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Joule Thomson CO₂ Cooling Under Different Regime Heating by Adjacent Layers

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Abstract

Heat exchange with surrounding formations and Joule–Thomson cooling during CO₂ injection into deep saline aquifers and depleted hydrocarbon reservoirs can lead to substantial declines in well injectivity. This work addresses these challenges by introducing an analytical model for non-isothermal CO₂ injection that accounts for both JT cooling and inter-formation heat exchange, assuming that heat transfer begins upon arrival of the temperature front rather than the gas–water front, as adopted in earlier models. An exact 1D solution is derived, providing closed-form expressions for temperature and pressure profiles. Model performance is evaluated through comparison with an exact 2D solution obtained from reservoir energy conservation. The new formulation demonstrates markedly improved accuracy over the previous model. The solution predicts a temperature drop from the injection temperature at the wellbore to a minimum at the temperature front, followed by a rapid rise back to the initial reservoir temperature. Mapping the evolving temperature and pressure profiles onto a (T, p) phase diagram enables assessment of hydrate-formation risk and identification of the distance from the injection well where hydrates may form.

Introduction

The reduction of greenhouse gas emissions, and specifically carbon dioxide (CO₂), is crucial for addressing global climate change (Andersen et al., 2024). A leading strategy involves capturing CO₂ from point sources or directly from the atmosphere and storing it in geological formations like depleted oil and gas reservoirs or deep saline aquifers. Depleted hydrocarbon reservoirs are particularly attractive due to established characterization and proven containment but have operational challenges related to Joule–Thomson (JT) cooling during injection, which can be significant due to low residual pressures after depletion (Maloney & Briceno, 2009).

When CO_2 is injected into a depleted reservoir at substantially lower pressure than the injection wellbore, it undergoes adiabatic expansion, resulting in JT cooling. This cooling can be magnified by additional effects such as water evaporation into the CO_2 stream, possibly leading to freezing of pore fluids and formation of gas hydrates. Such conditions can reduce well injectivity and lead to operational issues, including blockage in the wellbore or reservoir (Hajiabadi et al., 2021).

Mathematical modelling tools are essential for predicting and managing JT-induced cooling. Modern approaches simulate transient wellbore–reservoir coupling to identify sensitive operational conditions and develop effective mitigation strategies. These tools enable operators to anticipate hydrate formation or fluid freezing, design optimal injection rates, and schedule reservoir depressurization to maintain injectivity and operational safety (Hussain et al., 2016).

The aim of this work is delivering an analytical model for temperature distributions evaluation during geological CO_2 storage in deep saline aquifers and depleted gas and oil fields, in particularly accounting for the heat exchange between the reservoir and adjacent formations and Joule-Thomson effect of CO_2 cooling. The above effects can highly affect the reservoir behaviour during CO_2 injection with consequent effects on the storage capacity and well injectivity.

Temperature and pressure profiles provide an important information about the reservoir behaviour during CO_2 injection. Those correspond to timely paths in phase T-p diagrams (Fig. 1). Here the paths start at the wellbore and end up in situ the reservoir, usually within the drainage zone. If a path crosses phase envelope, hydrate formation occurs. Parametrisation of paths by radius for a fixed moment provides an important information on how far away from the injector the hydrate formation occurs, allowing to judge about the possible formation damage and well injectivity.

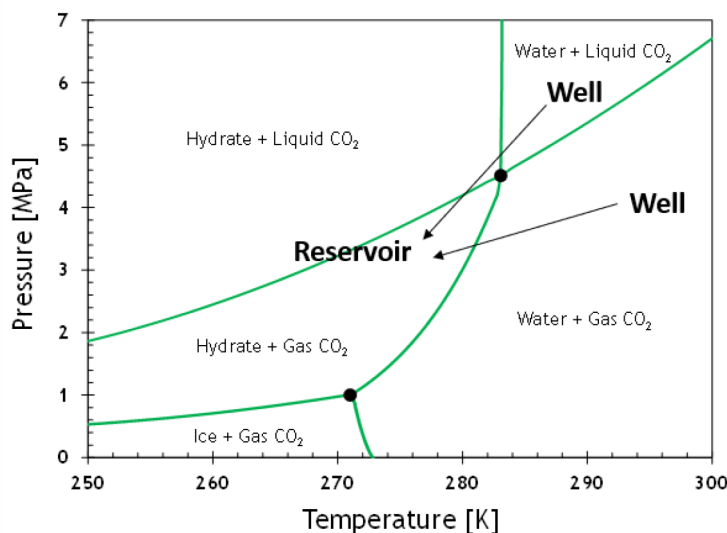


Figure 1—Water - CO_2 phase diagram (Voronov et al., 2016).

Fig. 2 shows the cooling effect on temperature profiles. Due to JT cooling in the injection well column, the depth temperature profile is located below the geothermal gradient (Fig. 2a). Fig. 2b shows the temperature distribution in the reservoir during CO_2 injection: counting from horizontal injector, in situ propagates gas flow zone determined by the injected temperature; ahead moves zone with cold CO_2 with almost initial reservoir temperature and ahead moves the displaced brine.

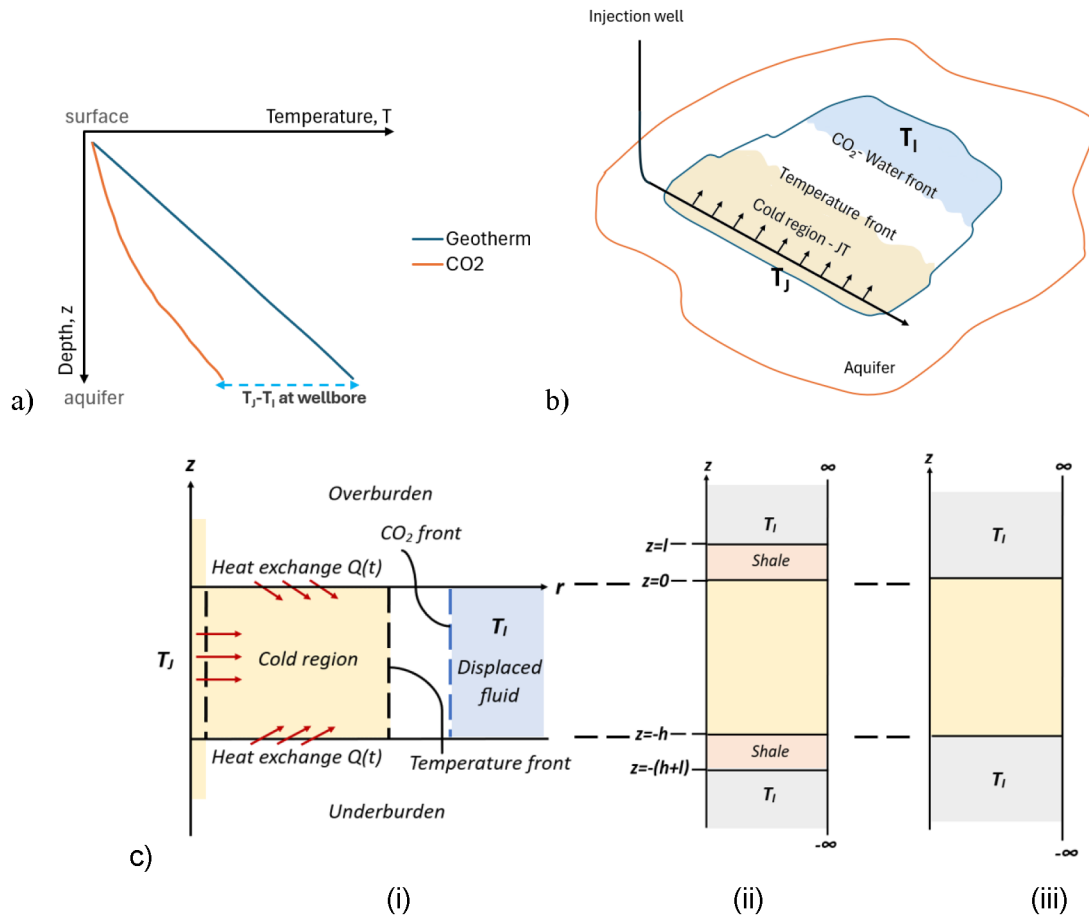


Figure 2—Schematic of CO₂ cooling during geological storage: (a) propagation of temperature profile from horizontal injector; (b) three zone structure within the formation (c) (i) three zone structure of the flow domain, (ii) geological structure for the steady-state heat exchange model, (iii) structure for the non-steady-state heat exchange model.

Analytical models can provide this information with less computational expenses and in shorter times than the numerical models. Besides, the analytical models yield significantly faster inverse solvers, where iterative procedures include forward runs; so, the inverse solutions for interpretation of lab corefloods and improved reservoir characterisation from field data are more effective under an analytical modelling. The analytical direct problem solutions usually regularise the ill-posed inverse problem, resulting in well-posed inverse solutions. This explains the intensive research on analytical modelling of CO₂ geo sequestration. Currently, numerous works are presented on analytical modelling of CO₂ injection. With respect to non-isothermal flows, below we present an incomplete list of exact and asymptotic solutions.

Mathias et al. (2010) obtained exact solution that accounts for Joule-Thomson effect but ignores heat exchange. This corresponds to thermally isolated reservoir and surrounding formations in Fig. 2c(i).

