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Conditions with Variable Leading Coefficients

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Higher-Order Parabolic Equations with VMO Assumptions and General Boundary Conditions with Variable Leading Coefficients

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We prove weighted mixed $L_p(L_q)$-estimates, with $p, q \in (1, \infty)$, and the corresponding solvability results for higher-order elliptic and parabolic equations on the half space $\mathbb{R}^{d+1}_+$ and on general $C^{2m-1,1}$ domains with general boundary conditions, which satisfy the Lopatinskii–Shapiro condition. We assume that the elliptic operators A have leading coefficients that are in the class of vanishing mean oscillations both in the time and the space variables and that the boundary operators have variable leading coefficients. The proofs are based on and generalize the estimates recently obtained by the authors in [6].

1   Introduction

In this paper, we study the higher-order parabolic equation

$$\begin{cases} u_t + (\lambda + A)u = f & \text{on} \quad (-\infty, T) \times \Omega \\ \mathrm{tr}_{\partial\Omega} B_j u = g_j & \text{on} \quad (-\infty, T) \times \partial\Omega, \ j = 1, \dots, m, \end{cases} \tag{1.1}$$

where Ω is a possibly unbounded $C^{2m-1,1}$ domain in $\mathbb{R}^d$, $T \in (-\infty, +\infty]$, "tr" denotes the trace operator, A is an elliptic differential operator of order $2m$, and (B_j) is a family of

differential operators of order $m_j < 2m$ for $j = 1, \ldots, m$. The leading coefficients of A are assumed to be in the class of vanishing mean oscillations (VMO) both in the time and space variables, while the operators B_j are assumed to have variable leading coefficients. In addition, we assume that near the boundary (A, B_j) satisfies the Lopatinskii–Shapiro condition. Roughly speaking, it is an algebraic condition involving the symbols of the principal part of the operators A and B_j with fixed coefficients, which is equivalent to the solvability of certain systems of ordinary differential equations. See for example [1, 20, 27, 33].

Below in Theorem 3.2, we establish weighted $L_p(L_q)$-estimates with $p, q \in (1, \infty)$ and the corresponding solvability results for (1.1) with time-dependent weights in the Muckenhoupt class. This generalizes the recent result obtained by the authors in [6, Theorem 3.5], where B_j are assumed to have constant leading coefficients and the equation is only considered in the half space.

In contrast to the case when A has uniformly continuous leading coefficients, the extension of the results in [6] to boundary operators with variable leading coefficients is nontrivial and does not follow from the standard perturbation argument. In fact, under the VMO assumption on the coefficients of A, in the case when the boundary operators have variable leading coefficients, to apply the method of freezing the coefficients as in [6, Lemma 4.6] one would need to show the mean oscillation estimates of [6, Lemma 4.5] for an equation with inhomogeneous boundary conditions. To the best of the authors' knowledge, this case is not covered by the known theory. Moreover, the well-known localization procedure (see for instance [3, Sec. 8]) does not seem to directly apply to the case $p \neq q$, since we would need a partition of unity argument in both t and x.

Our proof is based on a preliminary result for the case $p = q$, Lemma 4.1, in which the $L_q(L_q)$-estimate is shown as a combination of a recent result of Lindemulder in [19] and the estimates in [6], as well as the available extrapolation theory (see [2, Theorem 1.4] and [9, Theorem 2.5]) to extrapolate to $p \neq q$. We also use in a crucial way a version of the Fefferman–Stein sharp function theorem with A_p weights in spaces of homogeneous type, which was recently established in [9].

Research on $L_p(L_q)$-regularity for higher-order equations as (1.1) has been developed in the past decades by mainly two different approaches. On the one hand, a partial differential equation (PDE) approach has been developed by a series of papers by Krylov, Dong, and Kim. See for example [18], [7], [8], and references therein. In [8] a new technique was developed to produce mean oscillation estimates for equations in the whole and half spaces with the Dirichlet boundary condition, for $p = q$. These results had been extended by the same authors in [9] to mixed $L_p(L_q)$-spaces with

Muckenhoupt weights and small bounded mean oscillation (BMO) assumptions on the space variable, for any $p, q \in (1, \infty)$. It is worth noting that in all these references as well as others papers in the literature VMO coefficients were only considered for equations with specific boundary conditions (Dirichlet, Neumann, conormal, etc.).

On the other hand, $L_p(L_q)$-regularity can be viewed in a functional analytic approach as an application of a more general abstract result, namely that of maximal L_p-regularity. Maximal L_p-regularity means that, under certain assumption on g_j, for all $f \in L_p(\mathbb{R}, L_q(\mathbb{R}_+^d))$, the solution to the evolution problem (1.1) has the "maximal" regularity in the sense that u_t and Au are both in $L_p(\mathbb{R}, L_q(\mathbb{R}_+^d))$. We refer to [3, 23, 32], [14, 15], and [10, 11] for further information on autonomous and non-autonomous problems and applications to higher-order equations.

In [3], Denk, Hieber, and Prüss obtained $L_p(L_q)$-regularity for any $p, q \in (1, \infty)$ for autonomous, operator-valued parabolic problems on the half space and on domains with homogeneous boundary conditions of the Lopatinskii–Shapiro type. The leading coefficients of the operators involved are assumed to be bounded and uniformly continuous, and their proofs combine operator sum methods with tools from vector-valued harmonic analysis. These results were generalized in [4] by the same authors to $L_p(L_q)$-regularity for nonautonomous, operator-valued parabolic initial-boundary value problems with inhomogeneous boundary data, under the assumption that $t \to A(t)$ is continuous. See also Weidemaier [31] for the special case where $m = 1$, the coefficients are complex-valued coefficients, and $q \le p$. Later, Meyries and Schnaubelt in [22] further generalized the results of [4] to the weighted time-dependent setting, where the weights considered are Muckenhoupt power-type weights. See also [21]. Very recently, Lindemulder in [19] generalized the results of [22] to the setting of power weights both in the time and the space variables. In all these results, the leading coefficients of the operators are assumed to be bounded and uniformly continuous in both the time and space variables.

In this paper, we relax the assumptions on the coefficients of the operators involved to be VMO in the time and space variables, and with inhomogeneous general boundary operators having variable leading coefficients and satisfying the Lopatinskii–Shapiro condition. The main result of this paper is stated in Theorem 3.2 and in the elliptic setting in Theorem 3.3. The results presented here generalize the ones in [6], in which the boundary operators were assumed to have constant leading coefficients and only the half space setting was considered.

The paper is organized as follows. In Section 2 we give the necessary preliminary results and introduce the notation. In Section 3 we list the main assumptions on the

operators and state the main theorem. In Section 4 we prove an auxiliary lemma needed for the proof of the main theorem in the half-space case, which is given in Section 5. Finally, in Section 6 we prove the main theorem in the general case by using the estimates in the previous sections.

2   Preliminaries

In this section, we state some necessary preliminary results and introduce the notation used throughout the paper.

2.1 A_p-weights

Details on Muckenhoupt weights can be found in [13, Chapter 9] and [28, Chapter V].

A *weight* is a locally integrable function on $\mathbb{R}^d$ with $\omega(x) \in (0, \infty)$ for almost every $x \in \mathbb{R}^d$. The space $L_p(\mathbb{R}^d, \omega)$ is defined as all measurable functions f with

$$\|f\|_{L_p(\mathbb{R}^d, \omega)} = \left(\int_{\mathbb{R}^d} |f|^p \, \omega \, dx \right)^{\frac{1}{p}} < \infty \quad \text{if } p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\mathbb{R}^d, \omega)} = \text{ess. } \sup_{x \in \mathbb{R}^d} |f(x)|.$$

We recall the class of Muckenhoupt weights A_p for $p \in (1, \infty)$. A weight ω is said to be an A_p-*weight* if

$$[\omega]_p = [\omega]_{A_p} := \sup_B \left(\fint_B \omega(x) \, dx \right) \left(\fint_B \omega(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty.$$

Here the supremum is taken over all balls $B \subset \mathbb{R}^d$ and $\fint_B = \frac{1}{|B|} \int_B$. The extended real number $[\omega]_{A_p}$ is called the A_p-*constant*. In the case of the half space $\mathbb{R}^d_+$, we replace the balls B in the definition by $B \cap \mathbb{R}^d_+ =: B^+$ with center in $\overline{\mathbb{R}^d_+}$.

The following properties will be used. For the given $\omega \in A_p(\mathbb{R})$, an open interval $I \in \mathbb{R}$, and a measurable set $E \subset I$, it holds that

$$\omega(E)/\omega(I) \geq C\big(|E|/|I|\big)^p, \tag{2.1}$$

where $C > 0$ is a constant depending only on p and $[\omega]_{A_p}$ and $|E|$ is the Lebesgue measure of E. Moreover, using a reverse Hölder's inequality (see [13, Corollary 9.2.6]), there is a

positive number $\sigma_1 = \sigma_1(p, [\omega]_p)$ such that $p - \sigma_1 > 1$ and

$$\omega \in A_{p-\sigma_1}(\mathbb{R}).$$

Consequently, instead of (2.1), we have

$$\omega(E)/\omega(I) \geq C\big(|E|/|I|\big)^{p-\sigma_1}. \tag{2.2}$$

The celebrated result of Rubio de Francia (see [24–26], [12, Chapter IV]) allows one to extrapolate from weighted L_p-estimates for a single p to weighted L_q-estimates for all q. The proofs and statement have been considerably simplified and clarified in [2, Theorem 3.9]. The following version of the extrapolation theorem [2, Theorem 3.9] will be needed. Its main feature is that, to prove (2.4) for a given $\omega \in A_p, p \in (1, \infty)$, the estimate (2.3) as an assumption needs to hold only for a subset of A_{p_0}, not for all weights in A_{p_0}. We refer to [9, Theorem 2.5] for further details.

