Optimal Selection and Control in Monotone and Dynamic Games

Benenati, E.

DOI
10.4233/uuid:850417b5-9a30-4b22-82b1-d1475e2b1633

Publication date
2025

Document Version
Final published version

Citation (APA)
Benenati, E. (2025). *Optimal Selection and Control in Monotone and Dynamic Games*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:850417b5-9a30-4b22-82b1-d1475e2b1633

Optimal Selection and Control in Monotone and Dynamic Games

Emilio Benenati

Optimal Selection and Control in Monotone and Dynamic Games

Optimal Selection and Control in Monotone and Dynamic Games

Dissertation

for the purpose of obtaining the degree of doctor
at the Delft University of Technology,
by the authority of the Rector Magnificus, prof.dr.ir. T.H.J.J. van der Hagen,
chair of the Board for Doctorates,
to be defended publicly on
Wednesday, May 7, 2025, at 17:30

by

Emilio BENENATI

Master of Science in Robotics, Systems and Control,
Eidgenössische Technische Hochschule, Zürich, Switzerland
born in Catania, Italy.

This dissertation has been approved by the

promotor: dr.ir. S. Grammatico
promotor: prof.dr.ir. B. De Schutter

Composition of the doctoral committee:

Rector Magnificus,	chairperson
dr.ir. S. Grammatico,	Delft University of Technology, propotor
prof.dr.ir. B. De Schutter,	Delft University of Technology, promotor
Independent members:	
prof.dr.ir. T. Keviczky,	Delft University of Technology
prof.dr. T. Başar,	University of Illinois at Urbana-Champaign, USA
dr. K. Margellos,	University of Oxford, UK
prof.dr.ir. S.P. Hoogendoorn,	Delft University of Technology
Reserve member:	
prof.dr. M.M. de Weerdt,	Delft University of Technology

This research was supported by the European Research Council under research project COSMOS (802348).

Cover illustration by Adriano Di Mauro

An electronic version of this dissertation is available at
`http://repository.tudelft.nl`

This thesis has been completed in fulfillment of the requirements of the Dutch Institute of Systems and Control (DISC) for graduate study.

We step out of our solar system into the universe seeking only peace and friendship.
To teach, if called upon.
To be taught, if fortunate.

Kurt Wandheim,
as recorded on the Voyager golden record

Contents

Summary

Recent engineering developments have surrounded us with intelligent devices, which are required to autonomously take rational decisions while interacting with the physical world. These systems are increasingly widespread, interacting and interconnected, thus resulting in decision problems that involve multiple rational agents, generally with conflicting objectives and interrelated operating constraints. Currently relevant examples include autonomous driving, traffic routing, clearance of autonomous bidding markets, power consumption and production scheduling on the electricity grid, control of robotic swarms, and autonomous racing. The mathematical framework for formulating these problems is known as a game. Over the past decade, there has been significant progress in the development of algorithms that determine an action which is simultaneously rational (i.e. optimal) for each agent, namely, a generalized Nash equilibrium (GNE). This solution is particularly favorable as it is self-enforcing, in the sense that no decision maker can improve its payoff by unilaterally deviating from it. However, currently available algorithms for the computation of a GNE present numerous shortcomings, which limit their applicability to real-world non-cooperative decision processes and that we address in this thesis.

First of all, GNE problems typically admit multiple solutions, but currently available algorithms only compute an arbitrary, initialization-dependent GNE. In applications where a predictable and well-defined solution is necessary, it becomes important to select a specific GNE (among potentially infinitely many) that optimizes an arbitrary metric or a secondary, cooperative objective. We develop the first optimal GNE selection algorithms. We compare two different algorithm design methods, both developed under the framework of operator theory: the first, i.e. the hybrid steepest-descent method (HSDM), entails a gradient descent of the selection function with vanishing step size combined with a GNE seeking algorithm, while the second requires the solution of a sequence of Variational Inequalities (VIs) with a vanishing regularizing term. Both design methods lead to algorithms that are suitable to distributing the computation between a central node and the agents, and we include ad-hoc algorithms for the particular cases of aggregative and cocoercive games.

Secondly, we consider time-varying games, motivated by the need for algorithms that continuously monitor and control physical multiagent systems. In such games, the agents must track an evolving solution with limited computation time between the problem's updates. This scenario is particularly relevant when the agents are affected by disturbances whose time-scale is comparable to the algorithm convergence rate. The challenge lies in finding algorithms that exhibit fast convergence and a robustness property to external disturbances, both typically associated with a linear convergence rate. We derive and study the asymptotic tracking error of a fully-distributed algorithm (i.e., without a central coordinator) for GNE problems with linear equality constraints. For a time-varying GNE selection problem, we find the HSDM with constant stepsize to be linearly convergent to an approximate solution. We find that the approximation error can be controller by an appropriate choice of the stepsize and number of iterations per time step, and we derive a bound to the asymptotic tracking error.

Finally, again driven by the need of applying GNE seeking algorithms to the control of physical systems, we consider dynamic games, where the decision each agent has to take is a time sequence of inputs to a dynamical system. In this case, the coupling between the agents emerges not only through the objectives and constraints, but also through the system dynamics. Ideally, one should compute the GNE solution by predicting the dynamics over an indefinitely long horizon. This is typically computationally intractable, especially when constraints are present. We then approximate the infinite-horizon control sequence by recomputing at each time instant the solutions to a finite-time equilibrium problem, a method typically known as receding-horizon control (or model predictive control, in the single-agent case). We derive a novel characterization of the infinite horizon objective achieved by the Nash equilibrium trajectory, and we show that one can recover the infinite-horizon performance by including this expression in the agents' objectives as an additive terminal cost. With this result, we conclude asymptotic stability of the steady state under a receding-horizon game-theoretic control action. Compared to the literature, we do not assume stability of the uncontrolled plant, nor we introduce auxiliary constraints. Furthermore, we find that the asymptotic stability of the steady state can be obtained with a more generic terminal cost if the game is potential, as we demonstrate on a practical traffic routing application.

Samenvatting

Recente technologische ontwikkelingen hebben ons omringd met intelligente apparaten die zelfstandig rationele beslissingen moeten nemen tijdens interactie met de fysieke wereld. Deze systemen worden steeds wijdverspreider, reageren op elkaar en zijn onderling verbonden. Dit leidt tot besluitvormingsproblemen met meerdere rationele agenten die doorgaans conflicterende doelen en onderling afhankelijke operationele beperkingen hebben. Momenteel relevante voorbeelden omvatten autonoom rijden, verkeersroutering, afhandeling van autonome biedmarkten, planning van energieverbruik en energieproductie op het elektriciteitsnet, besturing van robotswarms en autonoom racen. Het wiskundige kader voor het formuleren van deze problemen staat bekend als een spel. In het afgelopen decennium is er aanzienlijke vooruitgang geboekt in de ontwikkeling van algoritmen die een actie kiezen die tegelijkertijd rationeel (d.w.z. optimaal) is voor elke agent, namelijk een generalized Nash equilibrium (GNE). Deze oplossing is bijzonder gunstig omdat deze zelfhandhavend is, in de zin dat geen enkele besluitvormer zijn opbrengst kan verbeteren door eenzijdig af te wijken. De momenteel beschikbare algoritmen voor het berekenen van een GNE hebben echter talrijke tekortkomingen, die hun toepasbaarheid op realistische niet-coöperatieve besluitvormingsprocessen beperken. Dit proefschrift gaat deze gebreken aanpakken.

Ten eerste hebben GNE-problemen doorgaans meerdere oplossingen, maar huidige algoritmen berekenen slechts een willekeurige GNE dat is afhankelijk van de initialisatie. In toepassingen waar een voorspelbare en goed gedefinieerde oplossing noodzakelijk is, wordt het belangrijk om een specifieke GNE te selecteren (uit mogelijk oneindig veel) die een willekeurige metriek of een secundair, coöperatief doel optimaliseert. Wij ontwikkelen de eerste algoritmen voor optimale GNE-selectie. We vergelijken twee verschillende methoden voor algoritmeontwerp, beide ontwikkeld binnen het kader van de operatortheorie: de eerste, de hybrid steepest-descent method (HSDM), omvat een gradient descent van de selectie-functie met een afnemende stapgrootte, gecombineerd met een GNE-zoekend algoritme, terwijl de tweede de oplossing van een reeks Variational Inequalities (VIs) met een afnemende regularisatieterm vereist. Beide ontwerpmethoden leiden tot algoritmen die geschikt zijn om de berekening te verdelen tussen een centraal knooppunt en de agenten, en we voegen ad-hoc algoritmen toe voor de bijzondere gevallen van aggregatieve en cocoercieve spellen.

Ten tweede beschouwen we tijdsafhankelijke spellen, gemotiveerd door de behoefte aan algoritmen die fysieke multi-agent systemen continu monitoren en besturen. In dergelijke spellen moeten de agenten een evoluerende oplossing volgen met beperkte rekentijd tussen de updates van het probleem. Dit scenario is met name relevant wanneer de agenten worden beïnvloed door verstoringen waarvan de tijdschaal vergelijkbaar is met de convergentiesnelheid van het algoritme. De uitdaging ligt in het

vinden van algoritmen die zowel snelle convergentie als robuustheid tegen externe verstoringen vertonen, beide doorgaans geassocieerd met een lineaire convergentiesnelheid. We leiden en bestuderen de asymptotische volgfout af van een volledig gedistribueerd algoritme (d.w.z. zonder centrale coördinator) voor GNE-problemen met lineaire gelijkheidsbeperkingen. Voor een tijdsafhankelijke GNE-selectieprobleem vinden we dat de HSDM met constante stapgrootte lineair convergeert naar een benaderde oplossing. We vinden dat de benaderingsfout kan worden beheerst door een geschikte keuze van de stapgrootte en het aantal iteraties per tijdstap, en we leiden een grens af voor de asymptotische volgfout.

Ten slotte richten we ons opnieuw op de toepassing van GNE-zoekende algoritmen voor de besturing van fysieke systemen. We bekijken dynamische spellen, waarin elke agent een tijdsreeks van inputs naar een dynamisch systeem moet bepalen als beslissingen. In dit geval ontstaat de koppeling tussen de agenten niet alleen door de doelen en beperkingen, maar ook door de systeemdynamica. Idealiter zou men de GNE-oplossing moeten berekenen door de dynamica over een oneindig lange horizon te voorspellen. Dit is doorgaans computationeel onhaalbaar, vooral wanneer er beperkingen zijn. We benaderen daarom de controlevolgorde voor een oneindige horizon door op elk tijdstip opnieuw de oplossingen te berekenen voor een evenwichtsprobleem over een eindige tijdshorizon, een methode die meestal bekend staat als receding-horizon control (of model predictive control, in het geval van een enkele agent). We leiden een nieuwe karakterisering af van het oneindige-horizon-doel behaald door de NE-traject, en tonen aan dat men de prestaties voor de oneindige horizon kan herstellen door deze uitdrukking op te nemen in de doelen van de agenten als een additieve terminalkost. Met dit resultaat concluderen we de asymptotische stabiliteit van de evenwichtstoestand onder een spel-theoretische regelactie voor receding-horizon controle. In vergelijking met de literatuur veronderstellen we noch de stabiliteit van de ongestuurde plant, noch introduceren we aanvullende beperkingen. Bovendien vinden we dat de asymptotische stabiliteit van de evenwichtstoestand kan worden verkregen met een meer algemene terminalkost als het spel potentieel is, zoals we aantonen in een praktische toepassing voor verkeersroutering.

Preliminaries

Basic notation

$:=$	equal to by definition
$\vert$	such that
$\in$	belongs to
$\notin$	does not belong to
$\exists$	there exists
$\forall$	for all
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\rightarrow$	maps to an element
$\rightrightarrows$	maps to a set

Sets, spaces and set operators notation

$\mathbb{N}$	set of whole numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{R}^n$	set of real n-dimensional vectors
$\mathbb{R}^{n\times m}$	set of real n by m matrices
$\mathbb{R}^n_\Psi$	space of n-dimensional real vectors equipped with the inner product $\langle\cdot,\cdot\rangle_\Psi$, where $\Psi \in \mathbb{R}^{n\times n}$ and $\Psi = \Psi^\top \succ 0$
$A\cup B$	union of the sets A and B
$A\cap B$	intersection of the sets A and B
$A\subset B$	A is a subset of B
$A\supset B$	A is a superset of B
$A\setminus B$	set of elements that are in A, but not in$B$
$A+B$	Minkowski sum of sets A and B
$A\times B$	Cartesian product of the sets A and B
$\displaystyle\prod_{i=1}^N A_i$	$:= A_1\times A_2\times\cdots\times A_N$
$\displaystyle\bigcup_{i=1}^N A_i$	$:= A_1\cup A_2\cup\cdots\cup A_N$
A^n	$:= \prod_{i=1}^N A$

Operations on vectors and matrices, norms

I_n	identity matrix of dimensions n. The dimension is suppressed when clear from context.
$\mathbf{0}_n$	zero vector of dimension n. The dimension is suppressed when clear from context.
$\mathbf{1}_n$	vector of dimension n with all elements 1. The dimension is suppressed when clear from context.
$(\cdot)^\top$	transpose.
$\mathrm{col}(M_1, \dots M_N)$	$:= \left[M_1^\top, \dots, M_N^\top\right]^\top$, where $M_i, ..., M_N$ are real vectors or matrices.
$\mathrm{col}(M_i)_{i\in\mathcal{I}}$	$:= \mathrm{col}(M_{i_1}, M_{i_2}, ..., M_{i_N})$, where $\mathcal{I} = \{i_1, ..., i_N\}$.
$\mathrm{diag}(M_1, \dots M_N)$	Block diagonal matrix with diagonal elements $M_1, ..., M_N$.
$\mathrm{diag}(M_i)_{i\in\mathcal{I}}$	$:= \mathrm{diag}(M_{i_1}, ..., M_{i_N})$, where $\mathcal{I} = \{i_1, ..., i_N\}$.
$\mathrm{avg}(v_1, \dots v_N)$	$:= \frac{1}{N}\sum_{i=1}^{N} v_i$, where $v_1, ..., v_N$ are real vectors.
$\mathrm{avg}(v_i)_{i\in\mathcal{I}}$	$:= \frac{1}{\lvert\mathcal{I}\rvert}\sum_{i\in\mathcal{I}} v_i$, where v_i is a real vector for each $i \in \mathcal{I}$.
M^{-1}	inverse of the non-singular matrix M
$M \succ 0$	the matrix $M = M^\top$ is positive definite
$M \succcurlyeq 0$	the matrix $M = M^\top$ is positive semidefinite
$\langle v, u\rangle$	Euclidean inner product, i.e. $v^\top u$
$\langle v, u\rangle_\Psi$	Weighted inner product, i.e. $v^\top \Psi u$, with $\Psi = \Psi^\top \succ 0$
$\lVert v\rVert$	Euclidean norm, i.e. $\sqrt{v^\top v}$
$\lVert v\rVert_\Psi$	Euclidean weighted norm, i.e. $\sqrt{v^\top \Psi v}$, with $\Psi = \Psi^\top \succ 0$
$\mathrm{dist}_\Psi(x, C)$	distance of x to the set C with respect to $\lVert\cdot\rVert_\Psi$, i.e. $\mathrm{dist}_\Psi(x, C) = \inf_{z\in C}\lVert x - z\rVert_\Psi$

Preliminaries on convex functions

Consider a scalar, convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a convex set $\mathcal{C}$. We denote:

∂f	subdifferential of f, i.e. $\partial f(x) = \{u \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, \langle y-x, u\rangle + f(x) \leq f(y)\}$
$\nabla f(x)$	gradient of f
$\nabla_{x_i} f(x)$	partial gradient of f with respect to x_i, where $x = \mathrm{col}(x_j)_{j\in\mathcal{I}}$ and $i \in \mathcal{I}$.
$Dg(x)$	Jacobian matrix of g
$\mathrm{prox}^{\Psi}_{\partial f}$	proximal operator of ∂f, that is, $\mathrm{prox}^{\Psi}_{\partial f}(x) := \underset{y\in\mathbb{R}^n}{\mathrm{argmin}}\, f(y) + \frac{1}{2}\|y-x\|^2_{\Psi}$
$\mathrm{N}_{\mathcal{C}}$	normal cone of the closed convex set $\mathcal{C} \subseteq \mathbb{R}^n$, that is, $N_{\mathcal{C}}(x) = \{u \in \mathbb{R}^n \mid \sup_{y\in\mathcal{C}} \langle y-x, u\rangle \leq 0\}$
$\mathrm{proj}^{\Psi}_{\mathcal{C}}$	projection of x to the set $\mathcal{C}$ for the norm $\|\cdot\|_{\Psi}$, that is, $\mathrm{proj}^{\Psi}_{\mathcal{C}}(x) := \underset{y\in\mathcal{C}}{\mathrm{argmin}}\, \|y-x\|^2_{\Psi}$
$\iota_{\mathcal{C}}$	indicator function of the set $\mathcal{C}$, that is, $\iota_{\mathcal{C}}(x) = \begin{cases} +\infty & \text{if } x \notin \mathcal{C}, \\ 0 & \text{if } x \in \mathcal{C}. \end{cases}$

A continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is σ-strongly convex with respect to a Ψ-weighted norm, with $\sigma > 0$ and $\Psi \succ 0$, if, for all $x, x' \in \mathrm{dom}\, f$,

$$f(x') \geq f(x) + \langle \nabla f(x), x'-x\rangle + \tfrac{\sigma}{2}\|x'-x\|^2_{\Psi}.$$

Additionally, f is convex if the previous inequality holds for $\sigma = 0$.

Operator theory notation and nomenclature

Consider the set-valued operator $F : \mathcal{X} \rightrightarrows \mathcal{Y}$. We denote:

Id	identity operator, i.e. $\mathrm{Id}(x) = x$
$\mathrm{dom}(F)$	$:=\{x \in \mathcal{X} \mid F(x) \neq \varnothing\}$
$\mathrm{zer}(F)$	$:=\{x \in \mathcal{X} \mid 0 \in F(x)\}$
$\mathrm{fix}(F)$	$:=\{x \in \mathcal{X} \mid x = F(x)\}$
$\mathrm{gph}(F)$	$:=\{x \in \mathcal{X}, y \in \mathcal{Y} \mid y \in F(x)\}$
F^{-1}	inverse operator, i.e. $\mathrm{gph}(F^{-1}) = \{(y,x) \mid (x,y) \in \mathrm{gph}(F)\}$
$\mathcal{R}_F$	$:= (\mathrm{Id} + F)^{-1}$, i.e., the resolvent operator of F

Furthermore, we classify F as:

monotone	if $\langle y-y', x-x' \rangle \geq 0, \quad \forall (x,y),(x',y') \in \mathrm{gph}(F)$
β-strongly monotone	if $F-\beta\mathrm{Id}$ is monotone
L-Lipschitz continuous	if there exists a constant $L>0$ such that $\|y-y'\| \leq L\|x-x'\| \ \forall (x,y),(x',y') \in \mathrm{gph}(F)$
nonexpansive	if 1-Lipschitz continuous
attracting nonexpansive	if it is nonexpansive with $\mathrm{fix}(F) \neq \varnothing$ and $\|F(x)-z\| < \|x-z\|, \quad \forall z \in \mathrm{fix}(F)$ and $\forall x \notin \mathrm{fix}(\mathcal{T})$
quasi-nonexpansive	if $\mathrm{fix}(F) \neq \varnothing$ and $\|F(x)-z\| \leq \|x-z\|, \forall z \in \mathrm{fix}(F)$ and $\forall x \in \mathbb{R}^n$
α-averaged nonexpansive	for $\alpha \in (0,1)$, if there exists $\mathcal{R}$ nonexpansive such that $F = (1-\alpha)\mathrm{Id} + \alpha\mathcal{R}$
β-cocoercive	if $\langle y'-y, x'-x \rangle \geq \beta\|y'-y\|, \quad \forall (x,y),(x',y') \in \mathrm{gph}(F)$
Firmly nonexpansive	if for all $(x,y),(x',y') \in \mathrm{gph}(F)$, $\|y-y'\|^2 + \|(x-y)-(x'-y')\|^2 \leq \|x-x'\|^2$.

1 Introduction

"We have found nothing you can sell. We have found nothing you can put to practical use. [...] We have satisfied nothing but curiosity, gained nothing but knowledge. To me, these are the noblest goals."

Becky Chambers, in "To be taught, if fortunate"

In the first chapter of this thesis, we introduce the generalized Nash equilibrium problem as a model for rational decision-making in a non-cooperative, multiagent setting with operational constraints. We illustrate the limitations of the models and solution algorithms typically found in the literature through relevant application examples. We emphasize the importance of selecting a specific Nash equilibrium when multiple are available, as well as the necessity of incorporating time dependency and the physical dynamics of the system in conjunction with the decision-making process. Finally, we outline the research questions arising from these considerations that motivate this thesis, and we present the structure of the thesis at the end of the chapter.

1

1.1 Game theory

The widespread diffusion of low-cost computing devices and the improvement in sensing and communication technologies have surrounded us with "intelligent" systems, which are able of complex calculations, of interacting with both the physical world and humans, as well as acquiring and broadcasting data in real time. Inevitably, given their pervasive presence, such cyberphysical systems often need to interact with one another, and are thus required to compute a complex, collective decision [1].

An illustrative example is that of personal vehicles, which are nowadays typically equipped with significant computing and sensing capabilities, as well as access to the internet. Consider multiple vehicles driving on the same road: each driver (whether human or artificial) has to continuously determine the control input of their vehicle. This does not only require each driver to take into account the physics of their own car, safety speed and distance limits, and the preferences of the eventual humans on board: it also requires them to monitor and react to the actions of the remaining vehicles. The available sensing and information processing technologies (radars, LIDARs, cameras, computer vision) enable a vehicle to determine the *current* position, velocity, and nature of the surrounding obstacles. However, a capable driver must also *predict* the intentions of the other drivers that populate the road, which in turn requires a *model* of their behavior. A sensible assumption on this behavior is that the remaining drivers are *rational*, meaning that they will also pursue the best possible course of action according to their own preferences and operating limits of the vehicle [2].

We typically refer to rational entities as *agents*. The problem of computing an optimal course of action while being influenced by other, equally rational, agents, is called a *game*. If the operating limits that must be respected are also influenced by the actions of the remaining agents, then the game is *generalized*, and we refer to such interdependent operating limits as *coupling constraints* (in opposition to *local* constraints). In the context of the multi-vehicle driving example above, the legal speed limit is a local constraint, as it does not depend on the speed or position of the other vehicles. Conversely, the safety distance is a coupling constraint, as the allowed position on the road of each vehicle depends on the location of the remaining vehicles.

Mathematically, a game can be formalized as an optimization problem in the decision variable x_i associated to the agent i, which is parametrized in the decision variables of the remaining agents, denoted as $\boldsymbol{x}_{-i}$, see e.g. [3]:

$$\forall i: \quad \min_{x_i \in \mathbb{R}^n} \; J_i(x_i, \boldsymbol{x}_{-i}) \tag{1.1a}$$

$$\text{s.t. } x_i \in \Omega_i(\boldsymbol{x}_{-i}). \tag{1.1b}$$

In (1.1), J_i is a local objective function, which encodes the control goals of agent i, and Ω_i is a mapping from the actions of the remaining agents to the set of feasible actions for agent i, and it encodes the operating constraints of the system. Typically, the problem in (1.1) is solved by finding a GNE. A GNE is a collective action, such that no agent i can improve its objective J_i without being able to change the action of the remaining agents. This model respects the multiagent setting (a driver cannot select the speed of the surrounding vehicles) and it characterizes a *non-cooperative* regime, as no agent will rationally hinder its own objective to advantage another agent. Returning to the example concerning driving, consider the instantaneous position p_i and velocity v_i of the vehicle as decision variables. Then, a possible choice for J_i which aims at maintaining the same speed of the vehicle in front v_j is

$$J_i(x_i, \boldsymbol{x}_{-i}) = (v_i - v_j)^2, \tag{1.2}$$

while a possible choice for Ω_i which requires the vehicle i to respect the speed limit v^{max} (local constraint) and the safety distance Δp^{safe} from the vehicle in front (coupling constraint) is

$$\Omega_i : \boldsymbol{x}_{-i} \mapsto \{v_i \in [0, v^{\text{max}}]\} \times \{p_j - p_i \in [\Delta p^{\text{safe}}, +\infty)\}. \tag{1.3}$$

1.2 GNE seeking in a dynamic environment

In recent years, the GNE seeking problem has attracted significant attention, and numerous, efficient algorithms for its solutions have been devised [4]. However, when it comes to deploying the decision process captured by the model in (1.1), the interface between the decision-making process and the physical world poses additional, relevant challenges, as it can already be acknowledged when taking into consideration the problem in (1.1) with the example objective and constraints in (1.2) and (1.3) as a model of an actual driving scenario.

First of all, the game in (1.1) ignores the physics of the vehicles. A more suitable model for each vehicle is, in fact, a *dynamical system*, which restricts the set of positions and velocities that the vehicle can reach in a given time to the solutions of an initial-value problem. An optimization problem constrained to the solutions of an initial-value problem is typically referred to as an optimal control problem (OCP), and the engineering interest in its multiagent extension has recently surged [2, 5–8], drawing a renovated interest to the *dynamic games* field [9–11]. It is well-known that a crucial design parameter, when modelling a control system as an OCP, is the *horizon* of the problem, that is, how far ahead in time is the solution to the initial-value problem simulated and optimized. Clearly, one would ideally predict the system evolution over an indefinitely long time, that is, over an *infinite horizon*. However, this results in an infinite number of optimization variables, thus rendering the dynamic game unsolvable, except in a handful of restrictive, particular cases. In the OCP case, an effective alternative is the celebrated model predictive control (MPC) method, which entails truncating the horizon and continuously re-solving the resulting finite horizon problem [12]. It is well known that, by an appropriate design of the objective and constraints, the MPC control coincides with the solutions to an infinite-horizon OCP [12, Exercise 2.7]: no such result is, however, currently available in the dynamic game case.

Secondly, the decision problem should be dependent on time: in fact, a driver must constantly react to external stimuli that modify the driving pattern, for instance, obstacles appearing on the road, a change in wind speed or an immediate safety concern. When the conditions change, then the problem in (1.1) is updated and a

reaction to the situation variation must be swiftly recomputed. The available GNE seeking algorithms are typically iterative and their convergence to the solution is only asymptotic. This implies that the exact computation of a GNE in a dynamic setting cannot be guaranteed, as the problem definition will need to be updated before the algorithm reaches convergence. Thus, a dynamic setting requires algorithms with guaranteed finite-time error bounds and which are able to follow the time evolution of a GNE.
Finally, the solution of the problem in (1.1) is in general non-unique. When multiple GNEs are available, the existing GNE seeking algorithms behave non-deterministically, in the sense that they compute an initialization-dependent, unspecified GNE. This can occur if agents' objectives are indifferent to certain decision variables, or if a variable has a range of values that doesn't affect the objective. By shifting perspectives, this ambiguity can be viewed as flexibility: among all possible GNEs (which are rational in a non-cooperative setting), the agents can resolve the ambiguity by employing the GNE that is optimal according to a secondary, cooperative goal. We can think of this setting as "conditioned cooperative decision-making", where the condition for cooperation is competitive efficiency. Let us consider again the usual driving example: the objective in (1.2) does not specify a preferred position for the vehicle on the road, as long as the safety constraints in (1.3) are met. Leveraging this flexibility, the drivers can agree to maintain specific positions that optimize road usage, in order to reduce the road occupancy. That is, provided that this does not compromise their competitive speed goals encoded in (1.2).
Given the aforementioned limitations, we identify the following research questions, which drive the development of this thesis

- Q1 How to deterministically compute a specific GNE, optimally selected according to a design criterion?
- Q2 How to track the (possibly selected) time-varying GNE of a game, by means of an algorithm which only employs a finite number of iterations?
- Q3 Is a dynamical system controlled by the receding-horizon solution of a generalized Nash equilibrium problem (GNEP) asymptotically stable?
- Q4 Can the solution to a finite-horizon GNEP problem involving a multiagent linear time-invariant (LTI) dynamical system match the one of an infinite-horizon dynamic game?

Besides the aforementioned example, similar game theoretic models and challenges emerge in other practical applications of contemporary interest:

- **Energy markets**: A contemporary challenge in the power industry, fueled by the recent liberalization process and the availability of small-scale renewable power generators and energy storage units, is the inclusion of retail-level consumers and producers of energy (prosumers) in the energy market operation. In this scenario, the prosumers bid on their energy consumption and production, while the system operator aims at clearing the market, that is, scheduling the energy prices and the generation at the available plants. The solution needs to be recomputed at regular intervals in order to balance unexpected variations in demand and/or power availability, while the inertia of the mechanical generators, as well as the storage charging state and degradation process, introduce a dynamics to the decision process. In this scenario, the flexibility of the prosumers can be exploited to select a GNE which provides some ancillary services to the grid. [13]

- **Autonomous racing**: Game theoretic models have been recently employed for improving the performance in both aerial and terrestrial drone racing [8, 14]. The drones need to take into account both the physics of the driven vehicle and the adversarial behavior and dynamics of the other agents. The intrinsic fast pace and aggressiveness of the maneuvers highlights the need for fast decision-making.

1.3 Thesis organization

The thesis is composed of two parts. We present the focus of each part and a summary of each chapter next, while a schematic representation of the thesis structure is depicted in Figure 1.1.

1.3.1 Part 1: Optimal Nash equilibrium selection and tracking

In the first part of this thesis, we focus on the problem of computing a (possibly optimal) GNE, while ignoring the dynamics of the plant. The algorithms here proposed determine a set-point for the multiagent system, with the implicit assumption that the underlying system is asymptotically stable (or stabilized) to the given set-point, as depicted in Figure 1.2. The time-varying component of the games, when considered, is addressed as an exogenous process which defines a novel GNEP at each time step, and the role of the algorithms here designed is to compute an approximation of the (actual) sequence of GNEs, as for Figure 1.3.

With the purpose of obtaining reliable and deterministic computation methods for games which allow for multiple GNEs, we design algorithms that guarantee the optimality of the computed equilibrium according to an arbitrary performance metric. We design solution algorithms that apply to generic monotone games with a convex selection objective (Chapters 2 and 4), as well as more specific algorithms for the particular case of aggregative games (Chapter 3), that is, games where the objective coupling is only determined by the average of the decision variables.

For time-varying problems, we study the tracking performance of the finite-iteration version of a selection algorithm in Chapter 2. With the aim of developing a faster solution method, and thus achieving better tracking performance, we find a novel linear convergence result for strongly monotone, equality constrained games, which works even without a central coordinator (Chapter 5).

- **Chapter 2** (addressing Q1, Q2)
  We derive the first optimal GNE selection algorithm for monotone games with a convex selection function. We proceed by first writing a generic (non-optimal)

1

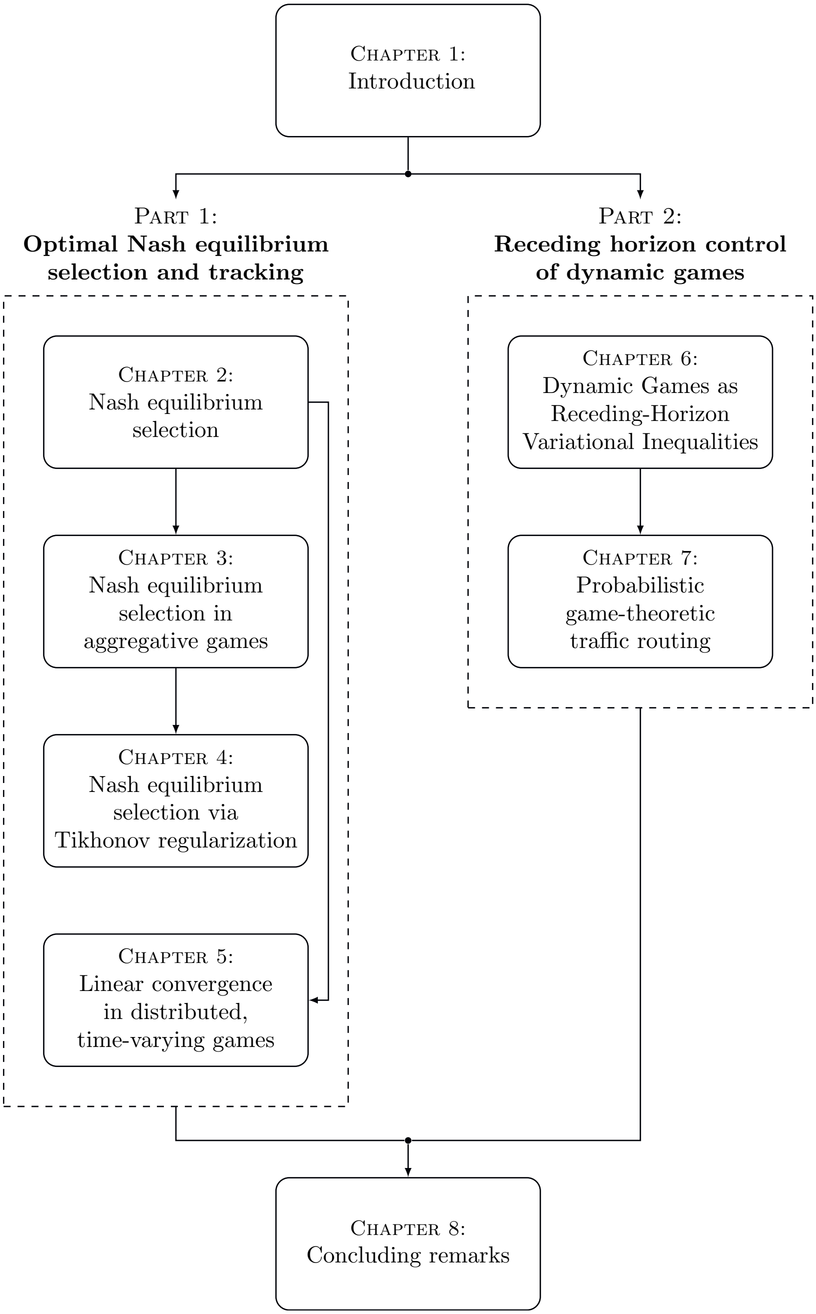


Figure 1.1: Structure of the thesis. Arrows indicate the suggested reading order.

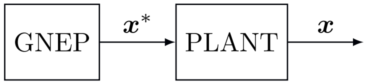


Figure 1.2: Model of the interaction between the GNEP problem solution and the multiagent system (plant) for Part 1 of this thesis. The solution of the GNEP $\boldsymbol{x}^*$ is used as set-point for the plant.

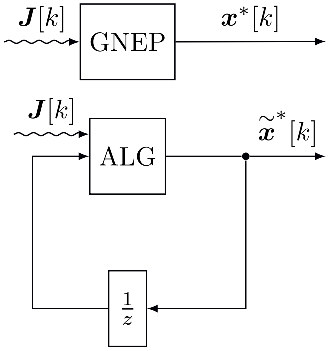


Figure 1.3: Tracking problem for a time-varying GNEP. At each iteration k, external factors determine the cost functions $\boldsymbol{J}[k]$ and, consequently, the GNE $\boldsymbol{x}^*[k]$ (assumed unique or optimally selected). The solution to the GNEP in Figure 1.2 is approximated by the output $\widetilde{\boldsymbol{x}}^*$ of the algorithm block ALG. Asymptotically, $\widetilde{\boldsymbol{x}}^*$ has to lie in a neighborhood of $\boldsymbol{x}^*$.

GNE seeking algorithm as a fixed-point operator. This allows us to cast the selection problem as an optimization problem constrained to the fixed point set of an operator, which is in turn typically solved by means of the HSDM [15] in the case of a non-expansive, quasi-shrinking operator. We show that the latter assumptions on the fixed point operator are satisfied by the forward-backward-forward (FBF) algorithm [16] and, for cocoercive operators, by the preconditioned forward-backward (pFB) algorithm [17]. Thus, we pair the HSDM to either of these well-known algorithms to derive two optimal GNE selection algorithms. We finally study the finite-horizon performance of the HSDM paired with the FBF algorithm, and we find an upper bound for the tracking error with respect to a time-varying optimal GNE, which improves with the number of allowed iterations-per-problem update. This chapter is partly based on the following publication:

[18] Benenati, E., Ananduta, W. and Grammatico, S. “Optimal Selection and Tracking Of Generalized Nash Equilibria in Monotone Games”. In: *IEEE Transactions on Automatic Control*, 68(12):7644–7659, (Dec. 2023)

- **Chapter 3** (addressing Q1)
  Building on the findings of Chapter 2, we design an optimal selection algorithm for linearly coupled games (which generalize aggregative games). By taking advantage of the particular structure of the problem, we pair the HSDM

1

with the preconditioned proximal-point (PPP) algorithm, which exhibits a fast convergence rate [19]. This chapter is partly based on the following publication:

[20] Benenati, E., Ananduta, W. and Grammatico, S. "On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games". In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 6389–6394, Cancun, Mexico, (Dec. 2022). IEEE

- **Chapter 4** (addressing Q1)
  In this chapter, we explore an alternative design method for optimal GNE-selection algorithms to the HSDM-based one presented in Chapters 2, 3. In particular, we cast the optimal selection problem as a variational inequality (VI) constrained to the set of solutions of a second VI, which coincides with the set of GNEs. We then employ the Tikhonov regularization method in [21] to derive a two-level solution algorithm, which requires the approximate solution of a VI at each iteration. Despite the bi-level structure of the algorithm, we show numerically that the convergence rate is comparable to the HSDM method paired with the FBF algorithm presented in Chapter 2. Theoretically, we show that the two methods are related, as a particular instance of the HSDM method and the Tikhonov regularization are, respectively, a forward-backward (FB) splitting and the resolvent operator for the same monotone inclusion. This chapter is partly based on the following publication:

  [22] Benenati, E., Ananduta, W. and Grammatico, S. "A Semi-Decentralized Tikhonov-Based Algorithm for Optimal Generalized Nash Equilibrium Selection". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 4243–4248, Singapore, Singapore, (Dec. 2023). IEEE

- **Chapter 5** (addressing Q2)
  In this chapter, we derive a fully distributed algorithm for the tracking of a time-varying GNE subject to only linear equality constraints. For this specific case, the proposed algorithm exhibits a linear convergence rate, resulting in a particularly straightforward and interpretable bound on the tracking error. This chapter is partly based on the following publication:

  [23] Bianchi, M., Benenati, E. and Grammatico, S. "Linear Convergence in Time-Varying Generalized Nash Equilibrium Problems". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 7220–7226, Singapore, Singapore, (Dec. 2023). IEEE

1.3.2 Part 2: Receding horizon control of dynamic games

In the second part of this thesis, we consider the problem of stabilizing a steady-state for a multiagent linear dynamical system with quadratic, competitive, general-sum objectives by computing the input to the system as a solution to a GNEP, as depicted in Figure 1.4. In particular, we focus on the design of the objective and constraints for the finite-horizon GNEP, such that its receding-horizon solution presents some

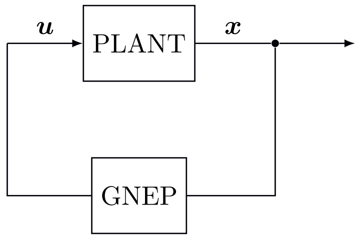


Figure 1.4: Model of the interaction between the GNEP problem solution and the multiagent system (plant) for Part 2 of this thesis. The solution to the GNEP is used as feedback input to stabilize the plant.

