssim b$ if there exists a $C > 0$ such that $a \leq Cb$. Once this inequality chain is proven we derive the lower bound on the critical exponent δ.

The first step in proving this inequality will be bounding cluster size. Let Λ be a finite subset of $\mathbb{Z}^d$. We define the median $M_\beta(\Lambda)$ as

$$M_\beta(\Lambda) := \min\{n \geq 0 : \mathbb{P}_\beta(|K_0(\Lambda)| \geq n) \leq \frac{1}{e}\}$$

A result of high importance from Tom Hutchcroft in the proofs to come will be Theorem 3.2.1 down below [16].

Theorem 3.2.1 (Universal tightness of maximum cluster size)**.** Let $G = (V, E, J)$ be a countable weighted graph, Then both inequalites

$$\mathbb{P}_\beta(|K_{max}(\Lambda)| \geq cM_\beta(\Lambda)) \leq e^{-\frac{c}{9}}$$
$$\text{and } \mathbb{P}_\beta(|K_v(\Lambda)| \geq cM_\beta(\Lambda)) \leq e\mathbb{P}_\beta(|K_u \cap \Lambda))e^{-\frac{c}{9}}$$

hold.

We assume a slightly stronger condition on δ:

$$\sum_{k=1}^{n} \mathbb{P}_{\beta_c}(|K_0| \geq k) \leq Cn^{1-\tau}$$

for some constant $C > 0$. This then implies

$$\mathbb{P}_\beta(|K_0| \geq n) = n^{-1} \sum_{k=1}^{n} \mathbb{P}_\beta(|K_0| \geq n) \leq n^{-1} \sum_{k=1}^{n} \mathbb{P}_\beta(|K_0| \geq k) \leq Cn^{-\tau}$$

Note that $C \geq 1$ always, as

$$\sum_{k=1}^{1} \mathbb{P}_{\beta_c}(|K_0| \geq k) = \mathbb{P}_{\beta_c}(|K_0| \geq 1) = 1 \leq C,$$

as the cluster K_0 always contains at least 1 element, that being 0 itself.

Lemma 3.2.2. *Assume that the exists a* $1 \leq C < \infty$ *such that* $\sum_{k=1}^{n} \mathbb{P}_{\beta_c}(|K_0| \geq k) \leq Cn^{1-\tau}$. *Let* $\Lambda \subset \mathbb{Z}^d$ *with* $|\Lambda| = n$. *Then*

$$M_\beta(\Lambda) \leq 3Cn^{\frac{1}{1+\tau}}.$$

Proof. Define

$$K_x(\Lambda) = \{y \in \Lambda : y \overset{\Lambda}{\leftrightarrow} x\}$$

the set of all vertices connected to x inside Λ. Write $D = 3C$. Then we have that for the expected number of vertices in sufficiently large clusters:

$$\mathbb{E}_{\beta_c}\left[\left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right\}\right|\right] = \sum_{x\in\Lambda} \mathbb{P}_{\beta_c}\left(|K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right) \tag{3.1}$$

$$\leq \sum_{x\in\Lambda} \mathbb{P}_{\beta_c}\left(|K_x| \geq Dn^{\frac{1}{1+\tau}}\right) \leq \sum_{x\in\Lambda} C\left(Dn^{\frac{1}{1+\tau}}\right)^{-\tau} \tag{3.2}$$

$$= CD^{-\tau}n^{1-\frac{\tau}{1+\tau}} = CD^{-\tau}n^{\frac{1}{1+\tau}}$$

Now, if there is some $x \in \Lambda$ such that $K_x(\Lambda) \geq Dn^{\frac{1}{1+\tau}}$, then clearly the largest component $K_{max}(\Lambda)$ satisfies $|K_{max}(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}$. Consequently, if $|K_{max}(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}$ then there are at least $Dn^{\frac{1}{1+\tau}}$ $x \in \Lambda$ satisfying $|K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}$. It follows that

$$Dn^{\frac{1}{1+\tau}}\mathbf{1}_{\left\{|K_{max}(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right\}} \leq \left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right\}\right|.$$

Therefore, if we take probabilities on both sides we obtain

$$\mathbb{P}_\beta\left(|K_{max}(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right) \leq \frac{1}{Dn^{\frac{1}{1+\tau}}}\mathbb{E}_\beta\left[\left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right\}\right|\right]. \tag{3.3}$$

Using the earlier proven inequality

$$\mathbb{E}_\beta\left[\left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right\}\right|\right] \leq CD^{-\tau}n^{\frac{1}{1+\tau}},$$

we obtain that

$$\mathbb{P}_\beta\left(|K_{max}(\Lambda)| \geq Dn^{\frac{1}{1+\tau}}\right) \leq \frac{1}{Dn^{\frac{1}{1+\tau}}}CD^{-\tau}n^{\frac{1}{1+\tau}} = CD^{-\tau-1}.$$

Substituting back in $D = 3C$ we obtain that

$$CD^{-\tau-1} = C(3C)^{-\tau-1} \leq C(3C)^{-1} = \frac{1}{3} < \frac{1}{e}.$$

By definition of the median $M_\beta(\Lambda)$, we have that

$$M_\beta(\Lambda) \leq 3Cn^{\frac{1}{1+\tau}}.$$

□

We are now ready to bound $\mathbb{E}_\beta[|K_0(\Lambda)|]$ from above.

Lemma 3.2.3. *Assume that the exists a $C > 0$ such that $\sum_{k=1}^{n}\mathbb{P}_{\beta_c}(|K_0| \geq k) \leq Cn^{1-\tau}$. Let $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = n$. Then there exists a constant $C_2(C,\tau)$ such that*

$$\mathbb{E}_{\beta_c}\left[|K_0(\Lambda)\right] \leq C_2 n^{\frac{1-\tau}{1+\tau}}.$$

Proof. For clarity, write $M = M_\beta(\Lambda)$. We then get that

$$\mathbb{E}_{\beta_c}\left[|K_0(\Lambda)\right] = \sum_{k=1}^{\infty}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq k) = \sum_{l=0}^{\infty}\sum_{k=1}^{M}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq lM + k)$$

where we split the single sum into two sums. The inner sum sums over the block $(lM, \ldots, (l+1)M]$. Therefore, by taking out the first term we have that

$$\sum_{l=0}^{\infty}\sum_{k=1}^{M}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq lM + k) = \sum_{k=1}^{M}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq k) + \sum_{l=1}^{\infty}\sum_{k=1}^{M}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq lM + k) \tag{3.4}$$

$$\leq CM^{1-\tau} + \sum_{l=1}^{\infty}M\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq lM) \tag{3.5}$$

where we the assumption that $\sum_{k=1}^{n}\mathbb{P}_{\beta_c}(|K_0| \geq k) \leq Cn^{1-\tau}$. Then we have that:

$$CM^{1-\tau} + \sum_{l=1}^{\infty}M\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq lM) \leq Cn^{1-\tau} + eM\sum_{l=1}^{\infty}\mathbb{P}_{\beta_c}(|K_0(\Lambda)| \geq M)e^{-\frac{l}{9}}$$

$$\leq CM^{1-\tau} + eMCM^{1-\tau}\sum_{l=1}^{\infty}e^{-\frac{l}{9}}$$

As $\sum_{l=1}^{\infty} e^{-\frac{l}{9}}$ converges, we have that for some C':

$$CM^{1-\tau} + eMCM^{1-\tau} \sum_{l=1}^{\infty} e^{-\frac{l}{9}} \leq C'M^{1-\tau} \leq C_2 n^{\frac{1-\tau}{1+\tau}}$$

Therefore we have that

$$\mathbb{E}_{\beta_c}[|K_0(\Lambda)|] \leq C_2 n^{\frac{1-\tau}{1+\tau}}.$$

□

The proofs for the upper bounds have thus far been for arbitrary finite sets $\Lambda \subset \mathbb{Z}^d$. For the lower bound we introduce more structure to our considered sets. Define

$$\Lambda_n = \{-n, \dots, n\}^d.$$

Now, we define the boundary of Λ_n:

$$\partial\Lambda_n = \{\{x, y\} \in E : x \in \Lambda_n, y \notin \Lambda_n\}.$$

We are particularly interested in the number edges that are part of this boundary *and* are part of the central cluster. For this we define $\phi_\beta(S)$,

$$\phi_\beta(S) := \sum_{x \in S} \sum_{y \notin S} \mathbb{P}_\beta(0 \overset{S}{\leftrightarrow} x)\mathbb{P}_\beta(x \sim y).$$

See Figure 3.2 below.

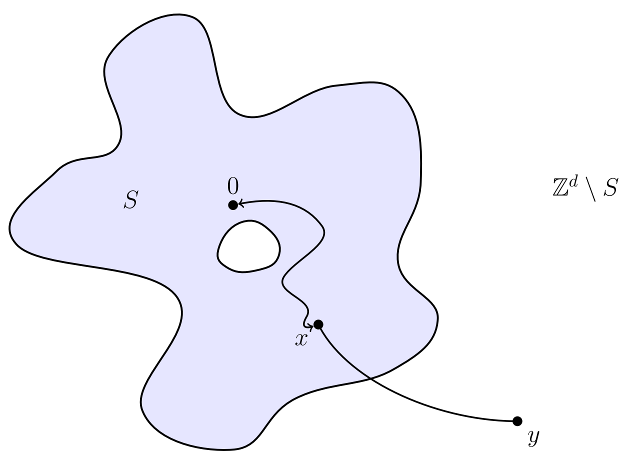


Figure 3.2: Illustration of a contribution to $\varphi_\beta(S)$. The point $x \in S$ is connected to the origin by a path contained entirely in S, written $0 \overset{S}{\leftrightarrow} x$, while the edge $\{x, y\}$ crosses from S to $\mathbb{Z}^d \setminus S$. The quantity $\varphi_\beta(S)$ is the expected number of such open crossing edges.

If we let $S = \Lambda_n$, this notion of $\phi_\beta(\Lambda_n)$ corresponds to our intuition of the expected number of edges in the boundary $\partial\Lambda_n$ of the central component.

