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Lost periodicity in N-continued fraction expansions

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Introduction

When we look at the decimal expansion of a real number this tells us something about the nature of that number. An eventually periodic expansion means that we have a rational number $\frac{p}{q} \in \mathbb{Q}$, and a non-periodic expansion means that we are dealing with an irrational number $r \in \mathbb{R}\backslash\mathbb{Q}$.

When we are looking at regular continued fraction expansions of numbers we can find a similar classification. A finite expansion means that we have a rational number $\frac{p}{q} \in \mathbb{Q}$. However, the irrational numbers $r \in \mathbb{R}\backslash\mathbb{Q}$ can be devided into two categories. We have regular continued fractions with an eventually periodic expansion, which means that we have a quadratic irrational, while a non-periodic expansion will mean that we are dealing with any irrational number $r \in \mathbb{R}\backslash\mathbb{Q}$ that is not a quadratic irrational.

But when dealing with N-continued fractions, which are a simple varriation of the regular continued fractions, we see a change in this classification. In this case it is not proven that all quadratic irrationals will have an eventually periodic expansion. This means that the N-continued fractions, while being build up in a very similar way as regular continued fractions, might have significant different properties. This thesis will not give a proof for the conjecture that there are quadratic irrationals without an eventually periodic N-continued fraction expansion, but it will try to make it plausible that this conjecture holds true.

In chapter 1 we will give a brief intorduction to continued fractions, and scetch the prove of why quadratic irrationals always have an eventully periodic regular continued fraction expansion. In chapter 2 we introduce the N-continued fraction expansion and end with stating the goal of my thesis. Finally chapter 3 gives some examples of how to construct the 2-continued fraction from the regular continued fraction. Here Raney's transducers are introduced, playing a key role in the reason why periodicity is probably lost in some cases. This is backed up by the statistical properties derived from computer computations, which are compared with properties yielded by the underlying ergodic system for the 2-expansion.

Contents

1 Regular Continued Fractions **7**
1.1 Continued fraction expansion . . . 7
1.2 The convergents $\frac{p_n}{q_n}$ . . . 8
1.3 Quadratic irrationals and periodicity . . . 9

2 N-continued fractions **11**
2.1 Modifying the original case . . . 11
2.2 New convergents $\frac{p_n}{q_n}$ . . . 11
2.3 Quadratic irrationals . . . 12

3 2-continued fraction expansion **13**
3.1 Simple approach . . . 13
3.2 Raney . . . 14
3.3 Examples . . . 15
3.4 Ergodic Properties . . . 16
3.5 Implementations . . . 17
3.6 Results . . . 18
3.7 Conclusion . . . 19

4 Appendix **21**
4.1 Using Raneys transducers in detail . . . 21
4.2 Approximating frequencies of the digits for some quadratic irrationals . . . 24
4.3 Matlab files used . . . 25

1 Regular Continued Fractions

1.1 Continued fraction expansion

When we write a real number x as a *regular continued fraction* or *RCF* it has the following form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}} \tag{1.1}$$

The numbers a_i are sometimes called the partial quotients, here simply called digits. The digits are restricted by the following: $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}_{\geq 1}$ for all $i \geq 1$. Furthermore, a rational number $\frac{p}{q} \in \mathbb{Q}$ has two finite RCF expansions, so $\frac{p}{q} = [a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$. Although both expressions represent the same number it is custom to use the first, and thus shorter, notation. A number $x \in \mathbb{R}\backslash\mathbb{Q}$ has an infinite RCF expansion, so we write $x = [a_0; a_1, a_2, \ldots]$.

In this thesis we are only interested in numbers $x \in [0, 1)$. We can make this restriction without loss of generality, because if we have a number y that lies outside the interval $[0, 1)$ and we want to find its RCF expansion we see that $y = \lfloor y \rfloor + (y - \lfloor y \rfloor) = a_0 + x$ for a certain $x = y - \lfloor y \rfloor \in [0, 1)$ and $a_0 = \lfloor y \rfloor$. So when we want to find the RCF of a number y we can always assume that the number we are actually interested in is a number $x \in [0, 1)$.

To find the RCF of a given number x we introduce the operator $T : [0, 1) \to [0, 1)$ wich is defined as:

$$T(x) := \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & ; x \in (0, 1) \\ 0 & ; x = 0 \end{cases}$$

Setting $a_1 := \left\lfloor \frac{1}{x} \right\rfloor$ we find:

$$x = \frac{1}{a_1 + T(x)}$$

Now if $T(x) \neq 0$ we let T work on the number $T(x)$, setting $a_2 := \left\lfloor \frac{1}{T(x)} \right\rfloor$ we find:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2(x)}}$$

Repeating this iterative process yields the expression:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{n-1} + \cfrac{1}{a_n + T^n(x)}}}} \tag{1.2}$$

Due to Euclid's Algorithm for rational numbers we find for certain $N \in \mathbb{N}$ that $T^N(x) = 0$. At that moment our iterative process stops and the complete RCF is constructed. For irrational numbers this iterative process will continue infinitely.

When we approximate a number $x = [a_0; a_1, a_2, \ldots]$ by computing only its first n digits and cut off the RCF (giving us a rational number) we call this the nth convergent of x and denote this by: $x_n = [a_0; a_1, a_2, \ldots, a_n]$. In fact one has that $x_n \to x$ as $n \to \infty$, which explains the limit notation (1.1). In the next section we will explain why and also how *fast* x_n converges to x as $n \to \infty$.

1.2 The convergents $\frac{p_n}{q_n}$

A 2×2 matrix can operate as a Möbius transformation on a number x as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{a \cdot x + b}{c \cdot x + d}$$

Which is also known as a fractional linear transformation. When we calculate the first n digits of a certain x we can define n matrices A_i by:

$$A_i := \begin{bmatrix} 0 & 1 \\ 1 & a_i \end{bmatrix}$$

When we let A_n operate on $T^n(x)$ we get the following expression:

$$A_n(T^n(x)) = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}(T^n(x)) = \frac{0 \cdot T^n(x) + 1}{1 \cdot T^n(x) + a_n} = \frac{1}{a_n + T^n(x)}$$

When we let A_{n-1} operate on $A_n(T^n(x))$ this yields:

$$A_{n-1}(A_n(T^n(x))) = \begin{bmatrix} 0 & 1 \\ 1 & a_{n-1} \end{bmatrix}\left(\frac{1}{a_n + T^n(x)}\right) = \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + T^n(x)}}$$

Repeating this process will give us:

$$A_1 A_2 \cdots A_n(T^n(x)) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; a_{n-1} + \cfrac{1}{a_n + T^n(x)}}}} \tag{1.3}$$

and from (1.2) we see that this representation equals x, therefore:

$$x = A_1 A_2 \cdots A_n(T^n(x)) \tag{1.4}$$

Now define the following n matrices by: $M_i := A_1 A_2 \cdots A_i$. We than find that: $M_n = A_1 A_2 \cdots A_n$ and $M_{n-1} = A_1 A_2 \cdots A_{n-1}$, so we see that: $M_n = M_{n-1} A_n$.

Matrix M_n can be written as: $M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}$, and because we have $M_n = M_{n-1} A_n$ we get the following equation:

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} = \begin{bmatrix} p_{n-1} & r_{n-1} + a_n p_{n-1} \\ q_{n-1} & s_{n-1} + a_n q_{n-1} \end{bmatrix}$$

This yields: $r_n = p_{n-1}$ en $s_n = q_{n-1}$, using this we also find:

$$\begin{aligned} p_n &= r_{n-1} + a_n p_{n-1} = p_{n-2} + a_n p_{n-1} \\ q_n &= s_{n-1} + a_n q_{n-1} = q_{n-2} + a_n q_{n-1} \end{aligned}$$

So we find that:

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}$$

With p_n and q_n given by the recurrence relation:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned} \tag{1.5}$$

However, to start this recurrence relation we need the values of p_0, p_1, q_0 and q_1. These we find using $M_1 = A_1$:

$$M_1 = A_1 \quad \Rightarrow \quad \begin{bmatrix} p_0 & p_1 \\ q_0 & q_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} p_0 = 0, & p_1 = 1 \\ q_0 = 1, & q_1 = a_1 \end{matrix}$$

The nth convergent is given by equation (1.3) when we choose $T^n(x)$ to be equal to 0. This gives us the following expression for x_n:

$$x_n = A_1 A_2 \cdots A_n(0) = M_n(0) = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}(0) = \frac{0 \cdot p_{n-1} + p_n}{0 \cdot q_{n-1} + q_n} = \frac{p_n}{q_n}$$

So one of the properties of the numbers p_n and q_n is that:

$$x_n = \frac{p_n}{q_n} \tag{1.6}$$

Now we can look at how fast x_n converges to x, because from (1.4) we see that:

$$x = M_n(T^n(x)) = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} (T^n(x)) = \frac{T^n(x) \cdot p_{n-1} + p_n}{T^n(x) \cdot q_{n-1} + q_n} \tag{1.7}$$