The heat exchange between the reservoir and adjacent layers significantly impacts non-isothermal transport in rocks (LaForce et al., 2014; Lawal, 2020). The exact solution for non-adiabatic flows in porous media with steady-state Newton-law heat exchange and Joule-Thomson effect was derived by Chesnokov et al. (2024). This solution corresponds to the reservoir surrounded by thin shales, where the temperature field and heat flux can be assumed to be quasi steady state (Fig. 2c(ii)).

However, temperature distribution in surrounded formations cannot be steady-state due to infinity of the heat transfer domain (Fig. 2c(iii)). Usually, a self-similar solution for heat transfer equation is used to introduce the heat exchange term into the heat conductivity equation. This solution contains a delay because heat exchange starts at the moment when the reservoir temperature differs from the initial reservoir temperature. Joule-Thomson effect associates temperature gradient with pressure gradient. Joule-Thomson

effect is significant for compressible CO₂ and is negligible for low-compressible brine, displaced in the process of CO₂ injection. Therefore, Chesnokov et al. (2025) assumed the following delay in heat-exchange term: the heat exchange starts at the moment of arrival of the displacement water-gas front. The analytical model has been derived. As it was expected, the model exhibit temperature variation behind the displacement water-gas front.

However, this model shows that temperature difference between the reservoir and initial temperature ahead of the temperature front is insignificant, due to low remote pressure gradient values. Introduction of the delay with respect to temperature front rather than displacement front becomes a next step in analytical non-isothermal flow modelling.

This motivated the current work, where we derive exact solution for CO₂ injection problem, where the heat exchange with adjacent layers starts at the moment of arrival of the temperature front. The solution accounts for Joule-Thomson effects and non-steady state heat exchange between the reservoir and semi-infinite adjacent formations. The derived explicit formula allows for the analysis of temporal evolution of temperature profiles. Unlike the steady state model, the unsteady-state model reveals no profile stabilization over time. The explicit formulae determines the temperature and pressure profiles at a fixed time, which calculates the path of radial dependency in CO₂-water phase diagram; the path connects the well and the reservoir. The analytical model allows for express method to plot temperature-pressure paths in phase diagram and diagnose hydrate formation. A new method of 1D model validation via using heat balance in the reservoir while comparing with 2D heat transfer problem is developed. The method exhibits significantly higher accuracy of the derived analytical model if compared with previous above-cited models. This validates the proposed analytical model for Joule-Thomson cooling during CO₂ injection into aquifers.

Governing equations

Main assumptions of the model

The model is based on the following assumptions: (i) one-dimensional radial, unsteady, single-phase, non-isothermal flow in an infinite, homogeneous formation; (ii) incompressible reservoir fluid ahead of the CO₂–water front, incompressible CO₂, and incompressible rock matrix; (iii) a uniform connate water saturation, S_{wi} , throughout the reservoir; (iv) a constant injection temperature for CO₂ within the wellbore; (v) constant Joule–Thomson coefficient, permeability, porosity, thermal conductivity, and heat capacity of both the rock and the fluids; (vi) CO₂ injection occurs at a fixed rate while the pressure at the drainage boundary remains constant; (vii) water vaporisation into the gas phase is neglected; (viii) fluid density and viscosity are assumed constant; and (ix) heat exchange with surrounding formations follows a self-similar vertical heat-transfer model, incorporating a delay associated with the arrival of the temperature front.

Mathematical model

Non-steady heat exchange occurs between the reservoir and the adjacent formations. Appendix A presents the solution to the classical one-dimensional vertical heat-transfer equation (Eq. A1) subject to the boundary conditions given in Eqs. (A3–A5). Evaluating the temperature solution $T(z, t)$ at $z = 0$ provides the temperature gradient used to derive the heat-exchange term (Eq. A7). In this formulation, the interface temperature T_i is replaced by the unknown reservoir temperature $T(r, t)$. The temperature ahead of the temperature front is equal to the initial reservoir temperature T_i ; therefore, heat exchange at position r begins only upon the arrival of the temperature front. Equation (A7) is applied assuming that the inlet temperature is not constant but instead equals the local reservoir temperature. For the steady-state heat-exchange model, the heat flux is determined using Newton's law of cooling, as described by Chesnokov et al. (2024) and expressed in Eq. (1). The time delays associated with the arrival of the temperature front and the CO₂ front, under both steady-state and non-steady-state conditions, are subsequently defined in Eqs. (2) and (3).

$$Q_{SS} = -\gamma \frac{T - T_I}{l} \quad (1)$$

$$Q_{NSST} = -\sqrt{\frac{\gamma[\phi\rho_w C_{pw} + (1-\phi)\rho_s C_{ps}]}{\pi}} \frac{T - T_I}{\sqrt{t - \frac{\pi r^2 - \pi r_w^2}{u_T}}} \quad (2)$$

$$Q_{NSS} = -\sqrt{\frac{\gamma[\phi\rho_w C_{pw} + (1-\phi)\rho_s C_{ps}]}{\pi}} \frac{T - T_I}{\sqrt{t - \frac{\pi r^2 - \pi r_w^2}{u_{CO2}}}} \quad (3)$$

where $Q_{SS}(\text{M T}^{-3})$ denotes the steady-state heat-exchange term, $Q_{nsst}(\text{M T}^{-3})$ denotes the non-steady-state heat-exchange term with delay in temperature front arrival, $Q_{nss}(\text{M T}^{-3})$ denotes the non-steady-state heat-exchange term with delay in CO_2 -water front arrival, $\gamma(\text{M L T}^{-3} \text{ K}^{-1})$ is the thermal conductivity of the surrounding formations, $l(\text{L})$ is the thickness of thin shales, ϕ is the porosity, $\rho_w(\text{M L}^{-3})$ is the water density, $C_{pw}(\text{L}^2 \text{ T}^{-2} \text{ K}^{-1})$ is the specific heat capacity of water, $\rho_s(\text{M L}^{-3})$ is the solid (rock) density, $C_{ps}(\text{L}^2 \text{ T}^{-2} \text{ K}^{-1})$ is the specific heat capacity of the solid phase, $T(\text{K})$ is the temperature, $T_I(\text{K})$ is the initial reservoir temperature, $r_w(\text{L})$ is the wellbore radius, $u_{CO2}(\text{L}^2 \text{ T}^{-1})$ represents the effective injection rate per unit reservoir thickness, and $u_T(\text{L}^2 \text{ T}^{-1})$ represents the effective velocity of the temperature front.

The influence of each heat-exchange model on the resulting temperature profiles is illustrated in Figure 3(b–d), with the adiabatic reference case shown in Figure 3(a). Each model modifies the profile shape by either generating a more pronounced temperature drop between the injection well and the advancing temperature front or altering how the temperature transitions back to the initial reservoir conditions at the front, either abruptly or gradually.

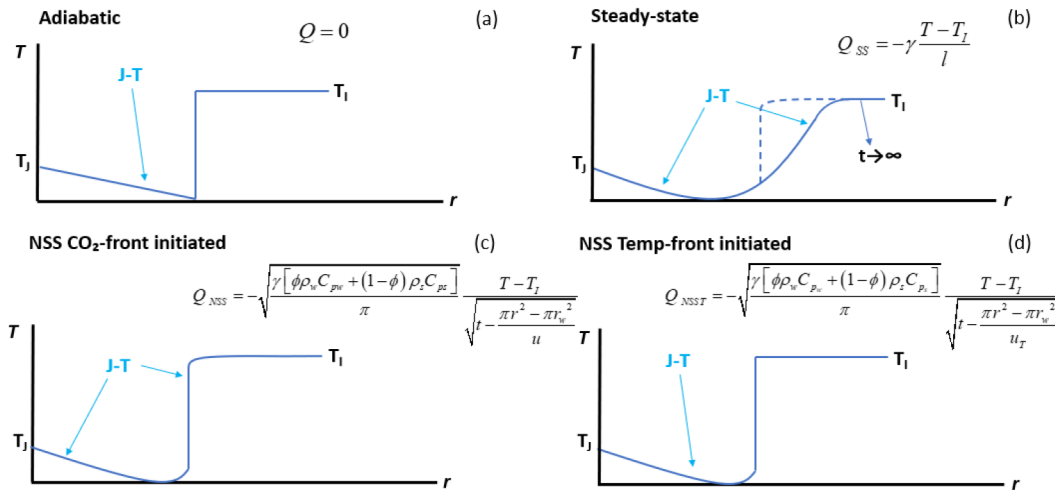


Figure 3—Temperature profiles for different heat exchange models.

Formulation of governing equations

The energy balance accounts for the heat capacities of the rock, CO_2 , and water, advective heat transport, the Joule–Thomson effect, and thermal exchange with the surrounding formations:

$$\left[(1-\phi)\rho_s C_{ps} + \phi(1-S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw} \right] h \frac{\partial T}{\partial t} + \rho_g C_{pg} \frac{q}{2\pi r} \frac{\partial}{\partial r} [T - \alpha_{JT} p] = Q \quad (4)$$

where S_{wi} is the connate water saturation, $\rho_g(\text{M L}^{-3})$ is the gas density, $C_{pg}(\text{L}^2 \text{ T}^{-2} \text{ K}^{-1})$ is the gas specific heat capacity, $h(\text{L})$ is the reservoir thickness, $q(\text{L}^3 \text{ T}^{-1})$ is the injection rate, $\alpha_{JT}(\text{M}^{-1} \text{ L T}^2 \text{ K})$ is the Joule–Thomson coefficient, and $p(\text{M L}^{-1} \text{ T}^{-2})$ is the fluid pressure.