In the critical phase it has been proven that $\phi_\beta(S)$ is always greater than 1. [10]

Theorem 3.2.4. Consider the long-range percolation model with paramters $\alpha > 0$ and $\beta \geq \beta_c$. Then for each finite subset $S \subset \mathbb{Z}^d$ we have that

$$\phi_\beta(S) := \sum_{x \in S} \sum_{y \notin S} \mathbb{P}_\beta(0 \overset{S}{\leftrightarrow} x)\mathbb{P}_\beta(x \sim y) \geq 1.$$

We will use this theorem to obtain a contradiction. We are now going to prove a lower bound on the expected cluster size $\mathbb{E}_\beta(|K_0(\Lambda_n)|)$

Lemma 3.2.5. *Let $\Lambda_k = \{-k, \dots, k\}^d$. We write*

$$K_0(\Lambda_k) := \{y \in \Lambda_k : 0 \overset{\Lambda_k}{\longleftrightarrow} y\}$$

for the set of vertices in Λ_k that are connected to 0 by an open path lying entirely inside Λ_k. Assume that the connection kernel satisfies

$$J(x, y) \asymp \|x - y\|^{-d\alpha}.$$

Let $n \in \mathbb{N}$ be arbitrary and fixed. If $d = 1$ and $\alpha \in (1, 2)$, or if $d > 1$ and $\alpha > 1$, then for every $\beta > 0$ there exists a constant $C_3 = C_3(\alpha, \beta, d) < \infty$, independent of n, such that there exists a $k \in \{1, \dots, n\}$ satisfying

$$\phi_\beta(\Lambda_k) := \sum_{x \in \Lambda_k} \sum_{y \notin \Lambda_k} \left(1 - e^{(-\beta J(x,y))}\right) \mathbb{P}_\beta\left(0 \overset{\Lambda_k}{\longleftrightarrow} x\right) \leq C_3 \mathbb{E}_\beta\left[|K_0(\Lambda_n)|\right] f(n, \alpha),$$

where

$$f(n, \alpha) = \begin{cases} n^{-d(\alpha-1)}, & 1 < \alpha < 1 + \frac{1}{d}, \\ n^{-1}\log(n), & \alpha = 1 + \frac{1}{d}, \\ n^{-1}, & \alpha > 1 + \frac{1}{d}. \end{cases}$$

Proof. Let $x \in \Lambda_n$. For the ease of notation denote the probability of a vertex connecting to the origin within the box Λ_n by $t_x = \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0)$. Then the expected size of the cluster is exactly the sum of these t_x:

$$\sum_{x \in \Lambda_n} t_x = \sum \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0) = \mathbb{E}_\beta[|K_0(\Lambda_n)|.]$$

Now, denote X_k as the number of open edges $\{x, y\}$ on the boundary $\partial\Lambda_k = \{\{x, y\} \in E : x \in \Lambda_k, y \notin \Lambda_k\}$ for which $x \leftrightarrow 0$ holds.

$$X_k = |\{\{x, y\} \in E : \{x, y\} \in \partial\Lambda_k \text{ and } x \overset{\Lambda_n}{\leftrightarrow} 0\}|$$

As the events $x \overset{\Lambda_k}{\leftrightarrow} 0$ and $\{x, y\} \in \partial\Lambda_k$ depend on disjoint sets of vertices, and edges are open independently we have that

$$\mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0, \{x, y\} \in \partial\Lambda_n) = \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0)\mathbb{P}_\beta(\{x, y\} \in \partial\Lambda_n).$$

Consequently we have that

$$\mathbb{E}_\beta(X_k) = \sum_{x \in \Lambda_k} \sum_{y \notin \Lambda_k} \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0)\mathbb{P}_\beta(x \sim y) = \phi_\beta(\Lambda_k).$$

We now average over $k \in \{1, \dots, n\}$. Since the minimum is bounded above by the average, there exists some $k \in \{1, \dots, n\}$ such that

$$\phi_\beta(\Lambda_k) = \mathbb{E}_\beta[X_k] \le \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}_\beta[X_j].$$

It therefore remains to bound the averaged quantity on the right-hand side. From $\mathbb{E}_\beta[X_k] = \phi_\beta(\Lambda_k)$ we get that

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}_\beta[X_j] = \frac{1}{n}\sum_{j=1}^{n} \phi_\beta(\Lambda_j) = \frac{1}{n}\sum_{j=1}^{n}\sum_{x\in\Lambda_j}\sum_{y\notin\Lambda_j} \mathbb{P}_\beta(x \overset{\Lambda_j}{\leftrightarrow} 0)\mathbb{P}_\beta(x \sim y).$$

We now reorder the summations. Fix $x \in \Lambda_n$. The event $x \overset{\Lambda_j}{\leftrightarrow} 0$ can only occur if $x \in \Lambda_j$, which holds only if

$$\|x\|_\infty \le j.$$

Indeed, if $\|x\|_\infty > j$, then $x \notin \Lambda_j$, and hence

$$\mathbb{P}_\beta(x \overset{\Lambda_j}{\leftrightarrow} 0) = 0.$$

Therefore it follows that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in\Lambda_j}\sum_{y\notin\Lambda_j} \mathbb{P}_\beta(x \overset{\Lambda_j}{\leftrightarrow} 0)\mathbb{P}_\beta(x \sim y) &= \frac{1}{n}\sum_{x\in\Lambda_n}\sum_{j=1\vee||x||_\infty}^{n}\sum_{y\notin\Lambda_j} \mathbb{P}_\beta(x \overset{\Lambda_j}{\leftrightarrow} 0)\mathbb{P}_\beta(x \sim y) \\ &\le \frac{1}{n}\sum_{x\in\Lambda_n} \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0)\sum_{j=1\vee||x||_\infty}^{n}\sum_{y\notin\Lambda_j} \mathbb{P}_\beta(x \sim y) \qquad (3.6)\end{aligned}$$

where in the last line we used that $\mathbb{P}_\beta(x \overset{\Lambda_j}{\leftrightarrow} 0) \le \mathbb{P}_\beta(x \overset{\Lambda_n}{\leftrightarrow} 0)$, and n does not depend on j.

Now, as our kernel J satisfies $J(x,y) \asymp ||x-y||^{da}$, there exists a $C > 0$ such that $J(x,y) \le C||x-y||^{-d\alpha}$. Therefore we have that

$$\mathbb{P}_\beta(x \sim y) = 1 - \exp(-\beta J(x,y)) \le \beta J(x,y) \le C\beta||x-y||^{-d\alpha}$$

using the standard inequality $1 - e^{-t} \le t$ for $t \ge 0$. Substituting in (3.6), and using the notation $t_x = \mathbb{P}_\beta(x \leftrightarrow 0)$ yields

$$\frac{1}{n}\sum_{x\in\Lambda_n} t_x \sum_{j=1\vee||x||_\infty}^{n}\sum_{y\notin\Lambda_j} \mathbb{P}_\beta(x \sim y) \le \frac{1}{n}\sum_{x\in\Lambda_n} t_x \sum_{j=1\vee||x||_\infty}^{n}\sum_{y\notin\Lambda_j} C\beta||x-y||^{-d\alpha}.$$

Now, consider the innermost sum

$$\sum_{y\notin\Lambda_j} C\beta\|x-y\|^{-d\alpha}.$$

We would like to write this as a sum over radii from x. For fixed $x \in \Lambda_j$, every $y \notin \Lambda_j$ satisfies

$$\|x-y\|_\infty \ge j - \|x\|_\infty + 1.$$

Indeed, the distance from x to the geometric boundary of Λ_j is $j-\|x\|_\infty$, and one additional lattice step is needed to leave the box. Therefore we have that

$$\sum_{y\notin\Lambda_j} C\beta\|x-y\|^{-d\alpha} \le \sum_{l\ge j+1-\|x\|_\infty}\ \sum_{y\in\mathbb{Z}^d:\|x-y\|_\infty=l} C\beta\|x-y\|^{-d\alpha}$$
$$= \sum_{l\ge j+1-\|x\|_\infty}\ \sum_{y\in\mathbb{Z}^d:\|x-y\|=l} C\beta l^{-d\alpha}$$

Now, from (A) we have that $|\{y\in\mathbb{Z}^d : \|x-y\|_\infty = l\}| \asymp l^{d-1}$. Therefore there exists a C' such that

$$\sum_{l\ge j+1-\|x\|_\infty}\ \sum_{y\in\mathbb{Z}^d:\|x-y\|=l} C\beta l^{-d\alpha} \le C' \sum_{l\ge j+1-\|x\|_\infty} l^{d(1-\alpha)-1}.$$

As $\alpha>1$, we have that $d(1-\alpha)-1<-1$ and thus the right side converges by comparing it to the p-series. Now,

$$C' \sum_{l\ge j+1-\|x\|_\infty} l^{d(1-\alpha)-1} \le C''(j+1-\|x\|)^{d(1-\alpha)}.$$

Thus we obtain the bound

$$\sum_{y\notin\Lambda_j} C\beta\|x-y\|^{-d\alpha} \le C''(j+1-\|x\|)^{d(1-\alpha)}.$$

Therefore:

$$\frac{1}{n}\sum_{j=1\vee||x||_\infty}^{n}\ \sum_{y\notin\Lambda_j} C\beta||x-y||^{-d\alpha} \le \frac{1}{n}\sum_{j=||x||_\infty}^{n} C''(j+1-\|x\|)^{d(1-\alpha)}$$
$$\le C''\frac{1}{n}\sum_{j=1}^{n+1} j^{d(1-\alpha)} \le \hat{C}f(n,\alpha)$$

for some $\hat{C}>0$.

So, finally we have that

$$\min_{k\in\{1,\ldots n\}} \mathbb{E}_\beta(X_k) \le \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}_\beta(X_j) \le \hat{C}f(n,\alpha)\sum_{x\in\Lambda_n} t_x = \hat{C}f(n,\alpha)\mathbb{E}_\beta(|K_0(\Lambda_n)|),$$

concluding the proof. □

Now that we have proven 3.2.5, we prove a small result following from this lemma.

Lemma 3.2.6. *Assume that $\beta_c<\infty$ and that the assumptions of Lemma 3.2.5 hold. Then there exists a constant $C_3<\infty$, independent of n, such that*

$$\mathbb{E}_{\beta_c}[|K_0(\Lambda_n)|] = \sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}(0\leftrightarrow x) \ge \frac{1}{C_3}f(n,\alpha)^{-1}.$$

C_3 is the same constant as in (3.2.5).

Proof. Assume for contradiction that

$$\mathbb{E}_{\beta_c}[|K_0(\Lambda_n)| < \frac{1}{C_3} f(n,\alpha)^{-1}.$$

By Lemma (3.2.5) we have that there exists a $k \in \{1, \ldots, n\}$ such that

$$\phi_\beta(\Lambda_k) \leq C_3 \mathbb{E}_\beta |K_0(\Lambda_n)|] f(n,\alpha).$$

But then under our assumption

$$\phi_{\beta_c}(\Lambda_k) \leq C_3 \mathbb{E}_{\beta_c} |K_0(\Lambda_n)|] f(n,\alpha) < 1,$$

which contradicts Theorem 3.2.4 which states that for β_c

$$\phi_\beta(\Lambda_k) \geq 1.$$

We conclude that

$$\mathbb{E}_{\beta_c}[|K_0(\Lambda_n)|] \geq \frac{1}{C_3} f(n,\alpha)^{-1}.$$

□

3.3 Proof of the lower bound

We are now ready to prove the core result in Theorem 3.1.1.