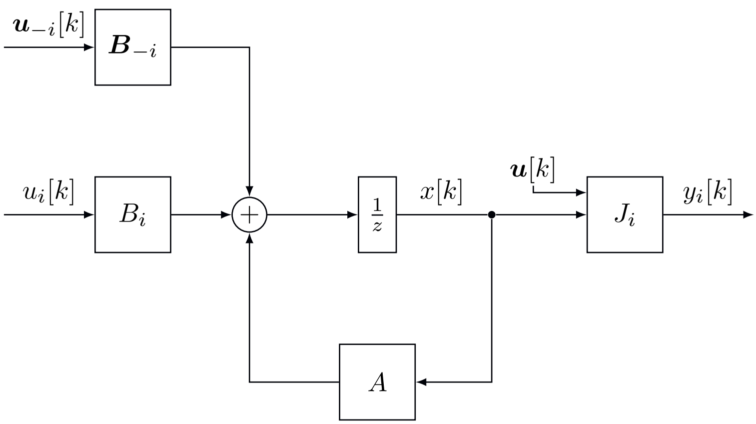


Figure 1.5: Optimal control problem solved by agent i when the GNEP in Figure 1.4 is an open-loop Nash equilibrium problem with linear dynamics. Given the input sequence of the remaining agents $\boldsymbol{u}_{-i}$, agent i chooses u_i to minimize some norm of the sequence y_i.

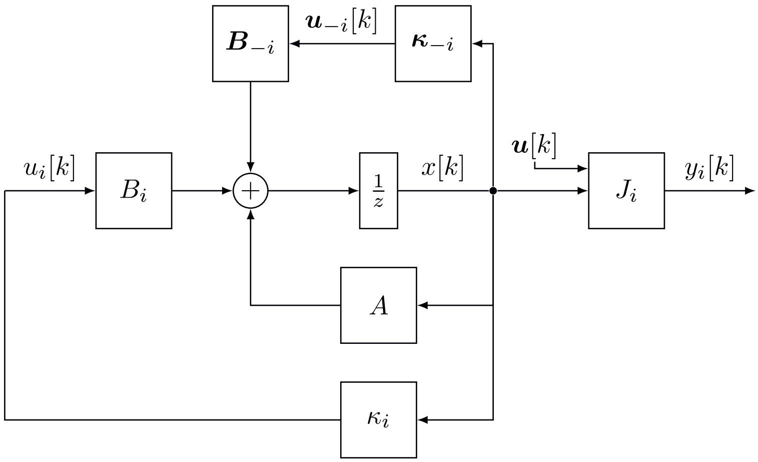


Figure 1.6: Optimal control problem solved by agent i when the GNEP in Figure 1.4 is a closed-loop Nash equilibrium problem setting with linear dynamics. Given the feedback control law of the remaining agents $\boldsymbol{\kappa}_{-i}$, agent i chooses κ_i so to minimize some norm of the sequence y_i.

stability guarantees (Chapter 6). We then explore an application to a vehicle traffic routing scenario (Chapter 7).

- **Chapter 6** (addressing Q3, Q4):
  In this chapter, we derive a novel method for computing the infinite-horizon Nash equilibrium (NE), and we establish a design criterion for the finite-horizon problem such that its solution matches the one of the infinite-horizon NE control. Building on these results, we derive a stability condition for the receding-horizon control action based on the continuous recomputation of the finite-horizon solution. We compare two different GNEP formulations, which emerge depending on the model that each agent has of their adversaries: the open-loop Nash equilibrium (ol-NE), depicted in Figure 1.5, where each agent treats the actions of others as an exogenous, unavoidable disturbance; and the closed-loop Nash equilibrium (cl-NE), depicted in Figure 1.6, where each agent considers the actions of others as part of the plant dynamics that can be influenced. This chapter is partly based on the following publication:

  [24] Benenati, E. and Grammatico, S. "Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities". In: *arXiv preprint 2408.15703*, (2024)

- **Chapter 7** (addressing Q3):
  In this chapter, we consider a routing problem for a population of vehicles. The routing systems have control authority on the probability distribution over which the vehicles draw their path. Through a nonlinear transformation of variables, we reformulate the problem as a GNEP, which we show to be monotone under less stringent conditions than those established in [25]. This

allows us to solve the problem by the well-known inertial forward-reflected backward (I-FoRB) algorithm. We alternatively cast the routing problem as a dynamic game and design a finite-horizon ol-NE-seeking problem, ensuring that the control action obtained from its receding-horizon solution renders the desired steady state asymptotically stable. This chapter is partly based on the following publication:

[26] Benenati, E. and Grammatico, S. "Probabilistic Game-Theoretic Traffic Routing". In: *IEEE Transactions on Intelligent Transportation Systems*, 25(10):13080–13090, (Oct. 2024)

Finally, in Chapter 8, we provide a summary of the main conclusions of this thesis, and we outline potential future research directions.

I

Optimal Nash equilibrium selection and tracking

2

Nash equilibrium selection

In a climate of suspicion, Isperia gathered the guilds to propose a radical idea: cooperation

Guild summit, Magic: the gathering

A fundamental open problem in monotone game theory is the computation of a generalized Nash equilibrium (GNE) among all the available ones, e.g. the optimal equilibrium with respect to a system-level objective. The existing GNE seeking algorithms have in fact convergence guarantees toward an arbitrary, possibly inefficient, equilibrium. In this chapter, we solve this open problem by leveraging results from fixed-point selection theory and in turn derive distributed algorithms for the computation of an optimal GNE in monotone games. We then extend the technical results to the time-varying setting and propose an algorithm that tracks the sequence of optimal equilibria up to an asymptotic error, whose bound depends on the local computational capabilities of the agents.

This chapter is partly based on Benenati, E., Ananduta, W. and Grammatico, S. "Optimal Selection and Tracking Of Generalized Nash Equilibria in Monotone Games". In: *IEEE Transactions on Automatic Control*, 68(12):7644–7659, (Dec. 2023).

2.1 Introduction

Numerous engineering systems of recent interest, such as smart electrical grids [27, 28], traffic control systems [25], and wireless communication systems [29–31] can be modelled as a *generalized game*, that is, a system of multiple agents aiming at optimizing their individual, inter-dependent objectives, while satisfying some common constraints. A typical operating point for these systems is the Generalized Nash Equilibrium (GNE), where no agent can unilaterally improve their objective function [3].

The recent literature has witnessed the development of theory and algorithms for computing a *variational* GNE (v-GNE) [32, 33], which exhibits desirable properties of fairness and stability. *Semi-decentralized* GNE seeking algorithms, where a reliable central coordinator gathers and broadcasts aggregate information, have been proposed for strongly monotone [34, 35] and merely monotone games [17, 36]. A breakthrough idea in [37], later generalized for non-strongly monotone games [16, 38, 39], enables a *distributed* computation of GNEs by exploiting a suitable consensus protocol [40], thus requiring a peer-to-peer information exchange.

Existing results present, however, two fundamental shortcomings that might limit their practical application. First, unless strong assumptions are considered (namely, strong monotonicity of the pseudogradient), a game may have infinitely many v-GNEs *and virtually all the existing algorithms provide no characterization of the equilibrium computed.* For instance, a Nash equilibrium can be arbitrarily inefficient with respect to system-level efficiency metrics (e.g., overall social cost) [41]. Such uncertainty on the obtained equilibrium is often unacceptable. A notable exception is the class of double-layer Tikhonov regularization algorithms, [42–44]. While the method in [42] works for generalized games, it only guarantees convergence to the minimum-norm solution. On the other hand, the equilibrium selection algorithms in [43, 44] solve at each (outer) iteration a regularized sub-problem where the objectives of the agents are augmented with a convex selection function to be optimized, weighted by a small parameter. However, the latter are only applicable to non-generalized games. In addition, the double-layer method in [45, 46] seeks the GNE closest to a desired strategy. It is important to note that double-layer algorithms require the exact solution of a sub-problem at each (outer) iteration, and thus they would require a virtually infinite amount of communications per outer iteration in a distributed setting. Recently, a single-layer algorithm based on a regularized projected-pseudogradient dynamics was proposed in [47], which however is only suitable for non-generalized games and requires nested vanishing stepsizes both on the pseudogradient and on the regularization. Second, decision-making agents often operate in a time-dependent environment and, due to the limited computation capabilities and to the time required to exchange information, it can be impossible to ensure a time-scale separation between the environment and the algorithm dynamics. This results in non-constant objectives and constraints between the discrete-time algorithmic iterations, as discussed in [48], and the references therein, for the particular case of optimization problems. Only few works, e.g., [49, 50], consider this setting in the case of game equilibrium problems and only with a strong monotonicity assumption on the game pseudogradient mapping.

Optimal equilibrium selection and tracking
We can formulate the first issue, identified in the seminal work [3, Sect. 6], as an *optimal GNE selection* problem, that is, the problem of computing a GNE of a game (among the potentially infinitely many) that satisfies a selection criterion. This criterion characterizes the desired equilibrium and can be formalized as a system-level *selection function* to be optimized over the set of GNEs. For example, the system-level objective of an electricity market can be to minimize the deviation from an efficient operating set-point [48]; for multiple autonomous vehicles, it can be to minimize the overall travel time of the network. Meanwhile, the second issue can be cast as an *optimal GNE tracking* problem, i.e., the problem of tracking the sequence of optimal GNEs of a time-varying game, with finite computation time and limited information on the future instances of the game available. As the GNE set is in general not a singleton, the tracking objective should be again chosen by means of a (time-varying) selection function. These problems, although of high practical interest, have never been addressed in the literature.

Under mild assumptions on the selection function, the optimal GNE selection problem in a monotone game is a special case of a variational inequality (VI) [32] defined over the set of v-GNEs. On the other hand, as shown in [16, 17, 36, 39], operator splitting techniques [51] can be leveraged to characterize v-GNEs as the zeros of a monotone operator and, in turn, as the fixed-point set of a suitable operator. Therefore, here we can cast the problem as that of fixed-point selection [15]. In the literature, e.g., [15, 52, 53], the latter can be solved by the hybrid steepest-descent method (HSDM), whose iterations depend on the fixed-point operator (whose definition depends on the primitives of the game) and the monotone operator that defines the VI, namely the gradient of the selection function in our setting.

Contributions
In the first part of this chapter (Sections 2.3 – 2.5), we propose the first single-layer distributed algorithms for solving the optimal GNE selection problem. Our method employs the Forward-Backward-Forward (FBF) operator [16] combined with the HSDM. We show that the proposed algorithm guarantees convergence to the *optimal* variational GNE (v-GNE) set in monotone games. Moreover, for a special class of monotone games, namely cocoercive games with affine coupling constraints, we also show that the preconditioned Forward-Backward (pFB) [36] can be paired with the HSDM to derive optimal GNE selection algorithms. Technically, our contribution is to show that these operators fulfill special properties that guarantee the convergence of the HSDM toward the solution set of the corresponding fixed-point selection VI. Compared to [42, 45, 46], our proposed algorithms significantly generalize the class of selection functions; additionally, our method works for generalized games and does not require solving an equilibrium problem at each iteration nor a vanishing stepsize on the pseudogradient dynamics.

In the second part of the chapter (Section 2.6), we formalize the optimal GNE tracking problem as a time-varying fixed-point selection problem. Thus, as a solution framework, we propose the *restarted HSDM*, which adapts its operators when the problem changes. In line with the results in the time-varying optimization literature [54, 55], we show convergence up to a tracking error which depends on the problem data and can be controlled by a suitable tuning of the algorithm parameters. Similarly to the equilibrium selection problem, the restarted HSDM works with the aforementioned fixed-point operators to solve the optimal GNE tracking problem for the corresponding classes of monotone games.

2.2 Mathematical preliminaries

In this section, we introduce the definition of *quasi-shrinking* operators, which were first considered in [15], and some relevant properties of this class of operators which are useful throughout this chapter.

Let C be a non-empty, closed, and convex subset of $\mathbb{R}^n$, $\mathcal{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ be quasi-nonexpansive under the Ψ-induced norm $\|\cdot\|_\Psi$ for some positive definite matrix Ψ, i.e., $\|\mathcal{T}(x) - z\|_\Psi \leq \|x - z\|_\Psi$, for all $z \in \operatorname{fix}(\mathcal{T}) \neq \varnothing$ and $x \in \mathbb{R}^n$. We define the distance of a point $x \in \mathbb{R}^n$ to C by

$$\operatorname{dist}_\Psi(x, C) := \inf_{z \in C} \|x - z\|_\Psi. \tag{2.1}$$

For $r \geq 0$, we define the sets

$$C^\Psi_{\geq r} := \{x \in \mathbb{R}^n \mid \operatorname{dist}_\Psi(x, C) \geq r\}, \tag{2.2}$$

$$A := \{r \in \mathbb{R}_{\geq 0} \mid \operatorname{fix}(\mathcal{T})^\Psi_{\geq r} \cap C \neq \varnothing\}. \tag{2.3}$$

Furthermore, we denote the indicator function of the set A as ι_A. Let us define the shrinkage function for the operator $\mathcal{T}$ under the norm $\|\cdot\|_\Psi$, which slightly generalizes [28, Def. 1], as follows:

$$D_\Psi(r) := \iota_A(r) + \inf_{x \in \operatorname{fix}(\mathcal{T})^\Psi_{\geq r} \cap C} \operatorname{dist}_\Psi(x, \operatorname{fix}(\mathcal{T})) - \operatorname{dist}_\Psi(\mathcal{T}(x), \operatorname{fix}(\mathcal{T})). \tag{2.4}$$

For $\Psi = I$, we suppress the subscript of D. The function D_Ψ has the properties stated next in Proposition 2.1 (see [56, Prop. 2.6] for the case $\Psi = I$).

Proposition 2.1. *Let Ψ be positive definite. For the function D_Ψ defined in (2.4), it holds that:*

1. *D_Ψ is positive semidefinite and non-decreasing;*
2. *$D_\Psi(\operatorname{dist}(x, \operatorname{fix}(\mathcal{T}))) \leq \|x - \mathcal{T}(x)\|_\Psi$ for all $x \in C$.*

Definition 2.1 (Quasi-shrinking)**.** *A quasi-nonexpansive operator $\mathcal{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is* quasi-shrinking *on a non-empty, closed, and convex set $C \subseteq \mathbb{R}^n$ if $\operatorname{fix}(\mathcal{T}) \cap C \neq \varnothing$ and $D(r) = 0 \Leftrightarrow r = 0$, where $D(r)$ is defined as in (2.4).*

Remark 2.1. *Suppose that a quasi-nonexpansive operator $\mathcal{T}$ is quasi-shrinking on C, i.e., $D(r) = 0 \Leftrightarrow r = 0$. Then, it also holds that $D_\Psi(r) = 0 \Leftrightarrow r = 0$, for any $\Psi \succ 0$.*

Example 2.1. *The Euclidean projection onto C, is quasi-shrinking and its shrinkage function (defined in (2.4)) is*

$$D(r) = \inf_{\{u \mid \mathrm{dist}(u,C) \geq r\}} \mathrm{dist}(u,C) - \underbrace{\mathrm{dist}(\mathrm{proj}_C(u),C)}_{=0} = r.$$

Finally, we identify a class of quasi-shrinking operators, as formally stated in Lemma 2.1, which generalizes the result in [56, Prop. 2.11] and is useful for our analysis.

Definition 2.2 (Demiclosed operator [51, Def. 4.26])**.** *Let $C \subseteq \mathbb{R}^n$ be a closed set. An operator $\mathcal{T}\colon C \to \mathbb{R}^n$ is demiclosed at $u \in \mathbb{R}^n$ if $\mathcal{T}(\boldsymbol{\omega}^\infty) = u$, for any sequence $(\boldsymbol{\omega}_k)_{k\in\mathbb{N}} \in C$ such that $\lim_{k\to\infty} \boldsymbol{\omega}_k = \boldsymbol{\omega}^\infty$ and $\lim_{k\to\infty} \mathcal{T}(\boldsymbol{\omega}_k) = u$.*

Lemma 2.1. *Let $\mathcal{T}$ be an operator with $\mathrm{fix}(\mathcal{T}) \neq \varnothing$. Let $\mathcal{T}_2$ be an operator such that $\mathrm{Id} - \mathcal{T}_2$ is demiclosed at 0 and such that $\mathrm{fix}(\mathcal{T}_2) \subseteq \mathrm{fix}(\mathcal{T})$. Assume that for any $\boldsymbol{\omega}^\star \in \mathrm{fix}(\mathcal{T})$,*

$$\|\mathcal{T}(\boldsymbol{\omega}) - \boldsymbol{\omega}^\star\|_\Psi^2 \leq \|\boldsymbol{\omega} - \boldsymbol{\omega}^\star\|_\Psi^2 - \gamma \|\boldsymbol{\omega} - \mathcal{T}_2(\boldsymbol{\omega})\|_\Psi^2, \tag{2.5}$$

for some $\gamma > 0$ and $\Psi \succ 0$. Then, $\mathcal{T}$ is quasi-shrinking on any compact convex set C such that $C \bigcap \mathrm{fix}(\mathcal{T}) \neq \varnothing$.

2.3 Problem formulation

2.3.1 Generalized Nash equilibrium problem

Let us consider N agents, denoted by the set $\mathcal{I} := \{1, 2, \ldots, N\}$, with inter-dependent optimization problems:

$$\forall i \in \mathcal{I}\colon \begin{cases} \min\limits_{x_i \in \mathcal{X}_i} & J_i(\boldsymbol{x}) := \ell_i(x_i) + f_i(\boldsymbol{x}) \quad (2.6\text{a}) \\ \text{s.t.} & \sum\limits_{j \in \mathcal{I}} g_j(x_j) \leq 0, \quad (2.6\text{b}) \end{cases}$$

where $x_i \in \mathbb{R}^{n_i}$ is the decision variable of agent i whereas $\boldsymbol{x} := \mathrm{col}(x_i)_{i\in\mathcal{I}} \in \mathbb{R}^n$, with $n = \sum_{i\in\mathcal{I}} n_i$, is a concatenated vector of the decision variables of all agents. Let us use $\boldsymbol{x}_{-i} = \mathrm{col}(x_j)_{j \in \mathcal{I}\setminus\{i\}}$ to denote the concatenated decision variables of all agents except agent i. Let $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ denote the local feasible set of x_i and $J_i\colon \mathbb{R}^n \to \mathbb{R}$ denote the cost function of agent i. Moreover, (2.6b) represents a separable coupling constraint where $g_j : \mathbb{R}^{n_j} \to \mathbb{R}^m$ is associated with agent j. We denote the collective feasible set of the game in (2.6) by

$$\boldsymbol{\Omega} := \prod_{i\in\mathcal{I}} \mathcal{X}_i \cap \Big\{ \boldsymbol{x} \mid \sum_{j\in\mathcal{I}} g_j(x_j) \leq 0 \Big\}. \tag{2.7}$$

Here, we look for equilibrium solutions to (2.6) where no agent has the incentive to unilaterally deviate, namely, generalized Nash equilibrium (GNE):

2

Definition 2.3. *A set of strategies* $\boldsymbol{x}^* := \mathrm{col}(x_i^*)_{i\in\mathcal{I}}$ *is a GNE of the game in* (2.6) *if* $\boldsymbol{x}^* \in \boldsymbol{\Omega}$ *and, for each* $i \in \mathcal{I}$,

$$J_i(\boldsymbol{x}^*) \le J_i(x_i, \boldsymbol{x}_{-i}^*), \tag{2.8}$$

for any

$$x_i \in \mathcal{X}_i \cap \Big\{ y \mid g_i(y) \le - \sum_{j\in\mathcal{I}\setminus\{i\}} g_j(x_j^*) \Big\}.$$

□

Furthermore, we focus on the class of jointly convex GNE problems and hence, consider the following assumptions on Problem (2.6) [36, Assms 1–2]. We note that [16, 17, 37–39] consider the case of affine constraint functions.

Assumption 2.1. *In* (2.6), *for each* $i \in \mathcal{I}$, *the functions* $f_i(\cdot, \boldsymbol{x}_{-i})$, *for any* $\boldsymbol{x}_{-i}$, *and* $g_i(\cdot)$ *are component-wise convex and continuously differentiable;* ℓ_i *is convex and lower semicontinuous. For each* $i \in \mathcal{I}$, *the set* $\mathcal{X}_i$ *is nonempty, compact, and convex. The global feasible set* $\boldsymbol{\Omega}$ *defined in* (2.7) *is non-empty and satisfies Slater's constraint qualification [51, Eq. (27.50)].*

Assumption 2.2. *The mapping*

$$F(\boldsymbol{x}) := \mathrm{col}\left(\left(\nabla_{x_i} f_i(\boldsymbol{x}) \right)_{i\in\mathcal{I}} \right), \tag{2.9}$$

with $(f_i)_{i\in\mathcal{I}}$ *as in* (2.6a), *is monotone.*

As in [16, 17, 36–39], we can formulate the problem of finding a GNE of the game in (2.6) as that of a monotone inclusion. To this end, we introduce the dual variable $\lambda_i \in \mathbb{R}^m$, for each $i \in \mathcal{I}$, to be associated with the coupling constraint (2.6b). Furthermore, we focus on a subset of GNEs, namely variational GNE (v-GNE)s, indicated by equal optimal dual variables, $\lambda_i^* = \lambda^*$, for all $i \in \mathcal{I}$. As discussed in [3, 33], a v-GNE enjoys several desirable properties, such as fairness and larger social stability than non-variational ones. Under Assumptions 2.1–2.2, the set of v-GNE of the game in (2.6) is non-empty [57, Prop. 12.11]. The Karush–Kuhn–Tucker (KKT) optimality conditions of a v-GNE of the game in (2.6), denoted by $\boldsymbol{x}^*$, are:

$$\forall i \in \mathcal{I}\colon \begin{cases} \mathbf{0} \in \mathrm{N}_{\mathcal{X}_i}(x_i^*) + \partial_{x_i} J_i(\boldsymbol{x}^*) + \langle \nabla g_i(x_i^*), \lambda^* \rangle, & \text{(2.10a)} \\ \mathbf{0} \in \mathrm{N}_{\mathbb{R}^m_{\ge 0}}(\lambda^*) - \sum_{j\in\mathcal{I}} g_j(x_j^*). & \text{(2.10b)} \end{cases}$$

To obtain a v-GNE via a fully distributed algorithm, we incorporate a consensus scheme on the dual variables. In the full information case, one typically assumes that there exists a communication network over which the agents exchange information to update their dual variables. Let us represent this communication network as an undirected graph $\mathcal{G}^\lambda = (\mathcal{I}, \mathcal{E}^\lambda)$ and assume that $\mathcal{G}^\lambda$ is connected. Furthermore, we denote the Laplacian of $\mathcal{G}^\lambda$ by $\mathcal{L}$ and the neighbors of agent i in $\mathcal{G}^\lambda$ by $\mathcal{N}_i^\lambda$, i.e., $\mathcal{N}_i^\lambda := \{ j \in \mathcal{I} \mid (i,j) \in \mathcal{E}^\lambda \}$. Additionally, let $\mathcal{N}_i^J$ denote the set of agents whose decision variable x_j influences the cost function J_i. For simplicity, we assume that $\mathcal{N}_i^J \subseteq \mathcal{N}_i^\lambda$.

Now, let us denote by $\nu_i \in \mathbb{R}^m$ the consensus variable of agent i, and $\boldsymbol{\omega} = (\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{n_\omega}$, where $\boldsymbol{\lambda} = \mathrm{col}(\lambda_i)_{i\in\mathcal{I}}$, $\boldsymbol{\nu} = \mathrm{col}(\nu_i)_{i\in\mathcal{I}}$, and $n_\omega = n + 2Nm$. Then, we can define the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as follows:

$$\mathcal{A}(\boldsymbol{\omega}) := \prod_{i\in\mathcal{I}} (\mathrm{N}_{\mathcal{X}_i} + \partial \ell_i)(x_i) \times \mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}(\boldsymbol{\lambda}) \times \{\mathbf{0}_{Nm}\}, \tag{2.11}$$

$$\mathcal{B}(\boldsymbol{\omega}) := \mathrm{col}\left(F(\boldsymbol{x}), (\mathcal{L} \otimes I_m)\boldsymbol{\lambda}, \mathbf{0}_{Nm}\right), \tag{2.12}$$

$$\mathcal{C}(\boldsymbol{\omega}) := \begin{bmatrix} \mathrm{col}\left(\langle \nabla g_i(x_i), \lambda_i \rangle\right)_{i\in\mathcal{I}} \\ -\mathrm{col}\left(g_i(x_i)\right)_{i\in\mathcal{I}} - (\mathcal{L} \otimes I_m)\boldsymbol{\nu} \\ (\mathcal{L} \otimes I_m)\boldsymbol{\lambda} \end{bmatrix}. \tag{2.13}$$

In turn, we can translate the generalized Nash equilibrium problem (GNEP) in (2.6) as a monotone inclusion problem, i.e.,

$$\text{find } \boldsymbol{\omega} \text{ such that } \boldsymbol{\omega} \in \mathrm{zer}(\mathcal{A} + \mathcal{B} + \mathcal{C}). \tag{2.14}$$

Similarly to [37, Thm. 2], we can show that for any $\boldsymbol{\omega}$ such that (2.14) holds, we obtain the pair $(\boldsymbol{x}, \lambda)$ that satisfies the KKT conditions in (2.10) if Assumptions 2.1-2.2 hold (see Appendix 2.B for details). The zero set of $\mathcal{A}+\mathcal{B}+\mathcal{C}$ is convex following its maximal monotonicity (Lemma 2.4 in Appendix 2.B) and [51, Prop. 23.39]. This result generalizes the known convexity of the solution set to a convex optimization problem [51, Prop. 11.6], which in our case is recovered by setting $f_i \equiv 0$ for all i. Additionally, since the set of v-GNE of the game is bounded as it is a subset of $\boldsymbol{\mathcal{X}}$, the set of solutions of the inclusion in (2.10) and the set $\mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})$ are bounded [58, Prop. 3.3].

2.3.2 Optimal equilibrium selection problem

The inclusion problem in (2.14) may have multiple solutions. In this section, we consider the problem of finding an equilibrium solution that minimizes a selection function, denoted by $\phi \colon \mathbb{R}^{n_\omega} \to \mathbb{R}$, i.e.,

$$\begin{cases} \operatorname*{argmin}_{\boldsymbol{\omega}\in\mathbb{R}^{n_\omega}} & \phi(\boldsymbol{\omega}) \\ \quad \text{s.t.} & \boldsymbol{\omega} \in \mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C}). \end{cases} \tag{2.15}$$

Some examples of selection functions are given as follows:

1. **Minimum norm solution.** Considering $\phi(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|^2$ defines the problem of finding a GNE and its corresponding dual variable where both have minimum norm.

2. **Agents' favorite operating points.** Suppose that each agent has a desired operating point, denoted by ω_i^{ref}, e.g. as discussed in [45, 46]. Then, we may impose a criterion function such that we obtain a GNE that is the closest to that operating point, i.e.,
$$\phi(\boldsymbol{\omega}) = \sum_{i\in\mathcal{I}} \|\omega_i - \omega_i^{\mathrm{ref}}\|^2.$$

2

3. **System level objectives.** In some engineering applications, such as in electrical networks, agents' operations heavily influence the reliability of the network, which is the responsibility of a network operator [59]. Thus, when the agents decide their operating points by solving a generalized Nash equilibrium problem, a system operator might enforce some objectives, associated with system reliability, such that the decisions of the agents are not only a GNE but also meet these objectives (see Section 2.7). An example of this objective can be of the form:
$$\phi(\boldsymbol{\omega}) = \|Q\boldsymbol{\omega} - \boldsymbol{\omega}^{\mathrm{ref}}\|^2,$$
for some $Q \succcurlyeq 0$.

In the remainder of the chapter, we consider the following technical assumption on the selection function, which, together with the convexity of $\mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})$, guarantees that the optimization problem in (2.15) is convex.

Assumption 2.3. *The function ϕ in* (2.15) *is continuously differentiable, convex, and has L_ϕ-Lipschitz continuous gradient.*

As a first step towards computing an optimal v-GNE, we leverage existing results to derive operators $\mathcal{T}$ with the property that

$$\boldsymbol{\omega} \in \mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C}) \Leftrightarrow \boldsymbol{\omega} \in \mathrm{fix}(\mathcal{T}), \tag{2.16}$$

and such that the Banach-Picard iteration of $\mathcal{T}$ [51, Sect. 5.2] guarantees convergence to a solution of the inclusion in (2.14). For instance, for cocoercive generalized games, a preconditioned forward-backward (pFB) operator presents the desired characteristics[37], whereas the forward-reflected-backward (FRB) operator [60] or the forward-backward-forward (FBF) operator [61] meets these requirements even for general monotone games. Furthermore, we require that the operator $\mathcal{T}$ in (2.16) can be evaluated in a distributed manner. By (2.16) and Assumption 2.3, the optimal equilibrium selection problem in (2.15) can be cast as a fixed-point selection variational inequality (VI):

$$\text{find } \boldsymbol{\omega}^\star \text{ s.t. } \inf_{\boldsymbol{\omega} \in \mathrm{fix}(\mathcal{T})} \langle \boldsymbol{\omega} - \boldsymbol{\omega}^\star, \nabla\phi(\boldsymbol{\omega}^\star)\rangle \geq 0. \tag{2.17}$$

2.4 Equilibrium selection algorithms

With the aim of solving the VI in (2.17), we consider the hybrid steepest-descent method (HSDM) fixed-point selection algorithm [15], which is defined by the following discrete-time dynamical system or iteration:

$$\boldsymbol{\omega}^{(k+1)} = \mathcal{T}(\boldsymbol{\omega}^{(k)}) - \beta^{(k)}\nabla\phi\big(\mathcal{T}(\boldsymbol{\omega}^{(k)})\big). \tag{2.18}$$

The HSDM can solve Problem (2.17) when $\mathcal{T}$ is quasi-nonexpansive and quasi-shrinking with bounded $\mathrm{fix}(\mathcal{T})$, as formally stated next.

Assumption 2.4. *The step size of the HSDM $\beta^{(k)}$ satisfies:*

(i) $\lim_{k\to\infty}\beta^{(k)} = 0$, $\sum_{k\geq 1}\beta^{(k)} = \infty$;

(ii) $\sum_{k\geq 1}(\beta^{(k)})^2 < \infty$. □

Remark 2.2. *The sequence $\beta^{(k)} = \beta_0/k^p$, for any $\beta_0 > 0$ and $p \in (1/2, 1]$, satisfies Assumption 2.4.* □

Assumption 2.5. *$\mathcal{T}$ is quasi-shrinking on a nonempty compact convex set C.* □

Lemma 2.2 (From [15, Thm. 5])**.** *Let Assumption 2.3 hold and $\Omega^\star$ be the set of solutions of the VI in* (2.17)*, with non-empty and bounded* $\mathrm{fix}(\mathcal{T})$*. Suppose that $\mathcal{T}$ satisfies Assumption 2.5 with compact convex set C such that $(\boldsymbol{\omega}^{(k)})_{k\geq 0} \subset C$. If the step size $\beta^{(k)}$ satisfies Assumption 2.4.(i), then the HSDM in* (2.18) *generates a sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \Omega^\star) = 0.$$

Therefore, our main technical task is to find a suitable operator $\mathcal{T}$ that can be evaluated in a distributed manner and that satisfies both (2.16) and Assumption 2.5, required for the convergence of the HSDM sequence.

Under mere monotonicity of the pseudogradient mapping (Assumption 2.2), perhaps the most obvious choice is the forward-reflected backward (FoRB) splitting, which, however, is not quasi-nonexpansive[1] (and, thus, it is not quasi-shrinking). Another viable option is the forward-backward-forward (FBF) splitting method [61], which works for v-GNE seeking in monotone games satisfying Assumptions 2.1–2.2, as shown in [16, 36]. As our first technical result, we show that the FBF algorithm satisfies both the desired property in (2.16) and Assumptions 2.5. To that end, firstly, we compactly state the FBF operator for (2.14), as follows:

$$\mathcal{T}_{\text{FBF}} := \big((\mathrm{Id} - \Psi^{-1}(\mathcal{B}+\mathcal{C})) \circ \mathcal{R}_{\Psi^{-1}\mathcal{A}} \circ (\mathrm{Id} - \Psi^{-1}(\mathcal{B}+\mathcal{C}))\big) + \Psi^{-1}(\mathcal{B}+\mathcal{C}), \quad (2.19)$$

where $\Psi \succ 0$ is a diagonal positive definite matrix and $\mathcal{R}_{\Psi^{-1}\mathcal{A}} := (1 + \Psi^{-1}\mathcal{A})^{-1}$ denotes the resolvent operator [51, Def. 23.1]. The FBF requires the forward operator, which is $(\mathcal{B}+\mathcal{C})$, to be Lipschitz continuous. A sufficient condition for this requirement is given in Assumption 2.6 (see Lemma 2.5 in Appendix 2.B).

Assumption 2.6. *The mapping $F(\boldsymbol{x})$ in* (2.9) *is L_F-Lipschitz continuous. Furthermore, for each $i \in \mathcal{I}$, the function g_i in* (2.6b) *has a bounded and $L_{\nabla g}$-Lipschitz continuous gradient.*

Under maximal monotonicity and Lipschitz continuity, it holds that $\mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C}) = \mathrm{fix}(\mathcal{T}_{\text{FBF}})$ (see Lemma 2.7 in Appendix 2.C). In addition, we define the step-size matrix

$$\Psi := \mathrm{diag}(\rho^{-1}, \tau^{-1}, \sigma^{-1}) \quad (2.20)$$

where $\rho = \mathrm{diag}((\rho_i I_{n_i})_{i\in\mathcal{I}})$, $\tau = \mathrm{diag}((\tau_i I_m)_{i\in\mathcal{I}})$, and $\sigma = \mathrm{diag}((\sigma_i I_m)_{i\in\mathcal{I}})$ need to be small enough with respect to the Lipschitz constant of $\mathcal{B}+\mathcal{C}$. A sufficient condition

[1] The FoRB iteration does not generate a Fejér monotone sequence [60, Prop. 2.3], implying that it is not quasi-nonxepansive and violates Definition 2.1.

on Ψ for the fixed-point iteration with $\mathcal{T}_{\text{FBF}}$ to converge is given in the following Assumption 2.7, also considered in [16, Assumption 2].

Assumption 2.7. *It holds that* $|\Psi^{-1}| \leq 1/L_B$, *where* $L_B > 0$ *is the Lipschitz constant of* $\mathcal{B}+\mathcal{C}$ *and* Ψ *reads as in* (2.20).

We are now ready to present the distributed FBF for seeking an optimal v-GNE based on the selection function $\phi(\boldsymbol{\omega})$ via the HSDM, as shown in Algorithm 1. To have a convergence guarantee as stated in Lemma 2.2, the FBF operator must satisfy Assumption 2.5. Let us show that this is the case in the following lemma.

Lemma 2.3. *Let Assumptions 2.1, 2.2, 2.6 and 2.7 hold. The operator* $\mathcal{T}_{\text{FBF}}$ *in* (2.19), *where* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$ *are defined in* (2.11)–(2.13) *and* Ψ *is defined in* (2.20), *is quasi-shrinking on any compact convex set* C *such that* $C \cap \operatorname{fix}(\mathcal{T}_{\text{FBF}}) \neq \varnothing$.

Thus, we can show that Algorithm 1 generates a sequence that converges toward the solution set of the optimal GNE selection problem in (2.17), as stated next.

Theorem 2.1. *Let Assumptions 2.1–2.4 and 2.6–2.7 hold. Let* $\Omega^\star$ *be the set of solutions to Problem* (2.17) *with* $\mathcal{T} = \mathcal{T}_{\text{FBF}}$ *defined in* (2.19), *where* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$ *are defined in* (2.11)–(2.13). *Furthermore, let* $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$, *where* $\boldsymbol{\omega}^{(k)} = (\boldsymbol{x}^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\nu}^{(k)})$, *be the sequence generated by Algorithm 1. Then,* $\lim_{k\to\infty} \operatorname{dist}(\boldsymbol{\omega}^{(k)}, \Omega^\star) = 0$, *and* $(\boldsymbol{x}^{(k)})_{k\in\mathbb{N}}$ *converges to an optimal v-GNE of the game in* (2.6).

Remark 2.3. *A central coordinator and step 5 of Algorithm 1 are not needed if* ϕ *is a separable function, i.e.,* $\phi(\boldsymbol{\omega}) = \sum_{i\in\mathcal{I}} \phi_i(\omega_i)$. *In this case, step 6 can be immediately executed by using local information* $(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)})$ *only, as long as each agent* i *knows the gradient* $\nabla\phi_i$.

Algorithm 1 Optimal v-GNE selection via FBF and HSDM

Initialization: $x_i^{(0)} \in \mathcal{X}_i$, $\lambda_i^{(0)} \in \mathbb{R}^m_{\geq 0}$, and $\nu_i^{(0)} \in \mathbb{R}^m$, $\forall i \in \mathcal{I}$.
Iteration of each agent $i \in \mathcal{I}$**.**

1. Receives $x_j^{(k)}$ from agent $j \in \mathcal{N}_i^J$ and $\lambda_j^{(k)}, \nu_j^{(k)}$ from agent $j \in \mathcal{N}_i^\lambda$.
2. Updates:

$$\widetilde{x}_i^{(k)} = \operatorname{prox}^{\rho_i}_{\ell_i + \iota_{\mathcal{X}_i}} \left(x_i^{(k)} - \rho_i \big(\nabla_{x_i} f_i(\boldsymbol{x}^{(k)}) + \nabla g_i(x_i^{(k)})^\top \lambda_i^{(k)} \big) \right),$$
$$\widetilde{\lambda}_i^{(k)} = \operatorname{proj}_{\mathbb{R}_{\geq 0}} \left(\lambda_i^{(k)} + \tau_i \Big(g_i(x_i^{(k)}) + \sum_{j \in \mathcal{N}_i^\lambda} \big(\nu_i^{(k)} - \nu_j^{(k)} - \lambda_i^{(k)} + \lambda_j^{(k)} \big) \Big) \right),$$
$$\widetilde{\nu}_i^{(k)} = \nu_i^{(k)} - \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} \big(\lambda_i^{(k)} - \lambda_j^{(k)} \big).$$

3. Receives $\widetilde{x}_j^{(k)}$ from agent $j \in \mathcal{N}_i^J$ and $\widetilde{\lambda}_j^{(k)}, \widetilde{\nu}_j^{(k)}$ from agent $j \in \mathcal{N}_i^\lambda$.
4. Updates:

$$\mathring{x}_i^{(k)} = \widetilde{x}_i^{(k)} - \rho_i \Big(\nabla_{x_i} f_i(\widetilde{\boldsymbol{x}}^{(k)}) - \nabla_{x_i} f_i(\boldsymbol{x}^{(k)}) + \nabla g_i(\widetilde{x}_i^{(k)})^\top \widetilde{\lambda}_i^{(k)} - \nabla g_i(x_i^{(k)})^\top \lambda_i^{(k)} \Big),$$
$$\mathring{\lambda}_i^{(k)} = \widetilde{\lambda}_i^{(k)} + \tau_i \Big(g_i(\widetilde{x}_i^{(k)}) - g_i(x_i^{(k)}) + \sum_{j \in \mathcal{N}_i^\lambda} \big(\widetilde{\nu}_i^{(k)} - \nu_i^{(k)} - \widetilde{\nu}_j^{(k)} + \nu_j^{(k)} - \widetilde{\lambda}_i^{(k)} + \lambda_i^{(k)} + \widetilde{\lambda}_j^{(k)} - \lambda_j^{(k)} \big) \Big),$$
$$\mathring{\nu}_i^{(k)} = \widetilde{\nu}_i^{(k)} - \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} \Big(\widetilde{\lambda}_i^{(k)} - \lambda_i^{(k)} - \widetilde{\lambda}_j^{(k)} + \lambda_j^{(k)} \Big).$$

5. Sends $(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)})$ to a coordinator and receives back $\nabla_{\omega_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)})$, where $\omega_i = (x_i, \lambda_i, \nu_i)$.
6. Updates:

$$(x_i^{(k+1)}, \lambda_i^{(k+1)}, \nu_i^{(k+1)}) = (\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)}) - \beta^{(k)} \nabla_{\omega_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)}). \tag{2.21}$$

2

2.5 Optimal equilibrium selection in cocoercive games

In this section, we discuss a special class of monotone games, namely cocoercive games with affine coupling constraints. These games arise as a generalization of the widely studied class of strongly monotone games [37],[35]. Differently from the strong monotonicity assumption, however, cocoercivity alone does not guarantee uniqueness of the v-GNE. The class of cocoercive games is characterized by the following Assumptions 2.8 and 2.9.