Darcy velocity is defined as:

$$u = \frac{q}{2\pi rh} = -\frac{K_{rgwi}k}{\mu} \frac{\partial p}{\partial r} \quad (5)$$

where $\mu(\text{M L}^{-1} \text{T}^{-1})$ is the viscosity of CO_2 , K_{rgwi} is the end-point relative permeability of CO_2 , and $k(\text{L}^2)$ is the reservoir permeability.

Furthermore, the effective CO_2 injection rate per unit reservoir thickness is defined as:

$$u_{CO2} = \frac{q}{(1-S_{wi})\phi h} \quad (6)$$

and the effective velocity at which the thermal front propagates through the formation is given by:

$$u_T = \frac{q\rho_g C_{pg}}{\left[(1-\phi)\rho_s C_{ps} + \phi(1-S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw}\right]h} = 2Vu_{CO2} \quad (7)$$

The boundary condition at the wellbore corresponds to CO_2 injection at a constant temperature:

$$T(r=r_w) = T_J \quad (8)$$

The initial temperature condition in the reservoir is given by:

$$T(t=0) = T_I \quad (9)$$

Ahead of the temperature front, the fluid is incompressible and unaffected by the Joule–Thomson effect, so the temperature remains at its initial value. Heat exchange with the surrounding formations begins only upon the arrival of the temperature front, as detailed in [Appendix A](#). The boundary condition applied at the heat front is:

$$T\left(\sqrt{\frac{u_T t}{\pi}} + r_w^2, t\right) = T_I \quad (10)$$

Analytical solution

The exact solution to the problem defined by [Eq. \(4\)](#) and [Eqs. \(8–10\)](#) is obtained by introducing dimensionless variables as defined in [Appendix B](#), while the full derivation in linear coordinates (x, t) is provided in [Appendix C](#). The energy balance equation ([Eq. 4](#)) applies only in the region behind the propagating temperature front, where heat exchange and JT cooling occur. Ahead of the front, the reservoir remains undisturbed, and the temperature is equal to the initial value T_I .

The solution is given in [Eq. \(11\)](#):

$$T(r, t) = \begin{cases} T_I + (T_I - T_J) \left\{ v \left[Ei \left(\frac{w}{\sqrt{mt-b(\pi r^2 - \pi r_w^2)}} \pi r_w^2 \right) - Ei \left(\frac{w}{\sqrt{mt-b(\pi r^2 - \pi r_w^2)}} \pi r^2 \right) \right] - e^{\frac{w}{\sqrt{mt-b(\pi r^2 - \pi r_w^2)}} \pi r_w^2} \right\} e^{-\frac{w}{\sqrt{mt-b(\pi r^2 - \pi r_w^2)}} \pi r^2}, & t > \frac{\pi r^2 - \pi r_w^2}{2Vu_{CO2}} \\ T_I, & t < \frac{\pi r^2 - \pi r_w^2}{2Vu_{CO2}} \end{cases} \quad (11)$$

With the following dimensional constants:

$$w = \sqrt{\lambda [\phi \rho_w C_{pw} + (1-\phi)\rho_s C_{ps}]} \quad (12)$$

$$m = \frac{q^2 \rho_g C_{pg}}{(1-S_{wi})\phi h} \quad (13)$$

$$b = \frac{q}{(1-S_{wi})^2 \phi^2} \left[(1-\phi)\rho_s C_{ps} + \phi(1-S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw} \right] \quad (14)$$

$$v = \frac{\alpha_{JT} q \mu}{4\pi k h K_{rgwi} (T_I - T_J)} \quad (15)$$

Validation by comparison with 2D heat conductivity

The analytical model is validated by comparing the solution presented in Section 2.4 with the quasi-2D heat-conduction solution derived in [Appendix D](#). Using the reservoir temperature profile from [Eq. \(11\)](#), the heat exchange with the surrounding formations is obtained using [Eq. \(2\)](#). In the quasi-2D formulation, [Eq. \(D2\)](#) specifies the boundary condition as the heat flux at the interface between the reservoir and the adjacent formations. The vertical heat-exchange solution, $T_D(z_D, t_D)$, is given in [Eq. \(D9\)](#), and the corresponding interface temperature is obtained by evaluating this expression at $z_D = 0$, as shown in [Eq. \(D10\)](#). The analytical model is validated by comparing this interface temperature ([Eq. D10](#)) with the dimensionless reservoir temperature profile given in [Eq. \(C9\)](#).

The base case for all subsequent calculations uses the parameter values listed in [Table 1](#).

Table 1—Model parameters used in this study.

Parameter	Symbol	Base case	Minimum	Maximum	Unit
Half of reservoir thickness	h	91	1	100	m
Porosity	ϕ	0.11	0.1	0.3	[-]
Absolute permeability	k	2×10^{-15}	4×10^{-16}	6×10^{-14}	m ²
End point relative permeability	K_{rgwi}	1	1	1	[-]
Well radius	r_w	0.1	0.05	0.13	m
Rock density	ρ_s	2600	2270	3200	kg/m ³
Water density	ρ_w	992	527	1000	kg/m ³
CO ₂ density	ρ_f	141.4	0.5	1236	kg/m ³
CO ₂ viscosity	μ_f	16.7×10^{-5}	1.1×10^{-5}	2.7×10^{-4}	Pa.s
Heat capacity of rock	c_s	1000	776	1215	J/kg/K
Heat capacity of water	c_w	4037	3965.1	4335.4	J/kg/K
Heat capacity of CO ₂	c_f	904	709	1476	J/kg/K
Joule Thomson coefficient	α_{JT}	10.2×10^{-6}	7×10^{-6}	1.08×10^{-5}	K/Pa
Adjacent formation heat conductivity	γ	2.5	1	3.7	W/m/K
Well pressure	p_w	50×10^5	50×10^5	50×10^5	Pa
Reservoir temperature	T_i	378.15	294.15	398.15	K
Injection temperature	T_j	293.15	293.15	293.15	K
Injection time	t_j	10	0.00274	20	Years
Half of CO ₂ injection flow rate	q	0.02	0.005	0.04	m ³ /s
Connate water saturation	S_{wi}	0.2	0.2	0.4	[-]

[Figure 4](#) (al), (bl), and (cl) illustrate the model's sensitivity to variations in the dimensionless groups A , B , and V , respectively, which are further discussed in Section 4. The corresponding absolute errors are shown in [Figure 4](#) (a2), (b2), and (c2). In each case, the deviation between $T_{1D}(r_D, t_D)$ and $T_{2D}(z_D = 0, t_D)$ reaches its maximum at the temperature front and then rapidly decreases over a short distance away from it. This behaviour arises from the step-like nature of the temperature profiles, where the largest discrepancy occurs in the vicinity of the sharp transition at the front.