Proposition 3.3.1. Assume that $\beta_c < \infty$ and that the assumptions of Lemma 3.2.5 hold. Suppose furthermore that there exist constants $C < \infty$ and $\delta > 0$ such that

$$\sum_{m=1}^{N} \mathbb{P}_{\beta_c}(|K_0| \geq m) \leq C N^{1-\frac{1}{\delta}}$$

for all $N \in \mathbb{N}$. Then

$$\delta \geq \frac{d + (d(\alpha-1) \wedge 1)}{d - (d(\alpha-1) \wedge 1)}.$$

Proof. Write $\tau = \frac{1}{\delta}$. By assumption,

$$\sum_{m=1}^{N} \mathbb{P}_{\beta_c}(|K_0| \geq m) \leq C N^{1-\tau}.$$

Using the finite-volume cluster estimate from Lemma 3.2.3, there exists a constant $C' < \infty$ such that

$$\mathbb{E}_{\beta_c}\left[|K_0(\Lambda_n)|\right] \leq C' n^{d\frac{1-\tau}{1+\tau}}.$$

On the other hand, Lemma 3.2.6 gives

$$\mathbb{E}_{\beta_c}\left[|K_0(\Lambda_n)|\right] \geq \frac{1}{C_3} f(n,\alpha)^{-1}.$$

Combining the two inequalities yields

$$C' n^{d\frac{1-\tau}{1+\tau}} \geq \frac{1}{C_3} f(n,\alpha)^{-1}.$$

Recall that, in our notation,

$$f(n,\alpha) = \begin{cases} n^{-d(\alpha-1)}, & 1 < \alpha < 1 + \frac{1}{d}, \\ \dfrac{\log n}{n}, & \alpha = 1 + \frac{1}{d}, \\ n^{-1}, & \alpha > 1 + \frac{1}{d}. \end{cases}$$

Thus

$$f(n,\alpha)^{-1} = n^{d(\alpha-1)\wedge 1 + o(1)}.$$

Therefore the inequality

$$C' n^{d\frac{1-\tau}{1+\tau}} \geq \frac{1}{C_3} f(n,\alpha)^{-1}$$

implies

$$d\frac{1-\tau}{1+\tau} \geq d(\alpha-1) \wedge 1.$$

Substituting $\tau = 1/\delta$, we obtain

$$d\frac{1-\frac{1}{\delta}}{1+\frac{1}{\delta}} = d\frac{\delta-1}{\delta+1} \geq d(\alpha-1)\wedge 1.$$

Let $s = d(\alpha-1)\wedge 1$. Then

$$d\frac{\delta-1}{\delta+1} \geq s.$$

Since $s < d$, rearranging $d(\delta-1) \geq s(\delta+1)$ gives

$$\delta \geq \frac{d+s}{d-s}.$$

Hence

$$\delta \geq \frac{d+s}{d-s} = \frac{d + (d(\alpha-1)\wedge 1)}{d - (d(\alpha-1)\wedge 1)}.$$

This proves the desired lower bound. □

Chapter 4

Efficient sampling of long-range percolation graphs

To be able to estimate the critical parameter β_c, the percolation probability $\theta(\beta)$ and the critical cluster size decay exponent δ we first need an efficient algorithm to sample the long-range percolation graph on a finite box $\Lambda_L = \{0, \dots, L-1\}^d$. We call the model restricted to this box *finite size*. In this chapter we construct an efficient algorithm based on *geometric skipping* [6]. The implementation used in this thesis can be found in [32].

4.1 Overview of the sampling algorithm

Consider the d-dimensional box of sidelength L

$$V = \{0, \dots L-1\}^d.$$

Let

$$\mathcal{E} = \{\{x, y\} : x, y \in V, x \neq y\}$$

be set of all possible undirected edges on V. We declare any edge $\{x, y\} \in \mathcal{E}$ to be open independently with probability

$$p_{xy} = \mathbb{P}_\beta(x \sim y) = 1 - \exp\left(-\beta ||x-y||^{-d\alpha}\right). \tag{4.1}$$

We write E the set of open edges. We wish to efficiently sample a realization of the corresponding random long-range percolation graph

$$G = (V, E).$$

A naïve approach would be to enumerate all edges $\{x, y\} \in \mathcal{E}$ and sample each edge independently according to the probability p_{xy}. However, since

$$|\mathcal{E}| \asymp |V|^2$$

this approach requires quadratic runtime in the number of vertices.

In this chapter, we develop a more efficient sampling algorithm by exploiting the translation invariance of the model. The resulting algorithm achieves expected runtime

$$\begin{aligned} O(|V|) \qquad & \text{if } \alpha > 1, \\ O\left(|V| \log |V|\right) \qquad & \text{if } \alpha = 1, \\ O\left(|V|^{2-\alpha}\right) \qquad & \text{if } \alpha < 1. \end{aligned}$$

The key observation we leverage is that the probability p_{xy} depends only on the displacement vector

$$v = x - y.$$

Consequently, all edges with the same displacement vector are sampled with the same probability. Rather than considering every edge individually, we group edges according to their displacement vector and sample them collectively using a geometric skipping procedure.

4.2 Grouping

We define the set of admissible displacement vectors by

$$H = \left\{ v \in \{-L+1, \ldots, L-1\}^d \setminus \{0\} : v \succ 0 \right\}. \tag{4.2}$$

Where we write $x \succ 0$ if for $i = \min\{k : v_k \neq 0\}, v_i > 0$, that is the first nonzero entry in the vector is positive. We say that x is *lexicographically positive*. Restricting to vectors in H avoids double counting, since each undirected edge $\{x, y\}$ corresponds to exactly one displacement vector $v \in H$. We define the sets of valid origins for $v \in H$ as

$$S_v = \{x \in V : x + v \in V\}$$

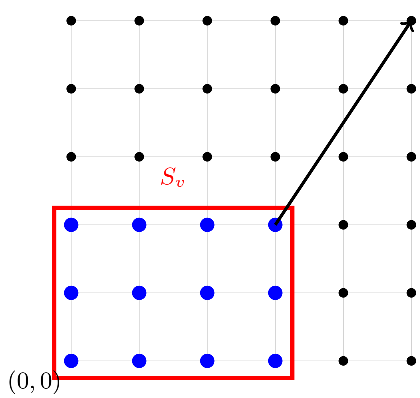


Figure 4.1: An example of the valid origin set S_v on $V = \{0, 1, 2, 3, 4, 5\}^2$, with $v = (2, 3)$. An example vector starting at $x = (3, 2)$ showcases that $x + v$ is still contained in V

Finally, the edge-occupation probability depends only on the displacement vector and is given by

$$p_v = 1 - \exp\left(-\beta \|v\|^{-d\alpha}\right).$$

All edges $\{x, y\} \in E$ with displacement $v \in H$ can now be sampled independently only considering the set S_v.

Lemma 4.2.1 (Finite geometric skipping)**.** *Let $X_1, \dots, X_M$ be i.i.d. Bernoulli(p) random variables. The set*

$$\{i \in \{1, \dots, M\} : X_i = 1\}$$

can be generated by successively sampling independent geometric(p) waiting times $T_1, T_2, \dots$, forming partial sums $S_k = T_1 + \cdots + T_k$, and retaining precisely those $S_k \leq M$.

Proof. In an infinite i.i.d. Bernoulli(p) sequence, the waiting times between successive successes are independent geometric(p) random variables. Therefore the success indices are exactly the partial sums $S_k = T_1 + \cdots + T_k$. Restricting this process to the finite set $\{1, \dots, M\}$ gives precisely the Bernoulli success set in the finite window. □

From Lemma 4.2.1 it follows that instead of sampling every edge independently, it suffices to sample geometrically for each set of valid origins S_v according to its probability p_v.

There is however one last hurdle that we need to overcome, namely ordering the elements in the valid origin set S_v. For this we will use a type of mixed radix decoding.

Definition 4.2.1. Let $m_1, \dots, m_d \in \mathbb{Z}_{>0}$ and $M = \prod_{i=1}^{d} m_i$. Let $t \in \{0, \dots, M-1\}$. We define its *mixed radix decoding* as the unique coordinate tuple $(x_1, \dots, x_d) \in \prod_{k=1}^{d}\{0, \dots, m_k - 1\}$ whose components are given explicitly by

$$x_k = \left\lfloor \frac{t}{\prod_{j=k+1}^{d} m_j} \right\rfloor \pmod{m_k},$$

where we adopt the empty product convention $\prod_{j=d+1}^{d} m_j = 1$.

Mixed radix decoding can be done in $O(d)$ time and is bijective [25].

Proposition 4.2.2. Let $v \in H$ and let $S_v = \{x \in V : x + v \in V\}$. Then the elements of S_v can be indexed by $\{0, \dots, |S_v| - 1\}$, with decoding cost $O(d)$ per index.

Proof. The condition $x \in S_v$ is equivalent to

$$a_k \leq x_k < b_k, \qquad k = 1, \dots, d,$$

where

$$a_k = \max(0, -v_k), \qquad b_k = \min(L, L - v_k).$$

Thus S_v is a rectangular box with side lengths $m_k = b_k - a_k$ and cardinality $M = \prod_{k=1}^{d} m_k$. Every $t \in \{0, \dots, M-1\}$ has a unique mixed-radix expansion with radices $m_1, \dots, m_d$. Decoding this expansion and shifting by $(a_1, \dots, a_d)$ gives a bijection from $\{0, \dots, M-1\}$ to S_v. Since one division and remainder operation is performed per coordinate, the decoding cost is $O(d)$. □

The final step is to verify that the grouped sampling procedure reproduces the desired finite-volume long-range percolation distribution.

Proposition 4.2.3 (Correctness of the grouped sampler)**.** For each $v \in H$, suppose the algorithm applies finite geometric skipping with parameter p_v to the ordered set S_v. Then the resulting edge set has the same distribution as the long-range percolation edge set on V, where each admissible edge $\{x, x + v\}$ is independently present with probability p_v.

Proof. For fixed $v \in H$, the set S_v contains exactly the origins x such that both x and $x+v$ lie in V. Hence the admissible edges with displacement v are in one-to-one correspondence with S_v.

By Lemma 4.2.1, geometric skipping with parameter p_v generates a subset of S_v with the same distribution as independent Bernoulli(p_v) sampling over all elements of S_v. Therefore, for fixed v, all edges of displacement v are sampled independently with the correct probability.

The sets of unordered edges corresponding to different displacements $v \in H$ are disjoint, because the lexicographic positivity condition selects exactly one of $y-x$ and $x-y$. Since the algorithm uses independent geometric samples for each displacement, the sampled edge sets for different v are independent. Hence the union over all $v \in H$ has precisely the desired finite-volume long-range percolation distribution. □

4.3 Pseudocode

We now summarize the exact sampling procedure in Algorithm 1. Here $\text{Decode}_v(q)$ denotes the mixed-radix map from $\{0, \ldots, |S_v|-1\}$ to S_v.

Algorithm 1 Exact sampling of long-range percolation on $V = \{0, \ldots, L-1\}^d$

Input: d, L, β, α
Output: edge set E
$E \leftarrow \emptyset$
$H \leftarrow \{v \in \{-L+1, \ldots, L-1\}^d \setminus \{0\} : v \succ 0\}$
for $v \in H$ **do**
  $p_v \leftarrow 1 - \exp(-\beta \|v\|^{-d\alpha})$
  for $k = 1, \ldots, d$ **do**
    $a_k \leftarrow \max(0, -v_k), \quad b_k \leftarrow \min(L, L - v_k), \quad m_k \leftarrow b_k - a_k$
  end for
  $M \leftarrow \prod_{k=1}^{d} m_k, \quad t \leftarrow -1$
  while true **do**
    sample $G \sim \text{Geom}(p_v)$ and set $t \leftarrow t + G$
    if $t \geq M$ **then**
      break
    end if
    $i \leftarrow \text{Decode}_v(t)$
    $E \leftarrow E \cup \{\{i, i+v\}\}$
  end while
end for
return E

4.4 Theoretical runtime

We now present the proof for efficient runtime for this algorithm.