Assumption 2.8 ([17, Assm. 5])**.** *The mapping F in* (2.9) *is η-cocoercive.*

Assumption 2.9 ([17, Eq. 3])**.** *For each $i \in \mathcal{I}$, the function g_i in* (2.6b) *is affine, i.e., $g_i(x_i) := A_i x_i - b_i$, for some matrix $A_i \in \mathbb{R}^{m \times n_i}$ and vector $b_i \in \mathbb{R}^m$.*

For this particular class of games, the preconditioned forward-backward (pFB) splitting [37] can efficiently compute a v-GNE. We note that, although [37] considers games with strongly monotone pseudogradient, the pFB splitting only requires cocoercivity of the forward operator [51, Thm. 26.14]. Compared with the FBF, the pFB has the advantages of only having one communication round per iteration (as opposed to two) and larger step size bounds. A numerical performance comparison is provided in [16]. Given the particular structure of the coupling constraint as stated in Assumption 2.9, we can rewrite the operators in (2.14) as follows:

$$\mathcal{A}(\boldsymbol{\omega}) := \prod_{i \in \mathcal{I}} (\mathrm{N}_{\mathcal{X}_i} + \partial \ell_i)(x_i) \times \mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}(\boldsymbol{\lambda}) \times \{\mathbf{0}_{Nm}\}, \tag{2.22}$$

$$\mathcal{B}(\boldsymbol{\omega}) := \mathrm{col}(F(\boldsymbol{x}), (\mathcal{L} \otimes I_m)\boldsymbol{\lambda} + \boldsymbol{b}, \mathbf{0}_{Nm}), \tag{2.23}$$

$$\mathcal{C}(\boldsymbol{\omega}) := \mathrm{col}(\boldsymbol{A}^\top \boldsymbol{\lambda}, -\boldsymbol{A}\boldsymbol{x} - (\mathcal{L} \otimes I_m)\boldsymbol{\nu}, (\mathcal{L} \otimes I_m)\boldsymbol{\lambda}), \tag{2.24}$$

where $\boldsymbol{A} = \mathrm{diag}((A_i)_{i \in \mathcal{I}})$ and $\boldsymbol{b} = \mathrm{col}((b_i)_{i \in \mathcal{I}})$. Thus, the pFB operator for the monotone inclusion in (2.14) based on the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in (2.22)–(2.24) is given by [37, Eq. (24)]:

$$\mathcal{T}_{\text{pFB}} := \mathcal{R}_{\Phi^{-1}(\mathcal{A}+\mathcal{C})} \circ (\mathrm{Id} - \Phi^{-1}\mathcal{B}), \tag{2.25}$$

where $\Phi \succ 0$ is a symmetric positive definite preconditioning matrix, defined as

$$\Phi := \Psi + \begin{bmatrix} 0 & -\boldsymbol{A}^\top & 0 \\ -\boldsymbol{A} & 0 & -\mathcal{L} \otimes I_m \\ 0 & -\mathcal{L} \otimes I_m & 0 \end{bmatrix},$$

where Ψ is as in (2.20). Then, we can have an extension of the pFB for the v-GNE optimal selection of cocoercive games, as stated in Algorithm 2. The step sizes in Ψ need to be small enough with respect to η and to the matrices defining the constraints, as highlighted in Assumption 2.10, which states the sufficient conditions for the convergence of the pFB (see [37, Eq. (27) and Thm. 3]).

Assumption 2.10. *For all $i \in \mathcal{I}$:*

(i) $\rho_i \leq (\max_{j=1,\dots,n_i} \sum_{k=1}^m |[A_i^\top]_{jk}| + \delta)^{-1}$;

(ii) $\tau_i \leq (\max_{j=1,\dots,n_i} \sum_{k=1}^{m} |[A_i]_{jk}| + 2|\mathcal{N}_i^{\lambda}| + \delta)^{-1}$;

(iii) $\sigma_i \leq (2|\mathcal{N}_i^{\lambda}| + \delta)^{-1}$, *where* $\delta > 1/(\min(\eta, (2\max_{i\in\mathcal{I}} |\mathcal{N}_i^{\lambda}|)^{-1}))$.

2

Theorem 2.2. *Let Assumptions 2.1–2.4, 2.6, and 2.8–2.10 hold. Let* $\Omega^\star$ *be the set of solutions to Problem* (2.17) *with* $\mathcal{T} = \mathcal{T}_{\mathrm{pFB}}$ *defined in* (2.25)*, where* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$ *are defined in* (2.22)–(2.24)*. Furthermore, let* $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$*, where* $\boldsymbol{\omega}^{(k)} = (\boldsymbol{x}^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\nu}^{(k)})$*, be the sequence generated by Algorithm 2. Then,* $\lim_{k\to\infty} \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \Omega^\star) = 0$*, and* $(\boldsymbol{x}^{(k)})_{k\in\mathbb{N}}$ *converges to an optimal v-GNE of the game in* (2.6)*.*

Algorithm 2 Optimal v-GNE selection via pFB and HSDM for linearly coupled cocoercive games

Initialization: $x_i^{(0)} \in \mathcal{X}_i$, $\lambda_i^{(0)} \in \mathbb{R}^m_{\geq 0}$, and $\nu_i^{(0)} \in \mathbb{R}^m$, $\forall i \in \mathcal{I}$.
Iteration of each agent $i \in \mathcal{I}$**.**

1. Receives $x_j^{(k)}$ from agent $j \in \mathcal{N}_i^J$ and $\lambda_j^{(k)}$ from agent $j \in \mathcal{N}_i^{\lambda}$.
2. Updates:

$$
\begin{aligned}
\mathring{x}_i^{(k)} &= \mathrm{prox}^{\rho_i}_{\ell_i + \iota_{\mathcal{X}_i}} \Big(x_i^{(k)} - \rho_i \big(\nabla_{x_i} f_i(\boldsymbol{x}^{(k)}) + A_i^\top \lambda_i^{(k)} \big) \Big), \\
\mathring{\nu}_i^{(k)} &= \nu_i^{(k)} - \sigma_i \sum_{j\in\mathcal{N}_i^{\lambda}} \big(\lambda_i^{(k)} - \lambda_j^{(k)} \big).
\end{aligned}
$$

3. Receives $\mathring{\nu}_j^{(k)}$ from agent $j \in \mathcal{N}_i^{\lambda}$.
4. Updates:

$$
\begin{aligned}
\mathring{\lambda}_i^{(k)} = \mathrm{proj}_{\mathbb{R}_{\geq 0}} \Big(\lambda_i^{(k)} &+ \tau_i \big(A_i (2\mathring{x}_i^{(k)} - x_i^{(k)}) - b_i \\
&+ \sum_{j\in\mathcal{N}_i^{\lambda}} (2\mathring{\nu}_i^{(k)} - 2\mathring{\nu}_j^{(k)} - \nu_i^{(k)} + \nu_j^{(k)} - \lambda_i^{(k)} + \lambda_j^{(k)}) \big) \Big).
\end{aligned}
$$

5. Sends $(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)})$ to a coordinator and receives back $\nabla_{\omega_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)})$, where $\omega_i = (x_i, \lambda_i, \nu_i)$.
6. Updates:

$$
(x_i^{(k+1)}, \lambda_i^{(k+1)}, \nu_i^{(k+1)}) = (\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)}) - \beta^{(k)} \nabla_{\omega_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)}). \tag{2.26}
$$

2

2.6 Online tracking of optimal GNEs

2.6.1 Online optimal equilibrium tracking problem

In the second part of this chapter, we consider the online GNE selection problem. Specifically, let us introduce the time-varying game:

$$\forall t \in \mathbb{N}, \forall i \in \mathcal{I}\colon \begin{cases} \min\limits_{x_i \in \mathcal{X}_{i,t}} & J_{i,t}(\boldsymbol{x}) \quad (2.27a) \\ \text{s.t.} & \sum\limits_{j \in \mathcal{I}} g_{j,t}(x_j) \leq 0, \quad (2.27b) \end{cases}$$

where t denotes the time index. The problem is time-varying in the sense that the objective functions of the agents, as well as the constraints, may vary over time. We assume that each instance of the games in (2.27) satisfies Assumptions 2.1 and 2.2. The time-varying GNE selection problem thus concerns the tracking of the sequence $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$:

$$\forall t \in \mathbb{N}\colon\ \boldsymbol{\omega}_t^\star := \begin{cases} \operatorname*{argmin}\limits_{\boldsymbol{\omega}} \ \phi_t(\boldsymbol{\omega}) \quad (2.28a) \\ \quad \text{s.t.}\ \boldsymbol{\omega} \in \operatorname{zer}(\mathcal{A}_t + \mathcal{B}_t + \mathcal{C}_t). \quad (2.28b) \end{cases}$$

The problems in (2.27) and (2.28) are a sequence in time of instances of (2.6) and (2.15), respectively. The operators $\mathcal{A}_t$, $\mathcal{B}_t$, and $\mathcal{C}_t$ are defined in (2.11)–(2.13), for the game in (2.27) at time step t. The agents need to compute the action $\boldsymbol{\omega}_{t+1}$, having only access to the game formulation up to time t. This setup describes the case in which the agents act in a variable environment with limited computation capabilities, so that they cannot compute the exact optimal selection before changes in the problem (either in the selection function or in the game) occur. The problem in (2.28) reduces to an online optimization problem for $|\mathcal{I}| = 1$, see for example [54] and the references therein. Inspired by the online optimization literature, we propose to track the solution sequence $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$ by computing at each time step t an approximate solution of the problem at time $t-1$. Such a solution principle is based on the assumption that $\boldsymbol{\omega}_{t-1}^\star$ contains information on $\boldsymbol{\omega}_t^\star$, which is a standard assumption in online optimization, see e.g. [48, Assm. 1], [55, Assm. 3.1], and [62, Assm. 3] and it is introduced next.

Assumption 2.11. *There exists $\delta \geq 0$ such that*

$$\sup_{t\in\mathbb{N}} \|\boldsymbol{\omega}_{t+1}^\star - \boldsymbol{\omega}_t^\star\| \leq \delta.$$

For every $t \in \mathbb{N}$, and under a suitable choice of the operator $\mathcal{T}_t$, such that

$$\boldsymbol{\omega} \in \operatorname{zer}(\mathcal{A}_t + \mathcal{B}_t + \mathcal{C}_t) \Leftrightarrow \boldsymbol{\omega} \in \operatorname{fix}(\mathcal{T}_t),$$

$\boldsymbol{\omega}_t^\star$ in (2.28) can be equivalently found as the solution of the time-varying fixed-point selection problem

$$\inf_{\boldsymbol{\omega} \in \operatorname{fix}(\mathcal{T}_t)} \langle \boldsymbol{\omega} - \boldsymbol{\omega}_t^\star, \nabla\phi_t(\boldsymbol{\omega}_t^\star)\rangle \geq 0. \tag{2.29}$$

The sequence $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$ is well-defined when, for each t, the solution of (2.28) is unique. Let us then introduce the following assumption, which guarantees uniqueness if $\mathrm{fix}(\mathcal{T}_t)$ is closed and convex for all t [63, Thm. 2.3.3]. This is the case, for example, when $\mathcal{T}_t$ is quasi-nonexpansive [15, Prop. 1a]:

Assumption 2.12. *The selection function $\phi_t\colon \mathbb{R}^{n_\omega} \to \mathbb{R}$ in* (2.29) *is continuously differentiable, σ-strongly convex, and has L_ϕ-Lipschitz continuous gradient for all $t \in \mathbb{N}$.*

Remark 2.4. *Under Assumption 2.12, if $\mathcal{T}_t = \mathcal{T}$, for all $t \in \mathbb{N}$, and the selection function at time t is the sampling of a function that varies continuously over time, then an estimate for δ in Assumption 2.11 can be found. In fact, let $\phi(\boldsymbol{\omega},t)$ be continuously differentiable, where we made the dependency on the (continuous) time index explicit. Then, we find by [64, Thm. 2F.7] that the mapping from t to the solution of* (2.29) *is locally Lipschitz continuous with Lipschitz constant $\sigma^{-1}|\nabla_t\phi(\boldsymbol{\omega}_t^\star,t)|$. Thus, if the time variation between two consecutive time steps t_1 and t_2 is small enough, δ can be estimated as $\sigma^{-1}|\nabla_t\phi(\boldsymbol{\omega}_{t_1}^\star,t_1)|(t_2-t_1)$. The solution mapping is in general discontinuous when $\mathcal{T}_t$ is time-varying; thus, a similar estimate cannot be found in the general case.*

In the remainder of this section, we build upon the results of Section 2.4 to derive an HSDM-inspired algorithm for tracking $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$.

2.6.2 The restarted hybrid steepest descent method

The existing results on the HSDM algorithm study the asymptotic behavior with vanishing step size $(\beta^{(k)})_{k\in\mathbb{N}}$ (see Assumption 2.4). However, in online scenarios, decision makers may not have the computational capability to exactly compute the fixed point of the algorithm, since that would require an infinite amount of iterations in a limited time span before a new instance of the problem becomes available. Thus, we propose and study the (approximate) convergence properties of an algorithm that only performs a finite number of HSDM iterations per time step. Consequently, the sequence of step sizes becomes truncated and a sequence of vanishing step sizes, which is required for the convergence of the HSDM, cannot be defined. We therefore simplify the analysis by considering a constant sequence of step sizes.

Let us introduce the restarted HSDM algorithm. Given an initial state $\boldsymbol{\omega}_1$, for each $t \in \mathbb{N}$, we propose the following:

$$\begin{aligned}\boldsymbol{y}^{(k+1)} &:= \begin{cases}\boldsymbol{\omega}_t, & \text{for } k=1,\\ \mathcal{T}_t(\boldsymbol{y}^{(k)}) - \beta\nabla\phi_t\big(\mathcal{T}_t(\boldsymbol{y}^{(k)})\big), & \text{for } k=2,\dots,K,\end{cases}\\ \boldsymbol{\omega}_{t+1} &:= \boldsymbol{y}^{(K+1)}.\end{aligned} \tag{2.30}$$

In words, at each time step t the auxiliary variable $\boldsymbol{y}^{(k)}$, with $k = 1,\dots K$, is updated with K iterations of the HSDM. Then, the decision variable at time step $t+1$ is obtained as $\boldsymbol{\omega}_{t+1} = \boldsymbol{y}^{(K+1)}$. The algorithm is then restarted when the information on the selection function and game for the next time step becomes available. Next, let us postulate the following technical assumptions:

Assumption 2.13. *There exists a compact set $\mathcal{Y}$ such that $\boldsymbol{\omega}_t^\star \in \mathcal{Y}$ for all $t \in \mathbb{N}$.*

Assumption 2.14. *There exists $U \geq 0$ such that $\sup_{\boldsymbol{\omega} \in \bigcup_{\tau \in \mathbb{N}} \mathrm{Im}(\mathcal{T}_\tau), t \in \mathbb{N}} \|\nabla \phi_t(\boldsymbol{\omega})\| \leq U$.*

2

The set $\mathcal{Y}$ introduced in Assumption 2.13 is only used in the analysis and its existence is practically reasonable, since we can assume that we do not aim at tracking a divergent sequence. Assumption 2.14 specifies an upper bound for the gradient of the selection function and is in line with the online optimization literature (see [65, Assm. 5], [62, Assm. 5], among others). As shown in Section 2.4, the HSDM method converges to the solution of a selection problem over the fixed point set of a quasi-shrinking operator. In the online scenario, assuming the operator $\mathcal{T}_t$ to be quasi-shrinking for all t is not enough, as the quasi-shrinking property might not hold asymptotically. Thus, we also postulate the technical Assumption 2.15.

Assumption 2.15. *(Uniformly quasi-shrinking operator) For any closed convex set C such that $C \cap \mathrm{fix}(\mathcal{T}_t) \neq \varnothing$, there exists $D : \mathbb{R}_{\geq 0} \to \mathbb{R}$ positive semidefinite such that $D_t(r) \geq D(r)$ for all $t \in \mathbb{N}$ and for all $r \geq 0$, where $D_t(\cdot)$ is the shrinkage function of $\mathcal{T}_t$ defined as in* (2.4).

Remark 2.5. *Assumptions 2.13, 2.14, and 2.15 are satisfied for example when at every time step t, the feasible set of Problem* (2.28) *is selected among the GNE sets of finitely many monotone games with a compact decision space. In fact, let $(\mathcal{T}_{\mathrm{FBF}}^h)_{h=1}^H$ be the set of FBF operators such that, for all $t \in \mathbb{N}$, there exists $h \in \{1, \ldots, H\}$ such that $\mathrm{zer}(\mathcal{A}_t + \mathcal{B}_t + \mathcal{C}_t) = \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}}^h)$. The operators $(\mathcal{T}_{\mathrm{FBF}}^h)_{h=1}^H$ are quasi-shrinking. Let us denote by $D^h(\cdot)$ the shrinkage function of $\mathcal{T}_{\mathrm{FBF}}^h$. As the minimum among a finite number of positive semi-definite functions is also finite, Assumption 2.15 is then satisfied with $D(r) = \min_{h \in \{1,\ldots,H\}} D^h(r)$. Furthermore, Assumption 2.13 holds with $\mathcal{Y} = \cup_{h=1}^H \mathrm{Im}(\mathcal{T}_{\mathrm{FBF}}^h)$, which is compact, and Assumption 2.14 holds as $\nabla \phi_t$ is L_ϕ−Lipschitz continuous for all t on a compact set.*

We find that the restarted HSDM (2.30) asymptotically tracks the solution trajectory of the online fixed point selection problem in (2.29), with an asymptotic error that can be controlled up to the variability of the problem, δ, via an appropriate choice of β and K, as shown in Theorem 2.3.

Theorem 2.3. *Let Assumptions 2.11–2.15 hold. Let the sequence $(\boldsymbol{\omega}_t)_{t \in \mathbb{N}}$ be generated by the restarted HSDM in* (2.30). *For any $\gamma > 0$, there exist $\beta \in (0, \frac{2\sigma}{L_\phi^2})$ and $\overline{K}$, such that, for all $K \geq \overline{K}$, the sequence $(\boldsymbol{\omega}_t)_{t \in \mathbb{N}}$ is bounded and*

$$\limsup_{t \to \infty} \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|^2 \leq \frac{(\gamma + \delta^2)}{1/2 - \alpha}, \tag{2.31}$$

where $\alpha = (1 - \tau(\beta))^K < \frac{1}{2}$ and $\tau(\beta) := 1 - \sqrt{1 - \beta(2\sigma - \beta L_\phi^2)} \in (0, 1)$.

Remark 2.6. *As it follows from the proof of Theorem 2.3 (see Remark 2.10 in the Appendix), to control the approximation error in (2.31), β must be chosen small so to obtain small values of γ. However, the value $\tau(\beta)$ tends to 0 for small values of*

β. This leads the denominator in (2.31) to be small for a small step size, unless the number of iterations K is increased. Therefore, a smaller step size leads to a better approximation error only if it is shouldered by an increase in the number of iterations of the algorithm per time step.

In the next section, we use the restarted HSDM to solve the online GNE tracking problem in (2.28).

2.6.3 Optimal equilibrium tracking algorithm

We recall from Section 2.4 that the set of v-GNEs for a monotone game can be characterized as the set of fixed points of the operator $\mathcal{T}_{\text{FBF}}$ defined in (2.19). Thus, for the time-varying game in (2.27) at time t, let $\mathcal{T}_{\text{FBF},t}$ be the FBF operator defined as:

$$\mathcal{T}_{\text{FBF},t} := \big(\mathrm{Id} - \Psi^{-1}(\mathcal{B}_t + \mathcal{C}_t)\big) \circ \mathcal{R}_{\Psi^{-1}\mathcal{A}_t} \circ \big(\mathrm{Id} - \Psi^{-1}(\mathcal{B}_t + \mathcal{C}_t)\big) + \Psi^{-1}(\mathcal{B}_t + \mathcal{C}_t), \tag{2.32}$$

where $\mathcal{A}_t$, $\mathcal{B}_t$, and $\mathcal{C}_t$ are those in Problem (2.28) and associated with the game in (2.27) at time t. The solutions of the time-varying GNE selection problem in (2.28) are equivalent to the solutions of (2.29), with $\mathcal{T}_t = \mathcal{T}_{\text{FBF},t}$ for all t. By Lemma 2.3, $\mathcal{T}_{\text{FBF},t}$, for each t, is a quasi-nonexpansive, quasi-shrinking operator. Therefore, the restarted HSDM algorithm in (2.30) can be employed for tracking the solution trajectory, with an asymptotic tracking error given by Theorem 2.3. We can then bound the asymptotic optimal GNE tracking error of Algorithm 3 by using Theorem 2.3, as formally stated next.

Corollary 2.1. *Let us consider the online GNE tracking problem in* (2.28) *for the time-varying game in* (2.27) *that satisfies Assumptions 2.1, 2.2, 2.6, for each $t \in \mathbb{N}$. Suppose that Assumptions 2.11 –2.14 hold and let $\mathcal{T}_t = \mathcal{T}_{\text{FBF},t}$ satisfy Assumption 2.15. Then, for any $\gamma > 0$ there exist $\beta \in (0, \frac{2\sigma}{L_\phi^2})$ and $\overline{K}$ such that, for any $K \geq \overline{K}$, the asymptotic tracking error of Algorithm 3 is given by* (2.31).

Remark 2.7. *The solution sequence computed by Algorithm 3, $(\boldsymbol{\omega}_t)_{t\in\mathbb{N}}$, can violate the constraints of the game in* (2.27)*. Such violation can be estimated using the Lipschitz continuity of $g_{i,t}$ for all $i \in \mathcal{I}$ (which follows from Assumption 1) and Theorem 2.3. Let us denote by L_g the maximum Lipschitz constant of $g_{i,t}$, for all $i \in \mathcal{I}$ and $t \in \mathbb{N}$, and by $(x_{i,t})_{i\in\mathcal{I}}$ the primal variables associated to $\boldsymbol{\omega}_t$. Then,*

$$\limsup_{t\to\infty} \sum_{i\in\mathcal{I}} g_{i,t}(x_{i,t}) \leq L_g \sqrt{\frac{\gamma + \delta^2}{1/2 - \alpha}}.$$

Remark 2.8. *The result of this section holds similarly if we substitute the FBF operator with the pFB operator in* (2.25)*, which is quasi-shrinking (see the proof of Theorem 2.2), for cocoercive games with affine coupling constraints.*

Algorithm 3 Optimal v-GNE tracking via FBF and HSDM

Initialization: $x_{i,0} \in \mathcal{X}_i$, $\lambda_{i,0} \in \mathbb{R}^m_{\geq 0}$, and $\nu_{i,0} \in \mathbb{R}^m$, $\forall i \in \mathcal{I}$.
Iteration at time $t \in \mathbb{N}_0$ **of each agent** $i \in \mathcal{I}$:

1. Receives $J_{i,t}(\cdot)$, $g_{i,t}(\cdot)$, and $\mathcal{X}_{i,t}$.
2. Assigns $\hat{x}_i^{(1)} \leftarrow x_{i,t}$, $\hat{\lambda}_i^{(1)} \leftarrow \lambda_{i,t}$, and $\hat{\nu}_i^{(1)} \leftarrow \nu_{i,t}$.
3. **For** $k = 1, \ldots, K$:

(i) Receives $\hat{x}_j^{(k)}$ from agent $j \in \mathcal{N}_i^J$ and $\hat{\lambda}_j^{(k)}, \hat{\nu}_j^{(k)}$ from agent $j \in \mathcal{N}_i^\lambda$.

(ii) Updates:

$$\begin{aligned}
\widetilde{x}_i^{(k)} &= \mathrm{prox}^{\rho_i}_{\ell_{i,t}+\iota_{\mathcal{X}_{i,t}}} \left(\hat{x}_i^{(k)} - \rho_i \left(\nabla_{x_i} f_{i,t}(\hat{\boldsymbol{x}}^{(k)}) + \nabla g_{i,t}(\hat{x}_i^{(k)})^\top \hat{\lambda}_i^{(k)} \right) \right), \\
\widetilde{\lambda}_i^{(k)} &= \mathrm{proj}_{\geq 0} \left(\hat{\lambda}_i^{(k)} + \tau_i \left(g_{i,t}(\hat{x}_i^{(k)}) + \sum_{j \in \mathcal{N}_i^\lambda} (\hat{\nu}_i^{(k)} - \hat{\nu}_j^{(k)} - \hat{\lambda}_i^{(k)} + \hat{\lambda}_j^{(k)}) \right) \right), \\
\widetilde{\nu}_i^{(k)} &= \hat{\nu}_i^{(k)} - \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} \left(\hat{\lambda}_i^{(k)} - \hat{\lambda}_j^{(k)} \right).
\end{aligned}$$

(iii) Receives $\widetilde{x}_j^{(k)}$ from agent $j \in \mathcal{N}_i^J$ and $\widetilde{\lambda}_j^{(k)}, \widetilde{\nu}_j^{(k)}$ from agent $j \in \mathcal{N}_i^\lambda$.

(iv) Updates:

$$\begin{aligned}
\mathring{x}_i^{(k)} =& \widetilde{x}_i^{(k)} - \rho_i \Big(\nabla_{x_i} f_{i,t}(\widetilde{\boldsymbol{x}}^{(k)}) - \nabla_{x_i} f_{i,t}(\hat{\boldsymbol{x}}^{(k)}) \\
& + \nabla g_{i,t}(\widetilde{x}_i^{(k)})^\top \widetilde{\lambda}_i^{(k)} - \nabla g_{i,t}(\hat{x}_i^{(k)})^\top \hat{\lambda}_i^{(k)} \Big), \\
\mathring{\lambda}_i^{(k)} =& \widetilde{\lambda}_i^{(k)} + \tau_i \Big(g_{i,t}(\widetilde{x}_i^{(k)}) - g_{i,t}(\hat{x}_i^{(k)}) \\
& + \sum_{j \in \mathcal{N}_i^\lambda} \left(\widetilde{\nu}_i^{(k)} - \hat{\nu}_i^{(k)} - \widetilde{\nu}_j^{(k)} + \hat{\nu}_j^{(k)} - \widetilde{\lambda}_i^{(k)} + \hat{\lambda}_i^{(k)} + \widetilde{\lambda}_j^{(k)} - \hat{\lambda}_j^{(k)} \right) \Big), \\
\mathring{\nu}_i^{(k)} =& \widetilde{\nu}_i^{(k)} - \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} \left(\widetilde{\lambda}_i^{(k)} - \hat{\lambda}_i^{(k)} - \widetilde{\lambda}_j^{(k)} + \hat{\lambda}_j^{(k)} \right).
\end{aligned}$$

(v) Sends $(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)})$ to a coordinator and receives $\nabla_{\boldsymbol{\omega}_i} \phi_t(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)})$, where $\boldsymbol{\omega}_i = (x_i, \lambda_i, \nu_i)$.

(vi) Updates:

$$(\hat{x}_i^{(k+1)}, \hat{\lambda}_i^{(k+1)}, \hat{\nu}_i^{(k+1)}) = (\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)}) - \beta \nabla_{\boldsymbol{\omega}_i} \phi_t(\mathring{x}_i^{(k)}, \mathring{\lambda}_i^{(k)}, \mathring{\nu}_i^{(k)}).$$

End For

4) Sets $x_{i,t+1} \leftarrow \hat{x}_i^{(K+1)}$, $\lambda_{i,t+1} \leftarrow \hat{\lambda}_i^{(K+1)}$, $\nu_{i,t+1} \leftarrow \hat{\nu}_i^{(K+1)}$.

2.7 Illustrative example

We consider a peer-to-peer electricity market clearing problem with operational constraints of the electrical network, adapted from [28]. We assume that each bus of a distribution network consists of one agent that has access to either a storage unit or a dispatchable generation unit. Each agent $i \in \mathcal{I}$ has decision authority on the power generated $p^{\mathrm{g}}_{i,h}$, the power bought from the main grid $p^{\mathrm{mg}}_{i,h}$, the power drawn from the storage unit $p^{\mathrm{st}}_{i,h}$, the power traded with the trading partners $p^{\mathrm{tr}}_{(i,j),h}$, $j \in \mathcal{N}_i$ and the phase at the bus $\theta_{i,h}$ over the horizon $h = 1,\dots,H$. Let us denote $\boldsymbol{x}_{i,h} = \mathrm{col}(p^{\mathrm{g}}_{i,h}, p^{\mathrm{mg}}_{i,h}, p^{\mathrm{st}}_{i,h}, (p^{\mathrm{tr}}_{(i,j),h})_{j\in\mathcal{N}_i}, \theta_{i,h})$, for all $i \in \mathcal{I}$ and $h = 1,\dots,H$, and denote $\boldsymbol{x}_i := \mathrm{col}((\boldsymbol{x}_{i,h})_{h=1,\dots,H})$, $\boldsymbol{x} := \mathrm{col}((\boldsymbol{x}_i)_{i\in\mathcal{I}})$. Each agent aims at minimizing its local cost function [28, Eq. (17)]:

$$J_i(\boldsymbol{x}) = \textstyle\sum_{h=1}^{H} f^{\mathrm{g}}_{i,h}(p^{\mathrm{g}}_{i,h}) + f^{\mathrm{tr}}_{i,h}\big((p^{\mathrm{tr}}_{(i,j),h})_{j\in\mathcal{N}_i}\big) + f^{\mathrm{mg}}_{i,h}(p^{\mathrm{mg}}_{i,h}, p^{\mathrm{mg}}_{-i,h}), \tag{2.33}$$

where $f^{\mathrm{tr}}_{i,h}$ encodes the cost or revenue of the trading with other agents and $f^{\mathrm{mg}}_{i,h}$ encodes the cost of purchasing energy from the main grid as in [28, Eq. (11)], while $f^{\mathrm{g}}_{i,h}$ is a linear function which encodes the cost of power generation. The local feasible sets $\mathcal{X}_i$, $i = 1,\dots,N$ include the satisfaction of the power demand at the bus, as well as the operating constraints of the generators and storage units. The shared constraints are of the form $g(\boldsymbol{x}) \le \mathbf{0}_{n_c}$, with g affine. They include the operating limits of the grid, the trading reciprocity $\{p^{\mathrm{tr}}_{(i,j),h} = -p^{\mathrm{tr}}_{(j,i),h}, \ \forall\, i \in \mathcal{N}, \ \forall\, j \in \mathcal{N}_i\}$ and the linearized power flow equations with DC approximation $\{p^{\mathrm{g}}_{i,h} + p^{\mathrm{st}}_{i,h} + \iota^{\mathrm{mg}}_i \sum_{j\in\mathcal{N}} p^{\mathrm{mg}}_{j,h} + \sum_{j\in\mathcal{B}_i} B_{ij}(\theta_{i,h} - \theta_{j,h}) = 0\}$, where $\iota^{\mathrm{mg}}_i \in \{0,1\}$ is 1 if and only if i is connected to the main grid, $\mathcal{B}_i$ is the set of buses that are connected to bus i on the electric grid and B is the susceptance matrix. We note that the game satisfies Assumptions 2.1 and 2.2. In addition, we consider the IEEE 13-bus distribution feeder for our numerical simulations, performed in Matlab.

We first simulate the day-ahead market clearing (with 24 hourly time steps) via the standard FBF-based algorithm, which can obtain a v-GNE, and Algorithm 1, which solves the optimal selection problem of this game. Specifically, we consider the GNE selection function:

$$\begin{aligned}\phi(\boldsymbol{x}) = \textstyle\sum_{h=1}^{H} \{ &\|\boldsymbol{p}^{\mathrm{g}}_h - \overline{\boldsymbol{p}}^{\mathrm{g}}\|^2_{Q_{\mathrm{d}}} + \|\boldsymbol{p}^{\mathrm{mg}}_h\|^2_{Q_{\mathrm{mg}}} + \|\boldsymbol{\theta}_h - \overline{\boldsymbol{\theta}}\|^2_{Q_\theta} \\ &+ \|G\boldsymbol{\theta}_h\|^2_{Q_{\mathrm{pf}}} + \|\boldsymbol{p}^{\mathrm{tr}}_h\|^2_{Q_{\mathrm{tr}}} + \|\boldsymbol{p}^{\mathrm{st}}_h\|^2_{Q_{\mathrm{st}}}\} + \|\boldsymbol{\lambda}\|^2_{Q_\lambda} + \|\boldsymbol{\nu}\|^2_{Q_\nu},\end{aligned} \tag{2.34}$$

where we denoted in bold the column stack of the respective variables for each agent and the matrices $Q_\star$ are diagonal positive definite. We choose $\overline{\boldsymbol{p}}^{\mathrm{g}}$ to be the column vector of the maximum generation production for each agent, in order to maximize the renewable energy production, and $\overline{\boldsymbol{\theta}}$ to be a vector which elements are all equal to the phase of the node connected to the main grid, in order to reduce the grid imbalances. The cost factors related to $\boldsymbol{p}^{\mathrm{mg}}, \boldsymbol{p}^{\mathrm{st}}, \boldsymbol{p}^{\mathrm{tr}}$ aim at reducing the burden on the transmission grid, increasing the lifespan of the storage units and reducing the load of the trading platform, respectively. The terms in $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ act as regularization of the dual variables. Finally, G is a matrix that maps the phase of the nodes to

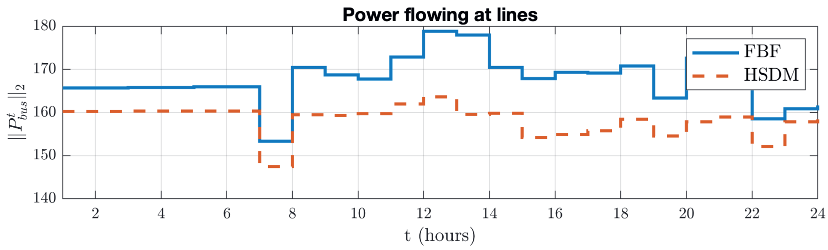
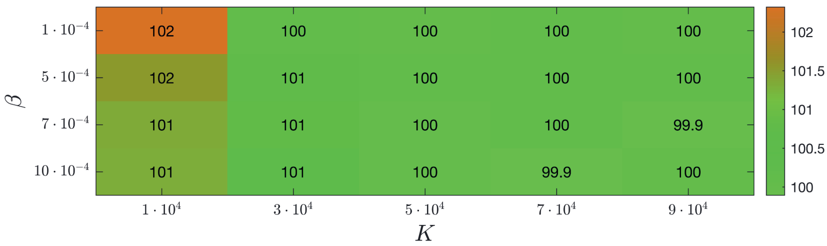


Figure 2.1: Total power flow achieved by the proposed algorithm compared to standard FBF in the day-ahead market scenario.

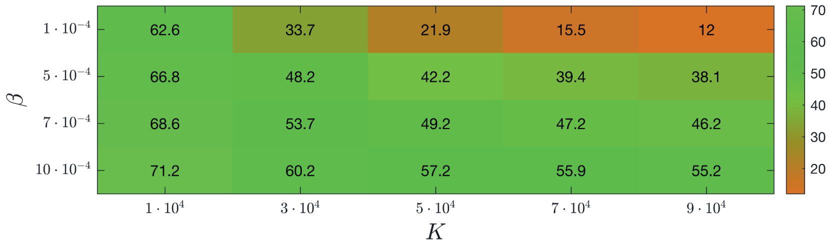
the power flowing through the lines. In this test, we aim at maximizing the lifespan of the grid lines by setting the non-zero elements of Q_{pf} to be large. We observe that, as expected, the solution obtained by Algorithm 1 achieves a 10.8% lower value of the selection function defined in (2.34) compared with the one achieved by the standard FBF, since the v-GNE computed by Algorithm 1 minimizes (2.34). In Figure 2.1, we observe that Algorithm 1 generates solutions with less congestion (power flow) than that of the standard FBF, as intended by the term of the selection function in (2.34) weighted by Q_{pf}.
Secondly, we test Algorithm 3 on a real-time market scenario, formulated as a time-varying game. Because of the variability along the day of the power demand, the local power balance constraint defined in [28, Eq. (6)] depends on t. The constraints of the game are therefore time-varying. We aim at computing a v-GNE that minimizes the power flowing on the line connecting buses 632 and 671 during peak hours. Thus, we consider (2.34) as the selection function at each t where the element of Q_{pf} related to this line is time-varying, i.e., it is set high between the peak hours, i.e., 8AM and 4PM. We note that this setup falls into the case considered in Remark 2.5, as only a finite umber of game instances are considered, whilst $(\phi_t)_t$ satisfies Assumption 2.12. The problem is solved every 15 minutes using the power balance constraints and selection function formulated at time-step t. After the computation is performed, the system implements the computed v-GNE at time $t+1$. The simulation is run over a 24 hour span, thus resulting in 96 consecutive instances of GNE selection problems. Due to the relatively short sampling time, the demand is not expected to vary a lot between two consecutive game instances. We can then consider Assumption 2.11 to be satisfied. We run the simulation for different values of the parameters K and β, and we compare the results with the baseline solution ($\boldsymbol{\omega}_t^{\mathrm{FBF}}$) obtained by running at each time-step the standard FBF algorithm for a limited ($9 \cdot 10^4$, that is, the largest K on which we tested the restarted HSDM) number of iterations. Figure 2.2a illustrates the relative average residual obtained by restarted HSDM with respect to the baseline solution, where the residual is computed as

$$\mathfrak{R}_{t+1}(\boldsymbol{\omega}_t) = \|\mathcal{T}_{\mathrm{FBF},t+1}(\boldsymbol{\omega}_t) - \boldsymbol{\omega}_t\|.$$

The residual provides a measure of the constraint satisfaction for the problem in (2.28), and we observe a comparable performance. However, our algorithm achieves a significant improvement on the selection function values, as shown in Figure 2.2b.



(a) Average residual with respect to the baseline sequence $\frac{\mathfrak{R}_{t+1}(\boldsymbol{\omega}_t)}{\mathfrak{R}_{t+1}(\boldsymbol{\omega}_t^{\mathrm{FBF}})}$ (%).



(b) Average cost improvement with respect to the baseline $\frac{\phi_{t+1}(\boldsymbol{\omega}_t)-\phi_{t+1}(\boldsymbol{\omega}_t^{\mathrm{FBF}})}{\phi_{t+1}(\boldsymbol{\omega}_t^{\mathrm{FBF}})}$ (%).

Figure 2.2: Algorithm performance for several restarted HSDM parameters.

Furthermore, increasing K might lead to a reduction in cost advantage, as outlined by Figure 2.2b, because for low values of K the solution approaches the unconstrained minimizer of ϕ_t, while for high values of K it approaches the minimizer within the v-GNE set. We also observe that a diminishing β implies a slower reduction of the cost function, which results in a higher cost as shown in Figure 2.2b. Each iteration of the algorithm is computed in approximately 15ms, thus in the considered 15 minutes time step an agent is able to compute circa $6 \cdot 10^4$ iterations. In the presented simulations, we consider larger values of K to better show the benefit of the increased number of iterations towards the tracking precision.