Theorem 2.1. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be a pair of measurable functions $p_0, p \in (1, \infty)$ and $\omega \in A_p$. Then there exists a constant $\Lambda_0 = \Lambda_0(p_0, p, [\omega]_{A_p}) \geq 1$ such that if

$$\|f\|_{L_{p_0}(\tilde{\omega})} \leq C\|g\|_{L_{p_0}(\tilde{\omega})} \tag{2.3}$$

for some constant $C \geq 0$ and for any $\tilde{\omega} \in A_{p_0}$ satisfying $[\tilde{\omega}]_{A_{p_0}} \leq \Lambda_0$, then we have

$$\|f\|_{L_p(\omega)} \leq 4C\|g\|_{L_p(\omega)}. \tag{2.4}$$

2.2 Function spaces and notation

In this section we introduce some function spaces and notation to be used throughout the paper.

We denote $D = -i(\partial_i, \ldots, \partial_d)$ and we consider the standard multi-index notation $D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

Denote

$$\mathbb{R}_+^d = \big\{x = (x_1, x') \in \mathbb{R}^d : x_1 > 0, \ x' \in \mathbb{R}^{d-1}\big\} \quad \text{and} \quad \mathbb{R}_+^{d+1} = \mathbb{R} \times \mathbb{R}_+^d.$$

The parabolic distance between $X = (t, x)$ and $Y = (s, y)$ in $\mathbb{R}_+^{d+1}$ is defined by

$$\rho(X, Y) = |x - y| + |t - s|^{\frac{1}{2m}}.$$

For a function f on $\mathcal{D} \subset \mathbb{R}_+^{d+1}$, we set

$$(f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(t, x) \, dx \, dt = \fint_{\mathcal{D}} f(t, x) \, dx \, dt.$$

For $m = 1, 2, \ldots$ fixed depending on the order of the equations under consideration, we denote by

$$Q_r(t, x) = (t - r^{2m}, t) \times B_r(x), \quad Q_r^+(t, x) = Q_r(t, x) \cap \mathbb{R}_+^{d+1}$$

the parabolic cylinders, where

$$B_r(x) = \left\{ y \in \mathbb{R}^d : |x - y| < r \right\} \subset \mathbb{R}^d$$

denotes the ball of radius r with center x. We use Q_r^+ to indicate $Q_r^+(0, 0)$. We also define

$$B_r^+(x) = B_r(x) \cap \mathbb{R}_+^d.$$

Let Ω be a domain in $\mathbb{R}^{d+1}$ and $T \in (-\infty, \infty]$. We denote

$$Q_r^{\Omega}(t, x) := Q_r(t, x) \cap (\mathbb{R} \times \Omega).$$

Note that when $\Omega = \mathbb{R}_+^d$, $Q_r^{\Omega}(t, x) = Q_r^+(t, x)$.

For $Q = Q_r^+(t, x)$ or $Q_r^{\Omega}(t, x)$, we define the mean oscillation of f on Q as

$$\mathrm{osc}(f, Q) := \fint_Q |f(s, y) - (f)_Q| \, ds \, dy$$

and we denote for $R \in (0, \infty)$,

$$(f)_R^{\sharp} := \sup_{(t,x) \in (-\infty, T) \times \Omega} \sup_{r \le R} \mathrm{osc}\left(f, Q_r^{\Omega}(t, x) \right).$$

Next, we introduce the function spaces that will be used in the paper. For $p \in (1, \infty)$ and $k \in \mathbb{N}_0$, we define the standard Sobolev space as

$$W_p^k(\Omega) = \left\{ u \in L_p(\Omega) : D^{\alpha} u \in L_p(\Omega) \quad \forall |\alpha| \le k \right\}.$$

For $p, q \in (1, \infty)$ and $-\infty \le S < T \le \infty$, we denote

$$L_p((S, T) \times \Omega) = L_p\left((S, T); L_p(\Omega) \right)$$

and mixed-norm spaces

$$L_{p,q}\left((S, T) \times \Omega\right) = L_p\left((S, T); L_q(\Omega)\right).$$

For parabolic equations, we denote for $k = 1, 2, \ldots,$

$$W_p^{1,k}\left((S, T) \times \Omega\right) = W_p^1\left((S, T); L_p(\Omega)\right) \cap L_p\left((S, T); W_p^k(\Omega)\right)$$

and mixed-norm spaces

$$W_{p,q}^{1,k}\left((S, T) \times \Omega\right) = W_p^1\left((S, T); L_q(\Omega)\right) \cap L_p\left((S, T); W_q^k(\Omega)\right).$$

We will use the following weighted Sobolev spaces. For $\omega \in A_p(\mathbb{R})$ we denote

$$L_{p,q,\omega}\left((S, T) \times \Omega\right) = L_p\left((S, T), \omega; L_q(\Omega)\right)$$

and

$$W_{p,q,\omega}^{1,k}((S, T) \times \Omega) = W_p^1\left((S, T), \omega; L_q(\Omega)\right) \cap L_p\left((S, T), \omega; W_q^k(\Omega)\right),$$

where by $f \in L_{p,q,\omega}((S, T) \times \Omega)$ we mean

$$\|f\|_{L_{p,q,\omega}((S,T)\times\Omega)} := \left(\int_S^T \left(\int_\Omega |f(t,x)|^q \, \mathrm{d}x\right)^{p/q} \omega(t) \, \mathrm{d}t\right)^{1/p} < \infty.$$

2.3   Interpolation and trace

The following function spaces from the interpolation theory will be needed. For more information and proofs we refer the reader to [21, 29, 30].

For $p \in (1, \infty)$ and $s = [s] + s_* \in \mathbb{R}_+ \backslash \mathbb{N}_0$, where $[s] \in \mathbb{N}_0$, $s_* \in (0, 1)$, we define the Slobodetskii space W_p^s by real interpolation as

$$W_p^s = \left(W_p^{[s]}, W_p^{[s]+1}\right)_{s_*,p}.$$

Let Ω be a $C^{2m-1,1}$ domain in $\mathbb{R}^d$ and $-\infty \leq S < T \leq \infty$. For $m \in \mathbb{N}$, $s \in (0, 1]$, and $\omega \in A_p(\mathbb{R})$, we consider weighted anisotropic spaces of the form

$$W_{p,\omega}^{s,2ms}((S, T) \times \Omega) = W_p^s\left((S, T), \omega; L_p(\Omega)\right) \cap L_p\left((S, T), \omega; W_p^{2ms}(\Omega)\right).$$

For $p \in (1, \infty)$, $q \in [1, \infty]$, $r \in \mathbb{R}$, $\omega \in A_p(\mathbb{R}^d)$, and X a Banach space, we introduce the Besov space $\mathcal{B}_{p,q}^r(\mathbb{R}^d)$ and the weighted X-valued Triebel–Lizorkin space $F_{p,q}^r(\mathbb{R}^d, \omega; X)$ as follows.

Let $\Phi(\mathbb{R}^d)$ be the set of all sequences $(\varphi_k)_{k \geq 0} \subset \mathcal{S}(\mathbb{R}^d)$ such that

$$\widehat{\varphi}_0 = \widehat{\varphi}, \quad \widehat{\varphi}_1(\xi) = \widehat{\varphi}(\xi/2) - \widehat{\varphi}(\xi), \quad \widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi),$$

where $k \geq 2$, $\xi \in \mathbb{R}^d$, and where the Fourier transform $\widehat{\varphi}$ of the generating function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfies $0 \leq \widehat{\varphi}(\xi) \leq 1$ for $\xi \in \mathbb{R}^d$ and

$$\widehat{\varphi}(\xi) = 1 \quad \text{if } |\xi| \leq 1, \quad \widehat{\varphi}(\xi) = 0 \quad \text{if } |\xi| \geq 3/2.$$

Definition 2.2.   Given $(\varphi_k)_{k \geq 0} \in \Phi(\mathbb{R}^d)$, we define the *Besov space* as

$$\mathcal{B}_{p,q}^r(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\mathcal{B}_{p,q}^r(\mathbb{R}^d)} := \left\| \left(2^{kr} \mathcal{F}^{-1}(\widehat{\varphi}_k \hat{f}) \right)_{k \geq 0} \right\|_{\ell_q(L_p(\mathbb{R}^d))} < \infty \right\},$$

and the weighted *X-valued Triebel–Lizorkin space* as

$$F_{p,q}^r(\mathbb{R}^d, \omega; X) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d, X) : \ \|f\|_{F_{p,q}^r(\mathbb{R}^d, \omega; X)} := \left\| \left(2^{kr} \mathcal{F}^{-1}(\widehat{\varphi}_k \hat{f}) \right)_{k \geq 0} \right\|_{L_p(\mathbb{R}^d, \omega; \ell_q(X))} < \infty \right\}.$$

Observe that $\mathcal{B}_{p,p}^r(\mathbb{R}^d) = F_{p,p}^r(\mathbb{R}^d)$ by Fubini's theorem. Moreover, we have the following equivalent definition of Slobodetskii space

$$W_p^s(\mathbb{R}^d) = \begin{cases} W_p^k(\mathbb{R}^d), & s = k \in \mathbb{N} \\ \mathcal{B}_{p,p}^s(\mathbb{R}^d), & s \in \mathbb{R}_+ \backslash \mathbb{N}_0. \end{cases}$$

Later on we will consider weighted X-valued Triebel–Lizorkin spaces on an interval $(-\infty, T) \subset \mathbb{R}$. We define these spaces by restriction.

Definition 2.3.   Let $T \in (-\infty, \infty]$ and let X be a Banach space. For $p \in (1, \infty)$, $q \in [1, \infty)$, $\omega \in A_p(\mathbb{R})$, and $r \in \mathbb{R}$ we denote by $F_{p,q}^r((-\infty, T), \omega; X)$ the collection of all restrictions of elements of $F_{p,q}^r(\mathbb{R}, \omega; X)$ on $(-\infty, T)$. If $f \in F_{p,q}^r((-\infty, T), \omega; X)$ then

$$\|f\|_{F_{p,q}^r((-\infty, T), \omega; X)} = \inf \|g\|_{F_{p,q}^r(\mathbb{R}, \omega; X)},$$

where the infimum is taken over all $g \in F_{p,q}^r(\mathbb{R}, \omega; X)$ whose restriction on $(-\infty, T)$ coincides with f.