Proposition 4.4.1 (Expected runtime of the exact sampler)**.** Fix $d \in \mathbb{N}$ and let $V = \{0, \dots, L-1\}^d$. For fixed $\beta > 0$, Algorithm 1 has expected runtime

$$\begin{aligned} O(|V|) \qquad & \text{if } \alpha > 1, \\ O(|V| \log |V|) \qquad & \text{if } \alpha = 1, \\ O\left(|V|^{2-\alpha}\right) \qquad & \text{if } \alpha < 1. \end{aligned}$$

Proof. For a fixed displacement $v \in H$, the algorithm considers all possible edges of the form $\{i, i+v\}$ that are contained in the box V. The number of such possible edges is

$$M_v = \prod_{k=1}^{d} m_k \leq |V|.$$

Each such edge is open independently with probability

$$p_v = 1 - \exp\left(-\beta \|v\|^{-d\alpha}\right).$$

The geometric skipping procedure samples exactly the open edges among these M_v possible edges as seen in (4.2.1). Hence the expected number of sampled edges for displacement v is $M_v p_v$. Using $1 - e^{-x} \leq x$ for $x \geq 0$, we obtain

$$p_v \leq \beta \|v\|^{-d\alpha}.$$

Therefore,

$$\mathbb{E}_\beta |E| = \sum_{v \in H} M_v p_v \leq |V| \sum_{v \in H} p_v \leq \beta |V| \sum_{v \in H} \|v\|^{-d\alpha}.$$

Since $H \subseteq \{-L+1, \dots, L-1\}^d \setminus \{0\}$, it remains to estimate

$$\sum_{v \in H} \|v\|^{-d\alpha}.$$

From Appendix A we have that the number of lattice points at distance r is $O(r^{d-1})$. Hence

$$\sum_{v \in H} \|v\|^{-d\alpha} \leq C \sum_{r=1}^{L} r^{d-1} r^{-d\alpha} = C \sum_{r=1}^{L} r^{d-1-d\alpha},$$

for some constant $C > 0$ depending only on d. Therefore, by comparison with the p-series we have that

$$\sum_{r=1}^{L} r^{d-1-d\alpha} = \begin{cases} O(1), & \text{if } \alpha > 1, \\ O(\log L), & \text{if } \alpha = 1, \\ O\left(L^{d(1-\alpha)}\right), & \text{if } \alpha < 1. \end{cases}$$

It follows that

$$\mathbb{E}|E| = \begin{cases} O(|V|), & \alpha > 1, \\ O(|V| \log L), & \alpha = 1, \\ O\left(|V| L^{d(1-\alpha)}\right), & \alpha < 1. \end{cases}$$

Since $|V| = L^d$, we have $\log L = O(\log |V|)$ and

$$|V|L^{d(1-\alpha)} = L^d L^{d(1-\alpha)} = L^{d(2-\alpha)} = |V|^{2-\alpha}.$$

Thus,

$$\mathbb{E}_\beta(|E|) = \begin{cases} O(|V|), & \alpha > 1, \\ O(|V| \log |V|), & \alpha = 1, \\ O\left(|V|^{2-\alpha}\right), & \alpha < 1. \end{cases}$$

It remains to account for the deterministic overhead. The number of displacements in H is $O(L^d) = O(|V|)$. For each displacement, computing p_v, the coordinate bounds, and M_v costs $O(d)$ time. Since d is fixed, this contributes $O(|V|)$ time.

Each sampled edge requires one geometric sample and one mixed-radix decoding step. The mixed-radix decoding costs $O(d)$ time, which is constant for fixed d. Therefore the expected cost of processing sampled edges is $O\left(\mathbb{E}_\beta(|E|)\right)$.

Combining the displacement overhead and the sampled-edge cost gives expected runtime

$$O(|V| + \mathbb{E}_\beta|E|),$$

which yields the stated bounds. □

Chapter 5

Finite-volume estimates of the percolation probability and critical quantities

In the previous chapter we developed an exact sampling algorithm for long-range percolation graphs on finite boxes. We now use this algorithm to study long-range percolation numerically. Since the infinite lattice $\mathbb{Z}^d$ cannot be simulated directly, we introduce finite-volume estimators for the percolation probability $\theta(\beta)$, the critical parameter β_c, and the critical cluster-size decay exponent δ.

This chapter is organized as follows. In Section 5.1 we introduce two estimators for $\theta(\beta)$ and prove that they converge to the infinite-volume percolation probability. In Section 5.2 we discuss two estimators for β_c based on finite-size behaviour. We then use these estimates for β_c to estimate the critical exponent δ in one dimension in Section 5.3. Finally, in Section 5.4 we present the simulation results and discuss their implications.

5.1 Estimating the theta function

We wish to estimate the probability that the cluster containing the origin is infinite, denoted by

$$\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty).$$

We introduce two finite-volume estimators for the percolation probability $\theta(\beta)$: a local estimator $\hat{p}_{N,L}$ and a global estimator $\hat{\mu}_{N,L}$. These two estimators both converge in probability to $\theta(\beta)$, where we say that a random variable X_n converges in probability to a value X if $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$ for any $\varepsilon > 0$.

The local estimator is defined by

$$\hat{p}_{N,L} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\{|K_{0,n}^L| \geq a_L\}},$$

where $K_{0,n}^L$ denotes the cluster containing the origin $0 = (\lfloor \frac{L}{2} \rfloor, \dots, \lfloor \frac{L}{2} \rfloor)$ in the k-th simulation on a finite box $\Lambda_L = \{0, \dots, L-1\}^d$. The term a_L is a size threshold satisfying $a_L \to \infty$

as $L \to \infty$ and $a_L = o(L^d)$. We call clusters $C^{(i)}$ on Λ_L that satisfy $|C^{(i)}| \geq a_L$ *large* or *macroscopic*. These macroscopic clusters serve as a proxy for the event that a cluster is infinite.

The global estimator is given by

$$\hat{\mu}_{N,L} = \frac{1}{N} \sum_{k=1}^{N} \sum_{C^{(i)} \in \mathcal{C}_k} \frac{|C^{(i)}|}{L^d} \mathbf{1}_{|C^{(i)}| \geq a_L},$$

where $\mathcal{C}_k$ denotes the set of connected components (clusters) in the k-th finite-volume realization. This estimator measures the total fraction of vertices contained in clusters whose size exceeds the cutoff a_L, thereby approximating the volume density of the supercritical (infinite-cluster) phase.

The pointwise estimator $\hat{p}_{N,L}$ relies on the fact that a cluster is unlikely be both large in the finite box Λ_L and not part of the infinite component in the supercritical regime. The global estimator $\hat{\mu}_{N,L}$ is based on the principle that in the supercritical regime, the infinite cluster is represented in finite volume by a macroscopic component occupying approximately a $\theta(\beta)$-fraction of the box.

5.1.1 Bounds on cluster size and convergence in probability

Prior to proving any convergence, we introduce the necessary theorems. The first result proven by Biskup we use is that as the scale L increases, the probability of a non-infinite cluster taking up a large fraction of the box Λ_L tends to 0. We modify his notation of the long-range percolation model to the one used in this thesis.

Theorem 5.1.1 (Macroscopic finite-box cluster in supercritical long-range percolation [3])**.** Consider long-range percolation on $\mathbb{Z}^d$. Assume that $\beta > \beta_c$ and $\alpha \in (1, 2)$ Let Λ_L denote the box of side length L, and let K_0^L be the set of vertices in Λ_L that are connected to 0 by an open path using only edges with both endpoints in Λ_L. Then there exists a constant $\rho > 0$ such that

$$\lim_{L \to \infty} \mathbb{P}\big(|K_0^L| < \rho|\Lambda_L|,\ |K_0| = \infty\big) = 0.$$

The second theorem we use to prove convergence is Theorem 5.1.2.

Theorem 5.1.2 (Largest component and negligible non-largest clusters [22])**.** Consider the long-range percolation model on $\mathbb{Z}^d$ with parameter $\alpha \in (1, 1+\frac{1}{d})$. Assume it is supercritical i.e. $\beta > \beta_c$. Let $\Lambda_L = \{0, \ldots, L-1\}^d$ be the finite box, and denote $n = |\Lambda_L|$. Then there exist constants $A, \zeta > 0$ such that

$$\mathbb{P}_\beta \left(|C^{(2)}| \leq A (\log n)^{\frac{1}{2-\alpha}} \right) \geq 1 - n^{-\zeta}.$$

Under the same conditions we have that

$$\frac{|C^{(1)}|}{|\Lambda_L|} \to \theta(\beta) \quad \text{in probability as } L \to \infty$$

5.1.2 Convergence of the pointwise estimator

We now prove that our pointwise estimator converges in probability to the percolation probability $\theta(\beta)$.

Theorem 5.1.3 (Convergence of the pointwise estimator probability)**.** Consider the long-range percolation model on $\mathbb{Z}^d$ with parameters $\beta \geq 0$, $\alpha \in (1,2)$. Let K_0 denote the (random) cluster containing the origin in $\mathbb{Z}^d$.

For $L \in \mathbb{N}$, define the finite-volume box

$$\Lambda_L := \{-L+1, \dots, L-1\}^d,$$

and let K_0^L be the cluster of the origin in the restricted graph on Λ_L, i.e.

$$K_0^L := \{x \in \Lambda_L : x \overset{\Lambda_L}{\longleftrightarrow} 0\}.$$

Let $(a_L)_{L\geq 1}$ be a sequence such that $a_L \to \infty$ and $a_L = o(L^d)$. Then

$$\mathbb{P}_\beta\big(|K_0^L| \geq a_L\big) \longrightarrow \mathbb{P}_\beta\big(|K_0| = \infty\big) = \theta(\beta), \qquad \text{as } L \to \infty.$$

Proof. We decompose the probability based on whether the origin belongs to a finite cluster or the infinite cluster in $\mathbb{Z}^d$:

$$\mathbb{P}_\beta\big(|K_0^L| \geq a_L\big) = \mathbb{P}_\beta\big(|K_0^L| \geq a_L,\, |K_0| < \infty\big) + \mathbb{P}_\beta\big(|K_0^L| \geq a_L,\, |K_0| = \infty\big).$$

We analyze the two terms on the right-hand side separately.

