
**A Mathematical Analysis of
A Belt System with A Low and
Time-Varying Velocity**

A Mathematical Analysis of A Belt System with A Low and Time-Varying Velocity

Proefschrift

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to N.A. Analis, G.E. Sulistyawan, and my family

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Chapter 1

Introduction

1.1 A brief historical background

Belts, tapes, and cables, just as beams, rods and strings have something in common, that is, the dimension in one direction, the so-called axial direction, is much larger than the dimensions in the other two directions. In many cases, just a one-dimensional approach can be used to describe a certain configuration involving these objects. The applications of belts, tapes, or cables to mechanical systems such as conveyor belts, magnetic tapes, or monocable ropeways are often referred to as axially moving materials or axially moving continua. Axially moving systems are present in a wide variety of engineering problems. Aerial cables, tramways, oil pipelines, magnetic tapes, power transmission belts, and band saw blades are just a few of the many examples of the applications of axially moving systems. The interest in studying axially moving systems is also motivated by the increased use of oil and water pipelines since early 1950 [1].

A model describing transversal vibrations of a moving strip was derived by Thurman and Mote [1]. This model can be used as a model of the transversal vibrations of tapes, fibers, belts and band saws. The following equations of motion are used in their paper:

$$\begin{aligned}w_{tt} + 2Vw_{xx} - (P_1^2 - V^2)w_{xx} &= (P_1^2 - 1 - \eta V^2)u_x u_{xx}, \\u_{tt} + 2Vu_{xt} - (1 - \kappa V^2)u_{xx} + P_0^2 u_{xxxx} &= \\(P_1^2 - 1 - \eta V^2) &\left(\frac{3}{2}u_x^2 u_{xx} + w_x u_{xx} + u_x w_{xx}\right),\end{aligned}\tag{1.1.1}$$

subjected to simply supported boundary conditions: $u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0$ and $w(0, t) = w(1, t) = 0$. In (1.1.1) u is the transversal displacement, w is longitudinal displacement, V is the (constant) belt speed and the parameters κ, η, P_0^2 , and P_1^2 are just constants which will be defined later on. A combination of the Linsted perturbation method and the averaging method was used in [1] to study (1.1.1). The focus of the research has been to study the relationship between the belt speed, the natural frequencies, and the importance of the nonlinearities in the model (1.1.1).

After Thurman and Mote, the transversal vibrations of a (constantly) moving material was studied by Wickert [2] and Pellicano and Vestroni [3]. Most of the studies concentrate on the axial-velocity-dependent natural frequencies and the existence of instabilities at critical velocities. The natural frequencies turn out to decrease with increasing belt speed.

In most of the references given above, the belt velocity is assumed to be constant. In reality, however, the systems are exposed to accelerating and decelerating motions due to some imperfections such as pulley excentricities. These disturbances can manifest in the form of external excitations as well as parametric excitations. The transversal vibrations of a string where one or both of its ends are harmonically excited was studied by Sack [4], Archibald and Emslie [5], Mote [6], Mahalingam [7] and very recently by van Horssen [8]. Van Horssen used the Laplace transform method to solve analytically the equation describing the transversal vibrations of a (constantly) moving string.

Miranker [9] was probably the first who derived the equation for the transversal vibrations of a tape moving with a time-dependent axial velocity. Recent studies of the transversal vibrations of a string or a beam moving with a time-dependent velocity was mainly done by Öz, Pakdemirli, and Boyaci in [10, 11, 12]. In [10] the vibrations of an axially moving beam with a time-dependent velocity was studied by Öz and Pakdemirli, while its associated string-like equation was studied in [11]. In [12] the authors continue to study a similar type of equation with an additional nonlinear term. A two-time-scales perturbation method was used in [10, 11, 12]. The solutions are then approximated by using a truncation method. In all of these three papers truncations to just one mode (without any justification) was applied. Since the solutions of the partial differential equations consist of infinitely many modes, this (extreme) truncation causes inaccurate results and many of the existing mode-interactions are lost. These observations were recently also made by Pellicano and Vestroni in [3].

1.2 Motivations

It was stated in the last paragraph of the previous section that an (extreme) truncation can lead to inaccurate approximations of the solution of the problem describing the transversal vibrations of a conveyor belt. The purpose in doing the present work is to investigate whether the truncation method can be applied or not, and how it should be applied to approximate the transversal vibrations of axially moving continua moving with time-dependent velocities. Extensions to the already existing literature will be presented. Not only linear but also nonlinear problems will be studied. In this thesis the study will be restricted to the transversal vibrations of a conveyor belt with a low and time-varying velocity.

1.3 Equations of motion

In this section the equations of motion describing the transversal and the longitudinal displacements of a conveyor belt will be derived. In Figure 1, a schematic model of a typical conveyor belt under consideration has been given. The equations of motion

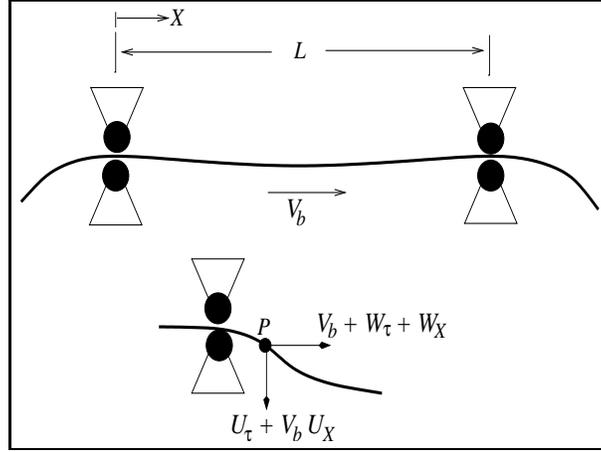


Figure 1.1: Schematic model of a conveyor belt, and velocity components at a point P on the belt.

for a belt system moving with a constant axial velocity have been derived in [1] using Hamilton's principle. For a time-varying velocity the same approach can also be applied with some modifications. A point particle P on the belt under consideration will have transversal and longitudinal velocities:

$$\begin{aligned} \frac{dU}{d\tau} &= \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial X} \frac{dX}{d\tau} \Leftrightarrow \frac{dU}{d\tau} = U_\tau + V_b(\tau)U_X, \\ \frac{dW}{d\tau} &= V_b(\tau) + W_\tau + V_b(\tau)W_X, \end{aligned} \quad (1.3.1)$$

respectively. Using these two velocities the kinetic energy of the belt is given by:

$$KE = \frac{1}{2} \rho A \int_0^L \left\{ (U_\tau + V_b U_X)^2 + [W_\tau + V_b(1 + W_X)]^2 \right\} dx, \quad (1.3.2)$$

and the potential energy is given by:

$$PE = \frac{1}{2} \int_0^L \left(\frac{1}{EA} \{ R_0 - EA + EA[(1 + W_X)^2 + U_X^2]^{\frac{1}{2}} \}^2 + EI U_{XX}^2 \right) dx, \quad (1.3.3)$$

with:

- ρ : the mass density of the belt,
- A : the cross-sectional area of the belt,
- $V_b(\tau)$: the belt velocity,
- E : the modulus of elasticity,
- R_0 : the constant tension in a dynamic equilibrium,
- I : the second moment of area with respect to the horizontal axis,
- $U(X, \tau)$: the transversal displacement of the belt,
- $W(X, \tau)$: the longitudinal displacement of the belt,
- X : the position along the horizontal axis,
- τ : the time, and
- L : the distance between the pulleys.

The Hamilton function $H(X, \tau, U_X, U_\tau, W_X, W_\tau, U_{XX})$ is defined by

$$\begin{aligned} & \frac{1}{2}\rho A \left\{ (U_\tau + V_b U_X)^2 + [W_\tau + V_b(1 + W_X)]^2 \right\} \\ & - \frac{1}{2} \left(\frac{1}{EA} \{ R_0 - EA + EA[(1 + W_X)^2 + U_X^2]^{\frac{1}{2}} \}^2 + EI U_{XX}^2 \right). \end{aligned} \quad (1.3.4)$$

Then according to Hamilton's principle, the equations of motion can be derived from $\frac{dI(\epsilon)}{d\epsilon} = 0$ with $\epsilon = 0$, where

$$I(\epsilon) = \int_{\tau_1}^{\tau_2} \int_0^L H(X, \tau, \bar{U}_X, \bar{U}_\tau, \bar{W}_X, \bar{W}_\tau, \bar{U}_{XX}) dXd\tau,$$

in which: $\bar{W}(X, \tau) = W(X, \tau) + \epsilon \xi(X, \tau)$, and $\bar{U}(X, \tau) = U(X, \tau) + \epsilon \zeta(X, \tau)$. The arbitrary functions $\xi(X, \tau)$ and $\zeta(X, \tau)$ have to satisfy:

$$\begin{aligned} \xi(0, \tau) = \xi(L, \tau) = \xi(X, \tau_1) = \xi(X, \tau_2) = 0, \quad \text{and} \\ \zeta(0, \tau) = \zeta(L, \tau) = \zeta(X, \tau_1) = \zeta(X, \tau_2) = 0. \end{aligned} \quad (1.3.5)$$

It then follows that

$$\begin{aligned} \frac{dI(\epsilon)}{d\epsilon} &= \int_{\tau_1}^{\tau_2} \int_0^L \frac{d}{d\epsilon} H(X, \tau, \bar{U}_X, \bar{U}_\tau, \bar{W}_X, \bar{W}_\tau, \bar{U}_{XX}) dXd\tau \\ &= \int_{\tau_1}^{\tau_2} \int_0^L \left\{ \frac{\partial H}{\partial \bar{W}_X} \frac{\partial \bar{W}_X}{\partial \epsilon} + \frac{\partial H}{\partial \bar{U}_X} \frac{\partial \bar{U}_X}{\partial \epsilon} + \frac{\partial H}{\partial \bar{W}_\tau} \frac{\partial \bar{W}_\tau}{\partial \epsilon} + \frac{\partial H}{\partial \bar{U}_\tau} \frac{\partial \bar{U}_\tau}{\partial \epsilon} + \right. \\ & \quad \left. \frac{\partial H}{\partial \bar{U}_{XX}} \frac{\partial \bar{U}_{XX}}{\partial \epsilon} \right\} dXd\tau, \\ &= \int_{\tau_1}^{\tau_2} \int_0^L \left\{ \frac{\partial H}{\partial \bar{W}_X} \xi_X + \frac{\partial H}{\partial \bar{U}_X} \zeta_X + \frac{\partial H}{\partial \bar{W}_\tau} \xi_\tau + \frac{\partial H}{\partial \bar{U}_\tau} \zeta_\tau + \frac{\partial H}{\partial \bar{U}_{XX}} \zeta_{XX} \right\} dXd\tau. \end{aligned} \quad (1.3.6)$$

So, $\frac{dI(0)}{d\epsilon} =$

$$\int_{\tau_1}^{\tau_2} \int_0^L \left\{ \frac{\partial H}{\partial \bar{W}_X} \xi_X + \frac{\partial H}{\partial \bar{U}_X} \zeta_X + \frac{\partial H}{\partial \bar{W}_\tau} \xi_\tau + \frac{\partial H}{\partial \bar{U}_\tau} \zeta_\tau + \frac{\partial H}{\partial \bar{U}_{XX}} \zeta_{XX} \right\} dXd\tau = 0. \quad (1.3.7)$$

Integrating (1.3.7) by parts and using (1.3.5) it then follows that (1.3.7) can be rewritten in:

$$\int_{\tau_2}^{\tau_1} \int_0^L \left\{ \xi \left[\frac{d}{dX} \left(\frac{\partial H}{\partial W_X} \right) + \frac{d}{d\tau} \left(\frac{\partial H}{\partial W_\tau} \right) \right] + \zeta \left[\frac{d}{dX} \left(\frac{\partial H}{\partial U_X} \right) + \frac{d}{d\tau} \left(\frac{\partial H}{\partial U_\tau} \right) - \frac{d^2}{dX^2} \left(\frac{\partial H}{\partial U_{XX}} \right) \right] \right\} dX d\tau = 0. \quad (1.3.8)$$

Since the functions $\eta(X, \tau)$ and $\zeta(X, \tau)$ are arbitrary it follows from (1.3.8) that

$$\begin{aligned} \frac{d}{dX} \left(\frac{\partial H}{\partial W_X} \right) + \frac{d}{d\tau} \left(\frac{\partial H}{\partial W_\tau} \right) &= 0, \\ \frac{d}{dX} \left(\frac{\partial H}{\partial U_X} \right) + \frac{d}{d\tau} \left(\frac{\partial H}{\partial U_\tau} \right) - \frac{d^2}{dX^2} \left(\frac{\partial H}{\partial U_{XX}} \right) &= 0. \end{aligned} \quad (1.3.9)$$

These equations are called the Euler-Lagrange equations. By substituting $H(X, \tau, U_X, U_\tau, W_X, W_\tau, U_{XX})$ as given by (1.3.4) into (1.3.9), the following equations are obtained:

$$\begin{aligned} \rho A W_{\tau\tau} + 2\rho A V_b W_{X\tau} + \rho A V_{b\tau} (1 + W_X) + (\rho A V_b^2 - EA) W_{XX} &= \\ (EA - R_0) \frac{(1 + W_X) U_X U_{XX} - U_X^2 W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}}, \\ \rho A U_{\tau\tau} + 2\rho A V_b U_{X\tau} + \rho A V_{b\tau} U_X + (\rho A V_b^2 - EA) U_{XX} + EI U_{XXXX} &= \\ (R_0 - EA) \frac{(1 + W_X)^2 U_{XX} - (1 + W_X) U_X W_{XX}}{[(1 + W_X)^2 + U_X^2]^{3/2}}. \end{aligned} \quad (1.3.10)$$

Using a Taylor series, the denominator in (1.3.10) can be approximated by:

$$\begin{aligned} [(1 + W_X)^2 + U_X^2]^{-3/2} &= 1 - 3W_X + 6W_X^2 - \frac{3}{2}U_X^2 - 10W_X^3 + \\ &\frac{15}{2}W_X U_X^2 + \mathcal{O}(4), \end{aligned} \quad (1.3.11)$$

where $\mathcal{O}(4)$ stands for terms of degree 4 or higher. Assuming that the displacements in the longitudinal direction are much smaller than the displacements in the transversal direction, that is, $W = \mathcal{O}(U^2)$ it follows from (1.3.11) that $[(1 + W_X)^2 + U_X^2]^{3/2} \approx 1 - 3W_X - \frac{3}{2}U_X^2$. Substitution of this approximation into (1.3.10) gives (approximately)

$$\begin{aligned} \rho A W_{\tau\tau} + 2\rho A V_b W_{X\tau} + \rho A V_{b\tau} (1 + W_X) + (\rho A V_b^2 - EA) W_{XX} &= \\ (EA - R_0) U_X U_{XX}, \\ \rho A U_{\tau\tau} + 2\rho A V_b U_{X\tau} + \rho A V_{b\tau} U_X + (\rho A V_b^2 - R_0) U_{XX} + EI U_{XXXX} &= \\ (EA - R_0) \left(\frac{3}{2} U_X^2 U_{XX} + W_X U_{XX} + U_X W_{XX} \right), \\ \tau > 0, \quad 0 < X < L. \end{aligned} \quad (1.3.12)$$

To put the equation of motion (1.3.12) into a non-dimensional form, the following substitutions are applied: $w(x, t) = \frac{W(X, \tau)}{L}$, $u(x, t) = \frac{U(X, \tau)}{L}$, $x = \frac{X}{L}$, $\beta^2 = \frac{T_0}{\rho A}$, $t = \frac{\beta\tau}{L}$, $V(t) = \frac{V_b(\tau)}{\beta}$, $P_0^2 = \frac{EI}{T_0 L^2}$, and $P_1^2 = \frac{EA}{T_0}$, where L is the distance between the two

pulleys which are assumed to be two simple supports, and T_0 is the initial tension which is related to R_0 through $R_0 = T_0 + \eta\rho AV_b^2$ with $0 \leq \eta \leq 1$. Substituting all those non-dimensional variables into (1.3.12) and letting $\kappa = 1 - \eta$ the following system of partial differential equations is then obtained:

$$\begin{aligned} w_{tt} + 2Vw_{xt} + V_t(1 + w_x) - (P_1^2 - V^2)w_{xx} &= (P_1^2 - 1 - \eta V^2)u_x u_{xx}, \\ u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - 1)u_{xx} + P_0^2 u_{xxx} &= \\ (P_1^2 - 1 - \eta V^2)\left(\frac{3}{2}u_x^2 u_{xx} + u_x w_{xx} + w_x u_{xx}\right), & \quad t > 0, \quad 0 < x < 1. \end{aligned} \quad (1.3.13)$$

The boundary conditions for the two simple supports are given by:

$$w(0, t) = w(1, t) = 0, \quad \text{and} \quad u(x, t) = u_{xx}(x, t) = 0 \quad \text{for} \quad x = 0, 1, \quad (1.3.14)$$

while the initial displacements and initial velocities are:

$$\begin{aligned} w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad u(x, 0) = u_0(x), \quad \text{and} \\ u_t(x, 0) = u_1(x). \end{aligned} \quad (1.3.15)$$

Throughout this thesis, it is assumed that the time-dependent velocity of the belt is given by $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$. The smallness of ϵ can be considered as a measure of the smallness of the belt velocity $V_b(t)$ compared to the wave velocity β . V_0 , α , and Ω are assumed to be constants.

1.4 An analytical approximation

In this thesis formal approximations of the solutions of (1.3.13) - (1.3.15) will be constructed. This formal approximation will satisfy the partial differential equations and the boundary conditions up to some order in ϵ . If a straight-forward ϵ -expansion is used to approximate the solutions, secular terms can occur in the approximations. To avoid these secular terms, a two-time-scales perturbation method is used. The first step in applying this perturbation method can be to transform the initial-boundary-value-problem (1.3.13) and (1.3.14) into the initial-value-problem by using the boundary conditions (1.3.14).

The boundary conditions (1.3.14) imply that the solutions $u(x, t)$ (and $w(x, t)$) should be extended as odd and 2-periodic functions in x , that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x). \quad (1.4.1)$$

This extension implies that all terms in (1.3.13)-(1.3.15) should be extended odd and 2-periodic functions in x . Particular attention needs to be paid to the terms containing u_x and u_{xt} since these terms are even with respect to x . To make these terms odd can

be accomplished by multiplying them with the following odd and 2-periodic function in x , $\mathcal{H}(x)$,

$$\mathcal{H}(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ -1 & \text{for } -1 < x < 0, \end{cases} = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin((2j+1)\pi x), \quad (1.4.2)$$

and $\mathcal{H}(x) = \mathcal{H}(x+2)$. It should be emphasized that in [10, 11, 12], the equations (1.3.13)-(1.3.15) have been solved incorrectly, that is, the solutions have never been extended properly and in fact the authors merely used a one mode approximation without any justification.

In the process of transforming (1.3.13)-(1.3.15) into a system of ordinary differential equations, the following identities can be used:

$$\begin{aligned} \int_0^1 \sin(n\pi x) \sin(k\pi x) dx &= \begin{cases} 0 & \text{for } k \neq n, \\ \frac{1}{2} & \text{for } k = n, \end{cases} \quad \text{and} \\ \int_0^1 \sin(n\pi x) \cos(k\pi x) dx &= \begin{cases} 0 & \text{for } k \pm n \text{ even,} \\ \frac{2n}{(n^2-k^2)\pi} & \text{for } k \pm n \text{ odd.} \end{cases} \end{aligned} \quad (1.4.3)$$

The results obtained by applying (1.4.1), (2.2.6), and (1.4.3) will be a set of ordinary differential equations in t . The solutions of these ordinary differential equations will be approximated by using a two-time-scales perturbation method. These two time scales are usually $t_0 = t$ and $t_1 = \epsilon t$. The introduction of these two time scales will give the following transformations:

$$\begin{aligned} \frac{d(\cdot)}{dt} &= \frac{\partial(\cdot)}{\partial t_0} + \epsilon \frac{\partial(\cdot)}{\partial t_1}, \\ \frac{d^2(\cdot)}{dt^2} &= \frac{\partial^2(\cdot)}{\partial t_0^2} + 2 \frac{\partial^2(\cdot)}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2(\cdot)}{\partial t_1^2}. \end{aligned} \quad (1.4.4)$$

The use of these transformations and the assumption that $u_n(t) = u_{n0}(t_0, t_1) + \epsilon u_{n1}(t_0, t_1) + \epsilon^2 u_{n2}(t_0, t_1) + \dots$ will give a set of ordered ordinary differential equations which can be solved one by one (if possible) systematically.

1.5 Outline of the Thesis

In Chapter 2 the linear string-like equation for the transversal vibrations of a conveyor belt with a low and time-varying velocity will be studied. The equation of motion describing this phenomenon is the second part of (1.3.13) with $P_0^2 = 0$ and $w(x, t)$ identically equal to zero. Stability of the approximations of the solutions is discussed as well as some similarities and differences between the present work and the results available in the literature.

In Chapter 3 the linear beam-like equation for the transversal vibrations of a conveyor belt will be studied. In this chapter the bending stiffness P_0^2 is no longer equal to

zero, which leads to a fourth order partial differential equation. It turns out that the dynamic behaviour of a conveyor belt system modelled by using a beam-like equation is considerably different from that modelled by using a string-like equation. For several values of Ω (the frequency fluctuation of the belt velocity) the vibration of the belt system are investigated. The most interesting cases for Ω can essentially be divided into the sum or difference type of two natural frequencies. Even more interesting, for special values of P_0^2 this sum and difference type can coincide giving rise to more complicated dynamical behaviour which has never been detected in the existing literature.

In Chapter 4 the two equations in (1.3.13) will be studied. In this chapter it is assumed that $P_0^2 = \mathcal{O}(1)$ and $P_1^2 = \mathcal{O}(\frac{1}{\epsilon})$ leading to the possibility of decoupling (1.3.13) into two uncoupled partial differential equations, through the application of the so-called Kirchhoff approach. It turns out that the nonlinear part of the equation stabilizes the solutions which are unstable in the linear case (see Chapter 2).

Another type of ordering assumptions on the transversal and longitudinal displacements, and on P_0^2 and P_1^2 will lead to another problem. The case where $P_0^2 = P_1^2 = \mathcal{O}(1)$ and $u(x, t) = \mathcal{O}(\sqrt{\epsilon}), w(x, t) = \mathcal{O}(\epsilon)$ will be the topic of Chapter 5. It will be shown in this chapter that $\mu^2 = P_0^2 \pi^2 = \frac{1}{2}$ is a special parameter value, in the sense that for $\mu^2 > \frac{1}{2}$ the dynamic behaviour of the system resembles the one as studied in Chapter 4, while for $\mu^2 < \frac{1}{2}$ a much more complicated behaviour will occur due to additional mode interactions.

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Chapter 2

On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part I: The String-like Case [‡]

Abstract. In this chapter initial-boundary value problems for a linear wave (string) equation are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed. The belt is assumed to move with a low and time-varying speed. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behavior of the conveyor belt. It also will be shown that the truncation method can not be applied to this problem in order to obtain approximations valid on long time scales.

2.1 Introduction

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [2] - [5] for an overview) and is still of interest today.

The main purpose of studying the dynamic behavior of a belt system is to know the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see [4]). Resonances that can cause severe vibrations can be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support component, and the increase of energy consumption of the system.

[‡]This chapter is a revised version of [1] *On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part I: The String-Like Case.*

Belt vibrations can be classified into two types, i.e. whether it is of a string-like type or of beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibrations of the belt system may be described as:

- string-like by

$$u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - c^2)u_{xx} = 0, \text{ and} \quad (2.1.1)$$

- beam-like (with a string effect) by

$$u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - c^2)u_{xx} + \frac{EI}{\rho A}u_{xxxx} = 0, \quad (2.1.2)$$

where:

- $u(x, t)$: the displacement of the belt in the vertical direction,
- V : the time-varying belt speed,
- c : the wave speed,
- E : Young's modulus,
- I : the moment of inertia with respect to the x (horizontal) axis,
- ρ : the mass density of the belt,
- A : the area of the cross section of the belt,
- κ : a constant representing the relative stiffness of the belt.
Its value is in $[0, 1]$,
- x : coordinate in horizontal direction,
- t : time, and
- L : the distances between the puleys.

The beam-like system with a low time-varying speed will be considered in Chapter 3. In this chapter we will study the string-like case where the belt velocity $V(t)$ is given by

$$V(t) = \epsilon(V_0 + \alpha \sin(\Omega t)), \quad (2.1.3)$$

where ϵ is a small parameter with $0 < \epsilon \ll 1$, and V_0 and α are constants with $V_0 > 0$ and $V_0 > |\alpha|$. The velocity variation frequency of the belt is given by Ω . In fact the small parameter ϵ indicates that the belt speed $V(t)$ is small compared to the wave speed c . The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. It will turn out that certain values of Ω can lead to complicated internal resonances of the belt system.

While for more accurate results, a non-linear model is required, it is not meaningless to investigate first a linear model. Knowledge about linear models is important in order to understand results found in non-linear models, especially for those cases which are

weakly non-linear. For non-linear models describing the dynamic behavior of belts, we refer the readers to [5], [7], and [8]. In [8] the role played by the external frequency of the non-constant belt velocity and the bending stiffness is studied. It is found that, as the bending stiffness tends to zero, the system behaves more like a string and its dynamics becomes more complicated than the beam-like system.

Most belt studies involve mainly belts moving with a constant velocity. Recently in a series of papers ([9] - [12]) several authors considered the vibrations of belts moving with time-dependent velocities and the vibrations of tensioned pipes conveying fluid with time-dependent velocities. In fact in ([9] - [12]) the equations (2.1.1) or (2.1.2) have been studied, where $V(t)$ as given by (2.1.3) belongs to and is included in the cases that have been studied in ([9] - [12]). To find approximations of the displacement of the belt in vertical direction the authors use in ([9] - [12]) the method eigenfunction expansions, the Galerkin truncation method, and the multiple-time-scales perturbation method as for instance described in [13, 14]. To apply the method of eigenfunction expansions and the perturbation method, special attention has to be paid to the $\mathcal{O}(\epsilon)$ terms involving u_x and u_{xt} in (2.1.1) or (2.1.2). To apply the truncation method the internal resonances between the vibration modes have to be studied. In ([9] - [12]) the $\mathcal{O}(\epsilon)$ terms in (2.1.1) or (2.1.2) involving u_x and u_{xt} are not treated correctly, and it is assumed in ([9] - [12]) that truncation to one mode (or a few modes) of the constant belt velocity system is allowed. In this chapter we will show that this truncation is not allowed. In [9, 11] no instabilities of the belt system (as described by (2.1.1)) were found using the truncation method when the velocity variation frequency Ω is equal to or close to the difference of two natural frequencies of the constant velocity system. In this chapter it will be shown that also instabilities can occur when Ω is equal to or close to the difference of two natural frequencies of the constant velocity system. In [5] and in ([15] - [19]) several remarks can be found on how and when truncation is allowed. In those papers weakly nonlinear problems for wave and for beam equations have been studied.

In this chapter we consider the vibrations of a belt modeled by a string moving with a non-constant velocity $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$, where V_0 , α , and Ω are constants with $V_0 > |\alpha|$. The velocity $V(t)$ can be considered as a periodically changing velocity such that the belt still moves in one direction. This variation in $V(t)$ can be considered as some kind of an excitation. In relation to excitations, some results in this area have been obtained in [20] and in [21]. In [20] problems for a string moving with a constant velocity are considered for which one of its ends (i.e. $x = L$) is subjected to an harmonic excitation. In [22], the vibrations of the string at $x = L$ is forced to be $u(x, t) = u_0 \cos \Omega t$. In [22] the author also studied the case where one end of the moving string is subjected to an harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied in [23]. In that paper a moving string model is used to study the transverse vibrations of power transmission chains. In all of these

papers ([20] - [23]), the belt velocity is assumed to be constant.

This chapter is organized as follows. In section 2.2, an equation to describe the transversal vibrations of a belt (which is modeled as a string) is derived. Here we assume that the belt moves with an arbitrary low velocity which is varied harmonically, i.e. $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$. In section 2.3 we study the energy and the boundedness of the solution of the problem as derived in section 2.2. In section 2.4 we discuss the application of the two time-scales perturbation method to solve the equation. It turns out that there are infinitely many values of Ω that can cause internal resonances. In this chapter we only investigate the resonance case $\Omega = \frac{c\pi}{L}$. All other resonance cases can be studied similarly. In this section it will also be shown that the truncation method can not be applied to this problem due to the distribution of energy among all vibration modes. In the last part of section 2.4 we also study a detuning case for the value $\Omega = \frac{c\pi}{L}$. Finally, in section 2.5 some remarks will be made and some conclusions will be drawn.

2.2 A string model

In this section the dynamic behavior of a conveyor belt which is modeled by a moving string is studied. Since the belt is assumed to move with a speed $V(t)$ (which explicitly

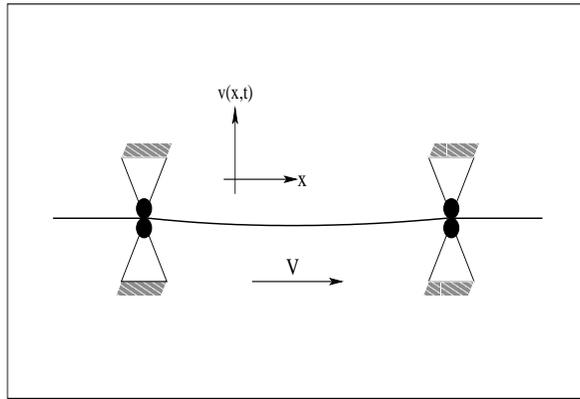


Figure 2.1: Conveyor belt system

depends on t) we obtain for the time-derivative of the transversal displacement $u(x, t)$ of the belt

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} = u_t + Vu_x, \quad (2.2.1)$$

and for the second order derivative with respect to time

$$\frac{D^2u}{Dt^2} = u_{tt} + 2Vu_{xt} + V^2u_{xx} + V_tu_x. \quad (2.2.2)$$

Accordingly, we have the following equation of motion

$$\begin{aligned} T_0 u_{xx} &= \rho \frac{D^2 u}{Dt^2}, \\ c^2 u_{xx} &= u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x, \end{aligned} \quad (2.2.3)$$

where $c = \sqrt{\frac{T_0}{\rho}}$, in which T_0 and ρ are assumed to be the constant tension and the constant mass-density of the string, respectively. At $x = 0$ and $x = L$ we assume that the string is fixed in vertical direction, where L is the distance between the pulleys.

For $V(t)$ we use $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$ with $V_0 > 0$ and $V_0 > |\alpha|$. This low velocity should be interpreted as low compared to the wave speed c of the belt. The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. Consequently (2.2.3) becomes:

$$c^2 u_{xx} - u_{tt} = \epsilon [\alpha \Omega \cos(\Omega t) u_x + 2(V_0 + \alpha \sin(\Omega t)) u_{xt}] + \epsilon^2 [V_0 + \alpha \sin(\Omega t)]^2 u_{xx}, \quad (2.2.4)$$

where the boundary and initial conditions are given by

$$\begin{aligned} u(0, t; \epsilon) &= u(L, t; \epsilon) = 0, \\ u(x, 0; \epsilon) &= f(x) \text{ and } u_t(x, 0; \epsilon) = g(x), \end{aligned} \quad (2.2.5)$$

where $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity of the belt, respectively. Throughout this chapter it is assumed that f and g are sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (2.2.4) - (2.2.5) exists. Moreover, it is assumed that all series representations for the solution v (and its derivatives), and for the functions f and g are convergent. In this section the initial-boundary value problem (2.2.4)-(2.2.5) for $u(x, t)$ will be reduced to a system of infinitely many ordinary differential equations. This system will be studied further in section 2.4 using a two-time scales perturbation method.

To satisfy the boundary conditions all functions should be expanded in Fourier-sin-series. Therefore, solution of the form $u(x, t; \epsilon) = \sum_{n=1}^{\infty} u_n(t; \epsilon) \sin(\frac{n\pi x}{L})$ is sought. This is an odd function in x , both with regard to $x = 0$ and $x = L$. All functions in the right hand side of (2.2.4) should be extended properly to make them odd with respect to $x = 0$ and $x = L$, and periodic with period $2L$ thereof. Note that this extension or expansion process is not applied in ([9]- [11]) causing the occurrence of incorrect results in the critical values of Ω .

To make the right hand side of (2.2.4) odd, terms which are not already in Fourier-sin-series form in x are multiplied with (see also [15, 18]):

$$\mathcal{H}(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ -1 & \text{if } -L < x < 0 \end{cases} = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right). \quad (2.2.6)$$

Substituting (2.2.6) into (2.2.4) results in

$$c^2 u_{xx} - u_{tt} = \epsilon \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right) [\alpha\Omega \cos(\Omega t)u_x + 2(V_0 + \alpha \sin(\Omega t))u_{xt}] + \epsilon^2(V_0 + \alpha \sin(\Omega t))^2 u_{xx}. \quad (2.2.7)$$

Substitution of $u(x, t) = \sum_{n=1}^{\infty} u_n(t; \epsilon) \sin\left(\frac{n\pi x}{L}\right)$ into (2.2.7) results in:

$$\begin{aligned} \sum_{n=1}^{\infty} \left(-\left(\frac{cn\pi}{L}\right)^2 u_n - \ddot{u}_n \right) \sin\left(\frac{n\pi x}{L}\right) &= \epsilon \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right) \\ &\left(\alpha\Omega \cos(\Omega t) \frac{n\pi}{L} u_n \cos\left(\frac{n\pi x}{L}\right) + 2(V_0 + \alpha \sin(\Omega t)) \frac{n\pi}{L} \dot{u}_n \cos\left(\frac{n\pi x}{L}\right) \right) - \\ &\epsilon^2 \sum_{n=1}^{\infty} (V_0 + \alpha \sin \Omega t)^2 \left(\frac{n\pi}{L}\right)^2 u_n \sin\left(\frac{n\pi x}{L}\right). \end{aligned} \quad (2.2.8)$$

By multiplying (2.2.8) with $\sin\left(\frac{k\pi x}{L}\right)$, and by integrating the so-obtained equation with respect to x from $x = -L$ to $x = L$, we obtain:

$$\begin{aligned} \ddot{u}_k + \left(\frac{ck\pi}{L}\right)^2 u_k &= \epsilon \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{2n}{(2j+1)L} \left[\alpha\Omega \cos(\Omega t)u_n + \right. \\ &\left. 2(V_0 + \alpha \sin(\Omega t))\dot{u}_n \right] + \epsilon^2(V_0 + \alpha \sin(\Omega t))^2 \left(\frac{k\pi}{L}\right)^2 u_k, \end{aligned} \quad (2.2.9)$$

where $\sum_1 = \sum_{k=n-(2j+1)}$, $\sum_2 = \sum_{k=2j+1+n}$, and $\sum_3 = \sum_{k=2j+1-n}$. Equation (2.2.9) will be studied further in section 2.4.

2.3 Energy and boundedness of the solution

We are going to use the concept of energy in many parts of the next sections. In this section we shall derive the energy of the moving string as modeled by the wave equation

$$c^2 u_{xx} = u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x. \quad (2.3.1)$$

By multiplying (2.3.1) with $(u_t + V u_x)$ we obtain after some elementary calculations

$$\begin{aligned} \left(\frac{1}{2} u_t^2 + u_t V u_x + \frac{1}{2} c^2 u_x^2 + \frac{1}{2} V^2 u_x^2 \right)_t + \\ \left(-c^2 u_x u_t - \frac{1}{2} c^2 V u_x^2 + V u_t^2 + V^2 u_x u_t + \frac{1}{2} V^3 u_x^2 - \frac{1}{2} V u_t \right)_x = 0. \end{aligned} \quad (2.3.2)$$

Integrating (2.3.2) with respect to x from $x = 0$ to $x = L$, and then integrating the so-obtained equation with respect to t from $t = 0$ to t , we obtain:

$$\int_0^L \left(\frac{1}{2} u_t^2 + V u_t u_x + \frac{1}{2} (c^2 + V^2) u_x^2 \right) \Big|_{t=0}^t dx = \frac{1}{2} \int_0^t (c^2 - V^2) V u_x^2 \Big|_{x=0}^L dt. \quad (2.3.3)$$

The energy $E(t)$ of the moving string is now defined to be:

$$E(t) = \frac{1}{2} \int_0^L ((u_t + Vu_x)^2 + c^2 u_x^2) dx. \quad (2.3.4)$$

So, (2.3.3) can be written as

$$\begin{aligned} E(t) - E(0) &= \frac{1}{2} \int_0^t (c^2 - V^2) V u_x^2|_{x=0}^L dt \\ \Leftrightarrow \frac{dE}{dt} &= \frac{1}{2} (c^2 - V^2) V (u_x^2(L, t) - u_x^2(0, t)) \leq MV, \end{aligned} \quad (2.3.5)$$

where M is the maximum of $\frac{1}{2}(c^2 - V^2)(u_x^2(L, t) - u_x^2(0, t))$, where we have assumed that $u(x, t)$ is two times continuously differentiable on $0 \leq x \leq L$ and $0 \leq t \leq T\epsilon^{-1}$ for some positive constant $T < \infty$. It follows from (2.3.5) that $\frac{dE}{dt} \leq \mathcal{O}(\epsilon)$ on $0 \leq t \leq T\epsilon^{-1}$ since V is $\mathcal{O}(\epsilon)$. And so, $E(t) - E(0) \leq \mathcal{O}(\epsilon t)$ on $0 \leq t \leq T\epsilon^{-1}$. The following estimate on $u(x, t)$ then also holds

$$\begin{aligned} |u(x, t)| &= \left| \int_0^x u_x(x, t) dx \right| \leq \int_0^x |u_x(x, t)| dx \\ &\leq \int_0^L |u_x(x, t)| dx \\ &\leq \sqrt{\int_0^L 1^2 dx} \sqrt{\int_0^L 2 \cdot \frac{1}{2} (c^2 u_x^2 + (u_t + Vu_x)^2) dx} \\ &= \sqrt{L} \sqrt{2E(t)}, \end{aligned} \quad (2.3.6)$$

on $0 \leq t \leq T\epsilon^{-1}$. We refer to [24] for more detailed descriptions of energetics of translating continua.

2.4 Application of the two time-scales perturbation method

Consider again equation (2.2.9). The application of a straight-forward expansion method to solve (2.2.9) will result in the occurrence of so-called secular terms which causes the approximations to become unbounded on long time-scales. To remove those secular terms, we introduce two *time-scales* $t_0 = t$ and $t_1 = \epsilon t$. The introduction of these two time-scales defines the following transformations:

$$\begin{aligned} u_k(t; \epsilon) &= v_k(t_0, t_1; \epsilon), & \frac{du_k(t; \epsilon)}{dt} &= \frac{\partial v_k}{\partial t_0} + \epsilon \frac{\partial v_k}{\partial t_1}, \\ \frac{d^2 u_k(t; \epsilon)}{dt^2} &= \frac{\partial^2 v_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 v_k}{\partial t_1^2}. \end{aligned} \quad (2.4.1)$$

By substituting (2.4.1) into (2.2.9) we obtain:

$$\begin{aligned} \frac{\partial^2 v_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \left(\frac{ck\pi}{L}\right)^2 v_k = \\ \epsilon \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{2n}{(2j+1)L} \left(\alpha \Omega \cos(\Omega t) v_n + 2[V_0 + \alpha \sin(\Omega t) \frac{\partial v_n}{\partial t_0}] \right) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (2.4.2)$$

Assuming that $v_k(t_0, t_1; \epsilon) = v_{k0}(t_0, t_1) + \epsilon v_{k1}(t_0, t_1) + \dots$, then in order to remove the secular terms up to $\mathcal{O}(\epsilon)$, we have to solve the following problems:

$$\begin{aligned} \mathcal{O}(1) : \frac{\partial^2 v_{k0}}{\partial t_0^2} + \left(\frac{ck\pi}{L}\right)^2 v_{k0} = 0, \\ \mathcal{O}(\epsilon) : \frac{\partial^2 v_{k1}}{\partial t_0^2} + \left(\frac{ck\pi}{L}\right)^2 v_{k1} = -2 \frac{\partial^2 v_{k0}}{\partial t_0 \partial t_1} + \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{2n}{(2j+1)L} \\ \left(\alpha \Omega \cos(\Omega t) v_{n0} + 2(V_0 + \alpha \sin(\Omega t)) \frac{\partial v_{n0}}{\partial t_0} \right). \end{aligned} \quad (2.4.3)$$

The $\mathcal{O}(1)$ problem has as solution

$$v_{k0}(t_0, t_1) = A_{k0}(t_1) \cos\left(\frac{ck\pi t_0}{L}\right) + B_{k0}(t_1) \sin\left(\frac{ck\pi t_0}{L}\right), \quad (2.4.4)$$

where A_{k0} and B_{k0} are still arbitrary functions that can be used to avoid secular terms in the solution of the $\mathcal{O}(\epsilon)$ -problem.