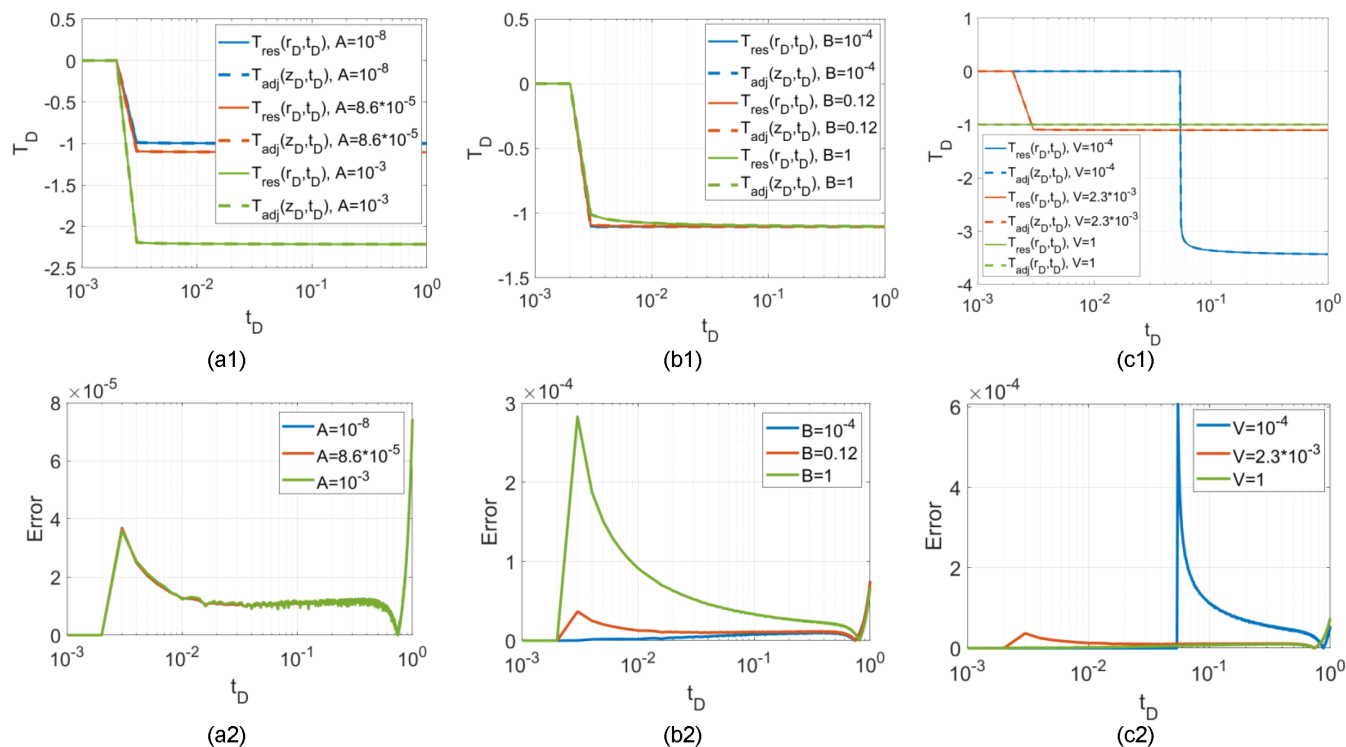


Figure 4—NSS model with heat exchange that starts at temperature front arrival - effects of dimensionless constants A, B, and V on temperature histories: (a1) sensitivity with respect JT number A; (b1) changing heat exchange number B; (c1) effects of varying temperature front velocity V; (a2), b2), and (c2) show the relative accuracy corresponding to (a1), (b1), and (c1), respectively. The reservoir point used is $r_D=0.002$. Solid curves $T_{res}(r_D, t_D)$ correspond to the analytical model with given heat exchange. Dashed curves $T_{adj}(z_D, t_D)$ reflect temperature at the inlet to vertical adjacent column on the reservoir top assuming the continuity of heat flux across the reservoir top.

The errors associated with this model are negligible, on the order of 10^{-5} to 10^{-4} , indicating that the model maintains high validity across all considered parameter ranges and throughout the entire duration of CO_2 injection.

Figure 5 presents the validity assessment of the NSS model in which heat exchange begins upon the arrival of the CO_2 front (Chesnokov et al., 2025). The figure illustrates the sensitivity of the model to parameters A, B, and V (Figure 5(a1), (b1), (c1)), as well as the corresponding error between the temperature profile within the reservoir and that within the adjacent formations (Figure 5(a2), (b2), (c2)). This NSS formulation produces results with errors on the order of 10^{-2} to 1. In contrast, the new NSS model developed in this work reduces the error to approximately 10^{-4} – 10^{-5} , representing an improvement of several orders of magnitude.

This substantial increase in accuracy confirms that the JT effect ahead of the temperature front can be neglected and that this region may be treated as remaining at its initial temperature. Applying this simplification significantly enhances the performance of the NSS heat-exchange model and ensures a much more reliable representation of heat transfer during CO_2 injection.

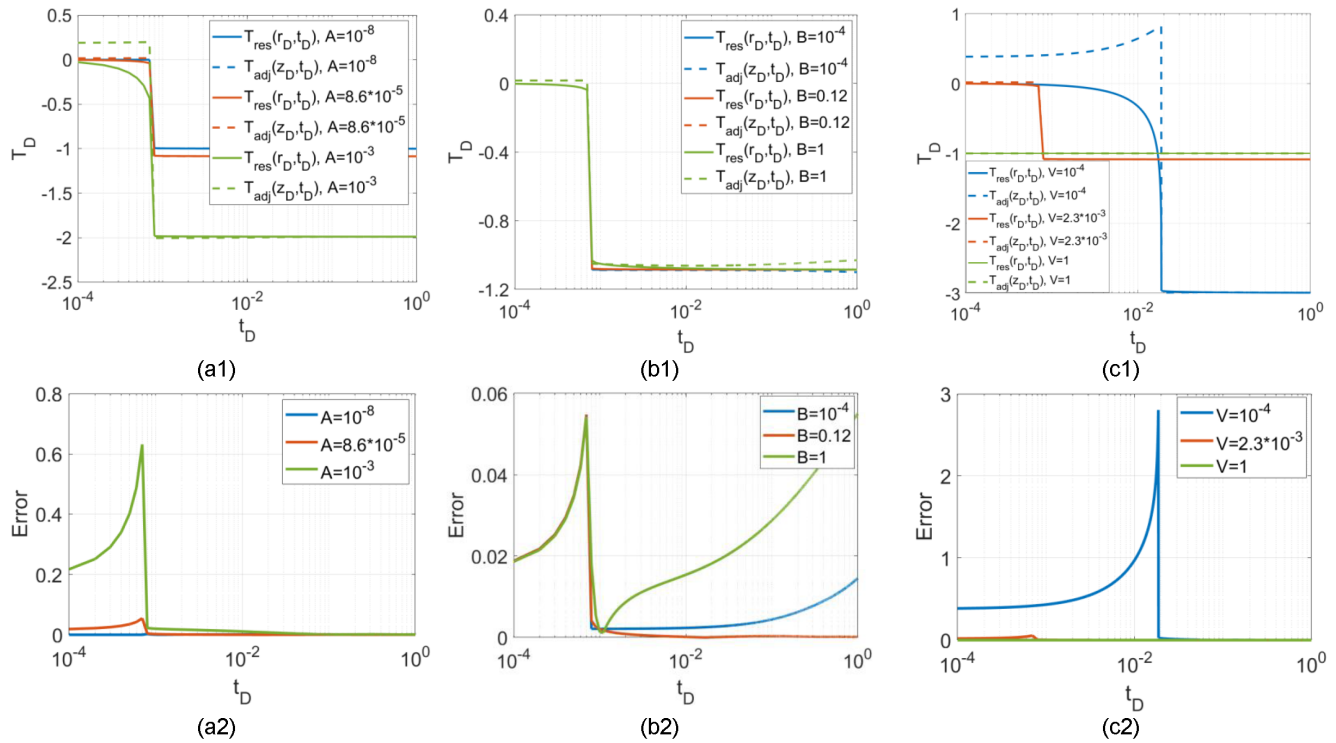


Figure 5—Effects of dimensionless constants A , B , and V on temperature histories for NSS model with heat exchange that starts at water-gas front arrival: (a1) sensitivity with respect JT number A ; (b1) changing heat exchange number B ; (c1) effects of varying temperature front velocity V ; (a2), (b2), and (c2) show the relative accuracy corresponding to (a1), (b1), and (c1), respectively. The reservoir point used is $r_D=0.002$. Solid curves $T_{res}(r_D, t_D)$ correspond to the analytical model with given heat exchange. Dashed curves $T_{adj}(z_D, t_D)$ reflect temperature at the inlet to vertical adjacent column on the reservoir top assuming the continuity of heat flux across the reservoir top.

Sensitivity study

Figure 4 illustrates the influence of the Joule–Thomson number A , the heat-exchange number B , and the temperature-front velocity V (see Eqs. B4–B6) on the temperature histories $T_{1D}(r_D, t_D)$ and $T_{2D}(z_D = 0, t_D)$ at a fixed radial position of $r_D = 0.002$. The sensitivity analysis is performed over parameter ranges selected to capture the physical impact of each dimensionless group. The JT number A varies from 10^{-8} to 10^{-3} , representing changes in the magnitude of Joule–Thomson cooling and its resulting influence on the temperature decline near the injection well. The heat-exchange number B spans 10^{-4} to 1, reflecting the contribution of non-steady-state heat transfer from the surrounding formations. The dimensionless temperature-front velocity V is varied over the range 10^{-4} to 1, capturing its effect on the rate at which the temperature front propagates into the reservoir. Across all parameter combinations, the deviation between the reservoir and boundary temperature profiles remains negligible, demonstrating the high accuracy and broad validity of the proposed heat-exchange formulation.

To further investigate parameter sensitivity in a phase-behaviour context, Figure 6 presents pressure-temperature (P–T) trajectories generated by combining the analytical solution in Eq. (11) with the pressure distribution obtained from Darcy's law (Eq. 5). These P–T trajectories are projected onto the CO₂–H₂O phase diagram (Figure 1), enabling assessment of hydrate-formation risk under varying dimensionless parameters. In Figure 6(a), the JT number A is varied between 10^{-8} and 10^{-4} . As A decreases toward zero, Joule–Thomson cooling becomes negligible, and the temperature drop near the wellbore is minimal, with the minimum temperature occurring only 3 m from the well.

