75

UNIVERSITY OF CALIFORNIA
INSTITUTE OF ENGINEERING RESEARCH
BERKELEY, CALIFORNIA

ON THE DAMPING FORCE AND ADDED MASS OF SHIPS HEAVING AND PITCHING

by

Fukuzo Tasai
Institute of Applied Mechanics Research
Kyushu University

Published in the Journal of the
Society of Naval Architects
of Japan

[Journal of Zôsen Kiôkai, 105 (July, 1959), 47-56]

Translated by Wen-Chin Lin.
Edited by William R. Porter.

Translation Distributed
Under Contract Number N-onr-222(30)

SERIES NO. 82

ISSUE NO. 15

DATE July, 1960

ON THE DAMPING FORCE AND ADDED MASS
OF SHIPS HEAVING AND PITCHING

by

Fukuzo Tasai
Institute of Applied Mechanics Research
Kyushu University

Published in the Journal of the
Society of Naval Architects
of Japan

[Journal of Zôsen Kiôkai, 105 (July, 1959), 47-56]

Translated by Wen-Chin Lin.
Edited by William R. Porter.

Translation Distributed
Under Contract Number N-onr-222(30)

University of California
Institute of Engineering Research
Berkeley, California

July 1960

1. Introduction

Since it is usually very difficult to calculate the damping force and added mass for heaving and pitching of three-dimensional ships having a given forward speed, the strip method is generally used. For this it is necessary to have information about the two-dimensional values for infinitely-long cylinders having the ship's cross-sections.

For the damping force one can use the source method of Havelock [1]. This method gives us an approximation to a certain degree, but according to the experiments of Golovato [2], it seems on the whole to show quite a large deviation from the values obtained from experiment. For a circular cylinder, the exact values have been obtained by Ursell [3]. By using another method, O. Grim [4] also obtained some quite accurate values for various cross-sections; furthermore, he found a good approximate method for cross-sections having their boundaries perpendicular to the water surface.

For the added mass, there is an exact calculation for circular cylinders by Ursell [3]. O. Grim also made calculations for a few cross-sections, but the results seem to be doubtful. A coefficient K_4 which takes account of the free surface is needed for calculating the added mass of ships; Korvin-Kroukovsky [5] used 0.75 for heaving and 1.20 for pitching. In [6], K_4 of Ursell's circular cylinder is used for other cross-sections. The same method has also been used by Professor Nakamura [7].

In this paper the method used by Ursell [3] is extended to calculate exactly the progressive-wave height and the added mass for several kinds of infinitely-long cylinders with boundaries perpendicular to the water surface in forced vertical oscillation. With the aid of the results obtained by this calculation, the damping forces and the added masses of the ships are calculated by the strip method, and then are compared with the experimental results of Golovato [2] and Gerritsma [8]. For the damping force, a more reasonable method of calculation is still under study. The current results are presented in this paper.

2. The Calculation of the Progressive-Wave Height and Added Mass Caused by the Forced Heaving of Cylinders.

2.1 Boundary Conditions and Basic Conditions.

Consider an infinitely long cylinder, with cross-section in the z-plane as shown in Fig. 1, making sinusoidal heaving oscillations of small amplitude in the y-direction. There are two

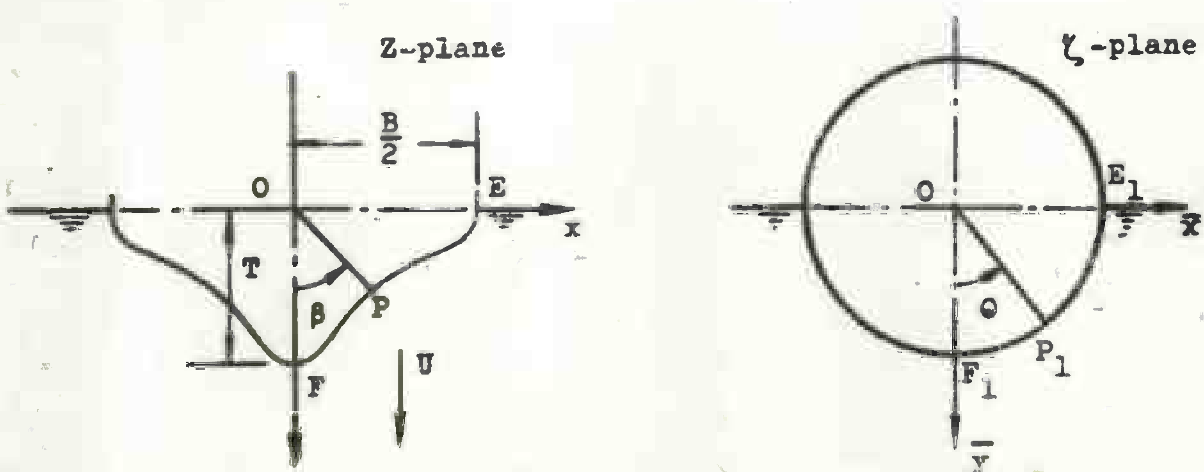

Fig. 1

kinds of waves caused by this heaving: one is a standing wave which decreases in amplitude rapidly with distance from the body, and the other is a regular progressive wave. If we neglect the viscosity and the surface tension of the water, the flow has a velocity potential φ and a stream function ψ and each of them satisfies Laplace's equation. The free-surface boundary condition will be:

$$K\phi+\partial\phi/\partial y=0 \quad (y=0,\ x>B/2)\ . \tag{1}$$

Here,

$$K=\omega^2/g\,;$$

and ω is the angular velocity of the circular motion corresponding to the heaving oscillation. The motion is symmetrical with respect to the y-axis. Next let us suppose that the axis of the cylinder, originally in the free surface (y=0), makes a small displacement $y_h = h\cos(\omega t+\varepsilon)$, and let

$$dy_h/dt=-h\omega\sin(\omega t+\varepsilon)\equiv U\ . \tag{2}$$

If h is small, the boundary condition of the cylinder at its average position (y=0) will be

$$\partial\phi/\partial\nu=U(\partial y/\partial\nu)\ . \tag{3}$$

Here ν is the outward normal to the boundary.

2.2 Mathematical Representation of the Shape of the Cross-Section.

Consider the figure formed by adding to the figure shown in Fig. 1 its reflection in the x-axis. The conformal mapping function which maps the region outside the figure in the z-plane onto the outside of the unit circle in the ζ-plane is

$$Z/M=\zeta+\sum_{n=1}^{\infty}a_{2n-1}\cdot\zeta^{-(2n-1)}\ . \tag{4}$$

If the sum terminates at n=2, then

$$Z/M=\zeta+a_1/\zeta+a_3/\zeta^3 \tag{5}$$

which represents the shape of the cross-sections used by Lewis [9] and Grim [4] for their calculations. In this paper, a calculation is made for Lewis cross-sections; however, for more general cases the calculation can be done similarly. For the Lewis cross-section, let

$$\zeta=ie^{\alpha}e^{-i\theta}$$

then

$$\left.\begin{aligned} x/M&=e^{\alpha}\sin\theta+a_1e^{-\alpha}\sin\theta-a_3e^{-3\alpha}\sin3\theta\\ y/M&=e^{\alpha}\cos\theta-a_1e^{-\alpha}\cos\theta+a_3e^{-3\alpha}\cos3\theta\end{aligned}\right\} \tag{6}$$

At the boundary of the cross-section, put $\alpha=0$, so that

$$\left.\begin{aligned} x_0/M&=(1+a_1)\sin\theta-a_3\sin3\theta\\ y_0/M&=(1-a_1)\cos\theta+a_3\cos3\theta\end{aligned}\right\} \tag{7}$$

Let

B = beam at the water surface.