From the $\mathcal{O}(\epsilon)$ problem it can readily be seen that there are infinitely many values of Ω that can cause internal resonance. In fact these values are $(n+k)\frac{c\pi}{L}$, $(n-k)\frac{c\pi}{L}$, $(k-n)\frac{c\pi}{L}$, and $-(n+k)\frac{c\pi}{L}$, where $k = n - 2j - 1$, or $k = 2j + 1 - n$, or $k = n + 2j + 1$ (see also the summations in (2.2.9)). It is also easy to see that these values for Ω are always odd multiples of $\frac{c\pi}{L}$ (or are in an $\mathcal{O}(\epsilon)$ -neighbourhood of these odd multiples). In [9] and [11] the critical values of Ω are found to be even multiples of the natural frequency. These incorrect results in [9] and [11] for $\mathcal{O}(\epsilon)$ belt velocities are due to the fact that certain terms in the PDE (that is, terms involving u_x and u_{xt} in (2.2.4)) are not extended or expanded correctly.

To show how the secular terms can be eliminated we will consider three cases: $\Omega = \frac{c\pi}{L}$, $\Omega = \frac{c\pi}{L} + \epsilon\delta$, and the case that Ω is not in a neighborhood of an odd multiple of $\Omega = \frac{c\pi}{L}$.

2.4.1 Case 1: $\Omega = \frac{c\pi}{L}$

In appendix 1 it has been shown for $\Omega = \frac{c\pi}{L}$ what equations $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy such that the approximations of the solution of the problem do not contain secular terms. It turns out that A_{k0} and B_{k0} have to satisfy:

$$\frac{dA_{k0}}{dt_1} = (k+1)B_{(k+1)0} + (k-1)B_{(k-1)0},$$

$$\frac{dB_{k0}}{d\bar{t}_1} = -(k+1)A_{(k+1)0} - (k-1)A_{(k-1)0}, \quad (2.4.5)$$

where $\bar{t}_1 = \frac{\alpha}{L}t_1$, and $k = 1, 2, 3, \dots$. For $\Omega = m\frac{c\pi}{L}$ where m is odd the same analysis as presented in appendix 1 can be followed. It then follows that A_{k0} and B_{k0} have to satisfy ($k = 1, 2, 3, \dots$):

$$\begin{aligned} \frac{dA_{k0}}{d\bar{t}_1} &= \frac{(k+m)(2k+2m-1)}{m(2k+m)}B_{(k+m)0} + \frac{(k-m)(2k-2m+1)}{m(2k-m)}B_{(k-m)0}, \\ \frac{dB_{k0}}{d\bar{t}_1} &= -\frac{(k+m)(2k+2m-1)}{m(2k+m)}A_{(k+m)0} - \frac{(k-m)(2k-2m+1)}{m(2k-m)}A_{(k-m)0}. \end{aligned}$$

It should be noticed that for $m = 1$ this system of ordinary differential equations is reduced to system (2.4.5). In this section we will study system (2.4.5), which is a coupled system of infinitely many ordinary differential equations.

Application of the truncation method

First we will try to find an approximation of the solution of system (2.4.5) by using Galerkin's truncation method. So, we will use just some first few modes and neglect the higher order modes. For example, in the case we consider the first 3 modes, we obtain from (2.4.5):

$$\dot{X} = AX, \quad (2.4.6)$$

$$\text{where: } X = \begin{pmatrix} B_{10} \\ A_{10} \\ B_{20} \\ A_{20} \\ B_{30} \\ A_{30} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

and where \dot{X} represents the derivative of X with respect to \bar{t}_1 . This system has eigenvalues $2\sqrt{2}i$, $-2\sqrt{2}i$, and 0, all with multiplicity 2. Their associated eigenvectors are: $(0, 1, \sqrt{2}i, 0, 0, 1)$, $(1, 0, 0, -\sqrt{2}i, 1, 0)$, $(1, 0, 0, \sqrt{2}i, 1, 0)$, $(0, 1, -\sqrt{2}i, 0, 0, 1)$, $(-3, 0, 0, 0, 1, 0)$ and $(0, -3, 0, 0, 0, 1)$, respectively. The solution of (2.4.6) is then given by:

$$\begin{aligned} B_{10}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) - 3C_5, \\ A_{10}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) - 3C_6, \\ B_{20}(t_1) &= -\sqrt{2}C_1 \sin(2\sqrt{2}t_1) + \sqrt{2}C_2 \cos(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\ A_{20}(t_1) &= \sqrt{2}C_3 \sin(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\ B_{30}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) + C_5, \\ A_{30}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) + C_6, \end{aligned} \quad (2.4.7)$$

where C_1, C_2, \dots, C_6 are all constants of integration. Note that we have dropped all the bars in (3.4.2).

From the initial conditions (2.2.5), that is, $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ it follows that

$$f(x) = \sum_{k=1}^{\infty} u_k(0; \epsilon) \sin\left(\frac{k\pi x}{L}\right) \Leftrightarrow u_k(0; \epsilon) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx,$$

$$g(x) = \sum_{k=1}^{\infty} \dot{u}_k(0; \epsilon) \sin\left(\frac{k\pi x}{L}\right) \Leftrightarrow \dot{u}_k(0; \epsilon) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx.$$

Moreover, since $u_k(0; \epsilon) = v_k(0, 0; \epsilon) = v_{k0}(0, 0) + \epsilon v_{k1}(0, 0) + \dots$ and $\dot{u}_k(0; \epsilon) = \dot{v}_k(0, 0; \epsilon) = \dot{v}_{k0}(0, 0) + \epsilon \dot{v}_{k1}(0, 0) + \dots$ it follows that

$$v_{k0}(0, 0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad \dot{v}_{k0}(0, 0) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx. \quad (2.4.8)$$

From (3.3.5) and (2.4.8) we then obtain

$$A_{k0}(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad \text{and} \quad B_{k0}(0) = \frac{2}{ck\pi} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx. \quad (2.4.9)$$

Equation (2.4.9) can be used to calculate the constants in (3.4.2).

In summary, after all constants in (3.4.2) have been calculated, $v_{k0}(t_0, t_1)$ can be determined using (3.3.5). Then $u(x, t; \epsilon)$ can be approximated by $\sum_{k=1}^3 u_k(t; \epsilon) \sin\left(\frac{k\pi x}{L}\right)$.

For example, using 1, 2, or 3 modes, respectively, with $f(x) = \frac{-8}{\pi^3} \sin(\pi x)$, $g(x) = 0$, $c = L = 1$ we find as approximations for $u(x, t; \epsilon)$:

$$u(x, t; \epsilon) \approx \frac{-8}{\pi^3} \cos(\pi t_0) \sin(\pi x),$$

$$u(x, t; \epsilon) \approx \frac{-8}{\pi^3} \cos(\sqrt{2}t_1) \cos(\pi t_0) \sin(\pi x) + \frac{4\sqrt{2}}{\pi^3} \sin(\sqrt{2}t_1) \sin(2\pi t_0) \sin(2\pi x),$$

$$u(x, t; \epsilon) \approx \left(-\frac{2}{\pi^3} \cos(2\sqrt{2}t_1) - \frac{6}{\pi^3} \right) \cos(\pi t_0) \sin(\pi x) + \frac{2\sqrt{2}}{\pi^3} \sin(2\sqrt{2}t_1) \sin(2\pi t_0) \sin(2\pi x) + \left(\frac{-2}{\pi^3} \cos(2\sqrt{2}t_1) + \frac{2}{\pi^3} \right) \cos(3\pi t_0) \sin(3\pi x). \quad (2.4.10)$$

The graphs of these approximations for $u(x, t)$ for $x = 0.5$ and $\epsilon = 0.01$ are depicted in Figure 2.2.

For more than three modes, eigenvalues and eigenvectors become more and more difficult to compute by just using pencil and paper. Using the computer software package Maple, the eigenvalues of system (2.4.5) have been computed up to 20 modes and are listed in Table 1. From the table, it can be seen that the eigenvalues of the truncated system are always purely imaginary, each has multiplicity two, and for an odd number of modes we get an additional pair of zero eigenvalues. From the approximations (2.4.10) and from Table 1 it can readily be seen that the truncation method will not give accurate results on long time-scales, that is, on time-scales of order ϵ^{-1} . On the other hand it is well-known in mathematics that if the truncated system has only purely imaginary eigenvalues and/or eigenvalues equal to zero then no conclusions can be drawn for the infinite dimensional system.

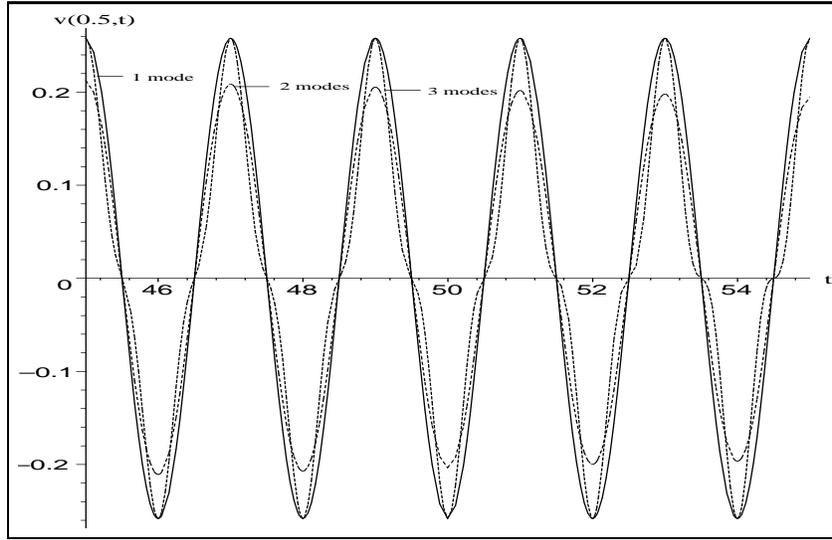


Figure 2.2: Approximations for $u(x, t)$ with initial displacement $f(x) = \frac{-8}{\pi^3} \sin(\pi x)$ and initial velocity $g(x) = 0$. The graphs are given for $x = 0.5, t \in [45, 55]$, and $\epsilon = 0.01$.

Analysis of the infinite dimensional system (2.4.5)

In the previous subsection we found that if system (2.4.5) is truncated then the eigenvalues of the truncated system are always purely imaginary or zero. In this section we shall show that the results obtained by applying the truncation method are not valid on time-scales of order ϵ^{-1} .

By putting $kB_{k0}(t_1) = Y_{k0}(t_1)$ and $kA_{k0}(t_1) = X_{k0}(t_1)$, system (2.4.5) becomes:

$$\frac{dY_{k0}}{dt_1} = k[-X_{(k+1)0} - X_{(k-1)0}], \quad \frac{dX_{k0}}{dt_1} = k[Y_{(k+1)0} + Y_{(k-1)0}], \quad (2.4.11)$$

for $k = 1, 2, 3, \dots$, and $X_{00} = Y_{00} = 0$.

Accordingly we also have:

$$\begin{aligned} Y_{k0} \dot{Y}_{k0} &= -k[Y_{k0}X_{(k+1)0} + Y_{k0}X_{(k-1)0}], \\ X_{k0} \dot{X}_{k0} &= k[X_{k0}Y_{(k+1)0} + X_{k0}Y_{(k-1)0}]. \end{aligned} \quad (2.4.12)$$

By adding both equations in (2.4.12), and then by taking the sum from $k = 1$ to ∞ we obtain:

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{d}{dt_1} (Y_{k0}^2 + X_{k0}^2) = \sum_{k=1}^{\infty} [X_{(k+1)0}Y_{k0} - Y_{(k+1)0}X_{k0}]. \quad (2.4.13)$$

By differentiating (2.4.13) with respect to t_1 we find (see also appendix 2)

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{d^2}{dt_1^2} (Y_{k0}^2 + X_{k0}^2) = 2 \sum_{k=1}^{\infty} (X_{k0}^2 + Y_{k0}^2), \quad (2.4.14)$$

| No. of modes | Eigenvalues of matrix A (all multiplicity 2) | Dimension eigen-space of A |
|--------------|--|------------------------------|
| 1 | 0 | 2 |
| 2 | $\pm\sqrt{2}i$ | 4 |
| 3 | $0, \pm 2\sqrt{2}i$ | 6 |
| 4 | $\pm 1.13i, \pm 4.33i$ | 8 |
| 5 | $0, \pm 2.30i, \pm 5.89i$ | 10 |
| 6 | $\pm 7.50i, \pm 1.00i, \pm 3.56i$ | 12 |
| 7 | $0, \pm 9.15i, \pm 2.05i, \pm 4.90i$ | 14 |
| 8 | $\pm 10.83i, \pm 0.93i, \pm 3.18i, \pm 6.30i,$ | 16 |
| 9 | $0, \pm 12.54i, \pm 1.89i, \pm 4.38i, \pm 7.74i$ | 18 |
| 10 | $\pm 14.26i, \pm 0.87i, \pm 5.65i, \pm 9.23i, \pm 2.93i$ | 20 |
| 11 | $0, \pm 16.01i, \pm 1.78i, \pm 4.05i, \pm 6.97i, \pm 10.76i$ | 22 |
| 12 | $\pm 17.76i, \pm 0.83i, \pm 2.76i, \pm 5.22i, \pm 8.33i, \pm 12.31i$ | 24 |
| 13 | $0, \pm 19.53i, \pm 1.70i, \pm 3.81i, \pm 6.45i, \pm 9.73i, \pm 13.88i,$ $\pm 19.53i$ | 26 |
| 14 | $\pm 21.31i, \pm 15.48i, \pm 0.80i, \pm 2.63i, \pm 4.92i, \pm 7.72i,$ $\pm 11.16i$ | 28 |
| 15 | $0, \pm 23.11i, \pm 17.10i, \pm 1.64i, \pm 3.63i, \pm 6.07i, \pm 9.03i,$ $\pm 12.63i$ | 30 |
| 16 | $\pm 24.91i, \pm 18.73i, \pm 0.78i, \pm 2.53i, \pm 4.68i, \pm 7.28i,$ $\pm 10.38i, \pm 14.11i$ | 32 |
| 17 | $0, \pm 26.71i, \pm 20.38i, \pm 1.58i, \pm 3.49i, \pm 5.79i, \pm 8.52i,$ $\pm 11.75i, \pm 15.62i$ | 34 |
| 18 | $\pm 28.53i, \pm 22.05i, \pm 0.75i, \pm 2.45i, \pm 4.50i, \pm 6.93i,$ $\pm 9.79i, \pm 13.16i, \pm 17.15i$ | 36 |
| 19 | $0, \pm 30.35i, \pm 23.72i, \pm 1.54i, \pm 3.37i, \pm 5.55i, \pm 8.12i,$ $\pm 11.10i, \pm 14.58i, \pm 18.70i$ | 38 |
| 20 | $\pm 32.18i, \pm 25.41i, \pm 0.73i, \pm 2.38i, \pm 4.34i, \pm 6.65i,$ $\pm 9.33i, \pm 12.43i, \pm 16.03i, \pm 20.27i$ | 40 |

Table 2.1: **Approximations of the eigenvalues of the truncated system (2.4.5).**

and so, by putting $\sum_{k=1}^{\infty} (X_{k0}^2 + Y_{k0}^2) = W(t_1)$ we finally obtain:

$$\frac{d^2 W(t_1)}{dt_1^2} - 4W(t_1) = 0. \quad (2.4.15)$$

The solution of (2.4.15) is $W(t_1) = K_1 e^{2t_1} + K_2 e^{-2t_1}$, where K_1 and K_2 are constants. Note that $W(t_1)$ is a first integral of system (2.4.5). K_1 and K_2 are both positive

numbers as is shown in the following calculation. From $W(t_1) = \sum_{k=1}^{\infty} [X_{k0}^2 + Y_{k0}^2]$ it follows that

$$W(0) = \sum_{k=1}^{\infty} [X_{k0}^2(0) + Y_{k0}^2(0)] \geq 0 \Rightarrow K_1 + K_2 \geq 0 \quad (2.4.16)$$

Differentiating $W(t_1)$ with respect to t_1 and then putting $t_1 = 0$ we get:

$$K_1 - K_2 = \sum_{k=1}^{\infty} [Y_{k0}(0)X_{(k+1)0}(0) - X_{k0}(0)Y_{(k+1)0}(0)]. \quad (2.4.17)$$

From (2.4.16) and (2.4.17) it then follows that

$$\begin{aligned} 2K_1 &= \sum_{k=1}^{\infty} \left[X_{k0}^2(0) + Y_{k0}^2(0) + Y_{k0}(0)X_{(k+1)0}(0) - X_{k0}(0)Y_{(k+1)0}(0) \right] \\ &= \frac{1}{2}X_{10}^2(0) + \frac{1}{2}Y_{10}^2(0) + \frac{1}{2}\left(X_{10}(0) - Y_{20}(0)\right)^2 + \frac{1}{2}\left(Y_{10}(0) + X_{20}(0)\right)^2 \\ &\quad + \frac{1}{2}\left(X_{20}(0) - Y_{30}(0)\right)^2 + \frac{1}{2}\left(Y_{20}(0) + X_{30}(0)\right)^2 + \dots + \\ &\quad \frac{1}{2}\left(X_{n0}(0) - Y_{(n+1)0}(0)\right)^2 + \frac{1}{2}\left(Y_{n0}(0) + X_{(n+1)0}(0)\right)^2 + \dots \\ &\geq 0. \end{aligned} \quad (2.4.18)$$

So, $K_1 \geq 0$ and 0 if and only if $X_{k0}(0) = Y_{k0}(0) = 0$ for each $k = 1, 2, 3, \dots$. Using a similar method, K_2 also can be shown to be a non-negative number. Consequently, $W(t_1)$ is, in general, non-negative and increases as t_1 increases. This behavior is different from the behavior of $A_{k0}(t_1)$ and $B_{k0}(t_1)$ as obtained by applying the truncation method. If we apply the truncation method, we merely obtain sin and cos functions for A_{k0} and B_{k0} while the energy (see next subsection) is described by exponential functions. This means that the approximations obtained by applying the truncation method to system (2.4.5) are not accurate on long time-scales, that is, on time-scales of order ϵ^{-1} .

The energy

The energy $E(t)$ of the conveyor belt system can also be approximated using the function $W(t_1)$. Since

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} u_k(t) \sin\left(\frac{k\pi x}{L}\right) \\ &= \sum_{k=1}^{\infty} \left[A_{k0}(t_1) \cos\left(\frac{ck\pi t}{L}\right) + B_{k0}(t_1) \sin\left(\frac{ck\pi t}{L}\right) \right] \sin\left(\frac{k\pi x}{L}\right) + \mathcal{O}(\epsilon) \end{aligned} \quad (2.4.19)$$

it follows that the energy $E(t)$ satisfies

$$\begin{aligned}
E(t) &= \frac{1}{2} \int_0^L [(u_t + Vu_x)^2 + c^2 u_x^2] dx \\
&= \frac{c^2 \pi^2}{4L} \sum_{k=1}^{\infty} k^2 \left[\left(-A_{k0} \sin\left(\frac{k\pi t}{L}\right) + B_{k0} \cos\left(\frac{k\pi t}{L}\right) \right)^2 + \right. \\
&\quad \left. \left(A_{k0} \cos\left(\frac{ck\pi t}{L}\right) + B_{k0} \sin\left(\frac{ck\pi t}{L}\right) \right)^2 \right] + \mathcal{O}(\epsilon) \\
&= \frac{c^2 \pi^2}{4L} \sum_{k=1}^{\infty} [(kA_{k0})^2 + (kB_{k0})^2] + \mathcal{O}(\epsilon) \\
&= \frac{c^2 \pi^2}{4L} \sum_{k=1}^{\infty} [X_{k0}^2 + Y_{k0}^2] + \mathcal{O}(\epsilon) \\
&= \frac{c^2 \pi^2}{4L} W(t_1) + \mathcal{O}(\epsilon) \\
&= \frac{c^2 \pi^2}{4L} (K_1 e^{2t_1} + K_2 e^{-2t_1}) + \mathcal{O}(\epsilon). \tag{2.4.20}
\end{aligned}$$

So, the energy increases, although it is bounded on a time-scale of order $\frac{1}{\epsilon}$.

2.4.2 Case 2: $\Omega = \frac{c\pi}{L} + \epsilon\delta$

In this section we will consider the detuning from $\Omega = \frac{c\pi}{L}$, that is, we will study the case $\Omega = \frac{c\pi}{L} + \epsilon\delta$ where $\delta = \mathcal{O}(1)$. In order to avoid secular terms in the approximation, it can be shown (the calculation are similar to those in section 4.1) that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\begin{aligned}
\frac{dA_{k0}}{dt_1} &= (k+1) \left[B_{(k+1)0} \cos(\delta \bar{t}_1) + A_{(k+1)0} \sin(\delta \bar{t}_1) \right] \\
&\quad + (k-1) \left[B_{(k-1)0} \cos(\delta \bar{t}_1) - A_{(k-1)0} \sin(\delta \bar{t}_1) \right], \\
\frac{dB_{k0}}{dt_1} &= -(k+1) \left[A_{(k+1)0} \cos(\delta \bar{t}_1) - B_{(k+1)0} \sin(\delta \bar{t}_1) \right] \\
&\quad - (k-1) \left[A_{(k-1)0} \cos(\delta \bar{t}_1) + B_{(k-1)0} \sin(\delta \bar{t}_1) \right], \tag{2.4.21}
\end{aligned}$$

for $k = 1, 2, 3, \dots$. It should be noticed that for $\delta = 0$ we obtain again system (2.4.5). For convenience, we will drop the bar from \bar{t}_1 .

The calculations as given on page 21 can be followed again, and we obtain:

$$\frac{d^2 W(t_1)}{dt_1^2} + (\delta^2 - 4)W(t_1) = D_1 \delta^2, \tag{2.4.22}$$

where $W(t_1)$ is defined as in section 4.1.2, and $D_1 = W(0)$. Elementary calculations then yield:

$$\text{for } |\delta| < 2 : W(t_1) = \frac{D_1}{4 - \delta^2} \left[4 \cosh(t_1 \sqrt{4 - \delta^2}) - \delta^2 \right] + \frac{D_2}{\sqrt{4 - \delta^2}} \sinh(t_1 \sqrt{4 - \delta^2}),$$

$$\begin{aligned}
&\text{for } |\delta| = 2 : W(t_1) = D_1 + D_2 t_1 + \frac{1}{2} D_1 \delta^2 t_1^2, \\
&\text{for } |\delta| > 2 : W(t_1) = \frac{D_1}{\delta^2 - 4} \left[\delta^2 - 4 \cos(t_1 \sqrt{\delta^2 - 4}) \right] + \frac{D_2}{\sqrt{\delta^2 - 4}} \sin(t_1 \sqrt{\delta^2 - 4}),
\end{aligned} \tag{2.4.23}$$

where $D_2 = \frac{dW(0)}{dt_1}$. The interesting features of these solutions are, that for $|\delta| < 2$, $W(t_1)$ (and so the energy) increases exponentially. For $|\delta| = 2$, $W(t_1)$ increases polynomially, and finally for $|\delta| > 2$, $W(t_1)$ is bounded due to the trigonometric functions.

2.4.3 Case 3: The non-resonant case

If Ω is not within an order ϵ -neighborhood of the frequencies that cause internal resonance, that is, not within an order ϵ -neighborhood of $m \frac{c\pi}{L}$ (with m odd) then $A_{k_0}(t_1)$ and $B_{k_0}(t_1)$ have to satisfy

$$\frac{dA_{k_0}}{dt_1} = 0, \quad \frac{dB_{k_0}}{dt_1} = 0, \tag{2.4.24}$$

in order to avoid secular terms. Consequently, $A_{k_0}(t_1)$ and $B_{k_0}(t_1)$ are constants, say $K1_{k_0}$ and $K2_{k_0}$. So, we have $u_{k_0}(t_0, t_1) = K1_{k_0} \cos(\frac{ck\pi t_0}{L}) + K2_{k_0} \sin(\frac{ck\pi t_0}{L})$. Since $u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(\frac{k\pi x}{L})$, where $u_k(t)$ is approximated by $v_{k_0}(t_0, t_1)$, it follows from the initial conditions $v(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ that

$$K1_{k_0} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad \text{and} \quad K2_{k_0} = \frac{2}{ck\pi} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx. \tag{2.4.25}$$

The energy $E(t)$ of the conveyor belt system for this case can be approximated from:

$$u(x, t) \approx \sum_{k=1}^{\infty} \left(K1_{k_0} \cos\left(\frac{ck\pi t_0}{L}\right) + K2_{k_0} \sin\left(\frac{ck\pi t_0}{L}\right) \right) \sin\left(\frac{k\pi x}{L}\right) + \mathcal{O}(\epsilon), \tag{2.4.26}$$

where $K1_{k_0}$ and $K2_{k_0}$ are given by (2.4.25). Then,

$$\begin{aligned}
E(t) &= \int_0^L (u_t^2 + c^2 u_x^2) dx + \mathcal{O}(\epsilon), \\
&= \sum_{k=1}^{\infty} \frac{(ck\pi)^2}{2L} (K1_{k_0}^2 + K2_{k_0}^2) + \mathcal{O}(\epsilon), \\
&= \frac{c^2 \pi^2}{2L} \sum_{k=1}^{\infty} k^2 (K1_{k_0}^2 + K2_{k_0}^2) + \mathcal{O}(\epsilon).
\end{aligned} \tag{2.4.27}$$

Using (2.4.25), we finally obtain:

$$\begin{aligned}
E(t) &= \frac{2c^2 L}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left[\int_0^L f'' \sin\left(\frac{k\pi x}{L}\right) dx \right]^2 + \\
&\quad \frac{2L^3}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{k^4} \left[\int_0^L g'' \sin\left(\frac{k\pi x}{L}\right) dx \right]^2 + \mathcal{O}(\epsilon) \\
&= \text{constant} + \mathcal{O}(\epsilon).
\end{aligned} \tag{2.4.28}$$

2.5 Conclusions

In this chapter we studied initial-boundary value problems which can be used as models to describe transversal vibrations of belt systems. The belt is assumed to move with a non-constant, low velocity $V(t)$, that is, $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, where $0 < \epsilon \ll 1$ and V_0, α, Ω are constants. Formal approximations of the solution of the initial-boundary value problem have been constructed. Also explicit approximations of the energy of the belt system are given. It turns out that there are infinitely many values of Ω giving rise to internal resonances in the belt system. These values for Ω are $m\frac{c\pi}{L} + \epsilon\delta$ where m is an arbitrary odd integer, $\frac{c\pi}{L}$ is the lowest natural frequency of the constant velocity system, and δ is a detuning parameter of $\mathcal{O}(1)$. For $\Omega = \frac{c\pi}{L} + \epsilon\delta$ (that is, $m = 1$) the problem has been studied completely. The following interesting results have been found: for $|\delta| < 2$ the energy of the belt system increases exponentially, for $|\delta| = 2$ the energy increases polynomially, and for $|\delta| > 2$ the energy is bounded and varies trigonometrically. When Ω is not in an order ϵ -neighborhood of $m\frac{c\pi}{L}$ (with m odd) the energy of the belt system is constant up to order ϵ . All the results found are valid on long time-scales, that is, on time-scales of order ϵ^{-1} .

One major conclusion of this chapter is that the truncation method can not be applied to obtain asymptotic results on long time-scales (that is, on time-scales of order ϵ^{-1}) when Ω is in an order ϵ -neighborhood of an odd multiple of the lowest natural frequency of the constant velocity system. Moreover, in this chapter we improve the (incorrect) results and applied methods as for instance given and used in ([9] - [12]) for low speed belt systems.

Appendix 1

To avoid secular terms in the approximation for $u(x, t; \epsilon)$ we will show in this appendix that the function $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\begin{aligned} \frac{dA_{k0}(t_1)}{dt_1} &= (k+1)B_{(k+1)0}(t_1) + (k-1)B_{(k-1)0}(t_1), \\ \frac{dB_{k0}(t_1)}{dt_1} &= -(k+1)A_{(k+1)0}(t_1) - (k-1)A_{(k-1)0}(t_1) \end{aligned} \quad (\text{A-1})$$

for $k = 1, 2, 3, \dots$. This can be derived as follows. After introducing a slow and a fast time in section 2.4, we obtain the equation (2.4.3) with $\Omega = \frac{c\pi}{L}$. The solution of the $\mathcal{O}(1)$ problem is $u_{k0}(t_0, t_1) = A_{k0}(t_1) \cos(\frac{ck\pi t_0}{L}) + B_{k0}(t_1) \sin(\frac{ck\pi t_0}{L})$, where A_{k0} and B_{k0} can be determined from the $\mathcal{O}(\epsilon)$ equation by removing terms in the right hand side of this equation that cause secular terms in $u_{k1}(t_0, t_1)$.

The first term in the right hand side of the $\mathcal{O}(\epsilon)$ equation causing secular terms is $-2\frac{\partial^2 u_{k0}}{\partial t_0 \partial t_1} = 2\frac{ck\pi}{L} [\frac{dA_{k0}}{dt_1} \sin(\frac{ck\pi t_0}{L}) + \frac{dB_{k0}}{dt_1} \cos(\frac{ck\pi t_0}{L})]$.

Taking apart those terms in the second term of the right hand side the $\mathcal{O}(\epsilon)$ equation that cause secular terms, we find:

$$\begin{aligned} & \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{2n\alpha\Omega}{(2j+1)L} \cos(\Omega t_0) u_{n0} = \\ & \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{2n\alpha\Omega}{(2j+1)L} \cos(\Omega t_0) \left[A_{n0}(t_1) \cos\left(\frac{cn\pi t_0}{L}\right) \right. \\ & \quad \left. + B_{n0}(t_1) \sin\left(\frac{cn\pi t_0}{L}\right) \right] = \frac{\alpha c\pi}{L^2} \cos\left(\frac{ck\pi t_0}{L}\right) \left[(k+1)A_{(k+1)0} \right. \\ & \quad \left. - (k-1)A_{(k-1)0} - \frac{k+1}{2k+1}A_{(k+1)0} - \frac{k-1}{2k-1}A_{(k-1)0} \right] + \\ & \quad \frac{\alpha c\pi}{L^2} \sin\left(\frac{ck\pi t_0}{L}\right) \left[(k+1)B_{(k+1)0} - (k-1)B_{(k-1)0} - \right. \\ & \quad \left. \frac{k+1}{2k+1}B_{(k+1)0} - \frac{k-1}{2k-1}B_{(k-1)0} \right] + \text{"terms not giving} \\ & \quad \text{rise to secular terms in } u_{k1} \text{"} \end{aligned}$$

Similarly we find for the third term:

$$\begin{aligned} & \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{4n}{(2j+1)L} (V_0 + \alpha \sin(\Omega t_0)) \frac{\partial u_{n0}}{\partial t_0} = \\ & \left[\sum_1 - \sum_2 - \sum_3 \right] \frac{4n}{(2j+1)L} (V_0 + \alpha \sin(\Omega t_0)) \frac{cn\pi}{L} \left[B_{n0} \cos\left(\frac{cn\pi t_0}{L}\right) - \right. \\ & \quad \left. A_{n0} \sin\left(\frac{cn\pi t_0}{L}\right) \right] = \frac{\alpha c\pi}{L^2} \cos\left(\frac{ck\pi t_0}{L}\right) \left[-2(k+1)^2 A_{(k+1)0} \right. \\ & \quad \left. - 2(k-1)^2 A_{(k-1)0} + \frac{2(k+1)^2}{2k+1} A_{(k+1)0} - \frac{2(k-1)^2}{2k-1} A_{(k-1)0} \right] + \\ & \quad \frac{\alpha c\pi}{L^2} \sin\left(\frac{ck\pi t_0}{L}\right) \left[-2(k+1)^2 B_{(k+1)0} - 2(k-1)^2 B_{(k-1)0} + \right. \\ & \quad \left. \frac{2(k+1)^2}{2k+1} B_{(k+1)0} - \frac{2(k-1)^2}{2k-1} B_{(k-1)0} \right] + \text{"terms not giving} \\ & \quad \text{rise to secular terms in } u_{k1} \text{"}. \end{aligned}$$

Collecting all terms in the right hand side of the $\mathcal{O}(\epsilon)$ equation containing $\cos\left(\frac{ck\pi t_0}{L}\right)$ and all terms containing $\sin\left(\frac{ck\pi t_0}{L}\right)$ and then setting their coefficients equal to 0 in order to remove the secular terms, we obtain (A-1).

Appendix 2

In this appendix we will show that:

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{d^2}{dt_1^2} (Y_{k0}^2 + X_{k0}^2) = 2 \sum_{k=1}^{\infty} (X_{k0}^2 + Y_{k0}^2). \quad (\text{A-2})$$

From (4.12) and (4.13) it follows that

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \frac{d}{dt_1} (Y_{k0}^2 + X_{k0}^2) &= \sum_{k=1}^{\infty} [Y_{k0} \dot{Y}_{k0} + X_{k0} \dot{X}_{k0}] \\ &= \sum_{k=1}^{\infty} [X_{(k+1)0} Y_{k0} - Y_{(k+1)0} X_{k0}]. \end{aligned}$$

Differentiating this expression with respect to t_1 , and using (4.11) we find:

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \frac{d^2}{dt_1^2} (Y_{k0}^2 + X_{k0}^2) &= \\ &= \sum_{k=1}^{\infty} [\dot{X}_{(k+1)0} Y_{k0} + X_{(k+1)0} \dot{Y}_{k0} - \dot{Y}_{(k+1)0} X_{k0} - Y_{(k+1)0} \dot{X}_{k0}] \\ &= \sum_{k=1}^{\infty} (k+1) [X_{k0}^2 + Y_{k0}^2] - \sum_{m=2}^{\infty} (m-1) [X_{m0}^2 + Y_{m0}^2] \\ &= 2(X_{10}^2 + Y_{10}^2) + \sum_{k=2}^{\infty} (k+1) [X_{k0}^2 + Y_{k0}^2] - \sum_{m=2}^{\infty} (m-1) [X_{m0}^2 + Y_{m0}^2] \\ &= 2(X_{10}^2 + Y_{10}^2) + \sum_{k=2}^{\infty} [(k+1) - (k-1)] [X_{k0}^2 + Y_{k0}^2] \\ &= 2(X_{10}^2 + Y_{10}^2) + \sum_{k=2}^{\infty} 2[X_{k0}^2 + Y_{k0}^2] = 2 \sum_{k=1}^{\infty} [X_{k0}^2 + Y_{k0}^2]. \end{aligned}$$

And so, (A-2) has been proved.

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Chapter 3

On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part II: The Beam-like Case [‡]

Abstract. In this chapter initial-boundary value problems for a beam equation (with string effect) are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed. Again it is assumed that the belt is moving with a low and time-varying velocity $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behavior of the belt. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or difference of any two natural frequencies of the system with zero belt velocity. For special values of the belt parameters these sum type and difference type of internal resonances coincide giving rise to even more complicated dynamical behaviour. Some examples (including detuning cases) will be studied in detail.

3.1 Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper sheet and web processes, fiber winding and band saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations.

Investigating transverse vibrations of a belt system is a challenging subject which

[‡]This chapter is a revised version of [1] *On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part II: The Beam-Like Case.*

has been studied for many years (see [2]- [5] for a recent overview) and is still of interest today. The vibrations can be classified into two types, i.e. whether it is of a string-like type or of beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibration of a belt system (with constant belt velocity V) can be modeled as:

- string-like by

$$u_{tt} + 2Vu_{xt} + (\kappa V^2 - c^2)u_{xx} = 0, \quad \text{and} \quad (3.1.1)$$

- beam-like (with a string effect) by

$$u_{tt} + 2Vu_{xt} + (\kappa V^2 - c^2)u_{xx} + \frac{EI}{\rho A}u_{xxxx} = 0, \quad (3.1.2)$$

where $V, \kappa, c, E, I, \rho,$ and A are constants as those explained in Chapter 1 and $u(x, t)$ is transversal vibrations.

The main purpose of studying the dynamic behavior of a belt system is to know the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see [4]). Resonances that can cause severe vibrations can be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support components, and the increase of energy consumption of the system.

In this chapter vibrations of a moving belt which is modeled by a beam equation with a string effect will be studied. The belt speed is considered to be time-varying and to be small compared to the wave speed. In [6] a string-like model for a similar belt system has been studied. It will turn out that the beam-like model and the string-like model give rise to different behaviour of the solutions. It is assumed that the low and time-varying belt speed $V(t)$ is given by $\epsilon(V_0 + \alpha \sin(\Omega t))$, where $\epsilon, V_0, \alpha,$ and Ω are constants with $0 < \epsilon \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter ϵ indicates that the belt speed $V(t)$ is small compared to the wave speed c . Recently the authors of [7] also studied the vibrations of an axially moving beam with a time-dependent velocity. As has been pointed out in [6] their application of the truncation method does not give approximations which are valid on long time-scales of order ϵ^{-1} . More results on axially moving strings and beams can also be found in [13], [14] and [15]. The variation in $V(t)$ can be considered as some kind of excitation. In relation to excitations, some results in this area have been obtained by Sack [8] and Archibald and Emslie [9]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e. $x = L$) is subjected to a harmonic excitation. In [8] the vibrations of the string at $x = L$ is forced to be $u(x, t) = u_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a

harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingam in [10]. A moving string model to study the transverse vibrations of power transmission chains has been used in [10]. In all of these works, the belt movement is assumed to be constant.

This chapter is organized as follows. In section 3.2 the equation of motion describing the dynamic behavior of a belt moving with a non-constant velocity is derived. The belt is assumed to be simply supported in vertical direction. Then in section 3.3, the two time-scales perturbation method is used to find approximations of the solution of the problem. It will turn out in section 3.3 that complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or the difference of any two natural frequencies ω_k and ω_n of the zero belt-velocity system. For special values of the belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. In section 3.4 of this chapter, the (difference type) case $\Omega = \omega_2 - \omega_1$ and the detuned case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one will be studied. While in section 3.5, the (sum type) cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$ will be considered. For some special values of the beam parameters the case (including detuning) for which a sum type and a difference type of internal resonance coincide will also be studied: that is, the case $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$. Finally in section 6 of this chapter some remarks and some conclusions will be drawn.