We will also use Besov spaces $\mathcal{B}_{p,p}^s(\partial\Omega)$ defined on $\partial\Omega$. We refer the reader to [30, Sec. 3.6] for the precise definition.

We will need the following spatial trace inequality. For full details about the proof, we refer the reader to [4, Lemma 3.5] for the unweighted setting and to [21, Lemma 1.3.11] where the weights considered are power-type weights in time. The restriction of power-type weights only plays a role at $t = 0$ in order to have a well-defined trace space. Thus, in the formulation below with $t \in \mathbb{R}$, the power-type weight can be replaced by any weight $\omega \in A_p(\mathbb{R})$; see for instance [19, Sec. 6.3] for details.

Lemma 2.4. Let $p \in (1, \infty)$, $\omega \in A_p(\mathbb{R})$, $m \in \mathbb{N}$, and $s \in (0, 1]$ so that $2ms \in \mathbb{N}$. Then the map

$$\text{tr}_{x_1=0} : W_{p,\omega}^{s,2ms}\left(\mathbb{R}_+^{d+1}\right) \hookrightarrow W_{p,\omega}^{s-\frac{1}{2mp},2ms-\frac{1}{p}}(\mathbb{R} \times \mathbb{R}^{d-1})$$

is continuous.

3   Assumptions and Main Result

In the sequel, we assume that Ω is a (possibly unbounded) $C^{2m-1,1}$ domain in $\mathbb{R}^d$. Let $T \in (-\infty, \infty]$, $p, q \in (1, \infty)$, and $m = 1, 2, \ldots$. We consider a $2m$-th order elliptic differential operator A given by

$$Au = \sum_{|\alpha| \leq 2m} a_\alpha(t, x)D^\alpha u,$$

where $a_\alpha : (-\infty, T) \times \Omega \to \mathbb{C}$. For $j = 1, \ldots, m$ and $m_j \in \{0, \ldots, 2m - 1\}$, we consider the boundary differential operators B_j of order m_j given by

$$B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(t, x)D^\beta u,$$

where $b_{j\beta} : (-\infty, T) \times \Omega \to \mathbb{C}$. For convenience, here and in the sequel, we denote $D_{x_j} = -i\frac{\partial}{\partial x_j}$.

We will give conditions on the operators A and B_j under which the $L_p(L_q)$-estimates hold for the solution to the parabolic problem

$$\begin{cases} u_t(t,x) + (A + \lambda)u(t,x) = f(t,x) & \text{in } (-\infty, T) \times \Omega \\ B_j u(t,x)\big|_{\partial\Omega} = g_j & \text{on } (-\infty, T) \times \partial\Omega, \ j = 1, \ldots, m. \end{cases} \tag{3.1}$$

We also consider the corresponding elliptic problem

$$
\begin{cases}
(A + \lambda)u = f & \text{in } \Omega \\
B_j u\big|_{\partial\Omega} = g_j & \text{on } \partial\Omega, \; j = 1, \dots, m,
\end{cases}
\tag{3.2}
$$

where the coefficients of the operators involved are functions independent on $t \in \mathbb{R}$ that is defined on Ω.

3.1   Assumptions on A and B_j

Denote

$$
A^H(t, x, D) := \sum_{|\alpha|=2m} a_\alpha(t, x) D^\alpha \quad \text{and} \quad B_j^H(D) := \sum_{|\beta|=m_j} b_{j\beta}(t, x) D^\beta
$$

to be the *principal part* of A and B_j, respectively, and

$$
A_\sharp(t, x, \xi) = \sum_{|\alpha|=2m} a_\alpha(t, x) \xi^\alpha
$$

to be the *principal symbol* of A. For any $(t_0, x_0) \in (-\infty, T) \times \Omega$ and in a coordinate system that will be specified later, taking the Fourier transform $\mathcal{F}_{x'}$ with respect to $x' \in \mathbb{R}^{d-1}$ and letting $v(x_1, \xi) := \mathcal{F}_{x'}(u(x_1, \cdot))(\xi)$, we obtain

$$
A^H(t_0, x_0, \xi, D_{x_1})v := \mathcal{F}_{x'}\left(A^H(t_0, x_0, D)u(x_1, \cdot) \right)(\xi)
$$

$$
= \sum_{k=0}^{2m} \sum_{|\beta|=k} a_{k,\beta}(t_0, x_0) \xi^\beta D_{x_1}^{2m-k} v
$$

and

$$
B_j^H(t_0, x_0, \xi, D_{x_1})v := \mathcal{F}_{x'}\left(B_j^H(t_0, x_0, D)u(x_1, \cdot) \right)(\xi)
$$

$$
= \sum_{k=0}^{m_j} \sum_{|\gamma|=k} b_{j,k,\gamma}(t_0, x_0) \xi^\gamma D_{x_1}^{m_j-k} v.
$$

We first introduce a parameter-ellipticity condition in the sense of [3, Definition 5.1].

$(E)_\theta$ Let $\theta \in (0, \pi)$. For any $t \in (\infty, T)$ and $x \in \Omega$, it holds that

$$
A_\sharp(t, x, \xi) \subset \Sigma_\theta, \quad \forall \, \xi \in \mathbb{R}^n, \; |\xi| = 1,
$$

where $\Sigma_\theta = \{ z \in \mathbb{C} \backslash \{0\} : \, |\arg(z)| < \theta \}$ and $\arg : \mathbb{C} \backslash \{0\} \to (-\pi, \pi]$.

Before stating the Lopatinskii–Shapiro condition, we need to introduce some notation. For each $\hat{x}_0 \in \partial\Omega$, we choose a coordinate system such that $\hat{x}_0$ is the origin and e_1 is the normal direction at $\hat{x}_0$ (Here $[e_j]_{j=1}^d$ denotes the standard basis of $\mathbb{R}^d$.). We assume that the (LS)$_\theta$–condition holds for any $t_0 \in (-\infty, T)$ and $x_0 \in B_{2R_0}(\hat{x}_0) \cap \Omega$ with respect to the above coordinate system, which can be stated as follows.

(LS)$_\theta$ For each $(h_1, \ldots, h_m)^T \in \mathbb{R}^{d-1}$, $\xi \in \mathbb{R}^m$, $\lambda \in \overline{\Sigma}_{\pi-\theta}$, $t_0 \in (-\infty, T)$, $x_0 \in B_{2R_0}(\hat{x}_0) \cap \Omega$, and $|\xi| + |\lambda| \neq 0$, the ordinary differential equation (ODE) problem in $\mathbb{R}_+$

$$\begin{cases} \lambda v + A^H(t_0, x_0, \xi, D_{x_1})v = 0, & x_1 > 0, \\ B_j^H(t_0, x_0, \xi, D_{x_1})v\big|_{x_1=0} = h_j, & j = 1, \ldots, m \end{cases}$$

admits a unique solution $v \in C^\infty(\mathbb{R}_+)$ such that $\lim_{x \to \infty} v(x) = 0$.

We now introduce a regularity condition on the leading coefficients, where ρ is a parameter to be specified.

Assumption 3.1. (ρ)   For $|\alpha| = 2m$, there exist a constant $R_0 \in (0, 1]$ such that $(a_\alpha)_{R_0}^\sharp \leq \rho$.

Throughout the paper, we impose the following assumptions on the coefficients of A and B_j.

(A)   For the multi-index α, the coefficients a_α are functions $(-\infty, T) \times \Omega \to \mathbb{C}$ and $\|a_\alpha\|_{L_\infty} \leq K$ and satisfy Assumption 3.1 (ρ) with a parameter $\rho \in (0, 1)$ to be determined later. Moreover, A satisfies condition $(E)_\theta$.

(B)   The coefficients $b_{j\beta} : (-\infty, T) \times \Omega \to \mathbb{C}$ satisfy

$$b_{j\beta} \in C^{\frac{2m-m_j}{2m}, 2m-m_j}\left((-\infty, T) \times \Omega\right), \quad \left\|b_{j\beta}\right\|_{C^{\frac{2m-m_j}{2m}, 2m-m_j}((-\infty,T)\times\Omega)} \leq K,$$

and

$$\lim_{|t|+|x| \to \infty} b_{j\beta}(t, x) = \overline{b}_{j\beta}.$$

The (LS)$_\theta$-condition is satisfied by $(A, \overline{B}_j)$ for any $A \in (E)_\theta$, where $\overline{B}_j$, $j = 1, \ldots, m$, are the boundary operators with coefficients $\overline{b}_{j\beta}$.

We can now state the 1st main result of this paper.