2.8 Conclusion

The optimal generalized Nash equilibrium selection problem in monotone games can be solved distributively by combining the hybrid steepest descent method with an appropriate fixed-point operator. The key requirement to guarantee convergence to the set of optimal generalized Nash equilibria is the quasi-shrinking property, which holds true for certain fixed-point operators. The hybrid steepest descent method can also be modified to track a time-varying optimal generalized Nash equilibria. The resulting approach is suitable for real-time decision-making in multiagent dynamic environments. Future works include i) improving the convergence rate of the proposed method via second-order information of the selection function and/or inertial terms, and ii) developing distributed Tikhonov-based methods for generalized Nash equilibrium selection problems as benchmarks for the proposed method.

Appendix

2.A Proof of Lemma 2.1

We prove by contradiction. We assume that there exists $r > 0$ such that $D_\Psi(r) = 0$. Thus, by the definition of D_Ψ in (2.4), there exists a sequence $(\boldsymbol{\omega}_k)_{k\in\mathbb{N}} \in (\mathrm{fix}(\mathcal{T})^\Psi_{\geq r})\bigcap C$ such that

$$\lim_{k\to\infty} \mathrm{dist}_\Psi(\boldsymbol{\omega}_k, \mathrm{fix}(\mathcal{T})) - \mathrm{dist}_\Psi(\mathcal{T}(\boldsymbol{\omega}_k), \mathrm{fix}(\mathcal{T})) = 0.$$

By the definition of projection, we have

$$\begin{aligned} \mathrm{dist}_\Psi(\mathcal{T}(\boldsymbol{\omega}_k), \mathrm{fix}(\mathcal{T})) &= \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\mathcal{T}(\boldsymbol{\omega}_k))\|_\Psi \\ &\leq \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi. \end{aligned} \tag{2.35}$$

By the quasi-nonexpansiveness of $\mathcal{T}$ (implied by (2.5)) and the latter inequality,

$$\begin{aligned} 0 &\leq \underbrace{\|\boldsymbol{\omega}_k - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi}_{=\mathrm{dist}_\Psi(\boldsymbol{\omega}_k, \mathrm{fix}(\mathcal{T}))} - \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi \\ &\leq \mathrm{dist}_\Psi(\boldsymbol{\omega}_k, \mathrm{fix}(\mathcal{T})) - \mathrm{dist}_\Psi(\mathcal{T}(\boldsymbol{\omega}_k), \mathrm{fix}(\mathcal{T})) \xrightarrow{k\to\infty} 0. \end{aligned}$$

It follows that

$$\lim_{k\to\infty} \|\boldsymbol{\omega}_k - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi - \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi = 0.$$

By (2.5), we then have that

$$\begin{aligned} \|\boldsymbol{\omega}_k - \mathcal{T}_2(\boldsymbol{\omega}_k)\|^2_\Psi &\leq \tfrac{1}{\gamma}(\|\boldsymbol{\omega}_k - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|^2_\Psi - \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|^2_\Psi) \\ &\leq \tfrac{2d}{\gamma}(\|\boldsymbol{\omega}_k - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi - \|\mathcal{T}(\boldsymbol{\omega}_k) - \mathrm{proj}^\Psi_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}_k)\|_\Psi), \end{aligned}$$

where the latter inequality follows from $a^2 - b^2 = (a-b)(a+b)$ for $a, b \in \mathbb{R}$ and where we substituted $d := \sup_{\boldsymbol{\omega}\in C} \|\boldsymbol{\omega}_k - \boldsymbol{\omega}\|_\Psi$, which is finite since the set C is compact. We conclude that

$$\lim_{k\to\infty} \|\boldsymbol{\omega}_k - \mathcal{T}_2(\boldsymbol{\omega}_k)\|^2_\Psi = 0. \tag{2.36}$$

By the Bolzano-Weierstrass theorem and the boundedness of $\boldsymbol{\omega}_k$, there exists a convergent subsequence $(\boldsymbol{\omega}_{k_l})_{l\in\mathbb{N}}$ with accumulation point $\boldsymbol{\omega}^\infty$. By (2.36), $\lim_{l\to\infty} \mathcal{T}_2(\boldsymbol{\omega}_{k_l}) = \boldsymbol{\omega}^\infty$. By the demiclosedness of $\mathrm{Id} - \mathcal{T}_2$ and by $\mathrm{fix}(\mathcal{T}_2) \subset \mathrm{fix}(\mathcal{T})$, $\boldsymbol{\omega}^\infty - \mathcal{T}_2(\boldsymbol{\omega}^\infty) = 0 \Rightarrow \boldsymbol{\omega}^\infty \in \mathrm{fix}(\mathcal{T}_2) \Rightarrow \boldsymbol{\omega}^\infty \in \mathrm{fix}(\mathcal{T})$. However, since $(\mathrm{fix}(\mathcal{T})^\Psi_{\geq r})\bigcap C$ is a closed set, then $\boldsymbol{\omega}^\infty \in \mathrm{fix}(\mathcal{T})^\Psi_{\geq r}$, which is in contradiction with $\boldsymbol{\omega}^\infty \in \mathrm{fix}(\mathcal{T})$. ■

2.B Properties of operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in (2.11)–(2.13)

Lemma 2.4. *Let Assumption 2.1 hold. Then, the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in* (2.11)–(2.13) *and the operator $\mathcal{A}+\mathcal{B}+\mathcal{C}$ are maximally monotone.* □

Proof. By Assumption 2.1, $\mathrm{N}_{\mathcal{X}_i}$ and $\partial\ell_i$ are maximally monotone [51, Thm. 20.25 & Example 20.26]. The operator $\mathcal{A}$ is thus maximally monotone by [51, Prop. 20.23 & Cor. 25.5]. The operator F is maximally monotone by Assumption 2.2 and by continuity in Assumption 2.6. Meanwhile, $\mathcal{L}$ is a linear positive semidefinite operator and, therefore, it is maximally monotone; thus, the operator $\mathcal{B}$ is maximally monotone. We can write $\mathcal{C}=\mathcal{C}_1+\mathcal{C}_2$, where $\mathcal{C}_1=\mathrm{col}((\langle\nabla_{x_i}g_i(x_i),\lambda_i\rangle)_{i\in\mathcal{I}},-(g_i(x_i))_{i\in\mathcal{I}},\mathbf{0}_{Nm})$ and $\mathcal{C}_2=\mathrm{col}(\mathbf{0}_n,-(\mathcal{L}\otimes I_m)\boldsymbol{\nu},(\mathcal{L}\otimes I_m)\boldsymbol{\lambda})$. The operator $\mathcal{C}_1$ is maximally monotone by continuity and by noting that, for any $\boldsymbol{\omega},\boldsymbol{\omega}'\in\mathbb{R}^n\times\mathbb{R}^{Nm}_{\geq 0}\times\mathbb{R}^{Nm}$,

$$
\begin{aligned}
\left\langle \mathcal{C}_1(\boldsymbol{\omega})-\mathcal{C}_1(\boldsymbol{\omega}'),\boldsymbol{\omega}-\boldsymbol{\omega}'\right\rangle &= \sum_{i\in\mathcal{I}}\left\langle g_i(x_i')-g_i(x_i)-\nabla_{x_i}g_i(x_i)^\top(x_i'-x_i),\lambda_i\right\rangle\\
&\quad+\sum_{i\in\mathcal{I}}\left\langle g_i(x_i)-g_i(x_i')-\nabla_{x_i}g_i(x_i')^\top(x_i-x_i'),\lambda_i'\right\rangle\\
&\geq 0,
\end{aligned}
$$

where the inequality follows by the convexity of g_i. As $\mathcal{C}_2$ is a linear skew-symmetric operator, it is maximally monotone [51, Ex. 20.35]. By invoking [51, Cor. 25.5], the result follows. ■

Lemma 2.5. *Let Assumptions 2.1 and 2.6 hold. Then the operators $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{B}+\mathcal{C}$, defined in* (2.12)–(2.13)*, are Lipschitz continuous.*

Proof. Due to Assumption 2.6, the operator $\mathcal{B}$ is L_F-Lipschitz continuous. Lipschitz continuity of $\mathcal{C}$ can be evaluated as follows. Similarly to the proof of Lemma 2.4, let us split $\mathcal{C}=\mathcal{C}_1+\mathcal{C}_2$. The operator $\mathcal{C}_2$ is Lipschitz continuous by linearity, while Lipschitz continuity of $\mathcal{C}_1$ is shown as follows. Let us denote the bound of $\nabla_{x_i}g_i(x_i)$ by $b_{\nabla g_i}$, i.e., $\|\nabla_{x_i}g_i(x_i)\|\leq b_{\nabla g_i}$ (c.f. Assumption 2.6) and the bound of λ_i by b_{λ_i}, for all $i\in\mathcal{I}$, which exists due to [58, Prop. 3.3]. For any $\boldsymbol{\omega},\boldsymbol{\omega}'\in\mathbb{R}^{n+2Nm}$,

$$
\begin{aligned}
\|\mathcal{C}_1(\boldsymbol{\omega})-\mathcal{C}_1(\boldsymbol{\omega}')\|^2 &\overset{\{1\}}{\leq} \sum_{i\in\mathcal{I}}\Big(2\|\nabla_{x_i}g_i(x_i)^\top(\lambda_i-\lambda_i')\|^2+\|g_i(x_i)-g_i(x_i')\|^2\\
&\quad+2\|(\nabla_{x_i}g_i(x_i)-\nabla_{x_i}g_i(x_i'))^\top\lambda_i'\|^2\Big)\\
&\overset{\{2\}}{\leq} \sum_{i\in\mathcal{I}}\Big(2\|\nabla_{x_i}g_i(x_i)^\top\|^2\|\lambda_i-\lambda_i'\|^2+b_{\nabla g_i}^2\|x_i-x_i'\|^2\\
&\quad+2\|\lambda_i'\|^2\|\nabla_{x_i}g_i(x_i)-\nabla_{x_i}g_i(x_i')\|^2\Big)\\
&\overset{\{3\}}{\leq} \sum_{i\in\mathcal{I}}\Big(2b_{\nabla g_i}^2\|\lambda_i-\lambda_i'\|^2+(2b_{\lambda_i}^2L_{\nabla g}^2+b_{\nabla g_i}^2)\|x_i-x_i'\|^2\Big)\\
&\leq \sum_{i\in\mathcal{I}}\max(2b_{\nabla g_i}^2,2b_{\lambda_i}^2L_{\nabla g}^2+b_{\nabla g_i}^2)\|\boldsymbol{\omega}_i-\boldsymbol{\omega}_i'\|^2,
\end{aligned}
$$

where $\{1\}$ follows by adding and subtracting the term $\nabla_{x_i}g_i(x_i)^\top\lambda_i'$ and by the bound $\|a+b\|^2\leq 2\|a\|^2+2\|b\|^2$; $\{2\}$ is obtained by the Cauchy-Schwartz inequality and by the fact that g_i is Lipschitz since it has a bounded gradient; $\{3\}$ is obtained by the Lipschitz continuity of $\nabla_{x_i} g_i$. Hence, $\mathcal{C}_1$ is $L_{\mathcal{C}_1}$-Lipschitz continuous, where $L_{\mathcal{C}_1} = \max_{i\in\mathcal{I}}(\max(2b_{\nabla g_i}, \sqrt{2b_{\lambda_i}^2 L_{\nabla g}^2 + b_{\nabla g_i}^2}))$. Since the sum of Lipschitz continuous operators is Lipschitz continuous, the result follows. ■

Lemma 2.6. *Let* $(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)$ *be a solution to the monotone inclusion in* (2.14)*. Then,* $(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)$ *is also a solution to the monotone inclusion in* (2.10)*.* □

Proof. The proof follows that of [37, Thm. 2(i)].

■

2.C Results and proofs of Section 2.3

The following lemma shows the equivalence between $\mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})$ and $\mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$.

Lemma 2.7. *Let Assumptions 2.1, 2.2, 2.6, and 2.7 hold. Furthermore, let* $\mathcal{T}_{\mathrm{FBF}}$ *be defined by* (2.19) *while* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$ *be defined in* (2.11)–(2.13)*. Then,* $\mathrm{fix}(\mathcal{T}_{\mathrm{FBF}}) = \mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})$*.* □

Proof. The proof is analogous to that of [16, Prop. 1]. ■

The following lemma is used to prove the quasi-shrinking property of the FBF operator (2.19).

Lemma 2.8. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *maximally monotone and* $\mathcal{B}$ *continuous. Let*

$$\mathcal{T} = \mathcal{R}_{\Psi^{-1}\mathcal{A}} \circ (\mathrm{Id} - \Psi^{-1}\mathcal{B}).$$

Then $\mathrm{Id} - \mathcal{T}$ *is demiclosed at* 0*.* □

Proof. Let us consider a sequence $(v_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} v_k = v, \qquad \lim_{k\to\infty} (\mathrm{Id}-\mathcal{T})(v_k) = 0.$$

We want to prove that $v - \mathcal{T}(v) = 0$ or, equivalently, $v \in \mathrm{fix}(\mathcal{T})$. Let us define $u_k := (\mathrm{Id}-\mathcal{T})(v_k)$. Then,

$$\begin{aligned}
v_k - u_k &= \mathcal{R}_{\Psi^{-1}\mathcal{A}}(\mathrm{Id} - \Psi^{-1}\mathcal{B})(v_k)\\
&\Leftrightarrow (\mathrm{Id} - \Psi^{-1}\mathcal{B})(v_k) \in (\mathrm{Id} + \Psi^{-1}\mathcal{A})(v_k - u_k)\\
&\Leftrightarrow v_k - \Psi^{-1}\mathcal{B}(v_k) + u_k - v_k \in \Psi^{-1}\mathcal{A}(v_k - u_k)\\
&\Leftrightarrow -\mathcal{B}(v_k) + \Psi u_k \in \mathcal{A}(v_k - u_k).
\end{aligned}$$

By the continuity of $\mathcal{B}$ and [51, Fact 1.19], we conclude that $\lim_{k\to\infty} -\mathcal{B}(v_k) + \Psi u_k = -\mathcal{B}(v)$. By [51, Prop. 20.37] and the monotonicity of $\mathcal{A}$, $\mathrm{gph}(\mathcal{A})$ is closed. Therefore, since $\lim_{k\to\infty} v_k - u_k = v$, we conclude that $-\mathcal{B}(v) \in \mathcal{A}(v)$. By [51, Prop. 26.1(iv)], we obtain $v \in \mathrm{fix}(\mathcal{T})$. ■

2

2.C.1 Proof of Lemma 2.3

By Lemmas 2.4 and 2.5, the operator $\mathcal{A}$ is maximally monotone whereas the operator $\mathcal{B}+\mathcal{C}$ is maximally monotone and Lipschitz continuous with Lipschitz constant denoted by L_B. Then, [16, Cor. 1] shows that $\mathcal{T}_{\text{FBF}}$ is quasi-nonexpansive under Assumption 2.7. Specifically, it holds that [16, Prop. 2]:

$$\|\mathcal{T}_{\text{FBF}}(\boldsymbol{\omega})-\boldsymbol{\omega}^\star\|_\Psi^2 \le \|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|_\Psi^2 - \frac{L_B^2}{\mu_{\min}(\Psi)^2}\|\widetilde{\boldsymbol{\omega}}-\boldsymbol{\omega}\|_\Psi^2, \tag{2.37}$$

where $\boldsymbol{\omega}^\star \in \operatorname{fix}(\mathcal{T}_{\text{FBF}})$, $\mu_{\min}(\Psi)$ is the smallest eigenvalue of Ψ and

$$\widetilde{\boldsymbol{\omega}} = (\mathrm{Id}+\Psi^{-1}\mathcal{A})^{-1}(\mathrm{Id}-\Psi^{-1}(\mathcal{B}+\mathcal{C}))(\boldsymbol{\omega}).$$

Finally, we prove that $\mathcal{T}_{\text{FBF}}$ is quasi-shrinking by invoking Lemma 2.1. Specifically, we choose

$$\mathcal{T}_2 = (\mathrm{Id}+\Psi^{-1}\mathcal{A})^{-1}(\mathrm{Id}-\Psi^{-1}(\mathcal{B}+\mathcal{C})).$$

By [51, Prop. 26.1(iv)] and Lemma 2.7, $\operatorname{fix}(\mathcal{T}_2)=\operatorname{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\operatorname{fix}(\mathcal{T}_{\text{FBF}})$. Moreover, Lemma 2.8 shows that $\mathrm{Id}-\mathcal{T}_2$ is demiclosed at 0 and (2.37) is indeed the inequality in (2.5) for $\mathcal{T}_{\text{FBF}}$. ■

Remark 2.9. *Although [16, Cor. 1] shows quasi-nonexpansiveness of $\mathcal{T}_{\text{FBF}}$ and [16, Prop. 2] shows the inequality in* (2.37) *for Problem* (2.6) *with a linear coupling constraint, these results also holds for nonlinear functions $g_i(x_i)$, for all $i \in \mathcal{I}$, as long as Assumption 2.6 holds, since the operator $\mathcal{C}$ in* (2.13) *remains Lipschitz continuous.*

2.C.2 Proof of Theorem 2.1

Let us introduce the following preliminary lemma:

Lemma 2.9. *Let Assumptions 2.1–2.4 and 2.6–2.7 hold. Then, the sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ generated by the HSDM method in* (2.18) *with $\mathcal{T}=\mathcal{T}_{\text{FBF}}$ in* (2.19)*, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined in* (2.11)–(2.13) *and Ψ is defined in* (2.20)*, is bounded, i.e., for any arbitrary $\boldsymbol{\omega}^\star \in \operatorname{fix}(\mathcal{T}_{\text{FBF}})$, it holds that $\|\boldsymbol{\omega}^{(k)}-\boldsymbol{\omega}^\star\| \le R(\boldsymbol{\omega}^\star)$, for some positive finite $R(\boldsymbol{\omega}^\star)$.*

Proof. Firstly, we show that, for an arbitrary $\boldsymbol{\omega}^\star \in \operatorname{fix}(\mathcal{T}_{\text{FBF}})$,

$$\|\mathcal{T}_{\text{FBF}}(\boldsymbol{\omega})-\boldsymbol{\omega}^\star\|_\Psi^2 < \|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|_\Psi^2, \tag{2.38}$$

for all $\boldsymbol{\omega} \notin \operatorname{fix}(\mathcal{T}_{\text{FBF}})$. To this end, let us recall the inequality (2.37) in the proof of Lemma 2.3, which holds under the considered assumptions:

$$\|\mathcal{T}_{\text{FBF}}(\boldsymbol{\omega})-\boldsymbol{\omega}^\star\|_\Psi^2 \le \|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|_\Psi^2 - (L_B/\mu_{\min}(\Psi))^2\|\widetilde{\boldsymbol{\omega}}-\boldsymbol{\omega}\|_\Psi^2,$$

where $\widetilde{\boldsymbol{\omega}} = (\mathrm{Id}+\Psi^{-1}\mathcal{A})^{-1}(\mathrm{Id}-\Psi^{-1}(\mathcal{B}+\mathcal{C}))(\boldsymbol{\omega}) =: \mathcal{T}_2(\boldsymbol{\omega})$. By [51, Prop. 26.1(iv)] and Lemma 2.7, $\operatorname{fix}(\mathcal{T}_2)=\operatorname{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\operatorname{fix}(\mathcal{T}_{\text{FBF}})$. Hence, $\widetilde{\boldsymbol{\omega}} \neq \boldsymbol{\omega}$ if $\boldsymbol{\omega} \notin \operatorname{fix}(\mathcal{T}_{\text{FBF}})$. We observe from the preceding inequality that when $\widetilde{\boldsymbol{\omega}} \neq \boldsymbol{\omega}$, (2.38) holds.

We now show that for any arbitrary fixed point $\boldsymbol{\omega}^\star \in \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$, there exists $R > 0$ such that

$$\inf_{\|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|\geq R}(\|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\| - \|\mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega})-\boldsymbol{\omega}^\star\|) > 0. \tag{2.39}$$

We proceed to prove (2.39) by contradiction, and thus we assume (2.39) to be false. By the nonexpansiveness of $\mathcal{T}_{\mathrm{FBF}}$, the left-hand side of (2.39) cannot be negative and it must be, for all $R > 0$,

$$\inf_{\|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|\geq R}(\|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\| - \|\mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega})-\boldsymbol{\omega}^\star\|) = 0. \tag{2.40}$$

Therefore, there exists a sequence $(\boldsymbol{\omega}_k)_{k\in\mathbb{N}}$ converging to $\overline{\boldsymbol{\omega}}$ such that $\|\boldsymbol{\omega}_k - \boldsymbol{\omega}^\star\| \geq R$ and

$$\lim_{k\to\infty} \|\boldsymbol{\omega}_k-\boldsymbol{\omega}^\star\| - \|\mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega}_k)-\boldsymbol{\omega}^\star\| = 0. \tag{2.41}$$

In particular, (2.41) must hold for $R > \sup_{\boldsymbol{y}\in\mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})} \|\boldsymbol{y}-\boldsymbol{\omega}^\star\| + \varepsilon$, which implies $\overline{\boldsymbol{\omega}} \notin \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$. Then, by (2.37), $\lim_{k\to\infty}\|\mathcal{T}_2(\boldsymbol{\omega}_k)-\boldsymbol{\omega}_k\| = 0$. As $\mathrm{Id}-\mathcal{T}_2$ is demiclosed at zero by Lemma 2.8, this implies the contradiction $\overline{\boldsymbol{\omega}} \in \mathrm{fix}(\mathcal{T}_2) \Rightarrow \overline{\boldsymbol{\omega}} \in \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$. The inequality in (2.39) is used in[66, Thm. 2] to prove the boundedness of the HSDM sequence with a nonexpansive operator $\mathcal{T}$ that satisfies (2.38). As (2.39) holds also for $\mathcal{T}_{\mathrm{FBF}}$, the same proof holds under the remaining assumptions: (i) $\nabla\phi$ is monotone and (Lipschitz continuous (Assumption 2.3), and (ii) the step size $\beta^{(k)}$ is non-summable but square summable (Assumption 2.4). ■

We are now ready to proceed with the proof of Theorem 2.1:

Proof. Let $\widetilde{\boldsymbol{\omega}}^{(k)} = (\widetilde{\boldsymbol{x}}^{(k)}, \widetilde{\boldsymbol{\lambda}}^{(k)}, \widetilde{\boldsymbol{\nu}}^{(k)})$ and $\mathring{\boldsymbol{\omega}}^{(k)} = (\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)})$, where $\widetilde{\boldsymbol{x}}^{(k)} = \mathrm{col}(\widetilde{x}_i)_{i\in\mathcal{I}}$ and the other variables are defined similarly. The updates of $\widetilde{\boldsymbol{\omega}}^{(k)}$ in Step 2 of Algorithm 1 can be compactly written as

$$\widetilde{\boldsymbol{\omega}}^{(k)} = (\mathrm{Id}+\Psi^{-1}\mathcal{A})^{-1}\big(\mathrm{Id}-\Psi^{-1}(\mathcal{B}+\mathcal{C})\big)(\boldsymbol{\omega}^{(k)}),$$

whereas the updates of $\mathring{\boldsymbol{\omega}}^{(k)}$ in Step 4 of Algorithm 1 can be compactly written as $\mathring{\boldsymbol{\omega}}^{(k)} = \widetilde{\boldsymbol{\omega}}^{(k)} - \Psi^{-1}(\mathcal{B}+\mathcal{C})(\widetilde{\boldsymbol{\omega}}^{(k)} - \boldsymbol{\omega}^{(k)})$, implying that $\mathring{\boldsymbol{\omega}}^{(k)} = \mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega}^{(k)})$ and the updates in (2.21) is compactly written as

$$\boldsymbol{\omega}^{(k+1)} = \mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega}^{(k)}) - \beta^{(k)}\nabla\phi(\mathcal{T}_{\mathrm{FBF}}(\boldsymbol{\omega}^{(k)})), \tag{2.42}$$

which is the HSDM applied to $\mathcal{T}_{\mathrm{FBF}}$. We can then invoke Lemma 2.2 to claim the hypothesis. By Lemma 2.7, $\mathrm{fix}(\mathcal{T}_{\mathrm{FBF}}) = \mathrm{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C})$; therefore $\mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$ is non-empty and bounded. Moreover, by Assumption 2.4, the step size $\beta^{(k)}$ meets the conditions in Lemma 2.2. Lemma 2.3 shows that $\mathcal{T}_{\mathrm{FBF}}$ is quasi-nonexpansive and quasi-shrinking on any bounded closed convex set, C such that $C \cap \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}}) \neq \varnothing$. On the other hand, Lemma 2.9 shows that the FBF-HSDM sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ obtained by the iterations in (2.42) is bounded, i.e., for any $\boldsymbol{\omega}^\star \in \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$, there exists a positive finite $R(\boldsymbol{\omega}^\star)$ such that $\|\boldsymbol{\omega}^{(k)} - \boldsymbol{\omega}^\star\| \leq R(\boldsymbol{\omega}^\star)$. Therefore, for an arbitrarily chosen $\boldsymbol{\omega}^\star \in \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}})$, we can construct the following bounded closed set $\mathfrak{B}(\boldsymbol{\omega}^\star) := \{x \in \mathrm{dom}(\mathcal{T}_{\mathrm{FBF}}) \mid \|x-\boldsymbol{\omega}^\star\| \leq R(\boldsymbol{\omega}^\star)\}$, on which the sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ lies. Moreover, we can observe that indeed $\mathfrak{B} \cap \mathrm{fix}(\mathcal{T}_{\mathrm{FBF}}) \neq \varnothing$, since $\boldsymbol{\omega}^\star \in \mathfrak{B}$ is a fixed point of $\mathcal{T}_{\mathrm{FBF}}$. Hence, $\mathcal{T}_{\mathrm{FBF}}$ is quasi-shrinking on $\mathfrak{B}$, which completes the proof. ■

2

2.D Proofs of Section 2.5

2.D.1 Proof of Theorem 2.2

First, we observe that in Algorithm 2, $\mathring{\boldsymbol{\omega}}^{(k)} = (\mathring{\boldsymbol{x}}^{(k)}, \mathring{\boldsymbol{\lambda}}^{(k)}, \mathring{\boldsymbol{\nu}}^{(k)})$ is updated by using $\mathcal{T}_{\text{pFB}}$ in (2.25), i.e., $\mathring{\boldsymbol{\omega}}^{(k)} = \mathcal{T}_{\text{pFB}}(\boldsymbol{\omega}^{(k)})$ [37, Section 4, Algorithm 1]. Hence, we can see that $\boldsymbol{\omega}^{(k)}$ is updated via the HSDM method, i.e.,

$$\boldsymbol{\omega}^{(k+1)} = \mathcal{T}_{\text{pFB}}(\boldsymbol{\omega}^{(k)}) - \beta^{(k)} \nabla\phi(\mathcal{T}_{\text{pFB}}(\boldsymbol{\omega}^{(k)})). \tag{2.43}$$

Similarly to the proof of Theorem 2.1, due to the boundedness of $\text{fix}(\mathcal{T}_{\text{pFB}}) = \text{zer}(\mathcal{A}+\mathcal{B}+\mathcal{C}) \neq \varnothing$ and the step size rule of $\beta^{(k)}$ in Assumption 2.4, we can invoke Lemma 2.2. Specifically, the operator $\mathcal{T}_{\text{pFB}}$ is averaged nonexpansive when Assumptions 2.1, 2.2, 2.6, and 2.8–2.10 hold [37, Thm. 3]. Therefore, $\mathcal{T}_{\text{pFB}}$ is also quasi-nonexpansive [51, Sect. 4.1]. By [51, Prop. 4.35 (iii)], the condition in (2.5) holds with $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_{\text{pFB}}$. By [51, Thm. 4.27], $\text{Id} - \mathcal{T}_{\text{pFB}}$ is demiclosed at 0. Therefore, by Lemma 2.1, $\mathcal{T}_{\text{pFB}}$ is quasi-shrinking on any closed bounded convex set whose intersection with $\text{fix}(\mathcal{T}_{\text{pFB}})$ is nonempty. Furthermore, since $\mathcal{T}_{\text{pFB}}$ is averaged nonexpansive, $\mathcal{T}_{\text{pFB}}$ is attracting. Therefore, by [66, Thm. 2] and due to the choice of the step size $\beta^{(k)}$ in Assumption 2.4, the sequence generated by (2.43) is bounded. Following the steps in the proof of Theorem 2.1, we can find a bounded set $\mathfrak{B}$ such that $\boldsymbol{\omega}^{(k)} \in \mathfrak{B}$ and $\mathcal{T}_{\text{pFB}}$ is quasi-shrinking on $\mathfrak{B}$. ∎

2.E Proofs of Section 2.6

2.E.1 Preliminary results

First, we show a series of preliminary results in Lemmas 2.10–2.13 that lead to the proof of Theorem 2.3. The proofs of this section are provided in the standard Euclidean norm for ease of notation. However, the case for any Ψ-induced norm, with $\Psi \succ 0$, follows verbatim. First, Lemma 2.10 shows the convergence of a particular sequence and can be regarded as a finite-iteration version of [15, Lem. 1].

Lemma 2.10. *Let $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be non-decreasing and non-negative. Let a sequence $(b^{(k)})_{k\in\mathbb{N}}$ be non-increasing, non-negative. Let $(a^{(k)})_{k\in\mathbb{N}} \subset [0,\infty)$ satisfy*

$$a^{(k+1)} \leq a^{(k)} - \psi(a^{(k)}) + b^{(k+1)}. \tag{2.44}$$

Let $K \in \mathbb{N}$. If there exists $\xi > 0$ such that $\psi(\xi) \geq \max\{2b^{(1)}, \frac{2}{K-1}a^{(1)}\}$, then

$$a^{(k)} \leq \xi + b^{(k)}, \quad \forall\, k \geq K. \tag{2.45}$$

Proof. Let us first show that there exists an $M \in \mathbb{N}$, $M \leq K$ such that $a^{(M)} \leq \xi$. We proceed by contradiction, assuming that $a^{(k)} > \xi$ $\forall k = 1, \ldots, K$. Then, by noting that $\psi(\cdot)$ is non-decreasing and that $\psi(\xi) \geq 2b^{(k)}$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} a^{(k+1)} &\leq a^{(k)} - \psi(a^{(k)}) + b^{(k+1)} \\ &\leq a^{(k)} - \psi(\xi) + \tfrac{1}{2}\psi(\xi) \\ &= a^{(k)} - \tfrac{1}{2}\psi(\xi). \end{aligned}$$

By iterating the latter relation and recalling that $\psi(\xi) \geq \frac{2}{K-1}a^{(1)}$, we find that

$$\begin{aligned} a^{(k+1)} &\leq a^{(1)} - \tfrac{k}{2}\psi(\xi) \\ &\leq a^{(1)} - \tfrac{k}{K-1}a^{(1)}. \end{aligned}$$

For $k = K$, we then obtain the contradiction $a^{(K+1)} < 0$. Thus, there exists $M \leq K$ such that $a^{(M)} \leq \xi$. We then proceed by induction to prove (2.45). Let us prove that, if $a^{(k)} \leq \xi + b^{(k)}$ then $a^{(k+1)} \leq \xi + b^{(k+1)}$ for all $k \geq M$. We distinguish two cases:
1) Case $a^{(k)} < \xi$. Then, by (2.44) and by the non-negativity of $\psi(\cdot)$, $a^{(k+1)} \leq a^{(k)} + b^{(k+1)} < \xi + b^{(k+1)}$.
2) Case $\xi \leq a^{(k)} \leq \xi + b^{(k)}$. Then, by the non-decreasing property of ψ, $a^{(k)} \geq \xi \Rightarrow \psi(a^{(k)}) \geq \psi(\xi)$. By the assumptions, $\psi(\xi) \geq 2b^{(1)}$ and by the non-incresing property of $(b^k)_{k\in\mathbb{N}}$, $2b^{(1)} \geq b^{(k)} + b^{(k+1)}$. We thus obtain $\psi(a^{(k)}) \geq b^{(k)} + b^{(k+1)}$. Substituting into (2.44) leads to

$$\begin{aligned} a^{(k+1)} &\leq a^{(k)} - \psi(a^{(k)}) + b^{(k+1)} \\ &\leq a^{(k)} - b^{(k)} - b^{(k+1)} + b^{(k+1)} \\ &= a^{(k)} - b^{(k)} \\ &\leq \xi. \end{aligned}$$

We conclude by induction that $a^{(k)} \leq \xi + b^{(k)}$ for all $k \geq M$ and, since $M \leq K$, the claim in (2.45) immediately follows. ∎

Lemma 2.11. *Let $\mathcal{T}$ be quasi-nonexpansive and $\mathcal{F}$ be strongly monotone, such that $\|\mathcal{F}(\boldsymbol{\omega})\| \leq U$, for all $\boldsymbol{\omega} \in \mathrm{im}(\mathcal{T})$. Let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ be generated from (2.18) with $\beta^{(k)} = \beta > 0$ for all k. Let $K \in \mathbb{N}$ and let $\boldsymbol{\omega}^\star$ be the solution of $\mathrm{VI}(\mathcal{F}, \mathrm{fix}(\mathcal{T}))$. If there exists ξ such that the shrinkage function $D(\cdot)$ of $\mathcal{T}$, defined in (2.4), satisfies $D(\xi) \geq \max\{2\beta U, 2\frac{\mathrm{dist}(\omega^{(1)},\mathrm{fix}(\mathcal{T}))}{K-1}\}$, then the following inequalities hold:*

$$\sup_{k\geq K} \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T})) \leq \xi + \beta U, \tag{2.46}$$

$$\sup_{k\geq K} \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\| \leq 2(\xi + \beta U), \tag{2.47}$$

$$\sup_{k\geq K} \langle \mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle \leq 3(\xi + \beta U)\|\mathcal{F}(\boldsymbol{\omega}^\star)\|. \tag{2.48}$$

Proof. (i.) For all k, it holds by the definition of distance and by the algorithm definition in (2.18) that:

$$\begin{aligned} \mathrm{dist}(\boldsymbol{\omega}^{(k+1)}, \mathrm{fix}(\mathcal{T})) &\leq \|\boldsymbol{\omega}^{(k+1)} - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\mathcal{T}(\boldsymbol{\omega}^{(k)}))\| \\ &= \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \beta\mathcal{F}(\mathcal{T}(\boldsymbol{\omega}^{(k)})) - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\mathcal{T}(\boldsymbol{\omega}^{(k)}))\| \\ &\leq \underbrace{\|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\mathcal{T}(\boldsymbol{\omega}^{(k)}))\|}_{=\mathrm{dist}(\mathcal{T}(\boldsymbol{\omega}^{(k)}), \mathrm{fix}(\mathcal{T}))} + \beta\|\mathcal{F}(\mathcal{T}(\boldsymbol{\omega}^{(k)}))\| \\ &\leq \mathrm{dist}(\mathcal{T}(\boldsymbol{\omega}^{(k)}), \mathrm{fix}(\mathcal{T})) + \beta U. \end{aligned} \tag{2.49}$$

2

Let us define $a^{(k)} := \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T}))$. Then, from (2.49) we find immediately $a^{(k+1)} - \beta U \leq \mathrm{dist}(\mathcal{T}(\boldsymbol{\omega}^{(k)}), \mathrm{fix}(\mathcal{T}))$. By the definition of shrinkage function in (2.4) and the latter inequality, we can write

$$\begin{aligned} D(a^{(k)}) \leq & a^{(k)} - \mathrm{dist}(\mathcal{T}(\boldsymbol{\omega}^{(k)}), \mathrm{fix}(\mathcal{T})) \\ \leq & a^{(k)} - a^{(k+1)} + \beta U \\ \Rightarrow \quad a^{(k+1)} \leq & a^{(k)} + \beta U - D(a^{(k)}), \end{aligned}$$

which defines a sequence of the kind in (2.44) with $\psi(\cdot) = D(\cdot)$ and $b^{(k)} = \beta U$ for all k. By Lemma 2.10, then $\mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T})) \leq \xi + \beta U$ for all $k \geq K$.

(ii.) By the triangle inequality, we can write $\|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\| \leq \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)})\| + \|\mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\|$. By quasi-nonexpansiveness of $\mathcal{T}$, we obtain, for all $k \geq K$,

$$\begin{aligned} \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)})\| \leq & \|\boldsymbol{\omega}^{(k)} - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)})\| \\ = & \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T})) \\ \Rightarrow \quad \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\| \leq & 2\mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T})). \end{aligned}$$

Finally, combining the last inequality and (2.46) yields (2.47).