3.2 A beam model

If the belt speed V is not constant but a function of t , then (3.1.2) becomes:

$$u_{tt} + (\kappa V^2 - c^2)u_{xx} + 2Vu_{xt} + V_t u_x + \frac{EI}{\rho A} u_{xxxx} = 0, \quad (3.2.1)$$

for $0 < x < L, t > 0$. The meanings of all the symbols have been explained in Chapter 2. Since the beam is assumed to be simply supported, it will follow that the boundary conditions are:

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0. \quad (3.2.2)$$

The initial values are given by:

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad (3.2.3)$$

where f is the initial displacement of the beam, and where g is the initial velocity of the beam. Considering the case where $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, in which V_0 and α are constants and $V_0 > |\alpha|$, equation (3.2.1) becomes:

$$u_{tt} - c^2 u_{xx} + \frac{EI}{\rho A} u_{xxxx} = -\epsilon \Omega \alpha \cos(\Omega t) u_x - 2\epsilon (V_0 + \alpha \sin(\Omega t)) u_{xt} + \epsilon^2 \kappa (V_0 + \alpha \sin(\Omega t))^2 u_{xx}. \quad (3.2.4)$$

It should be noticed that (3.2.4) is a subcase of a problem which has been studied by Öz and Pakdemirli in [7].

Solutions of the form $u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$ certainly satisfy the boundary conditions. There are two equivalent methods to determine what equations $u_n(t)$ for $n = 1, 2, 3, \dots$ have to satisfy. The first method is based on the principle of reflections. Using this method the initial-boundary value problem (3.2.1)-(3.2.4) is extended to an initial-value problem. Special attention has to be paid to the terms u_x and u_{xt} in the right-hand side of (3.2.4) when this method is applied. Since this method has already been applied in [6] (and also for instance in [11], [12]) we will now apply the other method which is based on the orthogonality properties of the set of functions $\sin\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, 3, \dots$ on $0 < x < L$. The following should be observed:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx = \begin{cases} 0 & \text{for } n \neq k, \\ \frac{1}{2}L & \text{for } n = k, \end{cases} \quad (3.2.5)$$

$$\text{and } \int_0^L \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx = \begin{cases} 0 & \text{for } n \pm k \text{ is even,} \\ \frac{-2Lk}{(n^2 - k^2)\pi} & \text{for } n \pm k \text{ is odd.} \end{cases} \quad (3.2.6)$$

Substituting $u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$ into (3.2.4) gives:

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\ddot{u}_n + \left\{ \left(\frac{cn\pi}{L}\right)^2 + \delta \left(\frac{n\pi}{L}\right)^4 \right\} u_n \right] \sin\left(\frac{n\pi x}{L}\right) = \\ -\epsilon \sum_{n=1}^{\infty} \frac{n\pi}{L} \left[\alpha \Omega \cos(\Omega t) u_n + 2(V_0 + \alpha \sin(\Omega t)) \dot{v}_n \right] \cos\left(\frac{n\pi x}{L}\right) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (3.2.7)$$

where $\delta = \frac{EI}{\rho A}$. Multiplying both sides of (3.2.7) with $\sin\left(\frac{k\pi x}{L}\right)$, and then integrating with respect to x from $x = 0$ to $x = L$ gives (using (3.2.5) and (3.2.6)):

$$\begin{aligned} \ddot{u}_k + \left\{ \left(\frac{ck\pi}{L}\right)^2 + \delta \left(\frac{k\pi}{L}\right)^4 \right\} u_k = \epsilon \sum_{n=1}^{\infty *} \frac{nk}{(n^2 - k^2)L} \left[4\alpha \Omega \cos(\Omega t) u_n + \right. \\ \left. 8(V_0 + \alpha \sin(\Omega t)) \dot{v}_n \right] + \mathcal{O}(\epsilon^2), \end{aligned} \quad (3.2.8)$$

where the $*$ in $\sum_{n=1}^{\infty}$ indicates that the summation is only carried out for $n \pm k$ is odd. For $t = 0$, $u_k(t)$ satisfies: $u_k(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx$, and $\dot{v}_k(0) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx$.

It should be observed that in order to obtain (3.2.8) the terms u_x and u_{xt} in (3.2.4) are in fact expanded in eigenfunctions (i.e. in $\sin\left(\frac{n\pi x}{L}\right)$) of the boundary-value problem (5.2.2), (3.2.4) with $\epsilon = 0$. In [7] these terms were not expanded accordingly (see also appendix 3). In the next sections approximations of the solutions of (3.2.8) will be constructed for different Ω -values.

3.3 Application of the two time-scales perturbation method

Due to occurrence of so-called secular terms a straightforward perturbation method can not be used to solve (3.2.8) approximately. For that reason a two-time-scales perturbation method (with time scales $t_0 = t$ and $t_1 = \epsilon t$) is used. The introduction of these two time scales defines the following transformations:

$$\begin{aligned} u_k(t; \epsilon) &= v_k(t_0, t_1; \epsilon), & \frac{du_k}{dt} &= \frac{\partial v_k}{\partial t_0} + \epsilon \frac{\partial v_k}{\partial t_1}, \\ \frac{d^2 u_k}{dt^2} &= \frac{\partial^2 v_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 v_k}{\partial t_1^2}. \end{aligned} \quad (3.3.1)$$

Substitution of (3.3.1) into (3.2.8) yields:

$$\begin{aligned} \frac{\partial^2 v_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 v_k}{\partial t_1^2} + \left[\left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right] v_k = \\ \epsilon \sum_{n=1}^{\infty *} \frac{nk}{(n^2 - k^2)L} \left[4\alpha\Omega v_n \cos(\Omega t_0) + 8(V_0 + \alpha \sin(\Omega t_0)) \frac{\partial v_n}{\partial t_0} \right] + \mathcal{O}(\epsilon^2), \end{aligned} \quad (3.3.2)$$

Assuming that $v_k(t_0, t_1; \epsilon) = v_{k0}(t_0, t_1) + \epsilon v_{k1}(t_0, t_1) + \epsilon^2 v_{k2}(t_0, t_1) + \dots$ then (3.3.2) becomes:

$$\begin{aligned} \left[\frac{\partial^2 v_{k0}}{\partial t_0^2} + \epsilon \frac{\partial^2 v_{k1}}{\partial t_0^2} + \mathcal{O}(\epsilon^2) \right] + 2\epsilon \left[\frac{\partial^2 v_{k0}}{\partial t_0 \partial t_1} + \epsilon \frac{\partial^2 v_{k1}}{\partial t_0 \partial t_1} + \mathcal{O}(\epsilon^2) \right] + \mathcal{O}(\epsilon^2) + \\ \left\{ \left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right\} (v_{k0} + \epsilon v_{k1} + \mathcal{O}(\epsilon^2)) = \\ \epsilon \sum_{n=1}^{\infty *} \frac{nk}{(n^2 - k^2)L} \left[4\alpha\Omega v_{n0} \cos(\Omega t_0) + 8(V_0 + \alpha \sin(\Omega t_0)) \frac{\partial v_{n0}}{\partial t_0} \right] + \mathcal{O}(\epsilon^2) \end{aligned} \quad (3.3.3)$$

By taking together terms of equal powers in ϵ from (3.3.3) the following $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ equations will be obtained:

$$\begin{aligned} \mathcal{O}(1) &: \frac{\partial^2 v_{k0}}{\partial t_0^2} + \left[\left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right] v_{k0} = 0, \\ \mathcal{O}(\epsilon) &: \frac{\partial^2 v_{k1}}{\partial t_0^2} + 2 \frac{\partial^2 v_{k0}}{\partial t_0 \partial t_1} + \left[\left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right] v_{k1} = \\ &\sum_{n=1}^{\infty *} \frac{nk}{(n^2 - k^2)L} \left[4\alpha\Omega v_{n0} \cos(\Omega t_0) + 8(V_0 + \alpha \sin(\Omega t_0)) \frac{\partial v_{n0}}{\partial t_0} \right]. \end{aligned} \quad (3.3.4)$$

The $\mathcal{O}(1)$ equation can be easily solved, yielding:

$$v_{k0} = A_{k0}(t_1) \sin(\omega_k t_0) + B_{k0}(t_1) \cos(\omega_k t_0), \quad (3.3.5)$$

where:

$$\omega_k^2 = \left(\frac{ck\pi}{L}\right)^2 + \delta \left(\frac{k\pi}{L}\right)^4, \quad B_{k0}(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx,$$

and

$$A_{k0}(0) = \frac{2}{\omega_k L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx. \quad (3.3.6)$$

The $A_{k0}(t_1)$ and $B_{k0}(t_1)$ in (3.3.5) are still arbitrary and can be used to avoid secular terms in the solution of the $\mathcal{O}(\epsilon)$ equation (3.3.4).

The $\mathcal{O}(\epsilon)$ equation now becomes

$$\frac{\partial^2 v_{k1}}{\partial t_0^2} + \omega_k^2 v_{k1} = -2\omega_k \left[\dot{A}_{k0}(t_1) \cos(\omega_k t_0) - \dot{B}_{k0}(t_1) \sin(\omega_k t_0) \right] +$$

$$\sum_{n=1}^{\infty} \frac{nk}{(n^2 - k^2)L} \left(4\alpha\Omega \cos(\Omega t_0) [A_{n0} \sin(\omega_n t_0) + B_{n0} \cos(\omega_n t_0)] \right.$$

$$\left. + 8(V_0 + \alpha \sin(\Omega t_0)) \omega_n [A_{n0} \cos(\omega_n t_0) - B_{n0} \sin(\omega_n t_0)] \right). \quad (3.3.7)$$

From (3.3.7) it can readily be seen that there are infinitely many values of Ω that can cause internal resonances. In fact these values are (in an $\mathcal{O}(\epsilon)$ neighbourhood of) $\omega_n + \omega_k, \omega_n - \omega_k, \omega_k - \omega_n$, and $-(\omega_n + \omega_k)$, where $k = n - 2j - 1$, or $k = 2j + 1 + n$ or $k = 2j + 1 - n$ for $j = 0, 1, 2, \dots$, (see also the summation in (3.3.7)). To show how the secular terms can be eliminated and how the belt system can behave, the (difference type) case $\Omega = \omega_2 - \omega_1$ and its detuned case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one will be studied in section 3.4 while the (sum type) cases $\Omega = \omega_2 + \omega_1, \Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$ will be considered in section 3.5. For some special values of the beam parameters the case (including detuning) for which a sum type and a difference type of internal resonance coincide will also be studied; that is the case $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$.

3.4 The case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$

In this section the case $\Omega = \omega_2 - \omega_1$, and the case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one will be studied.

3.4.1 The case $\Omega = \omega_2 - \omega_1$

It is shown in appendix 1 that for $\Omega = \omega_2 - \omega_1$ the equation $\Omega \pm \omega_n = \pm\omega_k$ only has the rather trivial solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if $\Omega + \omega_n = \omega_k$. Then, by taking apart those terms in the right-hand side of the $\mathcal{O}(\epsilon)$ -equation (3.3.7) that cause secular terms in $v_{k1}(t_0, t_1)$, it is found that A_{k0} and B_{k0} have to satisfy

$$\dot{A}_{10} = \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} B_{20}, \quad \dot{B}_{10} = \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} A_{20},$$

$$\dot{A}_{20} = \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_2 L} B_{10}, \quad \dot{B}_{20} = \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2 L} A_{10}, \quad (3.4.1)$$

and for $k \geq 3$: $\dot{A}_{k0} = \dot{B}_{k0} = 0$.

System (4.3.11) can readily be solved, yielding

$$\begin{aligned} A_{10}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_1}} B_{20}(0) \sin(\gamma t_1) + A_{10}(0) \cos(\gamma t_1), \\ B_{10}(t_1) &= \sqrt{\frac{\omega_2}{\omega_1}} A_{20}(0) \sin(\gamma t_1) + B_{10}(0) \cos(\gamma t_1), \\ A_{20}(t_1) &= A_{20}(0) \cos(\gamma t_1) - \sqrt{\frac{\omega_1}{\omega_2}} B_{10}(0) \sin(\gamma t_1), \quad \text{and} \\ B_{20}(t_1) &= B_{20}(0) \cos(\gamma t_1) + \sqrt{\frac{\omega_1}{\omega_2}} A_{10}(0) \sin(\gamma t_1), \end{aligned} \quad (3.4.2)$$

where $\gamma = \frac{2\alpha(\omega_1 + \omega_2)}{3L\sqrt{\omega_1\omega_2}}$ and for $k \geq 3$, $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. For $n \geq 1$, $A_{n0}(0)$ and $B_{n0}(0)$ can be determined from (3.3.6). From (3.3.7) a solution $v_{k1}(t_0, t_1)$ can now be obtained without unbounded terms (that is without secular terms). So, a formal approximation $v_{k0}(t_0, t_1) + \epsilon v_{k1}(t_0, t_1)$ of $u_k(t; \epsilon)$ has been constructed. And finally, an approximation $\sum_{k=1}^{\infty} v_{k0}(t_0, t_1) \sin(\frac{k\pi x}{L}) + \mathcal{O}(\epsilon)$ of the solution $u(x, t)$ of the initial-boundary value problem (3.2.1) - (5.2.3) with $\Omega = \omega_2 - \omega_1$ is obtained.

3.4.2 The detuning case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$

If the frequency of the belt velocity fluctuation is detuned by taking $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with $\phi = \mathcal{O}(1)$, then the $\mathcal{O}(\epsilon)$ -equation (3.3.7) becomes:

$$\begin{aligned} \frac{\partial^2 v_{k1}}{\partial t_0^2} + \omega_k^2 v_{k1} &= -2\omega_k \left[\dot{A}_{k0} \cos(\omega_k t_0) - \dot{B}_{k0} \sin(\omega_k t_0) \right] + \\ &\sum_{n=1}^{\infty} \frac{nk}{(n^2 - k^2)L} \left(4\alpha\Omega_0 \cos(\Omega t_0) [A_{n0} \sin(\omega_n t_0) + B_{n0} \cos(\omega_n t_0)] \right. \\ &\quad \left. + 8(V_0 + \alpha \sin(\Omega t_0)) \omega_n [A_{n0} \cos(\omega_n t_0) - B_{n0} \sin(\omega_n t_0)] \right), \end{aligned} \quad (3.4.3)$$

where $\Omega_0 = \omega_2 - \omega_1$. Now, it should be observed that in (3.4.3)

$$\begin{aligned} \cos(\Omega t) &= \cos((\omega_2 - \omega_1)t_0 + \phi t_1) = \cos(\Omega_0 t_0) \cos(\phi t_1) - \sin(\Omega_0 t_0) \sin(\phi t_1), \\ \text{and } \sin(\Omega t) &= \sin((\omega_2 - \omega_1)t_0 + \phi t_1) = \sin(\Omega_0 t_0) \cos(\phi t_1) + \cos(\Omega_0 t_0) \sin(\phi t_1). \end{aligned}$$

Then, by taking apart those terms in the right-hand side of (3.4.3) that give rise to secular terms in $v_{k1}(t_0, t_1)$, it is found that in order to avoid secular terms, $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\begin{aligned} \dot{A}_{10} &= -p \sin(\phi t_1) A_{20} + p \cos(\phi t_1) B_{20}, \\ \dot{B}_{10} &= -p \cos(\phi t_1) A_{20} - p \sin(\phi t_1) B_{20}, \\ \dot{A}_{20} &= -q \sin(\phi t_1) A_{10} - q \cos(\phi t_1) B_{10}, \\ \dot{B}_{20} &= q \cos(\phi t_1) A_{10} - q \sin(\phi t_1) B_{10}, \end{aligned} \quad (3.4.4)$$

and for $k \geq 3$

$$\dot{A}_{k0} = 0 \quad \text{and} \quad \dot{B}_{k0} = 0, \quad (3.4.5)$$

where

$$p = \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} \quad \text{and} \quad q = \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2 L}. \quad (3.4.6)$$

Notice that for $\phi = 0$, (3.4.4) is reduced to (4.3.11). In appendix 2, the solution of system (3.4.4) has been derived. It turns out that the solution of system (3.4.4) is:

$$\begin{aligned} A_{10}(t_1) &= K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1), \\ B_{10}(t_1) &= \frac{1}{pq\phi} [A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}], \\ A_{20}(t_1) &= \frac{-1}{p} [\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1)], \\ B_{20}(t_1) &= \frac{1}{p} [\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)], \end{aligned} \quad (3.4.7)$$

and $A_{k0} = A_{k0}(0)$, $B_{k0} = B_{k0}(0)$ for $k \geq 3$. Note also that in (3.4.7) $\phi \neq 0$. In (3.4.7), K_1, K_2, K_3 , and K_4 are constants of integration, p, q are given by (3.4.6), $\beta_1 = \sqrt{\frac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]}$, and $\beta_2 = \sqrt{\frac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}$. As in subsection 4.1, an approximation $\sum_{k=1}^{\infty} v_{k0}(t_0, t_1) \sin(\frac{k\pi x}{L}) + \mathcal{O}(\epsilon)$ of the solution $u(x, t)$ of the initial-boundary value problem (3.2.1) - (5.2.3) with $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ and $\phi = \mathcal{O}(1)$ has now been constructed.

For Ω in a neighbourhood of $\omega_2 - \omega_1$, now can be concluded (see also (3.4.7)) that no instabilities for the belt system will occur. A similar analysis as given in this section can be applied to other cases where Ω is of difference type (that is $\Omega = \omega_m - \omega_n$ for some m and n). However, in some of these cases instabilities can occur as will be explained in the next section.

3.5 Ω is a sum of two natural frequencies

It has been shown in section 3.3 that in order to remove secular terms, we have to solve the equation $\Omega \pm \omega_n = \pm \omega_k$, where $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $j = 0, 1, 2, \dots$. In the case $\Omega = \omega_2 - \omega_1$, it has been shown in section 3.4 that the only solutions of the problem $(\omega_2 - \omega_1) \pm \omega_n = \pm \omega_k$ (for an arbitrary value of $\mu^2 = \frac{\delta\pi^2}{c^2 L^2}$) are the trivial solutions $k = 1, n = 2$ and symmetrically $k = 2, n = 1$. For other values of Ω and for certain specific values of μ^2 solutions other than the trivial ones may occur. Cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$ will be considered in this section, other cases can be investigated similarly.

3.5.1 The case $\Omega = \omega_2 + \omega_1$

First, in order to solve the equation $\Omega \pm \omega_n = \pm\omega_k$, the following three cases have to be considered:

- i) $-\omega_n - \omega_k = \Omega$, which obviously has no solution since the right-hand side is positive while the left hand side is negative,
- ii) $\omega_n + \omega_k = \Omega$, which obviously has only the trivial solution $k = 2, n = 1$ or $k = 1, n = 2$,
- iii) $\omega_k - \omega_n = \Omega$ (or equivalently $\omega_n - \omega_k = \Omega$) which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = \Omega$, it follows that $k\sqrt{1+k^2\mu^2} = n\sqrt{1+n^2\mu^2} + 2\sqrt{1+4\mu^2} + \sqrt{1+\mu^2}$. Since $f(k) = k\sqrt{1+\mu^2k^2}$ is an increasing function it then follows from $k\sqrt{1+\mu^2k^2} > n\sqrt{1+\mu^2n^2}$ that $k > n \geq 1$. Then from $1 \leq n < k$ it follows that

$$\begin{aligned} k\sqrt{1+k^2\mu^2} &= n\sqrt{1+n^2\mu^2} + 2\sqrt{1+4\mu^2} + \sqrt{1+\mu^2} \\ &< n\sqrt{1+k^2\mu^2} + 2\sqrt{1+k^2\mu^2} + \sqrt{1+k^2\mu^2} \\ &\Rightarrow n < k < n + 3, \quad \Rightarrow k = n + 1 \quad \text{or} \quad k = n + 2. \end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $k = n + 1$. So, $\omega_k - \omega_n = \Omega = \omega_2 + \omega_1$ can only have solutions for $k = n + 1$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

| $\Omega = \omega_2 + \omega_1$ | | | $\Omega = 2\omega_1$ | | | $\Omega = \omega_3 + \omega_2$ | | | | | |
|--------------------------------|----------|----------|----------------------|----------|----------|--------------------------------|----------|----------|----------|----------|----------|
| k | n | μ^2 | k | n | μ^2 | k | n | μ^2 | k | n | μ^2 |
| 3 | 2 | - | 2 | 1 | 0.7143 | 6 | 5 | - | 4 | 1 | 0.3851 |
| 4 | 3 | 0.3851 | 3 | 2 | 0.1664 | 7 | 6 | - | 5 | 2 | 0.0732 |
| 5 | 4 | 0.1607 | 4 | 3 | 0.0773 | 8 | 7 | 0.4588 | 6 | 3 | 0.0349 |
| 6 | 5 | 0.0926 | 5 | 4 | 0.0451 | 9 | 8 | 0.2124 | 7 | 4 | 0.0211 |
| 7 | 6 | 0.0613 | 6 | 5 | 0.0297 | 10 | 9 | 0.1321 | 8 | 5 | 0.0143 |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |

Table 3.1: Values of k, n , and μ^2 for which $\omega_k = \omega_n + \Omega$ has solutions

Assuming that $\Omega \pm \omega_n = \pm\omega_k$ only has the trivial solutions ($k = 2$ and $n = 1$, and $k = 1$ and $n = 2$) it turns out that no secular terms occur in the solution of (3.3.7) if

$A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} B_{20}, & \dot{B}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} A_{20}, \\ \dot{A}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} B_{10}, & \dot{B}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} A_{10}, \end{aligned} \quad (3.5.1)$$

and for $k \geq 3$, $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of (3.5.1) can readily be determined, yielding

$$\begin{aligned} A_{10}(t_1) &= A_{10}(0) \cosh(r_1 t_1) - \sqrt{\frac{\omega_2}{\omega_1}} B_{20}(0) \sinh(r_1 t_1), \\ A_{20}(t_1) &= A_{20}(0) \cosh(r_1 t_1) - \sqrt{\frac{\omega_1}{\omega_2}} B_{10}(0) \sinh(r_1 t_1), \\ B_{10}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_1}} A_{20}(0) \sinh(r_1 t_1) + B_{10}(0) \cosh(r_1 t_1), \\ B_{20}(t_1) &= -\sqrt{\frac{\omega_1}{\omega_2}} A_{10}(0) \sinh(r_1 t_1) + B_{20}(0) \cosh(r_1 t_1), \end{aligned} \quad (3.5.2)$$

where $r_1 = \frac{2\alpha(\omega_2 - \omega_1)}{3L\sqrt{\omega_1\omega_2}}$, and for $k \geq 3$: $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. It is obvious from (3.5.2) that instabilities for the belt system will occur. When for instance $\mu^2 \approx 0.3851$ it turns out that $\Omega \pm \omega_n = \pm\omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of (3.3.7) it then turns out that A_{10}, B_{10}, A_{20} , and B_{20} still have to satisfy (3.5.1), and that

$$\begin{aligned} \dot{A}_{30} &= \frac{-12\alpha}{7L\omega_3} (\omega_3 + \omega_4) B_{40}, & \dot{B}_{30} &= \frac{12\alpha}{7L\omega_3} (\omega_3 + \omega_4) A_{40}, \\ \dot{A}_{40} &= \frac{-12\alpha}{7L\omega_4} (\omega_3 + \omega_4) B_{30}, & \dot{B}_{40} &= \frac{12\alpha}{7L\omega_4} (\omega_3 + \omega_4) A_{30}, \end{aligned} \quad (3.5.3)$$

and that for $k \geq 5$, $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of (3.5.3) can readily be determined, yielding

$$\begin{aligned} A_{30}(t_1) &= -\sqrt{\frac{\omega_4}{\omega_3}} B_{40}(0) \sin(\beta t_1) + A_{30}(0) \cos(\beta t_1), \\ A_{40}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_4}} B_{30}(0) \sin(\beta t_1) + A_{40}(0) \cos(\beta t_1), \\ B_{30}(t_1) &= \sqrt{\frac{\omega_4}{\omega_3}} A_{40}(0) \sin(\beta t_1) + B_{30}(0) \cos(\beta t_1), \\ B_{40}(t_1) &= \sqrt{\frac{\omega_3}{\omega_4}} A_{30}(0) \sin(\beta t_1) + B_{40}(0) \cos(\beta t_1), \end{aligned} \quad (3.5.4)$$

where $\beta = \frac{12\alpha(\omega_3 + \omega_4)}{7L\sqrt{\omega_3\omega_4}}$ and for $k \geq 5$: $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. Also for $\mu^2 \approx 0.3851$ it is obvious that instabilities for the belt system will occur. It should be observed that $\Omega = \omega_2 + \omega_1 = \omega_4 - \omega_3$ for $\mu^2 \approx 0.3851$. So, for special values of the beam parameters, also frequency Ω of difference type can lead to instabilities.

The detuning case $\Omega = \omega_2 + \omega_1 + \epsilon\phi$

In the detuning case $\Omega = \omega_2 + \omega_1 + \epsilon\phi$ secular terms will not occur if:

$$\begin{aligned}\dot{A}_{10} &= \frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1) \left[A_{20} \sin(\phi t_1) - B_{20} \cos(\phi t_1) \right], \\ \dot{B}_{10} &= -\frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1) \left[A_{20} \cos(\phi t_1) + B_{20} \sin(\phi t_1) \right], \\ \dot{A}_{20} &= \frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1) \left[A_{10} \sin(\phi t_1) - B_{10} \cos(\phi t_1) \right], \\ \dot{B}_{20} &= -\frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1) \left[A_{10} \cos(\phi t_1) + B_{10} \sin(\phi t_1) \right],\end{aligned}\tag{3.5.5}$$

and for $k \geq 3$, $\dot{A}_{k0} = 0$ and $\dot{B}_{k0} = 0$. In the following it will be assumed that $\alpha > 0$ (for $\alpha < 0$ the procedure is the same). By putting $p = \frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1)$ and $q = \frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1)$, and by differentiating \dot{A}_{10} once more it follows that

$$\ddot{A}_{10} = pqA_{10} - \phi\dot{B}_{10}.\tag{3.5.6}$$

By differentiating \ddot{A}_{10} twice and by making use of (3.5.6) it then follows that

$$A_{10}^{(4)} + (\phi^2 - 2pq)\ddot{A}_{10} + p^2q^2A_{10} = 0,\tag{3.5.7}$$

where $A_{10}^{(4)}$ is the fourth order derivative of A_{10} with respect to t_1 . This fourth order differential equation can be solved elementarily, and the following results are obtained:

- For $\phi^2 > 4pq$ the solutions of (3.5.5) will be stable,
- For $\phi^2 = 4pq$ the solutions of (3.5.5) will be unstable (due to linear term in t_1), and finally
- For $\phi^2 < 4pq$ the solutions of (3.5.5) will increase exponentially.

In Figure 3.1, the stability regions for system (3.5.5) in (α, ϕ) -plane for positive values of α have been given. The bifurcation lines are given by $\phi^2 = 4pq = k^2(L, \mu)\alpha^2$, where $k^2(L, \mu) = \frac{8(2\sqrt{1+4\mu^2} - \sqrt{1+\mu^2})^2}{9L^2\sqrt{1+4\mu^2}\sqrt{1+\mu^2}}$. From $\phi^2 = k^2\alpha^2$ it follows that $(\phi - k\alpha)(\phi + k\alpha) = 0$. The slope k is a function of L and μ and for fixed L it can be shown that $\frac{8}{9L^2} < k^2 < \frac{4}{L^2}$. Values of α and ϕ located in the regions *I* and *IV* lead to stable solutions of the system (3.5.5) while values of α and ϕ in the regions *II* and *III* (including the lines $\phi^2 = k^2\alpha^2$) lead to unstable solutions of system (3.5.5).

3.5.2 The case $\Omega = 2\omega_1$

As in subsection 3.5.1, in order to solve the equation $\Omega \pm \omega_n = \pm \omega_k$, again the following three cases have to be considered:

- i)* $-\omega_n - \omega_k = \Omega = 2\omega_1$, which obviously has no solutions,

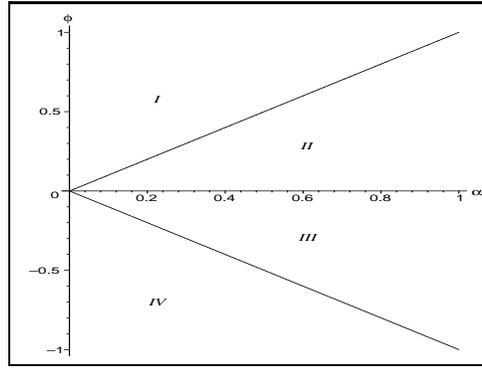


Figure 3.1: Areas of stability of system (3.5.5) for a specific value of $k = 1$.

ii) $\omega_n + \omega_k = \Omega = 2\omega_1$, which obviously only can be satisfied for $k = n = 1$. But since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$, it follows that there is in this case no solution.

iii) $\omega_k - \omega_n = \Omega = 2\omega_1$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = 2\omega_1$ it follows that $k\sqrt{1 + \mu^2 k^2} = n\sqrt{1 + \mu^2 n^2} + 2\sqrt{1 + \mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows from $k\sqrt{1 + \mu^2 k^2} > n\sqrt{1 + \mu^2 n^2}$ that $k > n \geq 1$. Then, from $1 \leq n < k$ it follows that

$$\begin{aligned} k\sqrt{1 + \mu^2 k^2} &= n\sqrt{1 + \mu^2 n^2} + 2\sqrt{1 + \mu^2} \\ &< n\sqrt{1 + \mu^2 k^2} + 2\sqrt{1 + \mu^2 k^2} \\ &\Rightarrow n < k < n + 2 \Rightarrow k = n + 1. \end{aligned}$$

So, $\omega_k - \omega_n = 2\omega_1$ can only have solutions for $k = n + 1$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

When μ^2 is not (in neighbourhood of) a value as listed in Table 1 then it easily follows from (3.3.7) that no secular terms occur in the solution if $\dot{A}_{k0} = \dot{B}_{k0} = 0$ for all $k \geq 1$. When for instance $\mu^2 \approx 0.7143$ it turns out that no secular terms occur in the solution of (3.3.7) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{-4\alpha}{3L\omega_1}(\omega_2 - \omega_1)B_{20}, & \dot{B}_{10} &= \frac{4\alpha}{3L\omega_1}(\omega_2 - \omega_1)A_{20}, \\ \dot{A}_{20} &= \frac{-8\alpha\omega_1}{3L\omega_2}B_{10}, & \dot{B}_{20} &= \frac{8\alpha\omega_1}{3L\omega_2}A_{10}, \end{aligned} \quad (3.5.8)$$

and for $k \geq 3$ $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of system (3.5.8) can readily be determined, yielding:

$$A_{10}(t_1) = -CB_{20}(0) \sin(\theta t_1) + A_{10}(0) \cos(\theta t_1),$$

$$\begin{aligned}
A_{20}(t_1) &= -\frac{1}{C}B_{10}(0)\sin(\theta t_1) + A_{20}(0)\cos(\theta t_1), \\
B_{10}(t_1) &= CA_{20}(0)\sin(\theta t_1) + B_{10}(0)\cos(\theta t_1), \\
B_{20}(t_1) &= \frac{1}{C}A_{10}(0)\sin(\theta t_1) + B_{20}(0)\cos(\theta t_1),
\end{aligned} \tag{3.5.9}$$

where $C^2 = \frac{\omega_2(\omega_2 - \omega_1)}{2\omega_1^2}$, $\theta = \frac{4\sqrt{2}\alpha}{3L}\sqrt{\frac{\omega_2 - \omega_1}{\omega_2}}$ and for $k \geq 3$: $A_{k0}(t_1) = A_{k0}(0)$, $B_{k0}(t_1) = B_{k0}(0)$. Obviously no instabilities for the belt system will occur when $\mu^2 \approx 0.7143$ or when μ^2 is not (in neighbourhood of) a value as listed in Table 1. It should be remarked that a similar analysis can be performed if $\Omega = 2\omega_N$ for some fixed $N > 1$.

3.5.3 The case $\Omega = \omega_3 + \omega_2$

As in the previous two subsections 3.5.1 and 3.5.2, again the following three cases have to be considered in order to solve the equation $\Omega \pm \omega_n = \pm\omega_k$. Those cases are:

- i) $-\omega_n - \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has no solution,
- ii) $\omega_n + \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has the trivial solutions $k = 2$ and $n = 3$, or $k = 3$ and $n = 2$. In this case additional solutions can only occur if $\omega_k + \omega_1 = \Omega = \omega_2 + \omega_3$ has a solution. In appendix 1 (see the case $\omega_k = \omega_n + \omega_2 - \omega_1$) it has been shown that this is not possible.
- iii) $\omega_k - \omega_n = \Omega = \omega_2 + \omega_3$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = \Omega$ it follows that $k\sqrt{1 + \mu^2 k^2} = n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows that $k > n$ and $k > 3$. Then, from $k > n$ and $k > 3$ it follows that

$$\begin{aligned}
k\sqrt{1 + \mu^2 k^2} &= n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2} \\
&< n\sqrt{1 + \mu^2 k^2} + 3\sqrt{1 + \mu^2 k^2} + 2\sqrt{1 + \mu^2 k^2} \\
&\Rightarrow n < k < n + 3 + 2 \Rightarrow k = n + 1 \text{ or } k = n + 2 \text{ or} \\
&\quad k = n + 3 \text{ or } k = n + 4.
\end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $n + 1$ or $n + 3$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

Assuming that $\Omega \pm \omega_n = \pm\omega_k$ only has the trivial solutions ($k = 2$ and $n = 3$, and $k = 3$ and $n = 2$) it turns out that no secular terms occur in the solution of (3.3.7) if

$A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned}\dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30}, & \dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30}, \\ \dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, & \dot{B}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20},\end{aligned}\quad (3.5.10)$$

and for $k = 1, 4, 5, 6, \dots$ $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solutions of system (3.5.10) can readily be determined, yielding:

$$\begin{aligned}A_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}B_{30}(0)\sinh(s_1t_1) + A_{20}(0)\cosh(s_1t_1), \\ A_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}B_{20}(0)\sinh(s_1t_1) + A_{30}(0)\cosh(s_1t_1), \\ B_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}A_{30}(0)\sinh(s_1t_1) + B_{20}(0)\cosh(s_1t_1), \\ B_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}A_{20}(0)\sinh(s_1t_1) + B_{30}(0)\cosh(s_1t_1),\end{aligned}\quad (3.5.11)$$

where $s_1 = \frac{6\alpha(\omega_3 - \omega_2)}{5L\sqrt{\omega_3\omega_2}}$, and for $k = 1, 4, 5, 6, \dots$ $A_{k0} = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. From (3.5.11) it is obvious that instabilities for the belt system will occur. When for instance $\mu^2 = 0.0732$ it turns out that $\Omega \pm \omega_n = \pm\omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of (3.3.7) it now turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\begin{aligned}\dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30} - \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)B_{50}, \\ \dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30} + \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)A_{50}, \\ \dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, & \dot{B}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20}, \\ \dot{A}_{50} &= -\frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)B_{20}, & \dot{B}_{50} &= \frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)A_{20},\end{aligned}\quad (3.5.12)$$

and $\dot{A}_{k0} = \dot{B}_{k0} = 0$ for $k = 1, 4, 6, 7, 8, \dots$. The solution of (3.5.12) can readily be determined, yielding

$$\begin{aligned}A_{20}(t_1) &= K_1 \sin(s_2t_1) + A_{20}(0) \cos(s_2t_1), \\ B_{20}(t_1) &= K_2 \sin(s_2t_1) + B_{20}(0) \cos(s_2t_1), \\ A_{30}(t_1) &= \frac{d_1 K_2}{s_2\omega_3} \cos(s_2t_1) - \frac{d_1 B_{20}(0)}{s_2\omega_3} \sin(s_2t_1) + \left(A_{30}(0) - \frac{d_1 K_2}{s_2\omega_3} \right), \\ B_{30}(t_1) &= \frac{d_1 K_1}{s_2\omega_3} \cos(s_2t_1) - \frac{d_1 A_{20}(0)}{s_2\omega_3} \sin(s_2t_1) + \left(B_{30}(0) - \frac{d_1 K_1}{s_2\omega_3} \right), \\ A_{50}(t_1) &= \frac{d_2 K_2}{s_2\omega_5} \cos(s_2t_1) - \frac{d_2 B_{20}(0)}{s_2\omega_5} \sin(s_2t_1) + \left(A_{50}(0) - \frac{d_2 K_2}{s_2\omega_5} \right), \text{ and} \\ B_{50}(t_1) &= \frac{-d_2 K_1}{s_2\omega_5} \cos(s_2t_1) + \frac{d_2 A_{20}(0)}{s_2\omega_5} \sin(s_2t_1) + \left(B_{50}(0) + \frac{d_2 K_1}{s_2\omega_5} \right),\end{aligned}\quad (3.5.13)$$

where $s_2 = \left[\left(\frac{10}{21} \right)^2 \frac{(\omega_2 + \omega_5)^2}{\omega_2 \omega_5} - \left(\frac{6}{5} \right)^2 \frac{(\omega_3 - \omega_2)^2}{\omega_3 \omega_2} \right]^{\frac{1}{2}} \frac{\alpha}{L}$, $K_1 = \frac{-1}{s_2} \left[\frac{d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0) \right]$, $K_2 = \frac{1}{s_2} \left[\frac{-d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0) \right]$, $d_1 = \frac{6\alpha(\omega_3 - \omega_2)}{5L}$, and $d_2 = \frac{10\alpha(\omega_5 + \omega_2)}{21L}$. And for $k = 1, 4, 6, 7, 8, \dots$, $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. From (3.5.13) it is obvious that now no instabilities for the belt system will occur. It should be observed that $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$ for $\mu^2 \approx 0.0732$. So, for special values of the beam parameters also frequencies Ω of sum type can lead to stable behaviour. To obtain more insight in the complicated dynamical behaviour of the belt system, in the next section the beam parameter μ^2 will be detuned (keeping Ω fixed).

