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Delft Institute of Applied Mathematics

Powerfactor improvement for a multiple frequency voltage source

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"Powerfactor improvement for a multiple frequency voltage source"

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Preface

This document is the result of a research by Gidius van de Kamp under the supervision of Jacob van der Woude. I hope this research helps providing new insights and asks relevant questions for future study.
First of all I thank my supervisor Jacob van de Woude for his help and feedback. Due to the Corona virus pandemic we had to find alternative ways to communicate instead of meeting in person. We had many useful Skype meetings, thank you very much.
The thesis of C. A. Bryan named *Compensator Design Extending Van der Woude-Jeltsema's orthogonal projection approach* was very interesting, as it asked the research questions I tried to answer. I would like to thank her for her research.
Because my thesis is written from an applied mathematics point of view, my report will start with some background in the field of physics.

1 Problem description

The power factor is a tool providing insight in the efficiency of an electrical circuit. The power factor is used in the design of Railway Power Flow Control System for instance [4]. The aim of this thesis is to study the power factor and compare different controllers to improve performance of the power factor. Power is the product of the current and the voltage.
The power factor is defined as a fraction of two powers.

$$PF = \frac{|P|}{S}. \tag{1}$$

The power factor PF depends on the powers P and S. P is the power consumed by the load called the active or real power, and S is the power which is provided by the source called the apparent power. A low PF means there is a transport of electricity not necessary for the load. The focus of the thesis is to find a lossless compensator to improve the PF for a simple RLC network as shown in figure 1. The compensator has to change the shape of the current in order to improve the power factor. The optimal shape will have the same shape as the voltage source.
In this report we will study the power factor in a network with an alternating current with multiple frequencies. This means we have a current and voltage that changes over time. We will note these as functions of time; $I(t)$ for the current and $V(t)$ for the voltage. $I_c(t)$ for the current we want to compensate in the controller. This thesis will not focus on finding this current, but on how to remove the unwanted current. The unwanted current is called reactive current.

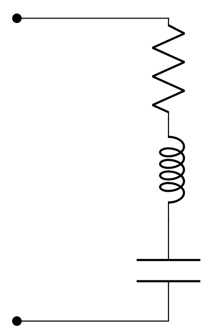


Figure 1: simple RLC-circuit

2 Functioning of a RLC netork

A RLC network consists of 3 elements that influence the current. These elements are called resistor, coil (or inductor) and a capacitor. All these elements depend on a value with an different unit. To note these values we use R for the resistor, L for a coil and C for the capacitor, hence the name RLC-network.

- The unit for the resistor is called Ohm (Ω).
- The unit for the coil is called Henry (H).
- The unit for the capacitor is called Farad (F)

The coil and capacitor are lossless, they influence the shape of the current but do not reduce it, whereas the resistor changes the electricity to heath. The change of current in a coil is described in the following equation:

$$V(t) = L\frac{I(t)}{dt}. \tag{2}$$

The change of current in a capacitor is described in the following equation:

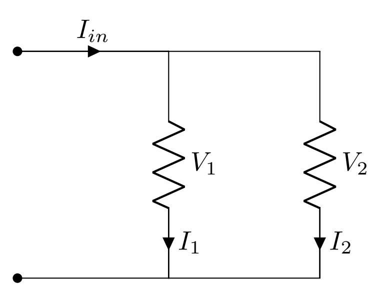
$$\frac{dV(t)}{dt} = CI(t). \tag{3}$$

The change of current in a resistor is described in the following equation:

$$V(t) = RI(t). \tag{4}$$

2.1 Kirchhoff

With the two laws of Kirchhoff one can find the current in a circuit consisting of multiple elements. One law states that in every node the current entering is equal to the current leaving. The other law states that the voltages in every loop together in a network is equal to zero. For both laws you have to take the direction of the electricity into account. [1](p.37-43)
For instance in this circuit:



The laws would give:

$$I_{in}(t) = I_1(t) + I_2(t) \tag{5}$$
$$V_1(t) - V_2(t) = 0. \tag{6}$$

2.2 Laplace transform

The Laplace transform is found to be useful for studying RLC circuits because of the differentiation and integration properties of the transformation. The transformation is from the t domain/ time domain to the s domain/ frequency domain. The Laplace transform is given by:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt. \tag{7}$$

One can transform the equations for the coil, capacitor and resistor and will obtain:

$$V(s) = sLI(s) \tag{8}$$

$$sV(s) = CI(s) \tag{9}$$

$$V(s) = RI(s) \tag{10}$$

These equations look more simple in the s domain than in the t domain.[1] In these equations we assumed the following initial conditions

$$V(0) = 0, I(0) = 0. \tag{11}$$

3 Possible improvements

The disadvantage of a small power factor is that there is a flow of electricity not used in the power consumed by the load. This is especially a problem for networks having a resistor in the cables to the actual load. In order to quantify what could be improved, we look at these two circuits:

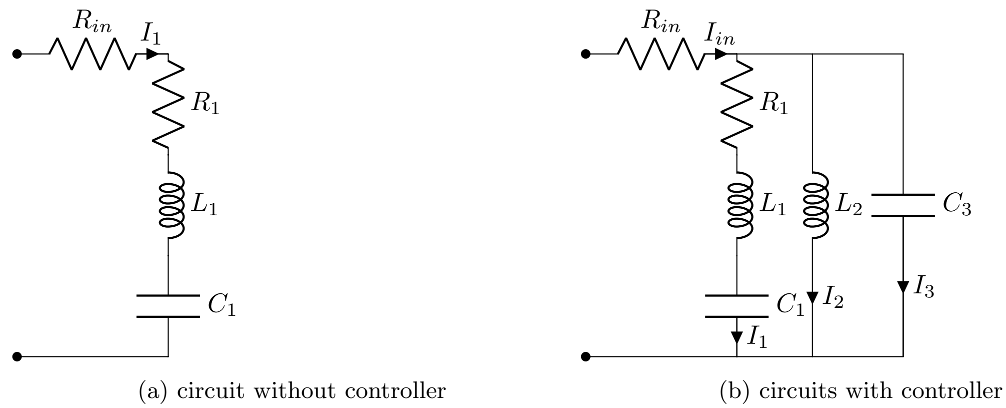


Figure 2: circuits that will studied

The circuit in figure 2a is a simple RLC-circuit with an extra resistor to simulate the loss in a cable. In figure 2b is the same RLC-circuit but with an added compensator, as proposed by Jacob van der Woude and Dimitri Jeltsema. [5]

3.1 Model

We will calculate the current in the two circuits from figure 2. To do this the Kirchhoff laws are useful.

