Hat Guessing

by

Lars van der Kuil

To obtain the degree of Master of Science
at the Delft University of Technology,
To be defended publicly at 13:45 on the 22nd of August, 2025.

Student number:	5337402	
Project duration:	January 14, 2025 – August 22, 2025	
Thesis committee:	Dr. A. Bishnoi,	TU Delft, daily supervisor
	Prof.Dr. D.C. Gijswijt	TU Delft, chair
	Dr. R.J. Fokkink	TU Delft

An electronic version of this thesis is available at `http://repository.tudelft.nl/`.

TU Delft

Summary

This thesis is about the following hat guessing game first described by Winkler [17]. Consider a group of n players situated at the vertices of a graph G. An adversary gives each player a hat coloured one of q possible colours. The players are unable to see the colour of their own hat, but each player can see the hat colour of their neighbours in G. The players are asked to simultaneously make a guess on the colour of their own hat, according to a predetermined guessing strategy based solely on the hat colours each player can see. The players win if at least one of them correctly guesses their hat colour, otherwise the adversary wins. Is it possible for the players to devise a strategy which guarantees they win regardless of how the adversary places the hats? Clearly, if the players have a winning strategy on the graph G for q colours, then the same strategy must also be winning for any number of colours less than q. This motivates us to define the *hat guessing number* $HG(G)$ of G, a parameter first introduced by Farnik [6].

Definition. *Let $HG(G)$ be the maximum q such that the players have a winning strategy when playing the hat guessing game on the graph G with q colours.*

Bosek et al. [3] showed an upper bound on the hat guessing number using a partition into independent sets. We show that the same bound holds when $V_1, \dots, V_l$ partition the vertices into sets that induce directed acyclic graphs. This implies that for any arc a of the complete graph K_n we have that $HG(K_n - a) = n-1$, whereas $HG(K_n) = n$. Furthermore, we give a family of graphs for which choosing a partition into the minimum number of independent sets may be arbitrarily worse than choosing a different partition into independent sets, when applying the bound.

Szczechla. [16] have shown the hat guessing number for cycles. They formulate strategies to show that $HG(C_4) \geq 3$ and $HG(C_{3n}) \geq 3$ in a complex coordinate system. We reformulate those same strategies and show that they are winning without using this coordinate system.

The winning strategy for C_4 can be generalised to obtain the following result. For an odd prime power q, let M a maximum matching in the complete graph K_{q+1}. Then $HG(K_{q+1} - M) = q$.

On a more general and technical note, we define a notion of equivalence of strategies to try revealing some of the inherent symmetries of the problem. For example, reassigning individual strategies according to an automorphism of the graph does not change whether the collective strategy is winning. We also obtain a notion of uniqueness of a winning strategy, which we conjecture to be a strong, but rare, property.

Contents

1
Introduction

This thesis is about the following hat guessing game first described by Winkler [17]. Consider a group of n players situated at the vertices of a graph G. An adversary gives each player a hat coloured one of q possible colours. The players are unable to see the colour of their own hat, but each player can see the hat colour of their neighbours in G. The players are asked to simultaneously make a guess on the colour of their own hat, according to a predetermined guessing strategy based solely on the hat colours each player can see. The players win if at least one of them correctly guesses their hat colour, otherwise the adversary wins. Is it possible for the players to devise a strategy which guarantees they win regardless of how the adversary places the hats?

Consider for example two people, Alice and Bob, that can see each other, so $G = K_2$. Each get a hat coloured either red or blue, so $q = 2$. Alice's strategy will be to guess that their hat has the same colour as Bob's hat. Bob's strategy will be to guess that their hat has the colour that Alice's hat does not have. This strategy ensures that either Alice or Bob correctly guesses the colour of their own hat, regardless of what the colours of the hats are. Indeed, if both hats are red or both hats are blue, then Alice guesses correctly, and if one of the hats is red and the other blue, then Bob guesses correctly. This strategy can be generalised to a strategy for the hat guessing game on the complete graph K_n with n colours.

Example 1.1. *Let $v_0, v_1, \ldots, v_{n-1}$ be the players situated on the vertices of K_n. Let $0, 1, \ldots, n-1$ be the possible colours of the hats. Given a colouring $c : \{v_0, v_1, \ldots, v_{n-1}\} \to \{0, 1, \ldots, n-1\}$, player v_i's strategy will be to guess that the colour of their hat is*

$$i - \sum_{j=0, j\neq i}^{n-1} c(v_j) \pmod{n}. \tag{1.1}$$

Note that player v_i guesses correctly precisely when $\sum_{j=0}^{n-1} c(v_j) \equiv i \pmod{n}$. Seeing as this must hold for some $i \in \{0, 1, \ldots, n-1\}$, we know that this player v_i will guess correctly.

Clearly, if the players have a winning strategy on the graph G for q colours, then the same strategy must also be winning for any number of colours less than q. This motivates us to define the *hat guessing number* $HG(G)$ of G, a parameter first introduced by Farnik [6].

Definition 1.2. *Let $HG(G)$ be the maximum q such that the players have a winning strategy when playing the hat guessing game on the graph G with q colours.*

In Chapter 2, we give an overview of known results on the hat guessing number. This is not an exhaustive overview, but meant as an introductory guide. In the rest of the thesis we go more in depth on some of the results and add our own remarks and analyses. Starting in Chapter 3 with a bound based on partitioning the vertices of a graph into independent sets. We show how to adapt this result for directed graphs and show that a minimum partition into independent sets is not necessarily the best choice of partition when applying the bound. Next, in Chapter 4, we look at the strategies that show the lower bound for the hat guessing number of cycles. The strategy on the 4-cycle can be generalised to show the hat guessing number of another family of graphs. For an odd prime power q, let M be a maximum matching in the complete graph K_{q+1}, then we show, in Chapter 5, that the hat guessing number of the graph $K_{q+1} - M$ obtained by removing the edges in M from K_{q+1} is $HG(K_{q+1} - M) = q$. In Chapter 6, we determine the hat guessing number for all on 5 or fewer vertices using the known literature. Finding the hat guessing numbers of all 6 vertex graphs remains an open problem. In Chapter 7 we explore the symmetries of the hat guessing game by introducing notions of equivalence and uniqueness of strategies. We conclude by discussing possible avenues for further research in Chapter 8.

2

Overview of Known Results

In this chapter we give an overview of results from the literature that the writer deemed significant. Some of these results will be discussed in more detail in later chapters. Firstly, in Section 2.1, we provide a more formal definition of the hat guessing number. Next, in Sections 2.2 and 2.3, we discuss general lower and upper bounds respectively. Note that the former are achieved via constructions and the latter via non-existence proofs. Finally, in Section 2.4, we discuss graphs for which we have exact results.

2.1. Formal Definition

We formalise the *hat guessing game* as follows. Let $G = (V, E)$ a graph, whose vertices are often referred to as players, and let q a positive integer. Then the hat guessing game on G with q available colours will be denoted $\langle G, q\rangle$. Let Γ a set of size q, whose elements are called colours. An element of Γ^V assigns a colour to each player and is thus a colouring of G. The colour assigned to a player is the colour of their hat. An individual strategy for player v is a function $f_v : \Gamma^V \to \Gamma$ which is required to only depend on the colours assigned to the neighbours of v. A collective strategy, or simply a *strategy*, is a collection consisting of an individual strategy for each player. A strategy f is *winning* if for any colouring $c \in \Gamma^V$ there exists a player $v \in V$ such that $f_v(c) = c_v$. A hat guessing game $\langle G, q\rangle$ is *winning* if there exists a winning strategy for the game and losing otherwise. The *hat guessing number* $HG(G)$ of a graph G is the largest q such that the game $\langle G, q\rangle$ is winning.

2.2. Lower Bounds

The most trivial, though not very useful, lower bound is that for any graph G the hat guessing number is $HG(G) \geq 1$. Seeing as any graph contains a vertex and there is just a single colour, the strategy where all players guess the one available colour is a winning strategy. In case G is an empty graph, this lower bound is tight.

Another trivial, but this time useful, lower bound follows from the observation that any strategy that is winning for a hat guessing game on a subgraph H of G is also winning for G. More precisely, if we have a winning strategy g for the game $\langle H, q\rangle$, then we can construct a strategy f for the game $\langle G, q\rangle$ by using the strategy g on the subgraph H and guessing an arbitrary colour for all vertices outside of H. This new strategy f is clearly winning, as the strategy g is winning. We obtain the following bound.

Theorem 2.1. *For a graph G and a subgraph $H \subseteq G$, we have $HG(G) \geq HG(H)$.*

By this theorem, any lower bound for a graph implies a lower bound on a lot of other graphs. For example, as shown in Chapter 1, the hat guessing number of a complete graph on n vertices is $HG(K_n) \geq n$ (Feige [7]), so we get a lower bound based on the size of a maximum clique in a graph.

Theorem 2.2. *For a graph G with clique number $\omega(G)$, we have $HG(G) \geq \omega(G)$.*

This bound is not tight in general. In fact, in general, the hat guessing number is not bound by any function of the clique number. Alon et al. [1] show that the hat guessing number of the complete r-partite graph is $HG(K_{n,\ldots,n}) \geq n^{\frac{r-1}{r}-o_n(1)}$, which approaches ∞ as $n \to \infty$, whereas $\omega(K_{n,\ldots,n}) = r$.

Another general lower bound is based on the lexicographic product of graphs. Consider a graph G and replace all of its vertices by copies of another graph H, where edges in G become complete connections between the corresponding copies of H. The result is the *lexicographic product* $G \times_L H$. Formally, its vertices are the pairs (u, v) of vertices from G and H respectively, where $(u_0, v_0) \sim (u_1, v_1)$ if and only if $u_0 \sim u_1$ or $u_0 = u_1$ and $v_0 \sim v_1$. Kokhas and Latyshev [14] [Theorem 3.2] replace a single vertex by H at a time and concludes a result about a variant of the hat guessing game where the number of available colours may be different per player. The special case of the following result where H is a clique was also shown by Gadouleau and Georgiou [9].

Theorem 2.3. *For graphs G and H, we have $HG(G \times_L H) \geq HG(G) \cdot HG(H)$.*

2.3. Upper Bounds

From the lower bound based on subgraphs, Theorem 2.1, it follows that the hat guessing number of a disconnected graph G is at least the maximum of the hat guessing numbers of the connected components. Now suppose that we have a strategy for the game $\langle G, q \rangle$ where q is strictly larger than the maximum of the hat guessing numbers of the connected components. Then each connected component of G can be coloured such that each of its players' guesses are fixed and incorrect. The concatenation of these colourings is a colouring of G where every player guesses incorrectly. We conclude that we need only consider connected graphs.