The detuned Case $\mu^2 \approx 0.0732$

In the previous subsection 3.5.3 it has been shown that if $\Omega = \omega_2 + \omega_3$ (and μ^2 is not in the neighbourhood of a value as listed in Table 1) then the belt system is unstable. For $\mu^2 \approx 0.0732$, however, the belt system is stable. To obtain more insight in this different behaviour μ^2 in a neighbourhood of 0.0732 will be detuned. Observe that $\mu^2 = \frac{\delta \pi^2}{c^2 L^2}$ with $\delta = \frac{EI}{\rho A}$. So, detuning of μ^2 can be achieved by detuning δ in the original PDE (3.2.4). For that reason it will be considered that $\mu^2 = \mu_{cr}^2 + \epsilon \psi$ with $\mu_{cr}^2 = 0.0732$ and ψ an arbitrary constant of $\mathcal{O}(1)$, and $\delta = \delta_{cr} + \epsilon \nu$ with $\psi = \frac{\pi^2}{c^2 L^2} \nu$ and $\mu_{cr}^2 = \frac{\pi^2}{c^2 L^2} \delta_{cr}$. The frequency Ω is kept fixed, that is, $\Omega = \sqrt{\left(\frac{2c\pi}{L} \right)^2 + \delta_{cr} \left(\frac{2\pi}{L} \right)^4} + \sqrt{\left(\frac{3c\pi}{L} \right)^2 + \delta_{cr} \left(\frac{3\pi}{L} \right)^4}$. It should be observed that this type of detuning is different from the one studied in section 3.4.2. By replacing δ in (3.2.4) by $\delta_{cr} + \epsilon \nu$ the same analysis as presented in section 3.2 and section 3.3 can be repeated. To avoid secular terms in the approximation it turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ now have to satisfy ($\phi = \frac{\pi^3 \nu}{cL^3}$, and $\bar{\omega}_k = k\sqrt{1 + \mu_{cr}^2 k^2}$ for $k = 2, 3$, and 5)

$$\begin{aligned}
\dot{A}_{20} &= -\frac{6\alpha}{5L\bar{\omega}_2}(\bar{\omega}_3 - \bar{\omega}_2)B_{30} - \frac{10\alpha}{21L\bar{\omega}_2}(\bar{\omega}_5 + \bar{\omega}_2)B_{50} - \frac{8\pi^4\phi}{\bar{\omega}_2 L^4}B_{20}, \\
\dot{B}_{20} &= -\frac{6\alpha}{5L\bar{\omega}_2}(\bar{\omega}_3 - \bar{\omega}_2)A_{30} + \frac{10\alpha}{21L\bar{\omega}_2}(\bar{\omega}_5 + \bar{\omega}_2)A_{50} + \frac{8\pi^4\phi}{\bar{\omega}_2 L^4}A_{20}, \\
\dot{A}_{30} &= -\frac{6\alpha}{5L\bar{\omega}_3}(\bar{\omega}_3 - \bar{\omega}_2)B_{20} - \frac{81\pi^4\phi}{2\bar{\omega}_3 L^4}B_{30}, \\
\dot{B}_{30} &= -\frac{6\alpha}{5L\bar{\omega}_3}(\bar{\omega}_3 - \bar{\omega}_2)A_{20} + \frac{81\pi^4\phi}{2\bar{\omega}_3 L^4}A_{30}, \\
\dot{A}_{50} &= -\frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)B_{20} - \frac{625\pi^4\phi}{2\bar{\omega}_5 L^4}B_{50}, \\
\dot{B}_{50} &= \frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)A_{20} + \frac{625\pi^4\phi}{2\bar{\omega}_5 L^4}A_{50}, \tag{3.5.14}
\end{aligned}$$

and for $k = 1, 4, 6, 7, 8, \dots$

$$\dot{A}_{k0} = -\frac{\nu}{2\bar{\omega}_k} \left(\frac{k\pi}{L} \right)^4 B_{k0}, \quad \text{and} \quad \dot{B}_{k0} = \frac{\nu}{2\bar{\omega}_k} \left(\frac{k\pi}{L} \right)^4 A_{k0}. \tag{3.5.15}$$

Obviously system (3.5.15) has bounded solution. The characteristic equation of system (3.5.14) is:

$$\lambda^6 + (1502.1631\phi^2 + 1.8787\eta^2)\lambda^4 + (.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)\lambda^2 + 874.7894\eta^4\phi^2 + .1876 \cdot 10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0, \quad (3.5.16)$$

where $\eta = \frac{\alpha}{L}$. By putting $\lambda^2 = a$ in (3.5.16) the following cubic equation for a is obtained

$$a^3 + (1502.1631\phi^2 + 1.8787\eta^2)a^2 + (.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)a + 874.7894\eta^4\phi^2 + .1876 \cdot 10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0. \quad (3.5.17)$$

Equation (3.5.17) can be solved by using the Cardano's formula. The *radicand* R (of the reduced form of the cubic equation (3.5.17) plays an important role in the solution-structure. When the radicand R is positive the reduced cubic equation of (3.5.17) will have one real, and two complex conjugate solutions. Since $a = \lambda^2$ it follows that at least two roots of the characteristic equation (3.5.16) will have a positive real part. Consequently the solution of system (3.5.14) will be unstable. For $R < 0$ the cubic equation (3.5.17) will have three distinct real roots, and for $R = 0$ there are three real roots of which two coincide. For $R \leq 0$ it requires an additional analysis to determine whether system (3.5.14) is stable or not.

In Figure 3.2 the bifurcation values of R as a function of ϕ and η have been given. In this figure it has been assumed that η and so α (the amplitude of the speed fluctuation) are positive. Similar results can be found for $\eta < 0$. When ϕ and η are in the areas *II* and *V* then the solutions of (3.5.17) are positive, leading to the unstable solutions for (3.5.14), whereas when ϕ and η are in the areas *I*, *III*, *IV* or *VI* the solutions of (3.5.17) will be negative leading to stable solutions for (3.5.14). When ϕ and η are exactly on the curves the solutions of (3.5.17) will be also negative which leads to stable solutions of (3.5.14).

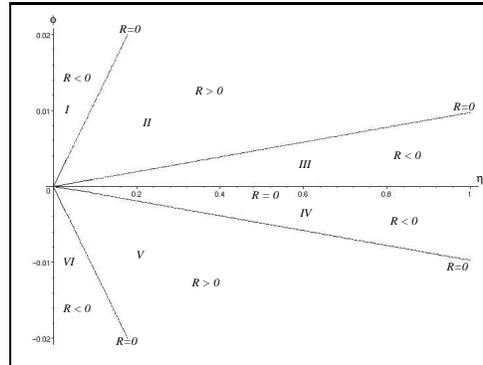


Figure 3.2: Bifurcation values of R as a function of η and ϕ .

3.6 Conclusions and remarks

In this chapter initial-boundary value problems for a beam equation are studied. The equations can be used as simple models to describe the vertical vibrations of a conveyor belt for which the time-varying belt velocity is small with respect to the wave speed. It is assumed that the belt velocity $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$ where ϵ , V_0 , α , and Ω are constants with $0 < \epsilon \ll 1$ and $|\alpha| < V_0$. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or difference of any two natural frequencies of the system for which the belt velocity is equal to zero. For special values of the belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. For both sum type and difference type of internal resonances instabilities for the belt system can occur.

In this chapter the following cases have been studied in detail with the following results:

- i)* $\Omega = \omega_2 - \omega_1$; interaction between the first and the second vibration modes; no instabilities for the belt system (also for the detuned case).
- ii)* $\Omega = \omega_2 + \omega_1$; interactions between the first and the second vibration modes, and for special values of the beam parameters (see Table 1) additional interactions; there will always be unstable behaviour of the belt system.
- iii)* The detuned case $\Omega = \omega_2 + \omega_1 + \epsilon\phi$; interactions occur between the first and the second vibration modes. Solutions will be unstable if $\phi^2 \leq 4pq$, while for $\phi^2 > 4pq$ the solutions are stable ($p = \frac{2\alpha}{3L\omega_1}(\omega_2 - \omega_1)$ and $q = \frac{2\alpha}{3L\omega_2}(\omega_2 - \omega_1)$).
- iv)* $\Omega = 2\omega_1$; only for special values of the beam parameters (see Table 1) there will be an interaction between two different vibration modes; there are no instabilities for the belt system.
- v)* $\Omega = \omega_2 + \omega_3$; interaction between the second and the third vibration modes, and for special values of beam parameters (see Table 1) there are additional interactions; in general there will be instabilities for the belt system. However, for special values of the beam parameters there can be stable behaviour of the belt system. When some of these beam parameters are detuned unstable behaviour can occur again (see the subsection of 3.5.3 where $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$ for $\mu^2 = \frac{EI\pi^2}{\rho Ac^2 L^2} \approx 0.0732$).

It is expected that for other values of Ω , the same techniques (as presented in this chapter) can be applied to determine the stability properties of the belt system.

Appendix

Appendix 1

In this appendix it will be shown that the equation $\Omega \pm \omega_n = \pm \omega_k$ with $\Omega = \omega_2 - \omega_1$ only has as solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if $\Omega + \omega_n = \omega_k$. To prove this, the following four cases have to be considered: $\omega_k = \omega_n + \omega_2 - \omega_1$, $\omega_k = -\omega_n + \omega_2 - \omega_1$, $-\omega_k = \omega_n + \omega_2 - \omega_1$, and $-\omega_k = -\omega_n + \omega_2 - \omega_1$. Note that $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$.

The case $\omega_k = \omega_n + \omega_2 - \omega_1$.

Since $\omega_k^2 = (\frac{ck\pi}{L})^2 + \delta(\frac{k\pi}{L})^4$, it follows from $\omega_k = \omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1, \quad (\text{A-1})$$

where $\mu^2 = \delta\frac{\pi^2}{c^2L^2}$. It can easily be shown that $f(k) = \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}}$ is an increasing function in k , and that $k \leq f(k) < k^2$. Then it follows from (A-1) that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} < \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}}. \quad (\text{A-2})$$

Since $f(k)$ is increasing in k it follows from the first inequality in (A-2) that $1 \leq n < k$. From the second inequality in (A-2) it then follows that

$$\begin{aligned} \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} &< \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} \\ &\Rightarrow k < n + 2. \end{aligned} \quad (\text{A-3})$$

Consequently, $k = n + 1$, and (A-1) becomes:

$$\frac{(n+1)\sqrt{1+\mu^2(n+1)^2}}{\sqrt{1+\mu^2}} - \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - \frac{\sqrt{1+\mu^2}}{\sqrt{1+\mu^2}}.$$

Denoting the left hand side of the last equation by $g(n)$ then the right hand side of the equation is just $g(1)$. It is not too difficult to show that $g(n)$ is an increasing function, so the last equation can only be satisfied if $n = 1$. Since $k = n + 1$ it follows that the only solution in this case is $k = 2$ and $n = 1$.

The case $\omega_k = -\omega_n + \omega_2 - \omega_1$

In this case it follows from $\omega_k = -\omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = -\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1. \quad (\text{A-4})$$

The only candidate for a solution of this equation is $n = 1$ since the left hand side is always positive while the right hand side is negative for $n \geq 2$. Accordingly, by substituting $n = 1$ into (A-4) it will follow that:

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 2. \quad (\text{A-5})$$

Now it should be observed that the left-hand side of (A-5) is between k and k^2 , and that the right-hand side is between 0 and 2. So, the only candidate for a solution is $k = 1$ (and $n = 1$). Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it easily follows that there are no solutions in this case.

The case $-\omega_k = \omega_n + \omega_2 - \omega_1$

In this case it follows from $-\omega_k = \omega_n + \omega_2 - \omega_1$ that

$$-\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1.$$

Now the left-hand side is always negative while the right-hand side is always positive. So, there are no solutions in this case.

The case $-\omega_k = -\omega_n + \omega_2 - \omega_1$

In this case it follows from $-\omega_k = -\omega_n + \omega_2 - \omega_1$ that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} = \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1.$$

By interchanging n and k , this case becomes the first case. So, the only solution in this case is $k = 1$ and $n = 2$.

This completes the proof of the statement at the beginning of this appendix.

Appendix 2

In this appendix the solutions of (3.4.4) will be determined, that is, the solutions of:

$$\begin{aligned} \dot{A}_{10} &= -p \sin(\phi t_1) A_{20} + p \cos(\phi t_1) B_{20}, \\ \dot{B}_{10} &= -p \cos(\phi t_1) A_{20} - p \sin(\phi t_1) B_{20}, \\ \dot{A}_{20} &= -q \sin(\phi t_1) A_{10} - q \cos(\phi t_1) B_{10}, \\ \dot{B}_{20} &= q \cos(\phi t_1) A_{10} - q \sin(\phi t_1) B_{10}, \end{aligned} \quad (\text{A-6})$$

where p and q are given by (3.4.6).

By differentiating the first and the second equation in (A-6) it will follow

$$\begin{aligned}\ddot{A}_{10} &= -p\phi \cos(\phi t_1)A_{20} - p \sin(\phi t_1)\dot{A}_{20} - p\phi \sin(\phi t_1)B_{20} + p \cos(\phi t_1)\dot{B}_{20} \\ &= \phi[-p \cos(\phi t_1)A_{20} - p \sin(\phi t_1)B_{20}] - p \sin(\phi t_1)[-q \sin(\phi t_1)A_{10} - \\ &\quad q \cos(\phi t_1)B_{10}] + p \cos(\phi t_1)[q \cos(\phi t_1)A_{10} - q \sin(\phi t_1)B_{10}] \\ &= \phi\dot{B}_{10} + pqA_{10}, \text{ and}\end{aligned}\tag{A-7}$$

$$\ddot{B}_{10} = -\phi\dot{A}_{10} + pqB_{10}.\tag{A-8}$$

Differentiating (A-7) and using (A-8), will result in:

$$A_{10}^{(3)} - pq\dot{A}_{10} = \phi\ddot{B}_{10} = -\phi^2\dot{A}_{10} + pq\phi B_{10},\tag{A-9}$$

and finally by differentiating (A-9) and using (A-7) it follows

$$A_{10}^{(4)} + (\phi^2 - 2pq)\ddot{A}_{10} + (pq)^2 A_{10} = 0.\tag{A-10}$$

The characteristic equation corresponding to (A-10) is $r^4 + (\phi^2 - 2pq)r^2 + (pq)^2 = 0$ with solutions $r_1 = \sqrt{\frac{1}{2}[2pq - \phi^2 + \sqrt{D}]}$, $r_2 = \sqrt{\frac{1}{2}[2pq - \phi^2 - \sqrt{D}]}$, $r_3 = -r_1$, and $r_4 = -r_2$ and where $D = \phi^4 - 4pq\phi^2$. Since p and q are of opposite sign it follows that $\phi^4 - 4pq\phi^2 > 0$ and $2pq - \phi^2 < 0$. Therefore, r_2 and r_4 are purely imaginary. And, since $\phi^2 - 2pq = \sqrt{(\phi^2 - 2pq)^2} = \sqrt{\phi^4 - 4pq\phi^2 + 4p^2q^2} > \sqrt{\phi^4 - 4pq\phi^2}$ then $|2pq - \phi^2| > \sqrt{\phi^4 - 4pq\phi^2}$. Accordingly r_1 and r_3 are also purely imaginary. So, all the solutions of the characteristic equation can be written in the form $r_1 = \beta_1 i$, $r_2 = \beta_2 i$, $r_3 = -r_1$, and $r_4 = -r_2$, where $\beta_1 = \sqrt{\frac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]}$ and $\beta_2 = \sqrt{\frac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}$. The solution of (A-10) now becomes:

$$A_{10}(t_1) = K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1),$$

where K_1, K_2, K_3 , and K_4 are constants of integration.

From (A-9) $B_{10}(t_1)$ can be derived, yielding

$$B_{10}(t_1) = \frac{1}{pq\phi}[A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}]; \quad \phi \neq 0.$$

From the first two equations in (A-6), A_{20} and B_{20} can now readily be determined, yielding

$$A_{20}(t_1) = \frac{-1}{p}[\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1),$$

$$B_{20}(t_1) = \frac{1}{p}[\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)].$$

So, the solutions of (3.4.4) have been derived.

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Chapter 4

On The Weakly Nonlinear, Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity †

Abstract. In this chapter the weakly nonlinear, transversal vibrations of a conveyor belt will be considered. The belt is assumed to move with a low and time-varying speed. Using Kirchhoff's approach a single equation of motion will be derived from a coupled system of partial differential equations describing the longitudinal and transversal vibrations of the belt. A two time-scales perturbation method is then applied to approximate the solutions of the problem. It will turn out that the frequencies of the belt speed fluctuations play an important role in the dynamic behaviour of the belt. It is well-known in linear systems that instabilities can occur if the frequency of the belt speed fluctuations is the sum of two natural frequencies. However, in the weakly nonlinear case as considered in this chapter this is no longer true. It turns out that the weak nonlinearity stabilizes the system.

4.1 Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper sheet and web processes, fiber winding and band saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations.

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [2] - [5] for a recent overview) and is still of

†This chapter is a revised version of [1] *On The Weakly Nonlinear, Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity*. *Nonlinear Dynamics*, **31**, 197-223.

interest today. In general, the studies about the dynamical behaviour of belt systems have been restricted to belts moving with a constant speed (see for instance [2] - [6]). Recently there are some studies about the transversal vibrations of belt systems moving with a non-constant speed (see for instance [7] - [13]). The vibrations of a belt system moving with a low non-constant velocity have been studied in [7], [8] and [9]. In [7] the belt vibrations have been modeled using a linear string-like equation while in [8] the vibrations have been modeled using a linear beam-like equation. The transversal vibrations of a belt system moving with an $\mathcal{O}(1)$ time-dependent speed have been studied in [10] and [11], while the associated nonlinear vibrations have been studied in [12] and [13]. A major drawback in the papers [10] - [13] which has been observed in [7] and [8], is the use of the truncation method (specifically the use of only one term). It has been pointed out in [2], [7] and [8] that a strong reduction in the phase space can lead to a poor description of the dynamic phenomena and in particular the use of only an one degree-of-freedom approximation can lead to errors in the spatial description and in the forecasting of the time evolution of the system. In [7] and [8] it has been shown that the truncation method as applied in [10] - [13] indeed leads to incorrect results for low speed belt systems on long timescales. A similar conclusion on the applicability of the truncation method to these type of problem can also be found in [14].

In this chapter the weakly nonlinear transversal vibrations of a moving belt will be studied. These vibrations are described by a single weakly nonlinear beam equation. Kirchhoff's approach has been used to obtain this single governing equation from the original coupled system of partial differential equations which describe the longitudinal and transversal vibrations of the belt. The belt speed is considered to be time-varying and to be small compared to the wave speed. It is assumed that the speed is $V(t) = \tilde{\epsilon}(V_0 + \alpha \sin(\Omega t))$, where $\tilde{\epsilon}$, V_0 , α , and Ω are all constants with $0 < \tilde{\epsilon} \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter $\tilde{\epsilon}$ indicates that the belt speed $V(t)$ is small compared to the wave speed. The variation in $V(t)$ may be due to the pulleys imperfection or some other sources of imperfection and it can be considered as some kind of excitation. In this chapter it is assumed that the displacement of the belt in the longitudinal and in the transversal directions are small.

In relation to excitations, some results in this area have been obtained by Sack [15] and Archibald and Emslie [16]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e. $x = L$) is subjected to an harmonic excitation. In [15] the vibrations of the string at $x = L$ is forced to be $v(x, t) = v_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingam in [17]. A moving string model has been used in [17] to study the transverse vibrations of power transmission chains.

In all of these works, the belt movement is assumed to be constant.

This chapter is organized as follows. In section 4.2 the coupled equations describing the motion of the belt system in longitudinal and in transversal direction are derived. These coupled partial differential equations are then reduced in section 4.3 to a single partial differential equation by applying Kirchhoff's approximation. In section 4.4 a two time-scales perturbation analysis of the equation as obtained in section 4.3 will be carried out. Some specific values of Ω , the frequency of the belt speed fluctuations, are used to demonstrate what kind of resonances can occur. Finally, in the last section some conclusions will be drawn and some remarks will be made.

4.2 Kirchhoff's approach

It has been derived in Chapter 1 that the equations of motion describing the transversal and longitudinal vibrations of a conveyor belt are respectively:

$$\begin{aligned} w_{tt} + 2Vw_{xt} + V_t(1 + w_x) - (P_1^2 - V^2)w_{xx} &= (P_1^2 - 1 - \eta V^2)u_x u_{xx}, \\ u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - 1)u_{xx} + P_0^2 u_{xxxx} &= \\ (P_1^2 - 1 - \eta V^2) \left(\frac{3}{2} u_x^2 u_{xx} + u_x w_{xx} + w_x u_{xx} \right), & \quad t \geq 0, 0 < x < 1. \end{aligned} \quad (4.2.1)$$

The boundary conditions for the two simple supports are given by:

$$w(0, t) = w(1, t) = 0, \quad \text{and} \quad u(x, t) = u_{xx}(x, t) = 0 \quad \text{for} \quad x = 0, 1, \quad (4.2.2)$$

while the initial displacements and initial velocities are:

$$\begin{aligned} w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad u(x, 0) = u_0(x), \\ \text{and} \quad u_t(x, 0) = u_1(x). \end{aligned} \quad (4.2.3)$$

In this chapter it will be assumed that u and V are $\mathcal{O}(\tilde{\epsilon})$, w is $\mathcal{O}(\tilde{\epsilon}^2)$, P_0^2 is $\mathcal{O}(1)$, and P_1^2 is $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$, where $\tilde{\epsilon}$ is a small parameter with $0 < \tilde{\epsilon} \ll 1$. Using these assumptions and following Kirchhoff's approach it will be shown in this section that the coupled system of PDEs (4.2.1) can be reduced to a single PDE for the transversal displacement $u(x, t)$.

Now, it should be observed that the equation for the longitudinal displacements $w(x, t)$ in (4.2.1) can be rewritten in:

$$w_{tt} + 2Vw_{xt} + V_t(1 + w_x) + V^2 w_{xx} = P_1^2 (w_x + \frac{1}{2} u_x^2)_x - (1 + \eta V^2) u_x u_{xx}. \quad (4.2.4)$$

Since u and V are $\mathcal{O}(\tilde{\epsilon})$, $w = \mathcal{O}(\tilde{\epsilon}^2)$, and $P_1^2 = \mathcal{O}(\frac{1}{\tilde{\epsilon}})$ then (4.2.4) up to order $\tilde{\epsilon}$ becomes:

$$\begin{aligned} P_1^2 (w_x + \frac{1}{2} u_x^2)_x = V_t &\Rightarrow P_1^2 (w_x + \frac{1}{2} u_x^2) = x V_t + f(t) \\ \Rightarrow P_1^2 \int_0^1 (w_x + \frac{1}{2} u_x^2) dx &= \frac{1}{2} V_t + f(t) \\ \Rightarrow f(t) = \frac{1}{2} \left(P_1^2 \int_0^1 u_x^2 dx - V_t \right), & \end{aligned} \quad (4.2.5)$$

where use has been made of the boundary conditions $w(0, t) = w(1, t) = 0$.

Similarly the equation for u in (4.2.1) can be rewritten in

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \left[P_1^2 \left\{ u_x \left(\frac{1}{2} u_x^2 + w_x \right)_x + u_{xx} \left(\frac{1}{2} u_x^2 + w_x \right) \right\} - 2V u_{xt} - V_t u_x \right] + \text{"h.o.t."}, \quad (4.2.6)$$

where *h.o.t.* stands for higher order terms. Substituting $w_x + \frac{1}{2}u_x^2$ from (4.2.5) into (4.2.6) gives:

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \left[\left(x - \frac{1}{2} \right) V_t u_{xx} - 2V u_{xt} + \frac{1}{2} P_1^2 u_{xx} \int_0^1 u_x^2 dx \right] + \text{"h.o.t."}, \quad (4.2.7)$$

where $u(x, t)$ additionally has to satisfy the boundary conditions (4.2.2) and the initial conditions (4.2.3).

When it is assumed that $P_1^2 \gg \mathcal{O}(\frac{1}{\epsilon})$ (instead of $P_1^2 = \mathcal{O}(\frac{1}{\epsilon})$) it follows from (4.2.4) that $(w_x + \frac{1}{2}u_x^2)_x = 0$ approximately. Following the same steps as given in (4.2.5) and (4.2.5) it then follows that $u(x, t)$ has to satisfy

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \left[-V_t u_x - 2V u_{xt} + \frac{1}{2} P_1^2 u_{xx} \int_0^1 u_x^2 dx \right] + \text{"h.o.t."}. \quad (4.2.8)$$

An equation similar to (4.2.8) has been studied in [13] using Galerkin's truncation method. In [7] and [8] it has been explained that for these type of equations many phenomena which are present in infinite dimensional systems can be lost in its finite dimensional approximations. In this chapter a justification of the applicability of the truncation method will be given by explicitly studying all (internal and external) resonances which are present in equation (4.2.7).

In (4.2.7) u, V , and P_1^2 are now replaced by \tilde{u}, \tilde{V} , and $\frac{1}{\epsilon} \tilde{P}_1^2$ respectively, where \tilde{u}, \tilde{V} and \tilde{P}_1^2 are of $\mathcal{O}(1)$. Equation (4.2.7) then becomes:

$$\tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2 \tilde{u}_{xxxx} = \tilde{\epsilon} \left[\left(x - \frac{1}{2} \right) \tilde{V}_t \tilde{u}_{xx} - 2\tilde{V} \tilde{u}_{xt} + \frac{1}{2} \tilde{P}_1^2 \tilde{u}_{xx} \int_0^1 \tilde{u}_x^2 dx \right] + \text{"h.o.t. in } \tilde{\epsilon}\text{"}, \quad 0 < x < 1, t > 0, \quad (4.2.9)$$

where $\tilde{u}(x, t)$ also has to satisfy the following boundary and initial values

$$\tilde{u}(x, t) = \tilde{u}_{xx}(x, t) = 0, \quad \text{for } x = 0 \quad \text{and} \quad x = 1, t \geq 0, \quad (4.2.10)$$

$$\tilde{u}(x, 0) = \tilde{u}_0(x), \quad \tilde{u}_t(x, 0) = \tilde{u}_1(x), \quad \text{for } t = 0, 0 < x < 1. \quad (4.2.11)$$

4.3 A perturbation analysis

In this section approximations of the solution $\tilde{u}(x, t)$ of the initial-boundary value problem (4.2.9)-(4.2.11) will be constructed. As mentioned in the introduction of this chapter it is assumed that the velocity $V(t) = \tilde{\epsilon} \tilde{V}(t)$ of the belt is given by

$$V(t) = \tilde{\epsilon} \tilde{V}(t) = \tilde{\epsilon} (V_0 + \alpha \sin(\Omega t)), \quad (4.3.1)$$

where $\tilde{\epsilon}$, V_0 , α , and Ω are all constants with $0 < \tilde{\epsilon} \ll 1$ and $V_0 > |\alpha|$. For special values of Ω it will turn out in this section that complicated resonances occur. Some of these cases for Ω will be studied in detail. Based on the boundary conditions (4.2.10) for $\tilde{u}(x, t)$ it follows that $\tilde{u}(x, t)$ can be written in the form: $\tilde{u}(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x)$. Since this series is odd and 2-periodic in x each term in (4.2.7) should be expanded odd with respect to $x = 0$ and $x = 1$ and 2-periodic in x . This is accomplished by multiplying each term in (4.2.9) which is not already odd in x , (i.e. terms like xu_{xx} and u_{xt}) with $\mathcal{H}(x)$ (see also [7], [19], [20]) where

$$\mathcal{H}(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ -1 & \text{for } -1 < x < 0, \end{cases} = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin((2j+1)\pi x), \quad (4.3.2)$$

and $\mathcal{H}(x) = \mathcal{H}(x+2)$. So, equation (4.2.9) then becomes on $-1 < x < 1$:

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2 \tilde{u}_{xxxx} = \tilde{\epsilon} \left[\tilde{V}_t \tilde{u}_{xx} \left(x\mathcal{H}(x) - \frac{1}{2} \right) - 2\tilde{V} \tilde{u}_{xt} \mathcal{H}(x) + \right. \\ \left. \frac{1}{2} \tilde{P}_1^2 \tilde{u}_{xx} \int_0^1 \tilde{u}_x^2 dx \right] + \text{"h.o.t. in } \epsilon". \end{aligned} \quad (4.3.3)$$

It can be shown elementarily that the Fourier series of $x\mathcal{H}(x)$ on $-1 < x < 1$ is

$$\frac{1}{2} - \sum_{j=0}^{\infty} \frac{4}{(2j+1)^2 \pi^2} \cos((2j+1)\pi x). \quad (4.3.4)$$

Substitution of (4.3.4) in (4.3.3) gives:

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2 \tilde{u}_{xxxx} = \tilde{\epsilon} \left[-4 \sum_{j=0}^{\infty} \frac{\cos((2j+1)\pi x)}{(2j+1)^2 \pi^2} \tilde{V}_t \tilde{u}_{xx} - 2\tilde{V} \tilde{u}_{xt} \mathcal{H}(x) \right. \\ \left. + \frac{1}{2} \tilde{P}_1^2 \tilde{u}_{xx} \int_0^1 \tilde{u}_x^2 dx \right] + \text{"h.o.t. in } \epsilon". \end{aligned} \quad (4.3.5)$$

Now by substituting $\tilde{V}(t)$ as given by (4.3.1) and the series $\tilde{u}(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x)$ for $\tilde{u}(x, t)$ into (4.3.5) and then by using the orthogonality properties of the Fourier sin-series on $-1 < x < 1$ it follows that u_k has to satisfy (for $k = 1, 2, 3, \dots$)

$$\begin{aligned} \ddot{u}_k + \omega_k^2 u_k = \tilde{\epsilon} \left[\sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n} \right] \frac{2n^2 \alpha \Omega \cos(\Omega t)}{(2j+1)^2} u_n - \\ 4\tilde{\epsilon} (V_0 + \alpha \sin(\Omega t)) \left[\sum_{k=2j+1+n} + \sum_{k=2j+1-n} - \sum_{k=n-2j-1} \right] \frac{n \dot{u}_n}{(2j+1)} - \\ \tilde{\epsilon} \frac{k^2 \tilde{P}_1^2 \pi^4}{4} u_k \left(\sum_{l=1}^{\infty} l^2 u_l^2 \right) + \mathcal{O}(\tilde{\epsilon}^2), \end{aligned} \quad (4.3.6)$$

where $\omega_k^2 = (k\pi)^2 + P_0^2 (k\pi)^4$. It should be observed that (4.3.6) is also obtained when (after the sin-series for $u(x, t)$ is substituted into (4.2.9)) equation (4.2.9) is multiplied with $\sin(k\pi x)$ and then integrated with respect to x from $x = 0$ to $x = 1$.

When a naive perturbation method is used secular terms will occur. To avoid these secular terms a two time-scales perturbation method will be used to solve (4.3.6) approximately. The introduction of two time-scales $t_0 = t$ and $t_1 = \tilde{\epsilon}t$ implies that

$$u_k(t) = \bar{u}_k(t_0, t_1), \quad \dot{u}_k = \frac{\partial \bar{u}_k}{\partial t_0} + \tilde{\epsilon} \frac{\partial \bar{u}_k}{\partial t_1}, \quad \ddot{u}_k = \frac{\partial^2 \bar{u}_k}{\partial t_0^2} + 2\tilde{\epsilon} \frac{\partial^2 \bar{u}_k}{\partial t_0 \partial t_1} + \tilde{\epsilon}^2 \frac{\partial^2 \bar{u}_k}{\partial t_1^2}.$$

For convenience the bar on $\bar{u}_k(t_0, t_1)$ will be dropped in the further analysis. Assuming that $u_k(t_0, t_1)$ can be written in the formal expansion $u_{k0} + \tilde{\epsilon}u_{k1} + \mathcal{O}(\tilde{\epsilon}^2)$ it then follows from the $\mathcal{O}(1)$ -terms and the $\mathcal{O}(\tilde{\epsilon})$ -terms in (4.3.6) that u_{k0} and u_{k1} have to satisfy:

$$\begin{aligned} \mathcal{O}(1) : \quad & \frac{\partial^2 u_{k0}}{\partial t_0^2} + \omega_k^2 u_{k0} = 0, \\ \mathcal{O}(\tilde{\epsilon}) : \quad & \frac{\partial^2 u_{k1}}{\partial t_0^2} + \omega_k^2 u_{k1} = -2 \frac{\partial^2 u_{k0}}{\partial t_0 \partial t_1} \\ & + \left[\sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n} \right] \frac{2n^2 \alpha \Omega u_{n0} \cos(\Omega t_0)}{(2j+1)^2} \\ & - \left[\sum_{k=2j+1+n} + \sum_{k=2j+1-n} - \sum_{k=n-2j-1} \right] \left(\frac{4n(V_0 + \alpha \sin(\Omega t))}{2j+1} \frac{\partial u_{n0}}{\partial t_0} \right) \\ & - \frac{\tilde{P}_1^2 k^2 \pi^4}{4} u_{k0} \left(\sum_{l=1}^{\infty} l^2 u_{l0}^2 \right), \end{aligned}$$

respectively. The solution of the $\mathcal{O}(1)$ problem is given by

$$u_{k0}(t_0, t_1) = A_{k0}(t_1) \sin(\omega_k t_0) + B_{k0}(t_1) \cos(\omega_k t_0), \quad (4.3.7)$$

where the functions $A_{k0}(t_1)$ and $B_{k0}(t_1)$ in (4.3.7) are still arbitrary and can be used to avoid secular terms in the $\mathcal{O}(\tilde{\epsilon})$ -problem for u_{k1} . By substituting $u_{k0}(t_0, t_1)$ into the $\mathcal{O}(\tilde{\epsilon})$ -problem it follows that

$$\begin{aligned} \frac{\partial^2 u_{k1}}{\partial t_0^2} + \omega_k^2 u_{k1} = & -2\omega_k [\dot{A}_{k0} \cos(\omega_k t_0) - \dot{B}_{k0} \sin(\omega_k t_0)] \\ & + \left[\sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n} \right] \frac{\alpha \Omega n^2}{(2j+1)^2} \left[A_{n0} \left\{ \sin((\omega_n + \Omega)t_0) \right. \right. \\ & \left. \left. + \sin((\omega_n - \Omega)t_0) \right\} + B_{n0} \left\{ \cos((\omega_n + \Omega)t_0) + \cos((\omega_n - \Omega)t_0) \right\} \right] \\ & + \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \frac{4n\omega_n V_0}{2j+1} \left[A_{n0} \cos(\omega_n t_0) - B_{n0} \sin(\omega_n t_0) \right] \\ & + \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \frac{2\alpha n \omega_n}{2j+1} \left[A_{n0} \left\{ \sin((\omega_n + \Omega)t_0) \right. \right. \\ & \left. \left. - \sin((\omega_n - \Omega)t_0) \right\} + B_{n0} \left\{ \cos((\omega_n + \Omega)t_0) - \cos((\omega_n - \Omega)t_0) \right\} \right] \\ & - \frac{k^2 \tilde{P}_1^2 \pi^4}{8} \left[A_{k0} \sin(\omega_k t_0) + B_{k0} \cos(\omega_k t_0) \right] \sum_{l=1}^{\infty} l^2 \left(A_{l0}^2 + B_{l0}^2 \right) + \end{aligned}$$

$$\begin{aligned}
& -\frac{k^2 \tilde{P}_1^2 \pi^4}{16} \sum_{l=1}^{\infty} l^2 (B_{l0}^2 - A_{l0}^2) \left[A_{k0} \left\{ \sin((2\omega_l + \omega_k)t_0) - \sin((2\omega_l - \omega_k)t_0) \right\} \right. \\
& \left. + B_{k0} \left\{ \cos((2\omega_l + \omega_k)t_0) + \cos((2\omega_l - \omega_k)t_0) \right\} \right] \\
& -\frac{k^2 \tilde{P}_1^2 \pi^4}{8} \sum_{l=1}^{\infty} l^2 A_{l0} B_{l0} \left[A_{k0} \left\{ \cos((2\omega_l - \omega_k)t_0) - \cos((2\omega_l + \omega_k)t_0) \right\} \right. \\
& \left. + B_{k0} \left\{ \sin((2\omega_l + \omega_k)t_0) + \sin((2\omega_l - \omega_k)t_0) \right\} \right]. \tag{4.3.8}
\end{aligned}$$

Now it can be seen from the right-hand side of (4.3.8) that secular terms (or equivalently resonances) will occur when $\omega_n \pm \Omega = \pm \omega_k$ or when $\omega_l = \omega_k$. In the following subsections, some cases will be studied in which resonances occur. In section 4.3.1 the case $\Omega \neq \pm \omega_k \pm \omega_n$ will be studied. In this case only internal resonances occur due to the nonlinear term in the PDE (4.2.9). In section 4.3.2 the case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ will be studied in which ϕ is a detuning parameter. This case is an example in which the frequency of the belt-velocity fluctuations is the difference of two natural frequencies of the constant belt-velocity problem. In section 4.3.3 and 4.3.4 the case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$ and $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ respectively will be studied. Again ϕ is a detuning parameter. These cases are examples in which the frequencies of the belt-velocity fluctuations are the sum of two natural frequencies of the constant belt-velocity problem.

4.3.1 The case where Ω causes no resonances

When $\Omega \neq \pm \omega_k \pm \omega_n$ (or not ϵ -close to these values) only internal resonances will occur due to the nonlinear term in the PDE (4.2.9). It can be shown elementarily from (4.3.8) that secular terms in u_{k1} can be avoided if A_{k0} and B_{k0} satisfy

$$\begin{aligned}
\dot{A}_{k0} &= -\frac{k^2 \tilde{P}_1^2 \pi^4}{32\omega_k} B_{k0} \left[k^2 (A_{k0}^2 + B_{k0}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\
\dot{B}_{k0} &= \frac{k^2 \tilde{P}_1^2 \pi^4}{32\omega_k} A_{k0} \left[k^2 (A_{k0}^2 + B_{k0}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \tag{4.3.9}
\end{aligned}$$

for $k = 1, 2, 3, \dots$. System (4.3.9) can be solved exactly by introducing polar coordinates, that is, $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$. System (4.3.9) in polar coordinates then becomes:

$$\dot{r}_k = 0, \quad \dot{\phi}_k = -\frac{k^2 \tilde{P}_1^2 \pi^4}{32\omega_k} \left(k^2 r_k^2 + 2 \sum_{l=1}^{\infty} l^2 r_l^2 \right). \tag{4.3.10}$$

From (4.3.10) it follows that

$$r_k(t_1) = r_k(0) \quad \text{and} \quad \phi_k(t_1) = -\frac{k^2 \tilde{P}_1^2 \pi^4}{32\omega_k} \left(k^2 r_k(0)^2 + 2 \sum_{l=1}^{\infty} l^2 r_l(0)^2 \right) t_1 + \phi_k(0),$$

for $k = 1, 2, 3, \dots$. The constants $r_k(0)$ and $\phi_k(0)$ follow from the initial values $A_{k0}(0)$ and $B_{k0}(0)$.