Theorem 3.2. Let $T \in (-\infty, \infty]$, $p, q \in (1, \infty)$, and $\omega \in A_p((-\infty, T))$. Let Ω be a $C^{2m-1,1}$-domain with the $C^{2m-1,1}$-norm bounded by K. There exists

$$\rho = \rho\left(\theta, m, d, K, p, q, [\omega]_p, b_{j\beta}\right) \in (0, 1)$$

such that under the assumptions (A), (B), and (LS)$_\theta$ for some $\theta \in (0, \pi/2)$, the following hold. There exists $\lambda_0 = \lambda_0(\theta, m, d, K, p, q, [\omega]_p, R_0, b_{j\beta}) \geq 1$ such that for any $\lambda \geq \lambda_0$ if

$$u \in W_p^1\left((-\infty, T), \omega; L_q(\Omega)\right) \cap L_p\left((-\infty, T), \omega; W_q^{2m}(\Omega)\right)$$

satisfies the problem (3.1), where $f \in L_{p,q,\omega}((-\infty, T) \times \Omega)$ and

$$g_j \in F_{p,q}^{k_j}\left((-\infty, T), \omega; L_q(\partial\Omega)\right) \cap L_p\left((-\infty, T), \omega; \mathcal{B}_{q,q}^{2mk_j}(\partial\Omega)\right)$$

with $k_j = 1 - m_j/(2m) - 1/(2mq)$, then it holds that

$$\|u_t\|_{L_p((-\infty,T),\omega;L_q(\Omega))} + \sum_{|\alpha| \leq 2m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p((-\infty,T),\omega;L_q(\Omega))}$$

$$\leq C\|f\|_{L_p((-\infty,T),\omega;L_q(\Omega))} + C \sum_{j=1}^m \|g_j\|_{F_{p,q}^{k_j}((-\infty,T),\omega;L_q(\partial\Omega)) \cap L_p((-\infty,T),\omega;\mathcal{B}_{q,q}^{2mk_j}(\partial\Omega))}, \tag{3.3}$$

where $C = C(\theta, m, d, K, p, q, [\omega]_p, b_{j\beta}) > 0$ is a constant. Moreover, for any $\lambda \geq \lambda_0$, $f \in L_{p,q,\omega}((-\infty, T) \times \Omega)$, and

$$g_j \in F_{p,q}^{k_j}\left((-\infty, T), \omega; L_q(\partial\Omega)\right) \cap L_p\left((-\infty, T), \omega; \mathcal{B}_{q,q}^{2mk_j}(\partial\Omega)\right),$$

there is a unique solution $u \in W_p^1((-\infty, T), \omega; L_q(\Omega)) \cap L_p((-\infty, T), \omega; W_q^{2m}(\Omega))$ to (3.1).

Using the same arguments as in [6, Theorem 3.6], from the a priori estimates for the parabolic equation in Theorem 3.2, we obtain the a priori estimates for the higher-order elliptic equation as well. The key idea is that the solutions to elliptic equations can be viewed as steady state solutions to the corresponding parabolic cases. The argument is quite standard, so we omit the proof. The interested reader can find more details in [9, Theorem 5.5] and [17, Theorem 2.6].

We state below the elliptic version of Theorem 3.2. In this case the coefficients of A and B_j are independent of t.

Theorem 3.3.   Let $q \in (1, \infty)$ and Ω be a $C^{2m-1,1}$-domain with the $C^{2m-1,1}$-norm bounded by K. There exists

$$\rho = \rho(\theta, m, d, K, q, b_{j\beta}) \in (0, 1)$$

such that under the assumptions (A), (B), and (LS)$_\theta$ for some $\theta \in (0, \pi/2)$ the following hold. There exists $\lambda_0 = \lambda_0(\theta, m, d, K, q, R_0, b_{j\beta}) \geq 0$ such that for any $\lambda \geq \lambda_0$ and $u \in W_q^{2m}(\Omega)$ satisfying (3.2), where $f \in L_q(\Omega)$ and $g_j \in \mathcal{B}_{q,q}^{2mk_j}(\partial\Omega)$ with $k_j = 1 - m_j/(2m) - 1/(2mq)$ it holds that

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)} + C \sum_{j=1}^m \|g_j\|_{\mathcal{B}_{q,q}^{2mk_j}(\partial\Omega)},$$

where $C = C(\theta, m, d, K, q, b_{j\beta}) > 0$ is a constant. Moreover, for any $\lambda \geq \lambda_0$ and $f \in L_q(\Omega)$ and $g_j \in \mathcal{B}_{q,q}^{2mk_j}(\partial\Omega)$, there is a unique solution $u \in W_q^{2m}(\Omega)$ to (3.2).

Note that in the case when Ω is bounded, for Theorem 3.3, the limit behavior of $b_{j\beta}$ in the assumption (B) is unnecessary.

Remark 3.4.

(i)   For notational simplicity, we consider the scalar case only. However, with the same proofs Theorem 3.2 and the corresponding elliptic results also hold if one considers finite-dimensional systems of operators.

(ii)   In [3, 4], [21, 22], and [19], the coefficients there considered are operator-valued, with values in a Banach space with the unconditional martingale difference (UMD) property (see [16] for details). In the unweighted setting our proofs refer to these results, and we therefore believe that it is possible to extend our results also to the case of operator-valued coefficients, with values in a Hilbert space or in a UMD-Banach space. We do not deal with these cases in order to not overburden the current paper. It would also be very interesting to see whether our results can be extended to mixed-order systems, say in the setting of [5]. This, however, is highly nontrivial and will be studied in our future work.

4   An Auxiliary Result

Throughout the section, we assume that A and B_j consist only of their principal part. Let

$$A_0 = \sum_{|\alpha|=2m} \bar{a}_\alpha D^\alpha$$

be an operator with constant coefficients satisfying $|\overline{a}_\alpha| \leq K$ and the condition $(E)_\theta$ with $\theta \in (0, \pi/2)$, and let

$$\overline{B}_j = \sum_{|\beta|=m_j} \overline{b}_{j\beta} D^\beta,$$

where the coefficients $\overline{b}_{j\beta}$ are also constants. In this section, we consider the problem

$$\begin{cases} u_t(t,x) + (\lambda + A_0)u(t,x) = f(t,x) & \text{in } (-\infty, T) \times \mathbb{R}^d_+ \\ \overline{B}_j u(t,x)\big|_{x_1=0} = g_j(t,x) & \text{on } (-\infty, T) \times \mathbb{R}^{d-1}. \end{cases} \tag{4.1}$$

We prove an auxiliary estimate, which is derived from a result in [19]. For a weight $\omega \in A_q(\mathbb{R})$, we denote in the following $L_{q,\omega}(\mathbb{R} \times \mathbb{R}^d_+) := L_q(\mathbb{R}, \omega; L_q(\mathbb{R}^d_+))$.

Lemma 4.1. Let $T \in (-\infty, +\infty]$, $q \in (1, \infty)$, and $\omega \in A_q(-\infty, T)$. Let A_0 and $\overline{B}_j$ be as above. Assume that, for some $\theta \in (0, \pi/2)$, $(A_0, \overline{B}_j)$ satisfies the $(LS)_\theta$-condition. Then for any $f \in L_{q,\omega}((-\infty, T) \times \mathbb{R}^d_+)$ and

$$g_j \in W_{q,\omega}^{k_j, 2mk_j}\left((-\infty, T) \times \mathbb{R}^{d-1}\right)$$

with $j \in \{1, \ldots, m\}$, $m_j \in \{0, \ldots, 2m-1\}$, $k_j = 1 - m_j/(2m) - 1/(2mq)$, and $u \in W_{q,\omega}^{1,2m}((-\infty, T) \times \mathbb{R}^d_+)$ satisfying (4.1) with $\lambda \geq 0$, we have

$$\|u_t\|_{L_{q,\omega}((-\infty,T)\times\mathbb{R}^d_+)} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{q,\omega}((-\infty,T)\times\mathbb{R}^d_+)}$$

$$\leq C\|f\|_{L_{q,\omega}((-\infty,T)\times\mathbb{R}^d_+)} + C\sum_{j=1}^m \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}((-\infty,T)\times\mathbb{R}^{d-1})}, \tag{4.2}$$

with $C = C(\theta, m, d, K, q, b_{j\beta}, [\omega]_q) > 0$. Moreover, for any $\lambda > 0$,

$$f \in L_{q,\omega}\left((-\infty, T) \times \mathbb{R}^d_+\right) \quad \text{and} \quad g_j \in W_{q,\omega}^{k_j, 2mk_j}\left((-\infty, T) \times \mathbb{R}^{d-1}\right)$$

with j, m_j, and k_j as above, there exists a unique solution $u \in W_{q,\omega}^{1,2m}((-\infty, T) \times \mathbb{R}^d_+)$ to (4.1).

Proof.   Consider first $T = \infty$. For any $\omega \in A_q(\mathbb{R})$, let $u \in W_{q,\omega}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}^d_+)$ be a solution to (4.1).

Decompose $u = v + w$, where

- $w \in W^{1,2m}_{q,\omega}(\mathbb{R}^{d+1}_+)$ is the solution to the inhomogeneous problem

$$\begin{cases} w_t + (A_0 + \lambda)w = f & \text{in } \mathbb{R} \times \mathbb{R}^d_+ \\ \overline{B}_j w\big|_{x_1=0} = 0 & \text{on } \partial\mathbb{R}^{d+1}_+, \ j = 1, \ldots, m \end{cases} \tag{4.3}$$

- $v \in W^{1,2m}_{q,\omega}(\mathbb{R}^{d+1}_+)$ is the solution to the homogeneous problem

$$\begin{cases} v_t + (A_0 + \lambda)v = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d_+ \\ \overline{B}_j v\big|_{x_1=0} = g_j(t,x) & \text{on } \mathbb{R} \times \mathbb{R}^{d-1}, \ j = 1, \ldots, m. \end{cases} \tag{4.4}$$

It follows directly from [6, Theorem 3.5 (i)] with $p = q$ that the solution $w \in W^{1,2m}_{q,\omega}(\mathbb{R}^{d+1}_+)$ of (4.3) satisfies

$$\|w_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha| \leq 2m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha w\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \leq C \|f\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \tag{4.5}$$

with $C = C(\theta, K, d, m, q, b_{j\beta}, [\omega]_q)$.