(iii) By the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} \langle \mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle = & \langle \mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle + \langle \boldsymbol{\omega}^{(k)} - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle \\ \leq & \|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\|\|\mathcal{F}(\boldsymbol{\omega}^\star)\| + \langle \boldsymbol{\omega}^{(k)} - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle. \end{aligned} \tag{2.50}$$

Based on (2.47), for all $k \geq K$, we can bound the first term on the right-hand side of (2.50) by $\|\mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^{(k)}\|\|\mathcal{F}(\boldsymbol{\omega}^\star)\| \leq 2(\xi + \beta U)\|\mathcal{F}(\boldsymbol{\omega}^\star)\|$ and rewrite the second term as

$$\begin{aligned} \langle \boldsymbol{\omega}^{(k)} - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle = & \langle \boldsymbol{\omega}^{(k)} - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)}), -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle \\ & + \langle \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle. \end{aligned}$$

We observe that the second addend is non-positive by the definition of VI solution. By applying the Cauchy-Schwarz inequality, the definition of projection, and (2.46), we obtain

$$\begin{aligned} \langle \mathcal{T}(\boldsymbol{\omega}^{(k)}) - \boldsymbol{\omega}^\star, -\mathcal{F}(\boldsymbol{\omega}^\star)\rangle \leq & 2(\xi + \beta U)\|\mathcal{F}(\boldsymbol{\omega}^\star)\| + \|\boldsymbol{\omega}^{(k)} - \mathrm{proj}_{\mathrm{fix}(\mathcal{T})}(\boldsymbol{\omega}^{(k)})\|\|\mathcal{F}(\boldsymbol{\omega}^\star)\| \\ = & 2(\xi + \lambda U)\|\mathcal{F}(\boldsymbol{\omega}^\star)\| + \mathrm{dist}(\boldsymbol{\omega}^{(k)}, \mathrm{fix}(\mathcal{T}))\|\mathcal{F}(\boldsymbol{\omega}^\star)\| \\ \leq & 3(\xi + \lambda U)\|\mathcal{F}(\boldsymbol{\omega}^\star)\| \end{aligned}$$

■

Lemma 2.12. *Let Assumptions 2.12–2.14 hold. For any $t \in \mathbb{N}$, let $\boldsymbol{\omega}_{t+1}$ be generated from the step at time t of the restarted HSDM algorithm in* (2.30). *Let $D_t(\cdot)$ be the shrinkage function of $\mathcal{T}_t$ as defined in* (2.4). *If there exists $\xi > 0$ such that*

$$D_t(\xi) \geq \max\left\{2\beta U, 2\frac{\mathrm{dist}(\boldsymbol{\omega}_t, \mathrm{fix}(\mathcal{T}_t))}{K-1}\right\}, \tag{2.51}$$

2

then,

$$\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 \leq (1-\tau(\beta))^K \|\boldsymbol{\omega}_t - \boldsymbol{\omega}^\star{}_t\|^2 + \gamma, \tag{2.52}$$

with

$$\gamma = \tfrac{\beta}{\tau(\beta)} U(6\xi + 11\beta U). \tag{2.53}$$

Proof. Let us define the operator $\mathcal{T}_t^\beta(\boldsymbol{\omega}) := \mathcal{T}_t(\boldsymbol{\omega}) - \beta\nabla\phi_t(\mathcal{T}_t(\boldsymbol{\omega}))$. By $\mathcal{T}_t(\boldsymbol{\omega}^\star{}_t) = \boldsymbol{\omega}^\star{}_t$ and by the definition of the algorithm in (2.30), $\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 = \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t(\boldsymbol{\omega}^\star{}_t)\|^2$. We sum and subtract $\beta\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)$ and substitute $\mathcal{T}_t^\beta$ to obtain

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 &= \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t(\boldsymbol{\omega}^\star{}_t) + \beta\nabla\phi_t(\boldsymbol{\omega}^\star{}_t) - \beta\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&= \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t) - \beta\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|^2.
\end{aligned}$$

Expanding the square {1}, expanding $\mathcal{T}_t^\beta$ {2}, and regrouping {3} leads to

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 &\overset{\{1\}}{=} \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 + \beta^2\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&\quad + 2\langle \mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t), -\beta\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\rangle \\
&\overset{\{2\}}{=} \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 + \beta^2\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&\quad - 2\beta\langle \mathcal{T}_t(\boldsymbol{y}^{(K)}) - \beta\nabla\phi_t(\mathcal{T}_t(\boldsymbol{y}^{(K)})) \\
&\quad - \mathcal{T}_t(\boldsymbol{\omega}^\star{}_t) + \beta\nabla\phi_t(\mathcal{T}_t(\boldsymbol{\omega}^\star{}_t)), \nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\rangle \\
&\overset{\{3\}}{=} \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 + \beta^2\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&\quad + 2\beta\langle \mathcal{T}_t(\boldsymbol{y}^{(K)}) - \boldsymbol{\omega}^\star{}_t, -\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\rangle \\
&\quad + 2\beta^2\langle \nabla\phi_t(\mathcal{T}_t(\boldsymbol{y}^{(K)})) - \nabla\phi_t(\boldsymbol{\omega}^\star{}_t), \nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\rangle.
\end{aligned} \tag{2.54}$$

We note that, by applying the Cauchy-Schwarz, the triangle inequalities and Assumption 2.14, we have $\langle \nabla\phi_t(\mathcal{T}_t(\boldsymbol{y}^{(K)})) - \nabla\phi_t(\boldsymbol{\omega}^\star{}_t), \nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\rangle \leq \|\nabla\phi_t(\mathcal{T}_t(\boldsymbol{y}^{(K)})) - \nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\| \leq (U + \|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|)\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|$. By (2.51) and Lemma 2.11, we can substitute in (2.54) the latter relation and the bound in (2.48) to obtain

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 &\leq \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&\quad + 6\beta(\xi + \beta U)\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\| + \beta^2(2U + 3\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|)\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\|.
\end{aligned}$$

Applying Assumption 2.14 and rearranging the terms leads to

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 &\leq \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 \\
&\quad + 6\beta(\xi + \beta U)\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\| + \beta^2 5U\|\nabla\phi_t(\boldsymbol{\omega}^\star{}_t)\| \\
&\leq \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 + \beta(6\xi + 11\beta U)U \\
&\leq \|\mathcal{T}_t^\beta(\boldsymbol{y}^{(K)}) - \mathcal{T}_t^\beta(\boldsymbol{\omega}^\star{}_t)\|^2 + \tau(\beta)\gamma.
\end{aligned} \tag{2.55}$$

By quasi-nonexpansiveness of $\mathcal{T}_t$ as well as strong monotonicity and Lipschitz continuity of $\nabla\phi_t$, we can apply [15, Lem. 4a] to obtain $\|\mathcal{T}_t^\beta(\boldsymbol{\omega}) - \mathcal{T}_t^\beta(\overline{\boldsymbol{\omega}})\| \leq$

2

$(1-\tau(\beta))\|\boldsymbol{\omega}-\overline{\boldsymbol{\omega}}\|$, for all $\boldsymbol{\omega} \in \operatorname{dom}(\mathcal{T}_t^{\beta}), \overline{\boldsymbol{\omega}} \in \operatorname{fix}(\mathcal{T}_t)$, which we substitute in (2.55) to obtain

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1}-\boldsymbol{\omega}^{\star}{}_t\|^2 &\leq (1-\tau(\beta))^2 \|\boldsymbol{y}^{(K)}-\boldsymbol{\omega}^{\star}{}_t\|^2+\tau(\beta)\gamma \\
&\leq (1-\tau(\beta)) \|\boldsymbol{y}^{(K)}-\boldsymbol{\omega}^{\star}{}_t\|^2+\tau(\beta)\gamma.
\end{aligned}$$

By iterating, we obtain

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1}-\boldsymbol{\omega}^{\star}{}_t\|^2 &\leq (1-\tau(\beta))^2 \|\boldsymbol{y}^{(K-1)}-\boldsymbol{\omega}^{\star}{}_t\|^2+(1-\tau(\beta))\,\tau(\beta)\gamma+\tau(\beta)\gamma \\
&\leq \ldots \leq (1-\tau(\beta))^K \|\boldsymbol{y}^{(1)}-\boldsymbol{\omega}^{\star}{}_t\|^2+\sum_{j=0}^{K-1}(1-\tau(\beta))^j\,\tau(\beta)\gamma \\
&\leq (1-\tau(\beta))^K \|\boldsymbol{y}^{(1)}-\boldsymbol{\omega}^{\star}{}_t\|^2+\sum_{j=0}^{\infty}(1-\tau(\beta))^j\,\tau(\beta)\gamma.
\end{aligned}$$

Applying the geometric series convergence and recalling from (2.30) that $\boldsymbol{y}^{(1)}=\boldsymbol{\omega}_t$ leads to (2.52). ■

The next lemma outlines a contraction property of the restarted HSDM to the solution sequence of Problem (2.29) up to an additive error, which can be controlled by an appropriate choice of the step size β and the number of iterations K.

Lemma 2.13. *Let Assumptions 2.12–2.15 hold. For any $t \in \mathbb{N}$, let $\boldsymbol{\omega}_{t+1}$ be generated by the restarted HSDM algorithm in (2.30). For any $\gamma>0$, there exist $K, \beta>0$, such that* (2.52) *holds.*

Proof. Let us consider $\xi := \frac{\gamma\sigma}{12U}$. Since $\mathcal{T}_t$ is quasi-shrinking, the shrinkage function D_t of $\mathcal{T}_t$ satisfies $D_t(\xi)>0$. Thus, there exist $\overline{\beta} \in (0, \frac{2\sigma}{L_\phi^2})$ and K such that, for any $\beta \in (0, \overline{\beta}]$, (2.51) holds.

It can be verified that $\lim_{\beta\to 0^+} \frac{\beta}{\tau(\beta)}=\frac{1}{\sigma}$. Then,

$$\lim_{\beta\to 0^+} \frac{\beta}{\tau(\beta)}(6\xi+11\beta U)U=\frac{6\xi U}{\sigma}=\frac{1}{2}\gamma. \tag{2.56}$$

We thus find $\beta \in (0, \overline{\beta}]$ small enough, such that

$$\frac{\beta}{\tau(\beta)}(6\xi+11\beta U)U \leq \gamma. \tag{2.57}$$

Hence, the hypothesis holds by invoking Lemma 2.12. ■

Remark 2.10. *From the proof of Lemma 2.13, as ξ (and thus $D_t(\xi)$) decreases with γ, it can be seen from* (2.51) *that for smaller values of γ a smaller step size β and a larger K are necessary.*

2.E.2 Proof of Theorem 2.3

We begin the proof by constructing a suitable step size $\overline{\beta}$ and number of iterations $\overline{K}$. We then proceed with proving that the statement holds for the chosen variables. Let us first define the auxiliary variable $\xi = \frac{\gamma\sigma}{12U}$. Following the steps in the proof of Lemma 2.13, we can choose a small enough $\overline{\beta} \in (0, \min\{\frac{2\sigma}{L_\phi^2}, \frac{D(\xi)}{2U}\})$, where $D(\cdot)$ is defined in Assumption 2.15, such that

$$\frac{\overline{\beta}}{\tau(\overline{\beta})}(6\xi + 11\overline{\beta}U)U \leq \gamma. \tag{2.58}$$

We now define $\alpha(K) := (1-\tau(\overline{\beta}))^K$. Since $\tau(\overline{\beta}) \in (0,1)$, α is decreasing with K. We can then choose K_1, such that $\alpha(K_1) < \frac{1}{2}$. Then, we define the mapping $a : \mathbb{N}_{\geq K_1} \to \mathbb{R}$

$$a(K) = \max\left\{\|\boldsymbol{\omega}_1\| + \sup_{\boldsymbol{\omega}\in\mathcal{Y}} \|\boldsymbol{\omega}\|, \sqrt{\frac{2\alpha(K)\delta^2+\gamma}{1-2\alpha(K)}}\right\}, \tag{2.59}$$

We can verify that $a(\cdot)$ is non-increasing. Consequently, the sequence $\left(\frac{2(a(K)+\delta)}{K-1}\right)_{K\geq K_1}$ is decreasing. We can then choose any sufficiently large $\overline{K} \geq K_1$, such that

$$D(\xi) \geq \frac{2(\overline{a}+\delta)}{\overline{K}-1}, \tag{2.60}$$

where $\overline{a} := a(\overline{K})$. We also define $\overline{\alpha} := \alpha(\overline{K})$. We now prove by induction that

$$\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}^\star\| \leq \overline{a} \quad \text{ for all } t > 1. \tag{2.61}$$

To that end, we first show that

$$\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}^\star\| \leq \overline{a} \Rightarrow \|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_t^\star\| \leq \overline{a}. \tag{2.62}$$

Let us then write

$$\begin{aligned}
\operatorname{dist}(\boldsymbol{\omega}_t, \operatorname{fix}(\mathcal{T}_t)) &\overset{\{1\}}{\leq} \|\boldsymbol{\omega}_t - \operatorname{proj}_{\operatorname{fix}(\mathcal{T}_t)}(\boldsymbol{\omega}_{t-1}^\star)\| \\
&\overset{\{2\}}{\leq} \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}^\star\| + \|\boldsymbol{\omega}_{t-1}^\star - \operatorname{proj}_{\operatorname{fix}(\mathcal{T}_t)}(\boldsymbol{\omega}_{t-1}^\star)\| \\
&\overset{\{3\}}{\leq} \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}^\star\| + \delta \qquad (2.63)\\
&d \leq \overline{a} + \delta, \qquad (2.64)
\end{aligned}$$

where $\{1\}$ follows from the definition of distance, $\{2\}$ from the triangle inequality and $\{3\}$ from Assumption 2.11 and $\boldsymbol{\omega}_t^\star \in \operatorname{fix}(\mathcal{T}_t)$. Then, by Assumption 2.15, by the choice $\overline{\beta} \leq \frac{D(\xi)}{2U}$ and (2.60), it holds that

$$\begin{aligned}
D_t(\xi) &\geq \max\left\{2\overline{\beta}U, \frac{2(\overline{a}+\delta)}{\overline{K}-1}\right\} \\
&\overset{(2.64)}{\geq} \max\left\{2\overline{\beta}U, \frac{2\operatorname{dist}(\boldsymbol{\omega}_t, \operatorname{fix}(\mathcal{T}_t))}{\overline{K}-1}\right\}.
\end{aligned} \tag{2.65}$$

2

By Lemma 2.12 and (2.58), we then have

$$\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_t^\star\|^2 \leq \overline{\alpha}\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|^2 + \gamma. \tag{2.66}$$

Applying on (2.66) the triangle inequality, the fact $(a+b)^2 \leq 2a^2 + 2b^2$ and Assumption 2.11 leads to

$$\begin{aligned}\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 &\leq 2\overline{\alpha}(\|\boldsymbol{\omega}_t - \boldsymbol{\omega}^\star{}_{t-1}\|^2 + \|\boldsymbol{\omega}_{t-1}^\star - \boldsymbol{\omega}_t^\star\|^2) + \gamma \\ &\leq 2\overline{\alpha}(\|\boldsymbol{\omega}_t - \boldsymbol{\omega}^\star{}_{t-1}\|^2 + \delta^2) + \gamma \\ &\leq 2\overline{\alpha}(\overline{a}^2 + \delta^2) + \gamma. \end{aligned} \tag{2.67}$$

Finally, by (2.59), it holds $\overline{a}^2 \geq \frac{2\overline{\alpha}\delta^2 + \gamma}{1 - 2\overline{\alpha}}$, which implies

$$2\overline{\alpha}(\overline{a}^2 + \delta^2) + \gamma \leq \overline{a}^2. \tag{2.68}$$

Thus, we obtain $\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}^\star{}_t\|^2 \leq \overline{a}^2$. We now continue the induction argument by proving

$$\|\boldsymbol{\omega}_2 - \boldsymbol{\omega}^\star{}_1\|^2 \leq \overline{a}^2. \tag{2.69}$$

From the triangle inequality and from (2.59), $\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}^\star{}_1\| \leq \|\boldsymbol{\omega}_1\| + \|\boldsymbol{\omega}^\star{}_1\| \leq \overline{a}$. From the definition of distance, we obtain

$$\operatorname{dist}(\boldsymbol{\omega}_1, \operatorname{fix}(\mathcal{T}_1)) \leq \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}^\star{}_1\| \leq \overline{a} \leq \overline{a} + \delta. \tag{2.70}$$

Then, by (2.60) and the choice $\overline{\beta} \leq \frac{D(\xi)}{2U}$, we have

$$\begin{aligned} D_t(\xi) \geq D(\xi) &\geq \max\left\{2\beta U, \tfrac{2(\overline{a}+\delta)}{\overline{K}-1}\right\} \\ &\geq \max\left\{2\beta U, \tfrac{2\operatorname{dist}(\boldsymbol{\omega}_1, \operatorname{fix}(\mathcal{T}_1))}{\overline{K}-1}\right\}. \end{aligned}$$

By Lemma 2.12 and (2.58), we find

$$\|\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1^\star\|^2 \leq \overline{\alpha}\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1^\star\|^2 + \gamma.$$

We then upper bound the right-hand side of the last inequality:

$$\|\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1^\star\|^2 \overset{(2.70)}{\leq} \overline{\alpha}\overline{a}^2 + \gamma \leq \overline{\alpha}(2\overline{a}^2 + 2\delta^2) + \gamma \overset{(2.68)}{\leq} \overline{a}^2.$$

Therefore, combining (2.62) and (2.69) leads to $\sup_{t>1} \|\boldsymbol{\omega}_t - \boldsymbol{\omega}^\star{}_{t-1}\| \leq \overline{a}$. Recalling that, from Assumption 2.13, $\boldsymbol{\omega}_t^\star \in \mathcal{Y}$ for all t, this immediately implies $\operatorname{dist}(\boldsymbol{\omega}_t, \mathcal{Y}) \leq \overline{a}$ for all $t > 1$, which proves that the sequence is bounded.

We now proceed with proving (2.31). We note that the relation in (2.66) holds for all t. We then observe that, by the triangle inequality, by $(a+b)^2 \leq 2a + 2b$, and by Assumption 2.11,

$$\begin{aligned}\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_{t+1}^\star\|^2 &\leq 2\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_t^\star\|^2 + 2\|\boldsymbol{\omega}_{t+1}^\star - \boldsymbol{\omega}_t^\star\|^2 \\ &\leq 2\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_t^\star\|^2 + 2\delta^2. \end{aligned}$$

By using (2.66) to upper bound $\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_t^\star\|^2$ and iterating, we find:

$$\begin{aligned}
\|\boldsymbol{\omega}_{t+1} - \boldsymbol{\omega}_{t+1}^\star\|^2 &\leq 2\overline{\alpha}\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|^2 + 2(\gamma + \delta^2) \\
&\leq (2\overline{\alpha})^2\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_{t-1}^\star\|^2 + 2(\gamma + \delta^2) + 2\overline{\alpha}(2\gamma + 2\delta^2) \\
&\leq \ldots \leq (2\overline{\alpha})^t\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1^\star\|^2 + \sum_{j=0}^{t-1}(2\overline{\alpha})^j(2\gamma + 2\delta^2).
\end{aligned}$$

By taking the limit for $t \to \infty$ and by applying the convergence of the geometric sequence, we obtain (2.31). ■

2.E.3 Proof of Corollary 2.1

Steps i–vi of Algorithm 3 are analogous to Steps 1–6 of Algorithm 1. Analogously to the proof of Theorem 2.1, we see that the variable $\boldsymbol{y}^{(k)} := (\hat{x}_i^{(k)}, \hat{\lambda}_i^{(k)}, \hat{\nu}_i^{(k)})$ is updated at each time step by K iterations of the HSDM:

$$\boldsymbol{y}^{(k+1)} = \mathcal{T}_{\text{FBF},t}(\boldsymbol{y}^{(k)}) - \beta\nabla\phi_t(\mathcal{T}_{\text{FBF},t}(\boldsymbol{y}^{(k)})), \quad k = 1, \ldots, K.$$

Then, the variable $\boldsymbol{\omega}_{t+1}$ is updated as $\boldsymbol{\omega}_{t+1} = \boldsymbol{y}^{(K+1)}$. Thus, we see that Algorithm 3 is a particular instance of the restarted HSDM algorithm (2.30). By Theorem 2.3, the tracking error is given by (2.31). ■

3

Nash equilibrium selection in aggregative games

-"I think we have different value systems."
-"Well, mine's better"

Douglas Adams, in "Mostly harmless"

Monotone aggregative games may admit multiple (variational) generalized Nash equilibria, yet currently there is no algorithm able to provide an apriori characterization of the equilibrium solution actually computed. In this chapter, we formulate for the first time the problem of selecting a specific variational equilibrium that is optimal with respect to a given objective function. We then propose a semi-decentralized algorithm for optimal equilibrium selection in linearly coupled aggregative games and prove its convergence.

This chapter is partly based on Benenati, E., Ananduta, W. and Grammatico, S. "On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games". In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 6389–6394, Cancun, Mexico, (Dec. 2022). IEEE.

3.1 Introduction

Multiple decision makers (agents) are engaged in an aggregative game when each agent aims at solving an optimization problem that is coupled with the strategies of the remaining agents through some aggregate effect, e.g. a congestion effect [4, pp. 90–92]. Interestingly, aggregative games effectively model various engineering problems, such as bandwidth allocation [67], power distribution [28], and vehicle traffic control [68]. In this context, a generalized Nash equilibrium (GNE) is a particularly favorable and stable set of strategies from which no agent has interest in unilaterally deviating.

GNEs are commonly studied under the monotone operator theory framework, see [4] for a general overview. When the game in consideratison is monotone, i.e., it has a jointly convex feasible set [3, Def. 3.6] and a monotone pseudogradient mapping, the GNE seeking problem can be solved via operator splitting techniques [51] in order to compute a variational GNE (v-GNE) [3, Def. 3.10]. For instance, [17] proposes multiple semi-decentralized methods, where decision makers do not communicate among each other but with a coordinator, for monotone aggregative games. Meanwhile, some works consider a particular class, namely strongly monotone aggregative games, which admit a unique v-GNE, and propose algorithms with a semi-decentralized [35] or distributed structure [69–71], where agents exchange information with each other. Recently, fully distributed algorithms have also been introduced for the class of monotone aggregative games [19, 39].

Merely monotone games are of particular interest, as they are more general than their strongly monotone counterpart and monotonicity is one of the weakest conditions under which globally convergent algorithmic solutions can be obtained [3, Sec. 5]. Nonetheless, they present additional hurdles to their solution. Among these complications, we focus on the non-uniqueness of v-GNE solutions. To the best of our knowledge, the existing GNE-seeking algorithms for monotone games, e.g. [16, 17, 19, 39], solely guarantee convergence to an arbitrary point in the v-GNE set, with no characterization of the computed equilibrium. A noteworthy exception is the Tikhonov method [42, 72] that is not based on operator splitting techniques and finds a minimum norm v-GNE. Therefore, an open challenge in the monotone game literature concerns finding an equilibrium, among infinitely many, with a desirable property, which is not necessarily the minimum norm as in [42, 72]. This problem is crucial from a practical standpoint as unpredictability of the computed equilibrium might lead to arbitrarily inefficient performance relatively to some system-level metric, e.g. social welfare.

In order to address this deficiency for the class of linearly coupled aggregative games, we pose the GNE selection problem, that is, the problem of computing a specific v-GNE such that it is optimal with respect to a convex selection function. The selection function encodes a preference criterion for the GNEs and it can be defined on the basis of a system-level performance metrics. For instance, in power distribution systems, a preference criterion can be the deviation from an operating set point desired by the system operator [73]. We cast the GNE selection problem as a variational inequality (VI) defined by the gradient of the selection function and the v-GNE set of the game.

Next, we propose an algorithmic solution to solve the GNE selection problem for the class of linearly coupled aggregative games. Our algorithm has a semi-decentralized structure in order to exploit the aggregative feature of the games. It is based on combining the preconditioned proximal point (PPP) method [17, 19], which provides fast convergence under the (non-strict) monotonicity assumption, with the hybrid steepest-descent method (HSDM) [15], which can solve fixed-point selection problems. We guarantee convergence to an optimal equilibrium by showing the equivalence of the GNE selection problem to that of fixed-point selection of the preconditioned proximal-point (PPP) operator. We then prove that the PPP operator satisfies the conditions under which the HSDM converges to a solution of the corresponding fixed-point selection problem, which, in turn, is also a solution of the GNE selection problem. Finally, we show the advantages of the proposed algorithm by comparing the performance of an optimal v-GNE computed by our proposed method with that of a non-characterized v-GNE obtained using the standard PPP algorithm in randomly generated numerical examples.

3.2 Optimal generalized Nash equilibrium selection problem in linearly coupled games

In this section, we specify the definitions of generalized Nash equilibrium (GNE) and optimal GNE selection to games where the agents' coupling in the objectives and constraints emerges through linear functions. These concepts were already introduced in Sections 2.3.1 and 2.3.2, respectively. We consider N agents, indexed by the set $\mathcal{I} := \{1, 2, \ldots, N\}$, which are engaged in a generalized game, i.e., each agent aims at solving an optimization problem which is coupled, both in the cost and in the constraints, with the decision variables of the remaining agents. Let us denote by $x_i \in \mathbb{R}^{n_i}$ the decision variable of agent i and by $\boldsymbol{x}_{-i} = \mathrm{col}(x_j)_{j\in\mathcal{I}\setminus\{i\}}$ the concatenated decision variables of all agents except agent i. Let us further denote the local feasible set of each agent i by $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and the cost function by $J_i\colon \mathbb{R}^n \to \mathbb{R}$, where $n := \sum_{i\in\mathcal{I}} n_i$. We focus on the class of linearly coupled games, where the cost function of each agent $i \in \mathcal{I}$ takes the form [17, Eq. (1)]:

$$J_i(x_i, \boldsymbol{x}_{-i}) := \ell_i(x_i) + \left\langle \textstyle\sum_{j\in\mathcal{I}\setminus\{i\}} C_{ij} x_j, x_i \right\rangle, \tag{3.1}$$

where ℓ_i denotes the local cost function of agent i. Furthermore, the matrix $C_{ij} \in \mathbb{R}^{n_i \times n_j}$ encodes the weight of the influence of the decision variable of agent j with respect to the cost function of agent i and is assumed to be local information known only by agent i. Additionally, we let the matrices $A_i \in \mathbb{R}^{m\times n_i}$ and the vectors $b_i \in \mathbb{R}^m$, for all $i \in \mathcal{I}$, encode a linear coupling constraint of the form $\sum_{i\in\mathcal{I}}(A_i x_i - b_i) \leq 0$. Therefore, the interdependent optimization problems are given by:

$$\forall i \in \mathcal{I}\colon \begin{cases} \min\limits_{x_i\in\mathcal{X}_i} & J_i(x_i, \boldsymbol{x}_{-i}) & \text{(3.2a)} \\ \text{s.t.} & \sum\limits_{j\in\mathcal{I}} (A_j x_j - b_j) \leq 0. & \text{(3.2b)} \end{cases}$$

The collective feasible set of the game in (3.2) is defined as

$$\boldsymbol{\Omega} := \boldsymbol{\mathcal{X}} \cap \Big\{ \boldsymbol{x} \mid \sum_{j \in \mathcal{I}} (A_j x_j - b_j) \le 0 \Big\}, \tag{3.3}$$

where $\boldsymbol{\mathcal{X}} := \prod_{i \in \mathcal{I}} \mathcal{X}_i$. Let us consider the following:

3

Assumption 3.1. *For each $i \in \mathcal{I}$, the function $\ell_i(x_i)$ in (3.1) is convex and lower semicontinuous. For each $i \in \mathcal{I}$, the set $\mathcal{X}_i$ in (3.2a) is nonempty, compact, and convex. The set $\boldsymbol{\Omega}$ in (3.3) satisfies Slater's constraint qualification condition.*

Assumption 3.2. *[19, Assm. 2] The matrices $C_{ij} \in \mathbb{R}^{n_i \times n_j}$ in (3.1), for all $i,j \in \mathcal{I}$, satisfy $C_{ij} = C_{ji}^\top$.*

Assumption 3.1 is standard (see e.g., [16, 17, 19]), while Assumption 3.2 is a technical assumption that implies that the game is potential [19, Lem. 1]. This assumption is necessary since our method relies on preconditioning [38]. This technique requires convergence to be proven on the norm induced by the preconditioning matrix, whose structure [19, Sec. IVB] makes Assumption 3.2 a necessary condition for such norm to be well-defined. The class of games described in (3.2) along with Assumptions 3.1–3.2 includes linearly-coupled aggregative games [17, Sec. IVB], obtained with $C_{ij} = \frac{1}{N} C$, for all $i,j \in \mathcal{I}$, and $\ell_i(x_i) = \overline{\ell_i}(x_i) + \frac{1}{N} x_i^\top C x_i$, implying that (3.1) can be written as

$$J_i(x_i, \boldsymbol{x}_{-i}) = \overline{\ell_i}(x_i) + \langle C \operatorname{avg}(x_j)_{j \in \mathcal{I}}, x_i \rangle, \qquad \forall i \in \mathcal{I}. \tag{3.4}$$

The solution to the game in (3.2) that we consider is a GNE, i.e., a set of decisions from which no agent finds an advantage in unilaterally deviating, as formally defined next.

Definition 3.1. *A set of strategies $\boldsymbol{x}^* := \operatorname{col}(x_i^*)_{i \in \mathcal{I}}$ is a GNE of the game in (3.2) if $\boldsymbol{x}^* \in \boldsymbol{\Omega}$ and, for each $i \in \mathcal{I}$,*

$$J_i(\boldsymbol{x}^*) \le J_i(x_i, \boldsymbol{x}_{-i}^*), \tag{3.5}$$

for any $x_i \in \mathcal{X}_i \cap \{y \mid A_i y - b_i \le -\sum_{j \in \mathcal{I} \setminus \{i\}} (A_j(x_j^*) - b_j)\}$.

As in [16, 17, 19, 39, 71] we focus on the computation of a subset of the GNEs of the problem in (3.2), namely, variational GNE (v-GNE)s, where, roughly speaking, each agent is penalized equally in meeting the coupling constraints. Under Assumption 3.1, a sufficient condition for the existence of a v-GNE is the monotonicity of the pseudodifferential/game mapping [57, Prop. 12.11], which is a standard assumption in the GNE seeking literature (see [17, Assm. 2], [37, Assm. 2], [19, Assm. 3] among others), and we postulate next:

Assumption 3.3. *The game mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$*

$$F(\boldsymbol{x}) := \begin{bmatrix} \partial_{x_1} J_1(x_1, \boldsymbol{x}_{-1}) \\ \vdots \\ \partial_{x_N} J_N(x_N, \boldsymbol{x}_{-N}) \end{bmatrix} = \operatorname{diag}(\partial_{\boldsymbol{x}_i} \ell_i)_{i \in \mathcal{I}} + \boldsymbol{C}, \tag{3.6}$$

where $\boldsymbol{C} \in \mathbb{R}^{n \times n}$ is a block matrix with $\mathbf{0}_{n_i \times n_i}$ on the i-th block element of the diagonal, and C_{ij} as defined in (3.1) *on the* (i,j)*-th block, is a maximally monotone operator.*

A v-GNE of the game in (3.2) can be characterized as the solution to a generalized variational inequality [57, Prop. 12.4] which, in turn, is solved by the pair $(\boldsymbol{x}, \lambda) =: \boldsymbol{\omega} \in \mathbb{R}^n \times \mathbb{R}^m$, where λ denotes the dual variable associated with the coupling constraint (3.2b), that satisfies the Karush–Kuhn–Tucker (KKT) optimality conditions [58, Thm. 3.1]:

3

$$\boldsymbol{\omega} \in \operatorname{zer}(\mathcal{T}_{\mathrm{e}}), \tag{3.7a}$$

$$\mathcal{T}_{\mathrm{e}}(\boldsymbol{\omega}) := \begin{bmatrix} \prod_{i \in \mathcal{I}} \mathrm{N}_{\mathcal{X}_i}(x_i) + F(\boldsymbol{x}) + \operatorname{col}(A_i^\top \lambda)_{i \in \mathcal{I}} \\ \mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda) - \sum_{i \in \mathcal{I}} (A_i x_i - b_i) \end{bmatrix}. \tag{3.7b}$$

Assumptions 3.1 and 3.3 are not enough to guarantee the uniqueness of the solution to the inclusion in (3.7a). Furthermore, we may want to select, among the potentially infinite equilibria of the game in (3.2), a v-GNE with some desirable features. Here, we propose to seek a v-GNE that optimizes a common objective function, denoted by ϕ. As pointed out in [45, 46], this objective function can be the distance from some desired strategies, or a norm as in the Tikhonov method [42]. In engineering applications, such functions can represent system-level objectives that the agents are willing to achieve, provided that these objectives do not strongly interfere with those of all agents. Thus, we formalize the GNE selection problem as follows:

$$\begin{cases} \underset{\boldsymbol{\omega} \in \mathbb{R}^n \times \mathbb{R}^m}{\operatorname{argmin}} & \phi(\boldsymbol{\omega}) \\ \text{s.t.} & \boldsymbol{\omega} \in \operatorname{zer}(\mathcal{T}_{\mathrm{e}}), \end{cases} \tag{3.8}$$

with $\mathcal{T}_e$ as in (3.7a) and the following assumption on the objective function:

Assumption 3.4. *The function ϕ in* (3.8) *is convex and differentiable. Its gradient $\nabla \phi$ is L_ϕ-Lipschitz continuous.*

We note that, by [17, Lemma 2], the operator $\mathcal{T}_{\mathrm{e}}$ in (3.8) is maximally monotone, which implies that the set $\operatorname{zer}(\mathcal{T}_{\mathrm{e}})$ is closed and convex [51, Prop. 23.39]. We conclude that under Assumption 3.4 the optimization problem in (3.8) is convex.

3.3 Optimal equilibrium selection algorithm

In this section, we present a semi-decentralized algorithm for selecting an optimal v-GNE of the game in (3.2). The algorithm is semi-decentralized in the sense that the agents locally compute the updates to their strategies and communicate them to an aggregator, which updates the dual and the aggregative variables and broadcasts them to the agents.

3.3.1 Semi-decentralized optimal equilibrium selection

We propose Algorithm 4, whose derivation is deferred to Section 3.3.2, for solving (3.8). The iterated steps of the algorithm are summarized as follows:

Algorithm 4 Optimal v-GNE selection for linearly coupled games

Initialization. Set $x_i^{(0)} \in \mathcal{X}_i$ and $\lambda_i^{(0)} \in \mathbb{R}^m_{\geq 0}$, for all $i \in \mathcal{I}$.
Iteration at stage $k \in \mathbb{N}$:

1. Each agent $i \in \mathcal{I}$ receives $x_j^{(k)}$, for all $j \in \mathcal{I}$, and $\lambda^{(k)}$ from the aggregator.
2. Each agent $i \in \mathcal{I}$ updates in parallel:

$$\mathring{x}_i^{(k)} = \underset{y \in \mathcal{X}_i}{\operatorname{argmin}}\; J_i(y, \boldsymbol{x}_{-i}^{(k)}) + \left\langle \lambda^{(k)}, A_i y \right\rangle + \frac{\rho_i}{2}\|y - x_i^{(k)}\|^2. \tag{3.9}$$

3. Each agent $i \in \mathcal{I}$ sends $\mathring{x}_i^{(k)}$ to the aggregator.
4. The aggregator updates:

$$\mathring{\lambda}^{(k)} = \operatorname{proj}_{\mathbb{R}^m_{\geq 0}} \left(\lambda^{(k)} + \tau \sum_{i \in \mathcal{I}} \left(A_i \mathring{x}_i^{(k)} - b_i \right) \right), \tag{3.10}$$

$$\lambda^{(k+1)} = \mathring{\lambda}^{(k)} - \beta^{(k)} \nabla_\lambda \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\lambda}^{(k)}). \tag{3.11}$$

5. Each agent receives $\nabla_{\boldsymbol{x}_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\lambda}^{(k)})$ from the aggregator.
6. Each agent updates:

$$x_i^{(k+1)} = \mathring{x}_i^{(k)} - \beta^{(k)} \nabla_{\boldsymbol{x}_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\lambda}^{(k)}), \tag{3.12}$$

   and sends $x_i^{(k+1)}$ to the aggregator.

Communication 1: Each player $i \in \mathcal{I}$ receives the updated dual variable $\lambda^{(k)}$ and the decision variables of the remaining agents $\boldsymbol{x}_{-i}^{(k)}$ to compute the current estimate of the local Lagrangian function

$$\Lambda(x_i, \boldsymbol{x}_{-i}^{(k)}, \lambda^{(k)}) = J_i(x_i, \boldsymbol{x}_{-i}^{(k)}) + \langle \lambda^{(k)}, A_i x_i \rangle.$$

Regularized optimal response: Each agent computes a strategy $\mathring{x}_i^{(k)}$ by finding the optimizer of $\Lambda(x_i, \boldsymbol{x}_{-i}^{(k)}, \lambda^{(k)})$, with a quadratic penalization term on the deviation from the current strategy weighted by the parameter ρ_i, as in (3.9).
Communication 2 and dual ascent: The proposed strategies $\mathring{x}_i^{(k)}$ are gathered by an aggregator, which computes the gradient $\nabla_{\boldsymbol{x}_i} \phi(\mathring{\boldsymbol{x}}^{(k)}, \mathring{\lambda}^{(k)})$ and updates the dual variable via a dual ascent step as in (3.10) with step size τ. The results are communicated back to the agents.
Step towards the optimal selection: The agents and the aggregator update the primal and the dual variables by performing a gradient descent step of ϕ with vanishing step size $\beta^{(k)}$ as in (3.11) and (3.12), respectively.

Remark 3.1. *If the matrices C_{ij} are all equal, then the local cost functions read as in* (3.4)*, thus they depend on the remaining agents only through the aggregate value* $\operatorname{avg}(\{x_j\}_{j\in\mathcal{I}})$*. In such case, the agents only suffice to receive* $\operatorname{avg}(\{x_j\}_{j\in\mathcal{I}})$ *and the gradient directions* $\nabla_{\boldsymbol{x}_i}\phi(\mathring{\boldsymbol{x}}^{(k)},\mathring{\lambda}^{(k)})$*.*

Remark 3.2. *If the selection function is separable, that is, $\phi(\boldsymbol{x})=\sum_{i\in\mathcal{I}}\phi_i(x_i,\lambda_i)$, then the second round of communications is not needed, as the step in* (3.12) *can be immediately computed using local information only.*

Under a choice of step sizes that satisfies Assumptions 3.5 and 3.6, let us now present the main result of this chapter in Theorem 3.1, which states that the sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ converges to the set of the optimal solutions to Problem (3.8), implying that $(\boldsymbol{x}^{(k)})_{k\in\mathbb{N}}$ converges to the optimal v-GNE set.

Assumption 3.5. *The step sizes ρ_i and τ satisfy:*

i) $\rho_i > \sum_{j\in\mathcal{I}\setminus\{i\}} \|C_{ij}\| + \|A_i^\top\|, \quad$ *for all* $i\in\mathcal{I}$,

ii) $\tau < \Big(\sum_{i\in\mathcal{I}}\|A_i\|\Big)^{-1}$.

Assumption 3.6. *The sequence $(\beta^{(k)})_{k\in\mathbb{N}}\in\mathbb{R}_{\geq 0}$ satisfies:*

i) $\lim_{k\to\infty}\beta^{(k)}=0$,

ii) $\sum_{k\geq 1}\beta^{(k)}=\infty$,

iii) $\sum_{k\geq 1}(\beta^{(k)})^2<\infty$.

Remark 3.3. *The sequence $\beta^{(k)}=\beta_0/k^\gamma$, for any $\beta_0>0$ and $\gamma\in(1/2,1]$, satisfies Assumption 3.6.*

Theorem 3.1. *Let Assumptions 3.1–3.6 hold. Let $\Omega^\star$ be the set of solutions to Problem* (3.8)*. Furthermore, let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$, where $\boldsymbol{\omega}^{(k)}=(\boldsymbol{x}^{(k)},\lambda^{(k)})$ be the sequence generated by Algorithm 4. Then, we have*

$$\lim_{k\to\infty}\operatorname{dist}(\boldsymbol{\omega}^{(k)},\Omega^\star)=0.$$

3.3.2 Algorithm derivation

As the first step towards obtaining Algorithm 4, let us consider the preconditioned proximal point (PPP) method, which was proposed in [17, 19] to solve the GNE seeking problem for the class of linearly coupled aggregative games. The preconditioned proximal-point (PPP) method can be compactly written as the iteration

$$\boldsymbol{\omega}^{(k+1)}=\mathcal{T}_{\mathrm{PPP}}(\boldsymbol{\omega}^{(k)}), \tag{3.13}$$

3

where the operator $\mathcal{T}_{\mathrm{PPP}}$ is defined as

$$\mathcal{T}_{\mathrm{PPP}}(\boldsymbol{\omega}) := (\mathrm{Id} + \Gamma^{-1}\mathcal{T}_{\mathrm{e}})^{-1}(\boldsymbol{\omega}), \tag{3.14}$$

with preconditioning matrix

$$\Gamma := \begin{bmatrix} \rho - \boldsymbol{C} & -\boldsymbol{A}^\top \\ -\boldsymbol{A} & \tau^{-1} I \end{bmatrix},$$

$\rho := \mathrm{diag}(\{\rho_i I_{n_i}\}_{i\in\mathcal{I}})$ and $\boldsymbol{A}^\top := \mathrm{col}(\{A_i^\top\}_{i\in\mathcal{I}})$. The operator $\mathcal{T}_{\mathrm{PPP}}$ enjoys the following property, which we leverage for proving Theorem 3.1:

Lemma 3.1. *If Assumptions 3.1, 3.2, 3.3, and 3.5 hold, then $\mathcal{T}_{\mathrm{PPP}}$ is attracting nonexpansive in the Γ-induced norm $\|\cdot\|_\Gamma$.*

We note that $\Gamma = \Gamma^\top$ is a necessary condition for the Γ-induced norm to be well-defined, which explains the need for Assumption 3.2. Next, we transform Problem (3.8) into a fixed-point selection problem:

Lemma 3.2. *Let Assumptions 3.1–3.5 hold. Problem* (3.8) *is equivalent to*

$$\text{find } \boldsymbol{\omega}^\star \text{ s.t. } \inf_{\boldsymbol{\omega}\in\mathrm{fix}(\mathcal{T}_{\mathrm{PPP}})} \langle \boldsymbol{\omega} - \boldsymbol{\omega}^\star, \nabla\phi(\boldsymbol{\omega}^\star)\rangle \geq 0. \tag{3.15}$$

By finding a connection between Problem (3.8) and a class of fixed-point selection problems that has been studied in the literature, e.g., [56, 74, 75], we can then resort to the algorithmic solutions available for the latter. Specifically, we consider the hybrid steepest-descent method (HSDM) [15]. As formally stated next, indeed Algorithm 4 is a particular instance of the HSDM.