4.3.2 The case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$

It has been shown at the end of section 4.3 that resonances will occur when $\omega_n \pm \Omega = \pm\omega_k$ or when $\omega_l = \omega_k$. In this section the case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ will be discussed where ϕ is a detuning parameter. By using this special value of Ω additional mode interactions will only occur between mode 1 and mode 2 as has been shown in [8]. Substituting $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ into (4.3.8), taking apart terms that cause resonances and setting these terms equal to zero to avoid secular terms, the following set of equations for $A_{k0}(t_1)$ and $B_{k0}(t_1)$ will be obtained:

$$\begin{aligned}
\dot{A}_{10} &= -\frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_1} [B_{20} \cos(\phi t_1) - A_{20} \sin(\phi t_1)] \\
&\quad - \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} B_{10} \left((A_{10}^2 + B_{10}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right), \\
\dot{B}_{10} &= \frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_1} [A_{20} \cos(\phi t_1) + B_{20} \sin(\phi t_1)] \\
&\quad + \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} A_{10} \left((A_{10}^2 + B_{10}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right), \\
\dot{A}_{20} &= -\frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_2} [A_{10} \sin(\phi t_1) + B_{10} \cos(\phi t_1)] \\
&\quad - \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} B_{20} \left(2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right), \\
\dot{B}_{20} &= \frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_2} [A_{10} \cos(\phi t_1) + B_{10} \sin(\phi t_1)] \\
&\quad + \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} A_{20} \left(2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right), \tag{4.3.11}
\end{aligned}$$

and (4.3.9) for $k = 3, 4, 5, \dots$. By introducing polar coordinates transformations in (4.3.9) for $k = 3, 4, 5, \dots$ and in (4.3.11), that is, $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$ it follows that

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_1} r_2 \sin(\phi_2 - \phi_1 + \phi t_1), \\
\dot{r}_2 &= -\frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_2} r_1 \sin(\phi_2 - \phi_1 + \phi t_1), \\
\dot{\phi}_1 &= -\frac{4\alpha(4\omega_1 - \omega_2)r_2}{9\omega_1 r_1} \cos(\phi_2 - \phi_1 + \phi t_1) - \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} \left(r_1^2 + 2 \sum_{l=1}^{\infty} l^2 r_l^2 \right),
\end{aligned}$$

$$\dot{\phi}_2 = -\frac{4\alpha(4\omega_1 - \omega_2)r_1}{9\omega_2r_2} \cos(\phi_2 - \phi_1 + \phi t_1) - \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} \left(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2 \right), \quad (4.3.12)$$

and $\dot{r}_k = 0$ for $k = 3, 4, 5, \dots$. To obtain (4.3.12) it has been assumed that $r_1 \neq 0$, and $r_2 \neq 0$. From (4.3.11) and (4.3.12) it can be seen that if there is no initial energy present in the k th mode, $k = 3, 4, 5, \dots$ then the energy in that mode will be zero up to $\mathcal{O}(\tilde{\epsilon})$ on time-scales of $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$. From (4.3.12) it can also be seen that if there is energy of $\mathcal{O}(1)$ present in the first mode then an $\mathcal{O}(1)$ part of this energy will be transferred to the second mode, and vice versa. This energy transport will take place on time-scales of $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$. In what follows it is assumed that there is energy present in each mode of vibration at $t = 0$. Since $\dot{r}_k = 0$ for $k = 3, 4, 5, \dots$ it then follows that $r_k(t_1) = r_k(0)$ for $t_1 > 0$. From the first two equations in (4.3.12) it follows that $\omega_1 r_1 \dot{r}_1 + \omega_2 r_2 \dot{r}_2 = 0$. This implies that $\omega_1 r_1^2 + \omega_2 r_2^2 = C$, where C is a constant of integration. In fact $r_k(t_1) = r_k(0)$ for $k = 3, 4, 5, \dots$, and $\omega_1 r_1^2 + \omega_2 r_2^2 = C$ are first integrals of the infinite dimensional system of ODEs (4.3.12). Now let $\Phi(t_1) = \phi_2(t_1) - \phi_1(t_1) + \phi t_1$. Then it can easily be deduced from (4.3.12) that

$$\begin{aligned} \dot{r}_1 &= \frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_1} \sqrt{\frac{C - \omega_1 r_1^2}{\omega_2}} \sin(\Phi), \\ \dot{\Phi} &= \phi + \frac{4\alpha}{9}(4\omega_1 - \omega_2) \left[\frac{r_2}{\omega_1 r_1} - \frac{r_1}{\omega_2 r_2} \right] \cos(\Phi) \\ &\quad + \tilde{P}_1^2 \pi^4 \left[\frac{1}{32\omega_1} \left(r_1^2 + 2 \sum_{l=1}^{\infty} l^2 r_l^2 \right) - \frac{1}{4\omega_2} \left(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2 \right) \right]. \end{aligned} \quad (4.3.13)$$

By introducing the following re-scalings $r_1(t_1) = \sqrt{\frac{c}{\omega_1}} R_1(s_2)$, $\Phi(t_1) = \Psi(s_2)$ with $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1 - \omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1\sqrt{1-R_1^2}}$, and by using the first integrals $r_k(t_1) = r_k(0)$ for $k = 3, 4, 5, \dots$, and $\omega_1 r_1^2 + \omega_2 r_2^2 = C$ it follows that (4.3.13) can be simplified to

$$\begin{aligned} \frac{dR_1}{ds_2} &= R_1(1 - R_1^2) \sin(\Psi), \\ \frac{d\Psi}{ds_2} &= (1 - 2R_1^2) \cos(\Psi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}, \end{aligned} \quad (4.3.14)$$

where $k_i = \frac{9\tilde{P}_1^2 \pi^4 \sqrt{\omega_1\omega_2}}{4\alpha(4\omega_1 - \omega_2)} \bar{k}_i$ for $i = 1, 2$ and $\bar{k}_1 = \left(\frac{3}{32\omega_1} - \frac{1}{4\omega_2} \right) \frac{C}{\omega_1} - \left(\frac{1}{4\omega_1} - \frac{3}{2\omega_2} \right) \frac{C}{\omega_2}$, and $\bar{k}_2 = \left(\frac{1}{4\omega_1} - \frac{3}{2\omega_2} \right) \frac{C}{\omega_2} + \frac{\phi}{\tilde{P}_1^2 \pi^2} + \left(\frac{1}{16\omega_1} - \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2$. Since α and ϕ are both arbitrary it follows that k_1 and k_2 are arbitrary. However, the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$, since for $k_1 < 0$ a simple rescaling ($\Psi := \Psi + \pi$, and $s_2 := -s_2$) leads again to system (4.3.14) with $k_1 \geq 0$ and $-\infty < k_2 < \infty$. It turns out that a first integral for (4.3.14) can also be obtained. To obtain this first integral it

should be observed from (4.3.14) that

$$\begin{aligned}
\frac{d\Psi}{dR_1} &= \frac{(1 - 2R_1^2) \cos(\Psi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}}{R_1(1 - R_1^2) \sin(\Psi)} \\
\Leftrightarrow \frac{\sin(\Psi) d\Psi}{dR_1} &= \frac{(1 - 2R_1^2) \cos(\Psi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}}{R_1(1 - R_1^2)} \\
\Leftrightarrow -\frac{d(\cos(\Psi))}{dR_1} &= \frac{1 - 2R_1^2}{R_1(1 - R_1^2)} \cos(\Psi) + \frac{(k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}}{R_1(1 - R_1^2)} \\
\Leftrightarrow \frac{d(\cos(\Psi))}{dR_1} + \frac{1 - 2R_1^2}{R_1(1 - R_1^2)} \cos(\Psi) &= -\frac{(k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}}{R_1(1 - R_1^2)} \tag{4.3.15}
\end{aligned}$$

which is a first order ODE in $\cos(\Psi)$. The solutions of this ODE(4.3.15) can readily be constructed, yielding

$$\begin{aligned}
\cos(\Psi) &= \frac{k_1}{3R_1 \sqrt{1 - R_1^2}} \left[R_1(1 - R_1^2)^{3/2} + \frac{2}{5}(1 - R_1^2)^{5/2} \right] \\
&\quad + \frac{k_2(1 - R_1^2)}{3R_1} + \frac{\tilde{C}}{R_1 \sqrt{1 - R_1^2}}, \tag{4.3.16}
\end{aligned}$$

where \tilde{C} is a constant of integration. In the following subsections an analysis of system (4.3.14) in the (R_1, Ψ) -phase plane will be given for different values of k_1 and k_2 with $k_1 \geq 0$ and $-\infty < k_2 < \infty$.

Equilibrium points of system (4.3.14)

The obvious equilibrium points of system (4.3.14) are $(R_1, \Psi) = (0, \pm \frac{n\pi}{2})$, and $(1, \pm \frac{n\pi}{2})$, with $n = 1, 3, 5, \dots$. The less obvious equilibrium points (R_1, Ψ) are given by $\Psi = m\pi$ with $m \in \mathbb{Z}$, where R_1 with $0 < R_1 < 1$ follows from $(1 - 2R_1^2) \cos(m\pi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2} = 0$. To determine the number of equilibrium points for a fixed value of m two cases have to be studied: (i) m is even, and (ii) m is odd. These two cases will now be studied.

(i) The case $\Psi = m\pi$ with m even

The R_1 -values in this case follow from

$$\begin{aligned}
1 - 2R_1^2 + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2} &= 0 \\
\Leftrightarrow \frac{1 - 2R_1^2}{R_1 \sqrt{1 - R_1^2}} + k_1 R_1^2 + k_2 &= 0 \\
\Leftrightarrow \frac{1 - R_1^2 - R_1^2}{R_1 \sqrt{1 - R_1^2}} + k_1 R_1^2 + k_2 &= 0 \\
\Leftrightarrow \frac{\sqrt{1 - R_1^2}}{R_1} - \frac{R_1}{\sqrt{1 - R_1^2}} + k_1 R_1^2 + k_2 &= 0 \\
\Leftrightarrow \frac{\sqrt{z - z^2}}{z} - \frac{z}{\sqrt{z - z^2}} + k_1 z + k_2 &= 0, \tag{4.3.17}
\end{aligned}$$

where $z = R_1^2$. To determine z from (4.3.17) is the same as determining the intersection point(s) of the following two curves: $y = k_1 z + k_2$, and $y = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$. For special values of k_1 and k_2 , these two curves are given in Figure 4.1. By varying k_1

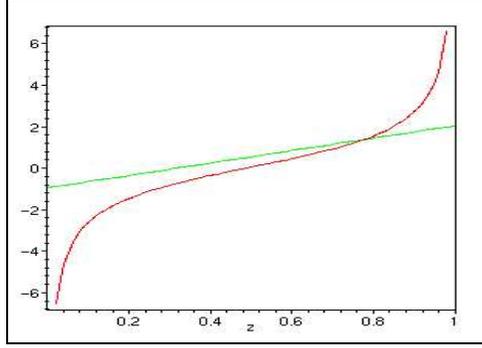


Figure 4.1: The curves $y = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$ and $y = k_1 z + k_2$ with $k_1 = 3$ and $k_2 = -1$.

and k_2 it is possible to obtain one, two, or three intersection points (i.e. equilibrium points). Observe also that as k_2 is getting larger, the intersection point tends to $z = 1$.

In the case that the straight line is tangent to the other curve, there will be two critical points. Assume that the straight line $y = k_1 z + k_2$ is tangent to $f(z) = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$ at the point $z = z_0$. It then follows that

$$\begin{aligned} k_1 &= f'(z_0) = \frac{-1}{2z_0(z_0 - 1)\sqrt{-z_0(z_0 - 1)}}, \\ k_2 &= f(z_0) - z_0 f'(z_0) = \frac{4z_0^2 - 6z_0 + 3}{2(z_0 - 1)\sqrt{-z_0(z_0 - 1)}} \\ &= (4z_0^2 - 6z_0 + 3)(-z_0)k_1. \end{aligned} \quad (4.3.18)$$

From the first equation in (4.3.18) z_0 can be determined, yielding

$$z_{0,1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \sqrt[3]{16/k_1^2}}, \quad (4.3.19)$$

and then from the second equation in (4.3.18) it follows that

$$k_{2_1} = (-4z_{0_1}^3 + 6z_{0_1}^2 - 3z_{0_1})k_1, \quad k_{2_2} = (-4z_{0_2}^3 + 6z_{0_2}^2 - 3z_{0_2})k_1. \quad (4.3.20)$$

From (4.3.19) and from $0 < z_0 < 1$ it follows that $1 - \sqrt[3]{\frac{16}{k_1^2}} \geq 0$. Since $k_1 \geq 0$ it then follows that $k_1 \geq 4$. In Figure 4.2 the curves in the (k_1, k_2) -plane (as defined by (4.3.19) and (4.3.20)) are given on which exactly two equilibrium points (R_1, Ψ) of system (4.3.14) can be found for $\Psi = m\pi$ with m even and fixed. Also in Figure 4.2 the region A-1 (A-3) is given in which exactly one (three) equilibrium point(s) of system (4.3.14) can be found for $\Psi = m\pi$ with m even and fixed.

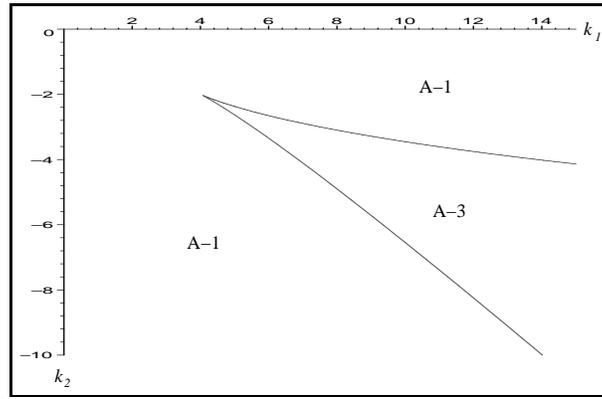


Figure 4.2: Bifurcation curve in the (k_1, k_2) -plane for the number of equilibrium points (R_1, Ψ) of system (4.3.14) with $\Psi = m\pi$, m even and fixed.

(ii) The case $\Psi = m\pi$ with m odd

The R_1 -values in this case follow from

$$-(1 - 2R_1^2) + (k_1R_1^2 + k_2)R_1\sqrt{1 - R_1^2} = 0, \quad (4.3.21)$$

which is equivalent to finding the intersection point(s) of the curves $y = -(k_1z + k_2)$ and $y = -\left(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}}\right)$, where $z = R_1^2$ (see also the previous case (i)). In this case always one equilibrium point will be found for $\Psi = m\pi$ with m odd and fixed since the straight line has a negative gradient.

The (R_1, Ψ) -phase plane of system (4.3.14)

In the previous subsection all equilibrium points of system (4.3.14) have been determined. In this subsection the orbits in the (R_1, Ψ) -phase plane for system (4.3.14) will be given for different values of k_1 and k_2 . In Figure 4.3 these orbits are presented. It can be seen from Figure 4.3 that for large values of the detuning parameter ϕ (that is, for large values of $|k_2|$) $R_1(s_2)$, and so $r_1(t_1)$ become constant. So, for large values of the detuning parameter ϕ the solutions of the "resonant" case (i.e. system (4.3.14)) tend to the solutions of the "non-resonant" case (i.e. system (4.3.9)). Figure 4.3 and the first integrals for system (4.3.14) also show that all solutions are bounded for this special value of $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$, which is of difference type. These results are in accordance with those obtained for the linearized problem (see [8]).

4.3.3 The case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$

At the end of section 4.3 it has been shown that resonances will occur when $\omega_n \pm \Omega = \pm\omega_k$, or when $\omega_l = \omega_k$. In this section the case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$, will be studied, where ϕ is again a detuning parameter. By using this special value of Ω additional mode interactions will only occur between mode 1 and 2 as has been shown

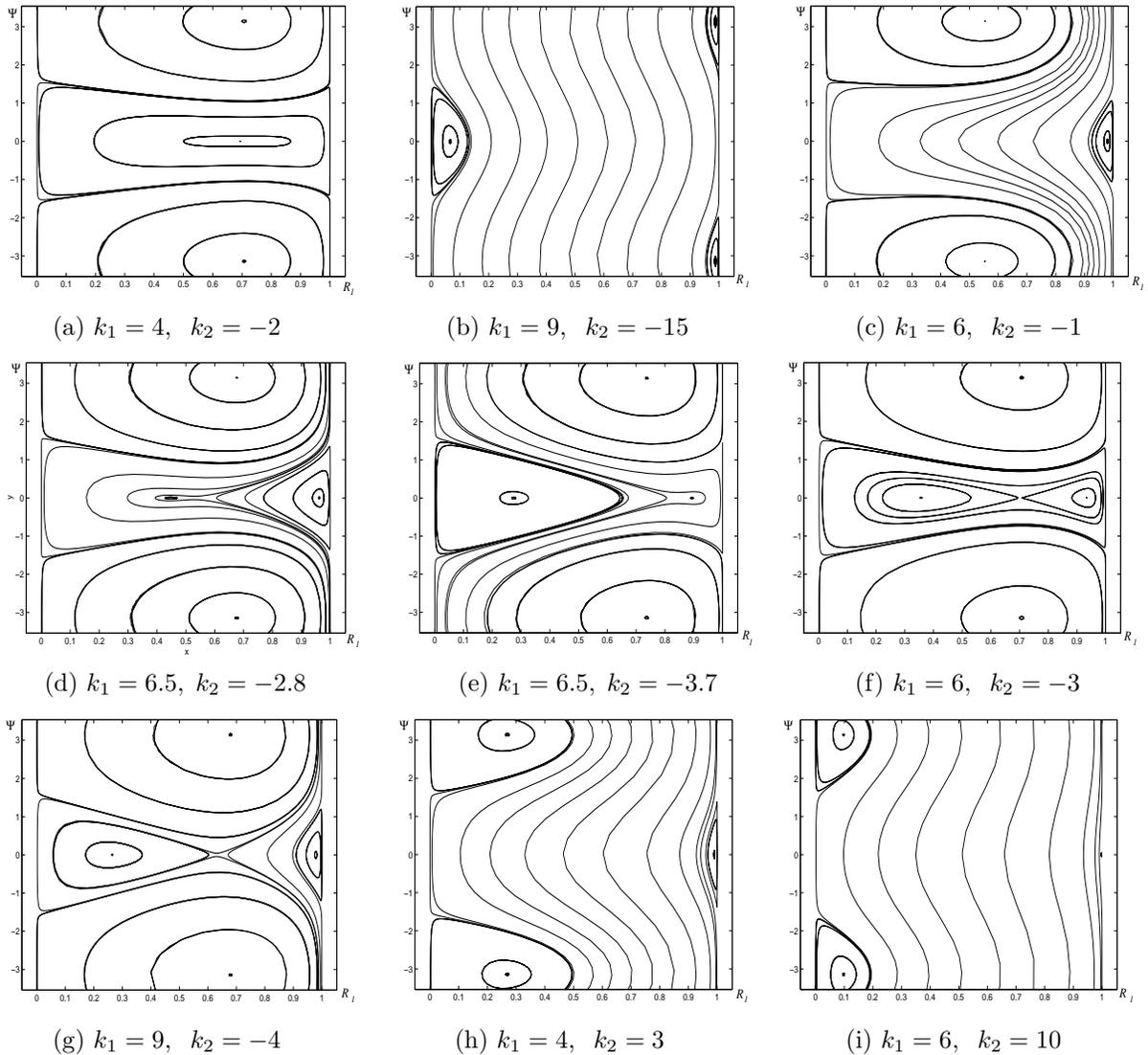


Figure 4.3: Orbits in the (R_1, Ψ) phase plane for system (4.3.14) for different values of k_1 and k_2 with $-\pi \leq \Psi \leq \pi$ (vertical axis) and $0 \leq R_1 \leq 1$ (horizontal axis).

in [8]. Substituting $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$ into (4.3.8), taking apart those terms that cause resonances, and setting these terms equal to zero to avoid secular terms, the following set of equations for $A_{k0}(t_1)$ and $B_{k0}(t_1)$ will be obtained:

$$\begin{aligned}
\dot{A}_{10} &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_1} [B_{20} \cos(\phi t_1) - A_{20} \sin(\phi t_1)] \\
&\quad - \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} B_{10} \left[A_{10}^2 + B_{10}^2 + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\
\dot{B}_{10} &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_1} [A_{20} \cos(\phi t_1) + B_{20} \sin(\phi t_1)] \\
&\quad + \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} A_{10} \left[A_{10}^2 + B_{10}^2 + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\
\dot{A}_{20} &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_2} [B_{10} \cos(\phi t_1) - A_{10} \sin(\phi t_1)] \\
&\quad - \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} B_{20} \left[2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\
\dot{B}_{20} &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_2} [A_{10} \cos(\phi t_1) + B_{10} \sin(\phi t_1)] \\
&\quad + \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} A_{20} \left[2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \tag{4.3.22}
\end{aligned}$$

and (4.3.9) for $k = 3, 4, 5, \dots$. By introducing polar coordinates in (4.3.9) for $k = 3, 4, 5, \dots$ and (4.3.11), that is, $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$ it follows that:

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_1} r_2 \sin(\phi_2 + \phi_1 + \phi t_1), & \dot{r}_2 &= \frac{4\alpha(\omega_2 + 4\omega_1)}{9\omega_2} r_1 \sin(\phi_2 + \phi_1 + \phi t_1), \\
\dot{\phi}_1 &= \frac{4\alpha(\omega_2 + 4\omega_1) r_2}{9\omega_1 r_1} \cos(\phi_2 + \phi_1 + \phi t_1) - \frac{\tilde{P}_1^2 \pi^4}{32\omega_1} \left[r_1^2 + \sum_{l=1}^{\infty} l^2 r_l^2 \right], \\
\dot{\phi}_2 &= \frac{4\alpha(\omega_2 + 4\omega_1) r_1}{9\omega_2 r_2} \cos(\phi_2 + \phi_1 + \phi t_1) - \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} \left[2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2 \right], \tag{4.3.23}
\end{aligned}$$

where $r_l^2 = A_{l0}^2 + B_{l0}^2$, and $\dot{r}_k = 0$ for $k = 3, 4, 5, \dots$. This implies that $r_k(t_1) = \tilde{K}$, where \tilde{K} is a constant. From the first two equations in (4.3.23) a first integral can again be derived, yielding $\omega_1 r_1^2 - \omega_2 r_2^2 = K$, where K is a constant of integration. As in the previous section it will turn out that a phase plane analysis can be performed. To give this analysis three cases have to be distinguished: (i) $K > 0$, (ii) $K = 0$, and (iii) $K < 0$.

The case $K > 0$

By using the first integrals and introducing $\Psi = \phi_2 + \phi_1 + \phi t_1$ a reduced system as in section 4.3.2 can be obtained from (4.3.23), that is;

$$\begin{aligned} \dot{r}_1 &= \frac{4\alpha}{9\omega_1}(4\omega_1 + \omega_2)\sqrt{\frac{\omega_1 r_1^2 - K}{\omega_2}} \sin(\Psi), \\ \dot{\Psi} &= \phi + \frac{4\alpha}{9}(4\omega_1 + \omega_2) \left[\frac{2\omega_1 r_1^2 - K}{\omega_1 \omega_2 r_1 \sqrt{\frac{\omega_1 r_1^2 - K}{\omega_2}}} \right] \cos(\Psi) - \tilde{P}_1^2 \pi^4 \left[\left(\frac{3}{32\omega_1} + \frac{1}{4\omega_2} \right) r_1^2 \right. \\ &\quad \left. + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{\omega_1 r_1^2 - K}{\omega_2} + \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 \right]. \end{aligned} \quad (4.3.24)$$

A further simplification in (4.3.24) can be made by introducing the re-scalings $r_1(t_1) = \sqrt{\frac{K}{\omega_1}} R_1(s_2)$, $s_1 = \frac{4\alpha}{9\sqrt{\omega_1 \omega_2}}(4\omega_1 + \omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1 \sqrt{R_1^2 - 1}}$ which results in:

$$\begin{aligned} \frac{dR_1}{ds_2} &= R_1(R_1^2 - 1) \sin(\Psi), \\ \frac{d\Psi}{ds_2} &= (2R_1^2 - 1) \cos(\Psi) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 - 1}, \end{aligned} \quad (4.3.25)$$

where $k_i = \frac{9\tilde{P}_1^2 \pi^4 \sqrt{\omega_1 \omega_2}}{4\alpha(\omega_2 + 4\omega_1)} \bar{k}_i$, for $i = 1, 2$, $\bar{k}_1 = \left(\frac{3}{32\omega_1} + \frac{1}{4\omega_2} \right) \frac{K}{\omega_1} + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{K}{\omega_2}$ and $\bar{k}_2 = \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 - \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{K}{\omega_2} - \frac{\phi}{\tilde{P}_1^2 \pi^2}$. For the same reasons as given in section 4.3.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. It should be observed that $K > 0$ implies that $R_1 > 1$. Using a similar method as described at the end of section 4.3.2, a first integral of (4.3.25) also can be derived, giving

$$\cos(\Psi) = \frac{1}{R_1 \sqrt{R_1^2 - 1}} \left[\frac{k_1}{7} R_1^7 + \frac{1}{5} (k_2 - k_1) R_1^5 - \frac{k_2}{3} R_1^3 + \hat{C} \right], \quad (4.3.26)$$

where \hat{C} is a constant of integration. The equilibrium points of system (4.3.25) have to satisfy $\frac{dR_1}{ds_2} = 0$ and $\frac{d\Psi}{ds_2} = 0$. Since $R_1 > 1$ in this case it follows for the equilibrium points that $\Psi = m\pi$ with $m \in \mathbb{Z}$ and R_1 has to satisfy

$$\pm(2R_1^2 - 1) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 - 1} = 0, \quad (4.3.27)$$

where the '+' sign is associated with $\Psi = m\pi$ and m even, and the '-' sign is associated with $\Psi = m\pi$ and m odd. Introducing $z = R_1^2$ (4.3.27) becomes

$$\pm \left(\frac{z}{\sqrt{z^2 - z}} + \frac{\sqrt{z^2 - z}}{z} \right) - (k_1 z + k_2) = 0. \quad (4.3.28)$$

The solution(s) of (4.3.28) will be the intersection point(s) of the curves given by $g_1(z) = \pm \left(\frac{z}{\sqrt{z^2 - z}} + \frac{\sqrt{z^2 - z}}{z} \right)$ and $g_2(z) = k_1 z + k_2$. In case $\Psi = m\pi$ and m even

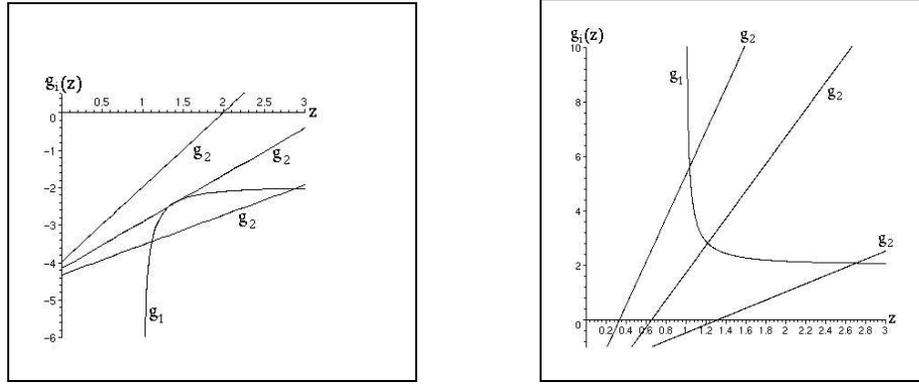


Figure 4.4: The functions $g_1(z) = \pm \left(\frac{\sqrt{z^2-z}}{z} + \frac{z}{\sqrt{z^2-z}} \right)$ and some functions $g_2(z) = k_1z + k_2$. In the left graph g_1 is given with the ‘-’ sign, and in the right graph g_1 is given with the ‘+’ sign.

always one intersection point will be found while in case $\Psi = m\pi$ and m odd zero, one or two intersection points can be found depending on the values of k_1 and k_2 (see also Figure 4.4). For $\Psi = m\pi$ with m odd exactly one intersection point will occur when the straight line is tangent to the other curve. Assume that the straight line $g_2(z) = k_1z + k_2$ is tangent to $g_1(z) = - \left(\frac{\sqrt{z^2-z}}{z} + \frac{z}{\sqrt{z^2-z}} \right)$ at the point $z = z_0$. It then follows that

$$\begin{aligned} k_1 &= g_1'(z_0) = \frac{1}{2}(z_0(z_0 - 1))^{-3/2}, \\ k_2 &= g_1(z_0) - z_0g_1'(z_0) = -(4z_0^3 - 6z_0^2 + 3z_0)k_1, \end{aligned} \quad (4.3.29)$$

where $z_0 > 1$. From the first equation in (4.3.29) it follows that $z_0 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \sqrt[3]{\frac{16}{k_1^2}}}$, and then from the second equation in (4.3.29) it follows how the curve in the (k_1, k_2) -plane is defined on which exactly one equilibrium point (R_1, Ψ) of system (4.3.25) can be found for $\Psi = m\pi$ with m odd and fixed. In Figure 4.5 this curve has been plotted. Also in Figure 4.5 the region A-0 and A-2 are given in which zero and exactly two equilibrium points, respectively, of system (4.3.25) can be found for $\Psi = m\pi$ with m odd and fixed. In Figure 4.6 some phase portraits of system (4.3.25) have been given for different values of k_1 and k_2 . It can also be seen in Figure 4.6 that all solutions for R_1 are bounded, and that for large $|k_2|$ -values (that is, for large values of the detuning parameter) the behaviour of the solutions of system (4.3.25) resembles the solutions of the ”non-resonant” system (4.3.9).

The case $K = 0$

By using the first integral $\omega_1 r_1^2 = \omega_2 r_2^2$ and by introducing $\Psi = \phi_2 + \phi_1 + \phi t_1$ a reduced system (as in section 4.3.2) can be obtained from (4.3.23), that is,

$$\dot{r}_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1 + \omega_2)r_1 \sin(\Psi),$$

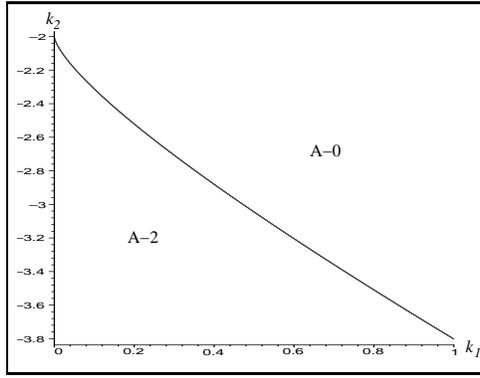


Figure 4.5: Bifurcation curve in the (k_1, k_2) -plane for the number of equilibrium points of system (4.3.25) with $\Psi = m\pi, m$ odd and fixed.

$$\begin{aligned} \dot{\Psi} = & \frac{8\alpha(4\omega_1 + \omega_2)}{9\sqrt{\omega_1\omega_2}} \cos(\Psi) - \tilde{P}_1^2 \pi^4 \left[\left\{ \frac{3}{32\omega_1} + \frac{1}{4\omega_2} + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{\omega_1}{\omega_2} \right\} r_1^2 \right. \\ & \left. + \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P}_1^2 \pi^4} \right]. \end{aligned} \quad (4.3.30)$$

A further simplification in (4.3.30) can be made by introducing the re-scaling $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}} (4\omega_1 + \omega_2)t_1$ which results in

$$\frac{dr_1}{ds_1} = r_1 \sin(\Psi), \quad \frac{d\Psi}{ds_1} = 2 \cos(\Psi) - (k_1 r_1^2 + k_2), \quad (4.3.31)$$

where $k_i = \frac{9\tilde{P}_1^2 \pi^4 \sqrt{\omega_1\omega_2}}{4\alpha(4\omega_1 + \omega_2)} \bar{k}_i$, for $i = 1, 2$, and $\bar{k}_1 = \frac{3}{32\omega_1} + \frac{1}{4\omega_2} + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{\omega_1}{\omega_2}$, $\bar{k}_2 = \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P}_1^2 \pi^4}$. For the same reasons as given in section 4.3.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. A first integral for system (4.3.31) can be computed as follows:

$$\begin{aligned} \frac{d\Psi}{dr_1} = \frac{2 \cos(\Psi)}{r_1 \sin(\Psi)} - \frac{k_1 r_1^2 + k_2}{r_1 \sin(\Psi)} & \Leftrightarrow \sin(\Psi) \frac{d\Psi}{dr_1} = \frac{2 \cos(\Psi)}{r_1} - \frac{k_1 r_1^2 + k_2}{r_1}, \\ \Leftrightarrow \frac{d \cos(\Psi)}{dr_1} + \frac{2 \cos(\Psi)}{r_1} = \frac{k_1 r_1^2 + k_2}{r_1}, \end{aligned} \quad (4.3.32)$$

which has as solution:

$$\cos(\Psi) = \frac{1}{r_1^2} \left[\frac{k_1}{4} r_1^4 + \frac{k_2}{2} r_1^2 + C^* \right], \quad (4.3.33)$$

where C^* is a constant of integration.

The equilibrium points of system (4.3.31) are given by $r_1 \sin(\Psi) = 0$ and $2 \cos(\Psi) - (k_1 r_1^2 + k_2) = 0$. Elementarily it can be shown that the equilibrium points (r_1, Ψ) of system (4.3.31) are:

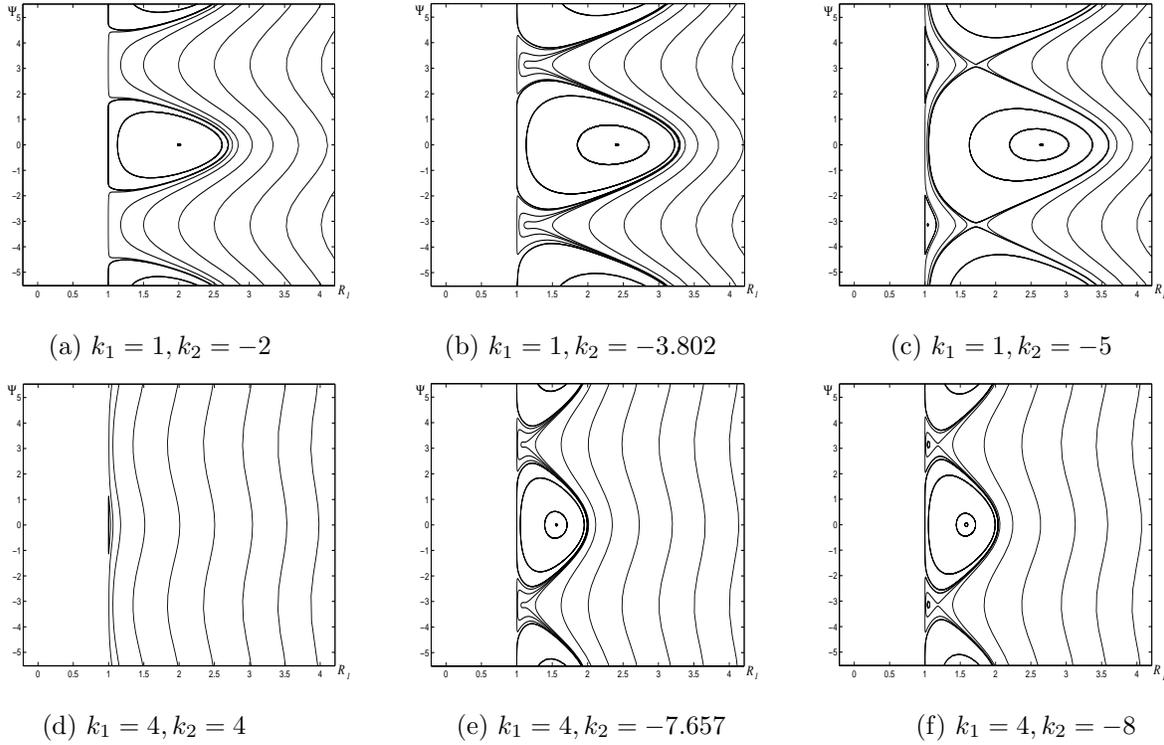


Figure 4.6: Phase portraits of system (4.3.25) for different values of k_1 and k_2 (case $K > 0$).

$$\begin{aligned}
 \text{for } k_2 \leq -2 & : (r_1, \Psi) = \left(\sqrt{\frac{-2-k_2}{k_1}}, m\pi \right) \text{ with } m \text{ odd, and} \\
 & (r_1, \Psi) = \left(\sqrt{\frac{2-k_2}{k_1}}, m\pi \right) \text{ with } m \text{ even.} \\
 \text{for } -2 \leq k_2 \leq 2 & : (r_1, \Psi) = (0, \Psi) \text{ with } \Psi \text{ given by } \cos(\Psi) = \frac{k_2}{2}, \text{ and} \\
 & (r_1, \Psi) = \left(\sqrt{\frac{2-k_2}{k_1}}, m\pi \right) \text{ with } m \text{ even.} \\
 \text{for } k_2 > 2 & : \text{ no equilibrium points.}
 \end{aligned}$$

In Figure 4.7 some phase portraits of system (4.3.31) have been given for different values of k_1 and k_2 . It can also be seen in Figure 4.7 (and from (4.3.33)) that all solutions for r_1 are bounded, and that for large $|k_2|$ -values (that is, for large values of the detuning parameter) the behaviour of the solution of system (4.3.31) resembles the behaviour of the solutions of the "non-resonant" system (4.3.9).

The case $K < 0$

From the first two equations in (4.3.23) a first integral $\omega_1 r_1^2 - \omega_2 r_2^2 = K$ can be derived. Substituting $K = -F$, with $F > 0$ into this first integral $\omega_2 r_2^2 = \omega_1 r_1^2 + F$ is obtained. By using this first integral and the other first integrals $r_k(t_1) = r_k(0)$ for $k > 2$, and by introducing $\Phi = \phi_2 + \phi_1 + \phi t_1$ the following reduced system will be obtained:

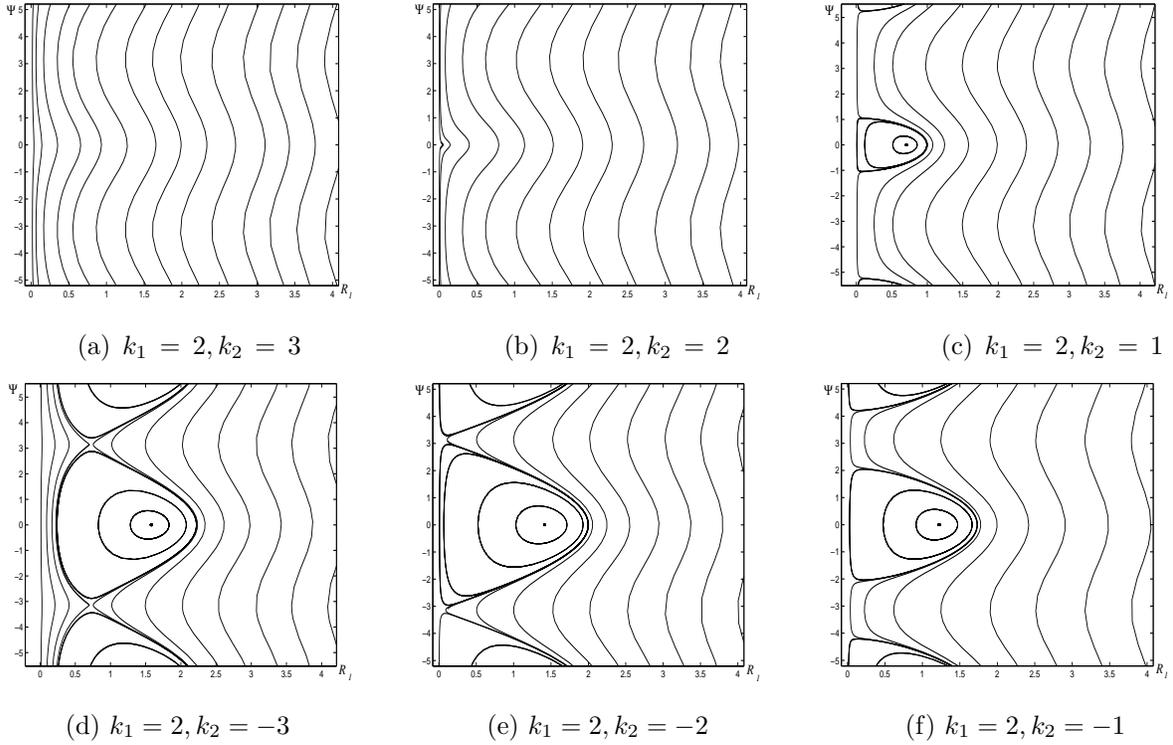


Figure 4.7: Phase portraits of system (4.3.31) for different values of k_1 and k_2 (case $K = 0$).

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha}{9\omega_1}(4\omega_1 + \omega_2) \sqrt{\frac{\omega_1 r_1^2 + F}{\omega_2}} \sin(\Phi), \\
\dot{\Phi} &= \phi + \frac{4\alpha}{9}(4\omega_1 + \omega_2) \left[\frac{2\omega_1 r_1^2 + F}{\omega_1 \omega_2 r_1 \sqrt{\frac{\omega_1 r_1^2 + F}{\omega_2}}} \right] \cos(\Phi) - \tilde{P}_1^2 \pi^4 \left[\left(\frac{3}{32\omega_1} + \frac{1}{4\omega_2} \right) r_1^2 \right. \\
&\quad \left. + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{\omega_1 r_1^2 + F}{\omega_2} + \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 \right]. \tag{4.3.34}
\end{aligned}$$

By introducing the following re-scalings $r_1(t_1) = \sqrt{\frac{F}{\omega_1}} R_1(s_2)$, $\Phi(t_1) = \Psi(s_2)$ with $s_1 = \frac{4\alpha}{9\sqrt{\omega_1 \omega_2}}(4\omega_1 + \omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1 \sqrt{R_1^2 + 1}}$ system (4.3.34) becomes:

$$\begin{aligned}
\frac{dR_1}{ds_2} &= R_1(R_1^2 + 1) \sin(\Psi), \\
\frac{d\Psi}{ds_2} &= (2R_1^2 + 1) \cos(\Psi) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 + 1}, \tag{4.3.35}
\end{aligned}$$

where $k_i = \frac{9\tilde{P}_1^2 \pi^4 \sqrt{\omega_1 \omega_2}}{4\alpha(4\omega_1 + \omega_2)} \bar{k}_i$ for $i = 1, 2$, and $\bar{k}_1 = \left[\left(\frac{3}{32\omega_1} + \frac{1}{4\omega_2} \right) + \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{\omega_1}{\omega_2} \right] \frac{F}{\omega_1}$ and $\bar{k}_2 = \left(\frac{1}{4\omega_1} + \frac{3}{2\omega_2} \right) \frac{F}{\omega_2} + \left(\frac{1}{32\omega_1} + \frac{1}{4\omega_2} \right) \sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P}_1^2 \pi^4}$. For the same reasons as given

in section 4.3.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. Using a similar method as described at the end of section 4.3.2 a first integral of (4.3.35) can be derived, yielding

$$\cos(\Psi) = \frac{1}{R_1 \sqrt{R_1^2 + 1}} \left[\frac{k_1}{4} R_1^4 + \frac{k_2}{2} R_1^2 + C^{**} \right], \quad (4.3.36)$$

where C^{**} is a constant of integration.