3.1.1 Model without controller

For the circuit in figure 2a we have the following equations;

$$V_{in} = I_1(R_{in} + R_1) + \frac{Q_1}{C_1} + L_1\frac{dI_1}{dt}. \tag{12}$$

Q is the Electric charge in the condensator. The unit for the electric charge is called Coulomb (C).

$$\frac{dQ_1}{dt} = I_1. \tag{13}$$

Written as a first order system of differential equations, gives:

$$\frac{dI_1}{dt} = \frac{1}{L_1}V_{in} - \frac{1}{L_1}I_1(R_{in} + R_1) - \frac{Q_1}{C_1L_1} \tag{14}$$

$$\frac{dQ_1}{dt} = I_1. \tag{15}$$

If you use;

$$Q_1 = \int I_1 dt \tag{16}$$

you can solve the current with the method of undetermined coefficients.

3.1.2 Model with controller

For the circuit in figure 2b we have more equations. We obtain the following equations;

$$L_1\frac{dI_1}{dt} + I_1R_1 + \frac{Q_1}{C_1} = \frac{Q_3}{C_3} \tag{17}$$

$$\frac{dQ_1}{dt} = I_1 \tag{18}$$

$$\frac{dQ_3}{dt} = I_3 = I_{in} - I_1 - I_2 = \frac{V_{in} - \frac{Q_3}{C_3}}{R_{in}} - I_1 - I_2 = \frac{V_{in}}{R_{in}} - \frac{Q_3}{C_3R_{in}} - I_1 - I_2 \tag{19}$$

$$L_2\frac{dI_2}{dt} = \frac{Q_3}{C_3}. \tag{20}$$

Written as a first order system of differential equations, gives;

$$\frac{dI_1}{dt} = -\frac{I_1R_1}{L_1} - \frac{Q_1}{C_1L_1} + \frac{Q_3}{C_3L_1} \tag{21}$$

$$\frac{dQ_1}{dt} = I_1 \tag{22}$$

$$\frac{dQ_3}{dt} = \frac{V_{in}}{R_{in}} - \frac{Q_3}{C_3 R_{in}} - I_1 - I_2 \tag{23}$$

$$\frac{dI_2}{dt} = \frac{Q_3}{C_3 L_2}. \tag{24}$$

In this circuit the method of undetermined coefficients does not seem to be useful.

3.2 PF as function of Rin

With the models found in the previous section we can find the PF for different values of R_{in}. We will do this using a method based on complex Fourier coefficients described in *An Orthogonal Projection Method for Computing Active, Reactive, and Scattered Power and its Application to Compensator Design* by Jacob van der Woude and Dimitri Jeltsema. From now on referred to as the Fourier method. In this method one writes a current as a vector with complex numbers. With these vectors and the system of equations for the current one can find the current by solving one matrix equation.

We study the relation of the PF and the R_{in}, for the following cases;

case 1:

$$R_1 = 1, L_1 = \frac{1}{2}, C_1 = \frac{2}{3}, \text{ with or without controller: } L_2 = \frac{4}{3}, C_3 = \frac{1}{4}. \tag{25}$$

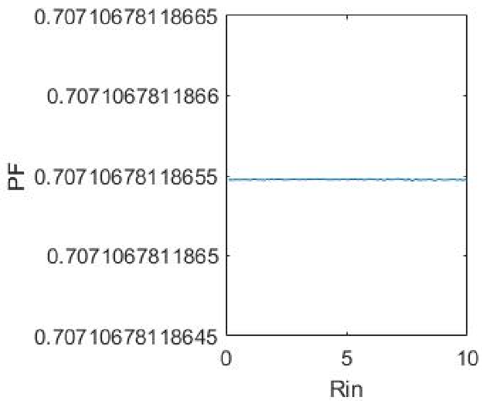
case 2:

$$R_1 = 1, L_1 = \frac{1}{2}, C_1 = \frac{2}{7} \text{ with or without controller: } L2 = \frac{20}{9}, C3 = \frac{3}{30}. \tag{26}$$

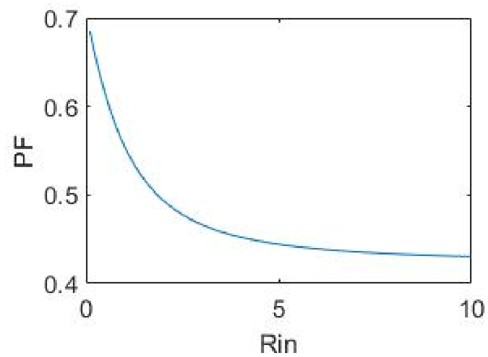
With the source voltage given by:

$$V(t) = \sqrt{2}(100\sin(t) + 100\cos(3t)). \tag{27}$$

These controllers are designed for the circuit without the R_{in} element. In figure 3 and 4 we see the relation between the PF and the value of R_{in}. For case 1 there seems to be no influence of the R_{in} element. The outcome here is the same as in the case when there is no R_{in} at all. However in case 2 there seems to be a influence of R_{in} on the PF. In figure 3b we see that for case 2 without controller the PF seems to decrease for increasing R_{in}. For case 2 with controller the PF decreases first and then increases. The controller seems not to work as great in case 2 as it does in case 1. There seems to be something special about case 1.

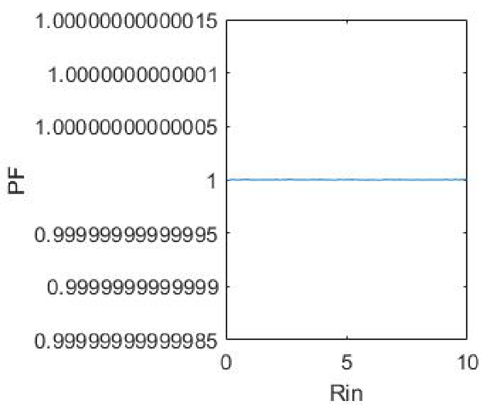


(a) PF for case 1.

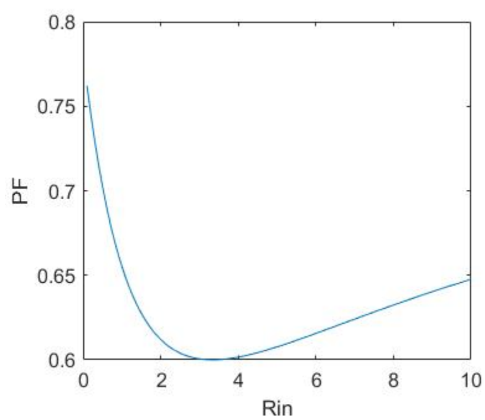


(b) PF for case 2.

Figure 3: Relation PF and R_{in} without a controller.



(a) PF for case 1.



(b) PF for case 2.

Figure 4: Relation PF and R_{in} with a controller.