Lemma 3.3. *Let Assumptions 3.1–3.6 hold. Then, Algorithm 4 and the HSDM iterations, for all $k \in \mathbb{N}$,*

$$\boldsymbol{\omega}^{(k+1)} = \mathcal{T}_{\mathrm{PPP}}(\boldsymbol{\omega}^{(k)}) - \beta^{(k)}\nabla\phi(\mathcal{T}_{\mathrm{PPP}}(\boldsymbol{\omega}^{(k)})), \tag{3.16}$$

with $\mathcal{T}_{\mathrm{PPP}}$ as defined in (3.14)*, are equivalent.*

3.4 Illustrative example

We illustrate the advantages of the proposed methodology on a numerical example. Let $N = 6$ agents compete over the usage of 3 utilities, where the cost of each utility grows linearly with its aggregate usage. This setting is modelled by the game in (3.2) with local cost function given by (3.4) and C diagonal, whose non-zero elements are randomly sampled from the uniform distribution with support set $[0,1]$. Let $\ell_i(x_i) = q^\top x_i$ represent the cost of agent i incurred by employing the contended utilities, where q is randomly sampled from the uniform distribution with support set $[-10,0]$. Let $\mathcal{X}_i = \Pi_{j=1}^3 [a_{i,j}, 100]$, where $a_{i,j}$ is drawn from the uniform distribution with support set $[-1,1]$. The shared constraints in (3.2b) are given by $A_i = I_3$ and

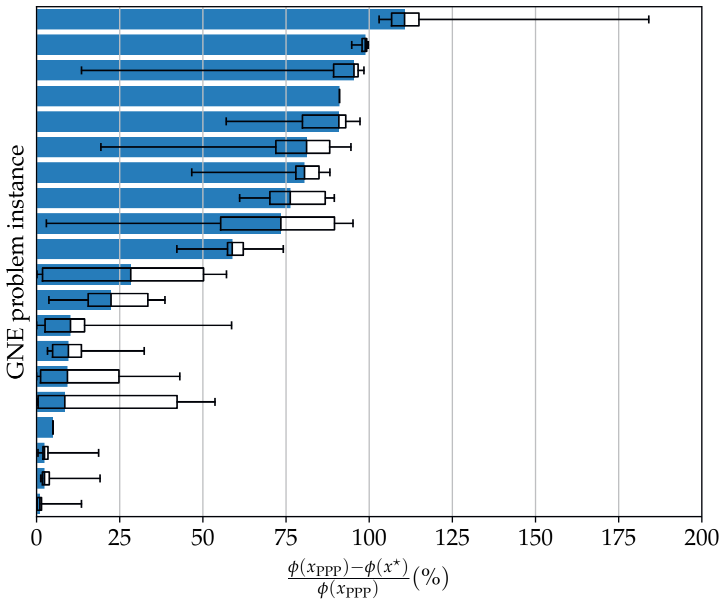
$b_i = 5\mathbf{1}_3$ for all $i \in \mathcal{I}$, that is, the sum of each utility is less than 5. Finally, the selection function ϕ is given by

$$\phi(\boldsymbol{x}) = \sum_{i \in \mathcal{R}} \|x_i\|_{Q_i}^2 + \widetilde{q}_i^\top x_i, \tag{3.17}$$

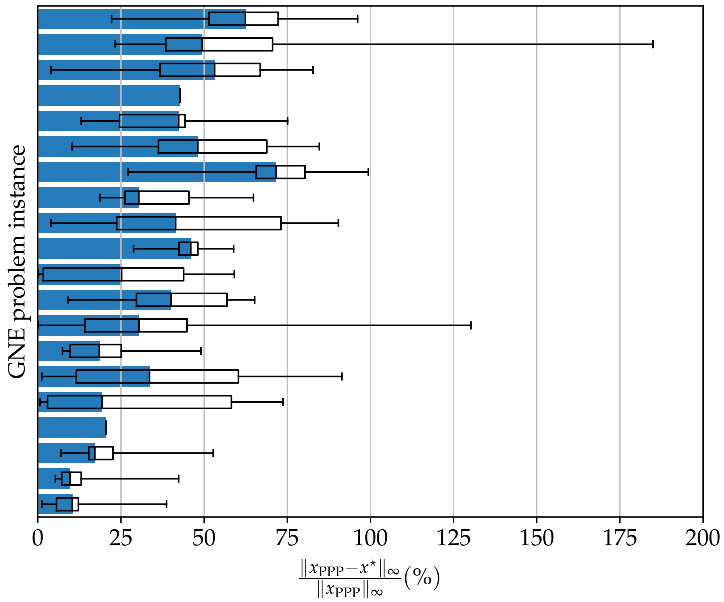
where $\mathcal{R} \subset \mathcal{I}$ is randomly generated with $|\mathcal{R}| = 2$, Q_i is a diagonal matrix whose nonzero elements are randomly drawn from the uniform distribution with support set $[1,2]$ and the elements of $\widetilde{q}_i$ are randomly drawn from the uniform distribution with support set $[-1,1]$. This choice for ϕ includes the case where the equilibrium point is selected in order to minimize a weighted distance from a reference point for the decision variables of 2 randomly selected agents. We sample 20 GNE selection problems. For each of the problem, we compare the result obtained by Algorithm 4 with the outcome of the standard PPP method [17, Alg. 6], which is obtained from Algorithm 4 by fixing $\beta^{(k)} = 0$ in (3.11) and (3.12). On the other hand, for Algorithm 4, we set $\beta^{(k)} = \beta_0 / k^\gamma$, with $\beta_0 = 0.1$ and γ selected from the set $\{0.6, 0.8, 1\}$ (see Remark 3.3). For each problem and each value of γ, both algorithms are run 20 times from a randomly generated initial condition. The results in Figure 3.4.1a show that the GNEs computed by the standard PPP are most of the time suboptimal with respect to ϕ in (3.17) as the values of the selection function are higher from those of the GNEs computed by Algorithm 4. As can also be seen from Figure 3.4.1a, the advantage gained by Algorithm 4 depends on the primitives of the problem and the performance of the standard PPP algorithm with respect to the selection function are strongly dependent on the initial condition. Figure 3.4.1b shows that the distance between the equilibrium points is not correlated with the reduction in the value of ϕ. Figure 3.4.2 compares the convergence rates to the set of GNEs by means of the residual r of the KKT conditions in (3.7a) defined as

$$r(\boldsymbol{\omega}^{(k)}) = \left\| \boldsymbol{\omega}^{(k)} - \begin{bmatrix} \mathrm{proj}_{\boldsymbol{\mathcal{X}}} \left(\boldsymbol{x}^{(k)} - F(\boldsymbol{x}^{(k)}) - \mathrm{col}(A_j^\top \lambda)_{j \in \mathcal{I}} \right) \\ \mathrm{proj}_{\mathbb{R}^m_{\geq 0}} \left(\lambda^{(k)} + \sum_{i \in \mathcal{I}} (A_i x_i^{(k)} - b_i) \right) \end{bmatrix} \right\|_\infty .$$

Figure 3.4.2 shows that Algorithm 4 presents slower convergence to the set of GNEs compared to PPP. This is expected since, although the updates in (3.9) and (3.10) lead the decision variables to the set of GNEs, the gradient step in (3.11) and (3.12) may lead the decision variables away from it until the step size $\beta^{(k)}$ is small enough; thus slowing down the convergence. Such an observation hints, as possible future research directions, the exploration of higher-order or accelerated methods inspired by the HSDM to achieve a faster convergence to the optimal GNE. Moreover, we observe from Figure 3.4.2 that a smaller value of γ results in a slower convergence to the GNE set (the residual) (see the top plot of Figure 3.4.2). This is due to the fact that a smaller value of γ implies a slower convergence of the diminishing step size $\beta^{(k)}$ to 0. This increases the weight of the gradient descent steps in (3.11)–(3.12) during the transient, which further slows the convergence down. On the other hand, it is observed that a high value of γ, despite having a fast residual convergence, might result in a slow convergence of the objective function $\phi(\boldsymbol{x}^{(k)})$ to the optimal value (see the bottom plot of Figure 3.4.2). This trade-off suggests that a careful choice of the step size $\beta^{(k)}$ is crucial for the performance of the algorithm.



(a) Reduction in the selection function value.



(b) Distance between the computed solutions.

Figure 3.4.1: Simulation results for the GNEs computed by Algorithm 4 ($\boldsymbol{x}^\star$) using $\gamma = 0.6$ compared to the standard PPP algorithm ($\boldsymbol{x}_{\mathrm{PPP}}$).

3

3

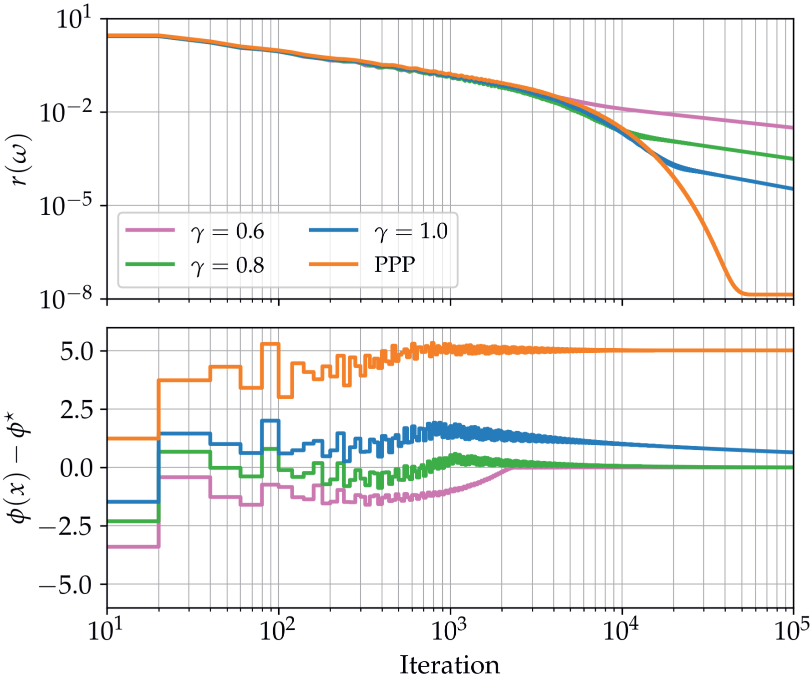


Figure 3.4.2: Comparison of Algorithm 1 for various values of γ with the standard PPP algorithm in terms of the convergence of the residual r (top plot) and the objective function ϕ (bottom plot). Here, $\phi^\star$ is the optimal ϕ computed by Algorithm 4 for $\gamma = 0.6$ (which obtains the minimum cost). Each line represents the average of simulation results from all the 20 GNE selection problems with 20 randomly sampled initial conditions.

3.5 Conclusion

For linearly coupled aggregative games with multiple equilibria, it is possible to select a particular solution that optimizes a convex preference function. The equilibrium selection can be achieved via a semi-decentralized computation by combining an instance of the preconditioned proximal point algorithm with the hybrid steepest descent method. This framework can be exploited to enforce a system-level objective among the set of equilibrium strategies. We identify as future work the extension of the results to more general monotone games, the characterization of the convergence rate, and the dependency of the latter on the design parameters.

Appendix

3.A Proof of Lemma 3.1

The matrix Γ is positive definite under Assumptions 3.2 and 3.5, following the generalized Gerschgorin disc theorem [76, Thm. 2] as in [17, Lemma 8]. It can be proven that this, together with the maximal monotonicity of $\mathcal{T}_{\mathrm{e}}$, implies that $\Gamma^{-1}\mathcal{T}_{\mathrm{e}}$ is maximally monotone in the Γ-induced norm $\|\cdot\|_{\Gamma}$, and the proof follows verbatim the one of [37, Lemma 7(*ii*)]. By noting that $\mathcal{T}_{\mathrm{PPP}}$ is the resolvent of the operator $\Gamma^{-1}\mathcal{T}_{\mathrm{e}}$ as defined in [51, Def. 23.1], it follows from [51, Prop. 23.8] that $\mathcal{T}_{\mathrm{PPP}}$ is firmly nonexpansive. From the existence of a v-GNE [57, Prop. 12.11], $\mathrm{fix}(\mathcal{T}_{\mathrm{PPP}}) \neq \varnothing$ and, therefore, the claim follows immediately by applying [51, Remark 4.36]. ∎

3.B Proof of Lemma 3.2

By Assumption 3.5, Γ is positive definite and $\Gamma^{-1}\mathcal{T}_{\mathrm{e}}$ is maximally monotone (see Appendix 3.A). Therefore, it holds that [51, Prop. 23.38]:

$$\boldsymbol{\omega} \in \mathrm{zer}(\mathcal{T}_{\mathrm{e}}) \Leftrightarrow \boldsymbol{\omega} \in \mathrm{fix}(\mathcal{T}_{\mathrm{PPP}}). \tag{3.18}$$

Since ϕ is differentiable by Assumption 3.4, the claim follows immediately as (3.15) is the stationary point problem associated to (3.8) [63, Sec. 1.3.1] . ∎

3.C Proof of Lemma 3.3

Let $\boldsymbol{y} := [\boldsymbol{p}, d]^\top = \mathcal{T}_{\mathrm{PPP}}(\boldsymbol{\omega}^{(k)})$. From (3.14), we obtain

$$\mathcal{T}_{\mathrm{e}}(\boldsymbol{y}) \ni \Gamma(\boldsymbol{\omega}^{(k)} - \boldsymbol{y}).$$

Substituting the definition of $\mathcal{T}_{\mathrm{e}}$, we obtain, for $\boldsymbol{p}$,

$$\left(\prod_{i\in\mathcal{I}} \mathrm{N}_{\mathcal{X}_i} + F\right)(\boldsymbol{p}) \ni (\rho - \boldsymbol{C})(\boldsymbol{x}^{(k)} - \boldsymbol{p}) - \boldsymbol{A}^\top \lambda^{(k)}. \tag{3.19}$$

By the definition of J_i in (3.1) and rearranging (3.19), we obtain

$$\forall\, i \in \mathcal{I}:\ p_i + \left(\mathrm{N}_{\mathcal{X}_i} + \rho_i^{-1}\partial\ell_i\right)(p_i) \ni x_i^{(k)} - \rho_i^{-1}\left(\sum_{j\in\mathcal{I}\setminus\{i\}} C_{ij}x_j^{(k)} + A_i^\top\lambda^{(k)}\right).$$

Following $\mathrm{N}_{\mathcal{X}_i} = \partial\iota_{\mathcal{X}_i}$ and [51, Prop. 16.44], then the update in (3.13), for each agent $i \in \mathcal{I}$, reads as

$$p_i = \mathrm{prox}_{\iota_{\mathcal{X}_i} + \rho_i^{-1}\ell_i}\left(x_i^{(k)} - \rho_i^{-1}\left(A_i^\top\lambda^{(k)} + \sum_{j\in\mathcal{I}\setminus\{i\}} C_{ij}x_j^{(k)}\right)\right),$$

which is equivalent to the update in (3.9) by the definition of proximal operator; thus, $\boldsymbol{p} = \mathring{\boldsymbol{x}}^{(k)}$. We similarly prove that the expression for $d = \mathring{\lambda}^{(k)}$ is equivalent to the update in (3.10). We can then finally observe that (3.11) and (3.12), for all $i \in \mathcal{I}$, are an expanded form of (3.16). ∎

3.D Proof of Theorem 3.1

We note that $\boldsymbol{\Omega}$ is bounded by Assumption 3.1. Thus, the set of v-GNEs is bounded as it is a subset of $\boldsymbol{\Omega}$ and it is nonempty under Assumption 3.3 [57, Prop. 12.11]. Under Assumption 3.1, the set of dual variables that solve (3.7a) is bounded [58, Prop. 3.3], thus $\mathrm{zer}(\mathcal{T}_{\mathrm{e}})$ (as well as $\mathrm{fix}(\mathcal{T}_{\mathrm{PPP}})$, from (3.18)) is nonempty and bounded. From Lemma 3.1, $\mathcal{T}_{\mathrm{PPP}}$ is attracting nonexpansive. Therefore, the iteration in (3.16), which is equivalent to Algorithm 4 by Lemma 3.3, satisfies all the assumptions of [66, Thm. 3] and the result follows. ■

4

Nash equilibrium selection via Tikhonov regularization

-"There's some sort of danger centering at the selenium pool. It increases as [the robot] approaches, and at a certain distance from it the third law of robotics [self-preservation] exactly balances the second law [obedience]."
- "So he follows a circle around the selenium pool, staying on the locus of all points of potential equilibrium."[...]
- "How about working the other way? If we increase the danger, we increase the third law's potential and drive him backward."

Isaac Asimov, in "Runaround"

To optimally select a generalized Nash equilibrium, in this chapter, we consider a semi-decentralized algorithm based on a double-layer Tikhonov regularization algorithm. Technically, we extend the Tikhonov method for equilibrium selection to generalized games. Next, we couple such an algorithm with the preconditioned forward-backward splitting, which guarantees linear convergence to a solution of the inner layer problem and allows for a semi-decentralized implementation. We then establish a conceptual connection and draw a comparison between the considered algorithm and the hybrid steepest descent method, the other known distributed approach for solving the equilibrium selection problem.

This chapter is partly based on Benenati, E., Ananduta, W. and Grammatico, S. "A Semi-Decentralized Tikhonov-Based Algorithm for Optimal Generalized Nash Equilibrium Selection". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 4243–4248, Singapore, Singapore, (Dec. 2023). IEEE.

4.1 Introduction

Several multiagent decision processes can be modelled as a game, that is, a set of inter-dependent optimization problems. In particular, if the agents are coupled not only through their respective objective functions, but also through a shared constraint set, then we label the setting as a *generalized* game. Application examples for generalized games include traffic routing [25], peer-to-peer energy markets [13] and cognitive radio networks [31]. A typical solution paradigm is the generalized Nash equilibrium (GNE), that is, an optimal situation for each agent given the decisions of the remaining agents, and especially the sub-class of variational GNEs (v-GNEs), which has recently received widespread attention due to its stability properties [3].
Various efficient variational GNE (v-GNE) seeking algorithms, e.g. [16, 37, 39, 42, 77], have been developed for games that satisfy a monotonicity condition. Crucially, monotone games admit in general an infinite number of v-GNEs (unless a much more restrictive *strong* monotonicity condition is imposed). An appealing method to deal with the non-uniqueness of the solution is to select a GNE that optimizes some desirable system-level objective, as proposed in [21, 43]. The selection algorithms in [21, 43] use Tikhonov's regularization method and build on the literature of Variational Inequalities (VIs) to cast the selection problem as a variational inequality (VI)-constrained VI, which is solved by finding a sequence of approximate solutions to regularized games. The algorithms presented in §2 and §3, instead, rely on fixed point selection theory and use the hybrid steepest descent method (HSDM) [15], which pairs an appropriate nonexpansive operator with a gradient descent. While the latter is recently proposed as distributed algorithms for GNE selection, the former works for non-generalized games only. Our main contribution consists in devising a Tikhonov-based algorithm for equilibrium selection in generalized, monotone games. The proposed algorithm is semi-decentralized, in the sense that communication with a central coordinator is required, but its only duty is to broadcast signals aimed at optimizing the system-level objective function. Technically, we cast the GNE selection problem as a VI-constrained VI, which can be solved via a sequence of regularized sub-problems. Compared to [21], we propose to solve the resulting regularized sub-problems via the preconditioned forward-backward or preconditioned forward-backward (pFB) [37], which has linear convergence rate and allows one to distribute the computation burden among the agents. Compared with the Tikhonov-based GNE seeking algorithms in [42, 77] that compute the minimum-norm v-GNE, our proposed algorithm works for general convex selection functions. Secondly, we find a theoretical connection between the proposed Tikhonov method and the hybrid steepest-descent method (HSDM). Although neither method generalizes the other, the HSDM can be cast as a forward-backward step towards the solution of the Tikhonov regularized problem. Finally, in Section 4.5, we compare the two methods numerically.

4.2 Equilibrium selection as a variational inequality

In this section, we provide an interpretation of the generalized Nash equilibrium (GNE) selection problem introduced in Section 2.3.2 as a variational inequality (VI) constrained to the solution set of another VI. We consider the multi-agent decision process in which each of N agents aims at solving an optimization problem over the decision variables $x_i \in \mathcal{X}_i \subset \mathbb{R}^{n_i}$, where $i \in \{1,\dots,N\} =: \mathcal{I}$. Let us denote $n := \sum_{i\in\mathcal{I}} n_i$, $\mathcal{I}_{-i} := \mathcal{I}\setminus\{i\}$, $\boldsymbol{x}_{-i} := \mathrm{col}(x_j)_{j\in\mathcal{I}_{-i}}$ and $\boldsymbol{x} := \mathrm{col}(x_j)_{j\in\mathcal{I}}$. Crucially, the decision problem associated with agent i is coupled to the decision variables of the remaining agents both through the objective function $J_i : \mathbb{R}^n \to \mathbb{R}$, and some constraints. We consider $m \in \mathbb{N}$ constraints of the form

$$\sum_{i\in\mathcal{I}} A_i x_i \le b, \tag{4.1}$$

4

where $A_i \in \mathbb{R}^{m\times n_i}$, for each $i \in \mathcal{I}$, and $b \in \mathbb{R}^m$. This problem is commonly referred to as a *generalized game* and we formalize it as follows:

$$\forall i \in \mathcal{I}: \begin{cases} \min\limits_{x_i\in\mathcal{X}_i} & J_i(x_i,\boldsymbol{x}_{-i}) & \text{(4.2a)} \\ \text{s.t.} & A_i x_i \le b - \sum\limits_{j\in\mathcal{I}_{-i}} A_j x_j. & \text{(4.2b)} \end{cases}$$

By defining $\boldsymbol{\mathcal{X}} := \Pi_{i\in\mathcal{I}}\mathcal{X}_i$, the collective feasible set of (4.2) is

$$\Gamma := \boldsymbol{\mathcal{X}} \cap \{\boldsymbol{x} \mid \text{(4.1) holds true}\}.$$

We address the problem in (4.2) by examining generalized Nash equilibria, which are points from which no agent has an incentive to deviate unilaterally:

Definition 4.1. *A set of decision variables $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ is a GNE for the game in* (4.2) *if, for each $i \in \mathcal{I}$,*

$$J_i(x_i^*,\boldsymbol{x}_{-i}^*) \le J_i(x_i,\boldsymbol{x}_{-i}^*),$$

for any $x_i \in \mathcal{X}_i \cap \{y \in \mathbb{R}^{n_i} \mid A_i y \le -\sum_{j\in\mathcal{I}_{-i}} A_j x_j^*\}$.

Existence of a GNE is guaranteed [57, prop. 12.11] under the following, standard assumptions [16, 39]:

Assumption 4.1. *For each i, $J_i(\cdot,\boldsymbol{x}_{-i})$ is convex and continuously differentiable for any $\boldsymbol{x}_{-i}$.*

Assumption 4.2. *For all $i \in \mathcal{I}$, $\mathcal{X}_i$ is compact and convex; $\Gamma \neq \varnothing$ and it satisfies Slater's constraint qualification.*

Assumption 4.3. *The pseudogradient of the game in* (4.2)

$$F(\boldsymbol{x}) := \mathrm{col}\left(\nabla_{x_i} J_i(x_i,\boldsymbol{x}_{-i})\right)_{i\in\mathcal{I}}$$

is monotone and L_F-Lipschitz continuous.

For solving the problem in (4.2), we assume that the agents can exchange information over an undirected, connected communication network. We denote the set of neighbours of agent i in this network by $\mathcal{N}_i$. For simplicity, we consider the case where, for all $i \in \mathcal{I}$, J_i depends on x_i and the decision variables of (a subset of) $\mathcal{N}_i$, so that each agent is able to evaluate its cost function by communicating with their neighbours. Additionally, each agent maintains a local estimate of the dual variable λ_i for the shared constraints in (4.1) and an auxiliary variable ν_i to achieve consensus on dual variable estimates. We then define the following extended Karush-Kuhn-Tucker (Karush–Kuhn–Tucker (KKT)) operator [37], which includes both the optimality conditions for the problem in (4.2) and the consensus condition:

$$\begin{aligned}
\mathcal{T}^{\mathrm{KKT}}(\boldsymbol{\omega}) &:= \mathcal{A}(\boldsymbol{\omega}) + \mathcal{B}(\boldsymbol{\omega}) + \mathcal{C}(\boldsymbol{\omega}), \\
\mathcal{A}(\boldsymbol{\omega}) &:= \mathrm{N}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{x}) \times \mathrm{N}_{\mathbb{R}^{|\mathcal{I}|m}_{\geq 0}}(\boldsymbol{\lambda}) \times \{\mathbf{0}_{|\mathcal{I}|m}\}, \\
\mathcal{B} &:= \begin{bmatrix} F(\boldsymbol{x}) \\ -\overline{\mathcal{L}}\boldsymbol{\lambda} \\ 0 \end{bmatrix}, \\
\mathcal{C}(\boldsymbol{\omega}) &:= \begin{bmatrix} \boldsymbol{A}^\top \boldsymbol{\lambda} \\ b - \boldsymbol{A}\boldsymbol{x} - \overline{\mathcal{L}}\boldsymbol{\nu} \\ \overline{\mathcal{L}}\boldsymbol{\lambda} \end{bmatrix},
\end{aligned} \tag{4.3}$$

where $\boldsymbol{\omega} = \mathrm{col}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$, $\overline{\mathcal{L}} = \mathcal{L} \otimes I_m$ and $\mathcal{L}$ is the Laplacian matrix of the communication graph. A subset of the GNEs of the game in (4.2) is characterized by the zero set of $\mathcal{T}^{\mathrm{KKT}}$ [37, Thm. 2]. These are called variational GNEs (v-GNEs) [3, Def. 3.10] and have been extensively studied with multiple efficient computation algorithms available. However, Assumptions 4.1–4.3 alone do not guarantee that $\mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$ is a singleton. Most of the algorithms in the literature compute an unspecified variational GNE (v-GNE) among the possibly infinitely many. In contrast, our approach focuses on finding an optimally selected v-GNE according to the selection function ϕ, in the sense that we aim at solving

$$\min_{\boldsymbol{\omega}} \ \phi(\boldsymbol{\omega}) \tag{4.4a}$$

$$\text{s.t. } \boldsymbol{\omega} \in \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}}). \tag{4.4b}$$

Assumption 4.4. *The selection function ϕ is convex, continuously differentiable, and coercive. Furthermore, its gradient is $L_{\nabla\phi}$-Lipschitz continuous.*

By [51, Prop. 23.39], (4.4a) is a convex optimization problem under Assumptions 4.1, 4.3 and 4.4. However, $\mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$ can seldom be written in a closed form and, thus, (4.4a) cannot be solved by standard optimization algorithms. To derive an algorithmic solution, we note that, by defining $\boldsymbol{\Omega} := \boldsymbol{\mathcal{X}} \times \mathbb{R}^{Nm}_{\geq 0} \times \mathbb{R}^{Nm}$ and by [63, Eq. (1.1.3)],

$$\boldsymbol{\omega}^* \in \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}}) \Leftrightarrow \boldsymbol{\omega}^* \in \mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{B} + \mathcal{C}). \tag{4.5}$$

Following the equivalence between convex optimization problems and VIs [63, Sec. 1.3.1], we then recast (4.4a) as:

$$\mathrm{VI}\big(\mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{B} + \mathcal{C}), \nabla\phi\big). \tag{4.6}$$

As discussed in [21], we can solve (4.6) by finding the ε_k-approximate solutions to a sequence of regularized sub-problems, indexed by $k \in \mathbb{N}$:

$$\mathrm{VI}\big(\boldsymbol{\Omega}, \mathcal{B}+\mathcal{C}+\gamma_k \nabla\phi + \alpha(\mathrm{Id}-\boldsymbol{\omega})\big), \quad \boldsymbol{\omega} \in \boldsymbol{\Omega}. \tag{4.7}$$

The regularization weights α, $(\gamma_k)_{k\in\mathbb{N}}$ and tolerances $(\varepsilon_k)_{k\in\mathbb{N}}$ are chosen according to the following criteria:

Assumption 4.5. *The parameter α is positive; $(\gamma_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ are positive and non-negative sequences of real numbers, respectively, such that $\sum_{k\in\mathbb{N}}\gamma_k = \infty$, $\lim_{k\to\infty}\gamma_k = 0$ and $\varepsilon_k = 0$ for all $k > K$, where $K \in \mathbb{N}$.*

By denoting $\overline{\gamma} := \sup_{k\in\mathbb{N}}\gamma_k$, we observe the following properties of the operators that define (4.7):

Lemma 4.1. *Let Assumptions 4.3–4.5 hold true. For any $k \in \mathbb{N}$, $\alpha > 0$, and $\boldsymbol{\omega} \in \boldsymbol{\Omega}$:*

1. *$\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi+\alpha(\mathrm{Id}-\boldsymbol{\omega})$ is α-strongly monotone,*

2. *$\mathcal{B}+\gamma_k\nabla\phi+\alpha(\mathrm{Id}-\boldsymbol{\omega})$ is L_G- Lipschitz continuous, where $L_G := \max(L_F, 2|\mathcal{N}_i|_{i\in\mathcal{I}}) + \overline{\gamma}L_{\nabla\phi}+\alpha$.*

By applying [63, Cor. 2.2.5] and [63, Thm. 2.3.3a], we conclude that the regularized problem in (4.7) admits a unique solution. Proposition 4.1, which follows directly from [21, Thm. 2], formalizes a prototypical algorithmic solution to the problem in (4.6):

Proposition 4.1. *Let Assumptions 4.1–4.5 hold. Let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}} \in \boldsymbol{\Omega}$ and, for every k, $\boldsymbol{\omega}^{(k+1)}$ be the ε_k-approximate solution of the VI in (7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. Then, the sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ is bounded and each of its limit points is a solution of* (4.6)*.*

4.3 Semi-decentralized equilibrium selection

In view of Proposition 4.1, next we derive an algorithm for generating a sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ such that, for all k, $\boldsymbol{\omega}^{(k+1)}$ is a ε_k-approximate solution to (4.7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. The exact solution, denoted by $\boldsymbol{\omega}_k^*$, satisfies the monotone inclusion

$$0 \in \big(\mathcal{A}+\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi+\alpha(\mathrm{Id}-\boldsymbol{\omega}^{(k)})\big)(\boldsymbol{\omega}_k^*). \tag{4.8}$$

We then apply the preconditioned forward-backward (pFB) method, proposed in [37] in the context of decentralized GNE seeking, for solving (4.8). Let us define the following matrix:

$$\Psi = \mathrm{diag}(\boldsymbol{\rho}^{-1}, \boldsymbol{\tau}^{-1}, \boldsymbol{\sigma}^{-1}), \tag{4.9}$$

where $\boldsymbol{\rho} := \mathrm{diag}(\rho_i I_{n_i})_{i\in\mathcal{I}}$, $\boldsymbol{\tau} := \mathrm{diag}(\tau_i I_{n_i})_{i\in\mathcal{I}}$, $\boldsymbol{\sigma} := \mathrm{diag}(\sigma_i I_{n_i})_{i\in\mathcal{I}}$ collect the step sizes associated with the primal, dual and auxiliary variables, respectively, chosen according to the following design criterion:

4

Assumption 4.6. *Let*

$$
\begin{aligned}
r_i^{\mathrm{x}} &:= \max_{j=1,\dots,n_i} \sum_{k=1}^{m} |[A_i]_{jk}|,\\
r_i^{\lambda} &:= \max_{j=1,\dots,n_i} \sum_{k=1}^{m} |[A_i]_{jk}| + 2|\mathcal{N}_i|,\\
r_i^{\nu} &:= 2|\mathcal{N}_i|.
\end{aligned}
$$

Furthermore, let $r = \max_{i\in\mathcal{I}}(r_i^{\mathrm{x}}, r_i^{\lambda}, r_i^{\nu})$ *and* $\delta > \max(\frac{L_G^2}{\alpha}, 2r)$*, with* L_G *defined as in Lemma 4.1. For all* $i \in \mathcal{I}$*:*

1. $(2\delta - r_i^{\mathrm{x}})^{-1} \leq \rho_i \leq (\delta + r_i^{\mathrm{x}})^{-1}$;
2. $(2\delta - r_i^{\lambda})^{-1} \leq \tau_i \leq (\delta + r_i^{\lambda})^{-1}$;
3. $(2\delta - r_i^{\nu})^{-1} \leq \sigma_i \leq (\delta + r_i^{\nu})^{-1}$.

Let us also define the preconditioning matrix

$$
\Phi = \Psi + \begin{bmatrix} \mathbf{0} & -\boldsymbol{A}^\top & \mathbf{0} \\ -\boldsymbol{A} & \mathbf{0} & -\overline{\mathcal{L}} \\ \mathbf{0} & -\overline{\mathcal{L}} & \mathbf{0} \end{bmatrix}. \tag{4.10}
$$

Lemma 4.2. *Under Assumption 4.6,* $\Phi \succcurlyeq \delta I$ *and* $\frac{\delta}{\|\Phi\|} \geq \frac{1}{2}$.

The preconditioned forward-backward (pFB) operator for the inclusion in (4.8) reads as

$$
\mathcal{T}_k^{\mathrm{pFB}} = \big(\mathrm{Id} + \Phi^{-1}(\mathcal{A}+\mathcal{C})\big)^{-1} \circ \Big(\mathrm{Id} - \Phi^{-1}\big(\mathcal{B} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega}^{(k)})\big)\Big). \tag{4.11}
$$

The following result formalizes the convergence of the fixed-point iteration generated by $\mathcal{T}_k^{\mathrm{pFB}}$ to the solution of (4.7).

Lemma 4.3. *Let Assumptions 4.1–4.6 hold and* $\boldsymbol{y}^0 \in \boldsymbol{\Omega}$*. Then, for all* $k \in \mathbb{N}$*, the sequence* $(\boldsymbol{y}^{(t)})_{t\in\mathbb{N}}$ *generated by the fixed-point iteration*

$$
\boldsymbol{y}^{(t+1)} = \mathcal{T}_k^{\mathrm{pFB}}(\boldsymbol{y}^{(t)}) \qquad \forall\, t \in \mathbb{N}, \tag{4.12}
$$

where $\mathcal{T}_k^{\mathrm{pFB}}$ *is defined in* (4.11)*, converges linearly in the* Φ*-induced norm to* $\boldsymbol{\omega}_k^*$ *in* (4.8) *and*

$$
\|\boldsymbol{y}^{(t)} - \boldsymbol{\omega}_k^*\|_\Phi \leq \|\boldsymbol{y}^{(t+1)} - \boldsymbol{y}^{(t)}\|_\Phi / (1-\beta), \tag{4.13}
$$

with $\beta := \big(1 + \frac{L_G^2}{\delta^2} - \frac{2\alpha}{\|\Phi\|}\big) < 1$.

Remark 4.1. *Differently from the pFB operator in [37], the one in* (4.11) *has additional regularization terms and thus achieves the desired linear rate.*

The linear convergence of $\mathcal{T}^{\mathrm{pFB}}k$ guarantees that, for $\varepsilon_k > 0$, the iteration $\boldsymbol{\omega}^{(k+1)}$ in Proposition 4.1 is found within a finite number of inner iterations, whose termination can be based on a simple stopping criterion derived from (4.13). The proposed method, which results from the expansion of the pFB operator, is illustrated in Algorithm 5, where we use t as the inner iteration index.

Proposition 4.2. *Let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ be generated by Algorithm 5. Under Assumptions 4.1–4.6, for each k, $\boldsymbol{\omega}^{(k+1)}$ is an ε_k-approximate solution of (4.7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$ and, if $\varepsilon_k > 0$, the condition in (4.17) is verified in a finite number of steps.*

Remark 4.2. *For Assumption 4.5 in Proposition 4.1 to be satisfied, the termination condition for the inner iterations needs to eventually ensure $\varepsilon_k = 0$. This is a stringent requirement, as the pFB algorithm only achieves an exact solution asymptotically. The authors of [21], alternatively, consider the less restrictive condition $\lim_{k\to\infty} \frac{\varepsilon_k}{\gamma_k} = 0$ in the case where the definition set of the VI-constrained VI is compact. Although $\boldsymbol{\Omega}$ does not satisfy this condition as the dual variables belong to an unbounded set, in practice the dual variables are bounded in view of [58, Prop. 3.3].*

Algorithm 5 Tikhonov-pFB for optimal GNE selection

Initialization. Let α, $(\varepsilon_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ satisfy Assumption 4.5. Let $\boldsymbol{\omega}^{(0)} \in \Pi_{i\in\mathcal{I}}(\mathcal{X}_i) \times \mathbb{R}_{\geq 0}{}^{|\mathcal{I}|m} \times \mathbb{R}^{|\mathcal{I}|m}$. Let $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}$ satisfy Assumption 4.6.
Outer iteration: for $k \in \mathbb{N}_0$

1. Each agent $i \in \mathcal{I}$ sets:

$$(x_i^{(k,0)}, \lambda_i^{(k,0)}, \nu_i^{(k,0)}) \leftarrow y_i^{(k,0)} \leftarrow \omega_i^{(k)}. \tag{4.14}$$

2. **Inner iteration: for** $t \in \mathbb{N}_0$
   -For each agent $i \in \mathcal{I}$**:**

   (a) Receive $x_j^{(k,t)}$, $\lambda_j^{(k,t)}, \nu_j^{(k,t)}$ from agent $j \in \mathcal{N}_i$ and $\nabla\phi_{\omega_i}(\boldsymbol{y}^{(k,t)})$ from the coordinator.

   (b) Update:

$$\begin{aligned} x_i^{(k,t+1)} =& \mathrm{proj}_{\mathcal{X}_i}\Big[x_i^{(k,t)} - \rho_i\big(\nabla_{x_i} J_i(\boldsymbol{x}^{(k,t)}) \\ &+ A_i^\top \lambda_i^{(k,t)} + \gamma_k \nabla_{x_i}\phi(\boldsymbol{y}^{(k,t)}) + \alpha(x_i^{(k,t)} - x_i^{(k,0)})\big)\Big], \end{aligned} \tag{4.15a}$$

$$\begin{aligned} \nu_i^{(k,t+1)} =& \nu_i^{(k,t)} - \sigma_i\Big(\sum_{j\in\mathcal{N}_i}(\lambda_i^{(k,t)} - \lambda_j^{(k,t)}) \\ &+ \alpha(\nu_i^{(k,t)} - \nu_i^{(k,0)}) + \gamma_k \nabla_{\nu_i}\phi(\boldsymbol{y}^{(k,t)})\Big). \end{aligned} \tag{4.15b}$$

   (c) Receive $\nu_j^{(k,t+1)}$ from agent $j \in \mathcal{N}_i$.

   (d) Update:

$$\begin{aligned} \lambda_i^{(k,t+1)} =& \mathrm{proj}_{\mathbb{R}^m_{\geq 0}}\Big[\lambda_i^{(k,t)} + \tau_i\Big(-\nabla_{\lambda_i}\phi(\boldsymbol{y}^{(k,t)}) + \alpha(\lambda_i^{(k,t)} - \lambda_i^{(k,0)}) \\ &+ A_i(2x_i^{(k,t+1)} - x_i^{(k,t)}) - b_i \\ &+ \sum_{j\in\mathcal{N}_i}\big(2\nu_i^{(k,t+1)} - 2\nu_j^{(k,t+1)} - \nu_i^{(k,t)} + \nu_j^{(k,t)} - \lambda_i^{(k,t)} + \lambda_j^{(k,t)}\big)\Big)\Big]. \end{aligned} \tag{4.16}$$

   -Coordinator:

   (a) Set $\boldsymbol{y}^{(k,t+1)} \leftarrow (\boldsymbol{x}^{(k,t+1)}, \boldsymbol{\lambda}^{(k,t+1)}, \boldsymbol{\nu}^{(k,t+1)})$.

   (b) Communicate $\nabla_{\omega_i}\phi(\boldsymbol{y}^{(k,t+1)})$ to each agent $i \in \mathcal{I}$.