Step 1: The finite cluster contribution.
Because the restricted graph on Λ_L contains only a subset of the edges available in $\mathbb{Z}^d$, any path in Λ_L is also a valid path in $\mathbb{Z}^d$. Thus, $K_0^L \subseteq K_0$, which immediately implies $|K_0^L| \leq |K_0|$. Using this monotonicity, we can bound the first term:

$$\mathbb{P}_\beta\big(|K_0^L| \geq a_L,\, |K_0| < \infty\big) \leq \mathbb{P}_\beta\big(a_L \leq |K_0| < \infty\big).$$

Since $|K_0|$ is an almost surely finite random variable on the event $\{|K_0| < \infty\}$, and we are given that $a_L \to \infty$ as $L \to \infty$, the tail probability of this finite variable vanishes:

$$\lim_{L\to\infty} \mathbb{P}_\beta\big(a_L \leq |K_0| < \infty\big) = 0.$$

Step 2: The infinite cluster contribution.
Next, consider the event where the origin belongs to the infinite cluster, $\{|K_0| = \infty\}$. Clearly, if $\beta \leq \beta_c$, then

$$\mathbb{P}_\beta\big(|K_0^L| \geq a_L,\, |K_0| = \infty\big) = 0 = \theta(\beta),$$

and we are done. Therefore restrict ourselves to the supercritical phase $\beta > \beta_c$, where $\theta(\beta) > 0$. Consider

$$\theta(\beta) = \mathbb{P}_\beta(|K_0^L| \geq a_L, |K_0| = \infty) + \mathbb{P}_\beta(|K_0^L| < a_L, |K_0| = \infty).$$

Using Theorem 5.1.1, combined with the fact that $\frac{a_L}{|\Lambda_L|} < \rho$ for large enough L and any $\rho > 0$, we obtain that

$$\begin{aligned}\theta(\beta) &= \lim_{L\to\infty} \mathbb{P}_\beta(|K_0^L| \geq a_L, |K_0| = \infty) + \mathbb{P}_\beta(|K_0^L| < a_L, |K_0| = \infty) \\ &= \lim_{L\to\infty} \mathbb{P}_\beta(|K_0^L| \geq a_L, |K_0| = \infty).\end{aligned}$$

And therefore we conclude that

$$\lim_{L\to\infty} \mathbb{P}_\beta\big(|K_0^L| \geq a_L\big) = \theta(\beta)$$

□

We now prove that the estimator $\hat{p}_{N,L}$ converges to our desired quantity $\theta(\beta)$.

Corollary 5.1.4. Consider the long-range percolation model on $\mathbb{Z}^d$ with parameters $\beta \geq 0$, $\alpha \in (1,2)$. Let $(a_L)_{L\geq 1}$ be a sequence such that $a_L \to \infty$ and $a_L = o(L^d)$. Then the estimator $\hat{p}_{N,L}$ converges in probability to the percolation probability $\theta(\beta)$:

$$\lim_{L\to\infty}\lim_{N\to\infty}\hat{p}_{N,L} = \lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{\{|K_0^L|\geq a_L\}} \xrightarrow{\mathbb{P}} \theta(\beta).$$

Proof. Let $\varepsilon > 0$ be given. We want to show that

$$\lim_{N,L\to\infty}\mathbb{P}_\beta\left(|\hat{p}_{N,L} - \theta(\beta)| > \varepsilon\right) = 0.$$

First, let $p_L = \mathbb{P}_\beta(|K_0^L| \geq a_L)$ denote the true probability that the restricted cluster size exceeds the threshold in a box of size L. By the triangle inequality, we can split the estimation error into a random fluctuation term and a deterministic term:

$$|\hat{p}_{N,L} - \theta(\beta)| \leq |\hat{p}_{N,L} - p_L| + |p_L - \theta(\beta)|.$$

From our previous result in Theorem 5.1.3, we know that $\lim_{L\to\infty} p_L = \theta(\beta)$. Therefore, for the given $\varepsilon > 0$, there exists a spatial cutoff $L_0(\varepsilon)$ such that for all $L > L_0(\varepsilon)$, the deterministic bias is bounded by half of our error allowance:

$$|p_L - \theta(\beta)| < \frac{\varepsilon}{2}.$$

Consequently, for any $L > L_0(\varepsilon)$, the event $\{|\hat{p}_{N,L} - \theta(\beta)| > \varepsilon\}$ implies that the random fluctuation term must exceed the remaining half of the allowance. That is:

$$\mathbb{P}_\beta\left(|\hat{p}_{N,L} - \theta(\beta)| > \varepsilon\right) \leq \mathbb{P}_\beta\left(|\hat{p}_{N,L} - p_L| > \frac{\varepsilon}{2}\right).$$

Since the indicator functions in the sum defining $\hat{p}_{N,L}$ are independent and identically distributed (i.i.d.) random variables with mean p_L, the expected value of our estimator is $\mathbb{E}_\beta[\hat{p}_{N,L}] = p_L$, and its variance scales inversely with the sample size N:

$$\mathrm{Var}(\hat{p}_{N,L}) = \frac{p_L(1-p_L)}{N} \leq \frac{1}{4N},$$

where we used the global maximum bound $p(1-p) \leq 1/4$ which holds for any probability $p \in [0,1]$.

Applying Chebyshev's Inequality to the random fluctuation term yields [15]:

$$\mathbb{P}_\beta\left(|\hat{p}_{N,L} - p_L| > \frac{\varepsilon}{2}\right) \leq \frac{\mathrm{Var}(\hat{p}_{N,L})}{(\varepsilon/2)^2} \leq \frac{1/4N}{\varepsilon^2/4} = \frac{1}{N\varepsilon^2}.$$

Combining our inequalities, we find that for all $L > L_0(\varepsilon)$ and any $N \geq 1$:

$$\mathbb{P}_\beta\left(|\hat{p}_{N,L} - \theta(\beta)| > \varepsilon\right) \leq \frac{1}{N\varepsilon^2}.$$

Taking the joint limit as both $N \to \infty$ and $L \to \infty$, the right-hand side goes to zero because it depends strictly on N. Thus:

$$\lim_{N,L\to\infty}\mathbb{P}_\beta\left(|\hat{p}_{N,L} - \theta(\beta)| > \varepsilon\right) = 0.$$

This establishes that $\hat{p}_{N,L} \xrightarrow{\mathbb{P}} \theta(\beta)$ as $N, L \to \infty$, completing the proof. □

5.1.3 Convergence of the global estimator

We now turn our attention to our second estimator $\hat{\mu}_{N,L}$.

Theorem 5.1.5. Consider the long-range percolation model on $\mathbb{Z}^d$ with parameters $\beta \geq 0$, $\alpha \in (1, 1+\frac{1}{d})$. Let $(a_L)_{L\geq 1}$ be a sequence such that $a_L \to \infty$ and $a_L = o(L^d)$. In addition, assume that

$$\frac{a_L}{(\log |\Lambda_L|)^{1/(2-\alpha)}} \to \infty,$$

Then it holds that

$$\mu_L = \sum_n \frac{|C_L^{(n)}|}{|\Lambda_L|} \mathbf{1}_{\{|C_L^{(n)}| \geq a_L\}} \to \theta(\beta) \text{ in probability}$$

Proof. First consider the case $\beta \leq \beta_c$. For $x \in \Lambda_L$, let K_x^L denote the cluster of x in the graph restricted to Λ_L, and let K_x denote its infinite-volume cluster. Since each vertex belongs to exactly one finite-volume cluster, we can write

$$\mu_L = \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \mathbf{1}_{\{|K_x^L| \geq a_L\}}.$$

Taking expectations gives

$$\mathbb{E}_\beta[\mu_L] = \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \mathbb{P}_\beta(|K_x^L| \geq a_L) \leq \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \mathbb{P}_\beta(|K_x| \geq a_L) = \mathbb{P}_\beta(|K_0| \geq a_L),$$

where in the inequality we used that $K_x^L \subseteq K_x$, followed by using the translation invariance. Because $a_L \to \infty$ and $\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty) = 0$, it follows that

$$\mathbb{E}_\beta[\mu_L] \leq \mathbb{P}_\beta(|K_0| \geq a_L) \to 0.$$

Since $\mu_L \geq 0$, Markov's inequality implies that for every $\varepsilon > 0$ [15],

$$\mathbb{P}_\beta(\mu_L > \varepsilon) \leq \frac{\mathbb{E}_\beta[\mu_L]}{\varepsilon} \to 0.$$

Thus for $\beta \leq \beta_c$

$$\mu_L \xrightarrow{\mathbb{P}} 0.$$

Now consider the supercritical case $\beta > \beta_c$. Write

$$\mu_L = \frac{|C_L^{(1)}|}{|\Lambda_L|} \mathbf{1}_{\{|C_L^{(1)}| \geq a_L\}} + \sum_{n\geq 2} \frac{|C_L^{(n)}|}{|\Lambda_L|} \mathbf{1}_{\{|C_L^{(n)}| \geq a_L\}}.$$

We denote the second term by

$$R_L := \sum_{n\geq 2} \frac{|C_L^{(n)}|}{|\Lambda_L|} \mathbf{1}_{\{|C_L^{(n)}| \geq a_L\}}.$$

We first consider the largest-component term. By Theorem 5.1.2,

$$\frac{|C_L^{(1)}|}{|\Lambda_L|} \xrightarrow{\mathbb{P}} \theta(\beta).$$

Since $\beta > \beta_c$, we have $\theta(\beta) > 0$. Moreover, since $a_L = o(|\Lambda_L|)$, we have $\frac{a_L}{|\Lambda_L|} \to 0$. Therefore, for all sufficiently large L,

$$\frac{a_L}{|\Lambda_L|} < \frac{\theta(\beta)}{2}.$$

It follows that

$$\mathbb{P}_\beta(|C_L^{(1)}| < a_L) \le \mathbb{P}_\beta\left(\frac{|C_L^{(1)}|}{|\Lambda_L|} < \frac{\theta(\beta)}{2}\right) \to 0.$$

Hence

$$\mathbf{1}_{\{|C_L^{(1)}| \ge a_L\}} \xrightarrow{\mathbb{P}} 1.$$

By Slutsky's theorem [20], it follows that

$$\frac{|C_L^{(1)}|}{|\Lambda_L|}\mathbf{1}_{\{|C_L^{(1)}| \ge a_L\}} \xrightarrow{\mathbb{P}} \theta(\beta).$$

It remains to show that

$$R_L \xrightarrow{\mathbb{P}} 0.$$

If $|C_L^{(2)}| < a_L$, then every component except the largest has size strictly smaller than a_L. Hence no component in the sum defining R_L contributes, and so $R_L = 0$. Therefore, for every $\varepsilon > 0$,

$$\mathbb{P}_\beta(R_L > \varepsilon) \le \mathbb{P}_\beta(R_L > 0) \le \mathbb{P}_\beta(|C_L^{(2)}| \ge a_L).$$

By Theorem 5.1.2, in the regime $\alpha < 1 + 1/d$ there exist constants $A, \zeta > 0$ such that

$$\mathbb{P}_\beta\left(|C_L^{(2)}| \le A(\log|\Lambda_L|)^{1/(2-\alpha)}\right) \ge 1 - |\Lambda_L|^{-\zeta}$$

for all sufficiently large L. From our assumptions for a_L we also have that for all sufficiently large L,

$$a_L \ge A(\log|\Lambda_L|)^{1/(2-\alpha)}.$$

Consequently,

$$\mathbb{P}_\beta(|C_L^{(2)}| \ge a_L) \le |\Lambda_L|^{-\zeta} \to 0.$$

Thus

$$R_L \xrightarrow{\mathbb{P}} 0.$$

Combining the convergence of the largest-component term with $R_L \to 0$ in probability, we obtain

$$\mu_L = \frac{|C_L^{(1)}|}{|\Lambda_L|}\mathbf{1}_{\{|C_L^{(1)}| \ge a_L\}} + R_L \xrightarrow{\mathbb{P}} \theta(\beta).$$

□

Corollary 5.1.6. Consider the long-range percolation model on $\mathbb{Z}^d$ with parameters $\beta \ge 0$, $\alpha \in (1, 1 + \frac{1}{d})$. Let μ_L^n i.i.d. with the same distribution as μ_L in Theorem 5.1.5. Then the estimator

$$\hat{\mu}_{N,L} = \frac{1}{N}\sum_{n=1}^{N} \mu_L^n \to \theta(\beta) \text{ in probability}$$

Proof. As μ_L^n are all i.i.d. and each converge in probability to $\theta(\beta)$, we have that

$$\frac{1}{N}\sum_{n=1}^{N}\mu_L^n \to \theta(\beta) \text{ in probability}$$

□

5.2 Estimating the critical parameter

We now construct finite-volume estimators for the critical parameter β_c. Although the estimators for the percolation probability $\theta(\beta)$ introduced earlier are natural, they are not ideal for estimating β_c directly. Their convergence is only known pointwise in β, and in particular we do not have a known rate of convergence or a uniform convergence result near the critical point. Consequently, estimating β_c from level sets of $\theta(\beta)$ may introduce a systematic finite-size bias.