T = draft.

M = scale factor of the mapping.

Then

$$M=\frac{B}{2}\Big/(1+a_1+a_3) \tag{8}$$

and

$$\frac{B}{2}\Big/T=H_0=\frac{1+a_1+a_3}{1-a_1+a_3} \tag{9}$$

$$S=\frac{\pi}{2}\cdot\left(\frac{B}{2}\right)^2\cdot\frac{1-a_1^2-3a_3^2}{(1+a_1+a_3)^2},$$

$$\sigma=\frac{S}{B\cdot T}=\frac{\pi}{4}\cdot H_0\,\frac{1-a_1^2-3a_3^2}{(1+a_1+a_3)^2} \tag{10}$$

where

S = area of the cross-section,

σ = cross-section area coefficient. $\frac{S}{BT}$

By suitable choice of the values of a_1 and a_3 in equations (9) and (10) one can approximate the shape of a ship cross-section by a Lewis Form.

2.3 The Calculation.

The method of calculation has been shown briefly in the appendix. If we calculate $\bar{\Lambda}$ and K_4 for elliptic cylinders of H_o = 1.5 and compare them with the results shown in Figure 2 of Grim [4], we can obtain the results shown in Figure 2 of this paper. For $\bar{\Lambda}$, the values obtained by Grim by his accurate method are very close to those of this paper. The upper dotted-line represents the approximate values obtained by Grim and given in Fig. 2 of [4]. Grim also derived an approximate equation for the circular cylinder, but he did not present a similar equation for other kinds of cross-sections. If we apply Grim's approximate method to Lewis cross-sections, we get the following:

$$\bar{\Lambda}=2\xi_0\int_1^\infty \frac{\left(\frac{1+a_1}{\beta^2}+\frac{3a_3}{\beta^4}\right)}{1+a_1+a_3}\cos\left[\xi_0\left\{\frac{\beta^4+a_1\beta^2+3a_3}{(1+a_1+a_3)\beta^3}-1\right\}\right]\cdot d\beta \quad (11)$$

For a circular cross-section, we put $a_1=a_3=0$ and then

$$\bar{\Lambda}=2\xi_0\int_1^\infty \frac{\cos\xi_0(\beta-1)}{\beta^2}\,d\beta$$

which is the same as the result shown by Grim in the appendix of [4]. The broken line in Figure 2 shows the result of calculations

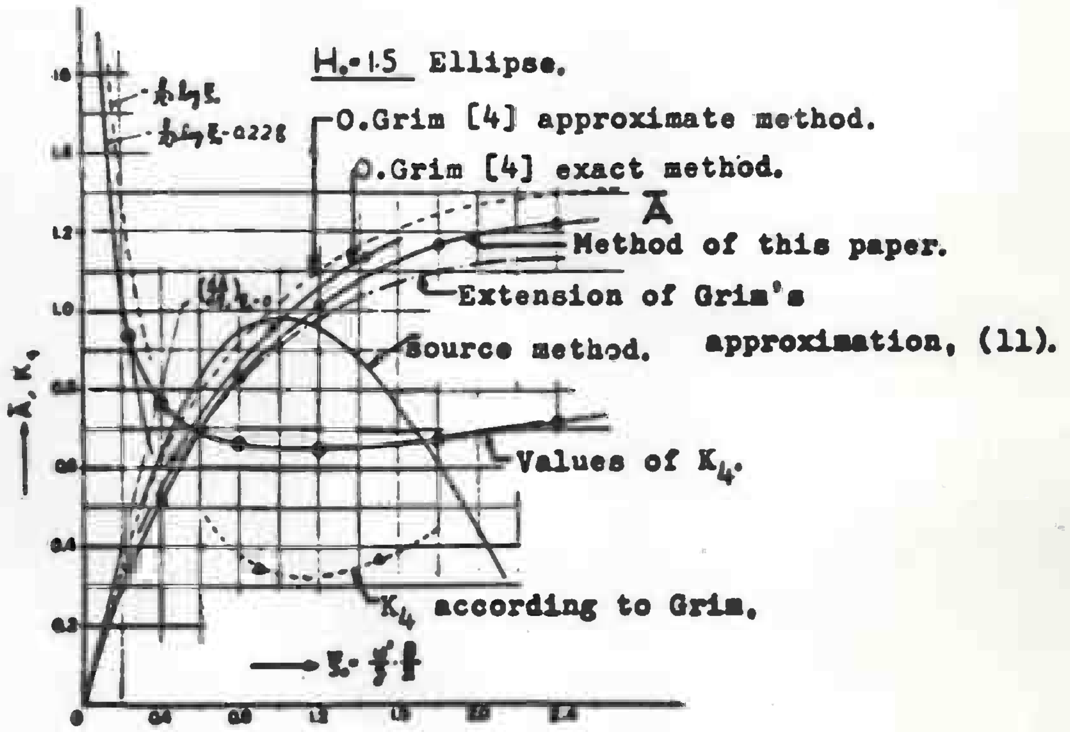
H.=1.5 Ellipse.
O.Grim [4] approximate method.
O.Grim [4] exact method.
Method of this paper.
Extension of Grim's approximation, (11).
Source method.
Values of K4.
K4 according to Grim.
Ā, K4

Fig. 2

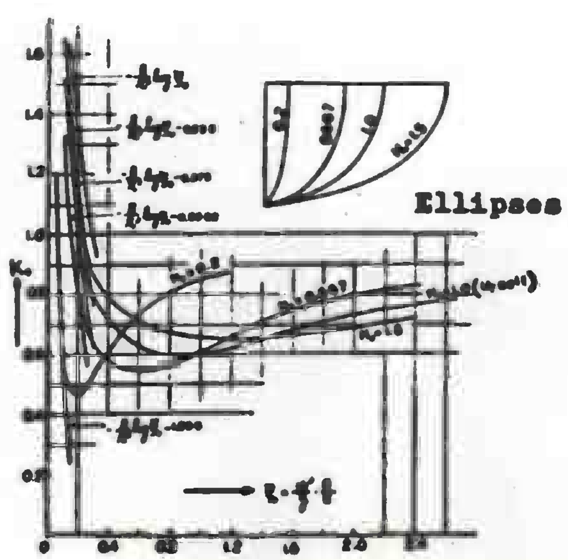
Ellipses

Fig. 3

by equation (11). These values deviate from the rigorous values of this paper up to 10% for $\xi_0 < 2.5$. The values of $\bar{A}$ obtained by the source method (of Havelock) are also shown in the figure, but the error is quite large. The calculated values of the free-surface coefficient K_4 for the added mass are shown by the curve with double circles. The values of Grim [4] are also shown in the figure and these values are quite small. Grim [4] stated that K_4 should approach $-\frac{8}{\pi^3}\log\xi_0$ in the neighborhood of $\xi_0 \rightarrow 0$. However, for ellipses, the value of K_4 as $\xi_0 \rightarrow 0$ was shown by Ursell [10] to be the following:

$$K_4 = -\frac{8}{\pi^3}\left[\log\xi_0 + \log\left(1+\frac{1}{H_0}\right) - 0.23\right] . \qquad (12)$$

From this equation, we get $K_4 = -\frac{8}{\pi^3}\log\xi_0 - 0.228$ for $H_0 = 1.5$. For $\xi_0 = 0.24$, the values of K_4 calculated in this paper are very close to the curve of the above equation.