Combining (1.13) and (1.7) we can find an expression for their difference:

$$\begin{aligned} |x - x_n| &= \left| \frac{T^n(x) \cdot p_{n-1} + p_n}{T^n(x) \cdot q_{n-1} + q_n} - \frac{p_n}{q_n} \right| \\ &= \left| \frac{q_n(T^n(x) \cdot p_{n-1} + p_n) - p_n(T^n(x) \cdot q_{n-1} + q_n)}{q_n(T^n(x) \cdot q_{n-1} + q_n)} \right| \\ &= \left| \frac{T^n(x)(q_n p_{n-1} - p_n q_{n-1}) + q_n p_n - p_n q_n}{q_n^2(T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1)} \right| \\ &= \left| \frac{T^n(x)(-1)^n}{q_n^2(T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1)} \right| \\ &= \frac{T^n(x)}{q_n^2 \left(T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1\right)} \end{aligned} \tag{1.8}$$

In the third step we used that: $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$. This result can easily be checked because:

$$q_n p_{n-1} - p_n q_{n-1} = \det(M_n) = \det(A_1 \cdots A_n) = \det(A_1) \cdot \ldots \cdot \det(A_n) = -1 \cdot \ldots \cdot -1 = (-1)^n$$

Because $T^n(x) < 1$ and $\left|T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1\right| \geq 1$ we can conclude from (1.8) that:

$$|x - x_n| < \frac{1}{q_n^2} \tag{1.9}$$

Now we can say something about the speed of convergence. We know that q_n is given by $q_n = a_n q_{n-1} + q_{n-2}$ where $q_0 = 1$ and $q_1 = a_1$. So the sequence $\{q_n\}_{n=1}^{\infty}$ grows the slowest when all a_i equal 1, givng us the sequence $1, 1, 2, 3, 5, 8, \ldots$ which is the Fibonacci sequence. So for any SCF we have that $q_n \geq F_n$ with F_n the nth Fibonacci number, therefore from (1.9) we see that:

$$|x - x_n| < \frac{1}{F_n^2} \tag{1.10}$$

And from (1.10) we can conclude that x_n converges exponentially fast to x, thus:

$$x_n \to x \quad \text{as} \quad n \to \infty$$

1.3 Quadratic irrationals and periodicity

A *quadratic irrational* is a solution $x \in \mathbb{R}\backslash\mathbb{Q}$ of a quadratic equation $ax^2 + bx + c = 0$ having $a, b, c \in \mathbb{Z}$. In fact we could take $a, b, c \in \mathbb{Q}$, but this would give us the same equations because we can always multiply the equation by a number $N \in \mathbb{N}\backslash\{0\}$ such that Na, Nb, Nc become integers again.

One surprising property of RCFs is the following theorem by Lagrange:

Theorem 1.3.1 *x is a quadratic irrational $\Leftrightarrow$ the regular continued fraction of x is eventually periodic*

We say that a RCF is *(eventually) periodic* when it has the form: $x = [a_0; a_1, a_2 \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+m}}]$, where the period is given by the sequence below the bar.

For the complete proof of Theorem 1.3.1 I would like to refer to [3], but here I will give you the idea for the proof of the implication from left to right. To prove that a quadratic irrational always gives a periodic RCF we will will use equation (1.4) and the fact that $A_1 A_2 \ldots A_n = M_n$. These two relations give the following equation:

$$x = M_n(T^n(x)) = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} (T^n(x)) = \frac{p_{n-1}T^n(x) + p_n}{q_{n-1}T^n(x) + q_n} \tag{1.11}$$

Using recurrence relation (1.5), this can be written as:

$$x = \frac{p_{n-1}T^n(x) + a_n p_{n-1} + p_{n-2}}{q_{n-1}T^n(x) + a_n q_{n-1} + q_{n-2}} = \frac{(T^n(x) + a_n)p_{n-1} + p_{n-2}}{(T^n(x) + a_n)q_{n-1} + q_{n-2}} \tag{1.12}$$

In a RCF $x = [a_0; a_1, a_2, \ldots, a_n, a_{n+1}, \ldots]$ we say that $a'_n := [a_n, a_{n+1}, \ldots]$ and call a'_n the *nth complete quotient* of the RCF. This a'_n is a RCF with a value greater than 1, having a_n as it's integer part and $\frac{1}{a'_{n+1}}$ as it's fractional part.
Because $\frac{1}{a'_{n+1}} = T^n(x)$ we can say that: $a'_n = a_n + T^n(x)$, so that equation (1.12) simplifies to:

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}} \tag{1.13}$$

Knowing that x is a quadratic irrational there are $a, b, c \in \mathbb{Z}$ such that x is the solution of the equation $ax^2 + bx + c = 0$. When we sutstitute (1.13) into this equation we find a new quadratic equation for a'_n:

$$A_n a'^2_n + B_n a'_n + C_n = 0 \tag{1.14}$$

where A_n, B_n, C_n are sums of products of $a, b, c, p_{n-1}, p_{n-2}, q_{n-2}, q_{n-2}$. This means that $A_n, B_n, C_n \in \mathbb{Z}$ again, so therefore a'_n is also a quadratic irrational.

Now in [3] some estimations for the absolute values of A_n, B_n, C_n are given and from this it is concluded in [3] that all of the A_n, B_n, C_n are bounded by a finite number $N \in \mathbb{N}$, and that this upper bound is independent of n. This means that there is only a limited amount of values the numbers A_n, B_n, C_n can take, and because there are infinitely many triplets (A_n, B_n, C_n) there must be at least one triplet (A, B, C) that occurs at least three times. So we find for certain $n_1 < n_2 < n_3$ that: $(A, B, C) = (A_{n_1}, B_{n_1}, C_{n_1}) = (A_{n_2}, B_{n_2}, C_{n_2}) = (A_{n_3}, B_{n_3}, C_{n_3})$, for wich we can find roots $a'_{n_1}, a'_{n_2}, a'_{n_3}$. But because a quadratic equation only has 2 different roots this means that at least two of those three roots are the same, therefore for example we have: $a'_{n_1} = a'_{n_2}$.
Now let $k = n_2 - n_1$ so that $a'_{n_1} = a'_{n_1+k}$, looking back at the definition of a'_n we find that:

$$a_{n_1} = a_{n_1+k},\ a_{n_1+1} = a_{n_1+1+k},\ \ldots,\ a_{n_1+k} = a_{n_1+2k},\ \ldots$$

So the RCF of a quadratic irrational is periodic.

2 N-continued fractions

Apart from the regular continued fraction expansion there are very many other continued fraction algorithms. In this thesis we consider a new algorith which is – to some extend – an easy and obvious generalization of the RCF. In spite of this, it turns out that some of the properties of the RCF might not hold for these new CF algorithms.

2.1 Modifying the original case

In this thesis we will look at a special kind of continued fractions. Let $N \in \mathbb{N}$ and $N \geq 1$, the new continued fraction is created with an operator $T_N : [0,1) \to [0,1)$ as follows:

$$T_N(x) := \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor & : x \in (0,1) \\ 0 & : x = 0 \end{cases}$$

Note that the case $N = 1$ is the RCF. For $x \in [0,1)\backslash\mathbb{Q}$, we than have, following the same steps as in the previous chapter:

$$x = \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \cfrac{N}{a_4 + \cfrac{N}{\ddots}}}}}$$

We will call this the *N-continued fraction expansion* or *N-CF* of x and write it in a shorter fashion as $x = [0; a_1, a_2, a_3, \ldots]_N$. Because for $x \in (0,1)$ we have that $\left\lfloor \frac{N}{x} \right\rfloor \geq N$ the N-CF has the property that all $a_i \geq N$.

Here again, for a number $x \in \mathbb{Q}$ the N-CF expansion is finite and for $x \in \mathbb{R}\backslash\mathbb{Q}$ the N-CF expansion will be infinite. When we only calculate the digits a_i for $i \leq n$ we call the number $x_n = [0; a_1, a_2, \ldots, a_n]_N$ the nth convergent of x.

2.2 New convergents $\frac{p_n}{q_n}$

For our N-CF we can find new p_n and q_n again. Given the first n digits we can define n matrices by $A_i := \begin{bmatrix} 0 & N \\ 1 & a_i \end{bmatrix}$. If we now let A_n operate on $T_N^n(x)$ we get the expression:

$$A_n(T_N^n(x)) = \begin{bmatrix} 0 & N \\ 1 & a_n \end{bmatrix}(T_N^n(x)) = \frac{0 \cdot T_N^n(x) + N}{1 \cdot T_N^n(x) + a_n} = \frac{N}{a_n + T_N^n(x)}$$

Following the same procedure as in the previous chapter we find that:

$$A_1 A_2 \cdots A_n(T_N^n(x)) = \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{\ddots \, a_{n-1} + \cfrac{N}{a_n + T_N^n(x)}}}}$$

Again we define the matrices: $M_i := A_1 A_2 \cdots A_i$.