Instead, we use quantities whose finite-volume behaviour changes qualitatively between the subcritical and supercritical regimes. These quantities should be interpreted as *pseudo-critical estimators*: for each finite system size they produce an estimate of the location of the transition, and the resulting estimates can then be compared across system sizes. Note that we do not prove rigourous behaviour for these estimators. We instead give heuristic arguments coupled with earlier simulation results to motivate their use.

5.2.1 Susceptibility-based estimation

A quantity that has qualitatively different behaviour in the subcritical and supercritical regimes is the susceptibility. In percolation models, the susceptibility is defined as the expected size of the cluster containing the origin [19],

$$\chi'(\beta) = \mathbb{E}_\beta\big[|K_0|\big].$$

For $\beta < \beta_c$, this quantity is finite, while it diverges as $\beta \uparrow \beta_c$ [1]. Moreover, for $\beta > \beta_c$, the probability that K_0 is infinite is positive, and hence $\chi'(\beta) = \infty$. Thus the usual susceptibility distinguishes the subcritical phase from the critical and supercritical regimes in the sense that it is finite in the subcritical regime, and infinite in the critical and supercritical regimes.

For numerical purposes, however, it is more useful to consider the finite-cluster susceptibility, where the infinite component is excluded:

$$\chi_{\text{fin}}(\beta) = \mathbb{E}_\beta\big[|K_0|\mathbf{1}_{\{|K_0|<\infty\}}\big].$$

For the related Bernoulli percolation model defined in Definition 2.3.1, it is known that the finite-cluster susceptibility is finite away from criticality and diverges at the critical parameter [30]. This behaviour motivates the use of a finite-volume analogue whose maximum is expected to occur near β_c.

We now define the finite-volume susceptibility estimator. Let Λ_L denote the finite box used in the simulation. For a given realization, let

$$C^{(1)}, C^{(2)}, \ldots$$

be the connected components in Λ_L, ordered by decreasing size, so that $|C^{(i)}| \geq |C^{(j)}|$ whenever $i \leq j$. In the supercritical regime, the largest component $C^{(1)}$ is interpreted as the finite-volume analogue of the infinite cluster. We therefore exclude it and define

$$\widehat{\chi}_{N,L}(\beta) = \left\langle \sum_{i\geq 2} |C^{(i)}|^2 \right\rangle,$$

where $\langle\cdot\rangle$ denotes averaging over the N independent realizations. Numerical simulations of long-range percolation inspired models have demonstrated that this finite-cluster susceptibility typically attains its maximum near the critical parameter [4, 5, 12].

Strictly speaking, the finite-volume susceptibility is obtained by normalizing by the volume:

$$\frac{1}{|\Lambda_L|} \left\langle \sum_{i\geq 2} |C^{(i)}|^2 \right\rangle.$$

In our estimator we omit this factor, since for fixed L it is independent of β and therefore does not change the location of the maximum.

We define the susceptibility-based estimator for the critical parameter by

$$\widehat{\beta}_c^{\chi}(L) = \arg\max_{\beta\in B} \widehat{\chi}_{N,L}(\beta),$$

where B is the finite grid of simulated β values.

5.2.2 Ratio estimation

The second estimator for the critical parameter β_c is based on a numerical study by Gori et al. of one-dimensional long-range percolation [13], where they study the ratio

$$Q_{N,L} = \left\langle \frac{\sum_j |C^{(j)}|^4}{\left(\sum_i |C^{(i)}|^2\right)^2} \right\rangle.$$

As before, $\langle\cdot\rangle$ denotes averaging over N realizations and $C^{(i)}$ are the clusters in a single realization.

Empirically, this quantity is expected to approach distinct limiting values in the subcritical, critical and supercritical regimes, which makes it useful for estimating β_c [13]. We give some additional motivation for the use of this ratio, but do not prove any concrete convergence. Consider a single realization of the long-range percolation model on Λ_L, with clusters

$$C^{(1)}, C^{(2)}, \ldots.$$

Define the weight assigned to the cluster $C^{(j)}$ by

$$w_j = \frac{|C^{(j)}|^2}{\sum_i |C^{(i)}|^2}.$$

Then $\sum_j w_j = 1$, and

$$Q_{N,L} = \left\langle \sum_j w_j^2 \right\rangle.$$

Thus $Q_{N,L}$ can be interpreted as the average concentration of the squared cluster-size mass. It is close to 1 when one cluster dominates the sum $\sum_i |C^{(i)}|^2$, and it is small when the contribution to $\sum_i |C^{(i)}|^2$ is spread over many clusters. To see this, consider the largest cluster $C^{(1)}$ for a single realization. Then

$$\sum_i |C^{(i)}|^4 \le (|C^{(1)}|)^2 \sum_i |C^{(i)}|^2,$$

and therefore

$$\frac{\sum_i |C^{(i)}|^4}{\left(\sum_j |C^{(j)}|^2\right)^2} \le \frac{(|C^{(1)}|)^2}{\sum_j |C^{(j)}|^2}.$$

The value on the right is small whenever the largest component $C^{(1)}$ does not dominate the sum $\sum_i |C^{(i)}|^2$. This is the expected behaviour in the subcritical regime, where the system consists of many finite clusters and the largest cluster is negligible compared to the whole system.

In the supercritical regime, on the other hand, the largest cluster is macroscopic, in the sense that the size of the largest cluster $C^{(1)}$ satisfies $|C^{(1)}| \approx \theta(\beta)|\Lambda_L|$, as seen in Theorem 5.1.2. If it dominates the second moment, then

$$\frac{|C^{(1)}|^4}{\left(\sum_j |C^{(j)}|^2\right)^2} \approx \frac{|C^{(1)}|^4}{|C^{(1)}|^4} = 1,$$

and thus in the supercritical regime $Q_{N,L}(\beta)$ is close to 1. Therefore we have that $Q_{N,L}$ distinguishes between the two phases by detecting whether the cluster-size moments are spread over many clusters or concentrated in one giant component.

This motivates the expected limiting behaviour

$$Q_{N,L}(\beta) \longrightarrow \begin{cases} 0, & \beta < \beta_c, \\ 1, & \beta > \beta_c, \end{cases}$$

as the system size $L \to \infty$ and $N \to \infty$.

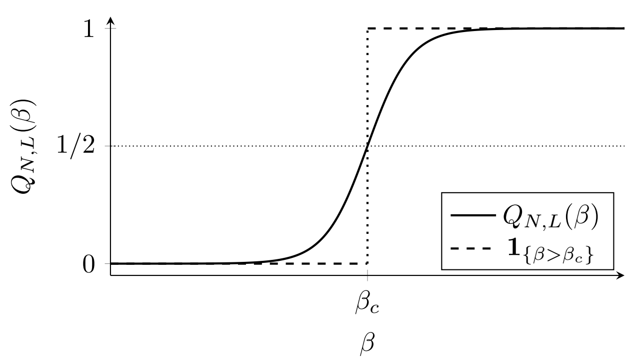


Figure 5.1: Expected qualitative behaviour of the ratio $Q_{N,L}(\beta)$. In the subcritical regime the second moment is spread over many finite clusters, giving a small ratio. In the supercritical regime the largest component dominates, giving a ratio close to one. This sketch is based on the heuristic behaviour, together with the results in the paper of Gori et al. [13].

Since the limiting curve is expected to approach a step function, we define a threshold estimator by

$$\hat{\beta}_c^Q(L) = \min\{\beta \in B : Q_{N,L}(\beta) \geq 0.5\}$$

The particular value 1/2 is not special; any fixed threshold in $(0, 1)$ would give the same limiting value if the crossover sharpens to a step function.

5.3 Estimating the critical exponent

With two estimators for the critical parameter β_c at our disposal, we now turn our attention to the cluster size critical exponent δ. For the long-range percolation model it is conjectured that at criticality β_c the cluster size distribution satisfies

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-\frac{1}{\delta}+o(1)}.$$

where $o(1)$ is some function tending to 0 as $n \to \infty$ [24]. For long-range percolation neither the existence nor the exact value of δ are known for general choices for the parameters α and d. However, if δ is well-defined, then from Chapter 3 we have bounds on δ.

The goal of this section is to investigate the possibility of estimating δ numerically using finite size approximations, and compare these estimates to the bounds in Chapter 3. This is done by first approximating the critical value β_c using the Ratio-based estimator $\hat{\beta}_c^Q$ defined in Section 5.2.2. Afterwards we estimate the tail probability

$$\mathbb{P}_{\beta_c}(|K_0^L| \geq n),$$

where $K_0(\Lambda_L)$ denotes the central cluster in $\Lambda_L = \{0, \ldots, L-1\}^d$. We then fit a power law using linear regression to estimate δ.

5.3.1 Behaviour of the estimator

The relation

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-\frac{1}{\delta}+o(1)}.$$

suggests that for large values of n, the log-log plot should be approximately a straight line with slope $-\frac{1}{\delta}$. However, the asymptotic regime where the log-log plot converges to a straight line is difficult to observe. On a finite scale we have that for large n, the finite size of the box suppresses the creation of large clusters. This suppression is for example seen in the paper of Roman et al. where they simulate the cluster size distribution for the related Bernoulli percolation model [29]. For small n the asymptotics might not have kicked in yet, and so small scale behaviour might dominate the asymptotic events. An example of both of these distorting effects can be seen in Figure 5.2 below.

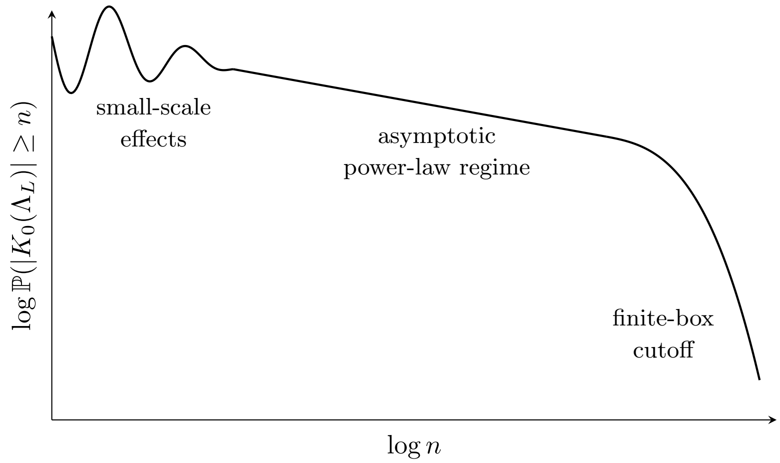


Figure 5.2: Expected qualitative behaviour of the cutoff probability $\mathbb{P}_\beta(|K_0(\Lambda_L)| \geq n)$ on a log-log scale. For small cluster sizes, finite-scale effects may cause irregular behaviour. In an intermediate regime, the curve approximately follows the expected linear decay corresponding to the critical exponent. For large cluster sizes, the finite simulation box suppresses large components, producing a downward bend.