Consider now (4.4). Since A_0 and $\overline{B}_j$ have constant coefficients, using a scaling $t \to \lambda^{-1}t$, $x \to \lambda^{-1/2m}x$, for a general $\lambda \in (0,1)$, we get that $\tilde{v}(t,x) := v(\lambda^{-1}t, \lambda^{-1/2m}x)$ satisfies

$$\begin{cases} \tilde{v}_t(t,x) + (1 + A_0)\tilde{v}(t,x) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d_+ \\ \overline{B}_j \tilde{v}(t,x)\big|_{x_1=0} = \tilde{g}_j(t,x) & \text{on } \mathbb{R} \times \mathbb{R}^{d-1}, \end{cases} \tag{4.6}$$

where

$$\tilde{g}_j(t,x) = \lambda^{-m_j/2m} g_j(\lambda^{-1}t, \lambda^{-1/2m}x).$$

Note that $\tilde{\omega}(t) := \omega(\lambda^{-1}t) \in A_q(\mathbb{R})$ and $[\tilde{\omega}]_q = [\omega]_q$. Applying [19, Lemma 6.6] to (4.6) with $p = q$ and $\gamma = 0$, we get that the solution $\tilde{v} \in W^{1,2m}_{q,\omega}(\mathbb{R}^{d+1}_+)$ to (4.6) satisfies

$$\|\tilde{v}_t\|_{L_{q,\tilde{\omega}}(\mathbb{R}^{d+1}_+)} + \|D^{2m}\tilde{v}\|_{L_{q,\tilde{\omega}}(\mathbb{R}^{d+1}_+)} \leq C \sum_{j=1}^m \|\tilde{g}_j\|_{W^{k_j,2mk_j}_{q,\tilde{\omega}}(\mathbb{R} \times \mathbb{R}^{d-1})}$$

with $C = C(\theta, m, d, K, q, b_{j\beta}, [\omega]_q)$. We remark that although the estimate is not explicitly stated in this reference, it can be extracted from the proof there.

Now scaling back and using Definition 2.2, it is easily seen that

$$\|v_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \|D^{2m}v\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \le C \sum_{j=1}^{m} \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\mathbb{R}\times\mathbb{R}^{d-1})},$$

with the constant C independent of $\lambda \in (0,1)$. Sending $\lambda \to 0$, we obtain that the above estimate holds when $\lambda = 0$. Finally, by applying an argument of S. Agmon as in [17, Theorem 4.1], from the above estimate with $\lambda = 0$ it follows that when $\lambda > 0$,

$$\|v_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha v\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \le C \sum_{j=1}^{m} \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\mathbb{R}\times\mathbb{R}^{d-1})} \tag{4.7}$$

with constant $C = C(\theta, m, d, K, q, b_{j\beta}, [\omega]_q)$. Since $u = w + v$, by (4.5) and (4.7) we get (4.2) with $T = \infty$ and $C = C(\theta, m, d, K, q, b_{j\beta}, [\omega]_q) > 0$. The solvability follows directly by the solvability argument in [6, Sec. 6] or the argument in [19, Lemma 6.6].

The proof for $T < \infty$ follows now the lines of [6, Lemma 4.1], so we omit the details. ∎

5   Proof of Theorem 3.2 when $\Omega = \mathbb{R}^d_+$

In this section, we prove Theorem 3.2 in the special case when $\Omega = \mathbb{R}^d_+$. The proof is divided into several steps. From Steps 1 to 3, we will assume $p = q \in (1,\infty)$ and we will show that the estimate (3.3) holds in this case. In Step 4, we will extrapolate the estimate from the previous steps to the case $p \ne q$ and complete the proof.

Proof of Theorem 3.2 when $\Omega = \mathbb{R}^d_+$.   It suffices to consider $T = \infty$. For the general case when $T \in (-\infty, \infty]$, we can follow the proof of [6, Lemma 4.1], so we omit the details.

Recall that the lower-order coefficients in A are bounded by K. By moving the terms $a_\alpha(t,x)D^\alpha u$ with $|\alpha| < 2m$ to the right-hand side of the equation and taking a sufficiently large λ, we may assume the lower-order coefficients of A to be all zero.

Denote

$$\tilde{Q}_r(t_0, x_0) = [t_0 - r^{2m}, t_0 + r^{2m}) \times C^+_{2r}(x_0),$$

where $C^+_{2r}(x_0)$ denotes a cube centered in x_0 having side-length $2r$ and axes parallel to the coordinate axes, intersected with the half space $\mathbb{R}^d_+$. As before, we use $\tilde{Q}_r$ to indicate $\tilde{Q}_r(0,0)$.

Let R be a large constant to be specified.

Step 1. We first consider the case $p = q$. We assume that there exists a constant $\Lambda_0 \geq 1$ such that $[\omega]_q \leq \Lambda_0$ and we assume that u is supported in $\overline{\mathbb{R}_+^{d+1}} \setminus \tilde{Q}_R$. Fix a point $(t_0, x_0) \in \overline{\mathbb{R}_+^{d+1}} \setminus \tilde{Q}_R$ and set

$$A(t_0, x_0)u = \sum_{|\alpha| = 2m} a_\alpha(t_0, x_0) D^\alpha u.$$

Decompose $u = u_1 + u_2$, where u_1 is a solution to

$$
\begin{cases}
\partial_t u_1 + (\lambda + A(t_0, x_0))u_1 = 0 & \text{in } \mathbb{R}_+^{d+1} \\[2mm]
\displaystyle\sum_{|\beta| = m_j} \bar{b}_{j\beta} D^\beta u_1 = -\sum_{|\beta| < m_j} b_{j\beta} D^\beta u \\[3mm]
\qquad + \displaystyle\sum_{|\beta| = m_j} (\bar{b}_{j\beta} - b_{j\beta}) D^\beta u + g_j & \text{on } \partial\mathbb{R}_+^{d+1},
\end{cases}
\tag{5.1}
$$

and u_2 is a solution to

$$
\begin{cases}
\partial_t u_2 + (\lambda + A)u_2 = f - (A - A(t_0, x_0))u_1 & \text{in } \mathbb{R}_+^{d+1} \\[2mm]
\displaystyle\sum_{|\beta| = m_j} \bar{b}_{j\beta} D^\beta u_2 = 0 & \text{on } \partial\mathbb{R}_+^{d+1}.
\end{cases}
\tag{5.2}
$$

By Lemma 4.1, we first solve (5.1). It follows from Lemma 2.4 that

$$
\|\partial_t u_1\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha| \leq 2m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}
$$

$$
\leq C \Big\| \sum_{|\beta| < m_j} b_{j\beta} D^\beta u \Big\|_{W_{q,\omega}^{\frac{2m-m_j}{2m}, 2m-m_j}(\mathbb{R}_+^{d+1})} + C \sum_{j=1}^m \|g_j\|_{W_{q,\omega}^{k_j, 2mk_j}(\partial\mathbb{R}_+^{d+1})}
$$

$$
+ C \Big\| \sum_{|\beta| = m_j} (\bar{b}_{j\beta} - b_{j\beta}) D^\beta u \Big\|_{W_{q,\omega}^{\frac{2m-m_j}{2m}, 2m-m_j}(\mathbb{R}_+^{d+1})}.
$$

Since $b_{j\beta}(t, x) \to \bar{b}_{j\beta}$ for $|t| + |x| \to \infty$, given $\varepsilon > 0$ and taking $R > 0$ large enough it holds that

$$
\sup_{(t,x) \in \overline{\mathbb{R}_+^{d+1}} \setminus \tilde{Q}_R} |\bar{b}_{j\beta} - b_{j\beta}(t, x)| < \varepsilon.
$$

This and the parabolic interpolation inequality yield that

$$
\begin{aligned}
&\|\partial_t u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le C \sum_{j=1}^m \|g_j\|_{W^{k_j,2mk_j}_{q,\omega}(\partial\mathbb{R}^{d+1}_+)} + C_K\Big(\varepsilon\|D^{2m}u\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \varepsilon\|u_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C_\varepsilon\|u\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}\Big).
\end{aligned}
$$

$$(5.3)$$

Now, [6, Theorem 3.5] with $w = 1$ applied to (5.2) yields that for $\lambda \ge \lambda_0$, where $\lambda_0 = \lambda_0(\theta, m, d, K, q, R_0, [\omega]_q, b_{j\beta}) > 0$, it holds that

$$
\begin{aligned}
&\|\partial_t u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le C\|f\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C\|(A - A(t_0, x_0))u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le C\|f\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C_K \sum_{|\alpha|\le 2m} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}
\end{aligned}
$$

$$(5.4)$$

provided that $\rho \le \bar{\rho}$, where $\bar{\rho} > 0$ is a constant depending only on θ, m, d, K, q, $[\omega]_q$, and $\bar{b}_{j\beta}$. Since $u = u_1 + u_2$, by (5.3) and (5.4), it follows that

$$
\begin{aligned}
&\|u_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le \|\partial_t u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\quad + \|\partial_t u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le \|\partial_t u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\quad + C_K \sum_{|\alpha|\le 2m} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C\|f\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \\
&\le C\|f\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C_K\|g_j\|_{W^{k_j,2mk_j}_{q,\omega}(\partial\mathbb{R}^{d+1}_+)} \\
&\quad + C_K\Big(\varepsilon\|D^{2m}u\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \varepsilon\|u_t\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C_\varepsilon\|u\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}\Big).
\end{aligned}
$$

Now taking ε small enough so that $C_K\varepsilon \le 1/2$ and $\lambda \ge \bar{\lambda} := \max\{\lambda_0, 2C_K C_\varepsilon\}$, we get (3.3) for u with support in $\overline{\mathbb{R}^{d+1}_+} \setminus \tilde{Q}_R$.

   Step 2. Let ε be a small constant to be specified. For any $(t_0, x_0) \in \overline{\tilde{Q}_{R+1}}$, by the stability of the $(LS)_\theta$-condition (see for instance [3, Remark 7.10]) and the

continuity of $b_{j\beta}$, there exists $r_{t_0,x_0} \in (0, R_0)$ such that the $(LS)_\theta$-condition is satisfied by $(A(t,x), B_j(t_0,x_0))$ for any $(t,x) \in \overline{\tilde{Q}_{r_{t_0,x_0}}(t_0,x_0)}$ and

$$\sup_{(t,x)\in\overline{\tilde{Q}_{r_{t_0,x_0}}(t_0,x_0)}} |b_{j\beta}(t_0,x_0) - b_{j\beta}(t,x)| < \varepsilon.$$

Assume that u is supported on $\tilde{Q}_{\kappa_{t_0,x_0}^{-2} r_{t_0,x_0}}(t_0,x_0)$, where κ_{t_0,x_0} is a large constant to be determined later. We only focus on the case when $x_0^1 \leq R_0$. The interior case $x_0^1 > R_0$ follows directly by [8, Sec. 5], since in this case there are no boundary conditions involved.