Theorem 2.4. *For a disconnected graph G with connected components $G_1, \ldots, G_k$, we have $HG(G) = \max_{1 \leq i \leq k}\{HG(G_i)\}$.*

The subgraph bound can also be reinterpreted as an upper bound. In this way, any upper bound on the hat guessing number of a graph will result in an upper bound on the hat guessing number of its subgraphs. For example, the hat guessing number of the complete graph is $HG(K_n) \leq n$ [4]. Seeing as any graph is the subgraph of a complete graph we obtain the following general upper bound.

Theorem 2.5. *For a graph G on n vertices, we have $HG(G) \leq n$.*

It turns out that complete graphs are the only graphs for which this bound is tight. The hat guessing number of the complete graph with an edge removed is $HG(K_n - e) \leq n - 1$. Seeing as any non-complete graph is the subgraph of a complete graph with an edge removed we obtain the following general upper bound.

Theorem 2.6. *For a non-complete graph G on n vertices, we have $HG(G) \leq n-1$.*

This result is a corollary of a result by Bosek et al. [3]. They use a counting argument to give the following bound on the hat guessing number using a partition into independent sets.

Theorem 2.7. *Let $V_1, \ldots, V_l$ a partition of the vertices of a graph G into l independent sets. Suppose that for some positive integer k we have*

$$l - \sum_{i=1}^{l} \left(\frac{k-1}{k}\right)^{|V_i|} < 1. \tag{2.1}$$

Then $HG(G) < k$.

One could, for example, take a partition of the vertices into as few independent sets as possible, thus certifying the chromatic number of the graph. This results in the following bound.

Theorem 2.8. *For a graph G on n vertices, for n large enough, and with chromatic number $\chi(G) \geq 2$, we have*

$$HG(G) \leq \frac{n}{\chi(G) \ln\left(\frac{\chi(G)}{\chi(G)-1}\right)}.$$

This choice of partition is not always optimal, which will be discussed in Chapter 3, where we will also take a closer look at the proof for the general bound and a version for directed graphs.

The hat guessing number can also be bound by the maximum degree Δ. The following theorem is considered part of the folklore and follows from Lovász's local lemma, see [6].

Theorem 2.9. *For a graph G with maximum degree Δ, we have $HG(G) < e\Delta$, where e is the base of the natural logarithm.*

However, one cannot bound the hat guessing number from below by the maximum degree. This is shown by the hat guessing number of a star graph S_n consisting of one vertex of degree n and n vertices of degree 1. We have $HG(S_n) \leq 2$. In fact, Alon et al. [1] have shown that degree 1 vertices do not increase the hat guessing number in general.

Theorem 2.10. *For a graph G with at least 2 edges and a vertex v of degree 1, we have $HG(G) \leq HG(G-v)$.*

Note that this bound is always tight by the subgraph bound, Theorem 2.1, since $G - v$ is a subgraph of G.

Another graph parameter we can use to bound the hat guessing number is the size of a minimum vertex cover. The following bound is implied by a theorem by Gadouleau [8].

Theorem 2.11. *For a graph G with minimum size of a vertex cover* $\tau(G)$*, we have* $HG(G) \leq 1 + \sum_{i=1}^{\tau(G)} i^i$.

The last general bound is tentatively related to the degeneracy of a graph and was shown by He and Li [11] (Lemma 4).

Theorem 2.12. *Let G be a graph with vertices ordered* $v_1, \dots, v_n$*, and define* t_i *for* $i = 1, \dots n$ *recursively by*

$$t_i = 1 + \prod_{\substack{v_j \sim v_i \\ j < i}} t_j,$$

where the empty product is 1. Then $HG(G) < \max\{t_1, \dots, t_n\}$.

Lastly, Gadouleau and Georgiou [9] showed that the complete bipartite graph $K_{n,m}$ has hat guessing number $HG(K_{n,m}) \leq \min\{n+1, m+1\}$.

2.4. Exact Results

From the above general results, we can immediately derive some exact values of hat guessing numbers. We have seen that $HG(K_n) \geq n$ and $HG(K_n) \leq n$ and thus $HG(K_n) = n$. We have also seen that $HG(K_n - e) \leq n - 1$ and $K_{n-1} \subseteq K_n - e$, so $HG(K_n - e) = n - 1$. Seeing as leaves do not increase the hat guessing number, it follows that $HG(T) = 2$ for any tree T.

There are also a number of graphs for which the exact value has been calculated using methods specific to those graphs. For example, He et al. [12] showed that the complete bipartite graph $K_{3,3}$ has hat guessing number $HG(K_{3,3}) = 3$, thereby showing that the above mentioned bound $HG(K_{n,m}) \leq \min\{n+1, m+1\}$ is not always tight.

The hat guessing number of cycle graphs is also known. We have the following theorem by Szczechla. [16]. In Chapter 4 we discuss the strategy that shows the lower bound.

Theorem 2.13. *For any positive integer* $n \geq 2$ *we have*

$$HG(C_n) = \begin{cases} 3 & \text{if } n = 4 \text{ or } n \equiv 0 \pmod 3, \\ 2 & \text{else.} \end{cases}$$

Cactus graphs are graphs where no two cycles share an edge and thus they are in a sense a combination of cycles and trees. This is reflected in their hat guessing number, as can be seen in the following theorem by Chizewer et al. [5].

Theorem 2.14. *For a cactus graph G we have*

$$HG(G) = \begin{cases} 4 & \text{if } G \text{ contains two } C_3, \\ 3 & \text{if } G \text{ contains two cycles or } C_4 \text{ or } C_{3n}, \text{ but no two } C_3, \\ 2 & \text{if } G \text{ contains at most one cycle, but no } C_4 \text{ or } C_{3n}. \end{cases} \tag{2.2}$$

The smallest example of a cactus graph with hat guessing number 4 is 2 copies of K_3 that share a vertex. This is an example of a windmill graph. In general, the windmill $W_{k,n}$ is the graph consisting of n copies of K_k sharing a vertex. He et al. [12] have shown the following two theorems on the hat guessing number of windmills.

Theorem 2.15. *For* $k \geq 2$, $n \geq \log_2(2k-2)$ *we have* $HG(W_{k,n}) = 2k-2$.

Theorem 2.16. *For* $n \geq 1$, $d \geq 2$ *we have* $HG(W_{d^n - d^{n-1}+1,n}) = d^n$.

They also showed that the general bound using the size of a minimum vertex cover, Theorem 2.11, turns out to be tight in the case of books. A book $B_{n,d}$ with n pages and a spine of size d is the complete connection between K_d and an independent set of n vertices. Note that $\tau(B_{n,d}) \leq d$, because the spine of the book connects to all edges. Also note that $\tau(B_{n,d}) \geq d$, because the spine with a single page form a $d+1$ clique. So, $\tau(B_{n,d})$. He et al. [12] give a strategy on books showing the corresponding lower bound, resulting in the following theorem.

Theorem 2.17. *For large enough* n *in terms of* d *we have* $HG(B_{n,d}) = 1 + \sum_{i=1}^{d} i^i$.

3

Partition Into Independent Sets Bound

In this chapter we take a more in depth look at the upper bound based on a partition into independent sets, Theorem 3.1. We have restated the theorem below for convenience. Firstly, in Section 3.1, we give the proof of this theorem in detail and show what this implies about the hat guessing game on directed graphs. Lastly, in Section 3.2, we give an example of a graph where taking a partition into independent sets that certify the chromatic number is not the best choice.

Theorem 3.1. *Let $V_1, \dots, V_l$ a partition of the vertices of a graph G into l independent sets. Suppose that for some positive integer k we have*

$$l - \sum_{i=1}^{l} \left(\frac{k-1}{k} \right)^{|V_i|} < 1. \tag{3.1}$$

Then $HG(G) < k$.

Note that this bound is generally not tight and in some cases quite bad. Consider for example an even cycle C_{2n} and recall that its hat guessing number is $HG(C_{2n}) \leq 3$. Let $v_1, v_2, \dots, v_{2n}$ be its vertices, where v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, 2n-1$ and v_{2n} is adjacent to v_1. We can partition the vertices into independent sets by taking all vertices with an even index in one set and all vertices with an odd index in another set. The theorem's condition then reads

$$2 - 2\left(\frac{k-1}{k} \right)^{n} < 1.$$

Solving for k gives the equivalent condition $k > (1 - 2^{-1/n})^{-1}$. However, as n approaches ∞, the right hand side approaches ∞ as well, whereas the hat guessing number of C_{2n} is constant.

3.1. Partition into DAGs

The hat guessing game still makes sense when the graph is a directed graph, though we need to replace neighbours by out-neighbours. That is to say, an individual strategy for a strategy may only depend on the out-neighbours of that vertex. When considering directed graphs, the partition into independent sets bound can be generalised to partitions into directed acyclic graphs (DAGs). To see this we need to take a closer look at the proof of the partition into independent sets bound. This proof can be seen as a generalisation of the upper bound of $HG(K_n)$, which we will give first.

Proof. $HG(K_n) < n+1$, [4] Fix some strategy for the game $\langle K_n, n+1\rangle$. For each vertex $v \in V$, it's guess gets fixed by a colouring of $V \setminus \{v\}$. So, there exist exactly $(n+1)^{n-1}$ colourings of V where v guesses correctly. By the union bound, there are at most $n(n+1)^{n-1}$ colourings where someone guesses correctly. However, there are $(n+1)^n$ ways to colour the vertices of K_n with $n+1$ colours and $(n+1)^n > n(n+1)^{n-1}$. Therefore, there exists a colouring such that each player guesses incorrectly and the game $\langle K_n, n+1\rangle$ is losing. □

Proof. of Theorem 3.1, [3] Fix some strategy for the game $\langle G, k\rangle$. Consider the players in an independent set V_i of G. Their guesses get fixed by a colouring of $V \setminus V_i$. So, there exist exactly $(k-1)^{|V_i|}k^{|V|-|V_i|}$ colourings where none of the players in V_i guess correctly. Conversely, there exist exactly $k^{|V|} - (k-1)^{|V_i|}k^{|V|-|V_i|}$ colourings where at least one player in V_i guesses correctly. By the union bound, there are at most

$$\sum_{i=1}^{l}\left(k^{|V|} - (k-1)^{|V_i|}k^{|V|-|V_i|}\right) = k^{|V|}\left(l - \sum_{i=1}^{l}\left(\frac{k-1}{k}\right)^{|V_i|}\right)$$

colourings where someone guesses correctly. Note that the total number of colourings of G with k colours is $k^{|V|}$. For a winning strategy to exist the total number of colourings has to be at most the number of colourings for which someone guesses correctly. If this is not the case, that is, when the condition in the theorem is satisfied, then no winning strategy can exist and $HG(G) < k$. □

Note that we use the fact that V_i is independent to bound the number of colourings where no player guesses correctly. A colouring of $V \setminus V_i$ fixes the guesses of the players in V_i, because the guesses of players in V_i may not depend on the colours of players in V_i. Therefore, each player in V_i has $k-1$ potential colours that make it guess incorrectly, for each colouring of $V \setminus V_i$.