4 Two frequency voltage

4.1 Transfer function

The transfer function is a useful analytic tool for finding the response in a system. We will use the Laplace transformation to find this function. We know the input, in this case the voltage, and the wanted output, being the current we want to remove. Dividing the input and output in the s domain gives a fraction which is called the transfer function. When dividing the voltage by the current one obtains a function that is called the impedance ($Z(s)$). When we divide the current by the voltage, the function is called admittance ($Y(s)$).[1] We will find the transfer function for a two frequency voltage. In section 5.4 we will see how this transfer function is realisable with a network.

$$V(t) = b_1 \cos(a_1 t) + b_2 \cos(a_2 t) \tag{28}$$

$$\mathcal{L}[V(t)] = V(s) = b_1 \frac{s}{s^2 + a_1^2} + b_2 \frac{s}{s^2 + a_2^2} = \frac{b_2 s(s^2 + a_1^2) + b_1 s(s^2 + a_2^2)}{(s^2 + a_2^2)(s^2 + a_1^2)} \tag{29}$$

$$v^{(1)}(t) = -b_1 a_1 \sin(a_1 t) - b_2 a_2 \sin(a_2 t) \tag{30}$$

$$v^{(3)}(t) = b_1 a_1^3 \sin(a_1 t) + b_2 a_2^3 \sin(a_2 t) \tag{31}$$

We can write the current we want to remove as a linear combination of the odd derivatives of the voltage.[5]

$$I_c(t) = \alpha v^{(1)}(t) + \beta v^{(3)}(t) \tag{32}$$

$$= (\beta b_1 a_1^3 - \alpha b_1 a_1) \sin(a_1 t) + (\beta b_2 a_2^3 - \alpha b_2 a_2) \sin(a_2 t). \tag{33}$$

$$I_c(s) = \mathcal{L}[I_c(t)] = (\beta b_1 a_1^3 - \alpha b_1 a_1) \frac{a_1}{s^2 + a_1^2} + (\beta b_2 a_2^3 - \alpha b_2 a_2) \frac{a_2}{s^2 + a_2^2} \tag{34}$$

$$= (\beta b_1 a_1^3 - \alpha b_1 a_1) \frac{a_1(s^2 + a_2^2)}{(s^2 + a_1^2)(s^2 + a_2^2)} + (\beta b_2 a_2^3 - \alpha b_2 a_2) \frac{a_2(s^2 + a_1^2)}{(s^2 + a_1^2)(s^2 + a_2^2)} \tag{35}$$

$$= \Big((\beta b_1 a_1^3 - \alpha b_1 a_1) a_1 (s^2 + a_2^2) + (\beta b_2 a_2^3 - \alpha b_2 a_2) a_2 (s^2 + a_1^2) \Big) \frac{1}{(s^2 + a_1^2)(s^2 + a_2^2)}. \tag{36}$$

$$Y(s) = \frac{I_c(s)}{V(s)} = \frac{(\beta b_1 a_1^3 - \alpha b_1 a_1) a_1 (s^2 + a_2^2) + (\beta b_2 a_2^3 - \alpha b_2 a_2) a_2 (s^2 + a_1^2)}{b_2 s(s^2 + a_1^2) + b_1 s(s^2 + a_2^2)}. \tag{37}$$

$$Z(s) = \frac{V(s)}{I_c(s)} = \frac{b_2 s(s^2 + a_1^2) + b_1 s(s^2 + a_2^2)}{(\beta b_1 a_1^3 - \alpha b_1 a_1) a_1 (s^2 + a_2^2) + (\beta b_2 a_2^3 - \alpha b_2 a_2) a_2 (s^2 + a_1^2)}. \tag{38}$$

We see that the numerator of $Z(s)$ is a polynomial, made of only odd powers of s. The denominator is a polynomial of only even powers of s. For $Y(s)$ we see the opposite. This seems to be a nice property. This will be explained in the next section.

4.2 Positive real function

The desired form of the transfer function is called positive real. This will guarantee that a configuration of inductors and capacitors that realises the transfer function actually exists. In Ch. 7 of *Passive Active and Digital Filters* we find a characterisation of positive real [3]:

A rational function represented in the form

$$F(s) = \frac{P(s)}{Q(s)} = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} \tag{39}$$

where $m_1(s), m_2(s)$ and $n_1(s), n_2(s)$ are the even and odd parts of the polynomials P(s) and Q(s), respectively, is positive real if and only if the following conditions are satisfied:

1. $F(s)$ is real when s is real.
2. $P(s) + Q(s)$ is strictly Hurwitz.
3. $m_1(j\omega)m_2(j\omega) - n_1(j\omega)n_2(j\omega) \geq 0$ for all ω.

These three conditions are important to check. For the transfer function, $Z(s)$, which we found in Ch 4.1, we have $m_1 = 0$ and $n_2 = 0$. This means that condition 3 is satisfied. Condition 1 is also satisfied. Condition 2 seems a bit more complicated. Also in chapter 7 we find: [3]

For $P(s) + Q(s)$ to be strictly Hurwitz, it is necessary and sufficient that the continued-fraction expansion

$$\left[\frac{m_1(s) + m_2(s)}{n_1(s) + n_2(s)}\right]^{\pm 1} = \alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{1}{\ddots + \cfrac{1}{\alpha_k s}}} \tag{40}$$

yields only real and positive α's, and does not terminate prematurely, i.e. k must equal the degree $m_1(s) + m_2(s)$ or $n_1(s) + n_2(s)$ whichever is larger.

For the $Z(s)$ we found in Ch4.1, we have;

$$Y(s) = \frac{1}{Z(s)} = \frac{m_1(s) + m_2(s)}{n_1(s) + n_2(s)}, \tag{41}$$

again for the fact that $m_1 = 0$ and $n_2 = 0$. For this $Z(s)$ condition 2 is exactly the same as having positive coefficients in the first Cauer form. If $Z(s)$ forms a partial faction decomposition with positive elements and does not terminate prematurely then so does $Z(s)^{-1}$. This means that if the voltage is described as multiple cosines and the unwanted current is removable, it is always possible with the first Cauer form. It also tells us that the first Cauer form will have at least the same amount of elements as the highest power found in $Z(s)$.