We write M_n as: $M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}$. Using $M_n = M_{n-1}A_n$ we now get the following equation:

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & N \\ 1 & a_n \end{bmatrix} = \begin{bmatrix} p_{n-1} & Nr_{n-1} + a_n p_{n-1} \\ q_{n-1} & Ns_{n-1} + a_n q_{n-1} \end{bmatrix}$$

This yields: $r_n = p_{n-1}$ and $s_n = q_{n-1}$, using this we also find:

$$p_n = Nr_{n-1} + a_n p_{n-1} = Np_{n-2} + a_n p_{n-1}$$
$$q_n = Ns_{n-1} + a_n q_{n-1} = Nq_{n-2} + a_n q_{n-1}$$

In this way we find, given the digits, a recurrence relation for the values of p_n and q_n, this time with starting values:

$$p_0 = 0, \quad p_1 = 2$$
$$q_0 = 1, \quad p_1 = a_1$$

Again the nth convergent is given by the quotient: $x_n = \frac{p_n}{q_n}$. We can check weather this converges to x again. As in the previous chapter we find:

$$|x - x_n| = \left| \frac{T^n(x)(q_n p_{n-1} - p_n q_{n-1}) + q_n p_n - p_n q_n}{q_n^2(T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1)} \right|$$

Because this time, again using the properties derived from the matrices M_n, we have $q_n p_{n-1} - p_n q_{n-1} = (-N)^n$ we find:

$$|x - x_n| = \frac{T^n(x) \cdot N^n}{q_n^2 \left(T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1\right)} \tag{2.1}$$

We still have that $T^n(x) < 1$ and $\left|T^n(x) \cdot \frac{q_{n-1}}{q_n} + 1\right| \geq 1$, we we can conclude from (2.1) that:

$$|x - x_n| < \frac{N^n}{q_n^2} \tag{2.2}$$

We know that q_n is given by $q_n = a_n q_{n-1} + Nq_{n-2}$ where $q_0 = 1$ and $q_1 = a_1$. So the sequence $\{q_n\}_{n=1}^{\infty}$ grows the slowest when all a_i equal N. In case $N = 1$ we get the Fibonacci sequence, for wich we have $\frac{F_n}{F_{n-1}} \to \varphi = \frac{\sqrt{5}+1}{2}$ as $n \to \infty$.

For $N > 1$ we don't get the Fibonacci sequence, so we need to prove that we still have convergence ourselves. To do this we look at the recurrence relation for q_n again, and this time we take all a_i to equal N. So q_n is given by $q_n = Nq_{n-1} + Nq_{n-2}$ where $q_0 = 1$ and $q_1 = N$. This means that for $n > 1$ we have: $q_n > Nq_{n-1}$, and because $q_1 = N$ we have $q_n > N^n$ for $n > 1$. So for $n > 1$ (2.2) becomes:

$$|x - x_n| < \frac{N^n}{(N^n)^2} = \frac{1}{N^n} \tag{2.3}$$

So from (2.3) we can conclude that x_n converges exponentially fast to x as $n \to \infty$.

2.3 Quadratic irrationals

In 2008, the Journal of Number Theory published an article of Ed Burger and some of his students about N-CFs, [1]. In that article they presented the following theorem:

Theorem 2.3.1 *For every real quadratic irrational α, there exist infinitely many integers N for which α can be expressed as a periodic N-continued fraction having period length one.*

This is an interesting theorem, but we can ask ourselves, do we have a periodic N-CF for every quadratic irrational, when N is fixed?

The estimations made in the proof of Theorem 1.3.1 make explicit use of the relation $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$. But in the case of a N-CF we find $p_n q_{n-1} - q_n p_{n-1} = (-N)^n$. This new relation makes that the values of A_n, B_n, C_n for $N \geq 2$ may grow to infinity as n goes to infinity. Therefore we can't say that there is a limited amount of different triplets anymore and the classical proof doesn't work.

In the start-up of my thesis research, Cor Kraaikamp came with the following:

Conjecture 2.3.2 *There are quadratic irrationals that for certain N have a non-periodic N-continued fraction.*

The following chapters will try to make it plausible that this conjecture might hold.

3 2-continued fraction expansion

Initially, we tried for several values of N to keep track of the A_n, B_n and C_n as used in Hardy and Wrights pfoof in [3] of Lagrange's result. Very quickly these values become extremely large, even if we divide by the gcd(A_n,B_n,C_n) in every step. To keep things a little under the lid, we choose N as small as possible; i.e. $N = 2$.

3.1 Simple approach

When we have a quadratic irrational x we can try to find its 2-CF expansion by using its RCF expansion. Given:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}}$$

We can multiply both numerator and denominator of the first fraction by 2, this gives us the following expression:

$$x = \cfrac{2}{2a_1 + 2\cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}} = \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}}$$

Now we multiply the most upper numenator that is not a 2 already and it's denominator by 2 again, yielding:

$$x = \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2a_3 + 2\cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}} = \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2a_3 + \cfrac{2}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}}$$

Repeating this process will result in the 2-CF expansion:

$$x = \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2a_3 + \cfrac{2}{a_4 + \cfrac{2}{2a_5 + \cfrac{2}{a_6 + \cfrac{2}{\ddots}}}}}}} \tag{3.1}$$

So we find that $x = [0; a_1, a_2, a_3, a_4, \ldots]_1 = [0; 2a_1, a_2, 2a_3, a_4, \ldots]_2$. But using this method a problem can arise. From our operator T_2 we saw that all the digits in a 2-CF should be 2 at least. But using the method described above the digits a_n with n even will not be modified. So when the RCF expansion has 1 as it's nth digit (with n even) then the CF expansion given by (3.1) can not be correct. So apparently we need to find a way for dealing with expressions like:

$$2 \cdot \cfrac{1}{1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}} \tag{3.2}$$

When we wouldn't have looked at the 2-CF expansion but at the N-CF expansion for $N \geq 3$ we would be in the same kind of situation, but than not only 1 would be a *forbidden* digit, but every other integer smaller than N would give rise to the same problem. But tackling the problem of multiplying a RCF by, for example, 7 takes a lot more work than multiplying it by 2.

To find a way to multiply a RCF by 2 we introduce Raney's Transducers.

3.2 Raney

In [5] A. Hurwitz published a method to multiply a RCF by 2. Although we could have used this result, I use a more general method described by George N. Raney in [7].

In short, Raney's trandsucers have as its imput a number x written in its RCF expansion and as output the RCF expansion of, for example, n times x, where n depends on which transducer you use. We are interesed in multiplying a RCF by 2 so we can use the simplest transducer.

They work as follows: first you convert the RCF into a so called R-L-word. This R-L-word is then used as input for an automata, wich gives us another R-L-word as output. Finally this output is converted back into a RCF, which is 2 times the RCF we started with.

The R-L-word is a string consisting of R's and L's only. To convert a RCF into such an R-L-word we look at the first digit of the RCF and write as many R's as the digits value. After this we look at the second digit and write as many L's as that digits value. The third digit tells us how many R's we get afther this, the forth how many L's and so on. To keep the notation short we write R^2 in stead of RR etc. So for example the RCF $[b_0; b_1, b_2, b_3, b_4, \ldots]$ would be converted into the word: $R^{b_0}L^{b_1}R^{b_2}L^{b_3}R^{b_4}\ldots$.

Now that we have the R-L-word our next step is using this word as the transducers input, so we get a new word as its output. How this works is described by Figure 3.1:

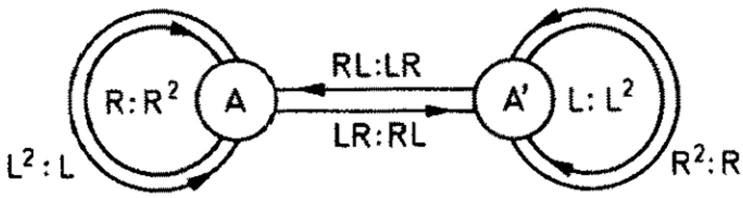

Figure 3.1: Raneys transcuser for $N = 2$, as presented in [7].

We start in state A and look at the first letter in the L-R-word, when we have a R as imput we receive R^2 as output, but when we have a L we need to use another letter to get an output. When this second letter is an L aswell we receive a single L as output, but when the second letter is a R we get RL as its output and we move to state A'. In this state we have a similar situation, a single L as input gives us L^2 as output. When we have a R we need to combine it with the next letter, another R gives a single R as output, a L as second letter gives LR as output and we are send back to state A again.

Although our input is an infinite word, we don't need to compile the complete word. This is because we certainly get a periodic output, we are constructing the RCF expansion of 2 times a quadratic irrational, wich is still a quadratic irrational, thus having a periodic RCF expansion. So we only need to take enough terms to find the periodicity of our output.