Now let V_i be such that the subgraph $G[V_i]$ of G on the vertices in V_i is a DAG. Let s be a sink of this DAG. A colouring of $V \setminus V_i$ no longer necessarily fixes the guesses of all vertices in V_i, but it does still fix the guess of s. So, given a colouring of $V \setminus V_i$, there are $k-1$ ways to colour s such that s guesses incorrectly. Now consider the set $V_i \setminus \{s\}$. It induces another DAG, which again has a sink, call it t. A colouring of $(V \setminus V_i) \cup \{s\}$ such that s guesses incorrectly still does not necessarily fix all the guesses of vertices in V_i, but it does fix the guess of t. So, given such a colouring, there are $k-1$ ways to colour t such that t guesses incorrectly. We can repeat this argument until we have emptied the set V_i, at which point we have found that there are $(k-1)^{|V_i|}k^{|V|-|V_i|}$ colourings where none of the players in V_i guess correctly. Seeing as this is the exact same number we found in the case where V_i was an independent set, we can continue the proof as before. This results in the following theorem.

Theorem 3.2. *Let $V_1, \ldots, V_l$ a partition of the vertices of a directed graph G into l sets that each induce a DAG. Suppose that for some positive integer k we have*

$$l - \sum_{i=1}^{l}\left(\frac{k-1}{k}\right)^{|V_i|} < 1. \tag{3.2}$$

Then $HG(G) < k$.

From the partition into independent sets bound, Theorem 3.1, it follows that for any edge e of K_n we have $HG(K_n - e) \leq n-1$, which implies that any non-complete graph on n vertices has hat guessing number at most $n-1$. Indeed, we have $n-2$ independent sets of size 1 and 1 independent set of size 2. For $k = n$, Condition (2.1) then reads $n-1-(n-2)\frac{n-1}{n} - \left(\frac{n-1}{n}\right)^2 = \frac{n^2-1}{n^2} < 1$, which clearly holds. Theorem 3.2 implies that this also holds in the case where e is an arc of K_n, thus implying that any non-complete directed or mixed graph on n vertices has hat guessing number at most $n-1$.

3.2. Chromatic Number Bound Not Optimal

As a special case of the partition into independent sets bound we have the chromatic number bound, Theorem 2.8, which we have restated below for convenience. One might wonder whether choosing a partition certifying the chromatic number is always the best choice of partition into independent sets. This turns out to not be the case, as we will show later in this section. The knowledge that the hat guessing number can be bound by a function of the chromatic number is valuable regardless.

Theorem 3.3. *([3]) For a graph G on n vertices, for n large enough, and with chromatic number $\chi(G) \geq 2$, we have*

$$HG(G) \leq \frac{n}{\chi(G)\ln\left(\frac{\chi(G)}{\chi(G)-1}\right)}.$$

Let us first consider a partition into independent sets $P = (V_1, \ldots, V_l)$ and a refinement Q of P. We will show that P gives bounds at least as good as Q using the partition into independent sets bound. Note that it suffices to show that for any positive integer k we have

$$|P| - \sum_{U \in P} \left(\frac{k-1}{k}\right)^{|U|} \leq |Q| - \sum_{U \in Q} \left(\frac{k-1}{k}\right)^{|U|}, \tag{3.3}$$

as this implies that condition (2.1) is easier to satisfy. Without loss of generality we may assume that $Q = (W_1, W_2, V_2, \ldots, V_l)$, where $W_1, W_2 \neq \varnothing$, $W_1 \cup W_2 = V_1$, and $W_1 \cap W_2 = \varnothing$. Equation (3.3) then reduces to

$$-1 - \left(\frac{k-1}{k}\right)^{|V_1|} \leq -\left(\frac{k-1}{k}\right)^{|W_1|} - \left(\frac{k-1}{k}\right)^{|W_2|} \text{ or equivalently}$$

$$\left(1 - \left(\frac{k-1}{k}\right)^{|W_1|}\right)\left(1 - \left(\frac{k-1}{k}\right)^{|W_2|}\right) \geq 0, \text{ as } |V_1| = |W_1| + |W_2|.$$

This holds, because $\frac{k-1}{k} \leq 1$.

This shows that we always want a minimal partition into independent sets, that is, we want a partition that is not a strict refinement of another partition into independent sets. It is not yet clear whether a minimum partition is optimal. The following example gives a graph with two minimal partitions, where the minimum partition, certifying the chromatic number, gives a worse bound than the other partition.

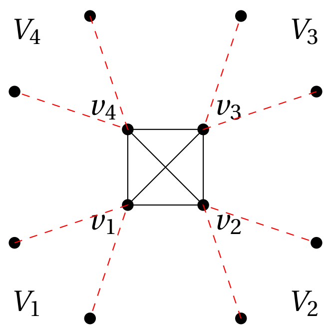


Figure 3.1: A sketch of the graph constructed from K_4 and $\varnothing_8$. The edges between K_4 and $\varnothing_8$ have been omitted, instead the red, dashed lines show the non-edges.

Example 3.4. *We are going to construct a graph G from a clique K_4 of size 4 and an independent set $\varnothing_8$ of size 8. Partition the vertices of $\varnothing_8$ into 4 sets of size 2, say $V(\varnothing_8) = V_1 \cup V_2 \cup V_3 \cup V_4$. Number the vertices of K_4 like $V(K_4) = \{v_1, v_2, v_3, v_4\}$. For $i \in [4]$, $u \in V(\varnothing_8)$ let $v_i \sim u$ if $u \notin V_i$.*

Note that the chromatic number $\chi(G)$ of G is at least 4, since G has K_4 as a subgraph. Furthermore, the sets $\{v_i\} \cup V_i$ for $i \in [4]$ are independent and partition $V(G)$, so $\chi(G) = 4$. Note that this is the only way to partition the vertices into 4 independent sets, since the v_i have to be in different sets and are each connected to all but the vertices in the corresponding V_i. For $k = 10$ condition (2.1) then reads $4 - \sum_{i=1}^{4} \left(\frac{9}{10}\right)^3 > 1.08 > 1$, so the condition is not satisfied and we can not draw a conclusion using Theorem 3.1.

Now consider the partition $V(G) = V(\varnothing_8) \cup \{v_1\} \cup \{v_2\} \cup \{v_3\} \cup \{v_4\}$. Clearly each of these sets is independent. For $k = 10$ condition (2.1) now reads $5 - \left(\frac{9}{10}\right)^8 - \sum_{i=1}^{4} \left(\frac{9}{10}\right)^1 < 0.97 < 1$, so the condition is satisfied and thus $HG(G) < 10$ by Theorem 3.1.

More generally one can construct a similar graph G_n from K_n and $\varnothing_{2n}$. This graph G_n has chromatic number n as certified by the unique partition into n independent sets $V(G_n) = \bigcup_{i=1}^{n} (\{v_i\} \cup V_i)$. Condition (2.1) then reads

$$n - n\left(\frac{k-1}{k}\right)^3 < 1. \tag{3.4}$$

Clearly, we can also partition the vertices into the $n+1$ independent sets $V(\varnothing_{2n})$ and $\{v_i\}$ for each $i \in [n]$. Condition (2.1) then reads

$$n + 1 - n\left(\frac{k-1}{k}\right) - \left(\frac{k-1}{k}\right)^{2n} < 1. \tag{3.5}$$

We may rewrite Condition (3.4) as

$$\frac{n-1}{n} < \left(\frac{k-1}{k}\right)^3.$$

This can be solved for k to obtain the equivalent condition

$$k > \frac{1}{1 - \left(\frac{n-1}{n}\right)^{1/3}}.$$

Let $K = \frac{1}{1-\left(\frac{n-1}{n}\right)^{1/3}}$, *then the minimum positive integer k for which condition* (3.4) *holds is* $k = \lceil K \rceil$.

Suppose that for some $\alpha < 1$ *we have that* (3.5) *holds for* $k = \alpha K$. *As n approaches* ∞, *we have that* K *approaches* ∞, *and thus* $\lim_{n\to\infty} \lceil K \rceil - \lceil \alpha K \rceil = \lim_{K\to\infty} (1-\alpha)K = \infty$, *since* $1-\alpha > 0$. *Therefore, the minimum partition can give an arbitrarily worse bound than the other minimal partition in* G_n *by taking n to be large enough.*

The existence of such an α *is suggested by the plot in Figure 3.2. The highlighted region is the collection of points* (n, α), *such that Condition* (3.5) *holds for* $k = \alpha K$. *Here n runs along the logarithmically scaled horizontal axis and* α *runs along the vertical axis. For example it seems that for* $\alpha = 0.9$ *and* $n \geq 4$ *the condition is satisfied. It turns out that for any* $\alpha > 0.7819$ *the condition is satisfied for n large enough. The precise calculations can be found in Appendix A.*

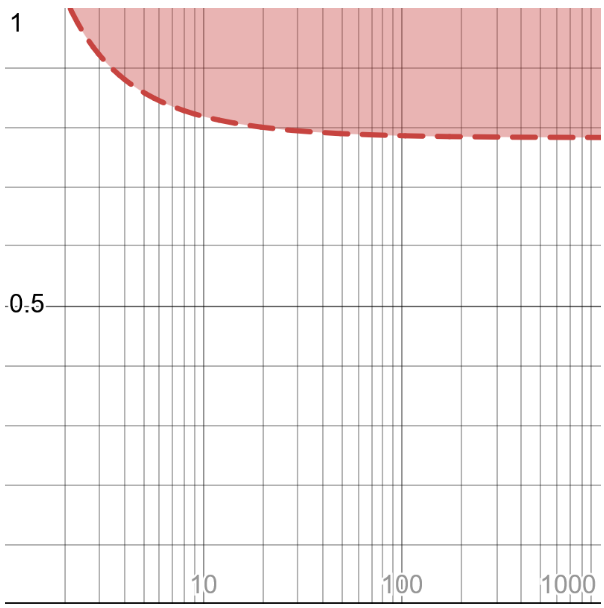


Figure 3.2: The points (n, α) such that Condition (3.5) holds for $k = \alpha K$.