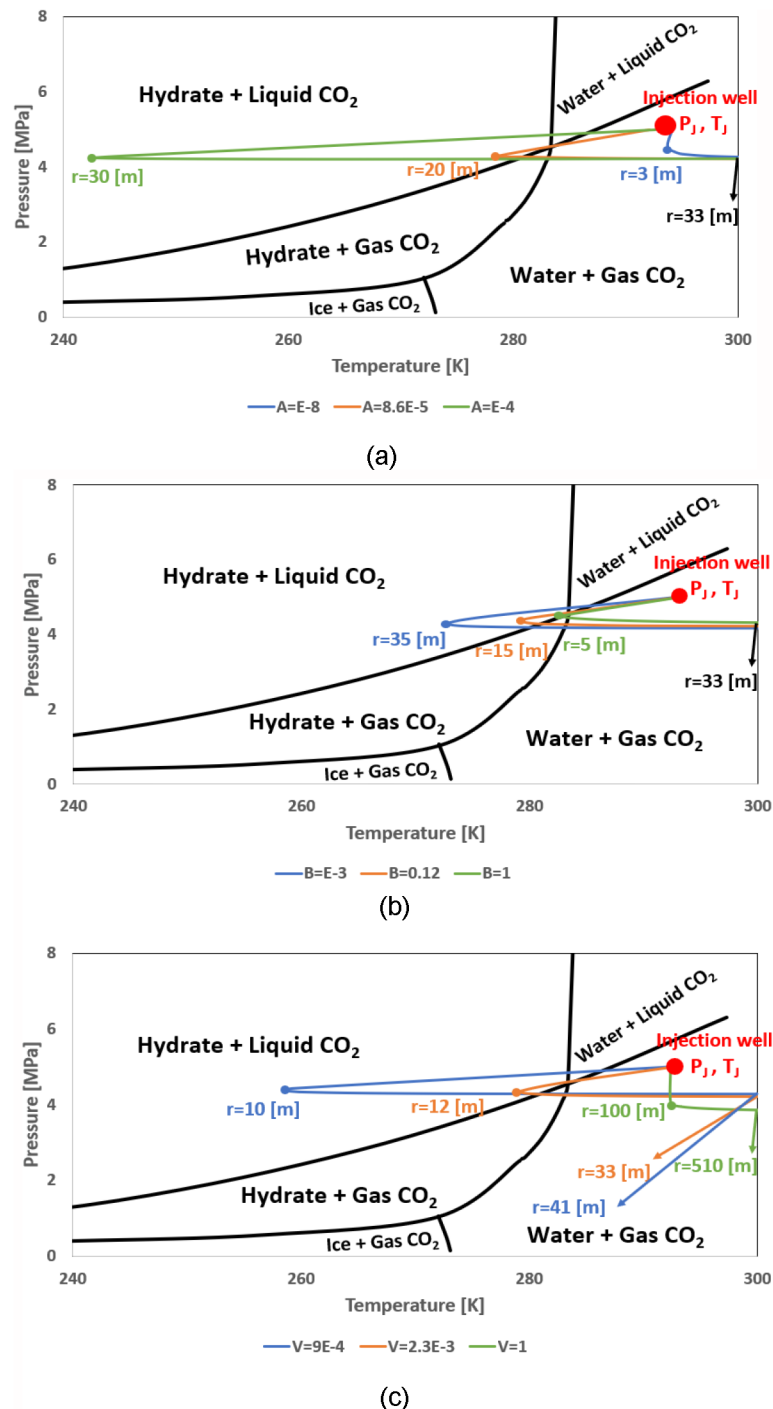


Figure 6—Rejection envelope at $t_D=1$ for hydrate formation at phase diagram (a) Effect of J-T parameter A (b) Effect of heat exchange parameter B , (c) Effect of heat front velocity parameter V .

Conversely, higher values of A intensify the cooling effect, shifting the minimum-temperature point to approximately 30 m from the well.

Figure 6(b) shows variations in the heat-exchange number B from 10^{-3} to 1. Among the three parameters, B exhibits the smallest influence on the P–T trajectories. Larger values of B , corresponding to more heat transfer from the surrounding formations, significantly counteract the cooling, preventing the system from entering the hydrate-formation region. For $B = 1$ (green curve), the trajectory remains entirely outside the hydrate stability zone, with the minimum temperature occurring only 5 m from the well. For low values of

B , however, weaker heat exchange results in deeper penetration into the hydrate region, with the minimum temperature occurring around 35 m from the well.

Figure 6(c) presents the effect of the temperature-front velocity V , varied from 9×10^{-4} to 1. Increasing V reduces the magnitude of the temperature drop, as a faster-moving temperature front allows less time for Joule–Thomson cooling to accumulate. At $V = 1$, only a pressure drop is observed at 100 m from the well, with negligible temperature change. In contrast, low values of V allow substantial cooling, leading to a pronounced temperature minimum at approximately 10 m from the well. Overall, higher values of V correspond to deeper front propagation and reduced cooling, indicating that an increase in CO_2 density and effective thermal capacity accelerates temperature transport through the formation.

Comparison between different models of heat exchange

Figure 7 compares temperature profiles at different times for three cases: non-steady-state (NSS) heat exchange initiated by the arrival of the temperature front (Eq. 11, solid lines), NSS heat exchange initiated by the arrival of the CO_2 front (Chesnokov et al. (2025), dashed lines), and steady-state (SS) heat exchange (Chesnokov et al. (2024), dotted lines). For times earlier than the SS stabilization time (15 years), the NSS and SS models differ only behind the temperature front, while remaining aligned ahead of it. At the SS stabilization time, the temperature front ceases to propagate, whereas the NSS models remain unsteady, resulting in lower temperatures. The NSS model with heat exchange at the CO_2 front propagates the furthest of the two, while the NSS model with heat exchange at the temperature front results in a sharp return to the initial temperature.

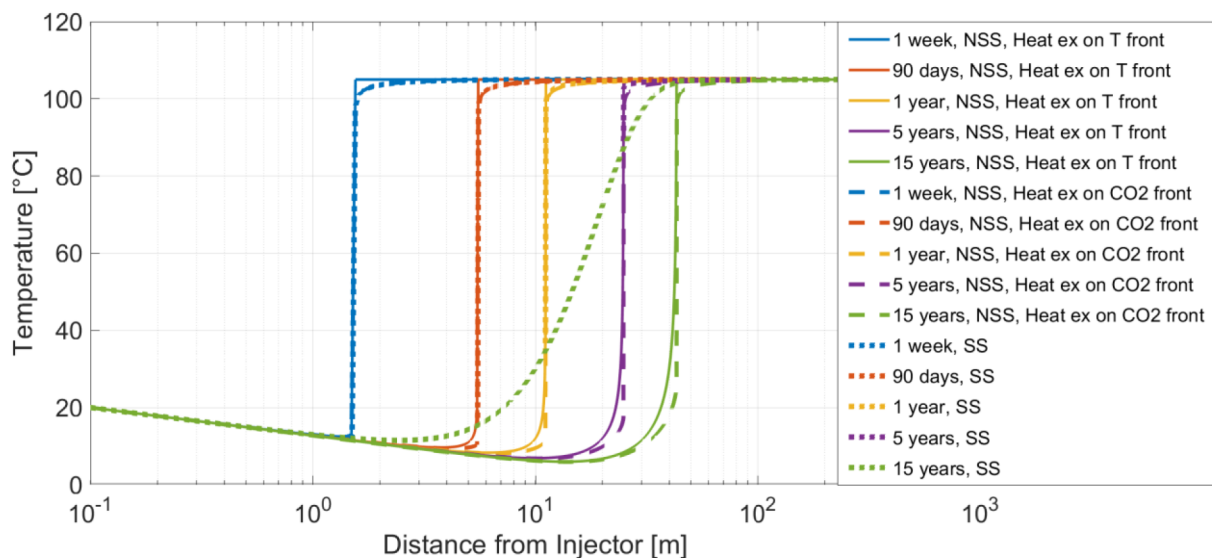


Figure 7—The impact of the NSS heat exchange model on the temperature front (solid lines) in comparison to the NSS heat exchange model on the CO_2 front (dashed lines, Chesnokov et al. (2025)) and SS heat exchange model (dotted lines, Chesnokov et al. (2024)) for the base case.

The injected CO_2 displaces water at the initial temperature T_i . As the gas expands into the lower-pressure domain, it cools due to expansion but is simultaneously heated by the surrounding rock and connate water, which are assumed to be in instantaneous thermal equilibrium with the gas. Non-steady-state heating from the surrounding formations contributes less heat than the steady-state model, leading to lower temperatures behind the cold front.

Ahead of the temperature front, both NSS models exhibit a positive temperature gradient until the SS model stabilizes (green dotted curve at 15 years) and the front disappears. In the region near the well, a negative gradient is observed due to quasi steady-state flow and the positive Joule–Thomson coefficient

of CO₂. At each radial position r_D , the temperature decreases over time, indicating that the JT effect dominates over heating from the surrounding formations. The slope of the temperature profiles near the well corresponds to the CO₂ JT coefficient, while further from the well, the slope deviates at the minimum temperature.

Hydrate formation prediction from the analytical model

This study presents a model that enables fast evaluation of depleted gas reservoirs for CO₂ storage, allowing estimation of CO₂ temperature profiles and assessment of hydrate formation risks. The analytical solution (Eq. 11) can also be used to optimize injection parameters, such as the maximum injection rate, to avoid excessively low temperatures and mitigate hydrate formation, which is critical for maintaining CO₂ injectivity.

Figure 8 presents the projection of the temperature-pressure (T-P) trajectory onto the CO₂–H₂O phase diagram for the adiabatic, steady-state (SS), and non-steady-state (NSS) heat-exchange models, from the wellbore to reservoir conditions. The position of the minimum temperature within the reservoir was determined for each model to evaluate the potential risk of hydrate formation near the injection well, which could impair injectivity. All models were simulated using the same reservoir and fluid properties, injection duration, and flow parameters.