The calculated values of A and K_4 of various Lewis cross-sections for $H_0 = 0.2, 0.667, 1.0, 1.25, 1.50$ are shown in Figures 4 to 12. Figure 3 shows how the values of K_4 of each kind of ellipse vary with different values of H_0. The approximate values from equation (11) are also shown in Figure 5. The error increases as ξ_0 becomes large and the cross-section becomes deeper. In Figures 8 through 12, for cases with $a_3 \neq 0$, the values of K_4 in the neighborhood of $\xi_0 \rightarrow 0$ were approximated by the values of K_4 for an elliptic cylinder of the same value of H_0.

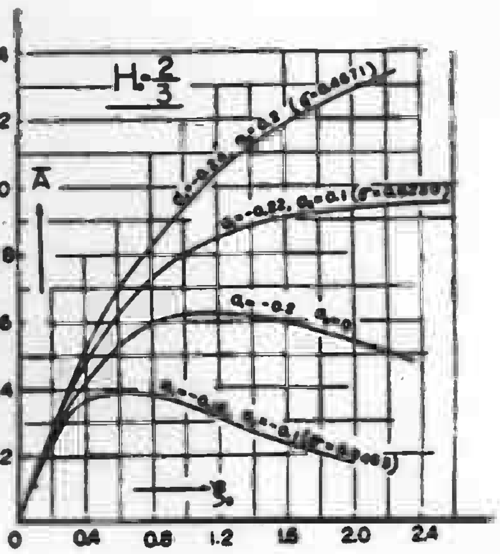

Fig. 4

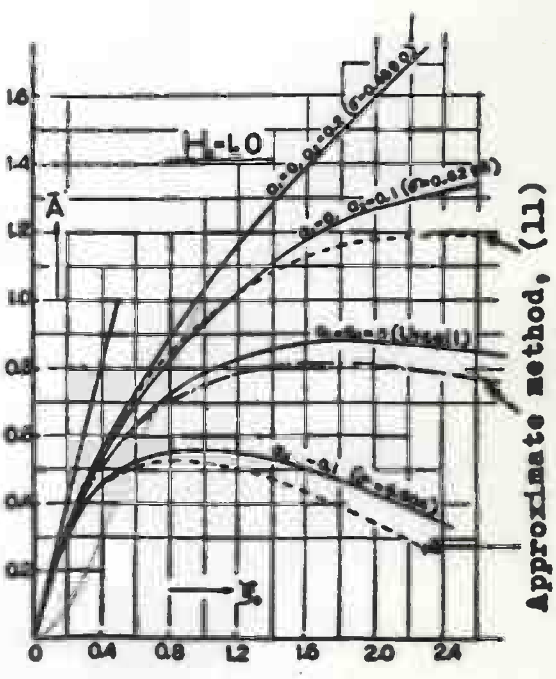

Fig. 5

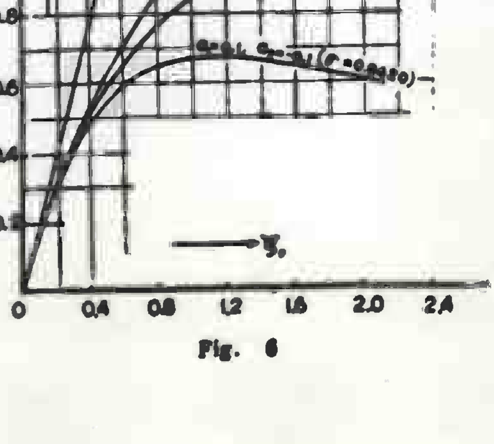

Fig. 6

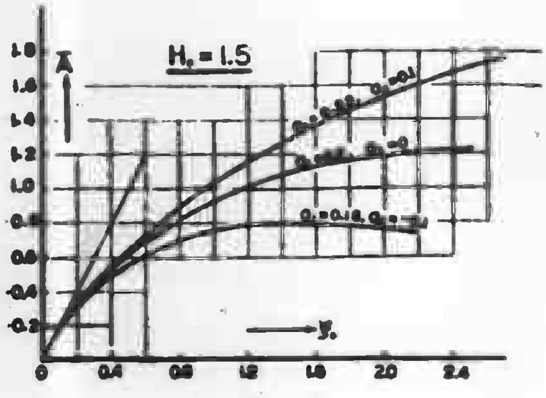

Fig. 7

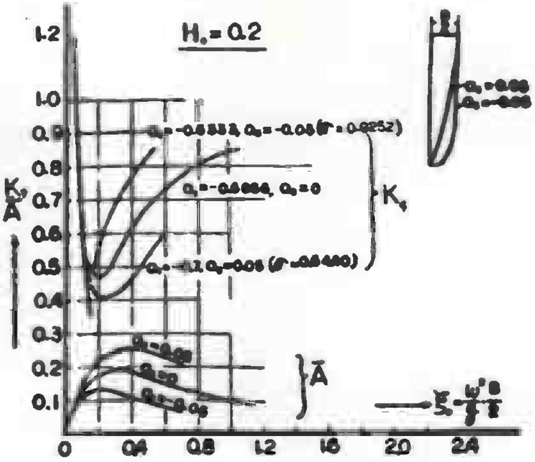

Fig. 8

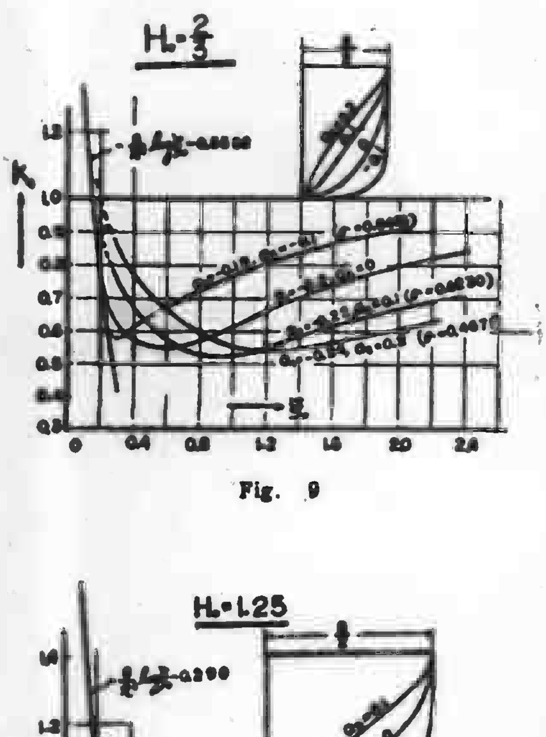

Fig. 9

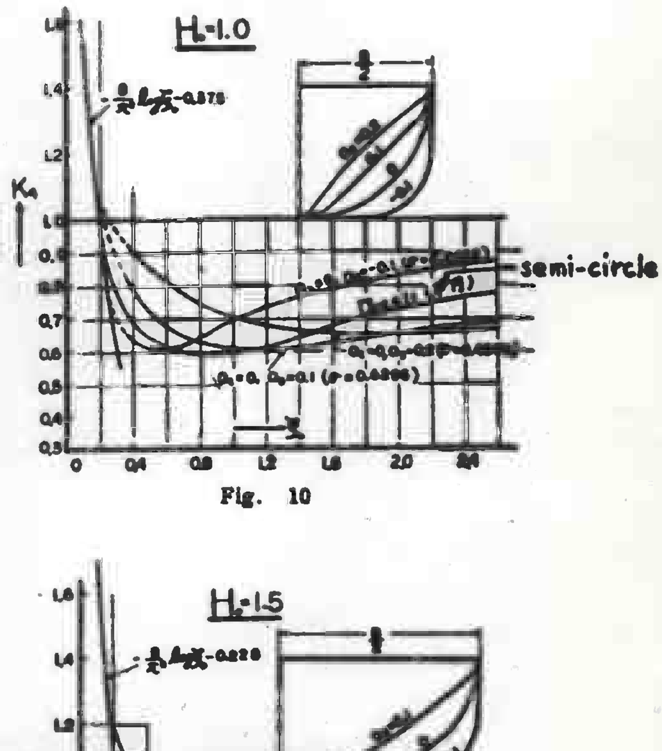

Fig. 10

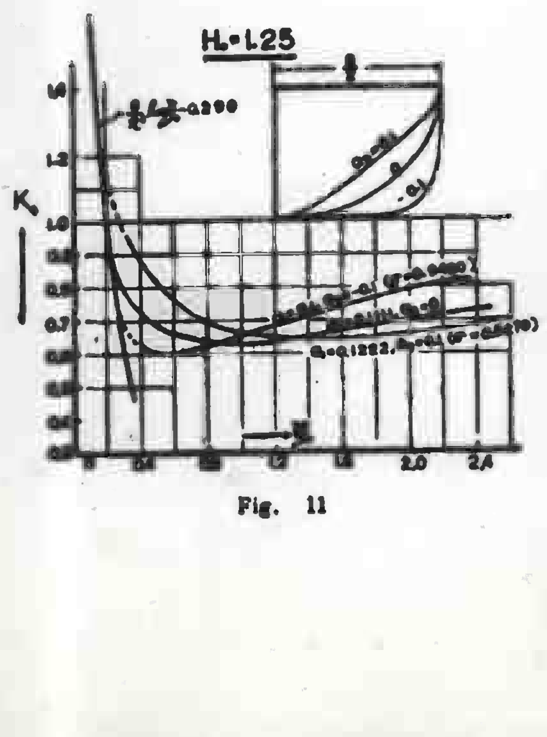

Fig. 11

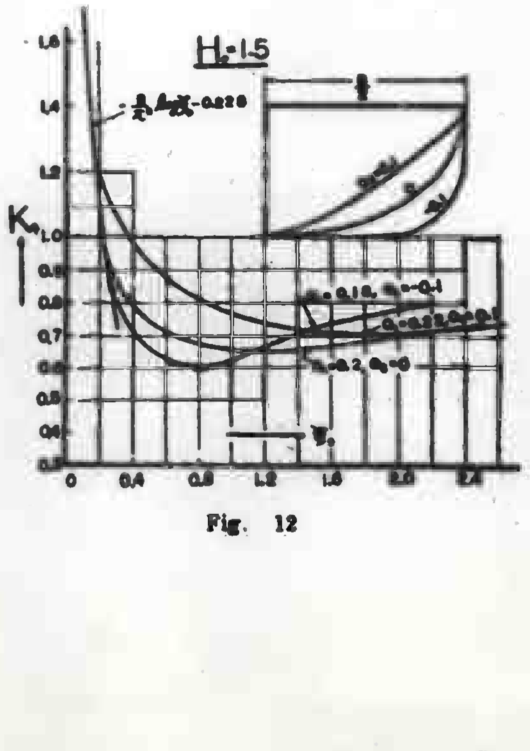

Fig. 12

3. Comparison of Model Experiments and Calculation.

The damping coefficients and the apparent masses of the ship models used in the experiments of Golovato [2] and Gerritsma [8] will be calculated by the strip method, using the calculated values of A and K_4 discussed above. Then the results will be compared with those obtained from the experiments. Let the x-axis be the direction of ship motion, the z-axis be the direction of gravitational force, the y-axis be horizontal, and locate the origin of the coordinate system in the midship-section at the L.W.L.

L = Length of the ship.

B* = Beam at the midship-section.

Δ = Displacement.

S = Cross-sectional area of the ship below the L.W.L. at a distance x from the midship-section.

B = Beam at the L.W.L. at a distance x from the midship-section.

m = Δ/g = mass of the ship.

I = Longitudinal moment of inertia of the cross-section.

N = Damping coefficient for heaving of the cylinder.

N_h, μ_z = Damping coefficient and added mass for pure heaving of the ship.

N_p, μ_φ = Damping coefficient and added moment of inertia for pure pitching of the ship.

Since $N = (\rho g^2/\omega^3) A^2$, its integration in the x-direction is

$$N_h = \int_{-L/2}^{L/2} N dx = \rho g^2 \int_{-L/2}^{L/2} \frac{A^2}{\omega^3} dx \tag{13}$$

$$N_p = \int_{-L/2}^{L/2} N \cdot x^2 \cdot dx = \rho g^2 \int_{-L/2}^{L/2} \frac{A^2}{\omega^3} \cdot x^2 \cdot dx \tag{14}$$

If we define the dimensionless coefficients

$$N_h' = \frac{N_h \cdot \sqrt{gL}}{\Delta}$$