4

Cycles

Let n a positive integer and consider the n-cycle C_n. Szczechla. [16] showed that the hat guessing number of C_n is as follows.

Theorem 4.1. *For any positive integer $n \geq 2$ we have*

$$HG(C_n) = \begin{cases} 3 & \text{if } n = 4 \text{ or } n \equiv 0 \pmod 3, \\ 2 & \text{else.} \end{cases}$$

Note that $HG(C_n) \geq 2$ follows from the fact that $HG(K_2) = 2$ and $K_2 \subseteq C_n$. So to show the theorem, Szczechla. [16] showed three things. Firstly, they gave a strategy for the hat guessing game on C_4 and C_{3n} with 3 colours. Secondly, they showed that no cycle has hat guessing number greater than 3. Lastly, they showed that no cycle other than C_4 and C_{3n} has hat guessing number greater than 2. The upper bounds are shown by categorising the potential strategies in a smart way. This is, however, quite specific for cycles and not very useful for more general graphs. For this reason, we will focus on the strategies that show the lower bounds for C_4 and C_{3n}.

4.1. The 4-Cycle

Recall from Example 1.1 in the introduction the strategy for the complete graph K_n. This strategy is based on the observation that for any $x \in \mathbb{Z}$ there exists an $i \in \{0, 1, \dots, n-1\}$ such that $x \equiv i \pmod n$. Taking $x = \sum_{j=0}^{n-1} c(v_j)$ then gives a set of n equations, one for each i, of which at least one must hold. Assigning each of these equations to one of the players gives the strategy.

We can do something similar for the 4-cycle. Its strategy is based on the following property of $\mathbb{F}_3$. Given $x, y \in \mathbb{F}_3$ one of the following holds $x = 0$, $y = 0$, $x + y = 0$, $x - y = 0$. Let the 4-cycle be given as in Figure 4.1, where the vertices v_A, v_B, v_C, v_D are coloured $A, B, C, D \in \mathbb{F}_3$ respectively. Take $x = A - (B + D)$ and $y = C - (B - D)$, then $x + y = B - (-A - C)$ and $x - y = D - (-A + C)$. Assign the following strategies to the vertices for the colouring $c = (A, B, C, D)$.

$$f_{v_A}(c) = B + D, \qquad f_{v_B}(c) = -A - C,$$
$$f_{v_C}(c) = B - D, \qquad f_{v_D}(c) = -A + C.$$

We see that v_A guesses correctly if $x = 0$, v_B if $x + y = 0$, v_C if $y = 0$, and v_D if $x - y = 0$. As established above, at least one of these must hold, and thus the strategy is winning. Similar sets of equations exist for other prime fields, these form the basis for Chapter 5.

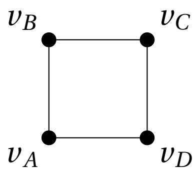


Figure 4.1: The 4-cycle with vertices v_A, v_B, v_C, v_D coloured $A, B, C, D \in \mathbb{F}_3$ respectively.

4.2. The $3n$-Cycle

Let $N = 3n$ and let $v_1, \ldots, v_N$ the vertices of C_N, where v_i is adjacent to v_{i+1} for $i = 1, \ldots, N-1$ and v_N is adjacent to v_1. Take as colour set $\mathbb{Z}_3 = \{-1, 0, 1\}$ and let $g : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ be given by

$$g(x, y) = \begin{cases} x + 1 & \text{if } x = y, \\ x & \text{if } x \neq y. \end{cases} \tag{4.1}$$

Then player v_i's strategy is given by $f_{v_i}(c_{v_{i-1}}, c_{v_{i+1}}) = g(c_{v_{i-1}}, c_{v_{i+1}})$ for $i = 2, \ldots, N-1$, player v_1's strategy is $f_{v_1}(c_{v_N}, c_{v_2}) = g(c_{v_N} + 1, c_{v_2})$, and player v_N's strategy is $f_{v_N}(c_{N-1}, c_1) = g(c_{N-1}, c_1 - 1)$.

To show that this strategy is winning we will attempt to construct a colouring that shows the contrary and see that we will always fail. Seeing as $g(x+1, y+1) = g(x, y) + 1$, we may pick the first colour c arbitrarily. From there we have three ways to pick the next colour, either c, $c+1$, or $c-1$. Consider a player v_i for $i = 2, \ldots, N-1$. Suppose we have picked $c_{v_{i-1}} = c$.

If we pick $c_{v_i} = c$, then we must also pick $c_{v_{i+1}} = c$, else v_i guesses correctly. We have now reached the same situation as before with an incremented index, so $c_{v_{i+2}} = c$, else v_{i+1} guesses correctly. This process repeats until all players with index at least $i-1$ have colour c. In case $i = 2$ all players have colour c and for $j = 2, \ldots, N-1$ player v_j guesses incorrectly. Furthermore, player v_1 guesses incorrectly, but player v_N guesses correctly.

If instead we pick $c_{v_i} = c + 1$, then we cannot pick $c_{v_{i+1}} = c$, else v_i guesses correctly. So either $c_{v_{i+1}} = c + 1$ or $c_{v_{i+1}} = c - 1$. The former results in the remaining colours being $c + 1$ analogous to the previous situation, where we started with $c_{v_i} = c$. The latter results in the same situation as we started in, that is, the colour was incremented by 1.

If instead we pick $c_{v_i} = c-1$, then either $c_{v_{i+1}} = c-1$, $c_{v_{i+1}} = c$, or $c_{v_{i+1}} = c+1$. In the first case we again continue with a constant sequence, in the second case we end up in the previous situation, where the colour was incremented by 1, and in the third case we are back in the situation we started in, that is, the colour was decremented by 1.

Given that we have coloured a vertex with colour x there are three ways to colour the next vertex, either $x \mapsto x-1$, $x \mapsto x+1$, or $x \mapsto x$. The first choice leaves all options open for the next vertex, the second choice prevents us from changing to the first choice, and the last choice forces us to continue as a constant sequence. These choices will be referred to respectively as *'down'*, *'up'*, and *'constant'*, where we cannot choose 'down' after either 'up' or 'constant' and we cannot choose 'up' after 'constant'.

Note that colouring v_1 and making these choices ensures that none of the players v_i for $i = 2,\ldots,N-1$ guess correctly, so it suffices to check for each starting choice and possible ending choice whether v_1 or v_N guesses correctly. We have already seen that starting with choosing 'constant' ends with 'constant' and results in player v_N guessing correctly. The rest of the possible starting and ending choices are given in Figure 4.2. The last two rows of the figure show the sequences for which neither v_1 nor v_N guess correctly in case $N \not\equiv 0 \pmod 3$.

$\cdots$	v_{N-1}	v_N	v_1	v_2	$\cdots$	guesses correctly
'constant'	0	0	0	0	'constant'	v_N
'constant'	0	0	0	1	'up'	v_N
'constant'	1	1	0	1	'up'	v_N
'constant'	-1	-1	0	1	'up'	v_1
'up'	1	-1	0	1	'up'	v_1
'constant'	0	0	0	-1	'down'	v_N
'constant'	1	1	0	-1	'down'	v_N
'constant'	-1	-1	0	-1	'down'	v_1
'up'	0	1	0	-1	'down'	v_1
'up'	1	-1	0	-1	'down'	v_1
'up'	-1	0	0	-1	'down'	v_N
'down'	-1	1	0	-1	'down'	v_1
'up'	0	1	0	1	'up'	× if $N \equiv 2 \pmod 3$
'down'	1	0	0	-1	'down'	× if $N \equiv 1 \pmod 3$

Figure 4.2: This table shows which of the players v_1 and v_N guess correctly for each possible start and end of sequences, where the players v_i for $i = 2,\ldots,N-1$ guess incorrectly. Without loss of generality $c_{v_1} = 0$. The last two rows show the sequences where neither v_1 nor v_N guess correctly in case $N \not\equiv 0 \pmod 3$.

5

Complete Graph Minus Maximum Matching

The main result of this chapter is regarding the hat guessing number of the following graph. Let q an odd prime power and let M be a maximum matching in K_{q+1}. We consider the graph $K_{q+1} - M$, that is, we remove the edges belonging to M. Note that the resulting graph can also be viewed as the complete $\binom{q+1}{2}$-partite graph with parts of size 2, that is $K_{q+1} - M = K_{2,\dots,2}$. Since this is a non-complete graph, we have that $HG(K_{q+1} - M) \le q$, by Theorem 2.6. It turns out that we have equality here.