Once we know what the output is we can convert this new R-L-word back into a RCF expansion in the same way as we got the R-L-word in the first case, so: $R^{c_0}L^{c_1}R^{c_2}L^{c_3}R^{c_4}\ldots$ would be converted into $[c_0; c_1, c_2, c_3, c_4, \ldots]$. The continued fraction we started with represented a number $x \in (\frac{1}{2}, 1)$, so our output will be a number

$2x \in (1,2)$, therefore we already know that $c_0 = 1$. So by using Raney's transducers we can now rewrite expression (3.2) as:

$$2 \cdot \cfrac{1}{1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}} = 1 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots}}}} \tag{3.3}$$

3.3 Examples

Our first example will use the positive solution of the quardatic equation $2\alpha^2 + 2\alpha - 1 = 0$, which is $\alpha = \frac{\sqrt{3}-1}{2} \approx 0.366$. The RCF of α is given by $[0; \overline{2,1}]_1$. To find its 2-CF we cannot simply use expression (3.1), because $a_2 = 1$, and we already saw that we are not allowed to leave that digit unchanged. So to find its 2-CF we will need to make use of Raney's transducer. But first we try to get as far as possible the simple way:

$$x = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} = \cfrac{2}{4 + 2\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} \tag{3.4}$$

So we need to find the RCF of $2 \cdot [0; \overline{1,2}]_1$, here I will only give the result, a detailed derivation is available in the appendix.

$$2 \cdot [0; \overline{1,2}]_1 = [1; \overline{2,6}]_1 \tag{3.5}$$

Filling in the result from (3.5) into (3.4) we can further compute the 2-CF:

$$x = \cfrac{2}{4 + 1 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{\ddots}}}}}} = \cfrac{2}{5 + \cfrac{2}{4 + \cfrac{2}{6 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{\ddots}}}}}} = \cfrac{2}{5 + \cfrac{2}{4 + \cfrac{2}{6 + \cfrac{2}{4 + \cfrac{2}{6 + \cfrac{1}{\ddots}}}}}} \tag{3.6}$$

We now see that, because 1 no longer occurs as a digit in the part that's still written as a RCF, we can continue indefinitely with multiplying the numerator and denominator by 2. So we find that the 2-CF of α is given by: $\alpha = [0; 5, \overline{4,6}]_2$. And it is periodic again.

As a second example we use the positive solution to the quadratic equation $3x^2 + 8x - 7 = 0$, which is $x = \frac{\sqrt{37}-4}{3} \approx 0.69$. The RCF of x is given by $[0; \overline{1,2,3}]_1$. To find its 2-CF we again cannot simply use expression (3.1), because $a_4 = 1$. So to find its 2-CF we will need to make use of Raney's transducer. But first we try to get as far as possible the simple way:

$$x = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}}}} = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{6 + 2\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}}}} \tag{3.7}$$

So we need to find the RCF of $2 \cdot [0; \overline{1,2,3}]_1$, here I will only give the result, a detailed derivation is available in the appendix.

$$2 \cdot [0; \overline{1,2,3}]_1 = [1; \overline{2,1,1,2,1,7}]_1 \tag{3.8}$$

Filling in the result from (3.8) into (3.7) we can compute the 2-CF a little bit further:

$$x = \cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{6+1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}} = \cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{7+\cfrac{2}{4+2\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}} \tag{3.9}$$

Here again we need to use Raney's transducer, this time to find an expression for $2 \cdot [0; \overline{1, 1, 2, 1, 7, 2}]_1$, again I will only show the result overhere:

$$2 \cdot [0; \overline{1, 1, 2, 1, 7, 2}]_1 = [1; \overline{6, 1, 3, 5}]_1 \tag{3.10}$$

Filling in the result from (3.10) into (3.9) we can again compute the 2-CF a little bit further:

$$x = \cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{7+\cfrac{2}{4+1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}} = \cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{7+\cfrac{2}{5+\cfrac{2}{12+2\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}} \tag{3.11}$$

Again we use Raney's transducer to deal with $2 \cdot [0; \overline{1, 3, 5, 6}]_1$ and find:

$$2 \cdot [0; \overline{1, 3, 5, 6}]_1 = [1; \overline{1, 1, 10, 3, 2, 1, 1, 1, 2, 13}]_1 \tag{3.12}$$

Going on with our proces we need to compute $2 \cdot [0; \overline{1, 10, 3, 2, 1, 1, 1, 2, 13, 1}]_1$ and find that:

$$2 \cdot [0; \overline{1, 10, 3, 2, 1, 1, 1, 2, 13, 1}]_1 = [1; \overline{1, 4, 1, 1, 1, 5, 3, 1, 27, 21, 1, 1, 2, 4, 2, 1, 6, 3}]_1 \tag{3.13}$$

Repeating this proces we will find that the length of the new periods rapidly increase, while they stay rich of 1's.

The reason for this is because while results as (3.10) don't change the period length too much we also have cases like (3.8) and (3.12). In the case of (3.8) we have an uneven period length, this means that to find periodicity in the new R-L-word we need to take at least 2 of the old periods in order to find a loop, because while starting with an L in the first period, the second period starts with an R, so this usually doesn't give the same output. In the case of (3.12) we do have an even period length, but while we build the new R-L-word and start with the first period in state A, we find that we end our first period in state A', so the second period won't give the same result as the first, and we need to use the second period to find a periodic output.

So in those two cases the new period lengt fluctuates around the doubble of the old period length. This makes it plausible that while using the transducer we only get longer and longer new periods and the proces never terminates, while getting into a loop only happens when the process terminates at a certain moment. So therefore I think that this number does not have a periodic 2-CF. Of course this is not a proof, but it does make Conjecture (2.3.2) more likely to be correct.

3.4 Ergodic Properties

In this section I will recall an old result and use this to give another indication that the 2-continued fraction expansion of $x = [0; \overline{1, 2, 3}]_1$ found in the previous section is probably not periodic. First I need to introduce some concepts from the ergodic theory. A probability space is given by a set X, a σ-algebra $\mathcal{F}$ and a measure μ.

Definition Let $(X, \mathcal{F}, \mu)$ be a probability space. A measurable transformation $T : X \to X$ is *measure preserving* with respect to μ, if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{F}$.

In our case X is the interval $[0,1)$ and $\mathcal{F}$ is the collection of all Borel measurable subsets $A \subset [0,1)$. In 1800 Gauss stated that in modern terminology T is measure preserving with respect to the *Gauss measure*:

$$\mu(A) := \frac{1}{\log 2} \int_0^1 f_A(x) \frac{dx}{1+x} \qquad \text{for} \qquad A \in \mathcal{F} \tag{3.14}$$

where $f_A(x)$ is the indicator function of A, i.e.:

$$f_A(x) = \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases} .$$

Using the Gauss measure we can determine for $n \in \mathbb{N}$ how often it occurs in the continued fraction expansion of a number $x \in [0,1)$. Lets use the following definition to make clear what we mean by *how often*:

Definition The *asymptotic frequenty* ϕ of digit n is given by:

$$\phi(n) := \lim_{m\to\infty} \frac{\sum_{k=1}^m I_n(a_k)}{m}$$

with I_n the indicator function: $I_n(a_k) = \begin{cases} 1 & : a_k = n \\ 0 & : a_k \neq n \end{cases}$.

Apart from some set of measure 0 we can use the Gauss measure to find the asymptotic frequenty of the digits of almost every x by using the Ergodic Theorem (see [2] theorem 3.1.7).

The Gauss measure μ was made for the RCF. To be able to use an expression like (3.14) for the N-CF we will need the following measure:

$$\mu_N(A) := \frac{1}{\log(\frac{N+1}{N})} \int_0^1 f_A(x) \frac{dx}{N+x} \qquad \text{for} \qquad A \in \mathcal{F} \tag{3.15}$$

Now let $N = 2$ and $n \in \mathbb{N}$ with $n \geq 2$. Let A_n such that for $x \in A_n$ we find $\lfloor \frac{2}{x} \rfloor = n$. This means that $n \leq \frac{2}{x} < n+1$ and thus $A_n = (\frac{2}{n+1}, \frac{2}{n}]$. To find the asymptotic frequenty of n we now calculate $\mu_2(A_n)$. Using equation (3.15) and only integrating over the area where f_{A_n} is not zero we find:

$$\begin{aligned} \phi(n) &= \frac{1}{\log(\frac{3}{2})} \int_{\frac{2}{n+1}}^{\frac{2}{n}} \frac{dx}{x+2} \\ &= \frac{1}{\log(\frac{3}{2})} \cdot \left(\log\left(\frac{2}{n}+2\right) - \log\left(\frac{2}{n+1}+2\right)\right) \\ &= \frac{1}{\log(\frac{3}{2})} \cdot \log\left(\frac{\frac{2}{n}+2}{\frac{2}{n+1}+2}\right) \\ &= \frac{\log\left(\frac{2n^2+4n+2}{2n^2+4n}\right)}{\log(\frac{3}{2})} \end{aligned} \tag{3.16}$$

Expression (3.16) gives us the asymptotic frequenties for all $n \geq 2$ in case we are dealing with the CF expansion of a typical number. Now we can compare this result with the 2-CF expansion of the number $x = [0; \overline{1,2,3}]_1$. But before we can do this we first need to calculate a lot of it's digits in order to be able to say things about its behaviour as the amount of digits tends to infinity.

3.5 Implementations

In order to calculate a lot of digits in a fast manner we first looked at the quadratic forms of the quadratic irrationals. Having a number x that satisfies: $A_1x^2 + B_1x + C_1 = 0$ we can use the computer to approximate x precisely enough to be able to calculate $a_1 := \lfloor \frac{N}{x} \rfloor$. Now we can calculate A_2, B_2, C_2 by taking sums of products of A_1, B_1, C_1 and a_1 such that $T_N(x)$ satisfies: $A_2T_N(x)^2 + B_2T_N(x) + C_2 = 0$. Because A_2, B_2, C_2 are sums of products of integers, they themselves are integers as well, so that the calculations are accurate enough with the computer in most cases. Now that we have a quadratic form for $T_N(x)$ we can find a precisely enough approximation for $T_N(x)$ in order to calculate $a_2 := \left\lfloor \frac{N}{T_N(x)} \right\rfloor$. And using A_2, B_2, C_2 and a_2 we can find a quadratic form for $T_N^2(x)$ and so on.