5.3.2 Methodology

For a finite box $\Lambda_L = \{0, \dots, L-1\}^d$, we estimate the finite-volume tail probability

$$p_L(n;\beta) = \mathbb{P}_\beta(|K_0(\Lambda_L)| \geq n).$$

by

$$\hat{p}_L(n;\beta) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\Lambda_L|} \sum_{C \in \mathcal{C}_L^{(i)}} |C|\, \mathbf{1}_{\{|C| \geq n\}},$$

where $\mathcal{C}_L^{(i)}$ denotes the set of clusters in the i-th simulation. This estimator works equivalently to sampling a vertex on Λ_L uniformly at random at looking at the cluster size.

The factor $|C|$ appears because we want to estimate the cluster-size distribution seen from a uniformly chosen vertex, rather than from a uniformly chosen cluster. Indeed, for a fixed realization with cluster set $\mathcal{C}_L$, the probability that a uniformly chosen vertex $v \in \Lambda_L$ belongs to a cluster C is $|C|/|\Lambda_L|$. Hence the probability that v belongs to a cluster of size at least n is

$$\sum_{C \in \mathcal{C}_L} \frac{|C|}{|\Lambda_L|} \mathbf{1}_{\{|C| \geq n\}} = \frac{1}{|\Lambda_L|} \sum_{C \in \mathcal{C}_L} |C| \mathbf{1}_{\{|C| \geq n\}}.$$

Averaging this quantity over independent simulations gives

$$\hat{p}_L(n;\beta) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\Lambda_L|} \sum_{C \in \mathcal{C}_L^{(i)}} |C| \mathbf{1}_{\{|C| \geq n\}}.$$

By translation invariance in the limit, the distribution of the cluster containing a uniformly chosen vertex in a large box is the finite-volume analogue of the distribution of the cluster

K_0 of the origin. Thus $\hat{p}_L(n;\beta)$ estimates $\mathbb{P}_\beta(|K_0| \geq n)$, up to finite-volume boundary effects.

To reduce noisy measurements, we only regress on $10 \leq n \leq L^{0.8d}$. Both the lower and upper bound have been arbitrarily chosen, and finding a more systematic way of choosing these bounds should be seen as a point of improvement for this model.

We then fit the linear regression model

$$\log \hat{p}(|C| \geq n) = -\hat{\tau} \log(n) + C$$

with intercept C and slope $\hat{\tau}$. We define the estimator for δ as

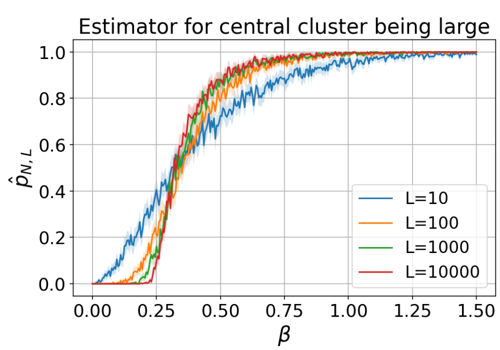
$$\hat{\delta} = \frac{1}{\hat{\tau}}$$

5.4 Simulation results

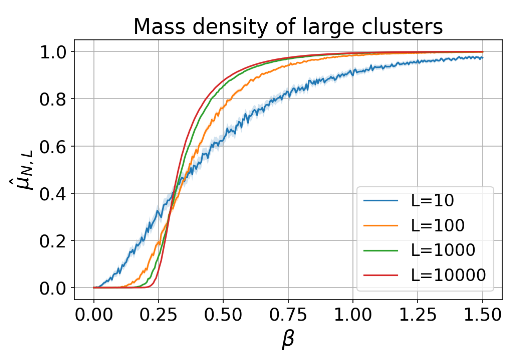
We now perform the simulations. Due to computational constraints we restrict ourselves to long-range percolation in one dimension. The code can be found in [32].

5.4.1 Percolation probability

We present our result for the estimates for the theta function, based on the two estimators defined in Section 5.1. We have performed the simulations in one dimension with parameters $\alpha = 1.5$, $N = 360$, $d = 1$ and $a_L = \sqrt{L^d}$.



(a) Local estimator $\hat{p}_{N,L}$.



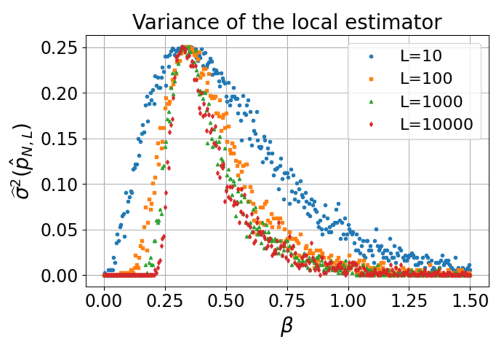
(b) Global estimator $\hat{\mu}_{N,L}$.

Figure 5.3: Resulting curves with empirical 0.95 confidence intervals for two estimators of supercritical behaviour. Figure (a) shows the local estimator $\hat{p}_{N,L}$ and Figure (b) shows the global mass estimator $\hat{\mu}_{N,L}$, for different values of L with parameters $\alpha = 1.5$, $N = 360$, $d = 1$, and $a_L = \sqrt{L^d}$.

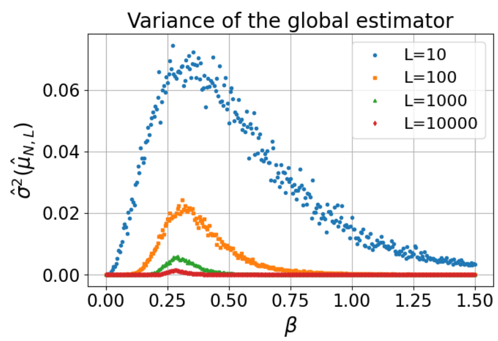
From our numerical simulations, we observe that the global estimator exhibits significantly lower variance compared to its local counterpart (see Figure 5.3a and Figure 5.3b), while both estimators appear to converge toward the same limiting deterministic curve. This is consistent with the theoretical convergence in probability we proved earlier in 5.1. In Figure 5.3b, we observe that for the curve with $L = 10000$ there are 3 regimes:

- **A subcritical regime** ($\beta \ll 0.3$), where the connection parameter is insufficient to sustain long-range order, keeping the probability of a large component near zero.
- **A transitional regime** ($\beta \approx 0.3$), where the estimators exhibit a sharp, non-linear increase. In this region, the empirical curves ascend rapidly as the network approaches the critical connectivity threshold, with the finite volume of the system smoothing out what would be a sharp threshold in the limit ($L \to \infty$).
- **A supercritical regime** ($\beta \gg 0.3$), where a dominant, macroscopic connected component emerges and spans the system, driving the estimators asymptotically toward the infinite-volume percolation probability $\theta(\beta)$.

From this one might conjecture that β_c can be found inside this transitional regime.



(a) Local estimator



(b) Global estimator

Figure 5.4: Empirical variance of the estimators for different system sizes L. On the left is the empirical variance $\widehat{\sigma}^2(\hat{p}_{N,L})$ of the local estimator $\hat{p}_{N,L}$. On the right is the empirical variance $\widehat{\sigma}^2(\hat{\mu}_{N,L})$ of the global estimator $\hat{\mu}_{N,L}$. The parameters used are $\alpha = 1.5$, $N = 360$, $d = 1$, and $a_L = \sqrt{L^d}$.

For the simulations, we have plotted the empirical variance curves above in Figure 5.4. Once again we observe that for the global estimator $\hat{\mu}_{N,L}$ the variance decreases as L increases, while the local estimator $\hat{p}_{N,L}$ does not benefit from a variance reduction as the system size L is increased.

Furthermore we can observe that around $\beta = 0.3$ the variance is largest for both models at all sizes. This coincides with our intuition of the regimes being more chaotic around the critical value β_c, in the sense that one might observe many different cluster sizes. For the local estimator $\hat{p}_{N,L}$ one observes that with increasing system sizes the peaks formed by the get steeper around $\beta = 0.3$, without decreasing as the system size increases. This suggests that the variance of the estimator $\hat{p}_{N,L}$ is actually independent of system size L around the critical value β.

5.4.2 Pseudo-critical estimates

We now present our simulation results for the estimates of the critical parameter β_c. We have performed the simulations in one dimension with parameters $\alpha = 1.5$, $N = 360$, $d = 1$. We take the domain of β to be $[0, 1.5]$ with stepsize $\Delta\beta = 0.005$.

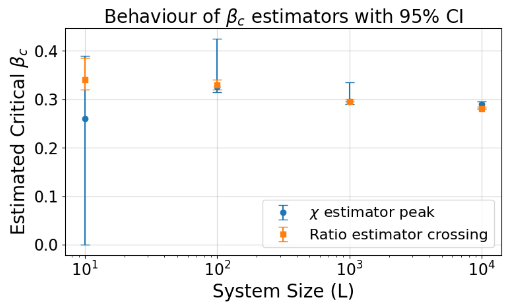
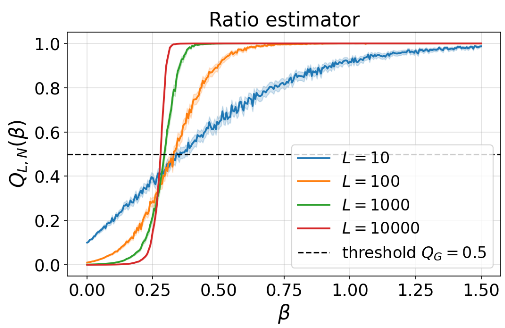
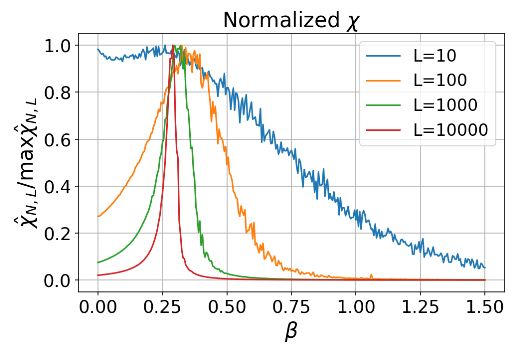


Figure 5.5: Finite-size estimates of β_c with their corresponding 0.95 empirical confidence intervals obtained from the susceptibility estimator $\hat{\beta}_c^{\chi}(L)$ and the ratio estimator $\hat{\beta}_c^{Q}(L)$ for $d = 1$, $\alpha = 1.5$, and $N = 360$.



(a) Ratio $Q_{N,L}(\beta)$ for different values of β, with the shaded regions denoting a 0.95 confidence interval.



(b) Estimated normalized susceptibility curves $\hat{\chi}_{N,L}(\beta)/\max_{\beta}\hat{\chi}_{N,L}(\beta)$ for different system sizes L, with appropriately scaled 0.95 confidence intervals.