The equilibrium points of system (4.3.35) have to satisfy $R_1(R_1^2 + 1) \sin(\Psi) = 0$ and $(2R_1^2 + 1) \cos(\Psi) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 + 1} = 0$. From the first equation it follows that $R_1 = 0$ or $\Psi = m\pi$ with $m \in \mathbb{Z}$. For $R_1 = 0$ it follows from the second equation that $\cos(\Psi) = 0 \Rightarrow \Psi = \frac{(2n+1)}{2}\pi$ with $n \in \mathbb{Z}$. For $\Psi = m\pi$ it follows from the second equation that

$$(-1)^m (2R_1^2 + 1) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 + 1} = 0. \quad (4.3.37)$$

Following the analysis as given in subsection 4.3.1 it can be shown elementarily that

- (i) for m even and fixed there will be always exactly one equilibrium point,
- (ii) for m odd and fixed it is possible to have zero, one, or two equilibrium point(s) depending on the values of k_1 and k_2 . In Figure 4.8 the bifurcation curve in the (k_1, k_2) -plane is given for which one equilibrium point occurs. Also in Figure 4.8 the regions A-0 and A-2 are given in which zero or two equilibrium points occur respectively.

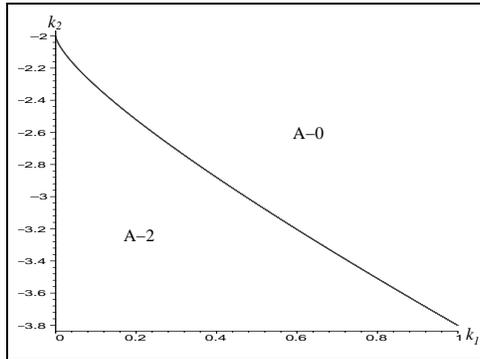


Figure 4.8: Bifurcation curve in the (k_1, k_2) -plane for the number of critical points of system (4.3.35) with $\Psi = m\pi$, m odd and fixed.

In Figure 4.9 some phase portraits of system (4.3.35) are given for different values of k_1 and k_2 . From these phase portraits and from (4.3.36) it can be deduced that R_1 remains bounded, and so, all solutions of the problem with $\Omega = \omega_2 + \omega_1 + \epsilon\phi$ will remain bounded. These results are different from the ones found in the linearized case (see [8]). For the problem under consideration it can be concluded that the nonlinear terms "stabilize" the conveyor belt system.

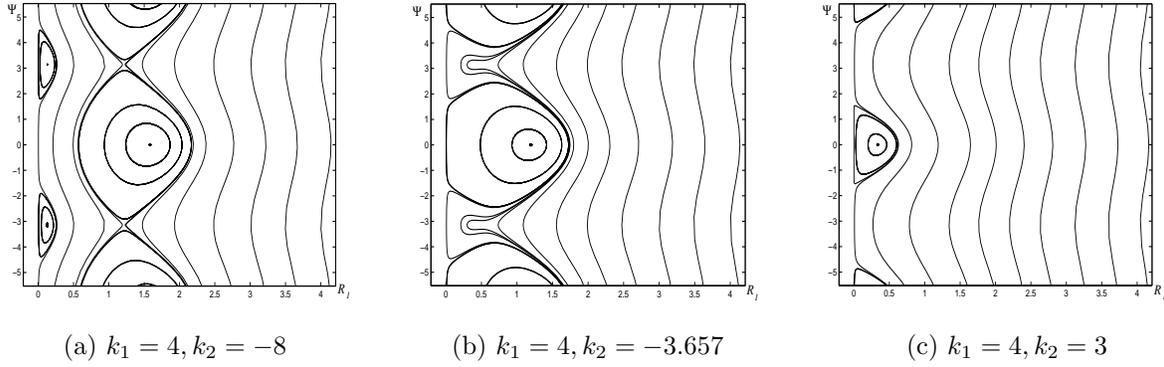


Figure 4.9: Phase portraits of system (4.3.35) for different values of k_1 and k_2 (case $K < 0$).

4.3.4 The case $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$

The linearized problem with $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ has been studied in [8]. It has been shown in [8] that for most parameter values only the second and the third mode will interact through an internal resonance and that for special values of the beam parameters there will be additional interactions. In this section it will be assumed that the beam parameters are such that only an interaction between the second and the third mode occurs due to velocity fluctuations with frequency $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$, where ϕ is a detuning parameter. In [8] it has been shown that for the linearized problem instabilities (that is, unbounded solutions) occur. For the nonlinear system (see (4.3.8)) with $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ it can again be shown that in order to remove secular terms that A_{k0} and B_{k0} have to satisfy:

$$\begin{aligned} \dot{A}_{20} &= \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)[B_{30} \cos(\phi t_1) - A_{30} \sin(\phi t_1)] \\ &\quad - \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} B_{20} \left[2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\ \dot{B}_{20} &= \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)[A_{30} \cos(\phi t_1) + B_{30} \sin(\phi t_1)] \\ &\quad + \frac{\tilde{P}_1^2 \pi^4}{4\omega_2} A_{20} \left[2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\ \dot{A}_{30} &= \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)[B_{20} \cos(\phi t_1) - A_{20} \sin(\phi t_1)] \\ &\quad - \frac{9\tilde{P}_1^2 \pi^4}{32\omega_3} B_{30} \left[9(A_{30}^2 + B_{30}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right], \\ \dot{B}_{30} &= \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)[A_{20} \cos(\phi t_1) - B_{20} \sin(\phi t_1)] \end{aligned}$$

$$+ \frac{9\tilde{P}_1^2 \pi^4}{32\omega_3} A_{30} \left[9(A_{30}^2 + B_{30}^2) + 2 \sum_{l=1}^{\infty} l^2 (A_{l0}^2 + B_{l0}^2) \right]. \quad (4.3.38)$$

and $\dot{A}_{k0} = 0$ and $\dot{B}_{k0} = 0$ for $k = 1, 4, 5, 6, \dots$. By introducing polar coordinates, that is, $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$ it follows that system (4.3.38) becomes

$$\begin{aligned} \dot{r}_2 &= \frac{12\alpha}{25\omega_2} (9\omega_2 + 4\omega_3) r_3 \sin(\phi_2 + \phi_3 + \phi t_1), \quad \dot{r}_3 = \frac{12\alpha}{25\omega_3} (9\omega_2 + 4\omega_3) r_2 \sin(\phi_2 + \phi_3 + \phi t_1), \\ \dot{\phi}_2 &= \frac{12\alpha}{25\omega_2} (9\omega_2 + 4\omega_3) \frac{r_3}{r_2} \cos(\phi_2 + \phi_3 + \phi t_1) - \frac{P_1^2 \pi^4}{4\omega_2} \left(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2 \right), \\ \dot{\phi}_3 &= \frac{12\alpha}{25\omega_3} (9\omega_2 + 4\omega_3) \frac{r_2}{r_3} \cos(\phi_2 + \phi_3 + \phi t_1) - \frac{9P_1^2 \pi^4}{32\omega_3} \left(9r_3^2 + 2 \sum_{l=1}^{\infty} l^2 r_l^2 \right), \end{aligned} \quad (4.3.39)$$

and $\dot{r}_{k0} = 0$ for $k = 1, 4, 5, 6, \dots$. It follows from the first two equations in (4.3.39) that $\omega_2 r_2 \dot{r}_2 - \omega_3 r_3 \dot{r}_3 = 0$ which leads to the first integral $\omega_2 r_2^2 - \omega_3 r_3^2 = \tilde{K}$, where \tilde{K} is a constant of integration.

Now it should be observed that system (4.3.39) and system (4.3.23) are of the same form. So, the analysis as presented in section 4.3.3 can be repeated leading to the same conclusions (see the end of section 4.3.3).

4.4 Conclusions and remarks

In this chapter a weakly nonlinear model describing the transversal vibrations of a conveyor belt with a low and time-varying velocity has been studied. The equations of motion have been derived using Hamilton's principle leading to a system of partial differential equations describing the longitudinal and the transversal displacements of the conveyor belt. Using Kirchhoff's assumption the system of partial differential equations has been reduced to a single fourth order, weakly nonlinear beam equation, which describes the transversal vibrations of the belt system. In the analysis it has been assumed that the belt moves with a time-varying velocity $V(t) = \tilde{\epsilon}(V_0 + \alpha \sin(\Omega t))$, where $\tilde{\epsilon}$, V_0 , and α are constants with $|\alpha| < V_0$ and $0 < \tilde{\epsilon} \ll 1$. The value of $\tilde{\epsilon}$ can be considered to be a measure of the smallness of the belt speed compared to the wave speed. Further it has been assumed that the vertical and the longitudinal displacement are of order $\tilde{\epsilon}$ and of order $\tilde{\epsilon}^2$ respectively, and that $P_0^2 = \frac{EI}{T_0 L^2}$ and $P_1^2 = \frac{EA}{T_0}$ are of order 1 and of order $\frac{1}{\tilde{\epsilon}}$ respectively. Complicated dynamical behaviour of the belt system occurs when the frequency Ω of the belt speed fluctuations is the sum or difference of any two natural frequencies of the belt system with velocity equal to zero. In [8] it has been shown for a linear model that the behaviour of the system will be unstable for frequencies Ω of sum type. In this chapter it has been shown for a weakly nonlinear model that the behaviour of the system will always be stable for $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$,

or $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$, or $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$, where ϕ is a detuning parameter. So, for $\Omega \approx \omega_1 + \omega_2$ and for $\Omega \approx \omega_2 + \omega_3$ it can be concluded that when weakly nonlinear terms are included in the model that the motion (which is still linearly unstable) does not blow up to infinity (as predicted by the linear theory) but remains bounded. It is expected that for other values of Ω the same techniques (as presented in this chapter) can be applied to determine the stability properties of the belt system. Finally it should be remarked that other order assumptions on the longitudinal and the vertical displacement, and on P_0^2 and P_1^2 lead to other model equations. These model problems will be the subject for future research. In particular the (from the point of view of applications) very interesting case $V(t) = \mathcal{O}(1)$, that is, the case for which the belt speed and the wave speed are of the same order of magnitude will be the subject for future research.

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Chapter 5

On The Weakly Nonlinear Transversal Vibrations of A Flexible, Non-stiff Conveyor Belt with A Low and Time-Varying Velocity

Abstract. In this paper the transversal vibrations of a conveyor belt are discussed. The belt is assumed to move with a low and time-varying velocity. It is also assumed that the displacements in the longitudinal direction are of order square of that in the transversal direction, while P_0^2 , the belt bending stiffness, and P_1^2 , the belt inverse static strain, are of the same order. With this set of assumptions Kirchhoff's approach (as has been used in [1]) can not be applied. The solutions of the problem have been approximated using a two time-scales perturbation method. It turns out that if $P_0^2 > \frac{1}{2\pi^2}$ the behaviour of the solutions of the problem studied in this paper resembles those studied in [1], where the Kirchhoff's approach has been used. If $P_0^2 < \frac{1}{2\pi^2}$ additional mode-interactions may arise. It turns out that the values of P_0^2 which cause resonances are clustering in the neighbourhood of $\frac{1}{2\pi^2}, \frac{1}{4\pi^2}, \frac{1}{6\pi^2}, \dots$. Therefore it can be expected that for these special values of P_0^2 , the solutions of the problem will be very complicated since a lot of modes will interact. For a resonant value of $P_0^2 = \frac{15}{154\pi^2}$ together with a resonant and a non-resonant case for Ω , the frequency of the velocity fluctuation, the problem has been studied thoroughly.

5.1 Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper

sheet and web processes, fiber winding and band saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations. Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [2] - [5] for a recent overview) and is still of interest today. In general, the studies about the dynamical behaviour of belt systems have been restricted to belts moving with a constant speed (see for instance [2] - [6]). Recently there are some studies about the transversal vibrations of belt systems moving with a non-constant speed (see for instance [7] - [13]). The vibrations of a belt system moving with a low non-constant velocity have been studied in [7], [8] and [9]. In [7] the belt vibrations have been modeled using a linear string-like equation while in [8] the vibrations have been modeled using a linear beam-like equation. The transversal vibrations of a belt system moving with an $\mathcal{O}(1)$ time-dependent speed have been studied in [10] and [11], while the associated nonlinear vibrations have been studied in [12] and [13]. In [10]- [13] the truncation method has been used, and in almost all cases the solutions are truncated to a single mode of vibration. In [2], [7], [8] and [14] it has been pointed out that a strong reduction in the phase space can lead to a poor description of the dynamic phenomena, and in particular the use of only one mode of vibration approximation can lead to errors in the spatial description and in the forecasting of the time evolution of the system. It has been shown in [7], [8] that the truncation method as applied in [10]- [13] indeed leads to inaccurate results for low speed belt system on long time-scales. A similar conclusion on the applicability of the truncation method to these type of problems can also be found in [14].

In this paper the transversal vibrations of a moving belt will be studied. These vibrations are described by a system of two weakly nonlinear partial differential equations. In [1] Kirchhoff's approach has been used to obtain a single governing equation from this coupled system of partial differential equations which describe the longitudinal and transversal vibrations of the belt. The use of Kirchhoff's approach becomes possible due to the assumption that the belt inverse static strain P_1^2 is very large (that is, $\mathcal{O}(\frac{1}{\epsilon})$ with $0 < \epsilon \ll 1$) and that the belt bending stiffness $P_0^2 = \mathcal{O}(1)$. The advantage of the use of the Kirchhoff's approach is that the problem under consideration can be decoupled into a problem in the transversal direction and into a problem in the longitudinal direction. The problem in the transversal direction is solved first and then the longitudinal problem can be solved in turn. In the case that the belt inverse static strain $P_1^2 = \mathcal{O}(1)$ and $P_0^2 = \mathcal{O}(1)$ the coupled system has to be considered. In this case the Kirchhoff's approach can not be used. The case that P_1^2 and P_0^2 are of the same order of magnitude will be studied in this chapter.

The belt speed is considered to be time-varying and to be small compared to the wave speed. It is assumed that the speed is $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, where ϵ , V_0 , α , and Ω are all constants with $0 < \epsilon \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter ϵ indicates that the belt speed $V(t)$ is small compared to the wave

speed. The variation in $V(t)$ may be due to the pulleys imperfection or some other sources of imperfection and it can be considered as some kind of excitation. In this paper it is assumed that the displacement of the belt in the longitudinal and in the transversal directions are small.

In relation to excitations, some results in this area have been obtained by Sack [15] and Archibald and Emslie [16]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e. $x = L$) is subjected to an harmonic excitation. In [15] the vibrations of the string at $x = L$ is forced to be $v(x, t) = v_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingam in [17]. A moving string model has been used in [17] to study the transverse vibrations of power transmission chains. In all of these works, the belt movement is assumed to be constant.

This paper is organized as follows. In section 5.2, the two partial differential equations describing the transversal and the longitudinal displacements of a conveyor belt are discussed. For the derivation of these equations, the authors refer to [1]. In section 5.3, a (coupled) system of ordinary differential equations is derived from the partial differential equations as obtained in section 5.2. By applying two time-scales perturbation method the system of ordinary differential equations will be studied. It will turn out that bounded solutions will occur only if one of the parameters Ω and P_1^2 are such that $\Omega \neq P_1 k \pi + \mathcal{O}(\epsilon)$ with $k \in \mathbb{N}^+$. In section 5.4 and section 5.5, the case $P_0^2 > \frac{1}{2\pi^2}$ and the case $P_0^2 < \frac{1}{2\pi^2}$ are considered respectively. For the case $P_0^2 < \frac{1}{2\pi^2}$ a special value of $P_0^2 = \frac{15}{154\pi^2}$ is studied in detail. Finally, some conclusions will given and some remarks will be made in section 5.7.

5.2 Equations of motion

The equations of motion describing the dynamical behaviour of a conveyor belt moving with a constant velocity have been derived in [18] using Hamilton's principle. A similar approach with some modifications can also be used to derive the equations of motion in the case that the velocity is a function of time, as has been shown in [7]. In those papers the equations of motion have been derived under the assumption that the displacements in the longitudinal direction are of order square of those in the transversal direction. Furthermore, terms of nonlinear degree higher than 3 have been neglected. In this paper, the equations of motion describing the dynamical behaviour of a conveyor belt are given by (the readers are referred to [7] for the derivation).

$$\begin{aligned} W_{tt} + 2VW_{xt} + V_t(1 + W_x) - (P_1^2 - V^2)W_{xx} &= (P_1^2 - 1 - \eta V^2)U_x U_{xx}, \\ U_{tt} + 2VU_{xt} + V_t U_x + (\kappa V^2 - 1)U_{xx} + P_0^2 U_{xxxx} &= \end{aligned}$$

$$(P_1^2 - 1 - \eta V^2) \left(\frac{3}{2} U_x^2 U_{xx} + U_x W_{xx} + W_x U_{xx} \right), \quad t \geq 0, 0 < x < 1, \quad (5.2.1)$$

where:

- $W(x, t)$ is the longitudinal displacement,
- $U(x, t)$ is the transversal displacement,
- P_0^2 is the dimensionless flexural rigidity of the belt ($P_0^2 = \frac{EI}{T_0 L^2}$),
- P_1^2 is the inverse static strain of the belt ($P_1^2 = \frac{EA}{T_0}$),
- E is the Young's modulus,
- I is the second moment of area with respect to the horizontal axis,
- A is the cross-sectional area of the belt,
- T_0 is the initial tension,
- L is the belt length,
- η is a support constant ($0 \leq \eta \leq 1$),
- κ is $1 - \eta$,
- x is the horizontal position, and
- t is time.

The boundary conditions for the two simple supports are given by:

$$W(0, t) = W(1, t) = 0, \quad \text{and} \quad U(x, t) = U_{xx}(x, t) = 0 \quad \text{for} \quad x = 0, 1, \quad (5.2.2)$$

while the initial displacements and initial velocities are:

$$W(x, 0) = W_0(x), \quad W_t(x, 0) = W_1(x), \quad U(x, 0) = U_0(x), \quad \text{and} \quad U_t(x, 0) = U_1(x). \quad (5.2.3)$$

In [1] P_1^2 is assumed to be much larger than P_0^2 , this leads to the possibility of using Kirchhoff's approximation. In this chapter P_0^2 and P_1^2 are assumed to be of $\mathcal{O}(1)$. Consequently, the coupled system of PDE's has to be considered in this case. The belt speed $V(t)$ is still assumed to be $\mathcal{O}(\epsilon)$.

5.3 The application of the two time-scales perturbation method

Assuming that $U(x, t)$ and $W(x, t)$ are small, that is, $U(x, t) = \sqrt{\epsilon} u(x, t)$, $W(x, t) = \epsilon w(x, t)$, where $u(x, t)$ and $w(x, t)$ are $\mathcal{O}(1)$ and $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, (5.2.1) can be transformed into:

$$\begin{aligned} w_{tt} - P_1^2 w_{xx} + \alpha \Omega \cos(\Omega t) - (P_1^2 - 1) u_x u_{xx} &= -\epsilon \left[2(V_0 + \alpha \sin(\Omega t)) w_{xt} \right. \\ &\quad \left. + \alpha \Omega w_x \cos(\Omega t) \right] + \mathcal{O}(\epsilon^2), \\ u_{tt} - u_{xx} + P_0^2 u_{xxxx} &= -\epsilon \left[2(V_0 + \alpha \sin(\Omega t)) u_{xt} + \alpha \Omega u_x \cos(\Omega t) \right. \\ &\quad \left. - (P_1^2 - 1) \left(\frac{3}{2} u_x^2 u_{xx} + u_x w_{xx} + w_x u_{xx} \right) \right] + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.3.1)$$

Based on the boundary conditions, the solutions are of the form $w(x, t) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x)$ and $u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x)$. By substituting the series for $w(x, t)$ and $u(x, t)$ into (5.3.1), multiplying each term with $\sin(k\pi x)$ and then by integrating the so-obtained equations with respect to x from $x = 0$ to $x = 1$, it follows that:

$$\begin{aligned}
& \ddot{w}_k + (kP_1\pi)^2 w_k + \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=n+m} + \sum_{k=m-n} - \sum_{k=n-m} \right] nm^2 u_n u_m \\
& + \left(1 + (-1)^{k+1}\right) \frac{2\alpha\Omega}{k\pi} \cos(\Omega t_0) = \epsilon \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \\
& \quad \left(\frac{4n\dot{u}_n(V_0 + \alpha \sin(\Omega t))}{2j+1} + \frac{2n\alpha\Omega w_n \cos(\Omega t)}{2j+1} \right), \\
& \ddot{u}_k + \omega_k^2 u_k = \epsilon \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \left(\frac{4n\dot{u}_n(V_0 + \alpha \sin(\Omega t))}{2j+1} \right. \\
& \quad \left. + \frac{2n\alpha\Omega u_n \cos(\Omega t)}{2j+1} \right) + \epsilon \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=n-m} - \sum_{k=n+m} - \sum_{k=m-n} \right] nm^2 (u_n w_m + u_m w_n) \\
& \quad - \epsilon \frac{3(P_1^2 - 1)\pi^4}{16} \left[2 \sum_{k=l+m-n} - 2 \sum_{k=n-m-l} + \sum_{k=l-n-m} + \sum_{k=n+m+l} - \sum_{k=n+m-l} \right] nml^2 u_n u_m u_l,
\end{aligned} \tag{5.3.2}$$

where $\omega_k^2 = (k\pi)^2 + P_0^2(k\pi)^4$. To obtain an approximate solution of (5.3.2), valid uniformly up to order ϵ , a two time-scale perturbation method will be used. So, two time-scales $t_0 = t, t_1 = \epsilon t$ are introduced. Further it is assumed that $w_k(t_0, t_1) = w_{k0}(t_0, t_1) + \epsilon w_{k1}(t_0, t_1) + \epsilon^2 w_{k2}(t_0, t_1) + \dots$ and $u_k(t_0, t_1) = u_{k0}(t_0, t_1) + \epsilon u_{k1}(t_0, t_1) + \epsilon^2 u_{k2}(t_0, t_1) + \dots$. Substituting these expansions for $w_k(t_0, t_1)$ and $u_k(t_0, t_1)$ into (5.3.2), and collecting terms of $\mathcal{O}(1)$, and of $\mathcal{O}(\epsilon)$ it follows that w_{k0}, w_{k1}, u_{k0} , and u_{k1} have to satisfy:

$$\begin{aligned}
\mathcal{O}(1) : & \frac{\partial^2 w_{k0}}{\partial t_0^2} + k^2 P_1^2 \pi^2 w_{k0} + \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=n+m} + \sum_{k=m-n} + \sum_{k=n-m} \right] nm^2 u_{n0} u_{m0} + \\
& \left(1 + (-1)^{k+1}\right) \frac{2\alpha\Omega}{k\pi} \cos(\Omega t_0) = 0, \quad \frac{\partial^2 u_{k0}}{\partial t_0^2} + \omega_k^2 u_{k0} = 0, \\
\mathcal{O}(\epsilon) : & \frac{\partial^2 w_{k1}}{\partial t_0^2} + k^2 P_1^2 \pi^2 w_{k1} = \\
& -2 \frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} - \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=n+m} + \sum_{k=m-n} - \sum_{k=n-m} \right] nm^2 (u_{n0} u_{m1} + u_{n1} u_{m0}) + \\
& \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \left(\frac{4n(V_0 + \alpha \sin(\Omega t_0))}{2j+1} \frac{\partial w_{n0}}{\partial t_0} + \frac{2n\alpha\Omega w_{n0} \cos(\Omega t_0)}{2j+1} \right), \\
& \frac{\partial^2 u_{k1}}{\partial t_0^2} + \omega_k^2 u_{k1} = -2 \frac{\partial u_{k0}}{\partial t_0 \partial t_1} +
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n} \right] \left(\frac{4n(V_0 + \alpha \sin(\Omega t_0))}{2j+1} \frac{\partial u_{n0}}{\partial t_0} + \frac{2n\alpha\Omega u_{n0} \cos(\Omega t_0)}{2j+1} \right) + \\
& \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=n-m} - \sum_{k=n+m} - \sum_{k=m-n} \right] nm^2 (u_{n0}w_{m0} + u_{m0}w_{n0}) - \\
& \frac{3(P_1^2 - 1)\pi^4}{16} \left[2 \sum_{k=l+m-n} - 2 \sum_{k=n-m-l} + \sum_{k=l-n-m} + \sum_{k=n+m+l} - \sum_{k=n+m-l} \right] nml^2 u_{n0}u_{m0}u_{l0}.
\end{aligned} \tag{5.3.3}$$

The $\mathcal{O}(1)$ equations have as solutions

$$\begin{aligned}
u_{k0}(t_0, t_1) &= A_{k0}(t_1) \sin(\omega_k t_0) + B_{k0}(t_1) \cos(\omega_k t_0), \\
w_{k0}(t_0, t_1) &= C_{k0}(t_1) \sin(kP_1\pi t_0) + D_{k0} \cos(kP_1\pi t_0) + \frac{2\alpha\Omega[1 + (-1)^{k+1}]}{k\pi(k^2P_1^2\pi^2 - \Omega^2)} \cos(\Omega t_0) \\
&+ \frac{(P_1^2 - 1)\pi^3}{2} \left[\sum_{k=m+n} + \sum_{k=m-n} + \sum_{k=n-m} \right] \left[\beta_1 \cos(\omega_n + \omega_m)t_0 + \beta_3 \sin(\omega_n + \omega_m)t_0 + \right. \\
&\left. \beta_2 \cos(\omega_n - \omega_m)t_0 + \beta_4 \sin(\omega_n - \omega_m)t_0 \right],
\end{aligned} \tag{5.3.4}$$

where $\beta_1 = \frac{nm^2(B_{n0}B_{m0} - A_{n0}A_{m0})}{2[k^2P_1^2\pi^2 - (\omega_n + \omega_m)^2]}$, $\beta_2 = \frac{nm^2(B_{n0}B_{m0} + A_{n0}A_{m0})}{2[k^2P_1^2\pi^2 - (\omega_n - \omega_m)^2]}$, $\beta_3 = \frac{nm^2(A_{n0}B_{m0} + B_{n0}A_{m0})}{2[k^2P_1^2\pi^2 - (\omega_n + \omega_m)^2]}$, and $\beta_4 = \frac{nm^2(A_{n0}B_{m0} - B_{n0}A_{m0})}{2[k^2P_1^2\pi^2 - (\omega_n - \omega_m)^2]}$. It should be observed that the $\mathcal{O}(1)$ solution for w_{k0} has been derived under the assumptions that Ω is not ϵ -close to $\pm kP_1\pi$ with k odd, and that $\pm\omega_n \pm \omega_m$ is not ϵ -close to $\pm P_1k\pi$ for $k = m+n$, or $k = m-n$, or $k = n-m$. It will turn out that on a time-scale of order $\mathcal{O}(\frac{1}{\epsilon})$ the transversal vibrations can be determined accurately by taking into account the motions in the longitudinal direction only up to $\mathcal{O}(1)$. Because of this fact and due to the complicated calculations to approximate longitudinal vibrations up to $\mathcal{O}(\epsilon)$, the longitudinal part is beyond the scope of this work.

With those assumptions mentioned above, the functions $A_{k0}(t_1), B_{k0}(t_1)$ are then determined by removing the secular terms occurring in the right hand side of the equation for u_{k1} . Now, by substituting u_{k0}, w_{k0} into the $\mathcal{O}(\epsilon)$ equation for u_{k1} , it will turn out that secular terms will occur due to terms containing $\cos(\omega_k t_0), \sin(\omega_k t_0), \sin(\Omega t_0) \frac{\partial u_{n0}}{\partial t_0}, u_{n0} \cos(\Omega t_0)$ and $u_{n0}u_{m0}u_{l0}$. It should be observed that the terms with $u_{n0}w_{m0}$ and $u_{m0}w_{n0}$ do not cause any resonances because of the assumptions mentioned above. Rewriting $\sin(\Omega t_0) \frac{\partial u_{n0}}{\partial t_0}$ and $u_{n0} \cos(\Omega t_0)$ shows that secular terms occur due to expressions such as $\sin((\Omega \pm \omega_n)t_0)$ and $\cos((\Omega \pm \omega_n)t_0)$, while from $u_{n0}u_{m0}u_{l0}$ secular terms occur due to expressions like $\sin((\omega_l + \omega_n - \omega_m)t_0), \sin((\omega_l + \omega_m - \omega_n)t_0), \sin((\omega_l + \omega_n + \omega_m)t_0), \sin((\omega_n + \omega_m - \omega_l)t_0), \cos((\omega_l + \omega_n - \omega_m)t_0), \cos((\omega_l + \omega_m - \omega_n)t_0), \cos((\omega_l + \omega_n + \omega_m)t_0),$ and $\cos((\omega_n + \omega_m - \omega_l)t_0)$. Therefore it follows that the speed fluctuations will cause resonances for $\Omega \pm \omega_n = \pm\omega_k$ for $k = n - 2j - 1$ or $k = n + 2j + 1$ or $k = 2j + 1 - n$, where $k, n \in \mathbb{N}^+$ and $j = 0, 1, 2, \dots$. While from the nonlinear part

resonances will occur if the following systems of equations have solutions:

$$(I) : \begin{cases} k &= m - n - l, \\ \omega_k &= \pm\omega_m \pm \omega_n \pm \omega_l, \end{cases} \quad (5.3.5)$$

$$(II) : \begin{cases} k &= m + n - l, \\ \omega_k &= \pm\omega_m \pm \omega_n \pm \omega_l, \end{cases} \quad (5.3.6)$$

$$(III) : \begin{cases} k &= m + n + l, \\ \omega_k &= \pm\omega_m \pm \omega_n \pm \omega_l, \end{cases} \quad (5.3.7)$$

It turns out (see Appendix) that (I) and (III) have no solutions at all, while for (II) it turns out that only $k = m+n-l, \omega_k = \omega_m + \omega_n - \omega_l$, and $k = m+n-l, \omega_k = \omega_m + \omega_n + \omega_l$ need to be considered. The problem $k = m + n - l, \omega_k = \omega_m + \omega_n - \omega_l$ has only the trivial solutions $k = m, n = l$, or $k = n, m = l$ for all $\mu^2 > 0$, where $\mu^2 = (P_0\pi)^2$. In the appendix it is also shown that the problem $k = m + n - l, \omega_k = \omega_m + \omega_n + \omega_l$ will only have solutions for specific values of μ^2 with $\mu^2 < \frac{1}{2}$.

5.4 The case where $(P_0\pi)^2 = \mu^2 > \frac{1}{2}$

Three cases for Ω will be considered in this section, namely:

1. The value of Ω does not lead to resonances,
2. Ω is a difference of two natural frequencies (that is, $\Omega = \omega_n - \omega_k$ for some n and k),
3. Ω is a sum of two natural frequencies (that is, $\Omega = \omega_n + \omega_k$ for some n and k).

5.4.1 The value of Ω does not cause resonances

In this case, secular terms will occur only due to the nonlinear terms. Also since $\mu^2 > \frac{1}{2}$, the only terms that give rise to resonances are the terms of the form $\sin((\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3})t_0)$ and $\cos((\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3})t_0)$ where $\gamma_1, \gamma_2, \gamma_3$ are l, m , or n . For this trivial case, it can be shown that secular terms can be removed if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{k0} &= -\frac{k\pi^6}{2\omega_k} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{k0} \left[\left(\frac{2}{k^2 P_1^2 \pi^2} + \frac{1}{k^2 P_1^2 \pi^2 - 4\omega_k^2} \right) r_k^2 + \sum_{n=1}^{\infty} L(k, n) r_n^2 \right] \\ &\quad - \frac{3(P_1^2 - 1)\pi^4 k^2}{128\omega_k} B_{k0} \left[3k^2 r_k^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{B}_{k0} &= \frac{k\pi^6}{2\omega_k} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{k0} \left[\left(\frac{2}{k^2 P_1^2 \pi^2} + \frac{1}{k^2 P_1^2 \pi^2 - 4\omega_k^2} \right) r_k^2 + \sum_{n=1}^{\infty} L(k, n) r_n^2 \right] \\ &\quad + \frac{3(P_1^2 - 1)\pi^4 k^2}{128\omega_k} A_{k0} \left[3k^2 r_k^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \end{aligned} \quad (5.4.1)$$

where $L(k, n) = \frac{kn^3(k+n)}{k^2P_1^2\pi^2 - (\omega_k - \omega_n)^2} + \frac{k^2n^2(k+n)}{k^2P_1^2\pi^2 - (\omega_k + \omega_n)^2}$ and $r_k^2 = A_{k0}^2 + B_{k0}^2$. Multiplying the first equation with A_{k0} and the second equation with B_{k0} and then by adding the so-obtained equations, it follows that $r_k \dot{r}_k = 0$. This means that $r_k(t_1)$ is constant and so $r_k(t_1) = r_k(0)$. By putting $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k \cos(\phi_k(t_1))$ it follows that $\dot{\phi}_k = -\frac{k\pi^6}{2\omega_k} \left(\frac{P_1^2-1}{2\sqrt{2}}\right)^2 \left[\left(\frac{2}{k^2P_1^2\pi^2} + \frac{1}{k^2P_1^2\pi^2 - 4\omega_k^2}\right) r_k^2(0) + \sum_{n=1}^{\infty} L(k, n) r_n^2(0) \right] - \frac{3(P_1^2-1)\pi^4 k^2}{128\omega_k} r_k^2 \left[3r_k^2(0) + 4 \sum_{n=1}^{\infty} n^2 r_n^2(0) \right]$. Assuming that $r_{k0}(0) \neq 0$ then $\phi_k(t_1) = \left[-\frac{k\pi^6}{2\omega_k} \left(\frac{P_1^2-1}{2\sqrt{2}}\right)^2 \left[\left(\frac{2}{k^2P_1^2\pi^2} + \frac{1}{k^2P_1^2\pi^2 - 4\omega_k^2}\right) r_k(0) + \sum_{n=1}^{\infty} L(k, n) r_n(0) \right] - \frac{3(P_1^2-1)\pi^4 k^2}{128\omega_k} r_k^2 \left[3r_k(0)^2 + 4 \sum_{n=1}^{\infty} n^2 r_n(0)^2 \right] \right] t_1 + \phi_k(0)$. Hence, up to $\mathcal{O}(\epsilon)$, $u_{k0}(t_0, t_1) = r_{k0}(t_1) \sin(\phi_k(t_1)) \sin(\omega_k t_0) + r_{k0}(t_1) \cos(\phi_k(t_1)) \cos(\omega_k t_0)$. Having found $u_{k0}(t_0, t_1)$, the solution $w_{k0}(t_0, t_1)$ also can be determined from the $\mathcal{O}(1)$ equation in (5.3.3). The solution $w_{k0}(t_0, t_1)$ will be bounded since no secular terms occur in the right hand side in (5.3.3).

5.4.2 The case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$, where the detuning parameter $\phi = \mathcal{O}(1)$

If $\Omega = \omega_2 - \omega_1$ then additional resonances will occur due to the external excitation, that is, from the terms $\sin((\Omega \pm \omega_n)t_0)$ and $\cos((\Omega \pm \omega_n)t_0)$ in (5.3.3). For this special value of Ω , additional mode-interactions will occur between mode 1 and mode 2. It turns out that secular terms can be removed if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy:

$$\begin{aligned} \dot{A}_{10} &= - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[B_{20} \cos(\phi t_1) \right. \\ &\quad \left. - A_{20} \sin(\phi t_1) \right] - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{10} \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n) r_n^2 \right] \\ &\quad - \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} B_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{B}_{10} &= \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[A_{20} \cos(\phi t_1) \right. \\ &\quad \left. + B_{20} \sin(\phi t_1) \right] + \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{10} \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n) r_n^2 \right] \\ &\quad + \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} A_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{A}_{20} &= - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2} + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[B_{10} \cos(\phi t_1) \right. \\ &\quad \left. + A_{10} \sin(\phi t_1) \right] - \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{20} \left[\left(\frac{1}{2P_1^2\pi^2} + \frac{1}{4P_1^2\pi^2 - 4\omega_2^2} \right) r_2^2 + \sum_{n=1}^{\infty} L(2, n) r_n^2 \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{3(P_1^2 - 1)\pi^4}{8\omega_2} B_{20} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\
\dot{B}_{20} = & \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2} + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[A_{10} \cos(\phi t_1) \right. \\
& \left. - B_{10} \sin(\phi t_1) \right] + \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{20} \left[\left(\frac{1}{2P_1^2\pi^2} + \frac{1}{4P_1^2\pi^2 - 4\omega_2^2} \right) r_2^2 + \sum_{n=1}^{\infty} L(2, n) r_n^2 \right] \\
& + \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} A_{20} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right], \tag{5.4.2}
\end{aligned}$$

and (5.4.1) for $k \geq 3$. It can be seen from (5.4.2) that if there is no initial energy present in the k -th mode with $k \geq 3$, then the energy in that mode will be zero up to $\mathcal{O}(\epsilon)$ on time-scales of $\mathcal{O}(\frac{1}{\epsilon})$. However, if there is initial energy present in the first mode, the energy will be transferred to the second mode and reverse energy will also be transferred from the second mode to the first mode. By introducing $A_{k0} = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0} = r_k(t_1) \cos(\phi_k(t_1))$, (5.4.2) can be transformed into:

$$\begin{aligned}
\dot{r}_1 = & \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{\omega_1} \left(\frac{\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] r_2 \sin(\phi_2 - \phi_1 + \phi t_1), \\
\dot{r}_2 = & - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{\omega_1} \left(\frac{\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] r_1 \sin(\phi_2 - \phi_1 + \phi t_1), \\
\dot{\phi}_1 = & - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{\omega_1} \left(\frac{\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \frac{r_2}{r_1} \cos(\phi_2 - \phi_1 + \phi t_1) \\
& - \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right] - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 \right. \\
& \left. + \sum_{n=1}^{\infty} \left(\frac{n^3(1+n)}{P_1^2\pi^2 - (\omega_1 - \omega_n)^2} + \frac{n^2(1+n)}{P_1^2\pi^2 - (\omega_1 + \omega_n)^2} \right) r_n^2 \right], \\
\dot{\phi}_2 = & - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2} + \frac{(P_1^2 - 1)\pi^2}{\omega_2} \left(\frac{\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \frac{r_1}{r_2} \cos(\phi_2 - \phi_1 + \phi t_1) \\
& - \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right] - \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\left(\frac{1}{2P_1^2\pi^2} + \frac{1}{4P_1^2\pi^2 - 4\omega_2^2} \right) r_2^2 \right. \\
& \left. + \sum_{n=1}^{\infty} \left(\frac{2n^3(2+n)}{4P_1^2\pi^2 - (\omega_2 - \omega_n)^2} + \frac{4n^2(2+n)}{4P_1^2\pi^2 - (\omega_2 + \omega_n)^2} \right) r_n^2 \right].
\end{aligned}$$

A first integral can be derived from the first two equations, that is, $\omega_1 r_1^2 + \omega_2 r_2^2 = K$ with K is a constant of integration. Now, by applying this first integral and some transformations, that is, $r_1 = \frac{K}{\omega_1} R_1$, $s_1 = \frac{2\alpha(\omega_1 + \omega_2)}{3\sqrt{\omega_1\omega_2}} t_1$ and $\frac{ds_2}{ds_1} = \frac{1}{R_1\sqrt{1-R_1^2}}$, the following reduced system of ODE's can be obtained:

$$\begin{aligned}
\frac{dR_1}{ds_2} &= R_1(1 - R_1^2) \sin(\Psi), \\
\frac{d\Psi}{ds_2} &= (1 - 2R_1^2) \cos(\Psi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}, \tag{5.4.3}
\end{aligned}$$

where $\Psi = \phi_2 - \phi_1 + \phi t_1$, $k_1 = \frac{\sqrt{\omega_1 \omega_2}}{N_1} \left(\frac{C_1 K}{\omega_1} - \frac{C_2 K}{\omega_2} \right)$, $k_2 = \frac{\sqrt{\omega_1 \omega_2}}{N_1} \left(\frac{C_2 K}{\omega_2} + C_n + \phi \right)$, $C_1 = 3(P_1^2 - 1)\pi^4 \left(\frac{7}{128\omega_1} - \frac{1}{8\omega_2} \right) + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left(\frac{1}{2\omega_1} \left[\frac{2}{P_1^2 \pi^2} + \frac{1}{P_1^2 \pi^2 - 4\omega_1^2} \right] + \frac{L(1,1)}{2\omega_1} - \frac{L(2,1)}{\omega_2} \right)$, $C_2 = 3(P_1^2 - 1)\pi^4 \left(\frac{1}{8\omega_1} - \frac{7}{8\omega_2} \right) + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left(\frac{L(1,2)}{2\omega_1} - \frac{L(2,2)}{\omega_2} - \frac{1}{\omega_2} \left[\frac{1}{2P_1^2 \pi^2} + \frac{1}{4P_1^2 \pi^2 - 4\omega_2^2} \right] \right)$, and $C_n = 3(P_1^2 - 1)\pi^4 \left(\frac{1}{32\omega_1} - \frac{1}{8\omega_2} \right) \sum_{n=3}^{\infty} n^2 r_n^2(0) + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n=3}^{\infty} \left(\frac{L(1,n)}{2\omega_1} - \frac{L(2,n)}{\omega_2} \right) r_n^2(0)$ where $L(k, n) = \frac{kn^3(k+n)}{k^2 P_1^2 \pi^2 - (\omega_k - \omega_n)^2} + \frac{k^2 n^2(k+n)}{k^2 P_1^2 \pi^2 - (\omega_k + \omega_n)^2}$ for $k = 1, 2$. It should be noticed that (5.4.3) is exactly the same as equation (38) in [1]. Therefore for the case where $\mu^2 > \frac{1}{2}$ and $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ together with some assumptions stated at the beginning of this paper, the transversal dynamics of a conveyor belt modeled by (5.2.1) is the same as that modeled by using equation (20) in [1] for which Kirchhoff's approximation has been used. Therefore, up to $\mathcal{O}(\epsilon)$ the displacements in the vertical direction will be bounded on time-scales $\mathcal{O}(\frac{1}{\epsilon})$. Finally, the solutions in the longitudinal direction, w_{k0} , can be determined by using the solutions in the vertical direction, u_{k0} , from the $\mathcal{O}(1)$ -equations in (5.3.3). Also w_{k0} will be bounded.