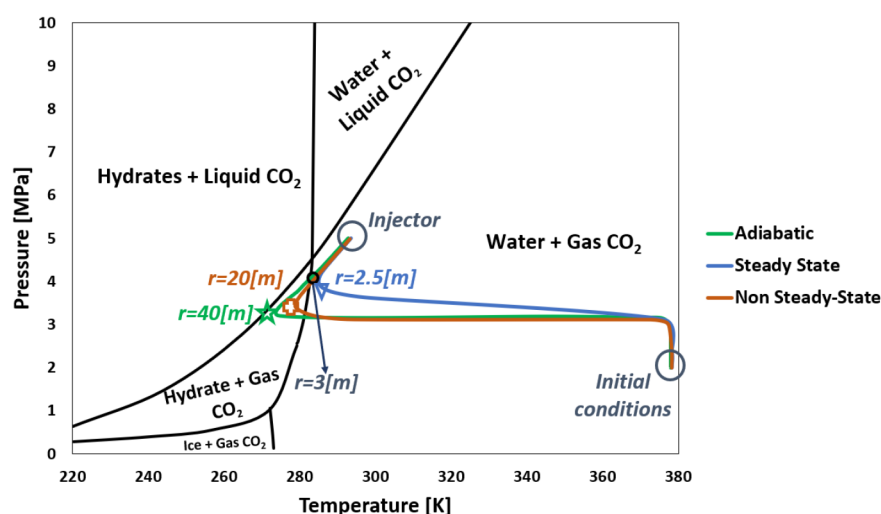


Figure 8—Phase diagram with the three model's trajectory, for a time of 1 PVI.

Both the adiabatic and NSS models produce pressure-temperature trajectories that enter the hydrate-formation region, indicating a potential risk of hydrate development near the injection well. In contrast, the SS model remains outside the hydrate zone due to the continuous heat supply from the surrounding formations. Under SS conditions, the minimum temperature is approximately 11°C, reached at a radial distance of about 2.5 m from the injection well. The adiabatic model exhibits the deepest penetration, while the NSS model reaches slightly shallower depths. For the base case parameters in Table 1, the NSS model attains its minimum pressure and temperature at $r = 20$ [m], the adiabatic model attains its minimum pressure at $r = 40$ [m], both crossing the hydrate phase boundary at $r = 3$ [m]. This indicates that any hydrates formed are unlikely to affect well injectivity. Hydrate formation at a distance from the well may, however, provide an additional CO₂ trapping mechanism (Wang et al., 2020).

The model was applied to field cases at the Gorgon CO₂ injection project in Western Australia and the Sleipner CO₂ storage project in Norway. Using publicly available data on reservoir characteristics and CO₂ injection parameters, the temperature profile from Eq. (11) was combined with the pressure distribution

from Darcy's law (Eq. 5) to generate pressure–temperature trajectories and predict system behaviour in each project.

Figure 9 shows the dimensionless T–P trajectories for selected times: $t_D = 0.01, 0.1, 1$ for Gorgon (blue) and $t_D = 1$ for Sleipner (green). For Gorgon, the high injection-well pressure results in hydrate formation across all salinity levels of the formation water, with minimum pressure locations occurring at 12 [m] for $t_D = 0.01$, 28 [m] for $t_D = 0.1$, and 63 [m] for $t_D = 1$.

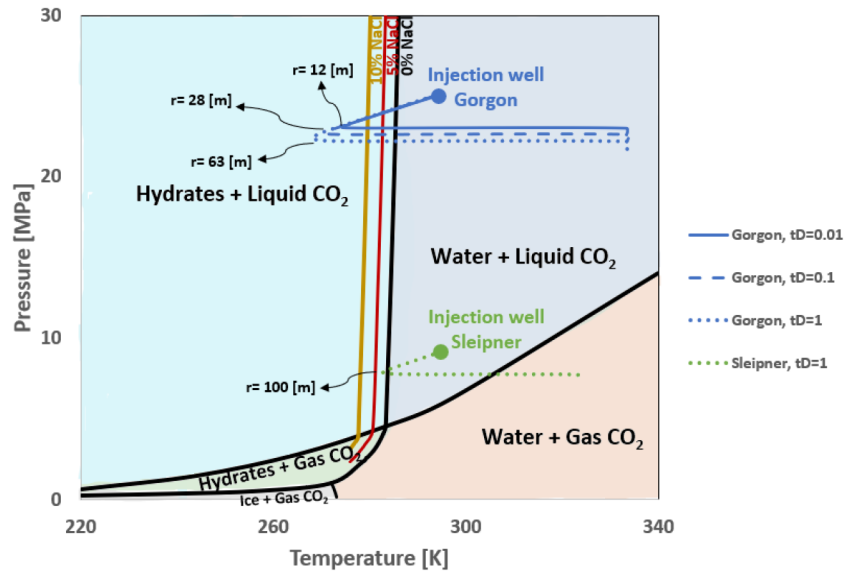


Figure 9—Phase diagram for a NSS temperature front model with trajectories for $t_D=0.01, 0.1, 1$ for the CO₂ storage project in Gorgon (blue) and Sleipner (green).

For Sleipner, the trajectory for 0% NaCl intersects the hydrate stability envelope at $t_D = 1$, while it remains outside the curves for hydrate formation at higher brine concentrations (5% and 10% NaCl). This indicates that hydrates are unlikely to form in saline formations, and for the 0% NaCl scenario, hydrate formation occurs only at approximately 100 [m] from the injection well.

In general, increasing NaCl concentration shifts the CO₂–H₂O hydrate boundary to lower temperatures, preventing hydrate formation under typical reservoir conditions. The Gorgon field, with its higher injection pressures, crosses the hydrate zone at all times, whereas the lower-pressure Sleipner field only enters the hydrate zone under low-salinity conditions.

Figure 10 illustrates the sensitivity of both the Gorgon and Sleipner projects to the dimensionless Joule–Thomson parameter A . The values of A for the Gorgon case are $1.67\text{e-}4$, $1.46\text{e-}3$ and $2.98\text{e-}3$, corresponding to injection rates of 35, 75 and 110 [kg/s]. The values of A for the Sleipner case are $5.32\text{e-}3$, $6.7\text{e-}3$ and $9.6\text{e-}3$, corresponding to injection rates of 30, 60 and 100 [kg/s]. The results show that the temperature–pressure trajectories in both systems are strongly influenced by the magnitude of A , highlighting the importance of the JT cooling effect in predicting hydrate behaviour. At low values of A , the Gorgon case no longer enters the hydrate stability region, indicating that weaker JT cooling suppresses hydrate formation near the wellbore. In contrast, the Sleipner case intersects the hydrate region for all examined values of A , reflecting its lower reservoir pressure and correspondingly higher susceptibility to cooling-induced hydrate formation. JT number A therefore plays a central role in determining whether the minimum-temperature trajectory crosses into the hydrate stability zone, and its influence is evident across both field scenarios.

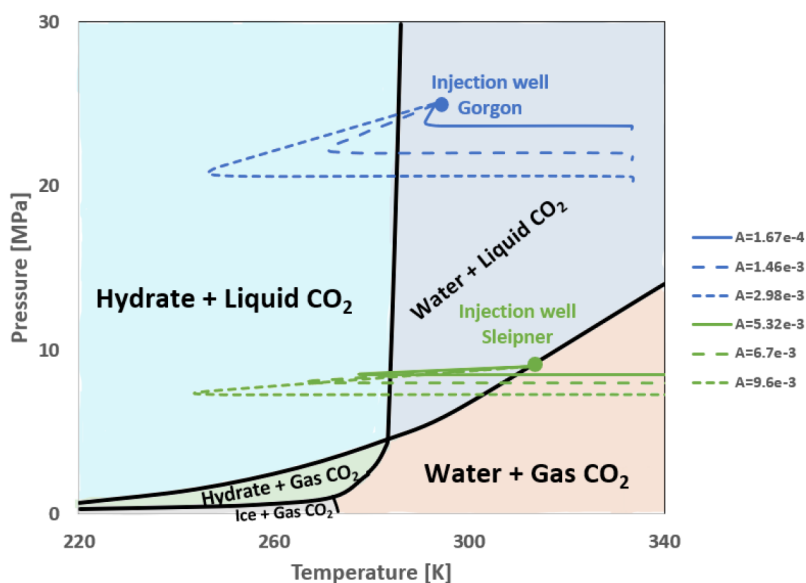


Figure 10—Phase diagram for a NSS temperature front model with trajectories for minimum, maximum and base case values of dimensionless parameter A , for the CO₂ storage project in Gorgon (blue) and Sleipner (green).

The information on hydrate formation around the injector, provided by the temperature-pressure profiles mapped in phase diagram can be essential for well injectivity forecast and management of injector. As shows the profiles in phase diagrams (Figures 9,10), the trajectories are located to the right of the phase envelope, not entering the hydrate zone at low injection rate. On the contrary, at high rates the trajectory passes the hydrate zones, yielding formation of the solid in the pore space, resulting in formation damage. The maximum rate where the hydrates do not form corresponds to touching the envelope by the P-T trajectory. So, the analytical model (Eq.11) allows determining the maximum hydrate-preventive rate.

Figure 10 shows how radius parametrises the trajectories in phase diagrams, $T=T(r,t)$ and $p=p(r,t)$. The radius where the trajectory crosses the envelope corresponds the minimum distance where the hydrates form. If the formation damage occurs far away from the injector, it wouldn't affect well injectivity but serve as an alternate physics mechanism of CO₂ geo sequestration. After injection stops, the hydrates dissociate, releasing CO₂ that follows a different migration pathway than during injection. This altered trajectory increases the range of CO₂ sequestration pathways, enabling additional CO₂ capture after injection has ended.

Discussion

The analytical model can be directly implemented in Excel or MATLAB and be used for estimates of temperature fields, benchmarking 3D numerical schemas and simulators. The implemented analytical model can determine maximum injection rate where hydrates do not form, pressure and temperature fields and the consequent formation damage.