$$N_\varphi' = \frac{N_\varphi \cdot \sqrt{gL}}{\Delta \cdot L^2}$$

then

$$N_h' = \frac{1}{C_b \cdot T}\sqrt{\frac{B^*}{L}} \cdot \int_{-L/2}^{L/2} \frac{\bar{A}^2}{(\xi_1^*)^2} dx \qquad (15)$$

$$N_\varphi' = \frac{1}{C_b \cdot T \cdot L^2}\sqrt{\frac{B^*}{L}} \int_{-L/2}^{L/2} \frac{\bar{A}^2}{(\xi_1^*)^2} \cdot x^2 dx \qquad (16)$$

where

C_b = block coefficient,

$\xi_1^* = \omega\sqrt{\frac{B^*}{g}}$

Since $\xi_0 = \frac{\omega^2}{g} \cdot \frac{B}{2} = \frac{1}{2}(\xi_1^*)^2 \cdot \left(\frac{B}{B^*}\right)$, ζ_0 can be determined from ζ_1^* and B/B^* of different cross-sections. With this ζ_0 we get A from the figure and obtain $\bar{A}^2/(\xi_1^*)^2$ for each cross-section.

Since $1/2 \cdot \rho\pi \cdot (B/2)^2 \cdot C_0 \cdot K_4$ is the added mass of the cylinder, μ_z is equal to

$$\mu_z = \frac{1}{2}\rho\pi \int_{-L/2}^{L/2} \left(\frac{B}{2}\right)^2 \cdot C_0 \cdot K_4 \cdot dx \qquad (17)$$

or, following Korvin-Kroukovsky [5], using S,

$$\mu_z = \rho \int_{-L/2}^{L/2} K_2 \cdot S \cdot K_4 \cdot dx \qquad (18)$$

K_2 for a Lewis form is $K_2 = (1 + a_1)^2 + 3a_3^2/(1 - a_1^2 - 3a_3^2)$ and generally it is determined by the Lewis-Prohaska [11] method.

Similarly,

$$\mu_\varphi = \rho \int_{-L/2}^{L/2} K_2 \cdot S \cdot K_4 \cdot x^2 dx \qquad (19)$$

Golovato [2] performed experiments for the mathematical hip-like shape given by Weinblum [12]. For this particular hape, each cross-section is wall-sided; furthermore, the mplitude of heaving is quite small, so that it is quite uitable to compare theoretical calculations and experimental esults. The value of H_o at the midship-section is $H_o^* = 1.25$, o that we calculate B, x, S, σ, and K_2 for the cross-sections f $H_o = 1.25, 1.0, 2/3, 0.2$, and also calculate ζ_o corresponding o ζ_1^* and obtain A, K_4 from the figure. Then N_h and μ_z are btained by graphical integration. ($\bar{A}$ is determined for σ by inter- olation and extrapolation.)

Figure 13 is obtained by plotting the calculated values of his paper in Figure 5 of Golovato [2]. Two sets of experimental alues, for Froude numbers 0.09 and 0.36, are also plotted in the same figure. (For another Froude number, the experimental points all between those of these two curves.) Compared to Grim's method, he calculated values of this paper are closer to the experimental alues. For K_z in Figure 14, the curve of the calculated values f this paper passes through the experimental points very well for $\zeta_1^o < 2.5$. The result of using K_4 for Ursell's semi-circle over he whole cross-section has also been shown in this figure.