5.4.3 The case $\Omega = \omega_2 + \omega_1 + \epsilon\phi$, where the detuning parameter $\phi = \mathcal{O}(1)$

For $\Omega = \omega_2 + \omega_1 + \epsilon\phi$ with the detuning parameter $\phi = \mathcal{O}(1)$, secular terms can be removed if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy:

$$\begin{aligned} \dot{A}_{10} &= - \left[\frac{2\alpha}{3\omega_1} (\omega_2 - \omega_1) + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2 \pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2 \pi^2 - \Omega^2} \right) \right] [B_{20} \cos(\phi t_1) \\ &\quad - A_{20} \sin(\phi t_1)] - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{10} \left[\left(\frac{2}{P_1^2 \pi^2} + \frac{1}{P_1^2 \pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n) r_n^2 \right] \\ &\quad - \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} B_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{B}_{10} &= - \left[\frac{2\alpha}{3\omega_1} (\omega_2 - \omega_1) + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2 \pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2 \pi^2 - \Omega^2} \right) \right] [A_{20} \cos(\phi t_1) \\ &\quad + B_{20} \sin(\phi t_1)] + \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{10} \left[\left(\frac{2}{P_1^2 \pi^2} + \frac{1}{P_1^2 \pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n) r_n^2 \right] \\ &\quad + \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} A_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{A}_{20} &= - \left[\frac{2\alpha}{3\omega_2} (\omega_2 - \omega_1) + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2 \pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2 \pi^2 - \Omega^2} \right) \right] [B_{10} \cos(\phi t_1) \\ &\quad - A_{10} \sin(\phi t_1)] - \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{20} \left[\left(\frac{1}{2P_1^2 \pi^2} + \frac{1}{4P_1^2 \pi^2 - 4\omega_2^2} \right) r_2^2 + \sum_{n=1}^{\infty} L(2, n) r_n^2 \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{3(P_1^2 - 1)\pi^4}{8\omega_2} B_{20} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\
\dot{B}_{20} = & -\left[\frac{2\alpha}{3\omega_2} (\omega_2 - \omega_1) + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] [A_{10} \cos(\phi t_1) \\
& + B_{10} \sin(\phi t_1)] + \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{20} \left[\left(\frac{1}{2P_1^2\pi^2} + \frac{1}{4P_1^2\pi^2 - 4\omega_2^2} \right) r_2^2 + \sum_{n=1}^{\infty} L(2, n) r_n^2 \right] \\
& + \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} A_{20} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right],
\end{aligned}$$

and (5.4.1) for $k \geq 3$. By using the transformation $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$, equation (5.4.4) can be transformed into:

$$\begin{aligned}
\dot{r}_1 = & -\frac{N_1}{\omega_1} r_2 \sin(\phi_2 + \phi_1 + \phi t_1), & \dot{r}_2 = & -\frac{N_1}{\omega_2} r_1 \sin(\phi_2 + \phi_1 + \phi t_1), \\
\dot{\phi}_1 = & -\frac{N_1 r_2}{\omega_1 r_1} \cos(\phi_2 + \phi_1 + \phi t_1) - \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right] \\
& - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n) r_n^2 \right], \\
\dot{\phi}_2 = & -\frac{N_1 r_1}{\omega_2 r_2} \cos(\phi_2 + \phi_1 + \phi t_1) - \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} \left[3r_2^2 + \sum_{n=1}^{\infty} n^2 r_n^2 \right] \\
& - \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\left(\frac{1}{2P_1^2\pi^2} + \frac{1}{4P_1^2\pi^2 - 4\omega_2^2} \right) r_2^2 + \sum_{n=1}^{\infty} L(2, n) r_n^2 \right], \quad (5.4.4)
\end{aligned}$$

where the assumptions that $r_i \neq 0$ for $i = 1, 2$ have been used, and $N_1 = \frac{2\alpha}{3}(\omega_2 - \omega_1) + (P_1^2 - 1)\pi^2 \left(\frac{\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right)$. A first integral can be obtained from the first and the second equation in (5.4.4), giving $\omega_1 r_1^2 - \omega_2 r_2^2 = K$. Three cases have to be considered: $K > 0$, $K = 0$, and $K < 0$.

The case $K > 0$

In this case, by using the first integral and several rescalings (namely $r_1 = \sqrt{\frac{K}{\omega_1}} R_1$, $s_1 = \frac{N_1}{\sqrt{\omega_1 \omega_2}} t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1 \sqrt{R_1^2 - 1}}$) system (5.4.4) can be reduced to:

$$\frac{dR_1}{ds_2} = -R_1(R_1^2 - 1) \sin(\Theta), \quad \frac{d\Theta}{ds_2} = (1 - 2R_1^2) \cos(\Theta) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 - 1}, \quad (5.4.5)$$

where $\Theta = \phi_2 + \phi_1 + \phi t_1$. Using the transformation $\Theta = \Psi \pm \pi$ system (5.4.5) can be transformed into:

$$\frac{dR_1}{ds_2} = R_1(R_1^2 - 1) \sin(\Psi), \quad \frac{d\Psi}{ds_2} = (2R_1^2 - 1) \cos(\Psi) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 - 1}, \quad (5.4.6)$$

where $k_1 = \left(\frac{C_1 K}{\omega_1} + \frac{C_2 K}{\omega_2}\right) \frac{\sqrt{\omega_1 \omega_2}}{N_1}$, $k_2 = -\left(C_n + \frac{C_2 K}{\omega_2}\right) \frac{\sqrt{\omega_1 \omega_2}}{N_1}$, $C_1 = 3(P_1^2 - 1)\pi^4 \left(\frac{7}{128\omega_1} + \frac{1}{8\omega_2} + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}}\right)^2 \left[\frac{1}{2\omega_1} \left(\frac{2}{P_1^2 \pi^2} + \frac{1}{P_1^2 \pi^2 - 4\omega_1^2}\right) + \frac{L(1,1)}{2\omega_1} + \frac{L(2,1)}{\omega_2}\right] + \frac{L(1,2)}{2\omega_1} + \frac{L(2,2)}{\omega_2}\right)$, and $C_n = \phi - \sum_{n=3}^{\infty} \left[\left(\frac{1}{8\omega_2} + \frac{1}{32\omega_1}\right)n^2 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}}\right)^2 \left(\frac{L(1,n)}{2\omega_1} + \frac{L(2,n)}{\omega_2}\right)\right] r_n^2(0)$. The parameters N_1 and $L(k, n)$ have been defined previously. This system of ODE's (5.4.6) is the same as (49) in [1] where Kirchhoff's approximation has been used. Therefore the analysis will be the same as the one in [1] and it will not be repeated here.

The case $K = 0$

Using the same method as that explained in section 4.3.1, the following reduced system of ODE's can be obtained from (5.4.4) for $K = 0$

$$\frac{dr_1}{ds_1} = -r_1 \sin(\Theta), \quad \frac{d\Theta}{ds_1} = -2 \cos(\Theta) - (k_1 r_1^2 + k_2), \quad (5.4.7)$$

where $\Theta = \phi_2 + \phi_1 + \phi t_1$, $k_1 = C_1 + \frac{C_2 \omega_1}{\omega_2}$ and $k_2 = -C_n$ (C_1, C_2 and C_n are defined in section 4.3.1). Again with the transformation $\Theta = \Psi \pm \pi$ (5.4.7) can be transformed into equation (55) of [1], that is,

$$\frac{dr_1}{ds_1} = r_1 \sin(\Psi), \quad \frac{d\Psi}{ds_1} = 2 \cos(\Psi) - (k_1 r_1^2 + k_2). \quad (5.4.8)$$

The analysis and its results can be obtained in [1], and it will not be repeated here.

The case $K < 0$

If $K < 0$ then by using the aforementioned transformations the following reduced system will be obtained from the system (5.4.4)

$$\begin{aligned} \frac{dR_1}{ds_2} &= -R_1(R_1^2 + 1) \sin(\Theta), \\ \frac{d\Theta}{ds_2} &= -(2R_1^2 + 1) \cos(\Theta) - (k_1 R_1^2 + k_2) R_1 \sqrt{R_1^2 + 1}, \end{aligned} \quad (5.4.9)$$

where $\Theta = \phi_2 + \phi_1 + \phi t_1$, $k_1 = \left(\frac{C_1}{\omega_1} + \frac{C_2}{\omega_2}\right) \frac{M \sqrt{\omega_1 \omega_2}}{N_1}$ and $k_2 = \left(\frac{C_2 M}{\omega_2} - C_n\right) \frac{\sqrt{\omega_1 \omega_2}}{N_1}$ with $M = -K > 0$ (C_1, C_2 and C_n are defined in section 4.3.1). Again this system of ODE's can be brought back into system (59) of [1] by using the transformation $\Theta = \Psi \pm \pi$. For analysis of this system, the reader is referred to [1].

From the analysis as presented in section 4 it can be concluded that in the case $(P_0 \pi)^2 = \mu^2 > \frac{1}{2}$ and $\Omega = \omega_2 \pm \omega_1 + \epsilon \phi$ the behaviour of the belt system as modeled by (5.2.1) is the same as that modeled in [1] (see subsection 4.3 of [1]) where Kirchhoff's approximation has been used. Hence, up to $\mathcal{O}(\epsilon)$ the solution will be bounded on time-scales of $\mathcal{O}(\frac{1}{\epsilon})$.

5.5 The case $(P_0\pi)^2 = \mu^2 < \frac{1}{2}$

For $\mu^2 < \frac{1}{2}$ additional mode-interactions may occur due to the nonlinear part of the equation. These additional interactions are never detected if Kirchhoff's approximation is applied. Since μ^2 can be small (because P_0^2 is usually small (see [5])) then the occurrence of these additional interactions can not be avoided. The occurrence of these additional interactions usually give rise to complicated problems, which are difficult to handle analytically. Two cases will be studied in this chapter. In this section the detuned case for $\mu^2 = \frac{15}{154}$ with a nonresonant value of Ω will be considered, and in section 6 the case $\mu^2 = \frac{15}{154}$ and the resonant value $\omega_2 - \omega_1 + \epsilon\phi$ for Ω will be considered.

5.5.1 The case $\mu^2 = \frac{15}{154} + \epsilon\zeta$ and Ω causes no resonances

From (5.3.6) and the appendix (see Figure 6, point P, $\lambda^2 = \frac{1}{\mu^2} = \frac{154}{15}$) it follows that for $\mu^2 = \frac{15}{154}$ the first, the third, and the fifth mode will interact since $\omega_5 = 2\omega_3 + \omega_1$. In what follows a detuned case of $\mu^2 = \frac{15}{154}$ will be chosen as an example to study the dynamic behaviour of the conveyor belt. Now, detuning μ^2 implies detuning P_0^2 in the original PDE's. Rewriting μ^2 as $\mu^2 = \mu_c^2 + \epsilon\zeta$ where $\mu_c^2 = \frac{15}{154}$ and $P_0^2 = P_{0c}^2 + \epsilon\sigma$ it follows from $\mu^2 = P_0^2\pi^2$ that $\mu_c^2 = P_{0c}^2\pi^2$ and $\zeta = \sigma\pi^2$.

It will turn out that by using $P_0^2 = P_{0c}^2 + \epsilon\sigma$ in the original PDE's, the application of the two time-scales perturbation method will give secular-free terms in the approximations if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= -\frac{\pi^4\sigma}{2\omega_1}B_{10} - \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_1} + \frac{810\pi^6}{2\omega_1[P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[B_{50} (B_{30}^2 - A_{30}^2) \right. \\ &\quad \left. + 2A_{30}B_{30}A_{50} \right] - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{10} \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n)r_n^2 \right] \\ &\quad - \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} B_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{B}_{10} &= \frac{\pi^4\sigma}{2\omega_1}A_{10} + \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_1} + \frac{810\pi^6}{2\omega_1[P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[A_{50} (B_{30}^2 - A_{30}^2) \right. \\ &\quad \left. - 2A_{30}B_{30}B_{50} \right] + \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{10} \left[\left(\frac{2}{P_1^2\pi^2} + \frac{1}{P_1^2\pi^2 - 4\omega_1^2} \right) r_1^2 + \sum_{n=1}^{\infty} L(1, n)r_n^2 \right] \\ &\quad + \frac{3(P_1^2 - 1)\pi^4}{128\omega_1} A_{10} \left[3r_1^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\ \dot{A}_{30} &= -\frac{81\pi^4\sigma}{2\omega_1}B_{30} - \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_3} + \frac{810\pi^6}{2\omega_3[9P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[B_{50} (B_{10}B_{30} \right. \\ &\quad \left. - A_{10}A_{30}) + A_{50} (B_{10}A_{30} + B_{30}A_{10}) \right] - \frac{3\pi^6}{2\omega_3} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{30} \left[\left(\frac{2}{9P_1^2\pi^2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{9P_1^2\pi^2 - 4\omega_3^2} r_3^2 + \sum_{n=1}^{\infty} L(3, n) r_n^2 \Big] - \frac{27(P_1^2 - 1)\pi^4}{128\omega_3} B_{30} \left[27r_3^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\
\dot{B}_{30} &= \frac{81\pi^4\sigma}{2\omega_1} A_{30} + \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_3} + \frac{810\pi^6}{2\omega_3[9P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[A_{50} (B_{10}B_{30} \right. \\
& - A_{10}A_{30}) - B_{50} (B_{10}A_{30} + B_{30}A_{10}) \Big] + \frac{3\pi^6}{2\omega_3} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{30} \left[\left(\frac{2}{9P_1^2\pi^2} \right. \right. \\
& + \frac{1}{9P_1^2\pi^2 - 4\omega_3^2} \Big) r_3^2 + \sum_{n=1}^{\infty} L(3, n) r_n^2 \Big] + \frac{27(P_1^2 - 1)\pi^4}{128\omega_3} A_{30} \left[27r_3^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\
\dot{A}_{50} &= -\frac{625\pi^4\sigma}{2\omega_1} B_{50} - \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_5} + \frac{810\pi^6}{2\omega_5[25P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[B_{10} (B_{30}^2 \right. \\
& - A_{30}^2) - 2A_{30}B_{30}A_{10} \Big] - \frac{5\pi^6}{2\omega_5} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{50} \left[\left(\frac{2}{25P_1^2\pi^2} + \frac{1}{25P_1^2\pi^2 - 4\omega_3^2} \right) r_5^2 \right. \\
& + \sum_{n=1}^{\infty} L(5, n) r_n^2 \Big] - \frac{75(P_1^2 - 1)\pi^4}{128\omega_5} B_{50} \left[75r_5^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \\
\dot{B}_{50} &= \frac{625\pi^4\sigma}{2\omega_1} A_{50} + \left[\frac{270(P_1^2 - 1)\pi^4}{128\omega_5} + \frac{810\pi^6}{2\omega_5[25P_1^2\pi^2 - 4\omega_3^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \right] \left[A_{10} (B_{30}^2 \right. \\
& - A_{30}^2) + 2A_{30}B_{30}B_{10} \Big] + \frac{5\pi^6}{2\omega_5} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{50} \left[\left(\frac{2}{25P_1^2\pi^2} + \frac{1}{25P_1^2\pi^2 - 4\omega_3^2} \right) r_5^2 \right. \\
& + \sum_{n=1}^{\infty} L(5, n) r_n^2 \Big] + \frac{75(P_1^2 - 1)\pi^4}{128\omega_5} A_{50} \left[75r_5^2 + 4 \sum_{n=1}^{\infty} n^2 r_n^2 \right], \tag{5.5.1}
\end{aligned}$$

and (5.4.1) for $k \neq 1, 3, 5$. By introducing $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$ it follows that (5.5.1) becomes:

$$\begin{aligned}
\dot{r}_1 &= -\frac{\tilde{K}_1}{\omega_1} r_3^2 r_5 \sin(2\phi_3 - \phi_5 + \phi_1), & \dot{r}_3 &= -\frac{\tilde{K}_3}{\omega_3} r_1 r_3 r_5 \sin(2\phi_3 - \phi_5 + \phi_1), \\
\dot{r}_5 &= \frac{\tilde{K}_5}{\omega_5} r_1 r_3^2 \sin(2\phi_3 - \phi_5 + \phi_1), \\
\dot{\phi}_1 &= -\frac{\pi^4\sigma}{2\omega_1} - \frac{\tilde{K}_1 r_5 r_3^2}{\omega_1 r_1} \cos(2\phi_3 - \phi_5 + \phi_1) - M_1 r_1^2 - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n=1}^{\infty} L(1, n) r_n^2 \\
& - \frac{12(P_1^2 - 1)\pi^4}{128\omega_1} \sum_{n=1}^{\infty} n^2 r_n^2, \\
\dot{\phi}_3 &= -\frac{81\pi^4\sigma}{2\omega_3} - \frac{\tilde{K}_3}{\omega_3} r_1 r_5 \cos(2\phi_3 - \phi_5 + \phi_1) - M_3 r_3^2 - \frac{3\pi^6}{2\omega_3} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n=1}^{\infty} L(3, n) r_n^2 \\
& - \frac{108(P_1^2 - 1)\pi^4}{128\omega_3} \sum_{n=1}^{\infty} n^2 r_n^2, \\
\dot{\phi}_5 &= -\frac{625\pi^4\sigma}{2\omega_5} - \frac{\tilde{K}_5 r_1 r_3^2}{\omega_5 r_5} \cos(2\phi_3 - \phi_5 + \phi_1) - M_5 r_5^2 - \frac{5\pi^6}{2\omega_5} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n=1}^{\infty} L(5, n) r_n^2
\end{aligned}$$

$$-\frac{300(P_1^2 - 1)\pi^4}{128\omega_5} \sum_{n=1}^{\infty} n^2 r_n^2, \quad (5.5.2)$$

where it has been assumed that r_1, r_3 and r_5 are not zero, $M_n = \frac{n\pi^6}{2\omega_n} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left(\frac{2}{n^2 P_1^2 \pi^2} + \frac{1}{n^2 P_1^2 \pi^2 - 4\omega_n^2} \right) + \frac{9n^4(P_1^2 - 1)\pi^4}{128\omega_n}$ and $\tilde{K}_n = \frac{270(P_1^2 - 1)\pi^4}{128} + \frac{810\pi^6}{2[nP_1^2 \pi^2 - 4\omega_n^2]} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2$ for $n = 1, 3, 5$. There are first integrals for system (5.5.2) connecting r_1, r_3 and r_5 , namely, $\frac{\omega_1}{K_1} r_1^2 + \frac{\omega_5}{K_5} r_5^2 = K_1$, and $\frac{\omega_1}{K_1} r_1^2 - \frac{\omega_3}{K_3} r_3^2 = K_3$. Consequently, it is also obvious that $\frac{\omega_3}{K_3} r_3^2 + \frac{\omega_5}{K_5} r_5^2 = K_1 - K_3$, where K_1 and K_3 are constants of integration. Notice that $K_1 > 0$, while K_3 can take any value, and $K_1 - K_3 > 0$ (implying that $\frac{K_3}{K_1} < 1$). By using these first integrals and denoting $\Psi = 2\phi_3 - \phi_5 + \phi_1$ system (5.5.2) can be reduced to a system consisting of only two ODE's, namely:

$$\begin{aligned} \dot{\rho}_1 &= -\frac{\tilde{K}}{\omega_1 \omega_3 \sqrt{\omega_5}} (\omega_1 \rho_1^2 - K_3) \sqrt{K_1 - \omega_1 \rho_1^2} \sin(\Psi), \\ \dot{\Psi} &= \left[4\omega_1^2 \rho_1^4 - (3K_1 + 2K_3)\omega_1 \rho_1^2 + K_1 K_3 \right] \frac{\tilde{K} \cos(\Psi)}{\omega_1 \omega_3 \sqrt{\omega_5} \rho_1 \sqrt{K_1 - \omega_1 \rho_1^2}} + \tilde{k}_1 \rho_1^2 + \tilde{k}_2, \end{aligned}$$

where $\rho_1 = \sqrt{\frac{1}{K_1}} r_1$, $\tilde{K} = \tilde{K}_3 \sqrt{\tilde{K}_1 \tilde{K}_5}$, $\tilde{k}_1 = N_1 \tilde{K}_1 + \frac{N_3 \tilde{K}_3 \omega_1}{\omega_3} - \frac{N_5 \tilde{K}_5 \omega_1}{\omega_5}$, $\tilde{k}_2 = \frac{N_5 K_1 \tilde{K}_5}{\omega_5} - \frac{N_3 K_3 \tilde{K}_3}{\omega_3} + \left(\frac{625}{2\omega_5} - \frac{1}{2\omega_1} - \frac{162}{2\omega_3} \right) \pi^4 \sigma + \left(\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right) (P_1^2 - 1) \pi^4 \sum_{n \neq 1, 3, 5}^{\infty} n^2 r_n^2(0) + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n \neq 1, 3, 5}^{\infty} \left[\frac{5L(5, n)}{2\omega_5} - \frac{L(1, n)}{2\omega_1} - \frac{6L(3, n)}{2\omega_3} \right] r_n^2(0)$, and $N_1 = M_5 - M_1 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 1)}{2\omega_5} - \frac{L(1, 1)}{2\omega_1} - \frac{6L(3, 1)}{2\omega_3} \right]$, $N_3 = 9M_5 - M_3 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 3)}{2\omega_5} - \frac{L(1, 3)}{2\omega_1} - \frac{6L(3, 1)}{2\omega_3} \right]$, $N_5 = M_5 + 25(P_1^2 - 1)\pi^4 \left(\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right) + \frac{5L(5, 5)}{2\omega_5} - \frac{L(1, 5)}{2\omega_1} - \frac{6L(3, 5)}{2\omega_3}$. Rescaling with $\rho_1 = \sqrt{\frac{K_1}{\omega_1}} R_1$, $s_1 = \frac{\tilde{K} t_1}{\omega_1 \omega_3 \sqrt{\omega_5}}$ and $s_2 = K_1^2 s_1$ will give:

$$\begin{aligned} \frac{dR_1}{ds_2} &= R_1(R_1^2 - 1)(R_1^2 - k_3) \sin(\Psi), \\ \frac{d\Psi}{ds_2} &= [4R_1^4 - (3 + 2k_3)R_1^2 + k_3] \cos(\Psi) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2}, \end{aligned} \quad (5.5.3)$$

where $k_3 = \frac{K_3}{K_1} < 1$, $k_1 = \frac{\tilde{k}_1}{\omega_1} \omega_3 \sqrt{\omega_1 \omega_5}$ and $k_2 = \frac{\tilde{k}_2}{K_1} \omega_3 \sqrt{\omega_1 \omega_5}$, and where $0 < R_1 < 1$.

5.5.2 Critical points of (5.5.3)

The most obvious critical point of (5.5.3) is $R_1^2 = k_3$, and by substituting this value of R_1 into the right hand side of the second equation in (5.5.3), Ψ can be determined. For $R_1^2 \neq k_3$ then the critical points have to satisfy $\sin(\Psi) = 0$, that is, $\Psi = n\pi, n \in \mathbb{Z}$. The second equation then becomes:

$$[4R_1^4 - (3 + 2k_3)R_1^2 + k_3](\pm 1) + (k_1 R_1^2 + k_2) R_1 \sqrt{1 - R_1^2} = 0,$$

$$\begin{aligned} \Leftrightarrow \pm \frac{4x^2 - (3 + 2k_3)x + k_3}{\sqrt{x - x^2}} &= -k_1x - k_2, \\ \Leftrightarrow \frac{4x^2 - (3 + 2k_3)x + k_3}{\sqrt{x - x^2}} &= \mp k_1x \mp k_2, \end{aligned} \quad (5.5.4)$$

where: $x = R_1^2$. The '+' sign corresponds to $\Psi = n\pi$ with n odd, while the '-' sign corresponds to $\Psi = n\pi$ with n even. If the left hand side of (5.5.4) is denoted by $y_1(x)$ and the right hand side by $y_2(x)$ then solutions of (5.5.4) will be the intersection points $y_1(x)$ and $y_2(x)$. In Figure 2, the graph of $y_1(x)$ has been drawn for different values of k_3 . To determine the solutions of (5.5.4), the parameter k_1 will be kept fixed, namely

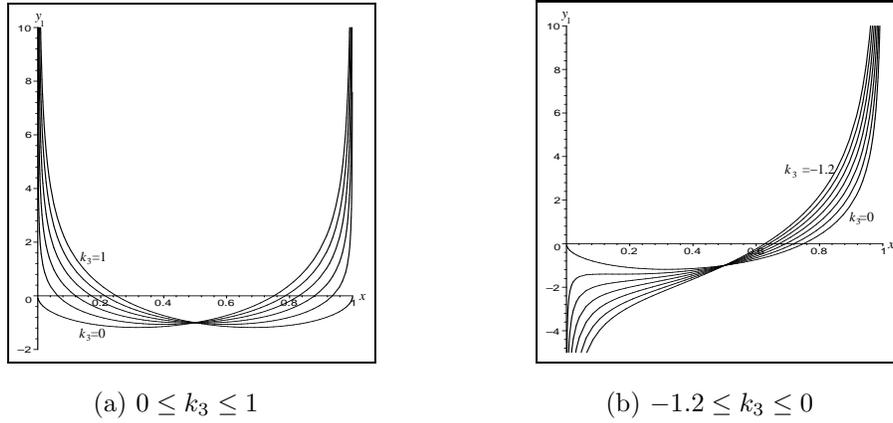


Figure 5.1: The curves of y_1

$k_1 = 1$. For other values of k_1 the method of analysis is the same. The values of k_2, k_3 will be varied with $k_3 < 1$. For $k_3 < 0$ it is possible to have 1, 2, or 3 critical points with $\Psi = n\pi$ for both n odd or even, while for $0 \leq k_3 < 1$ only 0, 1 or 2 critical points are possible. The set of parameter values (k_3, k_2) where $y_2(x)$ tangent to $y_1(x)$ gives a bifurcation diagram. This bifurcation diagram will now be determined.

The bifurcation diagram

The bifurcation diagrams depicting the change of the number of critical points of system (5.5.3) can be derived as follows. Let x_0 denote the point where $y_1(x)$ is tangent to $y_2(x)$. Then, it follows that

$$\frac{dy_1(x_0)}{dx} = \mp k_1, \text{ and } y_1(x_0) = y_2(x_0). \quad (5.5.5)$$

Since $k_1 = 1$ it follows from (5.5.5) that:

$$\frac{8x_0^3 - 12x_0^2 + 3x_0 + k_3}{2(x_0^2 - x_0)\sqrt{x_0 - x_0^2}} = \mp 1 \text{ and } k_2 = \pm x_0 \pm \frac{4x_0^2 - (3 + 2k_3)x_0 + k_3}{\sqrt{x_0 - x_0^2}}. \quad (5.5.6)$$

From (5.5.6), it follows that the parametric equations of the bifurcation curves are given by:

$$k_3 = 2(x_0 - x_0^2)\sqrt{x_0 - x_0^2} + 12x_0^2 - 8x_0^3 - 3x_0 \text{ and } k_2 = x_0 + \frac{4x_0^2 - (3 + 2k_3)x_0 + k_3}{\sqrt{x_0 - x_0^2}},$$

and

$$k_3 = 2(x_0^2 - x_0)\sqrt{x_0 - x_0^2} + 12x_0^2 - 8x_0^3 - 3x_0 \text{ and } k_2 = x_0 - \frac{4x_0^2 - (3 + 2k_3)x_0 + k_3}{\sqrt{x_0 - x_0^2}}. \tag{5.5.7}$$

The graphs of (5.5.7) in the (k_3, k_2) -plane have been drawn in Figure 2. In this Figure,

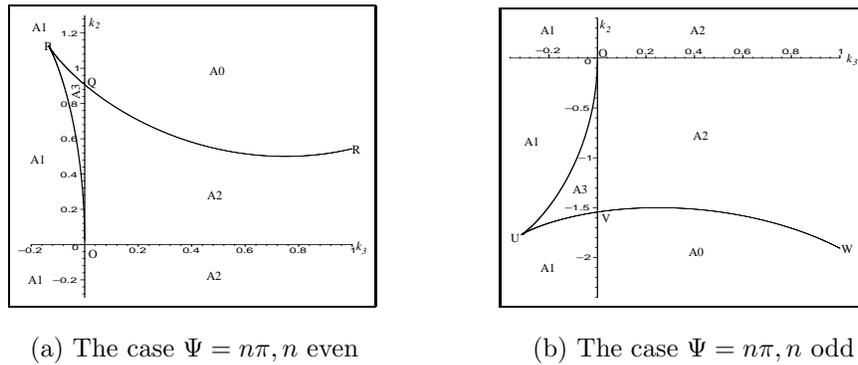


Figure 5.2: Bifurcation diagrams depicting the change in the number of critical points of (5.5.3) with $\Psi = n\pi, n \in \mathbb{Z}$.

if (k_3, k_2) is in region \mathbf{A}_i where $i=0,1,2,3$ then there will be i critical point(s) in system (5.5.3) on the line $\Psi = n\pi$ with n fixed. In Figure 2 (a), the parameter values (k_3, k_2) located on the curves OP, PQ, QO will give 2 critical points, the parameter values located on the curves QR and the k_2 -axis below O will give 1 critical point, while the parameter values on the k_2 -axis above the point Q will give 0 critical points. In Figure 2 (b), the parameter values on the curves OU, UV, and VO will give 2 critical points, the parameter values on curve VW and the k_2 -axis above the point O will give 1 critical point, and finally the parameter values on the k_2 -axis below the point V will give 0 critical points. In Figure 3 some phase planes of system (5.5.3) have been drawn for several values of k_2 and k_3 . In fact system (5.5.3) has a first integral which can be derived as follows. From (5.5.3) it follows that:

$$\begin{aligned} \frac{d\Psi}{dR_1} &= \frac{4\omega_1^2 R_1^4 - (3 + 2k_3)R_1^2 + k_3}{R_1(k_3 - R_1^2)(1 - R_1^2)} \cos(\Psi) + \frac{(k_1 R_1^2 + k_2)R_1 \sqrt{1 - R_1^2}}{R_1(k_3 - R_1^2)(1 - R_1^2) \sin(\Psi)}, \\ \Leftrightarrow \sin(\Psi) \frac{d\Psi}{dR_1} &= \frac{4R_1^4 - (3 + 2k_3)R_1^2 + k_3}{R_1(k_3 - R_1^2)(1 - R_1^2)} \cos(\Psi) + \frac{(k_1 R_1^2 + k_2) \sqrt{1 - R_1^2}}{(k_3 - R_1^2)(1 - R_1^2)} \\ \Leftrightarrow \frac{-dy(R_1)}{dR_1} &= F(R_1)y(R_1) + G(R_1), \end{aligned} \tag{5.5.8}$$

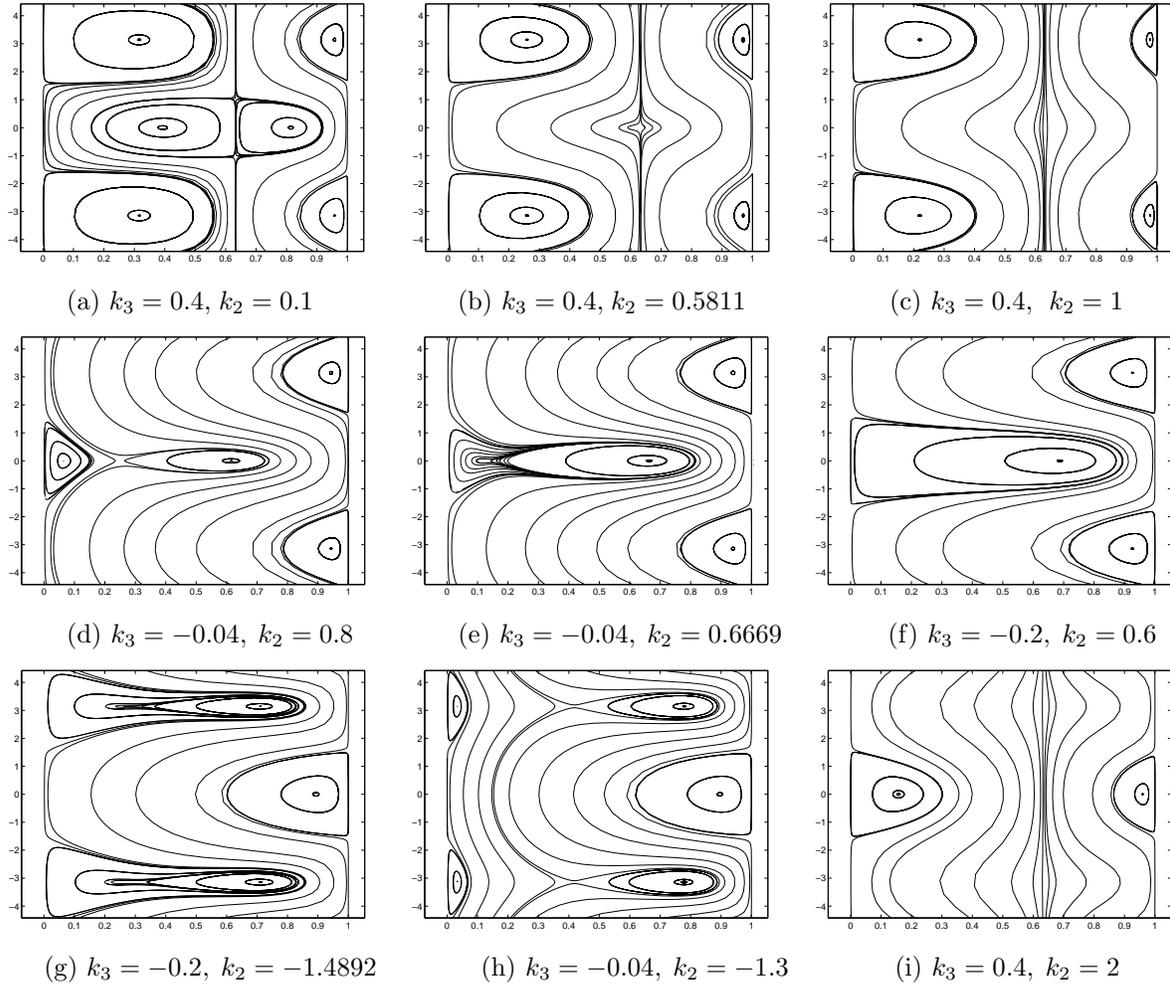


Figure 5.3: Some phase planes of system (5.5.3). The horizontal axis is R_1 while the vertical axis is Ψ .

where $y(R_1) = \cos(\Psi(R_1))$, $F(R_1) = \frac{4R_1^4 - (3+2k_3)R_1^2 + k_3}{R_1(k_3 - R_1^2)(1 - R_1^2)}$ and $G(r_1) = \frac{(k_1 R_1^2 + k_2)\sqrt{1 - R_1^2}}{(k_3 - R_1^2)(1 - R_1^2)}$.

This linear differential equation can readily be solved yielding:

$$\begin{aligned} \cos(\Psi) = & \left[R_1(k_3 - R_1^2)\sqrt{1 - R_1^2} \right] \left[-\frac{1}{2} \frac{k_1 + k_2}{(k_3 - 1)^2} \ln(1 - R_1^2) + \frac{1}{2} \frac{k_2 + k_1 k_3}{k_3(k_3 - 1)(R_1^2 - k_3)} \right. \\ & \left. + \frac{k_2}{k_3^2} \ln R_1 + \frac{1}{2} \frac{k_3 k_1 - k_2 - 2k_2 k_3}{k_3(k_3 - 1)^2} \ln |R_1^2 - k_3| + C^* \right], \end{aligned} \quad (5.5.9)$$

where C^* is a constant that can be determined from the initial condition.