Figure 5.6: Numerical ingredients used in the estimation of the critical exponent δ. The left figure shows the Ratio estimator used to estimate β_c, while the right figure are the susceptibility curves, normalized by the maximum value. The parameters used are $\alpha = 1.5$, $N = 360$, $d = 1$, and $a_L = \sqrt{L^d}$.

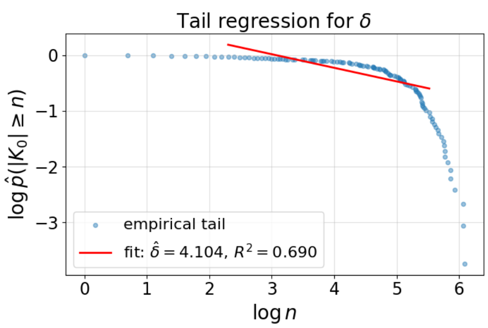
The curves for both behave according to theory, supporting the use of them. Figure 5.6a showcases the convergence to the indicator function $\mathbf{1}_{\{\beta>\beta_c\}}$, while in Figure 5.6b we can observe the stiff peaks around the supposed critical parameter β_c.

Both estimators depend on the discretisation of β, and the range chosen. From our simulations we clearly see that both estimators converge to a similar value as the system length L is increased. This reinforces the idea that both of them are valid estimators for β_c.

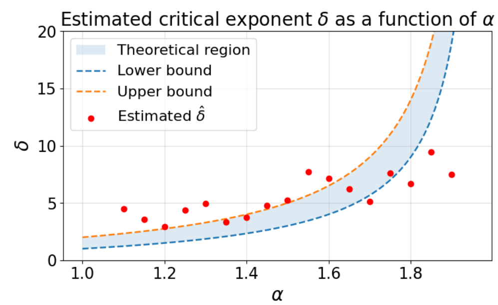
However, Figure 5.5 indicates that the maximal susceptibility estimator $\hat{\beta}_c^{\chi}(L)$ has substantially larger variance than the ratio estimator $\hat{\beta}_c^{Q}(L)$. A possible explanation is that a single large outlier in the susceptibility curve can move the location of the maximum far from its mean. The ratio-based estimator appears more stable in these simulations, although we do not prove a convergence rate for either estimator.

5.4.3 Critical cluster-size decay

We ran the model with parameters $d = 1, L = 1000, N = 360$. For the discretization of β we took a grid on $[0, 1.5]$ with stepsize $\Delta\beta = 0.005$. The parameter α was discretized on $[1.1, 1.9]$ with stepsize $\Delta\alpha = 0.05$. For each value of α, we first estimated the critical parameter β_c using the Ratio estimator defined in 5.2. We then estimated the empirical tail distribution at this value and performed the log-log regression described above.



(a) Linear regression fit used to estimate δ.



(b) Estimated critical exponents $\hat{\delta}$.

Figure 5.7: Estimation of the critical exponent δ. Figure (a) shows the empirical tail behaviour together with the linear regression fit on $[10, 251]$ for $\alpha = 1.5$. Figure (b) shows the resulting estimated critical exponents $\hat{\delta}$ with their empirical 0.95 confidence intervals.

Figure 5.7a shows the empirical tail distribution together with the fitted regression line. We see an approximately linear relation up to approximately $\log n = 4.5$. The curve supports the theory, however a clear separation between the small scale regime and asymptotic power regime as in the hypothetical Figure 5.2 cannot be seen. This suggests that either the small scale effects are negligible even at this small scale, or that we have not taken the system size to be sufficiently large and thus only see the small scale regime and the cutoff effects.

The resulting estimates of δ are shown in Figure 5.7b. The estimates for δ follow a general upward trend as α increases, however the values are not consistent with the theoretical bounds discussed in Chapter 3. This discrepancy is most pronounced at large values of α, where the values consistently undershoot.

There are two likely explanations for this discrepancy. First, the estimated value $\hat{\beta}_c$ may not be sufficiently close to the true critical value. Since the cluster-size tail is expected to follow a power law only at criticality, even a small bias in $\hat{\beta}_c$ can significantly affect the estimated slope.

Secondly the system size $L = 1000$ may be too small to observe the asymptotic regime. At this scale, the regression window may still be dominated by small-scale effects, while the larger cluster sizes are already affected by the finite-volume cutoff.

As a result, the fitted slope may not reflect the true critical exponent. The near-constant estimates of δ should therefore be interpreted as evidence that the current simulations do not yet reach the asymptotic critical regime, rather than as reliable estimates of the exponent.

Chapter 6

Conclusion and Discussion

The main question of this thesis was how the behaviour of the long-range percolation model can be described through critical parameters and critical exponents. We addressed this question by combining theoretical results with numerical simulations. In particular, we focused on the percolation probability

$$\theta(\beta) = \mathbb{P}_\beta(|K_0| = \infty),$$

the critical parameter β_c, and the critical exponent δ. The critical exponent δ describes the decay of the cluster-size distribution at criticality. If δ exists, it is expected to satisfy

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-\frac{1}{\delta}+o(1)}$$

In Chapter 3 we presented a proof that, if δ is well-defined for the long-range percolation model with parameters $\alpha \in (1, 2)$ and $d = 1$, or with $\alpha > 1$ and $d \geq 2$, then it must satisfy the lower bound

$$\delta \geq \frac{d + (d(\alpha - 1) \wedge 1)}{d - (d(\alpha - 1) \wedge 1)}.$$

In the one-dimensional case, we also compared our numerical estimates with the corresponding upper bound

$$\delta \leq \frac{\alpha + 1}{2 - \alpha}, \qquad 1 < \alpha < 2.$$

Our simulations for $\theta(\beta)$ and β_c are consistent with the theoretical phase-transition behaviour of the model. The estimates for δ are more delicate, since they are strongly affected by finite-size effects and by the choice of regression range.

A second main contribution of this thesis is the development of a numerical pipeline for estimating $\theta(\beta)$, β_c, and δ. We first constructed an efficient algorithm for sampling the long-range percolation graph on a finite vertex set in Chapter 4. The algorithm is based on geometric skipping: instead of checking all possible pairs of vertices, it skips over closed edges and samples only edges that are open. This substantially reduces the computational cost compared with a naive pairwise sampling approach.

Based on this sampling algorithm we constructed estimators for $\theta(\beta)$, β_c, and δ in Chapter 5. For the percolation probability $\theta(\beta)$, we introduced two estimators: a local estimator based on the cluster of the origin, and a global estimator based on the total mass of large clusters. Both estimators showed clear signs of convergence to $\theta(\beta)$. However, the local estimator generally had larger variance, since it uses only the cluster of a single vertex. The global estimator uses information from the whole simulated graph and therefore produced smoother estimates.

For the critical parameter β_c, we considered two estimators: a susceptibility-based estimator and a Ratio estimator. Both estimators converged to similar values as the system size was increased. However, the susceptibility-based estimator consistently had larger variance than the Ratio estimator. The Ratio estimator was therefore the more stable estimator in our simulations, and we used it as the main input for the estimation of δ.

Using the ratio estimator to obtain an estimate $\hat{\beta}_c$ of the critical parameter, we estimated the critical exponent δ by applying linear regression to the empirical cluster-size tail distribution on a log-log scale. This procedure is motivated by the expected critical behaviour

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) = n^{-\frac{1}{\delta}+o(1)}.$$

However, the resulting estimates are sensitive to both the accuracy of $\hat{\beta}_c$ and the domain on which the regression is performed. Small cluster sizes are affected by local finite-scale effects, while very large cluster sizes are suppressed by the finite simulation box. Consequently the estimated critical exponents for δ did not fully follow the theoretical bounds established earlier in Chapter 3.

There are three natural directions for future research based on the results of this thesis. First, most of our simulations were performed in one dimension due to computational constraints. It would be valuable to test whether the behaviour observed in this thesis persists in higher dimensions. Second, it would be interesting to study the estimators for β_c as functions of the system size L. Finite-size scaling is widely used to estimate critical values in models exhibiting phase transitions, and it may provide a useful framework for extrapolating finite-volume estimates of β_c in long-range percolation, under suitable assumptions; see, for example, [26]. Third, the estimation of δ could be improved by using a more systematic method for selecting the regression range. For instance, methods for fitting power-law distributions could be adapted to determine where small-scale effects become negligible; see [8].

Overall, the simulations support the theoretical picture of a phase transition in long-range percolation and show that finite-volume estimators can capture several important features of the model. At the same time, the simulation results also illustrate the difficulty of estimating critical exponents from finite systems. The estimation of δ is especially sensitive, because it depends on asymptotic cluster-size behaviour that can only be approximated on finite boxes.

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Appendix A

Growth of circles in $\mathbb{Z}^d$

Theorem A.0.1 (Growth of spheres in $\mathbb{Z}^d$)**.** Let $d \geq 1$ and let $r \geq 1$. Define

$$S_d(r) = \left|\{x \in \mathbb{Z}^d : \|x\|_1 = r\}\right|.$$

Then

$$S_d(r) \asymp r^{d-1}.$$

Proof. We first consider the case $d = 1$. Then

$$S_1(r) = |\{x \in \mathbb{Z} : |x| = r\}|.$$

Since $r \geq 1$, the only possibilities are $x = r$ and $x = -r$. Hence

$$S_1(r) = 2.$$

Therefore

$$S_1(r) \asymp 1 = r^0 = r^{d-1}.$$

Now let $d \geq 2$. A point $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ satisfies $\|x\|_1 = r$ precisely when

$$|x_1| + \cdots + |x_d| = r.$$

Define

$$a_i = |x_i|, \qquad i = 1, \ldots, d.$$

Then each a_i is a non-negative integer and

$$a_1 + \cdots + a_d = r.$$

Thus, after ignoring signs, counting points on the sphere is the same as counting the number of ways to distribute r indistinguishable balls among d distinguishable boxes. Here the i-th box corresponds to the absolute value of the i-th coordinate. By the stars and bars formula [11], the number of such choices is

$$\binom{r+d-1}{d-1},$$

counting the number of possible absolute value vectors

$$(|x_1|, \ldots, |x_d|).$$

It remains to account for the possible signs of the coordinates. If $a_i > 0$, then the coordinate x_i can be either a_i or $-a_i$. If $a_i = 0$, then there is only one possibility, namely $x_i = 0$. Therefore each absolute value vector corresponds to at least one and at most 2^d points of $\mathbb{Z}^d$. Hence

$$\binom{r+d-1}{d-1} \leq S_d(r) \leq 2^d \binom{r+d-1}{d-1}.$$

We now bound the binomial coefficient. For the upper bound, since $r \geq 1$, we have that $r + d - 1 \leq dr$, and therefore

$$\binom{r+d-1}{d-1} \leq (r+d-1)^{d-1} \leq d^{d-1} r^{d-1}.$$

For the lower bound, we use

$$\binom{r+d-1}{d-1} = \frac{(r+1)(r+2)\cdots(r+d-1)}{(d-1)!} \geq \frac{r^{d-1}}{(d-1)!}.$$

Combining the above inequalities gives

$$\frac{1}{(d-1)!} r^{d-1} \leq S_d(r) \leq 2^d d^{d-1} r^{d-1}.$$

Since d is fixed, this proves that

$$S_d(r) \asymp r^{d-1}.$$

$\square$