Gerritsma [8] conducted experiments with models of Todd's eries 60. Since $H_o^* = 1.25$ for this case also, the same method as used to perform the graphical integration as was used before. n Figures 15, 16, 17, 18, and 19, ω, represented by the abscissa f the axes, is plotted against N_h, N_p, μ_z, and μ_θ respectively.

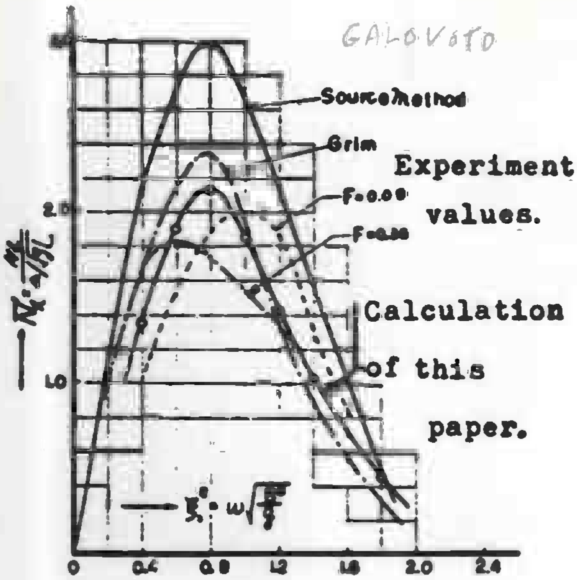

Fig 13

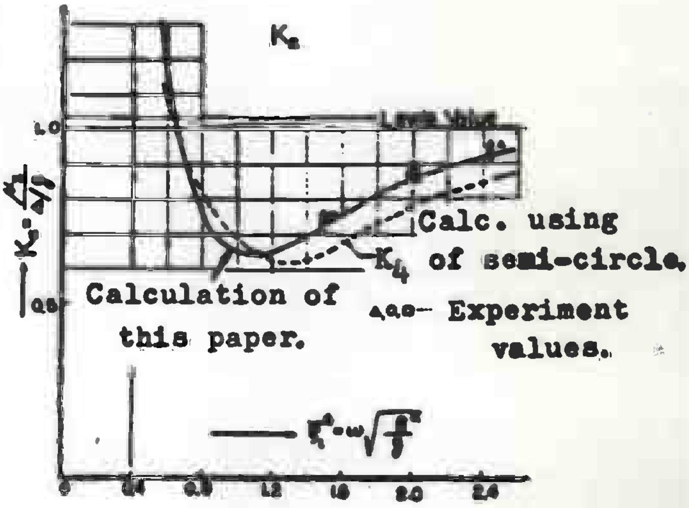

Fig. 14

Damping-Coefficient

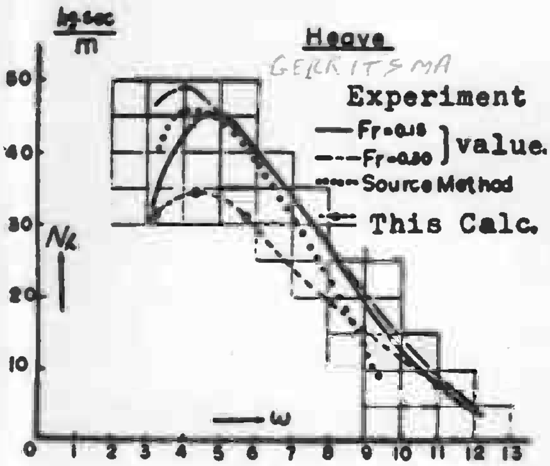

Fig. 15

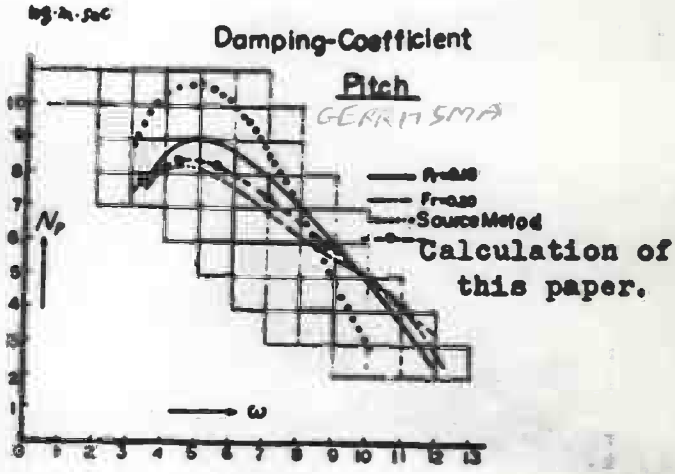

Fig. 16

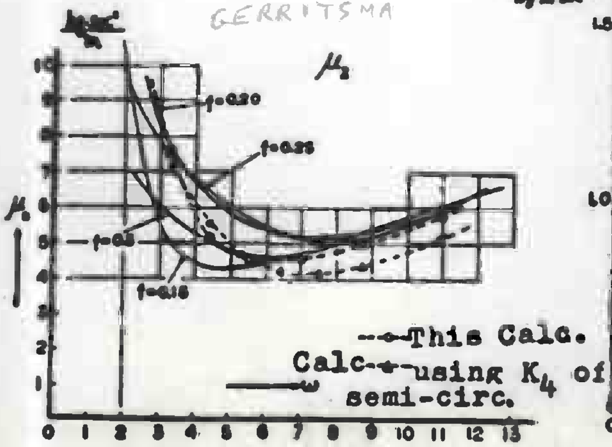

Fig. 17

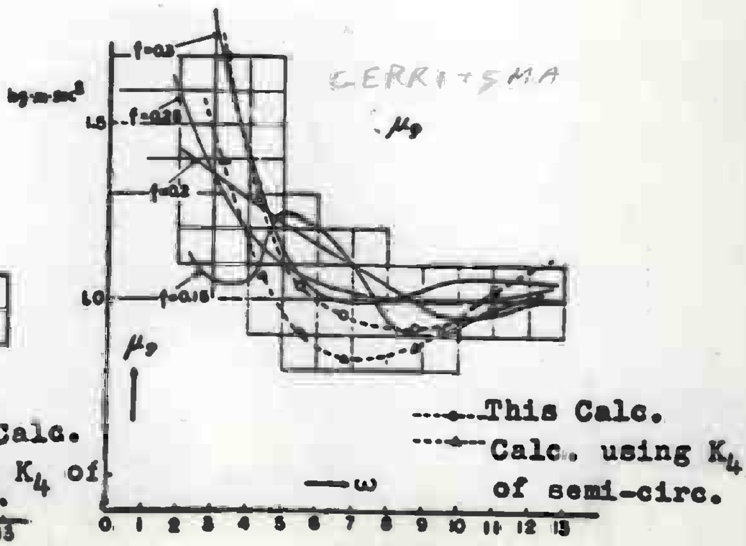

Fig. 18

'or the damping coefficient N_h, the calculated values of this paper ıre too small and the source method presents quite a good result. Ɔn the other hand, for N_p the calculated values of this paper give ı very good result but the source method gives values with a large leviation as a whole. With respect to the deviation between the experimental values and the calculation, we have first to take ıote of three-dimensional effects. For this there are calculations ›y Havelock [13] and Vossers [14]; furthermore, Newman [15] has ısed the three-dimensional source method of Havelock [1] to cal- :ulate the three-dimensional damping coefficient at zero velocity for the experimental model used by Gerritsma. The value of N_h ıccording to this latter calculation is about 20% greater than that ›btained by the two-dimensional strip method. (See Fig. 1 of [15].) :f we use this three-dimensional correction for the calculated ralues of this paper, we find them very close to the experimental ralues. Besides this three-dimensional effect, since the cross- ıections of the aft half of the models used by Gerritsma are not rall-sided but have a certain inclination at the L.W.L., some leviation from the theoretical calculation may be expected as a ıatter of course. When the amplitude of the oscillation is large, ıon-linear damping may also cause this deviation. Concerning these ›roblems, the matter is still under investigation, mainly by experimental means.