   (c) If the following is satisfied,

$$\|\boldsymbol{y}^{(k,t+1)} - \boldsymbol{y}^{(k,t)}\|_\Phi \leq (1-\beta)\varepsilon_k, \tag{4.17}$$

   terminate inner iteration. Each agent then sets

$$\omega_i^{(k+1)} = y_i^{(k,t+1)}. \tag{4.18}$$

4.4 Comparison with the hybrid steepest descent method

In §2 and §3, we take a different algorithm design path, where we reformulate Problem (4.4a) as a fixed point selection problem. This allows one to use the hybrid steepest descent method (HSDM) [15] to solve (4.4a). Specifically, one has to find a quasi-shrinking mapping $\mathcal{T}$ (Def. 2.1) such that $\operatorname{fix}(\mathcal{T}) = \operatorname{zer}(\mathcal{T}^{\text{KKT}})$. Then, the limit point of the sequence $(\boldsymbol{z}^{(k)})_{k\in\mathbb{N}}$, defined by

$$\boldsymbol{z}^{(k+1)} = \mathcal{T}(\boldsymbol{z}^{(k)}) - \gamma_k \nabla\phi(\mathcal{T}\boldsymbol{z}^{(k)}), \tag{4.19}$$

converges to the solution of (4.4a) if $(\gamma_k)_{k\in\mathbb{N}}$ is square-summable but non-summable [15, Thm. 5]. The vanishing weight on $\nabla\phi$ is reminiscent of the Tikhonov regularization introduced in Section 4.3. Indeed, the two methods are related, as shown next. Let us consider the exact solution to the VI in (4.7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. Then, from [63, Prop. 12.3.6],

$$\begin{aligned}\boldsymbol{\omega}^{(k+1)} &= \mathcal{J}_{\frac{1}{\alpha}(\mathcal{A}+\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi)}(\boldsymbol{\omega}^{(k)}) \\ &=: \mathcal{T}_k^{\text{TIK}}(\boldsymbol{\omega}^{(k)}).\end{aligned} \tag{4.20}$$

The properties of the hybrid steepest-descent method (HSDM) update in (4.19) depend on the choice of $\mathcal{T}$. Let us consider the particular case $\mathcal{T} = \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}$, which can be shown to be quasi-shrinking via Lemma 2.1. We rewrite (4.19) with this particular choice of $\mathcal{T}$ as:

$$\begin{aligned}\boldsymbol{v}^{(k+1)} &= \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}(\boldsymbol{z}^{(k)}), \\ \boldsymbol{z}^{(k+1)} &= \boldsymbol{v}^{(k+1)} - \gamma_k\nabla\phi(\boldsymbol{v}^{(k+1)}),\end{aligned} \tag{4.21}$$

where we introduced the auxiliary sequence $(\boldsymbol{v}^{(k)})_{k\in\mathbb{N}}$. By rearranging (4.21), we note that this sequence evolves as

$$\boldsymbol{v}^{(k+1)} = \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}} \circ (\mathrm{Id} - \gamma_k\nabla\phi)(\boldsymbol{v}^{(k)}) =: \mathcal{T}_k^{\text{HSDM}}(\boldsymbol{v}^{(k)}). \tag{4.22}$$

The operator $\mathcal{T}_k^{\text{HSDM}}$ in (4.22) corresponds to the forward-backward (FB) splitting for the inclusion

$$0 \in \mathcal{A} + \mathcal{B} + \mathcal{C} + \gamma_k\nabla\phi.$$

From [51, Prop. 26.1iv] and [51, Prop. 23.38],

$$\operatorname{fix}\left(\mathcal{J}_{\frac{1}{\alpha}(\mathcal{A}+\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi)}\right) = \operatorname{fix}\left(\mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}} \circ (\mathrm{Id} - \gamma_k\nabla\phi)\right),$$

which implies

$$\operatorname{fix}(\mathcal{T}_k^{\text{TIK}}) = \operatorname{fix}(\mathcal{T}_k^{\text{HSDM}}). \tag{4.23}$$

Thus, we conclude that both the Tikhonov update in (4.20) and the HSDM step in (4.19) apply at each step k a single update of a fixed point iteration, and the two operators in (4.23) have the same fixed point set. This analogy is only theoretical, as in practice the operator $\mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}$ cannot be implemented in a distributed fashion. Nevertheless, it outlines that both the Tikhonov method and the HSDM function by

	Tikhonov	Tikhonov (bounded set)	HSDM
param.	$\sum \gamma_k = \infty$, evt. $\varepsilon_k = 0$	$\sum \gamma_k = \infty$, $\lim_{k\to\infty} \frac{\varepsilon_k}{\gamma_k} = 0$	$\sum \gamma_k = \infty$, $\sum \gamma_k^2 < \infty$
t (#inner iterations)	evt. ∞	$t \to \infty$	1
ϕ coercive	yes	no	no

Table 4.4.1: Theoretical property differences of the Tikhonov method and HSDM for GNE selection.

the same underlying principle of tracking the solutions to a sequence of regularized problems. Moreover, we note that (4.23) does not hold for a generic choice of $\mathcal{T}$, thus the HSDM includes algorithms that are not covered by the Tikhonov method. On the other hand, the operator $\mathcal{T}_k^{\text{TIK}}$ cannot be rewritten in terms of an FB operator. Therefore, we should conclude that neither method is a generalization of the other. The key differences between the two frameworks are summarized in Table 4.4.1. In addition, we remark that the Tikhonov framework can be paired with any (splitting) methods for strongly monotone games to obtain a decentralized algorithm. Meanwhile, the HSDM requires methods for monotone games that are quasi-shrinking, such as the forward-backward-forward (FBF) splitting [51, Sec. 26.6]. Thus, Tikhonov-based methods can benefit from a larger pool of available algorithms.

4.5 Numerical simulations

We test the proposed algorithm on 100 game instances with 10 agents, where the pseudogradient is in the form

$$F(\boldsymbol{x}) = Q_F \boldsymbol{x} + \boldsymbol{c}_F.$$

The parameters $Q_F \succcurlyeq 0$ and $\boldsymbol{c}_F$ are randomly generated. We define the selection function

$$\phi(\boldsymbol{\omega}) = \|\boldsymbol{x}\|^2_{Q_\phi} + \boldsymbol{c}_\phi^\top \boldsymbol{x} + \theta(\|\boldsymbol{\lambda}\|^2 + \|\boldsymbol{\nu}\|^2),$$

where $Q_\phi \succcurlyeq 0$ and $\boldsymbol{c}_\phi$ are as well randomly generated and $\theta = 10^{-3}$. For all i, we define the local constraint set $\mathcal{X}_i = \{\mathbb{R}^5 : \|x_i\|_\infty \leq 1\}$. Furthermore, we let $A_i = I$, for all $i \in \mathcal{I}$, and $b = 2 \cdot \mathbf{1}_5$.We then set $(\gamma_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ in Algorithm 5 to

$$\gamma_k = 10^{-3} k^{-\xi}; \quad \varepsilon_k = \begin{cases} 10^{-3} k^{-\xi\zeta} & \text{if } 10^{-3} k^{-\xi\zeta} \geq \underline{\varepsilon} \\ 0 & \text{if } 10^{-3} k^{-\xi\zeta} < \underline{\varepsilon} \end{cases}$$

where $\underline{\varepsilon}$ is set to the computer numerical precision. The parameter ξ controls the decay of the regularization weight γ_k, while ζ controls the decay rate of ε_k with respect to γ_k. We evaluate Algorithm 5 in terms of the residual computed for each outer iteration k and inner iteration t as

$$\mathfrak{R}(t) = \|\boldsymbol{y}^{(k,t)} - (\mathrm{Id} + \mathcal{A})^{-1} \circ (\mathrm{Id} - \mathcal{B} - \mathcal{C})(\boldsymbol{y}^{(k,t)})\|,$$

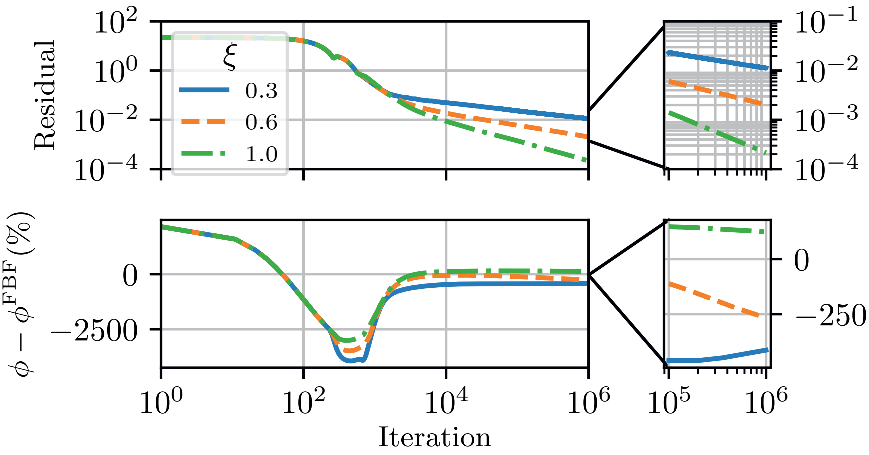


Figure 4.5.1: Average performance of Algorithm 5 for $\zeta = 2$ and $\alpha = 1$.

and in terms of the reduction of selection function ϕ with respect to the value obtained by the standard forward-backward-forward (FBF) algorithm (without optimal selection) [16, Alg. 2] in Figures 4.5.1–4.5.3. We observe that, for decreasing values of ξ, the algorithm achieves a lower selection function value and a larger residual (cf. Figure 4.5.1). This trade-off between convergence to a GNE (measured by the residual) and convergence to a ϕ-optimal point is expected, because a too slow decay of the regularization weight γ_k leads the algorithm to disregard the GNE seeking in order to compute the unconstrained optimal value of ϕ. Moreover, an increasing value of α improves the algorithm performance (cf. Figure 4.5.2). Finally, we observe that for increasing values of ζ, the algorithm reaches a higher residual and a lower selection function value (cf. Figure 4.5.3). In our experience, setting ζ too high might cause convergence failure as ε_k might become 0 before γ_k reaches a negligible value. In Figure 4.5.4 we compare Algorithm 5 with the HSDM paired with the FBF algorithm [20, Alg. 1]. The parameters for Algorithm 5 are chosen among the ones that performed reasonably well in both the performance metrics considered in Figures 4.5.1–4.5.3. We find the two algorithms to have similar convergence speed. One might find this surprising, as the Tikhonov method is double-layered; thus, one could expect slower convergence compared to the single-layer HSDM. This can be explained by noting that the slowdown caused by the double-layer iterations is compensated by the linear convergence of the pFB. In contrast, the HSDM uses the FBF, which only achieves sublinear convergence. Nevertheless, as observed in the first set of simulations, the Tikhonov method requires more careful parameter tuning than the HSDM.

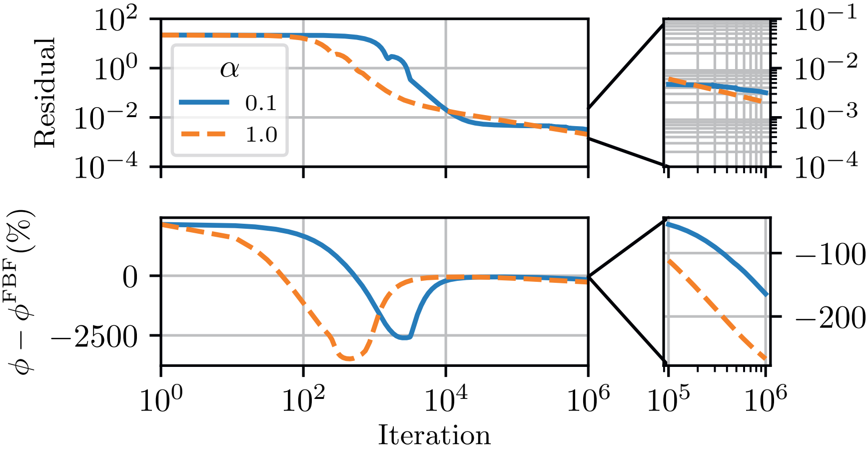


Figure 4.5.2: Average performance of Algorithm 5 for $\xi = 0.6$ and $\zeta = 2$.

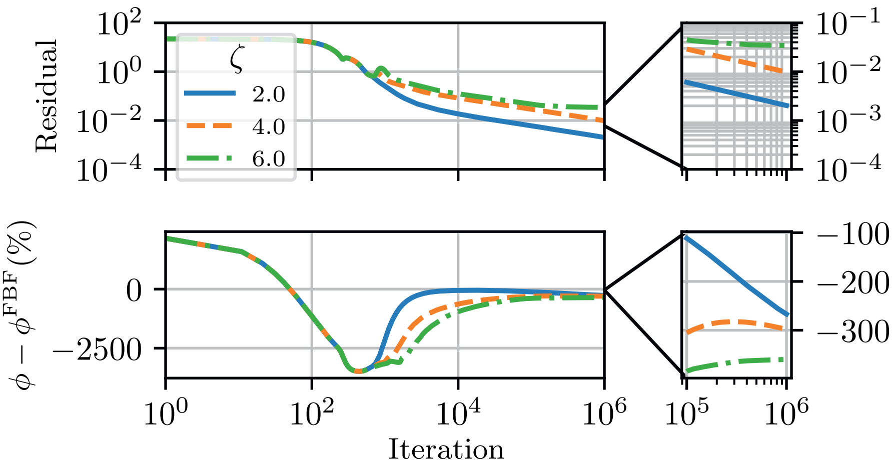


Figure 4.5.3: Average performance of Algorithm 5 for $\alpha = 1$ and $\xi = 0.6$.

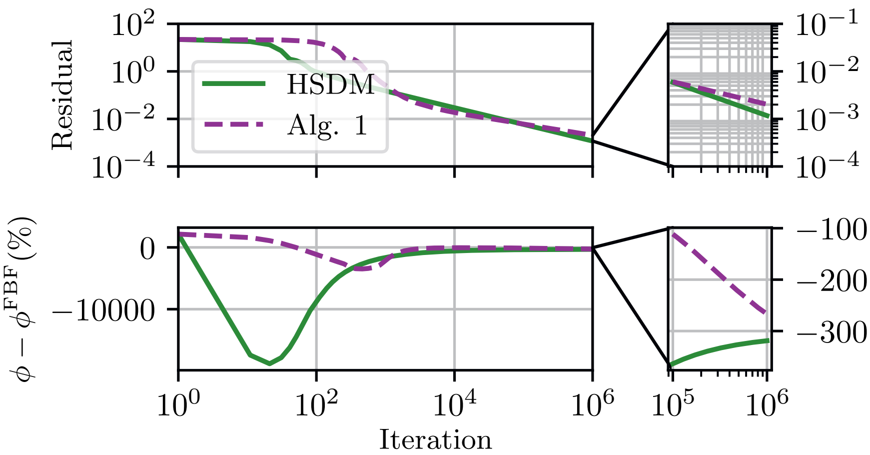


Figure 4.5.4: Comparison between Algorithm 5 with $\xi = 0.6$, $\zeta = 2$, $\alpha = 1$ and the HSDM-FBF method [20, Alg. 1]. The x-axis indicates the cumulative number of inner iterations for Algorithm 5.

4.6 Conclusion

The generalized Nash equilibrium selection problem can be solved with a semi-decentralized algorithm based on the Tikhonov regularization method combined with the preconditioned forward backward algorithm, which achieves linear convergence for the regularized sub-problems. Interestingly, both the Tikhonov regularization method and a particular instance of the hybrid steepest descent method seek at each iteration an approximate solution to the same regularized problem, indicating a conceptual connection. Although theoretically less practical (as shown in Table 4.4.1), the Tikhonov algorithm demonstrates comparable convergence performance to the state-of-the-art in our simulations.

Appendix

4.A Proof of Lemma 4.1

Lemma 4.1.1 is immediate from the monotonicity of $\mathcal{B}$ [37, Lem. 5], $\mathcal{C}$ [51, Ex. 20.35] and $\nabla\phi$ (Assumption 4.4). From Gerschgorin's theorem, $\|\overline{\mathcal{L}}\| \leq 2\max(|\mathcal{N}_i|_{i\in\mathcal{I}})$. Thus, $\mathcal{B}$ is $\max(L_F, 2|\mathcal{N}_i|_{i\in\mathcal{I}})$-Lipschitz continuous from Assumption 4.3. Lemma 4.1.2 then follows immediately. ■

4.B Proof of Proposition 4.1

Let us denote the solution set of (4.6) by $\mathcal{S}$. The proposition follows from [21, Thm. 2] if $\mathbf{\Omega}$ is closed and convex; $\mathcal{B}+\mathcal{C}$ is continuous and monotone; $\nabla\phi$ is continuous and monotone plus; $\mathcal{S}$ is bounded and not empty; and the set

$$\mathfrak{L}_1 := \{\boldsymbol{\omega}_2 \in \mathbf{\Omega} | \ \exists\ \boldsymbol{\omega}_1 \in \mathcal{S} \text{ s. t. } \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) > 0\}$$

is bounded. The conditions on $\mathbf{\Omega}$ and the continuity of $\mathcal{B}, \mathcal{C}, \nabla\phi$ follow from the assumptions. The monotonicity of $\mathcal{B}+\mathcal{C}$ is proven in Lemma 4.1. $\nabla\phi$ is monotone-plus from the convexity of ϕ and [51, Ex. 22.4i]. From [37, Thm. 2], $\mathrm{zer}(\mathcal{T}^{\mathrm{KKT}}) \neq \varnothing$. Let $\boldsymbol{\omega}_1 \in \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$ and consider

$$\mathfrak{L}_2 := \{\boldsymbol{\omega}_2 \in \mathbf{\Omega} | \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geq 0\}.$$

We apply the convexity inequality on ϕ to find

$$\phi(\boldsymbol{\omega}_1) - \phi(\boldsymbol{\omega}_2) \geq \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2), \quad \forall \boldsymbol{\omega}_2 \in \mathbf{\Omega}. \tag{4.24}$$

From the coercivity of ϕ, $\phi(\boldsymbol{\omega}_1) - \phi(\boldsymbol{\omega}_2) < 0$ for sufficiently large $\|\boldsymbol{\omega}_2\|$. Thus, $\mathfrak{L}_2$ is bounded. Therefore, from [63, Prop. 2.2.3], $\mathcal{S}$ is non-empty and compact. Consequently, as (4.24) holds for any $\boldsymbol{\omega}_1 \in \mathcal{S}$, we find for all $\boldsymbol{\omega}_2 \in \mathbf{\Omega}$,

$$\max_{\boldsymbol{\omega}_1 \in \mathcal{S}} \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \leq \phi^* - \phi(\boldsymbol{\omega}_2),$$

$\phi^* := \max_{\boldsymbol{\omega}_1 \in \mathcal{S}} \phi(\boldsymbol{\omega}_1)$. By the coercivity of ϕ, $\phi^* - \phi(\boldsymbol{\omega}_2) < 0$ for sufficiently large $\|\boldsymbol{\omega}_2\|$. Therefore, $\mathfrak{L}_1$ is bounded. ■

4.C Proof of Lemma 4.2

It follows from [37, Lemma 6] and Gerschgorin's theorem. ■

4.D Proof of Lemma 4.3

The operator $(\mathrm{Id} + \Phi^{-1}(\mathcal{A}+\mathcal{C}))^{-1}$ is non-expansive in the Φ-induced norm, following [37, Lem. 7ii]. Denote $G = \mathcal{B} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega}^{(k)})$. First, we observe

$$\|\Phi\| I \succcurlyeq \Phi \succcurlyeq \delta I \quad \Rightarrow \quad \|\Phi\| \|\boldsymbol{z}\|^2 \geq \|\boldsymbol{z}\|_\Phi^2 \geq \delta \|\boldsymbol{z}\|^2, \tag{4.25a}$$

$$\Phi \succcurlyeq \delta I \quad \Rightarrow \quad \Phi^{-1} \preccurlyeq \delta^{-1} I \quad \Rightarrow \quad \|\boldsymbol{z}\|_{\Phi^{-1}}^2 \leq \delta^{-1} \|\boldsymbol{z}\|^2, \tag{4.25b}$$

for any $\boldsymbol{z} \in \boldsymbol{\Omega}$. Furthermore, for any pair $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\Omega}$,

$$\begin{aligned}\left\langle \boldsymbol{z}-\boldsymbol{z}', \Phi^{-1}(G\boldsymbol{z}-G\boldsymbol{z}')\right\rangle_{\Phi} &\geq \langle \boldsymbol{z}-\boldsymbol{z}', \Phi^{-1}\gamma_k(\nabla\phi(\boldsymbol{z})-\nabla\phi(\boldsymbol{z}'))+\alpha\Phi^{-1}(\boldsymbol{z}-\boldsymbol{z}')\rangle_{\Phi} \\ &\geq \alpha\|\boldsymbol{z}-\boldsymbol{z}'\|^2 \\ &\geq \frac{\alpha}{\|\Phi\|}\|\boldsymbol{z}-\boldsymbol{z}'\|_{\Phi}^2. \end{aligned} \tag{4.26}$$

We use the fact that $\Phi^{-1}\mathcal{B}$ is cocoercive in the Φ-induced norm [37, Lem. 7i] (and, thus, monotone) in the first inequality, the monotonicity of $\nabla\phi$ in Euclidean norm in the second inequality, and (4.25a) in the third inequality. We then have that

4

$$\begin{aligned}\|\Phi^{-1}G(\boldsymbol{z})-\Phi^{-1}G(\boldsymbol{z}')\|_{\Phi}^2 &= \|G(\boldsymbol{z})-G(\boldsymbol{z}')\|_{\Phi^{-1}}^2 \\ &\overset{(4.25b)}{\leq} \frac{1}{\delta}\|G(\boldsymbol{z})-G(\boldsymbol{z}')\|^2 \\ &\leq \frac{L_G^2}{\delta}\|\boldsymbol{z}-\boldsymbol{z}'\|^2 \\ &\overset{(4.25a)}{\leq} \frac{L_G^2}{\delta^2}\|\boldsymbol{z}-\boldsymbol{z}'\|_{\Phi}^2, \end{aligned} \tag{4.27}$$

where we use the Lipschitz continuity of G (Lemma 4.1). By expanding the square and from (4.26) and (4.27), we have that

$$\begin{aligned}\|(\mathrm{Id}-\Phi^{-1}G)(\boldsymbol{z})-(\mathrm{Id}-\Phi^{-1}G)(\boldsymbol{z}')\|_{\Phi}^2 &\leq (1+\frac{L_G^2}{\delta^2}-\frac{2\alpha}{\|\Phi\|})\|\boldsymbol{z}-\boldsymbol{z}'\|_{\Phi}^2 \\ &= \beta\|\boldsymbol{z}-\boldsymbol{z}'\|_{\Phi}^2. \end{aligned}$$

From Assumption 4.6 and Lemma 4.2,

$$\frac{L_G^2}{\delta^2} < \frac{L_G^2\alpha}{L_G^2\delta} \leq \frac{2\alpha}{\|\Phi\|},$$

thus $\beta < 1$ and $(\mathrm{Id}-\Phi^{-1}G)$ is contractive. From [51, Prop. 26.1.iv] and [63, Eq. 1.1.3], $\mathrm{fix}(\mathcal{T}^{\mathrm{pFB}}{}_k) = \mathrm{zer}(\mathcal{A}+\mathcal{C}+G) = \mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{C}+G)$ and the claim follows from [51, Thm. 1.50]. ■

4.E Proof of Proposition 4.1

By expanding the operators that define $\mathcal{T}_k^{\mathrm{pFB}}$, (4.15a)–(4.16) are equivalent to $\boldsymbol{y}^{(k,t+1)} = \mathcal{T}^{\mathrm{pFB}}{}_k(\boldsymbol{y}^{(k,t)})$, for all $k \in \mathbb{N}$. From (4.17), (4.18), and Lemma 4.3,

$$\begin{aligned}\|\boldsymbol{\omega}^{(k+1)}-\boldsymbol{\omega}_k^*\|_{\Phi} &\leq \|\boldsymbol{y}^{(k,t+1)}-\boldsymbol{y}^{(k,t)}\|_{\Phi}/(1-\beta) \\ &\leq \varepsilon_k. \end{aligned}$$

Next, by triangle inequality, we have that

$$\begin{aligned}
\|\boldsymbol{y}^{(k,t+1)}-\boldsymbol{y}^{(k,t)}\|_\Phi &\leq \|\boldsymbol{y}^{(k,t+1)}-\boldsymbol{\omega}_k^*\|_\Phi+\|\boldsymbol{\omega}_k^*-\boldsymbol{y}^{(k,t)}\|_\Phi \\
&\overset{\{1\}}{\leq} (1+\beta)\|\boldsymbol{\omega}_k^*-\boldsymbol{y}^{(k,t)}\|_\Phi \\
&\overset{\{2\}}{\leq} \beta^t(\beta+1)\|\boldsymbol{\omega}_k^*-\boldsymbol{y}^{(k,0)}\|_\Phi,
\end{aligned}$$

where {1} follows from the contractivity of $\mathcal{T}^{\mathrm{pFB}}{}_k$ (see the proof of Lemma 4.3) and {2} from [51, Thm. 1.50iii]. Thus, if $\varepsilon_k > 0$, (4.17) holds for

$$t \geq \log_\beta\left(\frac{\varepsilon_k}{(\beta+1)\|\boldsymbol{\omega}_k^*-\boldsymbol{y}^{(k,0)}\|_\Phi}\right).$$

■

4

5

Linear convergence in distributed, time-varying games

-*"You don't think we're* networked, *do you?"*
-*"Well, I don't know. Are you?"*
-*"Gods around, no! Ugh! Can you imagine?"*
The robot's face was angular in its disgust.
-*"Would you want everybody else's thoughts in your head?"*

Becky Chambers, in "A psalm for the wild-built"

We study generalized games with full row rank equality constraints, and we provide a strikingly simple proof of strong monotonicity of the associated KKT operator. This allows us to show linear convergence to a variational equilibrium of the resulting primal-dual pseudo-gradient dynamics. Then, we propose a fully-distributed algorithm with linear convergence guarantee for aggregative games under partial decision information. Based on these results, we establish stability properties for online GNE seeking in games with time varying cost functions and constraints. Finally, we illustrate our findings numerically on an economic dispatch problem for peer-to-peer energy markets.

This chapter is partly based on Bianchi, M., Benenati, E. and Grammatico, S. "Linear Convergence in Time-Varying Generalized Nash Equilibrium Problems". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 7220–7226, Singapore, Singapore, (Dec. 2023). IEEE.

5.1 Introduction

Generalized Nash equilibrium (GNE) problems arise in many multiagent applications, where the agents are coupled not only because of their conflicting objectives, but also via shared constraints – operational limits of the system, that the agents should respect. Among others, GNE seeking is used in energy markets [28], radio communication [3] and formation control [78] problems.

The networked structure of these applications naturally calls for distributed solution methods. In fact, part of the recent literature focuses on semi-decentralized GNE seeking algorithms [17, 34], where the agents update their decisions locally, with the help of a coordinator that gathers and broadcasts information over the systems (a setup also named full-information scenario). Other works [40, 71, 79, 80] deal with applications where the agents can only rely on fully-distributed peer-to-peer communication and local data. In this so-called partial-decision information scenario, the agents compensate for the lack of global knowledge by estimating the unknown quantities and by embedding consensus dynamics in their local decision processes.

In both scenarios, to cope with the presence of coupling constraints and to distribute the computation among the agents, one should resort to Lagrangian reformulations. In fact, all the references above leverage primal-dual pseudo-gradient algorithms, aimed at solving the Karush–Kuhn–Tucker (KKT) optimality conditions of the GNE problem.

In general, primal-dual algorithms fail to achieve linear convergence, even for the class of strongly-monotone generalized games [17]. Importantly, together with linear convergence, some crucial input-to-state stability (ISS) properties of pseudo-gradient iterations are also not guaranteed. This lack of robustness is a critical issue for methods in the partial-decision information scenario, where convergence should be ensured despite the estimation error. To overcome this complication, vanishing step sizes can be used to drive the error to zero [39], at the price of slow convergence. Alternatively, several fixed-step algorithms for GNE seeking were derived based on operator-theoretic methods and on the use of preconditioning [40, 71, 79]. Unfortunately, this approach comes with important limitations, such as extending the analysis to time-varying setups.

For instance, there is no available fixed-step fully-distributed method to solve GNE problems when the agents can only exchange information over switching communication networks (while methods are available for games without coupling constraints [81]). Furthermore, in many decision processes with real-world applications, the cost functions of the agents and the system constraints can vary over time [82], for instance in cognitive radio networks and demand response in smart grids [83]. In such domains, linearly convergent algorithms become particularly desirable, as the solver needs to quickly update its solution in response to changes in the environment. Despite its practical relevance, there are very few works that study the online GNE problem. The chapter [84] proposes a regularized algorithm, which only achieves inexact convergence, and which is not fully-distributed. Instead, the authors of [49] develop an algorithm for the partial-decision information scenario, but that achieves sublinear regret only when the solution is asymptotically constant and for diminishing step sizes.

Contribution: In this chapter we study generalized games with full row rank coupling equality constraints –as those arising in resource allocation and transportation problems [85], where demand-matching [80] and flow [28] constraints are ubiquitous. For the first time, we show that, in this setup, linear convergence to a GNE can be achieved via primal-dual dynamics, both for the full- and partial-decision information scenario. Thanks to this result, we can also adapt the dynamics to online equilibrium seeking in time-varying games. Here we focus on the prominent class of aggregative games [17], for its desirable scalability properties, but the analysis carries over to generally-coupled costs. We summarize the novelties of our work as follows:

1. We provide a simple, constructive proof of the strong monotonicity of the KKT operator in games with full-row rank equality coupling constraints. As a consequence, we show linear convergence to a GNE of the pseudo-gradient ascent-descent method (Section 5.3);

2. We design a linearly convergent algorithm for GNE seeking in partial-decision information, via a tracking technique [39] that avoids the need for slack variables. Our proof is based on a change of coordinates and a small gain argument: due to its generality, the argument also applies to the case of (Q-connected) time-varying communication graphs (Section 5.4);

3. We exploit our linear convergence results to study the tracking properties of the proposed methods with respect to the solution of a game with time-varying costs and constraints. In particular, for the fully-distributed algorithm, we show that the extra error induced in the dynamic tracking procedure does not jeopardize stability (Section 5.5)

5

5.2 Mathematical setup

For clearness of exposition, we reintroduce the definition of generalized Nash equilibrium (GNE) presented in section 2.3.1. We consider a set of agents, $\mathcal{I} \coloneqq \{1,\dots,N\}$, where each agent $i \in \mathcal{I}$ shall choose its decision variable (i.e., strategy) $x_i \in \mathbb{R}^{n_i}$. Let $\boldsymbol{x} \coloneqq \mathrm{col}(x_i)_{i\in\mathcal{I}} \in \mathbb{R}^n$, with $n \coloneqq \sum_{i=1}^N n_i$. The goal of each agent $i \in \mathcal{I}$ is to minimize its objective function $J_i(x_i, \boldsymbol{x}_{-i})$, which depends on both the local variable x_i and on the decision variables of the other agents $\boldsymbol{x}_{-i} \coloneqq \mathrm{col}(x_j)_{j\in\mathcal{I}\setminus\{i\}}$. Furthermore, the feasible decisions of each agent depends on the action of the other agents via affine equality coupling constraints. Specifically, the feasible set is $\mathcal{X} \coloneqq \{x \in \mathbb{R}^n \mid Ax = b\}$, where $A \coloneqq [A_1,\dots,A_N]$ and $b \coloneqq \sum_{i=1}^N b_i$, $A_i \in \mathbb{R}^{m\times n_i}$ and $b_i \in \mathbb{R}^m$ being locally available information. The game is then represented by the inter-dependent optimization problems:

$$(\forall i \in \mathcal{I}) \quad \min_{y_i \in \mathbb{R}^{n_i}} \; J_i(y_i, \boldsymbol{x}_{-i}) \tag{5.1a}$$

$$\text{s.t. } (y_i, \boldsymbol{x}_{-i}) \in \mathcal{X}. \tag{5.1b}$$

The technical problem we consider here is the computation of a GNE, namely a set of decisions that simultaneously solve all the optimization problems in (5.1a).

Definition 5.1. *A collective strategy* $\boldsymbol{x}^\star = \operatorname{col}(x_i^\star)_{i\in\mathcal{I}}$ *is a generalized Nash equilibrium if, for all* $i \in \mathcal{I}$, $J_i\left(x_i^\star, \boldsymbol{x}_{-i}^\star\right) \leq \inf\{J_i\left(y_i, \boldsymbol{x}_{-i}^\star\right) \mid (y_i, \boldsymbol{x}_{-i}^\star) \in \mathcal{X}\}$. □

Next, we postulate some standard regularity and convexity assumptions for the constraint sets and cost functions.

Assumption 5.1 (Convexity)**.** *In* (5.1a)*,* $\mathcal{X}$ *is non-empty. For each* $i \in \mathcal{I}$, J_i *is continuous and* $J_i(\cdot, \boldsymbol{x}_{-i})$ *is convex and continuously differentiable for every* $\boldsymbol{x}_{-i}$. □

As common in the literature [86], [17], among all the GNEs, we focus on the subclass of variational GNEs (v-GNEs) [3, Def. 3.11], which are more economically justifiable, as well as computationally tractable [33]. Under Assumption 5.1 and defining the *pseudo-gradient* mapping of the game

$$F(\boldsymbol{x}) := \operatorname{col}\left(\nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i})\right)_{i\in\mathcal{I}}, \tag{5.2}$$

$\boldsymbol{x}^\star$ is a v-GNE of the game in (5.1a) if and only if there exists a dual variable $\lambda^\star \in \mathbb{R}^m$ such that the following Karush–Kuhn–Tucker (KKT) conditions are satisfied [3, Th. 4.8]:

$$\mathbf{0}_n \in F(\boldsymbol{x}^\star) + A^\top \lambda^\star, \quad \mathbf{0}_m \in -(A\boldsymbol{x}^\star - b). \tag{5.3}$$

Let us restrict our attention to strongly monotone games.

Assumption 5.2 (Strong monotonicity)**.** *The game mapping* F *in* (5.2) *is* μ_F*-strongly monotone and* ℓ_F*-Lipschitz continuous, for some* μ_F, $\ell_F > 0$. □

The strong monotonicity of F is sufficient to ensure existence and uniqueness of a v-GNE [63, Th. 2.3.3]. Strong monotonicity is a standard condition for algorithms with linear convergence. In addition, we make the following assumption.

Assumption 5.3 (Full rank constraints)**.** *A is full row rank.* $AA^\top \geq \mu_A I$, $\|A\| \leq \ell_A$ *for some scalars* $\mu_A, \ell_A > 0$. □

Assumption 5.3 postulates that there are no redundant constraints (or equivalently that redundant constraints are removed). This condition is well known in duality theory and optimization, as it ensures the uniqueness of dual solutions (as it can be inferred by (5.3)).
For ease of presentation, we will specialize our results to the prominent class of aggregative games, which arises in a variety of engineering applications, e.g., network congestion control and demand-side management [87]. In particular, we assume that the cost function J_i of each agent i depends only on the local decision x_i and on an aggregation value

$$\sigma(x) = \frac{1}{N} \sum_{i\in\mathcal{I}} \phi_i(x_i), \tag{5.4}$$

where $\phi_i : \mathbb{R}^{n_i} \to \mathbb{R}^q$ is a local function of agent i. In short, overloading the function J_i with some abuse of notation, we also write

$$J_i(x_i, x_{-i}) = J_i\big(x_i, \sigma(x)\big). \tag{5.5}$$

5.3 Linear convergence in generalized games

We start by showing that the KKT operator is strongly monotone, in a suitable norm, under Assumptions 5.2 and 5.3.

Lemma 5.1. *Let* $\boldsymbol{\omega} := \mathrm{col}(\boldsymbol{x}, \lambda)$*. The operator*

$$\boldsymbol{\omega} \mapsto \mathcal{A}(\boldsymbol{\omega}) = \begin{bmatrix} F(\boldsymbol{x}) + A^\top \lambda \\ -A\boldsymbol{x} + b \end{bmatrix} \tag{5.6}$$

is $\mu_{\mathcal{A}}$*-strongly monotone in* $\mathbb{R}^n_P$*, for some* $P \succ 0$ *and* $\mu_{\mathcal{A}} > 0$*.* □

Proof. For some $0 < \gamma < \ell_A$, let

$$P = \begin{bmatrix} I & \gamma A^\top \\ \gamma A & I \end{bmatrix} \succ 0. \tag{5.7}$$

For any $x, x^\star \in \mathbb{R}^n$ and $\lambda, \lambda^\star \in \mathbb{R}^m$, we have

$$\begin{aligned}
\langle \mathcal{A}(\boldsymbol{\omega}) - \mathcal{A}(\boldsymbol{\omega}^\star), \boldsymbol{\omega} - \boldsymbol{\omega}^\star \rangle_P =& \langle F(\boldsymbol{x}) - F(\boldsymbol{x}^\star), \boldsymbol{x} - \boldsymbol{x}^\star \rangle \\
&+ \gamma \langle F(\boldsymbol{x}) - F(\boldsymbol{x}^\star), A^\top(\lambda - \lambda^\star) \rangle \\
&+ \cancel{\langle A^\top(\lambda - \lambda^\star), \boldsymbol{x} - \boldsymbol{x}^\star \rangle} \\
&+ \gamma \langle A^\top(\lambda - \lambda^\star), A^\top(\lambda - \lambda^\star) \rangle \\
&- \gamma \langle A(\boldsymbol{x} - \boldsymbol{x}^\star), A(\boldsymbol{x} - \boldsymbol{x}^\star) \rangle \\
&- \cancel{\langle A(\boldsymbol{x} - \boldsymbol{x}^\star), \lambda - \lambda^\star \rangle} \\
\geq& (\mu_F - \gamma \ell_A^2) \| \boldsymbol{x} - \boldsymbol{x}^\star \|^2 \\
&- \gamma \ell_F \ell_A \| \boldsymbol{x} - \boldsymbol{x}^\star \| \| \lambda - \lambda^\star \| \\
&+ \gamma \mu_A \| \lambda - \lambda^\star \|^2
\end{aligned}$$

which is positive definite for $0 < \gamma < \frac{4\mu_F \mu_A}{\ell_F^2 \ell_A^2 + 4\mu_A \ell_A^2}$. The conclusion follows by equivalence of norms. ■

Based on Lemma 5.1 we can prove linear convergence of classic primal-dual iterations.