Theorem 5.1. *Let q an odd prime power and let M be a maximum matching in K_{q+1}. Then $HG(K_{q+1} - M) = q$.*

Proof. Consider the colours to be the elements of $\mathbb{F}_q$. We base our strategy on the fact that for any $x, y \in \mathbb{F}_q$ either $x = 0$, $y = 0$, or $x + i \cdot y = 0$ for some $i \in \mathbb{F}_q \setminus \{0\}$. Number the vertices v_α for $\alpha \in \mathbb{F}_q \cup \{\infty\}$ such that $\{v_\alpha, v_{-\alpha}\} \in M$ for $\alpha \in \mathbb{F}_q \setminus \{0\}$ and $\{v_0, v_\infty\} \in M$. Given a colouring c, let vertices v_0 and v_∞ respectively guess

$$f_{v_0}(c) = \sum_{\alpha \in \mathbb{F}_q \setminus \{0\}} \alpha \cdot c(v_\alpha) \quad \text{and} \quad f_{v_\infty}(c) = \sum_{\alpha \in \mathbb{F}_q \setminus \{0\}} c(v_\alpha).$$

Note that f_{v_0} does not depend on $c(v_\infty)$ and f_{v_∞} does not depend on $c(v_0)$. Set $x = c(v_0) - f_{v_0}(c)$ and $y = c(v_\infty) - f_{v_\infty}(c)$, then $x = 0$ corresponds to v_0 guessing correctly and $y = 0$ corresponds to v_∞ guessing correctly. For $i \in \mathbb{F}_q \setminus \{0\}$, let v_i guess its colour to be such that $x + i \cdot y = 0$. Substituting in the definitions of x and y and subsequently the definitions of $f_{v_0}(c)$ and $f_{v_\infty}(c)$, this corresponds with letting v_i guess

$$f_{v_i}(c) = (2i)^{-1} \left(c(v_0) + i \cdot c(v_\infty) - \sum_{\alpha \in \mathbb{F}_q \setminus \{0,i\}} (\alpha + i) \cdot c(v_\alpha) \right).$$

Note that f_{v_i} does not depend on $c(v_{-i})$, since the coefficient of $c(v_{-i})$ in the sum is $i - i = 0$. Hence the given strategy is a valid strategy. To see that the strategy is also winning, note that, as mentioned at the start of the proof, one of the following holds $x = 0$, $y = 0$, or $x + i \cdot y = 0$ for some $i \in \mathbb{F}_p \setminus \{0\}$. With the given choices of x and y, we see that v_0 guesses correctly precisely when $x = 0$, v_∞ guesses correctly precisely when $y = 0$, and, for $i \in \mathbb{F}_q \setminus \{0\}$, v_i guesses correctly precisely when $x + i \cdot y = 0$. □

Note that $K_{q+1} - M$ is $K_{\frac{q+1}{2}+1}$-free. Seeing as q is odd, we have that $q+1$ even. Suppose for the sake of contradiction that $K_{q+1} - M$ contains a copy K of $K_{\frac{q+1}{2}+1}$. There are $q+1-(\frac{q+1}{2}+1) = \frac{q+1}{2} - 1$ vertices $u \notin V(K)$. So there at most $\frac{q+1}{2} - 1$ vertices $v \in V(K)$ such that $\{u, v\} \in M$ for some $u \notin V(K)$. Hence there are at least 2 vertices in $V(K)$ not covered by M, a contradiction.

In [4], it is asked whether there exist K_q-free, q-solvable graphs which have a polynomial number of vertices in q. The above shows in particular that for odd primes p there exists a K_p-free, p-solvable graph with $p+1$ vertices. For a large enough positive integer q there exists a prime $p \in [q, q+q^{0.525}]$ by [2]. Then $q \leq p$, so $K_{p+1} - M$ is q-solvable, and $q \geq \frac{p+1}{2} + 1$, so $K_{p+1} - M$ is K_q-free. Furthermore, $K_{p+1} - M$ has $p+1 \leq q + q^{0.525} + 1$ vertices.

Gadouleau and Georgiou [9] show that there exist q-solvable graphs with arbitrarily low clique number, and at most a linear number of vertices in q. Furthermore, for any q divisible by 3, they show that there exists a q-solvable graph on $4q/3$ vertices with clique number $2q/3$. Compared to our result above, this guarantees a smaller, K_q-free, q-solvable graph in general. However, for q close to an odd prime power our result gives a smaller graph.

One might wonder if the clique number could provide an upper bound to the hat guessing number. However, Alon et al. [1] showed that the hat guessing number for the complete bipartite graph $K_{n,n}$ is $HG(K_{n,n}) = \Omega(n^{\frac{1}{2}-o(1)})$.

6

Smallest Cases

In this Chapter we will catalogue the hat guessing number for graphs on n vertices for $n \leq 5$. Recall from Chapter 2 that we need only consider connected graphs, Theorem 2.4. Similarly, we may omit graphs that contain a vertex of degree 1 and contain 2 or more edges, because the degree 1 vertex may be removed without changing the hat guessing number, Theorem 2.10. We discuss the graphs in order of increasing number of vertices.

6.1. $n = 1, 2,$ or 3 Vertices

For each of the cases $n = 1, 2, 3$ there is exactly one graph for which we need to determine the hat guessing number.

For $n = 1$ we only have the empty graph, which is the same as the complete graph K_1. Its hat guessing number is at least 1, because the player can guess the one available colour and this guess will be correct. Its hat guessing number is also at most 1, because a single guess cannot cover for 2 potential colours. So $HG(K_1) = 1$.

The only graphs on $n = 2$ vertices are the complete graph K_2 and the empty graph, the latter of which is disconnected. The hat guessing number of K_2 is $HG(K_2) = 2$. The lower bound is shown by the strategy in Example 1.1. The upper bound is shown in Section 3.1. In fact, these lower and upper bounds show the more general fact that the hat guessing number of the complete graph K_n on n vertices is $HG(K_n) = n$.

On $n = 3$ vertices the connected graphs are the 3-path P_3 and K_3, the former of which contains a vertex of degree 1 and has 2 edges. As noted above, the hat guessing number of K_3 is $HG(K_3) = 3$.

6.2. $n = 4$ Vertices

The graphs on $n = 4$ vertices we need to consider are K_4, C_4, and K_4 with a missing edge, denoted $K_4 - e$, see Figure 6.1. As usual, $HG(K_4) = 4$.

In Section 4.1 we have seen a winning strategy for the game $\langle C_4, 3\rangle$, so $HG(C_4) \geq 3$. Seeing as C_4 is not a complete graph, we get $HG(C_4) \leq 3$ from Theorem 2.6. So $HG(C_4) = 3$.

This is also a special case of both Theorem 2.13 and Theorem 5.1.

Lastly, the graph $K_4 - e$ shown in Figure 6.1 has hat guessing number $HG(K_4 - e) = 3$. The lower bound follows from the subgraph bound, Theorem 2.1, and any of the C_4 or K_3 subgraphs.

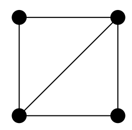


Figure 6.1: The complete graph on 4 vertices with a missing edge, denoted $K_4 - e$.

6.3. $n = 5$ Vertices

There are 11 connected graphs on $n = 5$ vertices with minimum degree 2. As usual, we have the complete graph K_5 with hat guessing number $HG(K_5) = 5$. From Theorem 2.13 it follows that the hat guessing number of the 5-cycle C_5 is $HG(C_5) = 2$. Removing an edge from K_5 results in $K_5 - e$, a non-complete graph, which still contains a K_4. So $HG(K_5 - e) = 4$. The graph in Figure 6.2 is another non-complete graph that contains K_4 as a subgraph. Therefore, this graph also has hat guessing number 4.

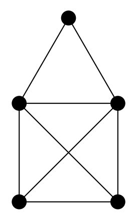
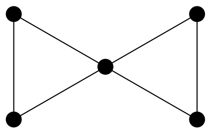


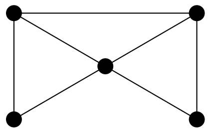
Figure 6.2: Another non-complete graph that contains K_4 as a subgraph.

The smallest example of a cactus graph is the hourglass, the result of gluing two copies of K_3 at a vertex, see Figure 6.3a. The hourglass is also known as the windmill $W_{3,2}$. Its hat guessing number is $HG(W_{3,2}) = 4$ and follows from either Theorem 2.14, as a cactus graph, or Theorem 2.16, as a windmill. There are two graphs on 5 vertices, which contain $W_{3,2}$ as a subgraph and do not contain K_4, see Figure 6.3b and Figure 6.3c. The former is constructed by adding any edge to $W_{3,2}$, the latter by adding another disjoint edge the result. The $W_{3,2}$ subgraph shows that their hat guessing number is at least 4. Seeing as neither is the complete graph on 5 vertices, both their hat guessing numbers are also at most 4, and thus they both have hat guessing number exactly 4.

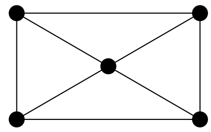
The last graph on 5 vertices for which the hat guessing number follows from general results is the complete bipartite graph $K_{2,3}$. It has hat guessing number $HG(K_{2,3}) = 3$. The lower bound follows from any of the C_4 subgraphs. The upper bound follows from the last upper bound mentioned in Section 2.3. This bound says that for any bipartite graph with parts of size m and n, the hat guessing number is at most $\min\{m+1, n+1\}$.



(a) The hourglass, also known as the windmill $W_{3,2}$, is the result of gluing 2 copies of K_3 at a vertex.



(b) The hourglass with an added edge.

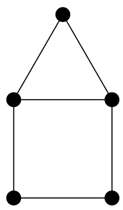


(c) The hourglass with two added edges.

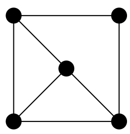
Figure 6.3: The hourglass and two related graphs.

The three graphs on 5 vertices that are left are the house, the broken wheel, and the book graph $B_{2,3}$, shown in Figure 6.4. All three of these have hat guessing number either 3 or 4. Each of them contains both C_4 and K_3 as a subgraph, either of which would show that the hat guessing number is at least 3. None of them is the complete graph, so their hat guessing numbers are at most 4.

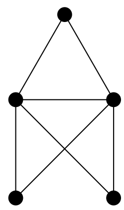
Gu et al. [10] have shown the hat guessing number of these three graphs. They determined a winning strategy for 4 colours on $B_{2,3}$, showing that $HG(B_{2,3}) = 4$. They also showed that there exists no strategy for 4 colours on the broken wheel, showing that its hat guessing number is 3. Seeing as the house is contained in the broken wheel, its hat guessing number is also 3.



(a) House



(b) Broken Wheel



(c) $B_{2,3}$

Figure 6.4: Three last cases

7

Equivalence of Strategies

Recall from Section 2.1 the formal definition of the hat guessing game. Note that we have not specified a set of colours Γ. Often Γ is taken to be $\{0, 1, \ldots, q-1\} =: [q]$, $\mathbb{Z}/q\mathbb{Z}$, or $\mathbb{F}_q$. Whether a hat guessing game is winning or not is not dependent on the chosen Γ.