4.3 Why not cosine and sine?

If the voltage was not written as the sum of cos and sin, this fraction would lose some wanted properties. say:

$$V(t) = b_1 \sin(a_1 t) + b_2 \sin(a_2 t), \tag{42}$$

then:

$$\mathcal{L}[V(t)] = V(s) = b_1 \frac{a_1}{s^2 + a_1^2} + b_2 \frac{a_2}{s^2 + a_2^2}. \tag{43}$$

$$V^{(1)}(t) = b_1 a_1 \cos(a_1 t) + b_2 a_2 \cos(a_2 t). \tag{44}$$

$$V^{(3)}(t) = -b_1 a_1^3 \cos(a_1 t) - b_2 a_2^3 \cos(a_2 t). \tag{45}$$

$$I_c(t) = \alpha V^{(1)}(t) + \beta V^{(3)}(t). \tag{46}$$

$$I_c(t) = (\alpha b_1 a_1 - \beta a_1^3 b_1) \cos(a_1 t) + (\alpha b_2 a_2 - \beta a_2^3 b_2) \cos(a_2 t). \tag{47}$$

$$\mathcal{L}[I_c(t)] = I_c(s) = (\alpha b_1 a_1 - \beta a_1^3 b_1) \frac{s}{s^2 + a_1^2} + (\alpha b_2 a_2 - \beta a_2^3 b_2) \frac{s}{s^2 + a_2^2}. \tag{48}$$

which gives:

$$\frac{I_c(s)}{V(s)} = \frac{(\alpha b_1 a_1 - \beta a_1^3 b_1) s(s^2 + a_2^2) + (\alpha b_2 a_2 - \beta a_2^3 b_2) s(s^2 + a_1^2)}{b_1 a_1 (s^2 + a_2^2) + b_2 a_2 (s^2 + a_1^2)}. \tag{49}$$

It seems there are no problems here, because we have a fraction where the odd powers are in the numerator and the even powers are in the denominator. So here we can find similar results. However a problem arises when:

$$V(t) = b_1 \sin(a_1 t) + b_2 \cos(a_2 t). \tag{50}$$

$$V^{(1)}(t) = a_1 b_1 \cos(a_1 t) - a_2 b_2 \sin(a_2 t). \tag{51}$$

$$V^{(3)}(t) = -a_1^3 b_1 \cos(a_1 t) + a_2^3 b_2 \sin(a_2 t). \tag{52}$$

$$I_c(t) = \alpha V^{(1)}(t) + \beta V^{(3)} \tag{53}$$

$$= (\alpha a_1 b_1 - \beta a_1^3 b_1) \cos(a_1 t) + (\beta a_2^3 b_2 - \alpha a_2 b_2) \sin(a_2 t). \tag{54}$$

$$V(s) = b_1 \frac{a_1}{s^2 + a_1^2} + b_2 \frac{s}{s^2 + a_2^2}(a_2 t). \tag{55}$$

$$I_c(s) = (\alpha a_1 b_1 - \beta a_1^3 b_1) \frac{s}{s^2 + a_1^2} + (\beta a_2^3 b_2 - \alpha a_2 b_2) \frac{a_2}{s^2 + a_2^2}. \tag{56}$$

$$\frac{I_c(s)}{V(s)} = \frac{(\alpha a_1 b_1 - \beta a_1^3 b_1) s(s^2 + a_2^2) + (\beta a_2^3 b_2 - \alpha a_2 b_2) a_2 (s^2 + a_1^2)}{b_1 a_1 (s^2 + a_2^2) + b_2 s (s^2 + a_1^2)}. \tag{57}$$

We see that this fraction loses the wanted property. Theoretically we can still solve this, but it will be more complicated. At first sight one may think a simple solution lays in the fact that:

$$\sin(t) = \cos(t - \frac{\pi}{2}). \tag{58}$$

However transforming this to the Laplace domain gives:

$$\mathcal{L}[\sin(t)] = \mathcal{L}[\cos(x - \frac{\pi}{2})] = e^{\frac{-\pi}{2} s} \frac{s}{s^2 + 1}, \tag{59}$$

where the $e^{-\frac{\pi}{2} s}$, would complicate matters. We could apply the same method just as easily for a sum of sin, but not as easily for a sum of both sin and cos.

4.4 Different controllers

We will study different controllers, that realise the transfer function found in Ch 4.1. The controllers we will study are called:

1. First Foster canonical form
2. Second Foster canonical form
3. First Cauer canonical form
4. Second Cauer canonical form

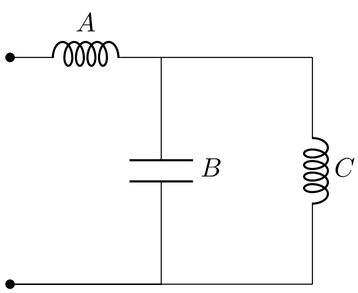
Again we will use chapter 7 from: *Passive Active and Digital Filters* by C. Wai-Kai.[3]
Because we didn't check condition 2 from chapter 4.2 yet, we still have no guarantee that the found elements will be positive. We will express the found elements in the controller in $b_1, a_1, b_2, a_2, \alpha$ and β. Remember that α and β depend on R, L, C of the circuit to be controlled.

4.4.1 First Foster canonical form

If we have a $Z(s)$ of this form:

$$Z(s) = Ds + \frac{Es}{s^2 + F}, \tag{60}$$

then this can be realised with this controller:



Where;

$$A = D, B = \frac{1}{E} \text{ and } C = \frac{E}{F}. \tag{61}$$

If we rewrite $Z(s)$ from Ch4.1 as a partial fraction decomposition, we find:

$$z(s) = \frac{V(s)}{I_c(s)} = \frac{(b_1 + b_2)s}{\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 \alpha b_1 - a_2^2 \alpha b_2} + \tag{62}$$

$$\frac{b_1 b_2 (a_1 - a_2)^2 (a_1 + a_2)^2 (a_1^2 \beta + a_2^2 \beta - \alpha) s}{(b_1 \beta (a_2^2 + s^2) a_1^4 + (\beta a_2^4 b_2 - \alpha (b_1 + b_2) a_2^2 - s^2 b_1 \alpha) a_1^2 + s^2 a_2^2 b_2 (a_2^2 \beta - \alpha))(\beta a_1^4 b_1 - a_1^2 \alpha b_1 + a_2^2 b_2 (a_2^2 \beta - \alpha))} \tag{63}$$

Rewriting gives:

$$Z(s) = \frac{(b_1 + b_2)}{\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 \alpha b_1 - a_2^2 \alpha b_2} s + \tag{64}$$

$$\frac{\frac{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)s}{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2)}}{(a_1^4 a_2^2 b_1\beta + a_1^2 a_2^4 b_2\beta - a_1^2 a_2^2\alpha b_1 - a_1^2 a_2^2\alpha b_2 + (a_1^4 b_1\beta + a_2^4 b_2\beta - a_1^2\alpha b_1 - a_2^2\alpha b_2)s^2)} \tag{65}$$

$$= \frac{(b_1 + b_2)}{\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2} s + \tag{66}$$

$$\frac{\frac{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)s}{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2)^2}}{\frac{a_1^4 a_2^2 b_1\beta + a_1^2 a_2^4 b_2\beta - a_1^2 a_2^2\alpha b_1 - a_1^2 a_2^2\alpha b_2}{(a_1^4 b_1\beta + a_2^4 b_2\beta - a_1^2\alpha b_1 - a_2^2\alpha b_2)} + s^2} \tag{67}$$