The calculated values of μ_z and μ_φ are very close to the experimental values except for small ω where the measurement is ıncertain. On the basis of the two experiments by Golovato and erritsma, it appears that μ_z and μ_φ obtained by the strip method ive very satisfactory values. From this fact we may conclude that xcept for small ζ_1^*, the three-dimensional influence on the pparent mass and moment of inertia is so small that we may disregard t for practical problems.

4. Conclusions.

The following conclusions are obtained from the above calcu- ations.

1) Since some of the $\bar{A}$ values in the figures of O. Grim [4] are oubtful, the equation used for its calculation has been shown as 11). The result obtained by Grim's approximate method using this quation is generally quite close to the more exact values of this aper. From comparison with the experiments of Golovato [2], it s seen that the calculated values of this paper are more accurate han those obtained by Grim's method. On the other hand, from omparison with the experiments of Gerritsma [2], we know that orrection should be made for three-dimensional effects, the ffect of non-wall-sided cross-sections, and the effect of finite mplitude when we perform the calculation for the damping coeffi- ient of the actual ship shape.

2) The value of K_4 varies with the various shapes of cross- ections. By using the values of this paper for cylinder, one may btain by the strip method very satisfactory values of μ_z and μ_φ or ships.

Many thanks are due Professor Watanabe for his valuable iggestions.

References

[1] T.H. Havelock. The damping of the heaving and pitching motion of a ship. Phil. Mag. (7) 33 (1942), 666-673.

[2] P. Golovato. The forces and moments on a heaving surface ship. J. Ship Res. 1 (1) (1957) 19-26, 54-55 = The David W. Taylor Model Basin, Washington, D.C., Rep. 1074 (1956) 44 pp.

[3] F. Ursell. On the heaving motion of a circular cylinder on the surface of a fluid. Quart. J. Mech. Appl. Math. 2 (1949), 218-231, and the discussion of [5].

[4] O. Grim. Berechnung der durch Schwingungen eines Schiffskörpers erzeugten hydrodynamischen Kräfte. Jbuch schiffbautech. Ges. 47 (1953), 277-296; Orörterung, 296-299.

[5] B.V. Korvin-Kroukovsky. Investigation of ship motions in regular waves. Trans. Soc. Naval Arch. Marine Engrs 63 (1955), 386-435.

[6] B.V. Korvin-Kroukovsky and W.R. Jacobs. Pitching and heaving motions of a ship in regular waves. Stevens Inst. Tech., Rep. 659 (1957), 26 pp. = Trans. Soc. Naval Arch. Marine Engrs 65 (1957), 590-632.

[7] Professor Shō-ichi Nakamura. Paper presented at the Kansai Section Spring Meeting, 1958.

[8] J. Gerritsma. Experimental determination of damping, added mass and added mass moment of inertia of a ship model. Intl. Shipbuilding Prog. 3 (1957), 505-519 = Netherlands' Research Centre T.N.O. for Shipbuilding and Navigation, Report 258 (1957) = Delft Shipbuilding Laboratory, Publication No. 8.

[9] F.M. Lewis. The inertia of the water surrounding a vibrating ship. Trans. Soc. Naval Arch. Marine Engrs 37 (1929) 1-17, Discussion 18-20.

[10] F. Ursell. On the rolling motion of cylinders in the surface of a fluid. Quart. J. Mech. Appl. Math. 2 (1949), 335-353.

[11] G. Weinblum and M. St.Denis. On the motion of ships at sea (Fig. 12: Sectional inertia coefficients C_z' as functions of the B/H ratio and section coefficient). Trans. Soc. Naval Arch. Marine Engrs 58 (1950), 232-248.

[12] G. Weinblum. Systematische entwicklung von Schiffsformen. Jbuch schiffbautech. Ges. 47 (1953), 186-210; Erörterung 210-215.

[13] T.H. Havelock. The damping of heave and pitch: a comparison of two-dimensional and three-dimensional calculations. Trans. Inst. Naval Arch. 98 (4) (1956), 464-468.

[14] G. Vossers. Discussion of [13].

[15] J.N. Newman. On the damping of pitch and heave. J. Ship Res. 1 (2) (1957), 48-53.

[16] F. Ursell. Short surface waves due to an oscillating immersed body. Proc. Roy. Soc. London. Ser. A. 222 (1953). 90-103.

[17] F. Ursell. Water waves generated by oscillating bodies. Quart. J. Mech. Appl. Math. 7 (1954), 427-437.

Appendix

For the conformal transformation of (5), if we put $K \cdot B/2 = \omega^2/g \cdot B/2 \equiv \xi_0$ then the free-surface boundary condition becomes

$$\xi_0 \varphi \left(\frac{e^{\alpha} - a_1 e^{-\alpha} - 3a_3 e^{-3\alpha}}{1+a_1+a_3} \right) \mp \frac{\partial \varphi}{\partial \theta} = 0 \quad \left(\theta = \pm \frac{\pi}{2} \right) \tag{20}$$

Consider the following potential function which satisfies $\nabla^2 \varphi = 0$, the boundary condition (20) and is symmetrical with respect to the y-axis:

$$\varphi_{2m} = e^{-2m\alpha} \cos 2m\theta + \frac{\xi_0}{1+a_1+a_3} \left[\frac{e^{-(2m-1)\alpha}}{2m-1} \cos(2m-1)\theta + \frac{a_1 e^{-(2m+1)\alpha}}{2m+1} \cos(2m+1)\theta - \frac{3a_3}{2m+3} e^{-(2m+3)\alpha} \cos(2m+3)\theta \right] \quad (m=1,2,3\cdots) \tag{21}$$

The corresponding stream function is

$$\psi_{2m} = e^{-2m\alpha} \sin 2m\theta + \frac{\xi_0}{1+a_1+a_3} \left[\frac{e^{-(2m-1)\alpha}}{2m-1} \sin(2m-1)\theta + \frac{a_1 e^{-(2m+1)\alpha}}{2m+1} \sin(2m+1)\theta - \frac{3a_3}{2m+3} e^{-(2m+3)\alpha} \sin(2m+3)\theta \right] \quad (m=1,2,3\cdots) \tag{22}$$

Both φ_{2m} and ψ_{2m} become 0 as $\alpha \to \infty$.

Let us suppose a two-dimensional source placed at the origin, following Ursell [3], in order to provide an expression representing progressive waves at infinity. For the stream function ψ one has

$$\left. \begin{aligned} \Psi_0 &= \frac{g\eta}{\pi\omega} [\Psi_c(K,x,y)\cos\omega t + \Psi_s(K,x,y)\sin\omega t] \\ \Psi_c &= \pi e^{-Ky} \sin Kx \\ \Psi_s &= \int_0^\infty \frac{e^{-kx}}{K^2+k^2} \{k \sin ky + K \cos ky\} dk - \pi e^{-Ky} \cos Kx \end{aligned} \right\} \tag{23}$$

or by changing the parameters,

$$\Psi_0=\frac{g\eta}{\pi\omega}\left[\Psi_c(\xi_0,a_1,a_3,\alpha,\theta)\cos\omega t+\Psi_s(\xi_0,a_1,a_3,\alpha,\theta)\sin\omega t\right] \tag{24}$$

where η is the amplitude of the progressive wave at infinity.