When we use this to calculate the digits for a RCF we know that the absolute values of the numbers A_n, B_n, C_n are bounded by a certain $N_0 \in \mathbb{N}$ for all n. For the N-CF with $N \geq 2$ we can not say this anymore. However, in some cases the numbers A_n, B_n, C_n (possibly devided by their GCD) are limited and we easily find the N-CF expansion for this number. In other cases the numbers A_n, B_n, C_n grow wild, and while using the computer we get roundoff errors because the integers get bigger than the computer precision can handle. In this case the quadratic forms are not accurate anymore and the digits we find are probably wrong.

Although this program turned out to be useless for finding the digits of the N-CF expansion of a quadratic irrational that seemed to be non-periodic, it was usefull for generating a list of quadratic forms for possible non-periodic N-CF expansions. To do this you first use the program to calculate the first 100 or 1000 digits the N-CF for a certain quadtratic irrational. After this you check whether the digits are eventually repeating a certain period or not.

So to deal with the problem of roundoff errors Raney's transducer came in. This was also the moment where we completely focussed on 2-CFs, because the transducer for multiplying by 2 is easier than for multiplying by bigger numbers.

Implementing the program that was going to produce a lot of digits turned out to be harder than exptected. To do this I first implemented a program that has as it's imput the period of the RCF. It would doubble the first digit and check whether the second digit was greater than 1. If this was the case the second digit is left unchanged and the program moves on to the next two digits (placing the two it 'used' at the end of the input period. In case the second digit was a 1 it had to use the period to find (using Raneys transducer) the RCF of two times the input RCF.

While at first this goes quite fast, the period length increases quite fast aswell. So, while using the new periods as input, the time it takes to find the next period after it finds a 1 as the second digit increases. Unfortunately the elapsed time increases exponentially with the period length, therefore the computation times run out of hand, and it's impossible to find a lot of digits in a reasonable time. For example: to find the first 109 digits of the 2-CF of $\frac{\sqrt{37}-4}{3}$ the program needs to run for a whole week!

Because 109 digits are far from enough to be able to talk about ergodic properties I had to edit the program. I did this by giving the program a maximum for the period length it could produce as output. So setting this maximum to, for example 40000, the program would stop using Raneys transducer when it found the whole new period or when the new periods length reached 40000. Because we terminate the process here, not finishing the actual new period we can't use that output as a periodic input anymore. However, this gives us a lot of digits to work with. So the program is given the fact that we don't have a real periodic input anymore, but just a truncation. The next time it needs to use Raneys transducer, it will therefore only compile the part of the given period, and stop when that is finished (or we hit the 40000 digits again). Also, the last computed digit might be wrong, because maybe we ended with a 4 while it had to be a 5 if you also used the next digit in the period (which we never knew). So after it finishes processing the input, the last digit of the output is thrown away. In this way, setting the maximum to 40000 we get about 28000 digits as our final output, and this is enough to be able to say something about its ergodic properties.

3.6 Results

Using the last Matlab program I made, described in the last paragraph of section 3.5, I calculated the first 70825 digits for the 2-CF for the number $x = [0; \overline{1,2,3}]_1 = \frac{\sqrt{37}-4}{3}$. Using our definition for the asymptotic frequenty of a number n we can use these 70825 digits to calculate our (approximate) asymptotic frequenty. The results can be found in Table 3.1.

n	2	3	4	5	6	7	8	9
theoretical using μ	.290489	.159172	.100679	.069478	.050853	.038840	.030638	.024787
$x = \frac{\sqrt{37}-4}{3}$	.291790	.159054	.099583	.069566	.050589	.039322	.031204	.024045

Table 3.1: Found asymptotic frequencies of the digits for $2 \leq n \leq 9$ using 70825 digits.

To verify that we weren't just lucky to find a single number that gets close to the theoretical asymptotic frequencies I picked 4 other quadratic irrationals that (based on the first 60 digits) seemed to be non-periodic. Because calculating more than 70000 digits took about 3.5 days I only calculated the first $\pm$ 28000 digits for those 4, which took about 4 hours. In order to check weather this approximation is much worse than the one with 70825 digits I also included $[0; \overline{1,2,3}]_1$ with only $\pm$ 28000 digits. The results can be found in Table 3.2.

n	2	3	4	5	6	7	8	9
theoretical using μ	.290489	.159172	.100679	.069478	.050853	.038840	.030638	.024787
$\frac{\sqrt{37}-4}{3}$	.292783	.160084	.100767	.069511	.049407	.040385	.031186	.024153
$\frac{\sqrt{569}-23}{10}$	.289744	.161513	.101920	.069210	.048976	.036285	.036284	.025560
$\frac{\sqrt{197}-13}{4}$	.290775	.161334	.101116	.071307	.052130	.037508	.031045	.023098
$\frac{\sqrt{577}-23}{8}$	.290787	.161663	.101862	.066032	.050135	.039584	.031122	.024111
$\frac{\sqrt{993}-9}{24}$	.287611	.158821	.101349	.069258	.051331	.038410	.030813	.025843

Table 3.2: Found asymptotic frequencies of the digits for $2 \leq n \leq 9$ using $\pm$ 28000 digits.

A more detailed table can be found in the Appendix.

Another way to look at these asymptotic frequencies is through a result shown by Khintchine [6]. He showed that for almost every RCF:

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = 1.7454066\ldots \tag{3.17}$$

For the five quadratic irrationals from Table 3.2 we can use the calculated digits to make approximations for Khintchines result for 2-CFs.

Instead of (3.17) we can also devide 1 by the sum over the reciprokes of all integers and multiply each reciproke by the frequency of its corresponding integer. We then get the following equality:

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = \frac{1}{\sum_{n=1}^{\infty} \frac{\phi(n)}{n}} \tag{3.18}$$

Using (3.18) and our theoretical found asymptotic frequencies we can find a result similar to (3.17) for the 2-CF, this gives:

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = 3.704751\ldots \tag{3.19}$$

The results can be found in the following table:

number of interest:	theoretical	$\frac{\sqrt{37}-4}{3}$	$\frac{\sqrt{37}-4}{3}$	$\frac{\sqrt{569}-23}{10}$	$\frac{\sqrt{197}-13}{4}$	$\frac{\sqrt{577}-23}{8}$	$\frac{\sqrt{993}-9}{24}$
number of ditits used:	∞	70825	28154	27974	28315	28245	28171
Khintchine number	3.704751	3.697957	3.686159	3.704431	3.691458	3.698910	3.722844

Table 3.3: Theoreical and numerical found numbers for equation (3.19).

3.7 Conclusion

Now we can come back to Conjecture (2.3.2). As said before, we haven't found any real proof, but we can surely say that this conjecture isn't just based on some guesswork.

In section 3.3, when rewriting the periodic RCF expansions of certain numbers into 2-CF expansions we see that, using Raney's transducers, the periods we work with get bigger and bigger as we calculate more digits. And it seems like we will never get to a point where we find periodicity again.

Also, in sections 3.4 and 3.6 we see that, using computer calculated digits for 2-CF expansions of certain quadratic irrationals, the digits seem to satifsy classical ergodic properties that only work for typical numbers. If the 2-CF expansions were periodic we probably wouldn't have found these results because when the digits follow a certain pattern it is very unlikely that they fit the theoretical asymptotic frequenty of the digits of a typical number so well.

So we can, thanks to the use of the transducers which equates to repeatedly doubling the RCF expansion, and backed up by the statistical properties that follow from the calculated digits of the 2-CF expansion, conclude that Conjecture (2.3.2) in all probability holds.

4 Appendix

4.1 Using Raneys transducers in detail

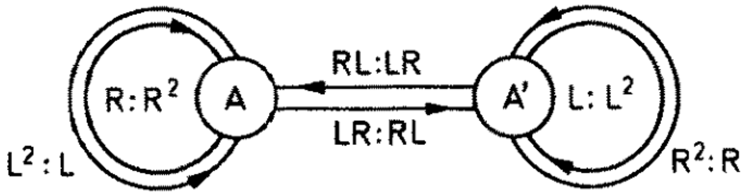

Figure 4.1: Raneys transcuser for $N = 2$, as presented in [7].

Result (3.5): finding $2 \cdot [0; \overline{1, 2}]_1$**.**

First we make the R-L-word:

$$R^0|L^1R^2|L^1R^2|L^1R^2|L^1R^2 \dots$$

Using this as our input we start in state A and the first letter we see is a single L, so we combine it with one R, this gives us as output RL and we go to state A'.

The situation up untill now: we are in state A', have as our input $R^1|L^1R^2|L^1R^2|L^1R^2 \dots$ and our output up untill now is RL.

Now we get the second R, and because we are in state A' we need to combine this with the L next to it to get LR as output and we go back to state A.