The orders of magnitude of the predicted parameters highly depend on brine salinity and pH, reservoir pressures and temperatures. So above numerical analysis must be performed at the high variety of those conditions.

Conclusions

Analytical 1D modelling of non-isothermal CO₂ injection into saline aquifer accounting for Joule-Thomson effect and heat exchange with adjacent formations, which assumes that the heat exchange starts at the moment of temperature front arrival, allows for the following conclusions.

The derived analytical model with unsteady-state heat exchange does not predict the temperature profile stabilisation, while the model with Newton's law of steady-state heat exchange exhibits asymptotic stabilisation of the temperature profile.

The method estimates the accuracy of the derived model as 10^{-5} – 10^{-4} , which validates the model and recommends its use for practical assessment of temperature distributions during geological CO₂ storage.

An analytical method for 1D thermal model validation by comparison with 2D heat transfer model via energy balance in the reservoir is developed. This method shows that the proposed model assuming the heat-exchange starting at the moment of temperature front arrival exhibits significantly higher accuracy than the previous model with delay in heat exchange of the gas-front arrival.

The sensitivity analysis demonstrates that the temperature profiles are strongly influenced by parameters A and V , while parameter B exhibits the smallest effect. Increasing A intensifies the Joule–Thomson cooling, leading to a deeper penetration into the hydrate-formation region and a larger temperature drop. Similarly, higher values of V result in a more pronounced pressure decline and lower temperatures occurring closer to the injection well, reflecting the impact of faster temperature-front propagation. In contrast, increasing B enhances heat transfer from the surrounding formations, which mitigates the cooling effect and produces only a minor temperature decline.

The analytical model allows for risk assessment of hydrate formation during CO₂ injection.

The analytical model is expressed by explicit formulae for timely temperature and pressure distributions and can be implemented in Excell or MATLAB. This provides an effective tool for temperature prediction and forecast of hydrate formation occurrence for a practical reservoir or production engineer involved in geological CO₂ storage in deep saline aquifers or depleted gas fields.

Nomenclature

a	Thermal diffusivity of the adjacent formations, [$\text{m}^2 \text{s}^{-1}$]
A	JT number, [-]
b	Dimensional constant, [$\text{kg m}^2 \text{K}^{-1} \text{s}^{-3}$]
B	Heat exchange number, [-]
C_p	Heat capacity, [$\text{J kg}^{-1} \text{K}^{-1}$]
C_{pg}	Heat capacity of gas, [$\text{J kg}^{-1} \text{K}^{-1}$]
C_{ps}	Heat capacity of solid, [$\text{J kg}^{-1} \text{K}^{-1}$]
C_{pw}	Heat capacity of water, [$\text{J kg}^{-1} \text{K}^{-1}$]
h	Half of formation thickness, [m]
k	Permeability, [m^2]
K_{rgwi}	End-point relative permeability of CO ₂ , [-]
m	Dimensional constant, [$\text{kg s}^{-4} \text{K}^{-1}$]
p	Pressure, [Pa]
p_w	Wellbore pressure, [Pa]
q	Half of injection rate, [$\text{m}^3 \text{s}^{-1}$]
Q	Heat exchange, [W m^{-2}]
Q_{ss}	Steady-state heat exchange, [W m^{-2}]
Q_{nss}	Non-steady-state heat exchange initiated by CO ₂ front arrival, [W m^{-2}]
Q_{nssT}	Non-steady-state heat exchange initiated by heat front arrival, [W m^{-2}]
r	Radial variable, [m]
r_w	Well radius, [m]
r_e	Position of gas front at t_j , [m]
r_f	Position of gas front, [m]
r_D	Dimensionless radial variable, [-]

r_{Dw}	Dimensionless well radius, [-]
S_{wi}	Connate water saturation, [-]
T	Temperature, [K]
T_I	Initial temperature, [K]
T_J	Injection temperature, [K]
T_D	Dimensionless temperature, [-]
T_{adj}	Temperature in adjacent formations, [-]
T_{res}	Temperature in reservoir, [-]
t	Time, [s]
t_j	Injection time, [s]
t_D	Dimensionless time, [-]
u	Darcy velocity, [m ² s ⁻¹]
u_{CO2}	Effective injection rate per unit of the formation thickness, [m ² s ⁻¹]
u_T	Effective velocity of temperature front, [m ² s ⁻¹]
v	Dimensionless constant, [-]
V	Temperature front velocity, [-]
w	Dimensional constant, [kg K ⁻¹ s ^{-3/2}]
x	Dimensionless linear variable, [-]
x_w	Dimensionless linear injection point, [-]
z	Vertical variable, [m]
z_D	Dimensionless vertical variable, [-]

Greek characters

α_{JT}	Joule-Thomson coefficient, [K Pa ⁻¹]
γ	Adjacent formations heat conductivity, [W m ⁻¹ K ⁻¹]
Δ	Time of temperature front delay, [s]
μ	Viscosity, [Pa s]
ρ_g	Gas density, [kg m ⁻³]
ρ_s	Solid density, [kg m ⁻³]
ρ_w	Water density, [kg m ⁻³]
ϕ	Porosity, [-]

Abbreviations

1D	One dimensional
2D	Two dimensional
JT	Joule–Thomson
NSS	Non-steady state
ODE	Ordinary differential equation
PDE	Partial differential equation
PVI	Pore volume injected
SS	Steady state
T-P	Temperature- Pressure

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Appendix A

To clarify the implementation of the heat-exchange term in Eq. (2), we examine the one-dimensional equation governing vertical heat transfer:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} \quad (\text{A1})$$

where a is the thermal diffusivity of the surrounding formations, defined as:

$$a = \frac{\gamma}{[\phi \rho_w C_{pw} + (1 - \phi) \rho_s C_{ps}]} \quad (\text{A2})$$

The initial condition for the vertical layers corresponds to the reservoir's initial temperature:

$$T(z, t = 0) = T_I \quad (\text{A3})$$

The temperature at the reservoir-formation interface is assumed constant, giving the boundary condition:

$$T(z = 0, t) = T_J \quad (\text{A4})$$

The boundary condition at infinity ensures that the formation temperature approaches the reservoir temperature at long times:

$$T(z \rightarrow \infty, t) = T_I \quad (\text{A5})$$

The solution to Eq. (A1) is self-similar and can be expressed as (Polyanin & Nazaikinskii, 2016):

$$T(z, t) = f\left(\frac{z}{2\sqrt{at}}\right) = T_J - (T_J - T_I) \operatorname{erf}\left(\frac{z}{2\sqrt{at}}\right) \quad (\text{A6})$$

The heat flux at the interface is obtained by differentiating Eq. (A6) with respect to z at $z = 0$:

$$Q = \gamma \left(\frac{\partial T}{\partial z} \right)_{z=0} = (T_I - T_J) \sqrt{\frac{\gamma [\phi \rho_w C_{pw} + (1 - \phi) \rho_s C_{pw}]}{\pi t}} \quad (\text{A7})$$

Assuming that heat exchange is delayed until the arrival of the temperature front, a time delay is introduced in Eq. (A7) as:

$$\Delta = t - \frac{\pi r^2 - \pi r_w^2}{u_T} \quad (\text{A8})$$

Substituting $T(r, t)$ and Eq. (A8) into Eq. (A7) yields the reformulated heat-exchange term presented in Eq. (2).

Appendix B

In this appendix, we introduce the dimensionless variables and constants used to solve the equation.

The position of the gas front, $r_f(t)$, is determined from Eq. (6). At $t = t_j$, the gas front reaches its position at the end of injection, denoted $r_e(L)$:

$$t_j = \frac{\pi r_e^2}{u}; r_e = \sqrt{\frac{u_{CO2} t_j}{\pi}} \quad (B1)$$

The following dimensionless variables are defined to formulate the energy-balance equation in dimensionless form:

$$T_D = \frac{T - T_I}{T_I - T_J}; t_D = \frac{t}{t_j}; r_D = \frac{r}{r_e}; r_{Dw} = \frac{r_w}{r_e} \quad (B2)$$

Substituting the variables in Eq. (B2) into Eq. (4) yields the dimensionless form of the energy-balance equation:

$$\frac{\partial T_D}{\partial t_D} + V \frac{1}{r_D} \frac{\partial T_D}{\partial r_D} + A \frac{1}{r_D^2} = -B \frac{T_D}{\sqrt{t_D - \frac{r_{Dw}^2}{2V}}} \quad (B3)$$

The corresponding dimensionless groups A , B , and V are defined as:

$$V = \frac{\rho_g C_{pg}(1 - S_{wi})\phi}{2[(1 - \phi)\rho_s C_{ps} + \phi(1 - S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw}]} \quad (B4)$$

$$A = \frac{\alpha_{JT} q \mu \rho_g C_{pg}(1 - S_{wi})\phi}{[(1 - \phi)\rho_s C_{ps} + \phi(1 - S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw}] 4\pi k h K_{rgwi}(T_I - T_J)} \quad (B5)$$

$$B = \sqrt{\frac{\gamma \mu [\phi \rho_w C_{pw} + (1 - \phi)\rho_s C_{ps}]}{[(1 - \phi)\rho_s C_{ps} + \phi(1 - S_{wi})\rho_g C_{pg} + \phi S_{wi}\rho_w C_{pw}]^2 h^2 \pi}} \quad (B6)$$

The dimensionless constant A is referred to as the Joule–Thomson (JT) number, B represents the heat-exchange number, and V corresponds to the dimensionless velocity of the thermal front. Substituting the dimensionless variables from Eq. (B2) into the boundary conditions in Eqs. (8–10) yields the dimensionless form of the wellbore boundary condition:

$$T_D(r_D = r_{Dw}, t_D) = -1 \quad (B7)$$

$$T_D(r_D = \sqrt{2V t_D + r_{Dw}^2}, t_D) = 0 \quad (B8)$$

By introducing the following change of variable:

$$x = r_D^2; x_w = r_{Dw}^2 \quad (B9)$$

and substituting the linear variable from Eq. (B9) into Eqs. (B3–B8), the problem is reformulated as a linear initial-boundary value problem for the PDE:

$$\frac{\partial T_D}{\partial t_D} + 2V \frac{\partial T_D}{\partial x} = -\frac{A}{x} - B \frac{T_D}{\sqrt{t_D - \frac{x - x_w}{2V}}} \quad (B10)$$

$$T(x = x_w, t_D) = -1 \quad (B11)$$

$$T(x = 2V t_D + x_w, t_D) = 0 \quad (B12)$$

Appendix C

The system of Eqs. (B10–B12) is solved using the method of characteristics (Polyanin & Zaitsev, 2003).