Similarly, we decompose $u = u_1 + u_2$, where u_1 is a solution to

$$\begin{cases} \partial_t u_1 + (\lambda + A(t_0,x_0))u_1 = 0 & \text{in } \mathbb{R}_+^{d+1} \\ \displaystyle\sum_{|\beta|=m_j} b_{j\beta}(t_0,x_0)D^\beta u_1 = -\sum_{|\beta|<m_j} b_{j\beta}(t,x)D^\beta u \\ \qquad + \displaystyle\sum_{|\beta|=m_j} (b_{j\beta}(t_0,x_0) - b_{j\beta}(t,x))D^\beta u + g_j & \text{on } \partial\mathbb{R}_+^{d+1}, \end{cases} \tag{5.5}$$

and u_2 is a solution to

$$\begin{cases} \partial_t u_2 + (\lambda + A)u_2 = f - (A - A(t_0,x_0))u_1 =: h & \text{in } \mathbb{R}_+^{d+1} \\ \displaystyle\sum_{|\beta|=m_j} b_{j\beta}(t_0,x_0)D^\beta u_2 = 0 & \text{on } \partial\mathbb{R}_+^{d+1}. \end{cases} \tag{5.6}$$

By Lemma 4.1, we first solve (5.5). It follows from Lemma 2.4 that

$$\|\partial_t u_1\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$\leq C \Big\| \sum_{|\beta|<m_j} b_{j\beta}D^\beta u \Big\|_{W_{q,\omega}^{\frac{2m-m_j}{2m},2m-m_j}(\mathbb{R}_+^{d+1})} + C \sum_{j=1}^m \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\partial\mathbb{R}_+^{d+1})}$$

$$+ C \Big\| \sum_{|\beta|=m_j} (b_{j\beta}(t_0,x_0) - b_{j\beta})D^\beta u \Big\|_{W_{q,\omega}^{\frac{2m-m_j}{2m},2m-m_j}(\mathbb{R}_+^{d+1})}$$

$$\leq C \sum_{j=1}^m \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\partial\mathbb{R}_+^{d+1})} + C_K \Big(\varepsilon \|D^{2m}u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \varepsilon \|u_t\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + C_\varepsilon \|u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} \Big). \tag{5.7}$$

where in the last estimate we used the parabolic interpolation inequality and the smoothness assumption of the coefficients $b_{j\beta}$.

In order to deal with (5.6), we exploit the property that u has a small support. We shall first establish mean oscillation estimates in

$$\mathcal{X} := \tilde{Q}_{\kappa_{t_0,x_0}^{-1} r_{t_0,x_0}}(t_0, x_0).$$

For this, we take a dyadic decomposition of $\mathcal{X}$ given by

$$\mathbb{C}_n = \big\{ Q^n = Q_i^n : i = (i_0, i_1, \ldots, i_d) \in \mathbb{Z}^{d+1},$$
$$i_0 = 0, \ldots, 2^{nm} - 1, i_k = 0, \ldots, 2^n - 1, k = 1, \ldots, d \big\},$$

where $n \in \mathbb{Z}$, and for $x_0 = (x_0^1, x_0') \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$ and $r_0 := \kappa_{t_0,x_0}^{-1} r_{t_0,x_0}$,

$$Q_i^n = \{t_0 - r_0^{2m} + r_0^{2m} 2^{-nm+1}([0,1) + i_0)\}$$
$$\times \big\{ \max(0, x_0^1 - r_0) + \min(r_0, x_0^1) 2^{-n}([0,1) + i_1) \big\}$$
$$\times \big\{ x_0' - r_0(1, \ldots, 1) + r_0 2^{-n+1}([0,1)^{d-1} + (i_2, \ldots, i_d)) \big\}.$$

Then for any $X \in \mathcal{X}$ and $Q^n \in C_n$ such that $X \in Q^n$, one can find $X_0 \in \mathcal{X}$ and the smallest $r \in (0, R_0)$ such that $Q^n \subset Q_r^+(X_0)$ and

$$\fint_{Q^n} |f(Y) - f_{|n}(X)| \, dY \le C \fint_{Q_r^+(X_0)} |f(Y) - (f)_{Q_r^+(X_0)}| \, dY \tag{5.8}$$

with $C = C(d, m)$, where $f_{|n}(X) = \fint_{Q^n} f(Y) \, dY$. For $x \in \mathcal{X}$, we define the dyadic sharp function of f by

$$f_{\mathrm{dy}}^\sharp(X) := \sup_{n < \infty} \fint_{Q^n \ni X} |f(Y) - f_{|n}(X)| \, dY.$$

We also define the parabolic maximal function of a function $f \in L_{1,\mathrm{loc}}(\mathbb{R}_+^{d+1})$ by

$$\mathcal{M}f(X) = \sup_{\substack{Q \in \mathcal{Q} \\ X \in Q}} \fint_Q |f(Y)| \, dY,$$

where

$$\mathcal{Q} = \Big\{ Q_r^+(t, x) : (t, x) \in \overline{\mathbb{R}_+^{d+1}}, \ r \in (0, \infty) \Big\}.$$

Recall that there is a positive number $\sigma_1 = \sigma_1(q, [\omega]_q)$ such that $q - \sigma_1 > 1$ and $\omega \in A_{q-\sigma_1}(\mathbb{R})$. We take $q_0, \mu \in (1, q)$ satisfying $q_0\mu = \frac{q}{q-\sigma_1} > 1$. Then it holds that

$$\omega \in A_{q-\sigma_1}(\mathbb{R}) = A_{q/(q_0\mu)}(\mathbb{R}) \subset A_{q/q_0}(\mathbb{R}). \tag{5.9}$$

Since u_2 satisfies (5.6), by (5.8) and the mean oscillation estimates of [6, Lemma 4.6] with $\kappa = \kappa_{t_0,x_0}$ and μ, ς satisfying $\frac{1}{\mu} + \frac{1}{\varsigma} = 1$,

$$\fint_{Q^n} |\partial_t u_2(Y) - (\partial_t u_2)_{|n}(X)| \, dY + \sum_{\substack{|\alpha| \leq 2m \\ \alpha_1 < 2m}} \lambda^{1-\frac{|\alpha|}{2m}} \fint_{Q^n} |D^\alpha u_2(Y) - (D^\alpha u_2)_{|n}(X)| \, dY$$

$$\leq C\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha u_2|^{q_0})_{Q^+_{\kappa_{t_0,x_0}r}(X_0)}^{\frac{1}{q_0}} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} (|h|^{q_0})_{Q^+_{\kappa_{t_0,x_0}r}(X_0)}^{\frac{1}{q_0}}$$

$$+ C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q_0\varsigma}} (|D^{2m} u_2|^{q\mu})_{Q^+_{\kappa_{t_0,x_0}r}(X_0)}^{\frac{1}{q_0\mu}}.$$

Taking the supremum with respect to all $Q^n \ni X$, $n \in \mathbb{Z}$, we see that for all $X \in \mathcal{X}$,

$$(\partial_t u_2)^\sharp_{\mathrm{dy}}(X) + \sum_{|\alpha| \leq 2m, \alpha_1 < 2m} \lambda^{1-\frac{|\alpha|}{2m}} (D^\alpha u_2)^\sharp_{\mathrm{dy}}(X)$$

$$\leq C\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} [\mathcal{M}(|D^\alpha u_2|^{q_0})(X)]^{\frac{1}{q_0}} \tag{5.10}$$

$$+ C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} [\mathcal{M}(|h|^{q_0})(X)]^{\frac{1}{q_0}} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q_0\varsigma}} [\mathcal{M}(|D^{2m} u_2|^{q_0\mu})(X)]^{\frac{1}{q_0\mu}}.$$

By taking the $L_{q,\omega}(\mathbb{R}^{d+1}_+)$-norms on both sides of (5.10) and applying Theorem 2.3 of [9] with $p = q$, we get for $C = C(\theta, d, m, K, q, [\omega]_q, b_{j\beta}, t_0, x_0)$,

$$\|\partial_t u_2\|_{L_{q,\omega}(\mathcal{X})} + \sum_{|\alpha| \leq 2m, \alpha_1 < 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_{q,\omega}(\mathcal{X})}$$

$$\leq C|I|^{-1} (\omega(I))^{1/q} \left(\|\partial_t u_2\|_{L_1(\mathcal{X})} + \sum_{|\alpha| \leq 2m, \alpha_1 < 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_1(\mathcal{X})} \right)$$

$$+ C\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \|h\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q_0\varsigma}} \|D^{2m} u_2\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)},$$

where $I := (t_0 - r_{t_0,x_0}^{2m}, t_0 + r_{t_0,x_0}^{2m})$ and we also used (5.9) and the weighted Hardy–Littlewood maximal function theorem to get, for instance,

$$\big\|[\mathcal{M}(D^{2m}u_2)^{q_0\mu}]^{\frac{1}{q_0\mu}}\big\|_{L_{q,\omega}(\mathcal{X})} = \big\|\mathcal{M}(D^{2m}u_2)^{q_0\mu}\big\|_{L_{q/(q_0\mu),\omega}(\mathbb{R}_+^{d+1})}^{\frac{1}{q_0\mu}}$$

$$\leq C\big\|(D^{2m}u_2)^{q_0\mu}\big\|_{L_{q/(q_0\mu),\omega}(\mathbb{R}_+^{d+1})}^{\frac{1}{q_0\mu}} = C\big\|D^{2m}u_2\big\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

with $C = C(d, q, [\omega]_q)$. Since

$$a_{\tilde{\alpha}\tilde{\alpha}}(t,x)D_{x_1}^{2m}u_2 = h - \partial_t u_2 - \sum_{|\alpha|=2m,\alpha_1<2m} a_\alpha(t,x)D^\alpha u_2 - \lambda u_2,$$

where $\tilde{\alpha} = (m, 0, \dots, 0)$, we get

$$\|\partial_t u_2\|_{L_{q,\omega}(\mathcal{X})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_2\|_{L_{q,\omega}(\mathcal{X})}$$

$$\leq C|I|^{-1}(\omega(I))^{1/q}\bigg(\|\partial_t u_2\|_{L_1(\mathcal{X})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_2\|_{L_1(\mathcal{X})}\bigg)$$

$$+ C\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})}\sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\big\|D^\alpha u_2\big\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\big\|h\big\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$+ C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\rho^{\frac{1}{q_0\varsigma}}\big\|D^{2m}u_2\big\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}. \tag{5.11}$$