The situation up untill now: we are in state A, have as our input $R^2|L^1R^2|L^1R^2 \dots$ and our output up untill now is $RLLR$.

There is R^2 left, so the output will be a R^4.

The situation up untill now: we are in state A, have as our input $L^1R^2|L^1R^2 \dots$ and our output up untill now is $RLLRR^4$.

What we see now is exactely what we had at the very start. So we found that the input $L^1R^2|L^1R^2$ gives us the following output: $RLLRR^4$. Because we are back in state A again the next 4 letters of the word will give exactelyt he same output. So:

$$R^0|L^1R^2|L^1R^2|L^1R^2|L^1R^2 \dots \quad \rightarrow \quad RLLRR^4|RLLRR^4 \dots$$

Rewriting the right part by moving the bars one place to the right yields:

$$R^1|L^2R^6|L^2R^6|L^2R^6 \dots$$

So we find that:

$$2 \cdot [0; \overline{1, 2}]_1 = [1; \overline{2, 6}]_1$$

Result (3.8): finding $2 \cdot [0; \overline{1, 2, 3}]_1$**.**

First we make the R-L-word:

$$R^0|L^1R^2L^3|R^1L^2R^3|L^1R^2L^3|R^1L^2R^3 \dots$$

Using this as our input we start in state A and the first letter we see is a single L, so we combine it with one R, this gives us as output RL and we go to state A'.

The situation up untill now: we are in state A', have as our input $R^1L^3|R^1L^2R^3|L^1R^2L^3|R^1L^2R^3\dots$ and our output up untill now is RL.

Now we get the second R, and because we are in state A' we need to combine this with the L next to it to get LR as output and we go back to state A.

The situation up untill now: we are in state A, have as our input $L^2|R^1L^2R^3|L^1R^2L^3|R^1L^2R^3\dots$ and our output up untill now is $RLLR$.

There is L^2 left, so the output will be a single L. After this we get a single R, wich gives R^2 as output. Next is L^2, wich gives us a single L and at last we see a R^3, so our output is R^6.

The situation up untill now: we are in state A, have as our input $L^1R^2L^3|R^1L^2R^3|L^1R^2L^3|R^1L^2R^3\dots$ and our output up untill now is $RLLRLR^2LR^6$.

What we see next is exactely what we had at the very start. So we found that the input $L^1R^2L^3|R^1L^2R^3$ gives us the following output: $RLLRLR^2LR^6$. Because we are back in state A again the next 6 letters of the word will give exactelyt he same output. So:

$$R^0|L^1R^2L^3|R^1L^2R^3|L^1R^2L^3|R^1L^2R^3\dots \quad\rightarrow\quad RLLRLR^2LR^6|RLLRLR^2LR^6\dots$$

Rewriting the right part by moving the bars one place to the right yields:

$$R^1|L^2R^1L^1R^2L^1R^7|L^2R^1L^1R^2L^1R^7\dots$$

So we found that:

$$2\cdot[0;\overline{1,2,3}]_1 = [1;\overline{2,1,1,2,1,7}]_1$$

Result (3.10): finding $2\cdot[0;\overline{1,1,2,1,7,2}]_1$.

Again we start with making the R-L-word:

$$R^0|L^1R^1L^2R^1L^7R^2|L^1R^1L^2R^1L^7R^2\dots$$

We use this as our input and start in state A. Again we find a single L, combining this with the R that follows we receive RL as our output and move to state A'

The situation up untill now: we are in state A', have as our input $L^2R^1L^7R^2|L^1R^1L^2R^1L^7R^2\dots$ and our output up untill now is RL.

Here we have L^2 as our input and we receive L^4. Next is a single R, so we combine it with an L, we receive LR as output and go back to state A again.

The situation up untill now: we are in state A, have as our input $L^6R^2|L^1R^1L^2R^1L^7R^2\dots$ and our output up untill now is RLL^4LR.

What's left is L^6, wich gives us L^3 and R^2 wich gives us R^4.

The situation up untill now: we are in state A, have as our input $L^1R^1L^2R^1L^7R^2|L^1R^1L^2R^1L^7R^2\dots$ and our output up untill now is $RLL^4LRL^3R^4$.

Now the circle is round, we have used a full period of the R-L-word and are in state A, so the next 6 letters will give us the exact same output as we just derived, therefore:

$$R^0|L^1R^1L^2R^1L^7R^2|L^1R^1L^2R^1L^7R^2\dots\rightarrow RLL^4LRL^3R^4|RLL^4LRL^3R^4\dots$$

Rewriting the right part by moving the bars one place to the right yields:

$$R^1|L^6R^1L^3R^5|L^6R^1L^3R^5\dots$$

So we find that:

$$2\cdot[0;\overline{1,1,2,1,7,2}]_1 = [1;\overline{6,1,3,5}]_1$$

Result (3.12): finding $2 \cdot [0; \overline{1,3,5,6}]_1$.

Again we start with making the R-L-word:

$$R^0|L^1R^3L^5R^6|L^1R^3L^5R^6|L^1R^3L^5R^6\ldots$$

We use this as our input and start in state A. Again we find a single L, combining this with the R that follows we receive RL as our output and move to state A'.

The situation up untill now: we are in state A', have as our input $R^2L^5R^6|L^1R^3L^5R^6|L^1R^3L^5R^6\ldots$ and our output up untill now is RL.

Now we have R^2 and receive a single R, followed by L^5, wich yields L^{10}, at last we have R^6 and receive R^3.

The situation up untill now: we are in state A', have as our input $L^1R^3L^5R^6|L^1R^3L^5R^6\ldots$ and our output up untill now is $RLRL^{10}R^3$.

Although we have completed one period of the R-L-word, and the next period starts (just like the first) with an L we can't say that we are finished already. This is because our first period we started in state A, while now we are in state A'. So we move on to the second period. We find the single L again, wich gives L^2 as output this time. Next is R^3, R^2 yields a single R, while the third R needs to be combined with an L to receive LR, and we go back to state A.

The situation up untill now: we are in state A, have as our input $L^4R^6|L^1R^3L^5R^6\ldots$ and our output up untill now is $RLRL^{10}R^3L^2RLR$.

Here we have L^4 left, wich gives us L^2. And last in this period we have R^6, wich gives us R^{12} as output.

The situation up untill now: we are in state A, have as our input $L^1R^3L^5R^6|L^1R^3L^5R^6\ldots$ and our output up untill now is $RLRL^{10}R^3L^2RLRL^2R^{12}$.

This time we are finished, because we are at the beginning of a new period again, starting with a L and we are in state A. So we now konw that:

$$R^0|L^1R^3L^5R^6|L^1R^3L^5R^6|L^1R^3L^5R^6\ldots \to RLRL^{10}R^3L^2RLRL^2R^{12}|RLRL^{10}R^3L^2RLRL^2R^{12}\ldots$$

Rewriting the right part by moving the bars one place to the right yields:

$$R^1|L^1R^1L^{10}R^3L^2R^1L^1R^1L^2R^{13}|L^1R^1L^{10}R^3L^2R^1L^1R^1L^2R^{13}\ldots$$

So we find that:

$$2 \cdot [0; \overline{1,3,5,6}]_1 = [1; \overline{1,1,10,3,2,1,1,1,2,13}]_1$$

4.2 Approximating frequencies of the digits for some quadratic irrationals

In section 3.6 five numbers were used to compute a large amount of the digits of their 2-CF expansion. These numbers are listed in Table 4.1 written in three different representations. The first one is in the most common representation (as the quadratic irrational they are). The second collum contains the equation each number satisfies, and the last collum gives the RCF expansion of each number.

Normal notation:	Quadratic form:	RCF
$\frac{\sqrt{37}-4}{3}$	$3x^2+8x-7=0$	$[0;\overline{1,2,3}]$
$\frac{\sqrt{569}-23}{10}$	$5x^2+23x-2=0$	$[0;\overline{11,1,2,2,23,2,2,1,11,4,1,2,5,1,1,1,1,5,2,1,4}]$
$\frac{\sqrt{197}-13}{4}$	$4x^2+26x-7=0$	$[0;\overline{3,1,6}]$
$\frac{\sqrt{577}-23}{8}$	$8x^2+46x-6=0$	$[0;\overline{7,1,5}]$
$\frac{\sqrt{993}-9}{24}$	$12x^2+9x-19=0$	$[0;\overline{1,15,7,1,4,2,1,1}]$

Table 4.1: Different representations of the used numbers.