Proof. Suppose, for the sake of contradiction, that the choice of Γ does matter. Then there exist a graph G, a positive integer q, and two colour sets Γ_1 and Γ_2 of size q such that the game $\langle G, q\rangle$ is winning with colour set Γ_1 with winning strategy f and losing with colour set Γ_2. Seeing as Γ_1 and Γ_2 are both of size q, there exists a bijection σ between them. Let $\tau : \Gamma_2^V \to \Gamma_1^V$ apply σ to each coordinate. Consider the strategy $\tau^{-1} \circ f \circ \tau$ for the game with colour set Γ_2. For a colouring c in Γ_2^V we have $(\tau^{-1} \circ f \circ \tau)_v(c) = \sigma^{-1}(f_v(\tau(c))) = \sigma^{-1}(\tau_v(c)) = \sigma^{-1}(\sigma(c_v)) = c_v$, where v is the player who guesses correctly with strategy f for the colouring $\tau(c)$. We see that v guesses correctly and thus $\tau^{-1} \circ f \circ \tau$ is a winning strategy for the game with colour set Γ_2, a contradiction. □

Instead of using a single universal set of colours Γ, one can also provide a set of colours Γ_v of size q for each player v. This also does not change whether a hat guessing game is winning or not. The proof is very similar to the one above, redefine τ to apply the bijection σ_v of the different Γ_v on the colour of player v. The rest of the proof follows as above. This inspires us to define a notion of equivalence of strategies.

Definition 7.1. *Two strategies f and g for a game $\langle G, q\rangle$ are said to be* equivalent *if there exists a bijection of colourings $\tau : \Gamma^V \to \Gamma^V$ that preserves winning strategies and $g = \tau^{-1} \circ f \circ \tau$.*

With *preserves winning strategies* we mean that for any strategy f for the game $\langle G, q\rangle$ we have that f is winning if and only if $\tau^{-1} \circ f \circ \tau$ is winning. Loosely speaking, conjugation by τ should preserve the property of 'winningness' of any strategy, which is the property we care about. Note that the composition and inverses of bijections that preserve winning strategies again preserve winning strategies.

In the proof above we have shown that for any strategy f we have $\tau^{-1} \circ f \circ \tau$ is winning whenever f is winning, where τ is a permutation of the colour set Γ. Seeing as τ^{-1} is also a permutation of Γ, the converse also holds. Therefore permutations of the colour set Γ are examples of bijections that preserve winning strategies. With the note below the proof we can generalise this to permutations of the individual colour set Γ_v for a player v. Another example is automorphisms of the graph G.

In this case τ permutes the entries according to an automorphism ϕ of G. Let f a winning strategy, let $c \in \Gamma^V$ a colouring, and let v be the player which guesses correctly for the colouring $\tau(c)$. Then player $\phi(v)$ guesses correctly for the strategy $\tau^{-1} \circ f \circ \tau$ and colouring c. Indeed, we have

$$(\tau^{-1} \circ f \circ \tau)_{\phi(v)}(c) = \tau^{-1}_{\phi(v)}(f(\tau(c))) = f_v(\tau(c)) = \tau_v(c) = c_{\phi(v)}.$$

The converse follows from the fact that τ^{-1} again permutes the entries according to the automorphism ϕ^{-1} of G.

Conjecture 7.2. *All bijections that preserve winning strategies are of the form $\tau \circ \pi$, where τ permutes entries according to an automorphism of G and π permutes the individual colour sets of each vertex.*

7.1. Uniqueness

Along with a notion of equivalence of strategies we also get a notion of uniqueness of strategies. One might wonder whether there exist graphs for which the hat guessing number is certified by a strategy which is unique up to equivalence. Showing that a strategy is the unique winning strategy might be useful in the following ways.

Conjecture 7.3. *Let G have hat guessing number $HG(G) \geq q$ as shown by some winning strategy for the game $\langle G, q\rangle$. Suppose this strategy is the unique winning strategy up to equivalence of strategies. Consider the hat guessing game on G where for one of the players there are $q+1$ available hat colours, whereas there are q available hat colours for each of the other players. This game is losing and thus $HG(G) = q$.*

Proof. Assume that Conjecture 7.2 holds.

Given a strategy f and a player v, we can count the number of colourings of the neighbourhood of v such that v guesses each of the colours. For a player v, let $M_v^f \in \mathbb{N}^{\Gamma_v}$ be the vector containing the results of this counting. Note that the specifics of M_v^f depend on which of the equivalent formulations for the given strategy f was chosen. Permuting the elements of Γ_v permutes the elements of M_v^f, though it does not influence the M_u^f for players $u \neq v$. To account for these permutations we redefine M_v^f to be the multiset containing the results of the counting. Automorphisms of the graph G change which multiset belongs to what player, but does not influence the multisets themselves. Under the assumption of Conjecture 7.2 the collection of multisets M_v^f uniquely determines f up to graph automorphisms of G.

Now let f be the unique winning strategy for the game $\langle G, q\rangle$. Suppose we have a winning strategy g for the game where player v has $q+1$ available hat colours. From g we can obtain a strategy h for the game $\langle G, q\rangle$ in the following way. Firstly, we remove a colour c_0 from the available hat colours Γ_v of v. Choose c_0 such that $M_v^g(c_0) \geq 1$. For $u \neq v$ let $h_u = g_u$, where the neighbours of v simply restrict their domains. At this point we may use graph automorphisms to redefine h in such a way that $M_u^h = M_u^f$ for all players $u \neq v$. If this is not possible, then h and f are already non-equivalent. If it is, then, for colourings c such that $g_v(c) = c_0$, we set $h_v(c) \neq c_0$ in such a way that $M_v^h \neq M_v^f$. This is always possible, since $M_v^g(c_0) \geq 1$. So, again h and f are non-equivalent strategies. Seeing as h is a restriction of g, h is winning. Therefore h and f are non-equivalent winning strategies for the game $\langle G, q\rangle$, a contradiction. □

Conjecture 7.4. *The graph G is also arc-critical, that is for any arc a of G we have* $HG(G - a) < q$.

Szczechla. [16] has shown that the winning strategies for 3 colours on the cycles C_{3n} for $n > 1$ are unique up to permutations of the individual colour sets. The game $\langle K_2, 2\rangle$ has the strategy from Example 1.1 as unique winning strategy up to graph automorphism, which can be seen by checking all 16 strategies. Seeing as Conjecture 7.2 holds in these cases, it follows that Conjecture 7.3 also holds in these cases. Conjecture 7.4 also holds in these cases.

The converses of Conjecture 7.3 and Conjecture 7.4 do not hold in general. As we have seen in Section 3.1, $HG(K_n - a) = n - 1$, whereas $HG(K_n) = n$. Furthermore, the proof for $HG(K_n) < n+1$ presented in Section 3.1 still holds when there are $n+1$ available hat colours for one of the players, while there are n available hat colours for each of the other players. However, the games $\langle K_3, 3\rangle$ and $\langle K_4, 4\rangle$ both have at least two non-equivalent winning strategies. For both games one of the strategies is the strategy from Example 1.1. By interpreting K_3 as C_3 we may use the strategy from Section 4.2 for the game $\langle K_3, 3\rangle$. By interpreting K_4 as $K_2 \times_L K_2$ we may use the strategy from Appendix B (a specific case of Theorem 2.3) for the game $\langle K_4, 4\rangle$. Below, we show that these strategies are indeed non-equivalent.

Proof: $\langle K_3, 3\rangle$ has two strategies. Let $f : \mathbb{F}_3^3 \to \mathbb{F}_3^3$ be the strategy from Example 1.1, that is,

$$f(c) = \begin{pmatrix} -c_1 - c_2 \\ 1 - c_0 - c_2 \\ 2 - c_0 - c_1 \end{pmatrix}.$$

Let $g : \mathbb{F}_3^3 \to \mathbb{F}_3^3$ be the strategy from Section 4.2, that is,

$$g(c) = \begin{pmatrix} h(c_2 + 1, c_1) \\ h(c_0, c_2) \\ h(c_1, c_0 - 1) \end{pmatrix}, \text{where } h(x, y) = \begin{cases} x + 1 & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}.$$

Suppose for the sake of contradiction that there exists a bijection $\tau : \mathbb{F}_3^3 \to \mathbb{F}_3^3$ that preserves winning strategies such that $g = \tau^{-1} \circ f \circ \tau$.

Note that f has order 3, that is $f^3 = f \circ f \circ f = \text{Id}$, where Id denotes the identity map $\text{Id} : \mathbb{F}_3^3 \to \mathbb{F}_3^3, c \mapsto c$. Indeed, we have

$$f^3(c) = f^2 \begin{pmatrix} -c_1 - c_2 \\ 1 - c_0 - c_2 \\ 2 - c_0 - c_1 \end{pmatrix} = f \begin{pmatrix} -c_0 + c_1 + c_2 \\ c_0 - c_1 + c_2 \\ c_0 + c_1 - c_2 \end{pmatrix} = c.$$

Seeing as $g = \tau^{-1} \circ f \circ \tau$, we get that

$$g^3 = (\tau^{-1} \circ f \circ \tau)^3 = \tau^{-1} \circ f^3 \circ \tau = \tau^{-1} \circ \tau = \text{Id}.$$

However, $g^3(0, 0, 0) = (2, 2, 2)$, a contradiction. □

Proof: $\langle K_4, 4\rangle$ *has two strategies.* Let $f : \mathbb{Z}_4^4 \to \mathbb{Z}_4^4$ be the strategy from Example 1.1, that is,

$$f(c) = \begin{pmatrix} -c_1 - c_2 - c_3 \\ 1 - c_0 - c_2 - c_3 \\ 2 - c_0 - c_1 - c_3 \\ 3 - c_0 - c_1 - c_2 \end{pmatrix}.$$

Let $g : (\mathbb{Z}_2^2)^4 \to (\mathbb{Z}_2^2)^4$ be the strategy from Appendix B, that is,

$$g(c) = \begin{pmatrix} g_{v_0}(c) \\ g_{v_1}(c) \\ g_{u_0}(c) \\ g_{u_1}(c) \end{pmatrix} = \begin{pmatrix} \left(-c_{v_1,0},\ -(1 - c_{u_0,0} - c_{u_1,0})c_{u_0,1} - (c_{u_0,0} + c_{u_1,0})c_{u_1,1}\right) \\ \left(1 - c_{v_0,0},\ -(1 - c_{u_0,0} - c_{u_1,0})c_{u_0,1} - (c_{u_0,0} + c_{u_1,0})c_{u_1,1}\right) \\ \left(-c_{u_1,0},\ 1 - (1 - c_{v_0,0} - c_{v_1,0})c_{v_0,1} - (c_{v_0,0} + c_{v_1,0})c_{v_1,1}\right) \\ \left(1 - c_{u_0,0},\ 1 - (1 - c_{v_0,0} - c_{v_1,0})c_{v_0,1} - (c_{v_0,0} + c_{v_1,0})c_{v_1,1}\right) \end{pmatrix}.$$

Suppose for the sake of contradiction that there exists a bijection $\tau : \mathbb{Z}_4^4 \to \mathbb{Z}_4^4$ that preserves winning strategies such that $g = \tau^{-1} \circ f \circ \tau$.