The stream function which satisfies the basic conditions and represents progressive waves at infinity is

$$\begin{aligned}(\pi\omega/g\eta)\psi=&\Psi_c(\xi_0,a_1,a_3,\alpha,\theta)\cos\omega t+\Psi_s(\xi_0,a_1,a_3,\alpha,\theta)\sin\omega t\\&+\cos\omega t\sum_{m=1}^{\infty}p_{2m}(\xi_0)\Big[e^{-2m\alpha}\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta\\&+\frac{a_1e^{-(2m+1)\alpha}}{2m+1}\sin(2m+1)\theta-\frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\sin(2m+3)\theta\Big\}\Big]\\&+\sin\omega t\sum_{m=1}^{\infty}q_{2m}(\xi_0)\Big[e^{-2m\alpha}\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{e^{-(2m-1)\alpha}}{2m-1}\sin(2m-1)\theta\\&+\frac{a_1e^{-(2m+1)\alpha}}{2m+1}\sin(2m+1)\theta-\frac{3a_3}{2m+3}e^{-(2m+3)\alpha}\sin(2m+3)\theta\Big\}\Big]\ .\end{aligned} \tag{25}$$

We assume this series is uniformly convergent for $\alpha \geq 0$

The stream function must satisfy the condition (3) on the boundary of the cylinder, $\alpha = 0$. Then (3) becomes

$$(-\partial\psi/\partial\theta)_{\alpha=0}=UM(\cos\theta+a_1\cos\theta-3a_3\cos 3\theta) \tag{26}$$

The following relation is obtained from (25) and (26) on the boundary of the cylinder, $\alpha = 0$:

$$\begin{aligned}(\pi\omega/g\eta)\psi_{\alpha=0}=&\Psi_{c0}(\xi_0,a_1,a_3,\theta)\cos\omega t+\Psi_{s0}(\xi_0,a_1,a_3,\theta)\sin\omega t\\&+\cos\omega t\sum_{m=1}^{\infty}p_{2m}(\xi_0)\Big[\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{\sin(2m-1)\theta}{2m-1}+\frac{a_1\sin(2m+1)\theta}{2m+1}-\frac{3a_3\sin(2m+3)\theta}{2m+3}\Big\}\Big]\\&+\sin\omega t\sum_{m=1}^{\infty}q_{2m}(\xi_0)\Big[\sin 2m\theta+\frac{\xi_0}{1+a_1+a_3}\Big\{\frac{\sin(2m-1)\theta}{2m-1}+\frac{a_1\sin(2m+1)\theta}{2m+1}-\frac{3a_3\sin(2m+3)\theta}{2m+3}\Big\}\Big]\\&=-(\pi\omega/g\eta)UM(\sin\theta+a_1\sin\theta-a_3\sin 3\theta)\ .\end{aligned} \tag{27}$$

Here, ψ_{co} and ψ_{so} are the values of ψ_c and ψ_s for $\alpha = 0$.

From equation (27) with θ = π/2 we get

$$\Psi_{c0}(\xi_0, a_1, a_6, \pi/2)\cos\omega t + \Psi_{s0}(\xi_0, a_1, a_6, \pi/2)\sin\omega t$$
$$+\cos\omega t\sum_{m=1}^{\infty} p_{2m}(\xi_0)\frac{\xi_0}{1+a_1+a_6}(-1)^{m-1}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_6}{2m+3}\right\}$$
$$+\sin\omega t\sum_{m=1}^{\infty} q_{2m}(\xi_0)\frac{\xi_0}{1+a_1+a_6}(-1)^{m-1}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_6}{2m+3}\right\}=-\left(\frac{\pi\omega}{g\eta}\right)UM(1+a_1+a_6)\,. \quad (28)$$

Use this equation and (27) to compare coefficients of the term cos ω t and equate coefficients to obtain

$$\Psi_{c0}(\xi_0, a_1, a_6, \theta)-\frac{\sin\theta+a_1\sin\theta-a_6\sin 3\theta}{1+a_1+a_6}\Psi_{c0}\left(\xi_0, a_1, a_6, \frac{\pi}{2}\right)$$
$$=-\sum_{m=1}^{\infty} p_{2m}(\xi_0)\Big[\sin 2m\theta+\frac{\xi_0}{1+a_1+a_6}\left\{\frac{\sin(2m-1)\theta}{2m-1}+\frac{a_1\sin(2m+1)\theta}{2m+1}-\frac{3a_6\sin(2m+3)\theta}{2m+3}\right\}$$
$$-\frac{\xi_0(-1)^{m-1}}{(1+a_1+a_6)^2}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_6}{2m+3}\right\}(\sin\theta+a_1\sin\theta-a_6\sin\theta)\Big]\,. \quad (29)$$

Define

$$f_{2m}(\xi_0, a_1, a_6, \theta)=-\Big[\sin 2m\theta+\frac{\xi_0}{1+a_1+a_6}\left\{\frac{\sin(2m-1)\theta}{2m-1}+\frac{a_1\sin(2m+1)\theta}{2m+1}-\frac{3a_6\sin(2m+3)\theta}{2m+3}\right\}$$
$$+\frac{\xi_0(-1)^{m}}{(1+a_1+a_6)^2}\left\{\frac{1}{2m-1}-\frac{a_1}{2m+1}-\frac{3a_6}{2m+3}\right\}(\sin\theta+a_1\sin\theta-a_6\sin 3\theta)\Big]\,. \quad (30)$$

Then (29) gives the first equation of (31) and a similar procedure with the coefficients of sin ωt gives the second equation:

$$\left.\begin{aligned}\Psi_{c0}(\xi_0, a_1, a_6, \theta)-\frac{\sin\theta+a_1\sin\theta-a_6\sin 3\theta}{1+a_1+a_6}\Psi_{c0}\left(\xi_0, a_1, a_6, \frac{\pi}{2}\right)&=\sum_{m=1}^{\infty} p_{2m}(\xi_0)\cdot f_{2m}(\xi_0, a_1, a_6, \theta)\\ \Psi_{s0}(\xi_0, a_1, a_6, \theta)-\frac{\sin\theta+a_1\sin\theta-a_6\sin 3\theta}{1+a_1+a_6}\Psi_{s0}\left(\xi_0, a_1, a_6, \frac{\pi}{2}\right)&=\sum_{m=1}^{\infty} q_{2m}(\xi_0)\cdot f_{2m}(\xi_0, a_1, a_6, \theta)\end{aligned}\right\} \quad (31)$$

Equation (31) is the relation used to determine $p_{2m}(\zeta_o)$ and $_{2m}(\zeta_o)$.

In equation (28), let

$$\left.\begin{aligned}\Psi_{c0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) + \sum_{m=1}^{\infty} p_{2m}(\xi_0)(-1)^{m-1}\frac{\xi_0}{1+a_1+a_3}\left\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\right\} = A_0(\xi_0)\\ \Psi_{s0}\left(\xi_0, a_1, a_3, \frac{\pi}{2}\right) + \sum_{m=1}^{\infty} q_{2m}(\xi_0)(-1)^{m-1}\frac{\xi_0}{1+a_1+a_3}\left\{\frac{1}{2m-1} - \frac{a_1}{2m+1} - \frac{3a_3}{2m+3}\right\} = B_0(\xi_0)\end{aligned}\right\} \quad (32).$$

Equation (28) then becomes

$$-(\pi\omega/g\eta)UM(1+a_1+a_3) = A_0(\xi_0)\cos\omega t + B_0(\xi_0)\sin\omega t \quad .$$

Then if we use (see (2) and (8))

$$U = -h\omega\sin(\omega t+\varepsilon),$$

$$M = \frac{B}{2}\Big/ 1+a_1+a_3$$

the ratio $\bar{A}$ of the amplitude of the progressive wave to the amplitude of heaving h is

$$\bar{A} = \frac{\eta}{h} = \frac{\pi\omega^2}{g}\cdot\frac{B}{2}\cdot\frac{1}{\sqrt{A_0^2+B_0^2}} = \frac{\pi\xi_0}{\sqrt{A_0^2+B_0^2}} \quad . \qquad (33)$$

If we put $a_1 = a_3 = 0$ in equations (29), (30), (31), and (32), we get the equations given for the semi-circle by Ursell [3].