Table 4.2 is an extended version of Table 3.2. This version contains more asymptotic frequencies. This table also shows how many digits were used to calculate these asymptotic frequencies.

	theoretical	$\frac{\sqrt{37}-4}{3}$	$\frac{\sqrt{37}-4}{3}$	$\frac{\sqrt{569}-23}{10}$	$\frac{\sqrt{197}-13}{4}$	$\frac{\sqrt{577}-23}{8}$	$\frac{\sqrt{993}-9}{24}$
# ditits used:	∞	70825	28154	27974	28315	28245	28171
$\phi(2)$	.290489	.291790	.292783	.289744	.290775	.290787	.287611
$\phi(3)$	.159172	.159054	.160084	.161513	.161334	.161663	.158821
$\phi(4)$	.100679	.099583	.100767	.101920	.101116	.101862	.101349
$\phi(5)$	.069478	.069566	.069511	.069210	.071307	.066032	.069258
$\phi(6)$	.050853	.050589	.049407	.048976	.052130	.050135	.051331
$\phi(7)$	.038840	.039322	.040385	.036285	.037508	.039584	.038410
$\phi(8)$	.030638	.031204	.031186	.031816	.031045	.031122	.030813
$\phi(9)$	.024787	.024045	.024153	.025560	.023098	.024111	.025843
$\phi(10)$	.020467	.021433	.021489	.018482	.020414	.020004	.021370
$\phi(11)$	.017187	.017734	.017546	.018160	.017341	.017844	.016578
$\phi(12)$	.014637	.014868	.014137	.013656	.013421	.015012	.014945
$\phi(13)$	.012615	.012439	.012751	.011404	.012361	.011861	.012176
$\phi(14)$	.010986	.010844	.010194	.011297	.010136	.010091	.010650
$\phi(15)$	.009653	.009869	.009199	.008973	.009607	.009170	.010046
$\phi(16)$	.008549	.008486	.008347	.008079	.008335	.008745	.008236
$\phi(17)$	.007624	.007342	.007175	.007579	.007947	.007506	.007987
$\phi(18)$	.006841	.006622	.006393	.007364	.006004	.005700	.006177
$\phi(19)$	.006173	.005874	.006393	.006256	.005404	.004992	.005502
$\phi(20)$	.005599	.006142	.005825	.005970	.006499	.006408	.005112

Table 4.2: Theoretical and numerical found asymptotic frequencies of the digits for $2 \leq n \leq 20$.

4.3 Matlab files used

Program to find a list of quadratic forms for possible non repetitive N-CF expansions:

```
N = 2; %teller
n = 1000; %aantal a_i te berekenen
aantalIteraties = 0;
indexABC = 0;
while (aantalIteraties < 10);
    good1 = 0;
    good2 = 0;
    while (good1 + good2 < 2)
        good1 = 0;
        k = floor(200*(rand()-.5));
        l = floor(200*(rand()-.5));
        m = floor(200*(rand()-.5));
        if (l^2-4*k*m > 0)
            if(sqrt(l^2-4*k*m)-floor(sqrt(l^2-4*k*m)) > 0)
                good1 = 1;
            end
        end
        % y and z are solutions
        y = (-l+sqrt(l^2-4*k*m))/(2*k);
        z = (-l-sqrt(l^2-4*k*m))/(2*k);
        if (0<y  && y<1)
            x = y;
        else
            x = z;
        end
        good2 = 0;
        if (0<x && x<1)
            good2 = 1;
        end
    end
    A = zeros(n+1,1);
    B = zeros(n+1,1);
    C = zeros(n+1,1);
    D = zeros(n+1,1);
    X = zeros(n+1,1);
    discr = zeros(n+1,1);

    A(1) = k;
    B(1) = l;
    C(1) = m;
    discr(1) = l^2-4*k*m;

    b = zeros(1,n+1);
    a = N;
    b(1) = floor(a/x);
    X(1) = x;

    for i = 1:n
        D(i) = gcd(C(i), gcd(2*b(i)*C(i)+a*B(i),a*b(i)*B(i)+A(i)*(a^2)+C(i)*(b(i)^2)));
        A(i+1) = C(i)/D(i);
        B(i+1) = (2*b(i)*C(i)+a*B(i))/D(i);
        C(i+1) = (a*b(i)*B(i)+A(i)*(a^2)+C(i)*(b(i)^2))/D(i);
        discr(i+1) = (B(i+1)^2-4*A(i+1)*C(i+1));
        newX = (-B(i+1)+sqrt(discr(i+1)))/(2*A(i+1));
        if ( newX > 0 && newX < 1 )
            X(i+1) = newX;
```

```
            b(i+1) = floor(a/X(i+1));
        else
            X(i+1) = (-B(i+1)-sqrt(discr(i+1)))/(2*A(i+1));
            b(i+1) = floor(a/X(i+1));
        end
    end
    Repetition = RepeatTester(b);
    if (Repetition == 0)
        indexABC = indexABC + 1;
        ABC(indexABC,:) = [k l m];
    end
    aantalIteraties = aantalIteraties+1;
end
zoekABC = ABC;
for vindKleine = 1:10
    sqrtsumABC = sum(zoekABC.^2,2);
    [MinSqrtSumABC IndSqrtSumABC] = min(sqrtsumABC);
    kleineABC(vindKleine,:) = zoekABC(IndSqrtSumABC,:);
    zoekABC(IndSqrtSumABC,:) = [100 100 100];
end
```

The function $RepeatTester()$ used by the previous program:

```
function res = RepeatTester(b)
repetition = 0;
NewB = b;
for i = 1:20
    NewB(i) = 0;
end
[MB(1) Imax(1)] = max(NewB);
NewB(Imax(1)) = 0;
repeat = 1;
RepCounter = 1;
while repeat
    RepCounter = RepCounter + 1;
    [MB(RepCounter) Imax(RepCounter)] = max(NewB);
    if (MB(RepCounter-1) == MB(RepCounter))
        fase(RepCounter-1) = Imax(RepCounter)-Imax(RepCounter-1);
        if (max(fase) == min(fase))
            repetition = 1;
            if (Imax(RepCounter) + max(fase) > size(b))
                repeat = 0;
            end
        else
            repetition = 0;
            repeat = 0;
        end
    else
        repeat = 0;
    end
    NewB(Imax(RepCounter)) = 0;
end
res = repetition;
```

Second program: to find unlimited digits, but works extreme slow:

```
VarB = [1 2 3]; %(periode van kettingbreuk)
PL = length(VarB);
PeriodLength(1) = PL;
L = -1;
R = 1;
k = 1;
while( k <= 100 ) %aantal te bepalen wijzergetallen
    NewB(k) = 2*VarB(1);
    k = k+1;
    if( VarB(2) > 1 )
        NewB(k) = VarB(2);
        clear TempB
        TempB = VarB;
        clear VarB
        VarB = TempB(3:length(TempB));
        VarB(length(VarB)+1) = TempB(1);
        VarB(length(VarB)+1) = TempB(2);
        k = k+1;
    else
        %gebruik Transcucer om van oude Rany woord een nieuwe te maken
        NewRanyWord = Transducer1(VarB);

        %gebruik nieuwe Rany woord
        NewB(k-1) = NewB(k-1)+1;
        clear VarB
        VarB = NewRanyWord([2:length(NewRanyWord)],2)';
        PL = length(VarB);
        PeriodLength(length(PeriodLength)+1) = PL;
    end
end
```