This would suggest the controller having the following A, B, C:

$$A = \frac{(b_1 + b_2)}{\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2} \tag{68}$$

$$B = \frac{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2)^2}{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)} \tag{69}$$

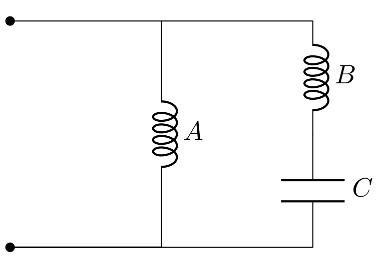
$$C = \frac{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)}{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2\alpha b_1 - a_2^2\alpha b_2)(a_1^4 a_2^2 b_1\beta + a_1^2 a_2^4 b_2\beta - a_1^2 a_2^2\alpha b_1 - a_1^2 a_2^2\alpha b_2)}. \tag{70}$$

4.4.2 Second Foster canonical form

If we have:

$$Y(s) = \frac{1}{Ds} + \frac{Es}{s^2 + F}, \tag{71}$$

then this can be realised with the controller:



Where:

$$A = D, B = \frac{1}{E} \text{ and } C = \frac{E}{F}. \tag{72}$$

If we rewrite $Y(s)$ as a partial fraction decomposition, we find:

$$Y(s) = \frac{a_1^2 a_2^2(a_1^2 b_1\beta + a_2^2 b_2\beta - \alpha b_1 - \alpha b_2)}{(a_1^2 b_2 + a_2^2 b_1)s} + \frac{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)s}{(a_1^2 b_2 + a_2^2 b_1)(a_1^2 b_2 + a_2^2 b_1 + b_1 s^2 + b_2 s^2)}. \tag{73}$$

$$Y(s) = \frac{a_1^2 a_2^2(a_1^2 b_1\beta + a_2^2 b_2\beta - \alpha b_1 - \alpha b_2)}{(a_1^2 b_2 + a_2^2 b_1)s} + \frac{b_1 b_2(a_1^6\beta - a_1^4 a_2^2\beta - a_1^2 a_2^4\beta + a_2^6\beta - a_1^4\alpha + 2a_1^2 a_2^2\alpha - a_2^4\alpha)s}{(a_1^2 b_2 + a_2^2 b_1)^2 + (a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2)s^2}. \tag{74}$$

$$Y(s) = \frac{a_1^2 a_2^2 (a_1^2 b_1 \beta + a_2^2 b_2 \beta - \alpha b_1 - \alpha b_2)}{(a_1^2 b_2 + a_2^2 b_1) s} + \frac{\frac{b_1 b_2 (a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2 a_1^2 a_2^2 \alpha - a_2^4 \alpha) s}{(a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2)}}{\frac{(a_1^2 b_2 + a_2^2 b_1)}{(b_1 + b_2)} + s^2}. \tag{75}$$

This would suggest a controller with A, B, C given by:

$$A = \frac{(a_1^2 b_2 + a_2^2 b_1) s}{a_1^2 a_2^2 (a_1^2 b_1 \beta + a_2^2 b_2 \beta - \alpha b_1 - \alpha b_2)} \tag{76}$$

$$B = \frac{(a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2)}{b_1 b_2 (a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2 a_1^2 a_2^2 \alpha - a_2^4 \alpha) s} \tag{77}$$

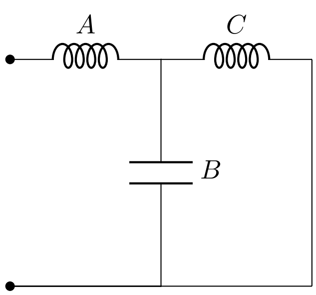
$$C = \frac{b_1 b_2 (a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2 a_1^2 a_2^2 \alpha - a_2^4 \alpha) s}{(a_1^2 b_2 + a_2^2 b_1)^2}. \tag{78}$$

4.4.3 First Cauer canonical form

If we have:

$$Z(s) = As + \cfrac{1}{Bs + \cfrac{1}{Cs}} \tag{79}$$

Than this is realisable with:



If we write $Z(s)$ as a continued fraction, we obtain:

$$Z(s) = \frac{(b_1 + b_2) s}{\beta(a_1^4 b_1 + a_2^4 b_2) - \alpha(a_1^2 b_1 - a_2^2 b_2)} + \tag{80}$$

$$\frac{1}{\frac{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 \alpha b_1 - a_2^2 \alpha b_2)^2 s}{b_1 b_2 (\beta(a_1^6 - a_1^4 a_2^2 - a_1^2 a_2^4 + a_2^6) - a_1^4 \alpha + 2 a_1^2 a_2^2 \alpha - a_2^4 \alpha)} + \frac{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 b_1 \alpha - a_2^2 b_2 \alpha) a_1^2 a_2^2 (a_1^2 b_1 \beta + a_2^2 b_2 \beta - \alpha b_1 - \alpha b_2)}{b_1 b_2 (a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2 a_1^2 a_2^2 \alpha - a_2^4 \alpha) s}}. \tag{81}$$

This leads to the controller with A, B, C given by:

$$A = \frac{(b_1 + b_2)}{\beta(a_1^4 b_1 + a_2^4 b_2) - \alpha(a_1^2 b_1 - a_2^2 b_2)} \tag{82}$$

$$B = \frac{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 \alpha b_1 - a_2^2 \alpha b_2)^2}{b_1 b_2(\beta(a_1^6 - a_1^4 a_2^2 - a_1^2 a_2^4 + a_2^6) - a_1^4 \alpha + 2a_1^2 a_2^2 \alpha - a_2^4 \alpha)} \tag{83}$$

$$C = \frac{b_1 b_2(a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2a_1^2 a_2^2 \alpha - a_2^4 \alpha)}{(\beta a_1^4 b_1 + \beta a_2^4 b_2 - a_1^2 b_1 \alpha - a_2^2 b_2 \alpha) a_1^2 a_2^2 (a_1^2 b_1 \beta + a_2^2 b_2 \beta - \alpha b_1 - \alpha b_2)}. \tag{84}$$

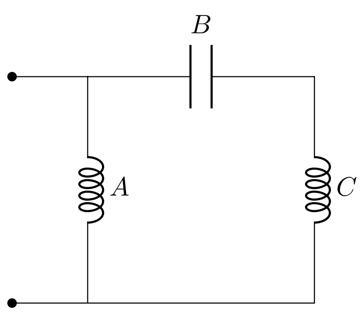
This controller has the same form as the one we found for the first Foster form. It also has the same A, B, C.