The potential φ corresponding to (25) is

$$\begin{aligned}(\pi\omega/g\eta)\varphi = {} & \Theta_c(\xi_0, a_1, a_3, \alpha, \theta)\cos\omega t + \Theta_s(\xi_0, a_1, a_3, \alpha, \theta)\sin\omega t\\ & + \cos\omega t\sum_{m=1}^{\infty} p_{2m}(\xi_0)\Bigg[e^{-2m\alpha}\cos 2m\theta + \frac{\xi_0}{1+a_1+a_3}\left\{\frac{e^{-(2m-1)\alpha}}{2m-1}\cos(2m-1)\theta\right.\\ & \qquad \left. + \frac{a_1 e^{-(2m+1)\alpha}}{2m+1}\cos(2m+1)\theta - \frac{3a_3 e^{-(2m+3)\alpha}}{2m+3}\cos(2m+3)\theta\right\}\Bigg]\\ & + \sin\omega t\sum_{m=1}^{\infty} q_{2m}(\xi_0)\Bigg[\quad '' \quad '' \quad '' \quad '' \quad '' \quad \Bigg]\end{aligned} \qquad (34)$$

where

$$\left.\begin{aligned} \Phi_c(\xi_0, a_1, a_3, \alpha, \theta) &= \pi e^{-Ky}\cos Kx \\ \Phi_s(\xi_0, a_1, a_3, \alpha, \theta) &= \pi e^{-Ky}\sin Kx - \int_0^\infty \frac{e^{-kx}}{K^2+k^2}\{k\cos ky - K\sin ky\}dk \end{aligned}\right\} . \tag{35}$$

From $p=-\rho\,\partial\phi/\partial t$ we get the pressure on the cylinder. (The density of the fluid is ρ.) The force in the y-direction acting on a unit length of the cylinder is of the form

$$F=\left(\frac{g\eta}{\pi}\right)\rho\cdot B[M_0\cos\omega t - N_0\sin\omega t] , \tag{36}$$

where M_0, N_0 are

$$\begin{aligned} M_0 &= \int_0^{\pi/2}\Phi_{s0}(\xi_0, a_1, a_3, \theta)\frac{\cos\theta + a_1\cos\theta - 3a_3\cos 3\theta}{1+a_1+a_3}d\theta \\ &+ \frac{1}{1+a_1+a_3}\left[\sum_{m=1}^{\infty}(-1)^{m-1}q_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m^2-9}\right)+\frac{\pi\xi_0}{4(1+a_1+a_3)}\{(1+a_1-a_1a_3)q_2-a_3q_4\}\right] \\ N_0 &= \int_0^{\pi/2}\Phi_{c0}(\xi_0, a_1, a_3, \theta)\frac{\cos\theta + a_1\cos\theta - 3a_3\cos 3\theta}{1+a_1+a_3}d\theta \\ &+ \frac{1}{1+a_1+a_3}\left[\sum_{m=1}^{\infty}(-1)^{m-1}\cdot p_{2m}\left(\frac{1+a_1}{4m^2-1}+\frac{9a_3}{4m^2-9}\right)+\frac{\pi\xi_0}{4(1+a_1+a_3)}\{(1+a_1-a_1a_3)p_2-a_3p_4\}\right] . \end{aligned} \tag{37}$$

The acceleration of the motion, from (28) and (31), is

$$\frac{d^2y_0}{dt^2}=\left(\frac{2g\eta}{\pi B}\right)\{A_0(\xi_0)\sin\omega t - B_0(\xi_0)\cos\omega t\} . \tag{38}$$

The component of the total force F that is opposite in phase to the acceleration acts as a force proportional to the added mass. This part is

$$\frac{g\eta}{\pi}\cdot\rho B\left(\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\right)\{A_0(\xi_0)\sin\omega t - B_0(\xi_0)\cos\omega t\} . \tag{39}$$

The ratio of (38) and (39) gives the added mass,

$$A.M=2\rho\cdot\left(\frac{B}{2}\right)^2\left(\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\right) . \qquad (40)$$

From Lewis [9], the added mass of a Lewis-form cylinder in an unbounded fluid is $1/2\cdot\rho\pi C_0(B/2)^2$ where

$$C_0=\frac{(1+a_1)^2+3a_3^2}{(1+a_1+a_3)^2} . \qquad (41)$$

To take account of the free-surface we use a coefficient K_4 representing its effect:

$$A.M=\frac{1}{2}\rho\pi\left(\frac{B}{2}\right)^2\cdot C_0\cdot K_4 . \qquad (42)$$

Then K_4 is obtained from

$$K_4=\frac{4}{\pi}\cdot\frac{M_0B_0+N_0A_0}{A_0^2+B_0^2}\cdot\frac{(1+a_1+a_3)^2}{(1+a_1)^2+3a_3^2} . \qquad (43)$$

The average work per cycle of the cylinder oscillation is $\rho g^2\eta^2/\pi^2\omega\cdot(M_0A_0-N_0B_0)$. Since this is equal to the energy propagated by waves to each side per unit time, $1/2\rho g^2\eta^2/\omega$ then $M_0A_0-N_0B_0=\pi^2/2$. This was used to check the numerical calculations.

The coefficients p_{2m}, q_{2m} are obtained from equation (31)

Let

$$\Psi_{c0}(\xi_0,a_1,a_3,\theta)-\frac{\sin\theta+a_1\sin\theta-a_3\sin 3\theta}{1+a_1+a_3}\Psi_{c0}(\xi_0,a_1,a_3,\frac{\pi}{2})\equiv H(\theta)$$

then $H(0)=H(\pi/2)=0$. For $0<\theta<\pi/2$

H(θ) has been expanded into non-orthogonal series:

$$H(\theta)=\sum_{m=1}^{\infty} p_{2m}(\zeta_0)\cdot f_{2m}(\zeta_0, a_1, a_3, \theta) \cdot$$

his should converge uniformly for $0\leq\theta\leq\frac{\pi}{2}$. In Ursell [3], 16], [17], it has been proved that in the case of the circular ylinder ($a_1 = a_3 = 0$) this series converges for all values of ζ_o. enerally it is difficult to determine the region of convergence or the case $a_1 \neq 0$, $a_3 \neq 0$. If we assume its convergence, use he terms to m = 6, and perform the actual calculation, the ignificant figures converge rapidly even when $\zeta_o \doteq 3$. In fact, f we perform the numerical calculation of H(θ) and $f_{2m}(\zeta_o, a_1, a_3, \theta)$ or $\theta = 10^\circ, 20^\circ, 30^\circ \ldots 80^\circ$ and apply the method of the least quares, we obtain simultaneous linear equations in six variables. rom these equations we can obtain $p_2, p_4, \ldots p_{12}$ and the corres-onding q_{2m}. For $H_o = 0.2$, the calculation has been done only or $\zeta_o < 1$.