Note that f has order 4, that is $f^4 = f \circ f \circ f \circ f = \mathrm{Id}$, where Id denotes the identity map $\mathrm{Id} : \mathbb{Z}_4^4 \to \mathbb{Z}_4^4, c \mapsto c$. Indeed, we have

$$f^4(c) = f^3 \begin{pmatrix} -c_1 - c_2 - c_3 \\ 1 - c_0 - c_2 - c_3 \\ 2 - c_0 - c_1 - c_3 \\ 3 - c_0 - c_1 - c_2 \end{pmatrix} = f^2 \begin{pmatrix} 2 - c_0 + 2c_1 + 2c_2 + 2c_3 \\ 0 + 2c_0 - c_1 + 2c_2 + 2c_3 \\ 2 + 2c_0 + 2c_1 - c_2 + 2c_3 \\ 0 + 2c_0 + 2c_1 + 2c_2 - c_3 \end{pmatrix} = f \begin{pmatrix} 2 + 2c_0 + c_1 + c_2 + c_3 \\ 1 + c_0 + 2c_1 + c_2 + c_3 \\ 0 + c_0 + c_1 + 2c_2 + c_3 \\ 3 + c_0 + c_1 + c_2 + 2c_3 \end{pmatrix} = c.$$

Therefore, the pre-image under f of c is $f^3(c)$, so f is surjective. It follows that f is bijective, as it maps a finite set unto itself. Seeing as τ is bijective and $g = \tau^{-1} \circ f \circ \tau$, we get that g must be bijective. However,

$$g \begin{pmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{pmatrix} = \begin{pmatrix} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{pmatrix} = g \begin{pmatrix} (0,0) \\ (0,1) \\ (0,0) \\ (0,1) \end{pmatrix},$$

a contradiction. □

8

Further Research

There are a lot of other questions to ask related to the hat guessing game. In this chapter we ask those questions and refer to some sources that try to answer these questions.

8.1. Variations

The hat guessing game has a number of rules that we could alter to obtain slightly different but related games. In this section we briefly discuss a few variations that arise this way.

The hat guessing game very naturally generalises to directed or mixed graphs. In this case players are able to see the hat colours of their out-neighbours. In Chapter 3 we have seen how one might generalise bounds on the hat guessing number to mixed graphs. Alon et al. [1] talk about the hat guessing number of multipartite directed cycles and Gadouleau [8] show an upper bound on the hat guessing number based on minimum feedback vertex sets.

Another natural variation of the game is having a different number of colours available for different players. For example, considering the path graph on 3 vertices P_3, there exists a winning strategy even when the middle player has a colour set of size 4 and the two outer players have colour sets of size 2. We say that the sequence $(2,4,2)$ is winning for P_3. In general, for a graph G, we consider sequences $h \in \mathbb{Z}^V$, sometimes referred to as *hatness functions*. The hat guessing number corresponds with the maximum integer q such that there exists a winning strategy on G for the constant q hatness function. This variation has been studied by Kokhas and Latyshev [14], resulting in a strategy for the lexicographic product of graphs, Theorem 2.3, and has been studied by Chizewer et al. [5], resulting in the hat guessing number of cactus graphs.

We can also help the players to win the game. For example, by giving them more than one guess. In this variation the players each guess a subset of size s of the colours and win if the hat colour of at least one of the players is in their guessed set. Bosek et al. [3] studied this variation and showed that the hat guessing number of graphs with sufficiently large genus and sufficiently large girth in its genus is bounded above by a simple function in s. Implying that the hat guessing number of planar graphs with girth at least 14 is at most 6.

Another way of making the game easier for the players is by restricting the colour arrangements available to the adversary. One could for example tell the players before hand that the hats are going to be placed according to a proper colouring, that is, no neighbouring players will have the same hat colour.

8.2. Edge-critical Graphs

A graph is said to be *edge-critical* for the hat guessing number if removing any edge from the graph reduces its hat guessing number by at least 1. For example, we have seen that the complete graph K_n has hat guessing number $HG(K_n) = n$, but removing an edge e results in a hat guessing number $HG(K_n - e) = n-1$, Theorem 2.6. Other examples include C_4, C_{3n}, and $W_{3,2}$. These graphs form the boundary between hat guessing numbers. Any graph containing an edge-critical graph G has hat guessing number at least $HG(G)$ and any graph strictly contained in G has hat guessing number less than $HG(G)$. Knowing precisely which graphs are edge-critical for a given hat guessing number one can determine the hat guessing number of any graph by checking what edge-critical graphs it contains and is contained in.

When considering directed and mixed graphs, one could ask: What graphs are *arc-critical*, that is, for what graphs does the hat guessing number strictly decrease when removing an arc? For example, we have seen that removing an arc a from K_n results in a graph with hat guessing number $HG(K_n - a) = n-1$, Theorem 3.2. Clearly, arc-critical graphs are also edge-critical. It is unclear, however, whether the converse should also hold.

8.3. Graph Operations

The main question for this section is as follows. How does the hat guessing number interact with graph operations? For example, let $\cup$ denote the disjoint union of graphs, then for any graphs G and H, we have seen in Theorem 2.4 that $HG(G \cup H) = \max\{HG(G), HG(H)\}$. Can we find similar relations for graph products like the Cartesian product, the tensor product, the strong product, and the lexicographic product?

We have seen in Theorem 2.3 that the lexicographic product provides a lower bound on the hat guessing number $HG(G \times_L H) \geq HG(G) \cdot HG(H)$. It is unclear when the corresponding upper bound holds. Given $HG(G \times_L H)$ and $HG(H)$, one can try constructing a strategy for the hat guessing game on G with $\frac{HG(G \times_L H)}{HG(H)}$ colours. Suppose that $HG(G \times_L H) \geq (HG(G)+1) \cdot HG(H)$, then the constructed strategy would imply that $HG(G) \geq HG(G)+1$, a contradiction. This still leaves a gap for the hat guessing number of the lexicographic product, though it might be easier to show.

Kokhas and Latyshev [14] consider the variation of the hat guessing game where different people have different size colour sets. In this context they show what happens if we glue two graphs at a vertex, if we blow up a vertex into a graph, or if we connect a new vertex in some specific ways. For what graphs do these imply something about the usual hat guessing number? What other changes can we make in the graph and say something about the hat guessing game?

8.4. Small Open Cases

As we have seen in Chapter 6, the hat guessing number is known for all graphs on 5 or fewer vertices. The natural next step is looking at the graphs on 6 vertices. The only new graphs for which the hat guessing number is known are the subgraphs of the complete bipartite graph $K_{3,3}$ with hat guessing number $HG(K_{3,3}) = 3$ [12]. Ignoring disconnected graphs and graphs with minimum degree 1, there are 48 graphs on 6 vertices for which the hat guessing number is currently unknown, all of which have hat guessing number either 3, 4, or 5.

8.5. Planar Graphs

Planar graphs are an interesting class of graphs when it comes to the hat guessing number. We have already seen that the hat guessing number of cycles is $HG(\text{cycles}) \leq 3$, [16], and more generally that the hat guessing number of cactus graphs is $HG(\text{cactus}) \leq 4$, [5]. It has been conjectured that the hat guessing number of all planar graphs is bounded above by a constant. Knierim et al. [13] have shown that the hat guessing number of outer planar graphs, a subclass of planar graphs, have hat guessing number $HG(\text{outer planar}) \leq 40$ and Latyshev and Kokhas [15] have constructed an example of an outer planar graph with hat guessing number at least 22.

A

Finding α for which (3.5) holds with $k = \alpha K$

Let $K = \frac{1}{1-\left(\frac{n-1}{n}\right)^{1/3}}$, that is K is such that $\frac{K-1}{K} = \left(\frac{n-1}{n}\right)^{1/3}$. We want to find α such that the following condition holds for $k = \alpha K$ and for n large enough, see Condition (3.5),

$$n+1-n\left(\frac{k-1}{k}\right)-\left(\frac{k-1}{k}\right)^{2n} < 1. \tag{A.1}$$

We will start by rewriting Condition (A.1), in view of which consider the following:

$$\begin{aligned}
\frac{\alpha K-1}{\alpha K} &= 1-\frac{1}{\alpha K} = \frac{1}{\alpha}\left(\alpha-\frac{1}{K}\right) \\
&= \frac{1}{\alpha}\left((\alpha-1)+\left(1-\frac{1}{K}\right)\right) \\
&= \frac{\alpha-1}{\alpha}+\frac{1}{\alpha}\left(\frac{K-1}{K}\right) \\
&= \frac{\alpha-1}{\alpha}+\frac{1}{\alpha}\left(\frac{n-1}{n}\right)^{1/3}.
\end{aligned}$$

Condition (A.1) then becomes

$$\frac{n}{\alpha}-\frac{n}{\alpha}\left(\frac{n-1}{n}\right)^{1/3}-\left(\frac{\alpha-1}{\alpha}+\frac{1}{\alpha}\left(\frac{n-1}{n}\right)^{1/3}\right)^{2n} < 0.$$

Define the functions f, g as

$$\begin{aligned}
f(n) &= 1-\left(1-\frac{1}{n}\right)^{1/3}, \\
g(n) &= \frac{n}{\alpha}f(n)-\left(1-\frac{1}{\alpha}f(n)\right)^{2n}.
\end{aligned}$$

Then Condition (A.1) corresponds to $g(n) < 0$. Note that it suffices to find α such that $\lim_{n\to\infty} g(n) < 0$, as we then get that the condition is satisfied for n large enough. In view of calculating this limit note that $\lim_{n\to\infty} f(n) = 0$, furthermore, in view of using L'Hôpital's rule, note that

$$\frac{\mathrm{d}}{\mathrm{d}n} f(n) = -\frac{1}{3n^2}\left(1-\frac{1}{n}\right)^{-2/3}.$$