4.4.4 Second Cauer canonical form

We can rewrite $z(s)$ in the form:

$$\frac{1}{\dfrac{1}{As} + \dfrac{1}{\dfrac{1}{Bs} + \dfrac{1}{\dfrac{1}{Cs}}}}. \tag{85}$$

This would imply a controller of this form:



With A, B, C given by:

$$A = \frac{a_1^2 b_2 + a_2^2 b_1}{a_1^2 a_2^2 (a_1^2 b_1 \beta + a_2^2 b_2 \beta - \alpha b_1 - \alpha b_2)} \tag{86}$$

$$B = \frac{b_1 b_2(a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2a_1^2 a_2^2 \alpha - a_2^4 \alpha)}{(a_1^2 b_2 + a_2^2 b_1)^2} \tag{87}$$

$$C = \frac{(a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2)}{b_1 b_2(a_1^6 \beta - a_1^4 a_2^2 \beta - a_1^2 a_2^4 \beta + a_2^6 \beta - a_1^4 \alpha + 2a_1^2 a_2^2 \alpha - a_2^4 \alpha)}. \tag{88}$$

4.4.5 Remark

Surprisingly these controllers do not consist of two but do consist of tree elements. One would expect the minimal controller to have the same amount of elements as the frequency. After all, this controller should be the minimal one. The found controller in second Foster canonical form is the same as the controller proposed by C. Bryan in the case $c_1 < 0$, where $I_c(t) = c_1 v^{(1)} + c_2 v^{(-1)}(t)$.[2] An explanation could be that for the voltage of two cos, one has always $c_1 < 0$. However more research is needed before concluding this.

5 R L C influence with a two frequencies voltage

We did not check condition 2 yet, but we learned that this is the same as having positive coefficients in the second Cauer form. Say;

$$V(t) = 100\sqrt{2}(\cos(t) + \cos(2t)). \tag{89}$$

We make two tables to see the amount of negative numbers in the Cauer form, to study the influence of R, L and C on when this method works.This method works when we have only positive numbers for the controller. When $R = 1$, we have the following amount of negative controller elements;

L=1	0	2	2	3	3
L=0.8	0	2	2	2	3
L=0.6	0	2	2	2	2
L=0.4	0	0	2	2	2
L=0.2	0	0	0	2	2
L=0	0	0	0	0	0
	C=0.2	C=0.4	C=0.6	C=0.8	C=1

We do the same for $R = 0.2$, and obtain:

L=1	0	2	2	2	3
L=0.8	0	3	2	2	2
L=0.6	0	0	3	2	2
L=0.4	0	0	0	3	2
L=0.2	0	0	0	0	0
L=0	0	0	0	0	0
	C=0.2	C=0.4	C=0.6	C=0.8	C=1

We can make similar tables for the other controller forms. For the other controllers we see different numbers for the controller elements, but we see the exact same amount of negative numbers for the controller elements as in the tables above.

6 Higher frequency voltage

In theory, we can apply the same approach for higher frequency voltage made from cos, however we will not show this in depth because the coefficients get absurdly large. Say we have a N frequency voltage input:

$$V(t) = \sum_{n=1}^{N} b_n \cos(a_n t). \tag{90}$$

$$\mathcal{L}[V(t)] = V(s) = \sum_{n=1}^{N} b_n \frac{s}{s^2 + a_n^2} = \frac{\sum_{n=1}^{N} b_n s \Pi_{m=1,m\neq n}(s^2 + a_m^2)}{\Pi_{n=1}^{N}(s^2 + a_n^2)}. \tag{91}$$

$$V^{(1)}(t) = \sum_{n=1}^{N} -b_n a_n \sin(a_n t). \tag{92}$$

$$V^{(3)}(t) = \sum_{n=1}^{N} b_n a_n^3 \sin(a_n t). \tag{93}$$

$$V^{(5)}(t) = \sum_{n=1}^{N} -b_n a_n^5 \sin(a_n t). \tag{94}$$

$$\vdots \tag{95}$$

$$I_c(t) = \alpha v^{(1)} + \beta v^{(3)} + \gamma v^{(5)} + \cdots + (-1)^N \omega v^{(2N-1)} \tag{96}$$

$$= \sum_{n=1}^{N} [(-\alpha a_n b_n + \beta a_n^3 b_n - \gamma a_n^5 b_n + \cdots + (-1)^N \omega a_n^{2N-1} b_n) \sin(a_n t)]. \tag{97}$$

$$\mathcal{L}[I(t)] = I(s) = \sum_{n=1}^{N} [(-\alpha a_n b_n + \beta a_n^3 b_n - \gamma a_n^5 b_n + \cdots + (-1)^N \omega a_n^{2N-1} b_n) \frac{a_i}{s^2 + a_i^2}] = \tag{98}$$

$$\Pi_{n=1}^{N}[\frac{1}{s^2 + a_n^2}] \cdot \sum_{n=1}^{N} [a_i(-\alpha a_n b_n + \beta a_n^3 b_n - \gamma a_n^5 b_n + \cdots + (-1)^N \omega a_n^{2N-1} b_n) \Pi_{m=1,m\neq n}^{N}(s^2 + a_m^2)]. \tag{99}$$

$$\frac{I(s)}{V(s)} = \frac{\sum_{n=1}^{N} [a_n(-\alpha a_n b_n + \beta a_n^3 b_n - \gamma a_n^5 b_n + \cdots + (-1)^N \omega a_n^{2N-1} b_n) \Pi_{m=1,m\neq n}^{N}(s^2 + a_m^2)]}{\sum_{n=1}^{N} b_n s \Pi_{m=1,m\neq n}^{N}(s^2 + a_m^2)}. \tag{100}$$

Also for this fraction, above are the even powers of s, and under the odd powers of s. We see that the highest power is $2N + 1$. This means the controller needs at least $2N + 1$ elements.

7 R L C influence with a three frequencies voltage

Again we will study the influence of R, L and C on the amount of positive coefficients found for the elements in the Cauer form. This time we use a 3 frequency voltage given by:

$$V(t) = 100\sqrt{2}(\cos(t) + \cos(2t) + \cos(3t)). \tag{101}$$

Say $R = 1$,

L=1	2	4	4	5	5
L=0.8	2	4	4	5	5
L=0.6	2	4	4	4	5
L=0.4	0	2	4	4	4
L=0.2	0	2	2	4	4
L=0	0	0	0	0	0
	C=0.2	C=0.4	C=0.6	C=0.8	C=1

Say $R = 0.2$,

L=1	2	3	4	4	5
L=0.8	2	3	4	4	4
L=0.6	5	2	3	4	4
L=0.4	0	2	4	5	5
L=0.2	0	0	2	2	2
L=0	0	0	0	0	0
	C=0.2	C=0.4	C=0.6	C=0.8	C=1

Also here we see for the other controllers different values for the controller elements but the same amount of negative values for the controller elements as in the tables. It seems the different compensators share the same influence of the R, L and C. Maybe the different controllers are equivalent.