Because $u_2 = u - u_1$, by applying the triangle inequality and Hölder's inequality, we estimate the 1st term on the right-hand side of (5.11) by

$$C|I|^{-1}(\omega(I))^{1/q}\bigg(\|\partial_t u_2\|_{L_1(\mathcal{X})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_2\|_{L_1(\mathcal{X})}\bigg)$$

$$\leq C|I|^{-1}(\omega(I))^{1/q}\bigg(\|\partial_t u\|_{L_1(\mathcal{X})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_1(\mathcal{X})}$$

$$+ \|\partial_t u_1\|_{L_1(\mathcal{X})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_1\|_{L_1(\mathcal{X})}\bigg)$$

$$\leq C|I|^{-1}(\omega(I))^{1/q}|I_1|(\omega(I_1))^{-1/q}\bigg(\|\partial_t u\|_{L_{q,\omega}(\mathcal{X})}$$

$$+ \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{q,\omega}(\mathcal{X})}\bigg) + C\|\partial_t u_1\|_{L_{q,w}(\mathcal{X})}$$

$$+ C\sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_1\|_{L_{q,w}(\mathcal{X})}, \tag{5.12}$$

where

$$I_1 := \left(t_0 - \left(\kappa_{t_0,x_0}^{-2} r_{t_0,x_0} \right)^{2m}, t_0 + \left(\kappa_{t_0,x_0}^{-2} r_{t_0,x_0} \right)^{2m} \right),$$

and we used the fact that u is supported on $\tilde{Q}_{\kappa_{t_0,x_0}^{-2} r_{t_0,x_0}}(t_0, x_0)$. Using (2.2),

$$|I|^{-1}(\omega(I))^{\frac{1}{q}} |I_1| (\omega(I_1))^{-\frac{1}{q}} \leq C(|I_1|/|I|)^{\frac{\sigma_1}{q}} \leq C\kappa_{t_0,x_0}^{-\frac{4m\sigma_1}{q}}. \tag{5.13}$$

By the triangle inequality, we estimate the remaining terms on the right-hand side of (5.11) by

$$C \left(\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} + \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q\varsigma}} \right) \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u_2 \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \left\| h \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$\leq C \left(\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} + \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q\varsigma}} \right) \left(\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \right.$$

$$\left. + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u_1 \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \right) + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \left\| h \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}. \tag{5.14}$$

Since $u = u_1 + u_2$ and $h = f - (A - A(t_0, x_0))u_1$, from (5.11), (5.12), (5.13), and (5.14) it follows that

$$\left\| u_t \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$\leq C \left\| \partial_t u_1 \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u_1 \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \left\| f \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C \left(\kappa_{t_0,x_0}^{-\left(1-\frac{1}{q_0}\right)} + \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \rho^{\frac{1}{q_0\varsigma}} + \kappa_{t_0,x_0}^{-\frac{4m\sigma_1}{q}} \right) \cdot$$

$$\left(\left\| \partial_t u \right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha| \leq 2m,} \lambda^{1-\frac{|\alpha|}{2m}} \| D^\alpha u \|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \right),$$

which combined with (5.7) yields

$$\|u_t\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$\leq C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \sum_{j=1}^{m} \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\partial\mathbb{R}_+^{d+1})} + C\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \|f\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$+ C_K C_\varepsilon \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}} \|u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + C\left(\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} + \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\rho^{\frac{1}{q_0\varsigma}} + \kappa_{t_0,x_0}^{-\frac{4m\sigma_1}{q}} + \varepsilon\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\right) \cdot$$

$$\left(\|\partial_t u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha|\leq 2m,} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}\right).$$

Now we take κ_{t_0,x_0} sufficiently large, ε sufficiently small, and then $\rho \leq \rho_{t_0,x_0}$ sufficiently small such that

$$C\left(\kappa_{t_0,x_0}^{-(1-\frac{1}{q_0})} + \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\rho^{\frac{1}{q_0\varsigma}} + \kappa_{t_0,x_0}^{-\frac{4m\sigma_1}{q}} + \varepsilon\kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\right) \leq 1/2,$$

and finally take $\lambda \geq \lambda_{t_0,x_0} := \max\{\lambda_0, 2C_K C_\varepsilon \kappa_{t_0,x_0}^{\frac{d+2m}{q_0}}\}$, we get (3.3) for u with support in $\tilde{Q}_{\kappa_{t_0,x_0}^{-2} r_{t_0,x_0}}(t_0,x_0)$ and

$$C = C(\theta, d, m, K, q, b_{j\beta}, t_0, x_0).$$

Observe that C, ρ_{t_0,x_0}, and λ_{t_0,x_0} all depend on (t_0,x_0). However, since $\overline{\tilde{Q}_{R+1}}$ is compact, we can apply a partition of the unity argument and get a uniform constant C. This will be done in the next step.

Step 3. Since $\overline{\tilde{Q}_{R+1}}$ is compact and

$$\overline{\tilde{Q}_{R+1}} \subset \bigcup_{(t_0,x_0)\in\overline{\tilde{Q}_{R+1}}} \tilde{Q}_{\kappa_{t_0,x_0}^{-2} r_{t_0,x_0}/2}(t_0,x_0),$$

there exists a finite number $N \in \mathbb{N}$ of points $(t_{0,i},x_{0,i}) \in \overline{\tilde{Q}_{R+1}}$, $i = 1, \ldots, N$ such that

$$\overline{\tilde{Q}_{R+1}} \subset \bigcup_{i=1}^{N} \tilde{Q}_{\kappa_{t_{0,i},x_{0,i}}^{-2} r_{t_{0,i},x_{0,i}}/2}(t_{0,i},x_{0,i}).$$

Take $\zeta_i \in C_0^\infty(\overline{\tilde{Q}_{\kappa_{t_{0,i},x_{0,i}}^{-2} r_{t_{0,i},x_{0,i}}}(t_{0,i}, x_{0,i})})$, $i = 1, \ldots, N$, such that $\zeta_i = 1$ on $\tilde{Q}_{\kappa_{t_{0,i},x_{0,i}}^{-2} r_{t_{0,i},x_{0,i}}/2}(t_{0,i}, x_{0,i})$, and $\zeta_0 \in C_0^\infty(\mathbb{R}_+^{d+1})$ such that

$$\zeta_0(x) = \begin{cases} 1 & x \in \overline{\mathbb{R}_+^{d+1}} \setminus \overline{\tilde{Q}_{R+1}} \\ 0 & x \in \overline{\tilde{Q}_R}. \end{cases}$$

Let $\overline{\zeta} = \sum_{i=0}^N \zeta_i^q \geq 1$ in $\overline{\mathbb{R}_+^{d+1}}$. Define $\eta_i = \zeta_i(\overline{\zeta})^{-1/q}$. Then, $\sum_{i=0}^N \eta_i^q = 1$ in $\mathbb{R}_+^{d+1}$.

Now we define

$$u_i(t, x) = u(t, x)\eta_i(t, x).$$

Observe that

$$\begin{cases} \partial_t u_i + (A + \lambda)u_i = f_i & \text{in } \mathbb{R}_+^{d+1} \\ \mathcal{B}_j u_i\big|_{x_1=0} = g_{j,i} & \text{on } \partial\mathbb{R}_+^{d+1}, \, j = 1, \ldots, m, \end{cases} \tag{5.15}$$

where by Leibnitz's rule

$$f_i = f\eta_i + u(\eta_i)_t + \sum_{|\alpha|=2m} \sum_{|\gamma| \leq 2m-1} \binom{\alpha}{\gamma} a_\alpha(t, x) D^\gamma u D^{\alpha-\gamma} \eta_i$$

and

$$g_{j,i} = g_j\eta_i + \sum_{1 \leq |\beta| \leq m_j} \sum_{|\tau| \leq |\beta|-1} \binom{\beta}{\tau} b_{j\beta}(t, x) D^\tau u D^{\beta-\tau} \eta_i\big|_{x_1=0}.$$

Now applying the result in *Step 2* to (5.15) we get for $i = 1, \ldots, N$,

$$\left\| (u_i)_t \right\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u_i \right\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$\leq C_i \left\| f_i \right\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + C_i \sum_{j=1}^m \left\| g_{j,i} \right\|_{W_{q,\omega}^{k_j, 2mk_j}(\partial\mathbb{R}_+^{d+1})}$$

with $C_i = C(\theta, d, m, K, q, [\omega]_q, b_{j\beta}, t_{0,i}, x_{0,i})$, provided that $\lambda \geq \lambda_{t_{0,i},x_{0,i}}$ and $\rho \leq \rho_{t_{0,i},x_{0,i}}$. Applying the result in *Step 1* to (5.15) with $i = 0$ we get a similar inequality, with