The function $Transducer1()$ used by the previous program:

```
function result = Transducer1(VarB)
PL = length(VarB);
L = -1;
R = 1;
%maak Rany woord
if( round(PL/2) == PL/2 )
    for i = 1:PL-1
        Cycle(i) = VarB(i+1);
    end
    Cycle(PL) = VarB(1);
    for i = 1:2:PL
        OldRanyWord(i,:) = [L,Cycle(i)];
        OldRanyWord(i+1,:) = [R,Cycle(i+1)];
end
else
    for i = 1:PL-1
        Cycle(i) = VarB(i+1);
    end
    Cycle(PL) = VarB(1);
    Cycle(PL+1:2*PL) = Cycle(1:PL);
    for i = 1:2:2*PL
        OldRanyWord(i,:) = [L,Cycle(i)];
        OldRanyWord(i+1,:) = [R,Cycle(i+1)];
    end
end
```

```
ORW = OldRanyWord;
%maak nieuw Rany woord
State = 1;
newInd = 1; %index NewRanyWord
NewRanyWord = [1 0];
j = 1; %index OldRanyWord
while( sum(OldRanyWord(:,2)) ~= 0 )
    if( OldRanyWord(j,2) > 0 ) %controleer of er nog letters over zijn om mee verder te werken
    if( OldRanyWord(j,1) == L )
    % L
        XL = OldRanyWord(j,2); %we hebben XL
        if( State == 2)
        % A'
            if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op een R
                newInd = newInd+1; %verhoog index van NewRanyWord met 1
                NewRanyWord(newInd,:) = [L 0]; %laat nieuw woord eindigen op L
            end %nieuwe woord eindigd nu op L
            NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 2*XL; %tel 2*XL bij nieuwe woord op
            OldRanyWord(j,2) = OldRanyWord(j,2) - XL; %haal XL van oude woord af
        else
        % A
            HXL = floor(XL/2);
            if( HXL > 0 ) %we hebben meer dan 1 L
                if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op een R
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [L 0]; %laat nieuw woord eindigen op L
                end %nieuwe woord eindgd nu op L
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + HXL; %tel HXL bij nieuwe woord op
                OldRanyWord(j,2) = OldRanyWord(j,2) - 2*HXL; %haal 2*HXL van oude woord af
            end
            if( OldRanyWord(j,2) == 1 ) %we hebben nog 1 L over
                if( NewRanyWord(newInd,1) == L ) %NewRanyWord eindigd nog op L
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [R 0]; %laat nieuwe woord op R eindigen
                end %nieuwe woord eindgd nu op R
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 1;
                newInd = newInd+1;
                NewRanyWord(newInd,:) = [L 1]; %plak RL achter nieuw woord
                OldRanyWord(j,2) = OldRanyWord(j,2)-1;
                OldRanyWord(j+1,2) = OldRanyWord(j+1,2)-1; %haal LR van oude woord af
                State = 2; %ga naar A'
            end
        end
    else
    % R
        XR = OldRanyWord(j,2); %we hebben XR
        if( State == 1 )
        % A
            if( NewRanyWord(newInd,1) == L ) %nieuwe woord eindig nog op een L
                newInd = newInd+1; %verhoog index van NewRanyWord met 1
                NewRanyWord(newInd,:) = [R 0]; %laat nieuwe woord eindigen op R
            end
            NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 2*XR; %tel 2*XR bij nieuwe woord op
            OldRanyWord(j,2) = OldRanyWord(j,2) - XR; %haal XR van oude woord af
            % als OldRanyWord nu 'opgebruikt' is dan zijn we klaar
        else
        % A'
        % als OldRanyWord bij laatse letter is moeten we verder omdat we in A willen eindigen
            if( j == length(OldRanyWord) )
                OldRanyWord(j+1:j+length(ORW),:) = ORW; %plak ORW achter OldRanyWord
            end
```

```
            % hier kan de periode flink oplopen
            HXR = floor(XR/2);
            if( HXR > 0 ) %we hebben meer dan 1 R
                if( NewRanyWord(newInd,1) == L ) %NewRanyWord eindigd nog op een L
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [R 0]; %laat nieuw woord eindigen op R
                end %nieuwe woord eindgd nu op R
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + HXR; %tel HXR bij nieuwe woord op
                OldRanyWord(j,2) = OldRanyWord(j,2) - 2*HXR; %haal 2*HXR van oude woord af
            end
            if( OldRanyWord(j,2) == 1 ) %we hebben nog 1 R over
                if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op R
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [L 0]; %laat nieuwe woord op L eindigen
                end %nieuwe woord eindgd nu op L
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 1;
                newInd = newInd+1;
                NewRanyWord(newInd,:) = [R 1]; %plak LR achter nieuw woord
                OldRanyWord(j,2) = OldRanyWord(j,2)-1;
                OldRanyWord(j+1,2) = OldRanyWord(j+1,2)-1; %haal RL van oude woord af
                State = 1; %ga naar A
            end
        end
    end
    end
    j = j+1;
end
if( NewRanyWord(newInd,1) == L ) %NewRanyWord eindigd nog op een L
    newInd = newInd+1; %verhoog index van NewRanyWord met 1
    NewRanyWord(newInd,:) = [R 0]; %laat nieuwe woord op R eindigen
end %nieuwe woord eindigd nu op R
NewRanyWord(length(NewRanyWord),2) = NewRanyWord(length(NewRanyWord),2)+1; %verhoog laatste R met 1
result = NewRanyWord;
```

Third program: to find limited digits, but works faster:

```
maxP = 40000;
cleanP = 1;
stop = 0;
VarB = [1 2 3];
PL = length(VarB);
PeriodLength(1) = PL;
L = -1;
R = 1;
k = 1;
while( ~stop ) %gaat zolang door tot er te weinig weizergetallen over zijn om mee te rekenen
    NewB(k) = 2*VarB(1);
    k = k+1
    if( VarB(2) > 1 )
        NewB(k) = VarB(2);
        clear TempB
        TempB = VarB;
        clear VarB
        VarB = TempB(3:length(TempB));
        if( cleanP )
            VarB(length(VarB)+1) = TempB(1);
            VarB(length(VarB)+1) = TempB(2);
        end
        k = k+1
    else
        %gebruik Transcucer om van oude Rany woord een nieuwe te maken
        if( cleanP )
            NewRanyWord = Transducer1(VarB);
        else
            NewRanyWord = Transducer2(VarB,maxP);
        end

        if( length(NewRanyWord) > maxP+1 || ~cleanP )
            cleanP = 0;
        end
        %gebruik nieuwe Rany woord
        if( cleanP )
            NewB(k-1) = NewB(k-1)+1;
            clear VarB
            VarB = NewRanyWord([2:length(NewRanyWord)],2)';
            PL = length(VarB);
            PeriodLength(length(PeriodLength)+1) = PL;
        else
            NewB(k-1) = NewB(k-1)+1;
            clear VarB
            VarB = NewRanyWord([2:min(length(NewRanyWord)-1,maxP)],2)';
            PL = length(VarB);
            PeriodLength(length(PeriodLength)+1) = PL;
        end
    end
    if( ~cleanP && length(VarB)<5 )
        stop = 1;
    end
end
```

The function $Transducer2()$ **also used by the previous program:**

```
function result = Transducer2(VarB,maxP)
PL = length(VarB);
```

```
L = -1;
R = 1;
%maak Rany woord
for i = 1:PL-1
    Cycle(i) = VarB(i+1);
end
Cycle(PL) = VarB(1);
for i = 1:PL
    if(mod(i,2))
        OldRanyWord(i,:) = [L,Cycle(i)];
    else
        OldRanyWord(i,:) = [R,Cycle(i)];
    end
end
%maak nieuw Rany woord
State = 1;
newInd = 1; %index NewRanyWord
NewRanyWord = [1 0];
j = 1; %index OldRanyWord
while( (j<length(OldRanyWord)) && (length(NewRanyWord) < maxP+1) )
    if( OldRanyWord(j,2) > 0 ) %controleer of er nog letters over zijn om mee verder te werken
    if( OldRanyWord(j,1) == L )
    % L
        XL = OldRanyWord(j,2); %we hebben XL
        if( State == 2)
        % A'
            if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op een R
                newInd = newInd+1; %verhoog index van NewRanyWord met 1
                NewRanyWord(newInd,:) = [L 0]; %laat nieuw woord eindigen op L
            end %nieuwe woord eindigd nu op L
            NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 2*XL; %tel 2*XL bij nieuwe woord op
            OldRanyWord(j,2) = OldRanyWord(j,2) - XL; %haal XL van oude woord af
        else
        % A
            HXL = floor(XL/2);
            if( HXL > 0 ) %we hebben meer dan 1 L
                if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op een R
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [L 0]; %laat nieuw woord eindigen op L
                end %nieuwe woord eindgd nu op L
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + HXL; %tel HXL bij nieuwe woord op
                OldRanyWord(j,2) = OldRanyWord(j,2) - 2*HXL; %haal 2*HXL van oude woord af
            end
            if( OldRanyWord(j,2) == 1 ) %we hebben nog 1 L over
                if( NewRanyWord(newInd,1) == L ) %NewRanyWord eindigd nog op L
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [R 0]; %laat nieuwe woord op R eindigen
                end %nieuwe woord eindgd nu op R
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 1;
                newInd = newInd+1;
                NewRanyWord(newInd,:) = [L 1]; %plak RL achter nieuw woord
                OldRanyWord(j,2) = OldRanyWord(j,2)-1;
                OldRanyWord(j+1,2) = OldRanyWord(j+1,2)-1; %haal LR van oude woord af
                State = 2; %ga naar A'
            end
        end
    else
    % R
        XR = OldRanyWord(j,2); %we hebben XR
        if( State == 1 )
        % A
```

```
            if( NewRanyWord(newInd,1) == L ) %nieuwe woord eindig nog op een L
                newInd = newInd+1; %verhoog index van NewRanyWord met 1
                NewRanyWord(newInd,:) = [R 0]; %laat nieuwe woord eindigen op R
            end
            NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 2*XR; %tel 2*XR bij nieuwe woord op
            OldRanyWord(j,2) = OldRanyWord(j,2) - XR; %haal XR van oude woord af
            % als OldRanyWord nu 'opgebruikt' is dan zijn we klaar
        else
        % A'
            HXR = floor(XR/2);
            if( HXR > 0 ) %we hebben meer dan 1 R
                if( NewRanyWord(newInd,1) == L ) %NewRanyWord eindigd nog op een L
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [R 0]; %laat nieuw woord eindigen op R
                end %nieuwe woord eindgd nu op R
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + HXR; %tel HXR bij nieuwe woord op
                OldRanyWord(j,2) = OldRanyWord(j,2) - 2*HXR; %haal 2*HXR van oude woord af
            end
            if( OldRanyWord(j,2) == 1 ) %we hebben nog 1 R over
                if( NewRanyWord(newInd,1) == R ) %NewRanyWord eindigd nog op R
                    newInd = newInd+1; %verhoog index van NewRanyWord met 1
                    NewRanyWord(newInd,:) = [L 0]; %laat nieuwe woord op L eindigen
                end %nieuwe woord eindgd nu op L
                NewRanyWord(newInd,2) = NewRanyWord(newInd,2) + 1;
                newInd = newInd+1;
                NewRanyWord(newInd,:) = [R 1]; %plak LR achter nieuw woord
                OldRanyWord(j,2) = OldRanyWord(j,2)-1;
                OldRanyWord(j+1,2) = OldRanyWord(j+1,2)-1; %haal RL van oude woord af
                State = 1; %ga naar A
            end
        end
    end
    end
    j = j+1;
end
result = NewRanyWord;
```

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