8 Example

Here we will show the different controllers one can make in the following case:

$$C = 0.4,\ L = 0.2,\ R = 1, \tag{102}$$

with the voltage:

$$v(t) = 100\cos(t) + 50\cos(2t) + 10\cos(3t). \tag{103}$$

The PF without any controller is 0.4989. The optimal PF is 0.8115 The current we want to remove is

$$I_c = \alpha v^{(1)} + \beta v^{(3)} + \gamma v^{(5)} \tag{104}$$

with;

$$\alpha = -0.40783, \beta = -0.042796, \gamma = -0.00063091. \tag{105}$$

First we find the transfer function.

$$Z(s) = \frac{V(s)}{I_c(s)}. \tag{106}$$

$$V(s) = \frac{100s}{s^2+1} + \frac{50s}{s^2+2^2} + \frac{10s}{s^2+3^2}. \tag{107}$$

$$I(s) = \frac{36.566}{s^2+1} + \frac{49.347}{s^2+2^2} + \frac{6.6386}{s^2+3^2}. \tag{108}$$

Gives:

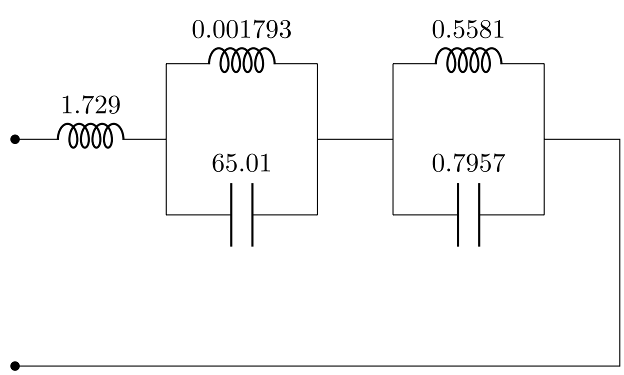
$$Z(s) = \frac{160s^5 + 1850s^3 + 4090s}{92.551s^4 + 1002.0s^2 + 1787.1}. \tag{109}$$

8.1 First Foster canonical form

$Z(s)$ as a partial fraction decomposition is:

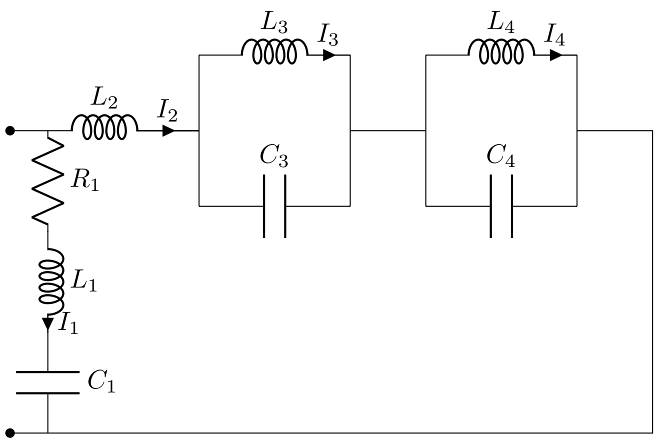
$$1.7288s + \frac{0.015383s}{s^2+8.5748} + \frac{1.2568s}{s^2+2.2518}. \tag{110}$$

Leads to the controller:



8.1.1 PF first Foster form controller

To obtain the improved PF we look for the current in the following circuit. We do this with the Fourier method.



The equations for this network are:

$$V_{in} = I_1 R_1 + \frac{dI_1}{dt} L_1 + \frac{Q_1}{C_1} \tag{111}$$

$$\frac{dQ_1}{dt} = I_1 \tag{112}$$

$$V_{in} = L_2 \frac{dI_2}{dt} + \frac{Q_3}{C_3} + \frac{Q_4}{C_4} \tag{113}$$

$$\frac{dQ_3}{dt} = I_2 - I_3 \tag{114}$$

$$\frac{dQ_4}{dt} = I_2 - I_4 \tag{115}$$

$$L_3 \frac{dI_3}{dt} = \frac{Q_3}{C_3} \tag{116}$$

$$L_4 \frac{dI_4}{dt} = \frac{Q_4}{C_4}. \tag{117}$$

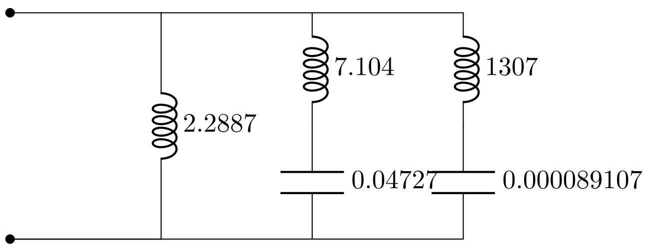
After implementing this in MATLAB we obtain a PF of 0.8115.

8.2 Second Foster canonical form

$1/Z(s)$ as a partial fraction decomposition is:

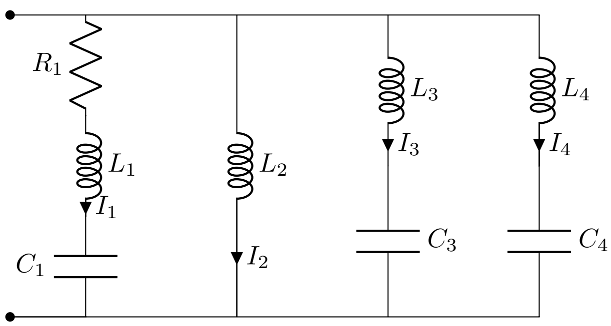
$$\frac{0.43693}{s} + \frac{0.14075s}{s^2 + 2.9776} + \frac{0.00076498s}{s^2 + 8.5849}. \tag{118}$$

Which leads to the controller:



8.2.1 PF second Foster canonical form

To obtain the improved PF we look for the current in the following circuit. We do this with the Fourier method.