5.6 The case $(P_0\pi)^2 = \mu^2 = \frac{15}{154}$ and $\Omega = \omega_2 - \omega_1 + \epsilon\phi$

The special values of $\Omega = \omega_2 - \omega_1$ will cause mode 2 and mode 1 to interact in addition to modes 1, 3, and 5. In this case, the secular terms in the approximation can be

removed if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy:

$$\begin{aligned}
 \dot{A}_{10} &= - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[B_{20} \cos(\phi t_1) \right. \\
 &\quad \left. - A_{20} \sin(\phi t_1) \right] - \frac{\tilde{K}_1}{\omega_1} \left[B_{50}(B_{30}^2 - A_{30}^2) + 2A_{30}B_{30}A_{50} \right] - M_1 B_{10} r_1^2 \\
 &\quad - \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{10} \sum_{n=1}^{\infty} L(1, n) r_n^2 - \frac{12(P_1^2 - 1)\pi^4}{128\omega_1} B_{10} \sum_{n=1}^{\infty} n^2 r_n^2, \\
 \dot{B}_{10} &= \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1} + \frac{(P_1^2 - 1)\pi^2}{2\omega_1} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[A_{20} \cos(\phi t_1) \right. \\
 &\quad \left. + B_{20} \sin(\phi t_1) \right] + \frac{\tilde{K}_1}{\omega_1} \left[A_{50}(B_{30}^2 - A_{30}^2) - 2A_{30}B_{30}B_{50} \right] - M_1 A_{10} r_1^2 \\
 &\quad + \frac{\pi^6}{2\omega_1} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{10} \sum_{n=1}^{\infty} L(1, n) r_n^2 + \frac{12(P_1^2 - 1)\pi^4}{128\omega_1} A_{10} \sum_{n=1}^{\infty} n^2 r_n^2, \\
 \dot{A}_{20} &= - \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2} + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[B_{10} \cos(\phi t_1) \right. \\
 &\quad \left. + A_{10} \sin(\phi t_1) \right] - M_2 B_{20} r_2^2 - \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 B_{20} \sum_{n=1}^{\infty} L(2, n) r_n^2 \\
 &\quad - \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} B_{20} \sum_{n=1}^{\infty} n^2 r_n^2, \\
 \dot{B}_{20} &= \left[\frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2} + \frac{(P_1^2 - 1)\pi^2}{2\omega_2} \left(\frac{2\alpha\Omega}{P_1^2\pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2\pi^2 - \Omega^2} \right) \right] \left[A_{10} \cos(\phi t_1) \right. \\
 &\quad \left. - B_{10} \sin(\phi t_1) \right] + M_2 A_{20} r_2^2 + \frac{\pi^6}{\omega_2} \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 A_{20} \sum_{n=1}^{\infty} L(2, n) r_n^2 \\
 &\quad + \frac{3(P_1^2 - 1)\pi^4}{8\omega_2} A_{20} \sum_{n=1}^{\infty} n^2 r_n^2, \tag{5.6.1}
 \end{aligned}$$

and (5.4.1) for $k \neq 1, 2, 3, 5$. The equations for \dot{A}_{k0} and \dot{B}_{k0} for $k = 3, 5$ can be taken from (5.5.1) with $\sigma = 0$.

By using polar coordinates, that is, $A_{k0}(t_1) = r_k(t_1) \sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1) \cos(\phi_k(t_1))$, two first integrals relating r_1, r_2, r_3 , and r_5 can be obtained, that is, $\omega_1 \rho_1^2 + \omega_2 \rho_2^2 - \omega_3 \rho_3^2 = K_1$ and $\omega_1 \rho_1^2 + \omega_2 \rho_2^2 + \omega_5 \rho_5^2 = K_2$ where K_1, K_2 are the constants of integration, and $r_1^2 = \tilde{K}_1 \rho_1^2, r_2^2 = \tilde{K}_1 \rho_2^2, r_3^2 = \tilde{K}_3 \rho_3^2$, and $r_5^2 = \tilde{K}_5 \rho_5^2$. From these first integrals it follows that r_1, r_2, r_3 , and r_5 will be bounded. The so-obtained first integrals can be used to reduce system (5.6.1) to a system of four first order ODE's, yielding:

$$\begin{aligned}
 \dot{\rho}_1 &= \frac{N_2 \rho_2}{\omega_1} \sin(\Phi) - \frac{\tilde{K} \sin(\Psi)}{\omega_1 \omega_3 \sqrt{\omega_5}} (\omega_1 \rho_1^2 + \omega_2 \rho_2^2 - K_1) \sqrt{K_2 - \omega_1 \rho_1^2 - \omega_2 \rho_2^2}, \\
 \dot{\rho}_2 &= -\frac{N_2}{\omega_2} \rho_1 \sin(\Phi),
 \end{aligned}$$

$$\begin{aligned}
\dot{\Phi} &= \left[\frac{\rho_2}{\omega_1 \rho_1} - \frac{\rho_1}{\omega_2 \rho_2} \right] N_2 \cos(\Phi) + \frac{\tilde{K}(\omega_1 \rho_1^2 + \omega_2 \rho_2^2 - K_1)}{\omega_1 \omega_3 \sqrt{\omega_5 \rho_1}} \sqrt{K_2 - \omega_1 \rho_1^2 - \omega_2 \rho_2^2} \cos(\Psi) \\
&\quad + \left[\frac{G_2 \tilde{K}_1}{\omega_1} + \frac{G_4 \tilde{K}_3}{\omega_3} - \frac{G_5 \tilde{K}_5}{\omega_5} \right] \omega_1 \rho_1^2 + \left[\frac{G_3 \tilde{K}_1}{\omega_2} + \frac{G_4 \tilde{K}_3}{\omega_3} - \frac{G_5 \tilde{K}_5}{\omega_5} \right] \omega_2 \rho_2^2 \\
&\quad + \left[G_1 - \frac{G_4 \tilde{K}_3}{\omega_3} K_1 + \frac{G_5 \tilde{K}_5}{\omega_5} K_2 \right], \\
\dot{\Psi} &= -\frac{N_2 \rho_2}{\omega_1 \rho_1} \cos(\Phi) + \left[\omega_1 \rho_1^2 (\omega_1 \rho_1^2 + \omega_2 \rho_2^2 - K_1) - (\omega_1 \rho_1^2 + \omega_2 \rho_2^2 - K_1) (K_2 - \omega_1 \rho_1^2 \right. \\
&\quad \left. - \omega_2 \rho_2^2) - 2\omega_1 \rho_1^2 (K_2 - \omega_1 \rho_1^2 - \omega_2 \rho_2^2) \right] \frac{\tilde{K} \cos(\Psi)}{\omega_1 \omega_3 \sqrt{\omega_5 \rho_1} \sqrt{K_2 - \omega_1 \rho_1^2 - \omega_2 \rho_2^2}} \\
&\quad + \left[\frac{G_6 \tilde{K}_1}{\omega_1} + \frac{G_8 \tilde{K}_3}{\omega_3} - \frac{G_9 \tilde{K}_5}{\omega_5} \right] \omega_1 \rho_1^2 + \left[\frac{G_7 \tilde{K}_1}{\omega_2} + \frac{G_8 \tilde{K}_3}{\omega_3} - \frac{G_9 \tilde{K}_5}{\omega_5} \right] \omega_2 \rho_2^2 \\
&\quad + \left[G_{10} - \frac{G_8 \tilde{K}_3 K_1}{\omega_3} + \frac{G_9 \tilde{K}_5 K_2}{\omega_5} \right], \tag{5.6.2}
\end{aligned}$$

where: $\Phi = \phi_2 - \phi_1 + \phi t_1$, $\Psi = 2\phi_3 + \phi_1 - \phi_5$, $N_2 = \frac{2\alpha}{3}(\omega_1 + \omega_2) + \frac{(P_1^2 - 1)\pi^2}{2} \left(\frac{2\alpha\Omega}{P_1^2 \pi^2 - \Omega^2} + \frac{2\alpha\Omega}{9P_1^2 \pi^2 - \Omega^2} \right)$, $G_1 = \phi + \frac{3}{8}\pi^2(P_1^2 - 1) \left(\frac{1}{4\omega_1} - \frac{1}{\omega_2} \right) \sum_{n \neq 1, 2, 3, 5}^\infty n^2 r_n^2 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \sum_{n \neq 1, 2, 3, 5}^\infty \left[\frac{L(1, n)}{2\omega_1} - \frac{L(2, n)}{\omega_2} \right] r_n^2$, $G_2 = M_1 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{L(1, 1)}{2\omega_1} - \frac{L(2, 1)}{\omega_2} \right] + \frac{3(p_1^2 - 1)\pi^4}{8} \left[\frac{1}{4\omega_1} - \frac{1}{\omega_2} \right]$, $G_3 = -M_2 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{L(1, 2)}{2\omega_1} - \frac{L(2, 2)}{\omega_2} \right] + \frac{12(P_1^2 - 1)\pi^4}{2} \left[\frac{1}{4\omega_1} - \frac{1}{\omega_2} \right]$, $G_4 = \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{L(1, 3)}{2\omega_1} - \frac{L(2, 3)}{\omega_2} \right] + \frac{27(P_1^2 - 1)\pi^4}{8} \left[\frac{1}{4\omega_1} - \frac{1}{\omega_2} \right]$, $G_5 = \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{L(1, 5)}{2\omega_1} - \frac{L(2, 5)}{\omega_2} \right] + \frac{75(P_1^2 - 1)\pi^4}{8} \left[\frac{1}{4\omega_1} - \frac{1}{\omega_2} \right]$, $G_6 = (P_1^2 - 1)\pi^4 \left[\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right] - M_1 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 1)}{2\omega_5} - \frac{L(1, 1)}{2\omega_1} - \frac{3L(3, 1)}{\omega_3} \right]$, $G_7 = 4(P_1^2 - 1)\pi^4 \left[\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right] + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 2)}{2\omega_5} - \frac{L(1, 2)}{2\omega_1} - \frac{3L(3, 2)}{\omega_3} \right]$, $G_8 = 9(P_1^2 - 1)\pi^4 \left[\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right] - 2M_3 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 3)}{2\omega_5} - \frac{L(1, 3)}{2\omega_1} - \frac{3L(3, 3)}{\omega_3} \right]$, $G_9 = 25(P_1^2 - 1)\pi^4 \left[\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right] + M_5 + \pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left[\frac{5L(5, 5)}{2\omega_5} - \frac{L(1, 5)}{2\omega_1} - \frac{3L(3, 5)}{\omega_3} \right]$, and $G_{10} = \sum_{n \neq 1, 2, 3, 5}^\infty \left[\pi^6 \left(\frac{P_1^2 - 1}{2\sqrt{2}} \right)^2 \left(\frac{5L(5, n)}{2\omega_5} - \frac{L(1, n)}{2\omega_1} - \frac{3L(3, n)}{\omega_3} \right) + n^2 (P_1^2 - 1)\pi^4 \left(\frac{300}{128\omega_5} - \frac{12}{128\omega_1} - \frac{216}{128\omega_3} \right) \right] r_n^2(0)$.

It should be noticed that in the process of deriving (5.6.2) it has been assumed that $r_k \neq 0$ for $k = 1, 2, 3, 5$. For $k \neq 1, 2, 3, 5$ it will follow that $r_k = r_k(0)$ and $\Phi_k = -\frac{3(P_1^2 - 1)\pi^4 k^2}{128\omega_k} \left[3r_{k_0}^2 + 4 \sum_{n=1}^\infty n^2 r_{n_0}^2 \right] t_1$.

It seems that this system of four first order ODE's can not be reduced any further analytically. Because of that, in what follows only numerical results will be presented.

In order to reduce the number of parameters the following scalings are introduced successively, that is, $\rho_1 = \sqrt{\frac{K_2}{\omega_1}} R_1$, $\rho_2 = \sqrt{\frac{K_2}{\omega_2}} R_2$, $s_1 = \frac{N_2 t_1}{\sqrt{\omega_1 \omega_2}}$, $s_2 = \frac{K_2 \tilde{K}}{N_2 \omega_3} \sqrt{\frac{\omega_2}{\omega_5}} s_1$ and $\frac{ds_3}{ds_2} = \frac{1}{R_1 R_2 \sqrt{1 - R_1^2 - R_2^2}}$. With these scalings it then follows from (5.6.2) that:

$$\frac{dR_1}{ds_3} = \eta R_1 R_2^2 \sqrt{1 - R_1^2 - R_2^2} \sin(\Phi) - (R_1^2 + R_2^2 - K)(1 - R_1^2 - R_2^2) \sin(\Psi),$$

$$\begin{aligned}
 \frac{dR_2}{ds_3} &= -\eta R_1^2 R_2 \sqrt{1 - R_1^2 - R_2^2} \sin(\Phi), \\
 \frac{d\Phi}{ds_3} &= \eta(R_2^2 - R_1^2) \sqrt{1 - R_1^2 - R_2^2} \cos(\Phi) + R_2(1 - R_1^2 - R_2^2)(R_1^2 + R_2^2 - K) \cos(\Psi) \\
 &\quad + (k_1 R_1^2 + k_2 R_2^2 + \delta_1) R_1 R_2 \sqrt{1 - R_1^2 - R_2^2}, \\
 \frac{d\Psi}{ds_3} &= -\eta R_2^2 \sqrt{1 - R_1^2 - R_2^2} \cos(\Phi) + [R_1^2(R_1^2 + R_2^2 - K) - 2R_1^2(1 - R_1^2 - R_2^2) \\
 &\quad - (R_1^2 + R_2^2 - K)(1 - R_1^2 - R_2^2)] R_2 \cos(\Psi) + (k_3 R_1^2 + k_4 R_2^2 + \delta_2) R_1 R_2 \sqrt{1 - R_1^2 - R_2^2},
 \end{aligned} \tag{5.6.3}$$

where $\eta = \frac{N_2 \omega_3}{K_2 K} \sqrt{\frac{\omega_5}{\omega_2}}$, $K = \frac{K_1}{K_2} < 1$, $k_1 = [\frac{G_2}{\omega_1} + \frac{G_4}{\omega_3} - \frac{G_5}{\omega_5}] \frac{\omega_3 \sqrt{\omega_1 \omega_5}}{K}$, $k_2 = [\frac{G_3}{\omega_2} + \frac{G_4}{\omega_3} - \frac{G_5}{\omega_5}] \frac{\omega_3 \sqrt{\omega_1 \omega_5}}{K}$, $k_3 = [\frac{G_6}{\omega_1} + \frac{G_8}{\omega_3} - \frac{G_9}{\omega_5}] \frac{\omega_3 \sqrt{\omega_1 \omega_5}}{K}$, $k_4 = [\frac{G_7}{\omega_2} + \frac{G_8}{\omega_3} - \frac{G_9}{\omega_5}] \frac{\omega_3 \sqrt{\omega_1 \omega_5}}{K}$, $\delta_1 = [\frac{G_1}{K_2} - \frac{G_4 K}{\omega_3} + \frac{G_5}{\omega_5}] \frac{K_2 \omega_1 \omega_3}{N_2 K} \sqrt{\omega_2 \omega_5}$, and $\delta_2 = [\frac{G_{10}}{K_2} + \frac{G_8 K}{\omega_3} + \frac{G_9}{\omega_5}] \frac{K_2 \omega_1 \omega_3}{N_2 K} \sqrt{\omega_2 \omega_5}$.

It should be noticed that if P_1^2 is fixed then all the k_1, k_2, k_3 and k_4 will be fixed, so the only parameters will be N_2, K, δ_1 and δ_2 . To give an illustration of the dynamical behaviour of (5.6.3) some projected trajectories in the (R_1, R_2) -plane for the case $P_1^2 = 100, \delta_1 = 1$, and $\delta_2 = 0$ have been given in Figure 4.

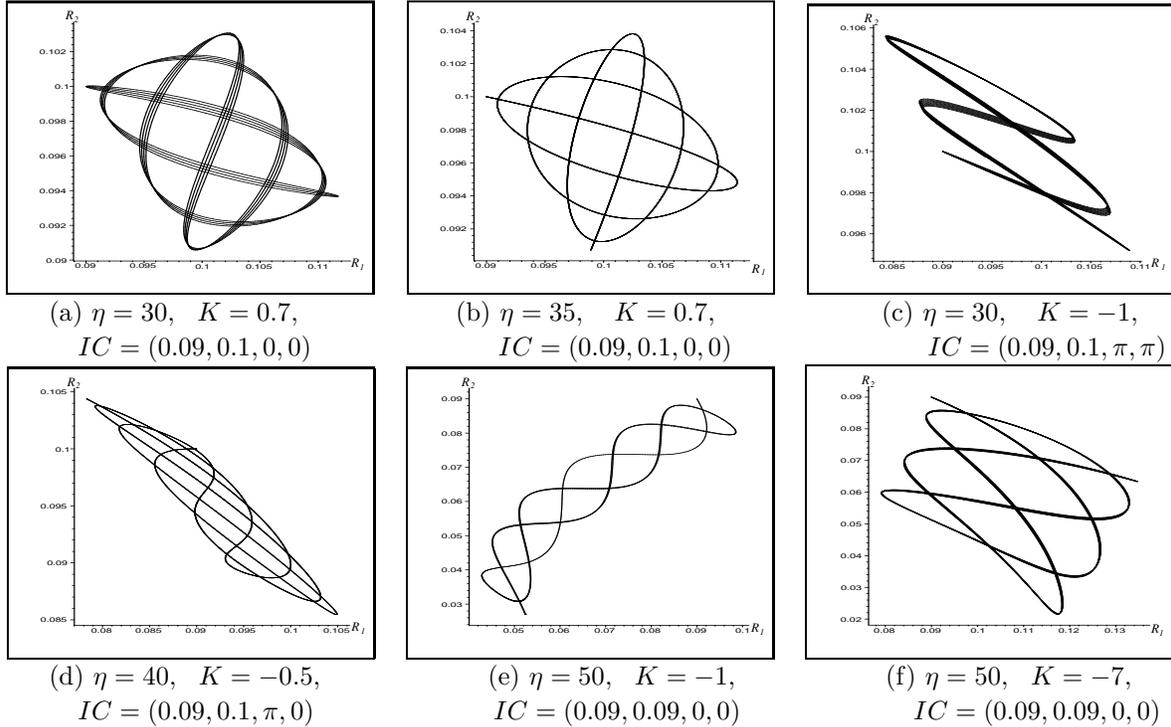


Figure 5.4: Some projected trajectories of system (5.6.3) for several values of η and K , and for several initial conditions $IC = (R_1(0), R_2(0), \Phi(0), \Psi(0))$.

5.7 Conclusions and remarks

In this paper a weakly nonlinear model describing the transversal and the longitudinal vibrations of a conveyor belt with a low and time-varying velocity has been studied. The model consists of a system of partial differential equations which can be derived by using Hamilton's principle.

In the analysis it has been assumed that the belt velocity is of order $\mathcal{O}(\epsilon)$, that is, $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, $|\alpha| < V_0$, $0 < \epsilon \ll 1$. The value of ϵ can be considered as a measure of the smallness of the belt velocity compared to the wave speed. Further, it has also been assumed that the longitudinal displacements are $\mathcal{O}(\epsilon)$, the transversal displacements are $\mathcal{O}(\sqrt{\epsilon})$, and P_0^2 and P_1^2 are $\mathcal{O}(1)$. Due to the assumption that P_1^2 is of order $\mathcal{O}(1)$ the Kirchhoff's approach can not be used.

It is found that if $\mu^2 = P_0^2 \pi^2 > \frac{1}{2}$ then the dynamical behaviour of the transversal vibrations of the belt will resemble those found in the case where Kirchhoff's approach has been applied. Whereas for $\mu^2 < \frac{1}{2}$ a more complicated behaviour will occur due to the additional mode interactions caused by the nonlinear part of the problem.

The case $\mu^2 = \frac{15}{154}$ has been studied in detail. For this special value of μ^2 it is found that the modes 1, 3, and 5 are interacting, while the other modes remain constant. When Ω causes no resonances, the detuned case $\mu^2 = \frac{15}{154} + \epsilon \zeta$ with $\zeta = \mathcal{O}(1)$ has been considered. Although the solutions in this case are always bounded, their behaviour may vary depending on ζ and the initial conditions. Also the resonant case $\Omega = \omega_2 - \omega_1$ and $\mu^2 = \frac{15}{154}$ has been considered. In this case the modes 1, 2, 3 and 5 are interacting while the other modes remain constant. The boundedness of the solutions for the modes 1, 2, 3 and 5 proved analytically. It is expected that for other values of μ^2 and Ω a similar analysis can be performed. In this work, focus has been mainly put to the transversal vibrations of the conveyor belt. This is not only due to the significant importance of the transversal vibrations compared to the longitudinal vibrations, but also due to the analytical complications in the study of the longitudinal vibrations.

Appendix

It has been stated in section 5.3 that the nonlinear part of the PDE's gives rise to resonances if the following systems of equations have solutions.

$$(I) : \begin{cases} k &= m - n - l, \\ \omega_k &= \pm \omega_m \pm \omega_n \pm \omega_l \end{cases} \quad (\text{A-1})$$

$$(II) : \begin{cases} k &= m + n - l, \\ \omega_k &= \pm \omega_m \pm \omega_n \pm \omega_l \end{cases} \quad (\text{A-2})$$

$$(III) : \begin{cases} k &= m + n + l, \\ \omega_k &= \pm \omega_m \pm \omega_n \pm \omega_l. \end{cases} \quad (\text{A-3})$$

It will be shown in this appendix that system *I* and system *III* have no solutions at all, while in system *II* the only cases that have to be considered are $k = m + n - l$, with $\omega_k = \omega_m + \omega_n - \omega_l$, $\omega_k = \omega_m + \omega_n + \omega_l$, $\omega_k = \omega_m - \omega_n + \omega_l$, and $\omega_k = -\omega_m + \omega_n + \omega_l$. The other equations either can be rewritten in these forms or do not have any solutions at all. It should be noticed that system *I* and system *III* are in fact equivalent, therefore only system *I* will be considered, and it will be shown that system *I* has no solutions.

System (A-1)

A. The case $\omega_k = \omega_m + \omega_n + \omega_l$

In this case $k = m - n - l$ implies that $m > k$. As a consequence $\omega_k = \omega_m + \omega_n + \omega_l$ can not be satisfied.

B. The case $\omega_k = \omega_m + \omega_n - \omega_l$

Since $k = m - n - l$ and $\omega_k = \omega_m + \omega_n - \omega_l$ then $m = k + n + l$ and

$$\begin{aligned} \omega_k + \omega_l &= \omega_m + \omega_n \Leftrightarrow \omega_{k+n+l} + \omega_n = \omega_k + \omega_l \\ \Leftrightarrow (k+n+l)\sqrt{1+(k+n+l)^2\mu^2} + n\sqrt{1+n^2\mu^2} &= k\sqrt{1+k^2\mu^2} + l\sqrt{1+l^2\mu^2}, \end{aligned}$$

where $\mu^2 = P_0^2\pi^2$. The last equation can not be satisfied since the left hand side is obviously larger than the right hand side.

C. The case $\omega_k = \omega_m - \omega_n + \omega_l$

Rewriting $\omega_k = \omega_m - \omega_n + \omega_l$ as $\omega_k + \omega_n = \omega_m + \omega_l$ and then by using the proof as given in part B this case similarly has no solution.

D. The case $\omega_k = \omega_m - \omega_n - \omega_l$

In this case $k = m - n - l$, and $\omega_k = \omega_m - \omega_n - \omega_l$ implies that $m = k + n + l$, and $\omega_m = \omega_k + \omega_n + \omega_l$. Obviously $\omega_{k+n+l} = \omega_k + \omega_n + \omega_l$ can not be satisfied.

E. The case $\omega_k = -\omega_m + \omega_n + \omega_l$

In this case $\omega_m = \omega_n + \omega_l - \omega_k$ with $m = k + n + l$. It is again obvious that $\omega_{k+n+l} = \omega_n + \omega_l - \omega_k$ can not be satisfied.

F. The case $\omega_k = -\omega_m + \omega_n - \omega_l$

This case is equivalent to $m = k + n + l$, and $\omega_m = \omega_n - \omega_l - \omega_k$. Since $m > n$ it is clear that there are no solutions.

G. The case $\omega_k = -\omega_m - \omega_n + \omega_l$

This case is equivalent to $\omega_{k+n+l} = \omega_l - \omega_n - \omega_k$, which is obviously can not be satisfied.

H. The case $\omega_k = -\omega_m - \omega_n - \omega_l$

This case certainly can not be satisfied since the left hand side is positive while the right hand side is negative. By using the same procedure as presented above, it can

also be shown that the case $-\omega_k = \pm\omega_m \pm \omega_n \pm \omega_l$ does not have any solutions. So, system I does not have any solutions.

Now, by rewriting system III as $m = k - n - l$ and $\pm\omega_m = \pm\omega_k \pm \omega_n \pm \omega_l$ then it can be seen that this system is equivalent to system I by interchanging the role of m and k . Consequently this system also does not have any solutions.

System (A-2)

As mentioned at the beginning of this appendix, in what follows only the cases $k = m + n - l$ with $\omega_k = \omega_m + \omega_n - \omega_l, \omega_k = \omega_m + \omega_n + \omega_l, \omega_k = \omega_m - \omega_n + \omega_l$, and $\omega_k = -\omega_m + \omega_n + \omega_l$ will be considered in this section. By using the method applied previously, other cases can be brought to these systems of equations or they do not have any solutions at all.

The case $\omega_k = \omega_m + \omega_n - \omega_l$

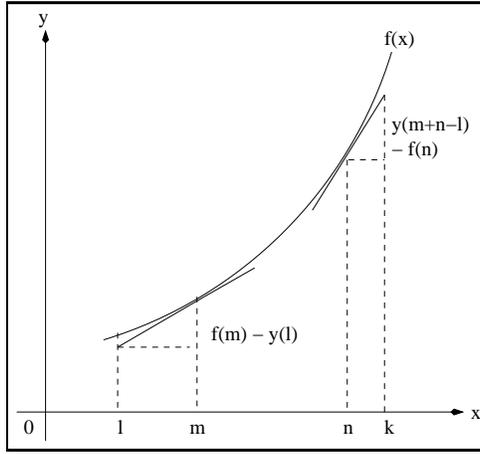
In this appendix it will be shown that the only solutions for the the Diophantine- like equation $k = m + n - l$ and $\omega_k = \omega_m + \omega_n - \omega_l$ for which $\mu^2 = P_0^2 \pi^2 > 0$ are the trivial solutions $k = m, n = l, k = n, m = l$ and $k = l = m = n$. From $\omega_k = \omega_m + \omega_n - \omega_l$ it follows that:

$$\begin{aligned} \sqrt{k^2 + k^4 \mu^2} &= \sqrt{m^2 + m^4 \mu^2} + \sqrt{n^2 + n^4 \mu^2} - \sqrt{l^2 + l^4 \mu^2}, \\ \Leftrightarrow k^2 \sqrt{1 + \lambda^2/k^2} &= m^2 \sqrt{1 + \lambda^2/m^2} + n^2 \sqrt{1 + \lambda^2/n^2} - l^2 \sqrt{1 + \lambda^2/l^2}, \end{aligned}$$

where $\lambda = 1/\mu$ and from $k = m + n - l$ it follows that $k + l = m + n$. For non-trivial solutions, without loss of generality, only the case where $k > n \geq m > l$ has to be considered. The other cases can be treated similarly. Now, let $f(x) = x^2 \sqrt{1 + \lambda^2/x^2}$ then

$$f'(x) = \frac{2x^2 + \lambda^2}{x \sqrt{1 + \frac{\lambda^2}{x^2}}}, \quad \text{and} \quad f''(x) = \frac{2x^2 + 3\lambda^2}{(x^2 + \lambda^2) \sqrt{1 + \frac{\lambda^2}{x^2}}}. \quad (\text{A-4})$$

Since the first and the second derivative of $f(x)$ are positive for $x > 0$ it follows that f is a convex function. In Figure 5.5 a typical example of a convex function f is given.

Figure 5.5: A typical graph of a convex function f .

Now, since $k = m + n - l$ and $\omega_k = \omega_m + \omega_n - \omega_l$ it follows that

$$\begin{aligned} \omega_k - \omega_n &= \omega_m - \omega_l \Leftrightarrow \omega_{m+n-l} - \omega_n = \omega_m - \omega_l \\ \Leftrightarrow f(m+n-l) - f(n) &= f(m) - f(l). \end{aligned} \quad (\text{A-5})$$

By using the tangent line at $x = n$, that is, $y(x) = f(n) + (x - n)f'(n)$ it follows that

$$\begin{aligned} y(m+n-l) &= f(n) + (m-l)f'(n) \Leftrightarrow y(m+n-l) - f(n) = (m-l)f'(n) \\ \Rightarrow f(m+n-l) - f(n) &> (m-l)f'(n). \end{aligned} \quad (\text{A-6})$$

On the other hand by using the tangent line at $x = m$, i.e., $y(x) = f(m) + (x - m)f'(m)$ it follows that:

$$\begin{aligned} y(l) &= f(m) + (l-m)f'(m) \Leftrightarrow y(l) - f(m) = -(m-l)f'(m) \Leftrightarrow \\ f(m) - y(l) &= (m-l)f'(m) \Rightarrow f(m) - f(l) < (m-l)f'(m). \end{aligned} \quad (\text{A-7})$$

Using (A-6) in (A-5) gives $f(m) - f(l) > (m-l)f'(n)$. Now, f is a convex function, that is, $f'' > 0$, therefore $f'(n) > f'(m)$ for $n > m$. Consequently, $f(m) - f(l) \gg (m-l)f'(m)$. But then this contradicts (A-7). Therefore, the system of equations $k = m + n - l$ and $\omega_k = \omega_m + \omega_n - \omega_l$, for all $\mu^2 > 0$ has no solutions other than the trivial ones, that is, $k = n, m = l$ or symmetrically $k = m, n = l$.

The case $\omega_k = \omega_m + \omega_n + \omega_l$

In this case $n = l$ and $k = m$, or $m = l$ and $k = n$ can not be solutions of

$$\begin{cases} k = m + n - l, \\ \omega_k = \omega_m + \omega_n + \omega_l, \end{cases} \quad (\text{A-8})$$

since the second equation will not be satisfied. Let's put $\nu = mn - ml - nl$. Now, if $\nu < 0$ then from $k^2 = m^2 + n^2 + l^2 + 2\nu$ it follows that $k^2 < m^2 + n^2 + l^2$. For $k = m + n - l$ the following cases have to be distinguished:

- If $k = \max\{k, l, m, n\}$ it then follows that

$$\begin{aligned}\omega_k = \omega_m + \omega_n + \omega_l &\Leftrightarrow k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} = m^2 \sqrt{1 + \frac{\lambda^2}{m^2}} + n^2 \sqrt{1 + \frac{\lambda^2}{n^2}} + l^2 \sqrt{1 + \frac{\lambda^2}{l^2}}, \\ \Rightarrow k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} &> m^2 \sqrt{1 + \frac{\lambda^2}{k^2}} + n^2 \sqrt{1 + \frac{\lambda^2}{k^2}} + l^2 \sqrt{1 + \frac{\lambda^2}{k^2}} \Leftrightarrow k^2 > m^2 + n^2 + l^2,\end{aligned}$$

which contradicts $k^2 < m^2 + n^2 + l^2$.

- If $k \neq \max\{k, l, m, n\}$ it then follows that $\omega_k = \omega_m + \omega_n + \omega_l \Leftrightarrow k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} = m^2 \sqrt{1 + \frac{\lambda^2}{m^2}} + n^2 \sqrt{1 + \frac{\lambda^2}{n^2}} + l^2 \sqrt{1 + \frac{\lambda^2}{l^2}}$ which obviously can not be satisfied.

So, in the case $\nu < 0$ then there are no solutions for (A-8).

In the case $\nu = 0$ (A-8) again has no solutions. This can be shown as follows. Since $mn - ml - nl = \nu = 0$ it then follows from $k = m + n - l$ that $k^2 = m^2 + n^2 + l^2$. Then, the second equation in (A-8) implies ($\lambda = \frac{1}{\mu}$):

$$\begin{aligned}\omega_k = \omega_m + \omega_n + \omega_l &\Leftrightarrow \sqrt{k^2 + \mu^2 k^4} = \sqrt{m^2 + \mu^2 m^4} + \sqrt{n^2 + \mu^2 n^4} + \sqrt{l^2 + \mu^2 l^4} \\ \Leftrightarrow k^2 \sqrt{\frac{1}{k^2} + \mu^2} &= m^2 \sqrt{\frac{1}{m^2} + \mu^2} + n^2 \sqrt{\frac{1}{n^2} + \mu^2} + l^2 \sqrt{\frac{1}{l^2} + \mu^2}, \\ \Leftrightarrow k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} &= m^2 \sqrt{1 + \frac{\lambda^2}{m^2}} + n^2 \sqrt{1 + \frac{\lambda^2}{n^2}} + l^2 \sqrt{1 + \frac{\lambda^2}{l^2}}, \\ \Rightarrow k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} &> m^2 \sqrt{1 + \frac{\lambda^2}{k^2}} + n^2 \sqrt{1 + \frac{\lambda^2}{k^2}} + l^2 \sqrt{1 + \frac{\lambda^2}{k^2}}, \\ \Rightarrow k^2 &> m^2 + n^2 + l^2,\end{aligned}\tag{A-9}$$

which contradicts $k^2 = m^2 + n^2 + l^2$. So, for $\nu = 0$ system (A-8) has no solutions. Now, using the estimate $1 < \sqrt{1 + \lambda^2/x^2} < 1 + \frac{1}{2}\lambda^2/x^2$, (where $\lambda = \frac{1}{\mu}$), it follows for $\nu > 0, k^2 > m^2 + n^2 + l^2$ that

$$\begin{aligned}k^2 < k^2 \sqrt{1 + \frac{\lambda^2}{k^2}} &< m^2 \left(1 + \frac{\lambda^2}{2m^2}\right) + n^2 \left(1 + \frac{\lambda^2}{2n^2}\right) + l^2 \left(1 + \frac{\lambda^2}{2l^2}\right) \\ \Rightarrow k^2 &< m^2 + n^2 + l^2 + \frac{3}{2}\lambda^2,\end{aligned}$$

and consequently

$$\begin{aligned}m^2 + n^2 + l^2 &< k^2 < m^2 + n^2 + l^2 + \frac{3}{2}\lambda^2, \\ \Leftrightarrow m^2 + n^2 + l^2 &< m^2 + n^2 + l^2 + 2\nu < m^2 + n^2 + l^2 + \frac{3}{2}\lambda^2 \\ \Leftrightarrow 0 < \nu &< \frac{3}{4}\lambda^2.\end{aligned}\tag{A-10}$$

Since ν is a positive integer, it then follows from (A-10) that solution only exist for $\lambda^2 > \frac{4}{3}$. Therefore it can be concluded that for $\lambda^2 \leq \frac{4}{3}$ or equivalently $\mu^2 \geq \frac{3}{4}$ system (A-8) has no solutions. In what follows it will be shown numerically that for various values of $\nu > 0$ the solutions of (A-8) only exist for $\lambda^2 > 2$ (or equivalently $\mu^2 < \frac{1}{2}$), and are clustering in the neighbourhood of $\lambda^2 = 2\nu = 2, 4, 6, \dots$. The results of these numerical computations are depicted in Figure 6 which are obtained by applying the following algorithm:

- Given l and $\nu \in \mathbb{N}$, compute m from $mn - ml - nl = \nu$, that is, $m = l + \frac{l^2 + \nu}{n - l}$,
- The values of n are determined such that $n - l$ divides $l^2 + \nu$. The Maple command `divisors` is used for this purpose. If d is this divisor of $l^2 + \nu$ then $n = l + d$. The value of k is then calculated from $k = m + n - l$,
- Finally, λ^2 is determined by solving the equation $\omega_k = \omega_m + \omega_n + \omega_l$ for λ^2 .

The case $\omega_k = \omega_m - \omega_n + \omega_l$

In this case, the only solutions for the system $k = m + n - l$ and $\omega_k = \omega_m - \omega_n + \omega_l$ are $k = m, l = n$ and $k = l = m = n$. For other values of k, l, m, n by assuming $k > m \geq n > l$ then $\omega_k = \omega_m - \omega_n + \omega_l$ can not be satisfied.

The case $\omega_k = -\omega_m + \omega_n + \omega_l$

In this case the only solutions for the system $k = m + n - l, \omega_k = -\omega_m + \omega_n + \omega_l$ are $k = n, m = l$ and $k = l = m = n$. For other values of k, l, m, n by assuming $k > m \geq n > l$ again $\omega_k = -\omega_m + \omega_n + \omega_l$ can not be satisfied.

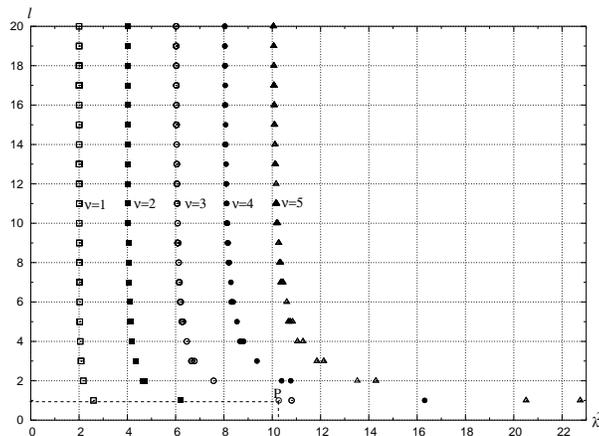


Figure 5.6: Relation between l and λ^2 for some values of ν .

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Summary

In this thesis a mathematical analysis has been given for model which describes the transversal vibrations of belt systems. The belt speed is assumed to be time-varying and to be small compared to the wave speed. Not only linear string-like or beam-like models but also nonlinear models have been studied. In all cases initial-boundary value problems are formulated, and are investigated by using multiple time-scales perturbation methods. Formal approximations of the solutions are constructed and it is shown whether or not mode interactions between vibration modes occur for specific values of the belt parameters. For some linear models instabilities in the solution occur, which disappear when nonlinear terms are included in the model. It is also shown for what parameter values in the nonlinear models a simplification in the formulation of the problem (based on Kirchhoff's assumption) can (or can not) be used.

Samenvatting

In dit proefschrift wordt een wiskundige analyse gegeven van modellen welke de transversale trillingen van transportbanden beschrijven. De transportband snelheid wordt verondersteld tijdsafhankelijk en klein ten opzichte van de golfsnelheid te zijn. Niet alleen lineaire golf-of balkachtige modellen maar ook niet-lineaire modellen worden bestudeerd. In alle gevallen worden begin-randwaarde problemen geformuleerd, en onderzocht door gebruik te maken van de meer-tijdschalen storings-methode. Formele benaderingen van de oplossingen worden geconstrueerd, en aangetoond wordt voor welke waarden van de transportband parameters interacties tussen verschillende trillingsvormen optreden (of niet). Voor enkele lineaire modellen verschijnen instabiliteiten in de oplossingen, welke verdwijnen als niet-lineaire termen in het model worden meegenomen. Ook wordt aangetoond voor welke parameter-waarden in de niet-lineaire modellen een vereenvoudiging in the formulering van het problem (gebaseerd op de veronderstelling van Kirchhoff) kan worden toegepast of niet.

Curriculum Vitae

Gede Suweken was born on November, 11, 1961 in Singaraja, Bali, Indonesia. After finishing his high school education in 1981 he continued studying Mathematics Education at the Faculty of Teacher Training and Education of the Udayana State University. He obtained his bachelor degree in 1985. In 1986 he was accepted as an academic staff member of the Mathematics Education Department of Udayana State University. In 1991 he had the opportunity to attend the so-called bridging courses in English and Mathematics at the Bandung Institute of Technology as a preparation to continue his study in Mathematics in Australia. He successfully finished this course and in 1992 he went to Adelaide, Australia to do a Graduate Diploma course in Mathematics and later obtained his Master degree in 1995. From 1998 until 2003 he was a Ph.D student at the Department of Applied Mathematical Analysis Delft University of Technology in The Netherlands.