Eq. (B10) in characteristic form is:

$$\frac{dt_D}{dx} = \frac{1}{2V} \quad (C1)$$

$$\frac{dT_D}{dx} + \frac{B}{2V} \frac{T_D}{\sqrt{t_D - \frac{x-x_w}{2V}}} = -\frac{A}{2Vx} \quad (C2)$$

As it follows from Eq. (C1), the characteristic curves of Eq. (B10) are straight lines:

$$t_D = \frac{x - x_w}{2V} + c_1 \quad (C3)$$

Solving ordinary differential equation (ODE) in Eq. (C2) by the method of integrating factor, we obtain a general form of temperature profile:

$$T_D(x, t_D) = c_2 e^{-\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x} - \frac{A}{2V} e^{-\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x} Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x\right) \quad (C4)$$

Solution in Domain I: For the region behind the temperature front, $t_D > \frac{x - x_w}{2V}$ the characteristic lines originate from the injection well. Substituting the boundary condition from Eq. (B11) into Eq. (C4) allows determination of the integration constant in this domain as:

$$c_2 = -e^{\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w} + \frac{A}{2V} Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w\right) \quad (C5)$$

Substituting Eq. (C5) into the general form in Eq. (C4), we obtain the temperature profile for the domain behind the temperature front:

$$T_D(x, t_D) = \left\{ \frac{A}{2V} \left[Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w\right) - Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x\right) \right] - e^{\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w} e^{-\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x} \right\} \quad (C6)$$

For the unperturbed region ahead of the thermal front, $t_D > \frac{x - x_w}{2V}$, the reservoir temperature remains at its initial state, i.e.,

$$T_D(x, t_D) = 0 \quad (C7)$$

The overall reservoir temperature profile in linear coordinates as:

$$T_D(x, t_D) = \begin{cases} \left\{ \frac{A}{2V} \left[Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w\right) - Ei\left(\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x\right) \right] - e^{\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x_w} e^{-\frac{B}{2V\sqrt{t_D - \frac{x-x_w}{2V}}}x} \right\}, & t_D > \frac{x - x_w}{2V} \\ 0, & t_D < \frac{x - x_w}{2V} \end{cases} \quad (C8)$$

By expressing x in terms of r_D from Eq. (B9) and substituting it into Eq. (C8), the dimensionless radial solution (C9) is obtained. Subsequently, substituting the dimensionless variables from Eqs. (B3–B6) into Eq. (C9) yields the corresponding dimensional solution given in Eq. (11).

$$T_D(r_D, t_D) = \begin{cases} \left[\frac{A}{2V} \left[Ei \left(\frac{B}{2V \sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_{Dw}^2 \right) - Ei \left(\frac{B}{2V \sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_D^2 \right) \right] - e^{-\frac{B}{2V \sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_{Dw}^2} \right] e^{-\frac{B}{2V \sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_D^2}, & t_D > \frac{r_D^2 - r_{Dw}^2}{2V} \\ 0, & t_D < \frac{r_D^2 - r_{Dw}^2}{2V} \end{cases} \quad (C9)$$

Appendix D

In this section, we validate the NSS heat-exchange model described by Eq. (2). This is achieved by comparing the temperature profile within the reservoir (Eq. C9) to the temperature at the interface between the reservoir and the surrounding formations.

Vertical heat conductivity in the adjacent formations is given in eq (A1).

The initial condition for the vertical layers is set equal to the reservoir's initial temperature:

$$T(z, t = 0) = T_I \quad (D1)$$

At the reservoir–formation interface, a Neumann boundary condition is applied, representing the heat flux at $z = 0$:

$$\gamma \frac{dT}{dz}(z = 0, t) = Q(t) \quad (D2)$$

Introducing the following dimensionless variables into Eqs. (A1), (D1–D2):

$$T_D = \frac{T - T_I}{T_I - T_J}; t_D = \frac{t}{t_j}; z_D = \frac{z}{\sqrt{at_j}}; \quad (D3)$$

The equations governing vertical heat conduction are expressed in dimensionless form as follows:

$$\frac{\partial T_D}{\partial t_D} = \frac{\partial^2 T_D}{\partial z_D^2} \quad (D4)$$

$$T_D(z_D, t_D = 0) = 0 \quad (D5)$$

$$\frac{dT_D}{dz_D}(z_D = 0, t_D) = -\sqrt{\frac{1}{\pi}} \frac{T_D}{\sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} \quad (D6)$$

$$\frac{dT_D(z_D = 0, t_D)}{dz_D} = \begin{cases} -\sqrt{\frac{1}{\pi}} e^{-\frac{B}{2V\sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_D^2} \left[\frac{A}{2V} \left[Ei\left(\frac{B}{2V\sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_{Dw}^2\right) - Ei\left(\frac{B}{2V\sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_D^2\right) \right] - e^{-\frac{B}{2V\sqrt{t_D - \frac{r_D^2 - r_{Dw}^2}{2V}}} r_{Dw}^2} \right], & t_D > \frac{r_D^2 - r_{Dw}^2}{2V} \\ 0, & t_D < \frac{r_D^2 - r_{Dw}^2}{2V} \end{cases} \quad (D7)$$

In Eq. (D7), the Neumann boundary condition captures the heat-flux dynamics resulting from temperature changes within the reservoir, as derived in Eq. (C9). Following Polyanin and Nazaikinskii (2016), a general solution to the problem defined by Eqs. (D4–D7) is expressed as:

$$T(z, t) = \frac{1}{2\sqrt{\pi at}} \int_0^\infty \left\{ e^{-\frac{(z-\xi)^2}{4at}} + e^{-\frac{(z+\xi)^2}{4at}} \right\} f(\xi) d\xi - \sqrt{\frac{a}{\pi}} \int_0^t \frac{e^{-\frac{z^2}{4a(t-\tau)}}}{\sqrt{t-\tau}} g(\tau) d\tau \quad (D8)$$

In this expression, $f(\xi)$ denotes the initial condition, while $g(\tau)$ represents the heat flux applied at the boundary. By substituting the variables from Eqs. (D4–D7) into Eq. (D8), the exact solution for vertical heat conduction is obtained:

$$T_D(z_D, t_D) = \begin{cases} \frac{1}{\pi} \int_0^{t_D} \frac{e^{-\frac{z_D^2}{4(t_D-\tau)}}}{\sqrt{t_D-\tau}} \left\{ \frac{e^{-\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V}}} \left[\frac{A}{2V} \left[Ei\left(\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_{Dw}^2\right) - Ei\left(\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_D^2\right) \right] - e^{-\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_{Dw}^2} \right] \right\} d\tau & , t_D > \frac{r_D^2 - r_{Dw}^2}{2V} \\ 0 & , t_D < \frac{r_D^2 - r_{Dw}^2}{2V} \end{cases} \quad (D9)$$

To determine the vertical temperature profile at the interface between the reservoir and the surrounding formation, we set $z_D = 0$ in Eq. (D9):

$$T_D(z_D = 0, t_D) = \begin{cases} \frac{1}{\pi} \int_0^{t_D} \frac{1}{\sqrt{t_D-\tau}} \left\{ \frac{e^{-\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V}}} \left[\frac{A}{2V} \left[Ei\left(\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_{Dw}^2\right) - Ei\left(\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_D^2\right) \right] - e^{-\frac{B}{2V\sqrt{\tau-\frac{r_D^2-r_{Dw}^2}}}{2V} r_{Dw}^2} \right] \right\} d\tau & , t_D > \frac{r_D^2 - r_{Dw}^2}{2V} \\ 0 & , t_D < \frac{r_D^2 - r_{Dw}^2}{2V} \end{cases} \quad (D10)$$

Comparing the reservoir temperature profile (Eq. C9) with the interface temperature profile (Eq. D10) at $r_D = 0.002$ enables the identification of the validity range for the heat-exchange model. The vertical interface temperature profile is denoted as $T_{adj}(z_D, t_D)$, while the reservoir temperature profile is represented as $T_{res}(r_D, t_D)$.