The equations for this network are:

$$V_{in} = I_1 R_1 + \frac{dI_1}{dt} L_1 + \frac{Q_1}{C_1} \tag{119}$$

$$\frac{dQ_1}{dt} = I_1 \tag{120}$$

$$V_{in} = L_2 \frac{dI_2}{dt} \tag{121}$$

$$V_{in} = L_3 \frac{dI_3}{dt} + \frac{Q_3}{C_3} \tag{122}$$

$$\frac{dQ_3}{dt} = I_3 \tag{123}$$

$$V_{in} = L_4 \frac{dI_4}{dt} + \frac{Q_4}{C_4} \tag{124}$$

$$\frac{dQ_4}{dt} = I_4. \tag{125}$$

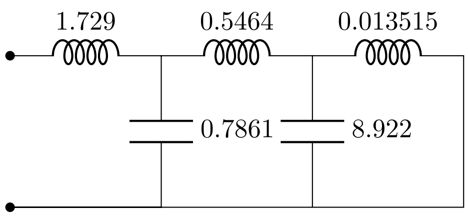
After implementing these equations in MATLAB we obtain PF= 0.8115.

8.3 First Cauer canonical form

$Z(s)$ as a simple continued fraction is:

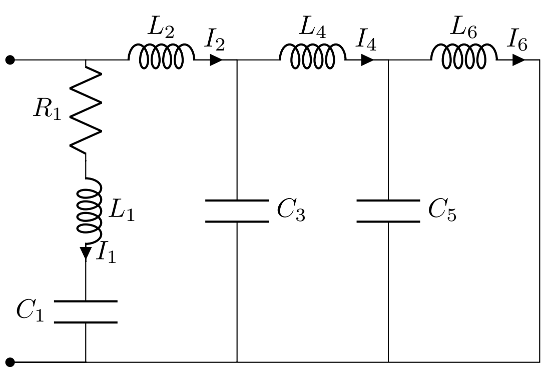
$$1.7288s + \cfrac{1}{0.78606s + \cfrac{1}{0.54640s + \cfrac{1}{8.9218s + \cfrac{1}{0.013515s}}}}. \tag{126}$$

Which leads to the controller:



8.3.1 PF first Cauer canonical form

To obtain the improved PF we look for the current in the following circuit. We do this with the Fourier method.



The equations for this network are:

$$V_{in} = I_1R_1 + \frac{dI_1}{dt}L_1 + \frac{Q_1}{C_1} \tag{127}$$

$$\frac{dQ_1}{dt} = I_1 \tag{128}$$

$$V_{in} = L_2\frac{dI_2}{dt} + \frac{Q_3}{C_3} \tag{129}$$

$$\frac{dQ_3}{dt} = I_2 - I_4 \tag{130}$$

$$\frac{Q_3}{C_3} = L_4 \frac{dI_4}{dt} + \frac{Q_5}{C_5} \tag{131}$$

$$\frac{dQ_5}{dt} = I_4 - I_6 \tag{132}$$

$$\frac{Q_5}{C_5} = L_6 \frac{dI_6}{dt}. \tag{133}$$

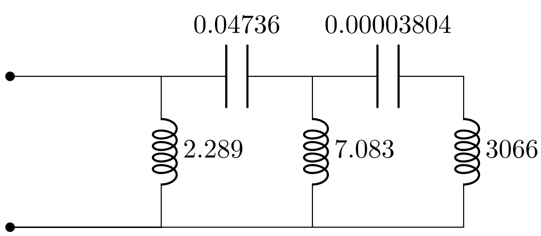
After implementing these equations in MATLAB with the Forier method we obtain PF= 0.8115.

8.4 Second Cauer canonical form

$Z(s)$ as a simple continued fraction of $1/s$ in stead of s is:

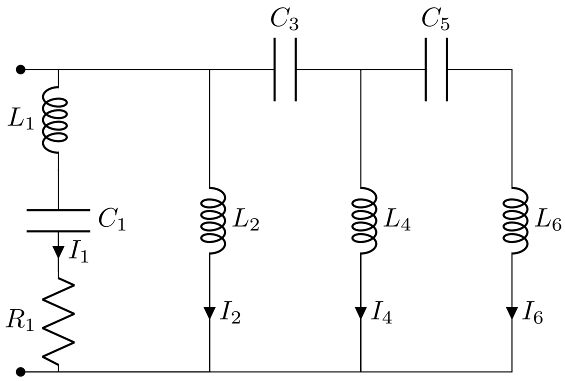
$$0.43693\frac{1}{s} + \cfrac{1}{21.116\frac{1}{s} + \cfrac{1}{0.14119\frac{1}{s} + \cfrac{1}{26291\frac{1}{s} + \cfrac{3066.2}{\frac{1}{s}}}}}. \tag{134}$$

Which leads to the controller:



8.4.1 PF second Cauer canonical form

To obtain the improved PF we look for the current in the following circuit. We do this with the Fourier method.



The equations for this network are:

$$V_{in} = L_1 \frac{dI_1}{dt} + \frac{Q_1}{C_1} + R_1 I_1 \tag{135}$$

$$\frac{dQ_1}{dt} = I_1 \tag{136}$$

$$V_{in} = L_2 \frac{dI_2}{dt} \tag{137}$$

$$V_{in} = \frac{Q_3}{C_3} + L_4 \frac{dI_4}{dt} \tag{138}$$

$$\frac{dQ_3}{dt} = I_4 + I_6 \tag{139}$$

$$\frac{dQ_5}{dt} = I_6 \tag{140}$$

$$V_{in} = L_6 \frac{dI_6}{dt} + \frac{Q_3}{C_3} + \frac{Q_5}{C_5} \tag{141}$$

After implementing these equations in MATLAB with the Forier method we obtain PF$= 0.8115$.

9 Discussion

In this thesis I did not focus on the initial conditions for the elements in the original circuit or in the controller. However we do know that these can be of importance.
In the example we saw that the values for the different forms differ as well. One could study which form controller is more efficient from an economical perspective.
By expressing the α and β in R, L and C their influence on the controllers could be studied in more depth. Personally, I think this will be interesting.
The restriction of only looking at voltage made of multiple *cos* is a restriction in usefulness. Maybe the problem without this restriction can be solved by studying the characteristics of positive real functions in depth. One could also study the problem with a different characteristic of positive real.

10 Conclusion

In cases that the voltage source is made of multiple cos's and we can compensate the current, it is always possible with the first Cauer form.
This is caused by the shape of the found transfer function $Z(s)$. This transfer function is a fraction where the odd powers of s are in the numerator, while in the denominator are only even powers of s. In this way the conditions 1 and 3 in chapter 4.2 are satisfied. For this special type of $Z(s)$ satisfying condition 2, is having positive coefficients in the First Cauer form, hence the conclusion.
In the tables in this thesis, it seems that when the first Cauer form works, the other forms are working as well.
It seems that the method used in this thesis to find a controller only works for small R, L and C.
In the example the improved PF is optimal, like one would expect.
I conclude that the method used in this thesis is working fine.

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