We calculate the limit of $g(n)$ in two steps. Firstly,

$$\begin{aligned}
\lim_{n\to\infty} g_1(n) &= \lim_{n\to\infty} \frac{n}{\alpha} f(n) \\
&= \frac{1}{\alpha} \lim_{n\to\infty} \frac{f(n)}{1/n} \\
&\overset{L'H}{=} \frac{1}{\alpha} \lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}n} f(n)}{-1/n^2} \\
&= \frac{1}{\alpha} \lim_{n\to\infty} \frac{1}{3}\left(1-\frac{1}{n}\right)^{-2/3} = \frac{1}{3\alpha},
\end{aligned}$$

where $L'H$ denotes an application of L'Hôpital's rule. Secondly,

$$\begin{aligned}
\lim_{n\to\infty} g_2(n) &= \lim_{n\to\infty} \left(1-\frac{1}{\alpha} f(n)\right)^{2n} \\
&= \exp\left(2 \lim_{n\to\infty} \frac{\log\left(1-\frac{1}{\alpha} f(n)\right)}{1/n}\right) \\
&\overset{L'H}{=} \exp\left(2 \lim_{n\to\infty} \frac{-\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}n} f(n)}{-\frac{1}{n^2}\left(1-\frac{1}{\alpha} f(n)\right)}\right) \\
&= \exp\left(-\frac{2}{\alpha} \lim_{n\to\infty} \frac{\frac{1}{3}\left(1-\frac{1}{n}\right)^{-2/3}}{\left(1-\frac{1}{\alpha} f(n)\right)}\right) = e^{-\frac{2}{3\alpha}},
\end{aligned}$$

where $L'H$ denotes an application of L'Hôpital's rule, $\exp(x) = e^x$, and log is the natural logarithm. Putting these together we obtain

$$\lim_{n\to\infty} g(n) = \lim_{n\to\infty} g_1(n) - \lim_{n\to\infty} g_2(n) = \frac{1}{3\alpha} - e^{-\frac{2}{3\alpha}}.$$

We find that $\lim_{n\to\infty} g(n) < 0$ whenever $\alpha > 0.781917$. Therefore, Condition (A.1) holds for example for $\alpha = 0.9$, $k = \alpha K$, and n large enough.

B

A Strategy on $K_2 \times_L K_2$

We present a strategy on $K_2 \times_L K_2$ based on [14]. Consider the colours to be the elements of $\mathbb{Z}_2^2$. Each player guesses the first coordinate of their colour according to a winning strategy for the game $\langle K_2, 2\rangle$ looking at the first coordinates of the players in the copy of K_2 they belong to. Let G and H be the copies of K_2. Given a colouring $c \in (\mathbb{Z}_2^2)^4$, let v, respectively u, be the player in G, respectively H, who correctly guesses the first coordinate of their colour.

View $G \times H$ as a copy of the graph K_2 with vertices G and H, see Figure B.1. The guess for the second coordinate is going to be the same among all players in the same copy of K_2.

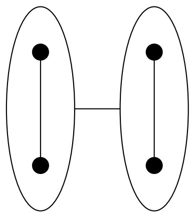


Figure B.1: The complete connection between two copies of K_2.

The players in G and H guess the second coordinate of their colour according to a winning strategy for the game $\langle K_2, 2\rangle$ on the big K_2. In this game, the colour of G, respectively H, is the second coordinate of v, respectively u. Note that the players in G know which of the players in H is the winner u, because they know which player has what strategy and can see the colours of the players in H. Similarly, the players in H know which of the players in G is v.

Seeing as we use a winning strategy for the game on the big K_2, either G or H correctly guesses their colour. Seeing as the colours of G and H are the second coordinates of v and u respectively, either v or u correctly guesses the second coordinate of their colour. Both v and u already correctly guess the first coordinate of their colour, therefore either v or u correctly guesses both coordinates of their colour, and thus the strategy is winning.

Let us explicitly construct this strategy $g : (\mathbb{Z}_2^2)^4 \to (\mathbb{Z}_2^2)^4$ using the strategy from Example 1.1 for the games $\langle K_2, 2\rangle$. Let v_0, v_1 be the players in G and u_0, u_1 the players in H. For a colouring $c \in (\mathbb{Z}_2^2)^4$, denote the first, respectively second, coordinate of a player w's colour as $c_{w,0}$, respectively $c_{w,1}$. As above, suppose that, given a colouring $c \in (\mathbb{Z}_2^2)^4$, v and u are the players who correctly guess the first coordinate of their colour in G and H respectively. Then g is as follows.

$$g(c) = \begin{pmatrix} g_{v_0}(c) \\ g_{v_1}(c) \\ g_{u_0}(c) \\ g_{u_1}(c) \end{pmatrix} = \begin{pmatrix} \left(-c_{v_1,0},\ -c_{u,1}\right) \\ \left(1 - c_{v_0,0},\ -c_{u,1}\right) \\ \left(-c_{u_1,0},\ 1 - c_{v,1}\right) \\ \left(1 - c_{u_0,0},\ 1 - c_{v,1}\right) \end{pmatrix}$$

Based on the strategies for the first coordinates we can determine explicitly what $c_{v,1}$ and $c_{u,1}$ are. Note that

$$\begin{aligned} v = v_0 &\iff c_{v_0,0} + c_{v_1,0} = 0, \\ v = v_1 &\iff c_{v_0,0} + c_{v_1,0} = 1, \\ u = u_0 &\iff c_{u_0,0} + c_{u_1,0} = 0, \\ u = u_1 &\iff c_{u_0,0} + c_{u_1,0} = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} c_{v,1} &= (1 - c_{v_0,0} - c_{v_1,0})c_{v_0,1} + (c_{v_0,0} + c_{v_1,0})c_{v_1,1}, \\ c_{u,1} &= (1 - c_{u_0,0} - c_{u_1,0})c_{u_0,1} + (c_{u_0,0} + c_{u_1,0})c_{u_1,1}. \end{aligned}$$

Substituting $c_{v,1}$ and $c_{u,1}$ into our expression for g we obtain

$$g(c) = \begin{pmatrix} g_{v_0}(c) \\ g_{v_1}(c) \\ g_{u_0}(c) \\ g_{u_1}(c) \end{pmatrix} = \begin{pmatrix} \left(-c_{v_1,0},\ -(1 - c_{u_0,0} - c_{u_1,0})c_{u_0,1} - (c_{u_0,0} + c_{u_1,0})c_{u_1,1}\right) \\ \left(1 - c_{v_0,0},\ -(1 - c_{u_0,0} - c_{u_1,0})c_{u_0,1} - (c_{u_0,0} + c_{u_1,0})c_{u_1,1}\right) \\ \left(-c_{u_1,0},\ 1 - (1 - c_{v_0,0} - c_{v_1,0})c_{v_0,1} - (c_{v_0,0} + c_{v_1,0})c_{v_1,1}\right) \\ \left(1 - c_{u_0,0},\ 1 - (1 - c_{v_0,0} - c_{v_1,0})c_{v_0,1} - (c_{v_0,0} + c_{v_1,0})c_{v_1,1}\right) \end{pmatrix}.$$

Bibliography

[1] N. Alon, O. Ben-Eliezer, C. Shangguan, and I. Tamo. The hat guessing number of graphs. *Journal of Combinatorial Theory, Series B*, 144:119–149, 2020. doi: 10.1016/j.jctb.2020.01.003.

[2] R.C. Baker, G. Harman, and J. Pintz. The difference between consecutive primes, ii. *Proceedings of the London Mathematical Society. Third Series*, 83, 11 2001. doi: 10.1112/plms/83.3.532.

[3] B. Bosek, A. Dudek, M. Farnik, J. Grytczuk, and P. Mazur. Hat chromatic number of graphs. *Discrete Mathematics*, 344(12):112620, 2021. doi: 10.1016/j.disc.2021.112620.

[4] S. Butler, M.T. Hajiaghayi, R.D. Kleinberg, and T. Leighton. Hat guessing games. *SIAM Journal on Discrete Mathematics*, 22(2), 2008. doi: 10.1137/060652774.

[5] J. Chizewer, I.M.J. McInnis, M. Sohrabi, and S. Kaistha. The hat guessing game on cactus graphs and cycles. *Discrete Mathematics*, 348(1), 2025. doi: 10.1016/j.disc.2024.114272.

[6] M. Farnik. *A hat guessing game*. PhD thesis, Jagiellonian University, 2015.

[7] U. Feige. You can leave your hat on (if you guess its color). Technical Report MCS04-03, The Weizmann Institute of Science, 2004.

[8] M. Gadouleau. Finite dynamical systems, hat games, and coding theory. *SIAM Journal on Discrete Mathematics*, 32(3):1922–1945, 2018. doi: 10.1137/15m1044758.

[9] M. Gadouleau and N. Georgiou. New constructions and bounds for winkler's hat game. *SIAM Journal on Discrete Mathematics*, 29(2):823–834, 2015. doi: 10.1137/130944680.

[10] Z. Gu, Y. Ido, and B. Przybocki. Hat-guessing on graphs. *surim.stanford.edu*, 2020.

[11] X. He and R. Li. Hat guessing numbers of degenerate graphs. *The Electronic Journal of Combinatorics*, 27(3):P3.58, 2020. doi: 10.37236/9449.

[12] X. He, Y. Ido, and B. Przybocki. Hat guessing on books and windmills. *The Electronic Journal of Combinatorics*, 29(1):P1.12, 2022. doi: 10.37236/10098.

[13] C. Knierim, A. Martinsson, and R. Steiner. Hat guessing numbers of strongly degenerate graphs. *SIAM Journal on Discrete Mathematics*, 37(2):1331–1347, 2023. doi: 10.1137/22M1479154.

[14] K. Kokhas and A. Latyshev. Cliques and constructors in "hats" game. *Journal of Mathematical Sciences*, 255:39–57, 2021. doi: 10.1007/s10958-021-05348-9.

[15] A. Latyshev and K. Kokhas. Hat guessing number for the class of planar graphs is at least 22. *Discrete Mathematics*, 347(3), 2024. doi: 10.1016/j.disc.2023.113820.

[16] W. Szczechla. The three colour hat guessing game on cycle graphs. *The Electronic Journal of Combinatorics*, 24(1):P1.37, 2017. doi: 10.37236/5135.

[17] P. Winkler. *Puzzlers Tribute: A Feast for the Mind*, chapter Games people don't play, pages 301–313. A K Peters/CRC Press, 2001.