$C_0 = C(\theta, d, m, K, q, [\omega]_q, \bar{b}_{j\beta})$. Observe that by the triangle inequality

$$\left\|\eta_i u_t\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\|\eta_i D^\alpha u\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$\leq \left\|(u_i)_t\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \left\|u(\eta_i)_t\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\|D^\alpha u_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ \sum_{|\alpha|=2m} \sum_{|\gamma|\leq 2m-1} \binom{\alpha}{\gamma} \lambda^{1-\frac{|\alpha|}{2m}} \left\|D^\gamma u D^{\alpha-\gamma} \eta_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)},$$

$$\left\|f_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} \leq \left\|f\eta_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \left\|u(\eta_i)_t\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ C_K \sum_{|\alpha|=2m} \sum_{|\gamma|\leq 2m-1} \binom{\alpha}{\gamma} \left\|D^\gamma u D^{\alpha-\gamma} \eta_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)},$$

and

$$\left\|g_{j,i}\right\|_{W^{k_j,2mk_j}_{q,\omega}(\partial\mathbb{R}^{d+1}_+)} \leq \left\|g_j\eta_i\right\|_{W^{k_j,2mk_j}_{q,\omega}(\partial\mathbb{R}^{d+1}_+)}$$

$$+ C_K \sum_{1\leq|\beta|\leq m_j} \sum_{|\tau|\leq|\beta|-1} \binom{\beta}{\tau} \left\|D^\tau u D^{\beta-\tau} \eta_i\right\|_{W^{\frac{2m-m_j}{2m},2m-m_j}_{q,\omega}(\mathbb{R}^{d+1}_+)},$$

where we used the boundedness of the coefficients a_α and $b_{j\beta}$. After taking the q-th power, and summing in $i = 0, 1, \ldots, N$, letting

$$C = C_0 + \sup_{i=1,\ldots,N} C(\theta, d, m, K, q, [\omega]_q, b_{j\beta}, t_{0,i}, x_{0,i})$$

and taking the q-th root, we get

$$\left\|u_t\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\|D^\alpha u\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$\leq C\left\|f\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)} + C \sum_{j=1}^{m} \left\|g_j\right\|_{W^{k_j,2mk_j}_{q,\omega}(\partial\mathbb{R}^{d+1}_+)} + C\left\|u\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ C_K \sum_{|\alpha|=2m} \sum_{|\gamma|\leq 2m-1} \binom{\alpha}{\gamma} \lambda^{1-\frac{|\alpha|}{2m}} \left\|D^\gamma u D^{\alpha-\gamma} \eta_i\right\|_{L_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

$$+ C_K \sum_{1\leq|\beta|\leq m_j} \sum_{|\tau|\leq|\beta|-1} \binom{\beta}{\tau} \lambda^{1-\frac{|\tau|}{m_j}} \left\|D^\tau u\right\|_{W^{\frac{2m-m_j}{2m},2m-m_j}_{q,\omega}(\mathbb{R}^{d+1}_+)}$$

with C uniform in $t_{0,i}, x_{0,i}$, provided that

$$\lambda \geq \lambda' := \max\{\bar{\lambda}, \lambda_{t_{0,i},x_{0,i}} : i = 1, \ldots, N\},$$

$$\rho \leq \rho' := \min\{\bar{\rho}, \rho_{t_{0,i},x_{0,i}} : i = 1, \ldots, N\}.$$

This, combined with interpolation estimates and taking ε small and λ large, gives (3.3) with $p = q$ and $\omega \in A_q(\mathbb{R})$ such that $[\omega]_q \leq \Lambda_0$, that is,

$$\|u_t\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})}$$

$$\leq C\|f\|_{L_{q,\omega}(\mathbb{R}_+^{d+1})} + C \sum_{j=1}^m \|g_j\|_{W_{q,\omega}^{k_j,2mk_j}(\partial\mathbb{R}_+^{d+1})}, \tag{5.16}$$

where $k_j = 1 - m_j/(2m) - 1/(2mq)$ and $C = C(\theta, m, d, K, q, \Lambda_0, b_{j\beta}) > 0$.

Step 4. We now extrapolate the estimate from the previous step to $p \neq q$. By (5.16) and Definition 2.2, we have that for all $\omega \in A_q(\mathbb{R})$ such that $[\omega]_{A_q} \leq \Lambda_0$ there exist constants $\lambda', \rho', C > 0$ depending on Λ_0 such that for any $\lambda \geq \lambda'$ and $\rho \leq \rho'$,

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|U_\alpha\|_{L_q(\mathbb{R},\omega)}$$

$$\leq C\|F\|_{L_q(\mathbb{R},\omega)} + C \sum_{j=1}^m \|G_{j,1}\|_{L_q(\mathbb{R},\omega)} + C \sum_{j=1}^m \|G_{j,2}\|_{L_q(\mathbb{R},\omega)},$$

where

$$U_\alpha = \|D^\alpha u\|_{L_q(\mathbb{R}_+^d)}, \quad F = \|f\|_{L_q(\mathbb{R}_+^d)},$$

$$G_{j,1} = \|2^{kk_j}\mathcal{F}^{-1}(\widehat{\varphi}_k\hat{g}_j)_{k\geq 0}\|_{\ell_q(L_q(\mathbb{R}^{d-1}))}, \quad G_{j,2} = \|g_j\|_{\mathcal{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1})}.$$

Since the above estimate holds for all of the A_q weights with uniformly bounded A_q-constant, ρ' and λ' can be chosen uniformly. Therefore, by the extrapolation result Theorem 2.1 it follows that, for all $\omega \in A_p$, there exist a constant C' depending on $[\omega]_p$ such that for all $\lambda \geq \lambda'$ and $\rho \leq \rho'$,

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|U_\alpha\|_{L_p(\mathbb{R},\omega)}$$

$$\leq C'\|F\|_{L_p(\mathbb{R},\omega)} + C' \sum_{j=1}^m \|G_{j,1}\|_{L_p(\mathbb{R},\omega)} + C' \sum_{j=1}^m \|G_{j,2}\|_{L_p(\mathbb{R},\omega)}.$$

This yields

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \left\| D^\alpha u \right\|_{L_p(\mathbb{R},\omega;L_q(\mathbb{R}^d_+))} \leq C \left\| f \right\|_{L_p(\mathbb{R},\omega;L_q(\mathbb{R}^d_+))}$$

$$+ C \sum_{j=1}^{m} \left\| g_j \right\|_{F_{p,q}^{k_j}(\mathbb{R},\omega;L_q(\mathbb{R}^{d-1})) \cap L_p(\mathbb{R},\omega;\mathcal{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1}))}$$

with $C = C(\theta, m, d, K, p, q, [\omega]_p, b_{j\beta})$. As $u_t = f - (\lambda + A)u$, the estimate (3.3) directly follows. Finally, the solvability follows from the a priori estimate (3.3), Lemma 4.1, and the method of continuity. The theorem is proved.   ∎

6 Proof of Theorem 3.2 for General $C^{2m-1,1}$ Domains

In this section, we complete the proof of Theorem 3.2 for general $C^{2m-1,1}$ domains by applying the technique of flattening the boundary.

Proof of Theorem 3.2.   For each $\hat{x}_0 \in \partial\Omega$, we find a coordinate system such that $\hat{x}_0$ is the origin and e_1 is the normal direction at $\hat{x}_0$. Since Ω is a $C^{2m-1,1}$ domain with the uniform $C^{2m-1,1}$-norm bounded by K, there is a function $\varphi \in C^{2m-1,1}(B'_{R_1})$ with $R_1 \in (0, R_0)$ independent of $\hat{x}_0$, where B'_{R_1} is the ball of radius R_1 centered at the origin in $\mathbb{R}^{d-1}$, such that $\|\varphi\|_{C^{2m-1,1}(B'_{R_1})} \leq K$,

$$\partial\Omega \cap B_{R_1} = \{x = (x_1, x') \in B_{R_1} : \ x_1 = \varphi(x')\},$$

and

$$\Omega \cap B_{R_1} = \{x \in B_{R_1} : \ x_1 > \varphi(x')\}.$$

Set

$$\Phi(x) := \begin{pmatrix} x_1 - \varphi(x') \\ x' \end{pmatrix}, \ \ x \in B,$$

$\Phi : B_{R_1} \rightarrow \Phi(B_{R_1})$, $x \mapsto y$. The differential operators A and B_j, $j = 1, \ldots, m$, are transformed into the operators

$$A^\Phi = \sum_{|\alpha| \leq 2m} a_\alpha^\Phi(t, y) D^\alpha, \ \ B_j^\Phi = \sum_{|\beta| \leq m_j} b_{j\beta}^\Phi(t, y) D^\beta$$

and act on functions defined on $\Phi(B_{R_1}) \cap \mathbb{R}^d_+$. As Φ is an isomorphism near the origin, the parameter ellipticity of A^Φ and, in particular, the condition $(E)_\theta$ are preserved under

the change of variables. Moreover, the transformed operators (A^Φ, B_j^Φ) satisfies the $(LS)_\theta$-condition on $\Phi(B_{R_1}) \cap \mathbb{R}_+^d$; see [33, Theorem 11.3]. Finally, it is easily seen that the leading coefficients of the new operator in the new coordinate system also satisfy Assumption 3.1 with a possibly different ρ, and the transformed function u^Φ satisfies

$$\begin{cases} \partial_t u^\Phi + (A^\Phi + \lambda) u^\Phi = f^\Phi & \text{in } (-\infty, T) \times \Phi(B_{R_1}) \cap \mathbb{R}_+^d \\ \text{tr}_{\mathbb{R}_+^d} B_j^\Phi u^\Phi = g_j^\Phi & \text{on } (-\infty, T) \times (\Phi(B_{R_1}) \cap \mathbb{R}^{d-1}), \ j = 1, \dots, m. \end{cases}$$

Therefore, the case when $p = q$ follows from the results in the previous section and a partition of the unity argument as in, for instance, [8, Theorem 6]. The general case is then derived from the case when $p = q$ and Theorem 2.1 as in Section 5. As before, the solvability follows from the a priori estimate and the method of continuity. The theorem is proved. ∎

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