Theorem 5.1 (GNE seeking in full-decision information)**.** *Let Assumptions 5.1-5.3 hold and* $\ell_{\mathcal{A}} := (\ell_F + \ell_A)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$*. For any* $0 < \alpha < \frac{2\mu_{\mathcal{A}}}{\ell_{\mathcal{A}}^2}$ *and any initial condition* $(\boldsymbol{x}^0, \lambda^0)$*, the iteration*

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \alpha \left(F(\boldsymbol{x}^k) + A^\top \lambda^k \right) \tag{5.8a}$$

$$\lambda^{k+1} = \lambda^k + \alpha \left(A\boldsymbol{x}^k - b \right) \tag{5.8b}$$

converges linearly to the unique solution $\boldsymbol{\omega}^\star = (\boldsymbol{x}^\star, \lambda^\star)$ *of the KKT conditions in* (5.3)*: for all* $k \in \mathbb{N}$

$$\| \boldsymbol{\omega}^{k+1} - \boldsymbol{\omega}^\star \|_P^2 \leq \rho \| \boldsymbol{\omega}^k - \boldsymbol{\omega}^\star \|_P^2, \tag{5.9}$$

where $\rho = 1 - 2\alpha\mu_{\mathcal{A}} + \alpha^2 \ell_{\mathcal{A}}^2 < 1$ *and* $\boldsymbol{\omega}^k = (\boldsymbol{x}^k, \lambda^k)$*.* □

Algorithm 6 Semi-decentralized GNE seeking

Iterate to convergence: for all $k \in \mathbb{N}$,

- Each $i \in \mathcal{I}$: receive λ^k, σ^k from coordinator; update

$$x_i^{k+1} = x_i^k - \alpha \left(\nabla_{x_i} J_i(x_i^k, \sigma^k) + A_i^\top \lambda^k \right)$$

- Coordinator: receive $\{\phi_i({x_i}^{k+1}), A_i x_i^k - b_i\}_{i \in \mathcal{I}}$; update

$$\begin{aligned}\lambda^{k+1} &= \lambda^k + \alpha \textstyle\sum_{i \in \mathcal{I}} (A_i x_i^k - b_i),\\ \sigma^{k+1} &= \tfrac{1}{N} \textstyle\sum_{i \in \mathcal{I}} \phi_i(x_i^{k+1})\end{aligned}$$

Proof. By Lemma 5.1, $\|\boldsymbol{\omega} - \alpha\mathcal{A}(\boldsymbol{\omega}) - (\boldsymbol{\omega}^\star - \alpha\mathcal{A}(\boldsymbol{\omega}^\star))\|_P^2 = \|\boldsymbol{\omega} - \boldsymbol{\omega}^\star\|_P^2 - 2\alpha\langle \mathcal{A}(\boldsymbol{\omega}) - \mathcal{A}(\boldsymbol{\omega}^\star), \boldsymbol{\omega} - \boldsymbol{\omega}^\star\rangle_P + \alpha^2 \|\mathcal{A}(\boldsymbol{\omega}) - \mathcal{A}(\boldsymbol{\omega}^\star)\|_P^2 \leq \rho \|\boldsymbol{\omega} - \boldsymbol{\omega}^\star\|_P^2$, and the conclusion follows because (5.8) can be rewritten as $\boldsymbol{\omega}^{k+1} = \boldsymbol{\omega}^k - \alpha\mathcal{A}(\boldsymbol{\omega}^k)$. ■

5

5.4 Partial decision information

In this section, we consider aggregative games in the so-called partial-decision information scenario, where there is no central coordinator, and the agents can only exchange information via peer-to-peer communication over a communication graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$, with weight matrix $W \in \mathbb{R}^{N \times N}$, and $w_{i,j} := [W]_{i,j} > 0$ if and only if (j,i) belongs to the set of edges $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{I}$.

Assumption 5.4 (Communication)**.** *The graph $\mathcal{G}$ is strongly connected. The weight matrix W satisfies:*

- Double stochasticity*:* $\mathbf{1}^\top W = \mathbf{1}^\top, W\mathbf{1} = W\mathbf{1}$*;*
- Self-loops*:* $w_{i,i} > 0$ *for all* $i \in \mathcal{I}$*.*

We denote $\theta := \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\|$*. Note that* $\theta < 1$ *following [88, Lemma 10.3]* □

To remedy the lack of global knowledge, we let each agent $i \in \mathcal{I}$ keep:

- $\hat{\sigma}_i \in \mathbb{R}^q$: estimate of the aggregation $\sigma(x)$;
- $\hat{\lambda}_i \in \mathbb{R}^m$: estimate of the dual variable λ;
- $\hat{r}_i \in \mathbb{R}^m$: estimate of the residual $Ax - b$;
- $z_i \in \mathbb{R}^m$: additional dual variable.

We make the following standard assumption.

Algorithm 7 Fully-distributed GNE seeking

Initialization: choose $\alpha > 0$ as in Theorem 5.2; for all $i \in \mathcal{I}$, set $x_i^0 \in \mathbb{R}^{n_i}$, $\hat{\sigma}_i^0 = \phi_i(x_i^0)$, $z_i^0 \in \mathbb{R}^m$, $\hat{\lambda}_i^0 = z_i^0$, $\hat{r}_i^0 = A_i x_i^0 - b_i$.
Iterate to convergence: for all $k \in \mathbb{N}$, for all $i \in \mathcal{I}$,

- Local variables update:

$$x_i^{k+1} = x_i^k - \alpha \left(\hat{F}_i(x_i, \hat{\sigma}_i^k) + A_i^\top \hat{\lambda}_i^k \right) \tag{5.11a}$$
$$z_i^{k+1} = z_i^k + \alpha N \hat{r}_i^k \tag{5.11b}$$

- Tracking: Agent i exchanges the variables $(\hat{\sigma}_i, \hat{\lambda}_i, \hat{r}_i)$ with its neighbors, and does

$$\hat{\sigma}_i^{k+1} = \textstyle\sum_{j \in \mathcal{N}_i} w_{i,j} \hat{\sigma}_j^k + \phi_i(x_i^{k+1}) - \phi_i(x_i^k) \tag{5.12a}$$
$$\hat{r}_i^{k+1} = \textstyle\sum_{j \in \mathcal{N}_i} w_{i,j} \hat{r}_j^k + A_i x_i^{k+1} - A_i x_i^k \tag{5.12b}$$
$$\hat{\lambda}_i^{k+1} = \textstyle\sum_{j \in \mathcal{N}_i} w_{i,j} \hat{\lambda}_j^k + z_i^{k+1} - z_i^k \tag{5.12c}$$

5

Assumption 5.5. *For each $i \in \mathcal{I}$, the function ϕ_i in* (5.4) *is continuously differentiable and ℓ_σ-Lipschitz continuous; furthermore, the mapping*

$$F_i(x_i, \hat{\sigma}_i) := \nabla_{x_i} J_i(x_i, \hat{\sigma}_i) + \frac{1}{N} [D\phi_i(x_i)]^\top \nabla_{\hat{\sigma}_i} J_i(x_i, \hat{\sigma}_i) \tag{5.10}$$

is $\widetilde{\ell}_\sigma$-Lipschitz continuous with respect to $\hat{\sigma}_i$. □

Our proposed dynamics are illustrated in Algorithm 7.

Let us define $\hat{\boldsymbol{\sigma}} = \operatorname{col}(\hat{\sigma}_i)_{i \in \mathcal{I}}$, $\hat{\boldsymbol{r}} = \operatorname{col}(\hat{r}_i)_{i \in \mathcal{I}}$, $\hat{\boldsymbol{\lambda}} = \operatorname{col}(\hat{\lambda}_i)_{i \in \mathcal{I}}$ and the extended game mapping

$$\boldsymbol{F}(\boldsymbol{x}, \hat{\boldsymbol{\sigma}}) := \operatorname{col}\left(F_i(x_i, \hat{\sigma}_i)\right)_{i \in \mathcal{I}}. \tag{5.13}$$

Note that

$$\boldsymbol{F}(\boldsymbol{x}, \mathbf{1} \otimes \sigma(x)) = F(\boldsymbol{x}). \tag{5.14}$$

The following lemma shows an invariance property for the tracking dynamics in Algorithm 7.

Lemma 5.2. *For all $k \in \mathbb{N}$, it holds that*

$$\begin{aligned} \overline{\lambda}^k &:= \operatorname{avg}(\hat{\lambda}_i^k)_{i \in \mathcal{I}} = \operatorname{avg}(z_i^k)_{i \in \mathcal{I}} \\ \overline{r}^k &:= \operatorname{avg}(\hat{r}_i^k)_{i \in \mathcal{I}} = \operatorname{avg}(A_i x_i^k - b_i)_{i \in \mathcal{I}} \\ \overline{\sigma}^k &:= \operatorname{avg}(\sigma_i^k)_{i \in \mathcal{I}} = \sigma(\boldsymbol{x}^k). \end{aligned}$$

Proof. Via induction, by the initialization and double stochasticity of W. ■

To study the convergence of Algorithm 7, we first need the following crucial reformulation.

Lemma 5.3. *The iteration in Algorithm 7 is equivalent to*

$$
\begin{aligned}
\begin{bmatrix} \boldsymbol{x}^{k+1} \\ \overline{\lambda}^{k+1} \end{bmatrix} &= \underbrace{\boldsymbol{\omega}^k - \alpha \mathcal{A}(\boldsymbol{\omega}^k)}_{:=\mathcal{B}_1(\boldsymbol{\omega}^k)} + \underbrace{\begin{bmatrix} \alpha\Big(F(\boldsymbol{x}^k) - \boldsymbol{F}(\boldsymbol{x}^k, \widetilde{\boldsymbol{\sigma}}^k + \mathbf{1}\otimes\sigma(\boldsymbol{x}^k)) - \boldsymbol{A}^\top \widetilde{\boldsymbol{\lambda}}^k\Big) \\ \mathbf{0} \end{bmatrix}}_{:=\mathcal{B}_2(\boldsymbol{\nu}^k)} \\
\begin{bmatrix} \widetilde{\boldsymbol{\sigma}}^{k+1} \\ \widetilde{\boldsymbol{r}}^{k+1} \\ \widetilde{\boldsymbol{\lambda}}^{k+1} \end{bmatrix} &= \underbrace{\begin{bmatrix} \boldsymbol{W}\widetilde{\boldsymbol{\sigma}}^k \\ \boldsymbol{W}\widetilde{\boldsymbol{r}}^k \\ \boldsymbol{W}\widetilde{\boldsymbol{\lambda}}^k \end{bmatrix}}_{:=\mathcal{B}_3(\boldsymbol{\chi}^k)} + \underbrace{\begin{bmatrix} \widetilde{\Pi}\operatorname{col}\left(\phi_i(x_i^{k+1}) - \phi_i(x_i^k)\right)_{i\in\mathcal{I}} \\ \widetilde{\Pi}\operatorname{col}\left(A_i(x_i^{k+1} - x_i^k)\right)_{i\in\mathcal{I}} \\ \alpha N \widetilde{\boldsymbol{r}}^k \end{bmatrix}}_{:=\mathcal{B}_4(\boldsymbol{\nu}^k)}
\end{aligned}
\tag{5.15}
$$

where, for all $k\in\mathbb{N}$, $\boldsymbol{\nu}^k = (\boldsymbol{\omega}^k, \boldsymbol{\chi}^k)$, $\boldsymbol{\omega}^k = (\boldsymbol{x}^k, \overline{\lambda}^k)$, $\boldsymbol{\chi}^k = (\widetilde{\boldsymbol{\sigma}}^k, \widetilde{\boldsymbol{r}}^k, \widetilde{\boldsymbol{\lambda}}^k)$, *and* $\widetilde{\Pi} := I - (\frac{1}{N}\mathbf{1}\mathbf{1}^\top)\otimes I_N$, $\boldsymbol{W} = W\otimes I$, $\boldsymbol{A} = \operatorname{diag}(A_i)_{i\in\mathcal{I}}$, *in the sense that the sequence*

$$(\boldsymbol{x}^k, \hat{\boldsymbol{\sigma}}^k, \hat{\boldsymbol{r}}^k, \hat{\boldsymbol{\lambda}}^k)_{k\in\mathbb{N}}$$

generated by Algorithm 7 and the sequence

$$(\boldsymbol{x}^k, \widetilde{\boldsymbol{\sigma}}^k + \mathbf{1}\otimes\sigma(\boldsymbol{x}^k), \widetilde{\boldsymbol{r}}^k + \mathbf{1}\otimes(A\boldsymbol{x}^k - b), \widetilde{\boldsymbol{\lambda}}^k + \mathbf{1}\otimes\overline{\lambda}^k)_{k\in\mathbb{N}}$$

generated by (5.15) *from the initial state*

$$
\begin{bmatrix} \overline{\lambda}^0 \\ \widetilde{\boldsymbol{\sigma}}^0 \\ \widetilde{\boldsymbol{r}}^0 \\ \widetilde{\boldsymbol{\lambda}}^0 \end{bmatrix} = \begin{bmatrix} \operatorname{avg}(\hat{\lambda}_i^0)_{i\in\mathcal{I}} \\ \hat{\boldsymbol{\sigma}}^0 - \mathbf{1}\otimes\sigma(\boldsymbol{x}^0) \\ \hat{\boldsymbol{r}}^0 - \mathbf{1}\otimes(A\boldsymbol{x}^0 - b) \\ \hat{\boldsymbol{\lambda}}^0 - \mathbf{1}\otimes\operatorname{avg}(\hat{\lambda}_i^0)_{i\in\mathcal{I}} \end{bmatrix}
\tag{5.16}
$$

coincide. □

Proof. We proceed by induction, noting that the initial step of the induction holds by the choice of initial state in (5.16). We first prove that $\widetilde{\boldsymbol{\sigma}}^{k+1} = \hat{\boldsymbol{\sigma}}^{k+1} - \mathbf{1}\otimes\sigma(\boldsymbol{x}^k + 1)$ holds, assuming that the equality holds for $k\in\mathbb{N}$:

$$
\begin{aligned}
\hat{\boldsymbol{\sigma}}^{k+1} - \mathbf{1}\otimes\sigma(\boldsymbol{x}^{k+1}) &\overset{(5.12a)}{=} \boldsymbol{W}\hat{\boldsymbol{\sigma}}^k + \operatorname{col}\left(\phi_i(x_i^{k+1}) - \phi_i(x_i^k)\right)_{i\in\mathcal{I}} - \mathbf{1}\otimes\sigma(\boldsymbol{x}^{k+1}) \\
&= \boldsymbol{W}\widetilde{\boldsymbol{\sigma}}^k + \boldsymbol{W}\big(\mathbf{1}\otimes\sigma(\boldsymbol{x}^k)\big) \\
&\quad + \operatorname{col}\left(\phi_i(x_i^{k+1}) - \phi_i(x_i^k)\right)_{i\in\mathcal{I}} - \mathbf{1}\otimes\sigma(\boldsymbol{x}^{k+1}) \\
&= \boldsymbol{W}\widetilde{\boldsymbol{\sigma}}^k + \operatorname{col}\left(\phi_i(x_i^{k+1}) - \phi_i(x_i^k)\right)_{i\in\mathcal{I}} \\
&\quad - \mathbf{1}\otimes\left(\sigma(\boldsymbol{x}^{k+1}) - \sigma(\boldsymbol{x}^k)\right) \\
&\overset{(5.4)}{=} \boldsymbol{W}\widetilde{\boldsymbol{\sigma}}^k + \big((I - \tfrac{\mathbf{1}\mathbf{1}^\top}{N})\otimes I_N\big)\operatorname{col}\left(\phi_i(x_i^{k+1}) - \phi_i(x_i^k)\right)_{i\in\mathcal{I}} \\
&\overset{(5.15)}{=} \widetilde{\boldsymbol{\sigma}}^{k+1}
\end{aligned}
$$

where we noted that $\boldsymbol{W}(\mathbf{1}\otimes y)=\mathbf{1}\otimes y$ for any y, following the row-stochastic property of W. With the same steps, one can show for the $\widetilde{\boldsymbol{r}}^k$ and $\widetilde{\boldsymbol{\lambda}}^k$ updates:

$$\begin{aligned}
\hat{\boldsymbol{r}}^{k+1}-\mathbf{1}\otimes(A\boldsymbol{x}^{k+1}-b)&=\widetilde{\boldsymbol{r}}^{k+1}\\
\hat{\boldsymbol{\lambda}}^{k+1}-\mathbf{1}\otimes\overline{\lambda}^{k+1}&=\boldsymbol{W}\widetilde{\boldsymbol{\lambda}}^k+\operatorname{col}(z_i^{k+1}-z_i^k)_{i\in\mathcal{I}}-\mathbf{1}\otimes(\overline{\lambda}^{k+1}-\overline{\lambda}^k)\\
&\overset{\text{Lem. 5.2}}{=}\boldsymbol{W}\widetilde{\boldsymbol{\lambda}}^k+\operatorname{col}(z_i^{k+1}-z_i^k)_{i\in\mathcal{I}}-\mathbf{1}\otimes\operatorname{avg}(z_i^{k+1}-z_i^k)_{i\in\mathcal{I}}\\
&\overset{(5.11b)}{=}\boldsymbol{W}\widetilde{\boldsymbol{\lambda}}^k+\alpha N\big(\hat{\boldsymbol{r}}^k-\mathbf{1}\otimes\operatorname{avg}(\hat{r}_i^k)_{i\in\mathcal{I}}\big)\\
&\overset{\text{Lem. 5.2}}{=}\boldsymbol{W}\widetilde{\boldsymbol{\lambda}}^k+\alpha N\big(\hat{\boldsymbol{r}}^k-\mathbf{1}\otimes(A\boldsymbol{x}^k-b)\big)\\
&\overset{(5.15)}{=}\widetilde{\boldsymbol{\lambda}}^{k+1}.
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
\overline{\lambda}^{k+1}&\overset{\text{Lem. 5.2}}{=}\operatorname{avg}(z_i^{k+1})_{i\in\mathcal{I}}\\
&\overset{(5.11b)}{=}\operatorname{avg}(z_i^k)_{i\in\mathcal{I}}+\alpha N\overline{r}^k\\
&\overset{\text{Lem. 5.2}}{=}\overline{\lambda}^k+\alpha\sum_{i\in\mathcal{I}}\big(A_ix_i^k-b_i\big),
\end{aligned}$$

which is exactly the update rule for $\overline{\lambda}^k$ in (5.15), and

$$\begin{aligned}
\boldsymbol{x}^{k+1}&\overset{(5.11a)}{=}\boldsymbol{x}^k-\alpha(\boldsymbol{F}(\boldsymbol{x}^k,\hat{\boldsymbol{\sigma}}^k)+\boldsymbol{A}^\top\hat{\boldsymbol{\lambda}})\\
&=\boldsymbol{x}^k-\alpha(F(\boldsymbol{x}^k)+A^\top\overline{\lambda}^k)+\alpha(F(\boldsymbol{x}^k)-\boldsymbol{F}(\boldsymbol{x}^k,\hat{\boldsymbol{\sigma}}^k)+A^\top\overline{\lambda}^k-\boldsymbol{A}^\top\hat{\boldsymbol{\lambda}})\\
&=\boldsymbol{x}^k-\alpha(F(\boldsymbol{x}^k)+A^\top\overline{\lambda}^k)+\alpha\big(F(\boldsymbol{x}^k)-\boldsymbol{F}(\boldsymbol{x}^k,\hat{\boldsymbol{\sigma}}^k)-\boldsymbol{A}^\top(\hat{\boldsymbol{\lambda}}^k-\mathbf{1}\otimes\overline{\lambda}^k)\big),
\end{aligned}\tag{5.17}$$

where the latter follows from $A^\top\overline{\lambda}^k=\boldsymbol{A}^\top(\mathbf{1}\otimes\overline{\lambda}^k)$. Note that $\hat{\boldsymbol{\lambda}}^k=\mathbf{1}\otimes\overline{\lambda}^k+\widetilde{\boldsymbol{\lambda}}^k$ from the induction assumption, thus (5.17) reads as the update rule for $\boldsymbol{x}^k$ in (5.15). ■

Theorem 5.2 (GNE seeking in partial-decision information)**.** *Let Assumptions 5.1-5.4 hold. Then, there is $\alpha_{\max}>0$ such that, for all $0<\alpha<\alpha_{\max}$, the iteration in* (5.15) *converges linearly to $\boldsymbol{\nu}^\star=(\boldsymbol{\omega}^\star,\mathbf{0})$, with $\boldsymbol{\omega}^\star=(\boldsymbol{x}^\star,\lambda^\star)$: for all $k\in\mathbb{N}$*

$$V(\boldsymbol{\nu}^{k+1})\le\eta V(\boldsymbol{\nu}^k),$$

where $V(\boldsymbol{\nu}):=\|\boldsymbol{\omega}-\boldsymbol{\omega}^\star\|_P^2+\|\boldsymbol{\chi}\|^2$, for some $\eta<1$. □

Proof. Following the initialization of Algorithm 7, (5.16) and the column-stochasticity of $\boldsymbol{W}$ in Assumption 5.4, it can be shown that $\operatorname{avg}(\chi_i^k)_{i\in\mathcal{I}}=\mathbf{0}$ for all $k\in\mathbb{N}$. The operator $\mathcal{B}_3$ in (5.15) is thus a contraction with factor θ by applying [88, Lemma 10.3]. Instead, $\mathcal{B}_1$ is a contraction for α small enough with factor $(1-2\alpha\mu_{\mathcal{A}}+\alpha^2\ell_{\mathcal{A}}^2)$, as in Theorem 5.1. The mappings $\mathcal{B}_2$ and $\mathcal{B}_4$ are Lipschitz continuous with constant proportional to α, by Assumption 5.5: this is clear for $\mathcal{B}_2$, while it is concluded for $\mathcal{B}_4$ by noting from the update of $\boldsymbol{x}^k$ in (5.15) that $\boldsymbol{x}^{k+1} - \boldsymbol{x}^k$ is given by a Lipschitz continuous mapping multiplied by α, which can then be substituted in the definition of $\mathcal{B}_4$. Furthermore, note that

$$\mathcal{B}_2(\boldsymbol{\omega}^*) = \mathbf{0} \tag{5.18}$$

and that

$$\boldsymbol{\omega}^\star = \mathcal{B}_1(\boldsymbol{\omega}^\star) \tag{5.19}$$

as it follows from the KKT conditions in (5.3). It is then immediate that

$$\mathcal{B}_4(\boldsymbol{\omega}^*) = \mathbf{0}. \tag{5.20}$$

Therefore, the proof can be carried out via standard small-gain arguments, and is hence only sketched here. By the Cauchy–Schwarz inequality, we can bound

$$\begin{aligned}\|\mathcal{B}_1(\boldsymbol{\omega}^k) + \mathcal{B}_2(\boldsymbol{\nu}^k) - \boldsymbol{\omega}^\star\|_P^2 \leq & \|\mathcal{B}_1(\boldsymbol{\omega}^k) - \boldsymbol{\omega}^\star\|_P^2 \\ & + 2\|\mathcal{B}_1(\boldsymbol{\omega}^k) - \boldsymbol{\omega}^\star\|_P \|\mathcal{B}_2(\boldsymbol{\nu}^k)\|_P + \|\mathcal{B}_2(\boldsymbol{\nu}^k)\|_P^2 \\ \leq & (1 - 2\alpha\mu_{\mathcal{A}} + \alpha^2 \ell_{\mathcal{A}}^2)\|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P^2 \\ & + \alpha \ell_1 \|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P \|\boldsymbol{\chi}^k\| + \alpha^2 \ell_2 \|\boldsymbol{\chi}^k\|^2\end{aligned}$$

where the latter follows from the aforementioned Lipschitz and contraction properties. Note that $\|\mathcal{B}_2(\boldsymbol{\nu}^k)\|$ is bounded independently of $\boldsymbol{x}^k$ (hence, of $\boldsymbol{\omega}^k$) following the definition of $\mathcal{B}_2$. We further bound:

$$\begin{aligned}\|\mathcal{B}_3(\boldsymbol{\chi}^k) + \mathcal{B}_4(\boldsymbol{\nu}^k)\|^2 \leq & \|\mathcal{B}_3(\boldsymbol{\chi}^k)\|^2 + 2\|\mathcal{B}_3(\boldsymbol{\chi}^k)\|\|\mathcal{B}_4(\boldsymbol{\omega}^k)\| + \|\mathcal{B}_4(\boldsymbol{\omega}^k)\|^2 \\ \leq & \|\mathcal{B}_3(\boldsymbol{\chi}^k)\|^2 \\ & + 2\|\mathcal{B}_3(\boldsymbol{\chi}^k)\|\|\mathcal{B}_4(\boldsymbol{\omega}^k) - \mathcal{B}_4(\boldsymbol{\omega}^\star)\| + \|\mathcal{B}_4(\boldsymbol{\omega}^k) - \mathcal{B}_4(\boldsymbol{\omega}^\star)\|^2 \\ \leq & \theta \|\boldsymbol{\chi}^k\|^2 + \alpha \ell_3 \|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P \|\boldsymbol{\chi}^k\| + \alpha \ell_4 \|\boldsymbol{\chi}^k\|^2 \\ & + \alpha^2 \ell_5 \|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P^2 + \alpha^2 \ell_6 \|\boldsymbol{\chi}^k\|^2,\end{aligned}$$

where we noted from the equivalence of norms that

$$\|\boldsymbol{\nu}^k - \boldsymbol{\nu}^\star\| = \leq \gamma_P \|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P + \|\boldsymbol{\chi}^k\|$$

for some $\gamma_P > 0$, and $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 > 0$ are parameters independent of α. Following the definition of V and the aforementioned bounds, we derive

$$\begin{aligned}V(\boldsymbol{\nu}^{k+1}) \leq & \begin{bmatrix}\|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P \\ \|\boldsymbol{\chi}^k\|\end{bmatrix}^\top M \begin{bmatrix}\|\boldsymbol{\omega}^k - \boldsymbol{\omega}^\star\|_P \\ \|\boldsymbol{\chi}^k\|\end{bmatrix} \\ \leq & \lambda_{\max}(M) V(\boldsymbol{\nu}^k),\end{aligned}$$

where

$$M := \begin{bmatrix} 1 - 2\alpha\mu_{\mathcal{A}} + \alpha^2(\ell_{\mathcal{A}} + \ell_5) & \alpha\frac{1}{2}(\ell_1 + \ell_3) \\ \alpha\frac{1}{2}(\ell_1 + \ell_3) & \theta + \alpha\ell_4 + \alpha^2(\ell_2 + \ell_6) \end{bmatrix}$$

and $\lambda_{\max}(M) = \|M\| < 1$ for small enough $\alpha > 0$. ■

Note that, for the special case $A = \mathbf{1}^\top$, a linearly convergent continuous-time method for GNE seeking in partial-decision information was studied in [80]. Yet, to our knowledge, Theorem 5.2 is the first result to ensure linear convergence in the case of more general full-rank constraints.

Remark 5.1. *The proof of Theorem 5.1 directly applies to the case of a time-varying graph with weight matrix W^k, provided that Assumption 5.4 holds for each $k \in \mathbb{N}$. With some modification, the argument can be extended also to the case of doubly stochastic graphs that are not strongly connected at each step, but such that $\|W^{kQ}W^{kQ+1}\dots W^{(k+1)Q} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\| \leq \theta < 1$, for a $Q > 0$ and all $k \in \mathbb{N}$.*

5.5 Equilibrium tracking in time-varying games

We now consider the case where the game in (5.1a) varies over time at a rate such that we can not assume a timescale separation between the game evolution and the GNE seeking iterations. For each time index $t \in \mathbb{N}$, the agents acquire a new instance of the game:

$$(\forall i \in \mathcal{I}) \;\; \min_{y_i \in \mathbb{R}^{n_i}} \;\; J_{i,t}(y_i, \boldsymbol{x}_{-i}) \quad \text{s.t. } (y_i, \boldsymbol{x}_{-i}) \in \mathcal{X}_t. \tag{5.21}$$

5

We consider the case when the constraints of the game vary only in their affine part, that is, $\mathcal{X}_t := \{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} = b^t\}$, and Assumptions 5.1–5.3 hold for each t. The games in (5.21) define a primal-dual GNE pair *sequence* $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$, corresponding to the zero set of the KKT operators $(\mathcal{A}_t)_{t\in\mathbb{N}}$ defined for all t as in (5.3), with F, b replaced respectively by $F^t := \mathrm{col}(\nabla_{x_i} J_i^t)_{i\in\mathcal{I}}$ and b^t. The GNE sequence is unique for each t following the strong monotonicity of $\mathcal{A}_t$ and [51, Ex. 22.12]. As the rate at which the problem varies is comparable to the agents' computation time, the agents can only compute an approximation of the GNE at time t before they are presented with a new instance of the problem. The goal of the agents is then to find a sequence $(\boldsymbol{\omega}^t)_{t\in\mathbb{N}}$ which asymptotically tracks relatively well the GNE sequence. We formulate the following assumption, which is standard in the literature of online optimization [48, Assm. 1], [82, Eq. 9] and is verified, for example, for games affected by a bounded process noise in the linear constraints [84, Lemma 5].

Assumption 5.6. *For some $\delta \geq 0$, it holds that the solution $\boldsymbol{\omega}_t^\star$ of the game in* (5.21) *satisfies*

$$\sup_{t\in\mathbb{N}} \|\boldsymbol{\omega}_{t+1}^\star - \boldsymbol{\omega}_t^\star\| \leq \delta.$$

Assumption 5.6 implies that the solution at time t is an approximate solution for the problem at time $t+1$. Given an estimate $\boldsymbol{\omega}_{t-1}$ of the solution at time for some time step t, we then propose to compute $\boldsymbol{\omega}_t$ by performing K iterations of the iteration in (5.8), warm-started at $\boldsymbol{\omega}_{t-1}$, that is:

$$\boldsymbol{y}_{t,0} = \boldsymbol{\omega}_{t-1} \tag{5.22a}$$

$$\boldsymbol{y}_{t,k+1} = \boldsymbol{y}_{t,k} - \alpha\mathcal{A}_t(\boldsymbol{y}_{t,k}) \qquad \text{for } k = 0, \dots, K-1 \tag{5.22b}$$

$$\boldsymbol{\omega}_t = \boldsymbol{y}_K, \tag{5.22c}$$

where $(\boldsymbol{y}_{t,k})_{k\in\{1,\dots,K\}}$ are auxiliary variables. The following lemma shows that, for an appropriately chosen step size, the proposed algorithm tracks the GNE trajectory up to an asymptotic error which depends on δ and K.

Theorem 5.3. *For any $0<\alpha<\frac{2\mu_{\mathcal{A}}}{\ell_{\mathcal{A}}^2}$, $\boldsymbol{\omega}_0$, $K\in\mathbb{N}_{>0}$, the sequence $(\boldsymbol{\omega}_t)_{t\in\mathbb{N}}$ generated by the iteration in* (5.22) *satisfies*

$$\limsup_{t\to\infty}\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_t^\star\|_P\le\frac{\rho^{K/2}}{1-\rho^{K/2}}\delta\sqrt{\lambda_{max}(P)},\tag{5.23}$$

where ρ is as in Theorem 5.1. □

Proof. Following Theorem 5.1, for $\alpha\le2\frac{\mu_{\mathcal{A}}}{\ell_{\mathcal{A}}^2}$,

$$\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_t^\star\|_P\le\rho^{K/2}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_t^\star\|_P.$$

From the latter, the triangle inequality and the fact $\|z\|_P^2\le\lambda_{\max}(P)\|z\|^2$ for all z:

$$\begin{aligned}\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_t^\star\|_P&\le\rho^{K/2}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_{t-1}^\star\|_P+\rho^{K/2}\|\boldsymbol{\omega}_t^\star-\boldsymbol{\omega}_{t-1}^\star\|_P\\&\le\rho^{K/2}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_{t-1}^\star\|_P+\rho^{K/2}\delta\sqrt{\lambda_{\max}(P)}.\end{aligned}$$

Iterating the latter t times, we obtain

$$\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_t^\star\|_P\le\rho^{Kt/2}\|\boldsymbol{\omega}_0-\boldsymbol{\omega}_0^\star\|_P+\sum_{\tau=1}^{t}\rho^{K\tau/2}\delta\sqrt{\lambda_{\max}(P)}.$$

Since $\rho^{K/2}<1$, the claim follows by the convergence of the geometric sequence. ■

Let us now turn our attention to the time-varying counterpart of the partial-decision information setup described in Section 5.4. Again, we consider aggregative games in the form

$$J_{i,t}(x_i,\boldsymbol{x}_{-i})=J_{i,t}\big(x_i,\sigma_t(\boldsymbol{x})\big)$$

where $\sigma_t(x):=\frac{1}{N}\sum_{i\in\mathcal{I}}\phi_{i,t}(x_i)$, and we postulate that $\phi_{i,t}$ satisfies Assumption 5.5 for all $I\in\mathcal{I},t\in N$. As in Section 5.4, we augment the state of each agent with an estimate of the aggregative variable $\sigma_t(\boldsymbol{x})$, of the dual variable and of the residual $A\boldsymbol{x}-b_t$. For every t, denote $\boldsymbol{\nu}_t^\star=(\boldsymbol{\omega}_t^\star,\mathbf{0})$, where $\boldsymbol{\omega}_t^\star=(\boldsymbol{x}_t^\star,\lambda_t^\star)$ is a primal-dual solution of the game at time t and the vector of zeros represents the target estimation error. We then define the reference trajectory as $(\boldsymbol{\omega}_t^\star)_{t\in\mathbb{N}}$. At each time-step, we propose to appropriately re-initialize the dynamic tracking of the estimated variables and, in the spirit of the iteration in (5.22), to apply a finite number of iterations of Algorithm 7. The resulting method is illustrated in Algorithm 8. We obtain the following counterpart of Lemma 5.2 for the re-initialized dynamic tracking.

Lemma 5.4. *For the sequence $(\boldsymbol{x}_t,\boldsymbol{z}_t,\hat{\boldsymbol{\sigma}}_t,\hat{\boldsymbol{r}}_t,\hat{\boldsymbol{\lambda}}_t)$ generated by Algorithm 8, for all $t\in\mathbb{N}$, it holds that*

$$\begin{aligned}\operatorname{avg}(\hat{\lambda}_{i,t})_{\in\mathcal{I}}&=\operatorname{avg}(z_{i,t})_{i\in\mathcal{I}},\\\operatorname{avg}(\hat{r}_{i,t})_{i\in\mathcal{I}}&=\operatorname{avg}(A_ix_{i,t}-b_{i,t})_{i\in\mathcal{I}},\\\operatorname{avg}(\hat{\sigma}_{i,t})_{i\in\mathcal{I}}&=\sigma_t(\boldsymbol{x}_t).\end{aligned}$$

Proof. Let us assume for some t:

$$\operatorname{avg}(\hat{r}_{i,t})_{i\in\mathcal{I}} = \operatorname{avg}(A_i x_{i,t} - b_{i,t})_{i\in\mathcal{I}}. \tag{5.25}$$

From the update step 4) and the re-initialization step 2), we obtain $x_{i,t} = \mathring{x}_{i,t}^{K} = \mathring{x}_{i,t+1}^{0}$ and

$$\begin{aligned}\operatorname{avg}(\mathring{r}_{i,t+1}^{0})_{i\in\mathcal{I}} &= \operatorname{avg}(\hat{r}_{i,t})_{i\in\mathcal{I}} - \operatorname{avg}(b_{i,t+1} - b_{i,t})_{i\in\mathcal{I}}\\ &\overset{(5.25)}{=} \operatorname{avg}(A_i \mathring{x}_{i,t+1}^{0} - b_{i,t+1})_{i\in\mathcal{I}}.\end{aligned}$$

Then, one can show with the same reasoning as in Lemma 5.2, that $\operatorname{avg}(\mathring{r}_{i,t+1}^{K})_{i\in\mathcal{I}} = \operatorname{avg}(A_i \mathring{x}_{i,t+1}^{K} - b_{i,t+1})_{i\in\mathcal{I}}$. From the latter and the update step 4),

$$\operatorname{avg}(\hat{r}_{i,t+1})_{i\in\mathcal{I}} = \operatorname{avg}(\mathring{r}_{i,t+1}^{K})_{i\in\mathcal{I}} = \operatorname{avg}(A_i \mathring{x}_{i,t+1}^{K} - b_{i,t+1})_{i\in\mathcal{I}}.$$

The result follows by induction and similarly for $\hat{\boldsymbol{\sigma}}_t, \hat{\boldsymbol{\lambda}}_t$. ∎

The re-inizialization of the dynamic tracking introduces an additional error term, which requires the following technical assumption on the time variation of the functions $(\phi_{i,t})_{t\in\mathbb{N}}$:

Assumption 5.7. *For some $\delta_\phi > 0$, it holds that*

$$\sup_{x,t} \|\operatorname{col}(\phi_{i,t}(x_i) - \phi_{i,t+1}(x_i))_{i\in\mathcal{I}}\| \leq \delta_\phi.$$

Consistently with the notation used in Lemma 5.3, we denote $\overline{\lambda}^t = \operatorname{avg}(\lambda_{i,t})_{i\in\mathcal{I}}$, $\widetilde{\boldsymbol{\sigma}}_t = \hat{\boldsymbol{\sigma}}_t - \mathbf{1}\otimes\sigma(\boldsymbol{x}_t)$, $\widetilde{\boldsymbol{r}}_t = \hat{\boldsymbol{r}}_t - \mathbf{1}\otimes(A\boldsymbol{x}_t - b_t)$, $\widetilde{\boldsymbol{\lambda}}_t = \hat{\boldsymbol{\lambda}}_t - \mathbf{1}\otimes\overline{\lambda}_t$

Theorem 5.4. *Let $\boldsymbol{\omega}^t = (\boldsymbol{x}_t, \overline{\lambda}_t, \widetilde{\boldsymbol{\sigma}}_t, \widetilde{\boldsymbol{r}}_t, \widetilde{\boldsymbol{\lambda}}_t)$. There exists α_{max} such that, for every $0 < \alpha < \alpha_{max}$, the sequence $(\boldsymbol{\omega}_t)_{t\in\mathbb{N}}$ generated by Algorithm 8 satisfies*

$$\limsup_{t\to\infty} \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|_Q \leq \tfrac{\eta^{K/2}}{1-\eta^{K/2}} \sqrt{\lambda_{max}(Q)}\big((1+N\ell_A)\delta + \delta_\phi\big)$$

where $Q = \operatorname{diag}(P/2, I)$, for some $\eta \in (0,1)$. □

Proof. Following the same steps as in Lemma 5.3, the inner iteration (Step 3) of Algorithm 8 can be shown to be equivalent to the iteration in (5.15), where $\boldsymbol{F}$ and ϕ_i are substituted by their time-varying counterpart. Now denote

$$\psi_t = \operatorname{col}\big(\mathbf{0}, \mathbf{0}, \operatorname{col}\big(\phi_{i,t}(x_{i,t}) - \phi_{i,t-1}(x_{i,t})\big)_{i\in\mathcal{I}}, \operatorname{col}(b_{i,t} - b_{i,t-1})_{i\in\mathcal{I}}, \mathbf{0}\big).$$

From Assumptions 5.6, 5.7 and from $Ax_t^\star = b_t$,

$$\begin{aligned}\|\psi_t\| &\leq \delta_\phi + \sum_{i\in\mathcal{I}} \|b_{i,t} - b_{i,t-1}\|\\ &\leq \delta_\phi + N\|b_t - b_{t-1}\| \leq \delta_\phi + N\ell_A\delta.\end{aligned} \tag{5.26}$$

From Theorem 5.2 and accounting for the re-initialization step, we find for every t:

$$\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|_Q \leq \eta^{K/2} \|\boldsymbol{\omega}_{t-1} + \psi_t - \boldsymbol{\omega}_t^\star\|_Q \tag{5.27}$$

for some $\eta \in (0,1)$. By the triangle inequality, Assumption 5.6 and (5.26), and from the fact $\|z\|_Q^2 \leq \lambda_{\max}(Q)\|z\|^2$, we have

$$\begin{aligned}
\|\boldsymbol{\omega}_{t-1} + \psi_t - \boldsymbol{\omega}_t^\star\|_Q &\leq \|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t^\star\|_Q + \|\psi_t\|_Q \\
&\leq \|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_{t-1}^\star\|_Q + \|\boldsymbol{\omega}_t^\star - \boldsymbol{\omega}_{t-1}^\star\|_Q + \|\psi_t\|_Q \\
&\leq \|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_{t-1}^\star\|_Q + \sqrt{\lambda_{\max}(Q)}((1+N\ell_A)\delta + \delta_\phi).
\end{aligned}$$

By substituting the latter in (5.27) and by iterating the resulting inequality, we obtain

$$\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_t^\star\|_Q \leq \eta^{Kt/2} \|\boldsymbol{\omega}_0 - \boldsymbol{\omega}_0^\star\|_Q + \textstyle\sum_{\tau=1}^{t} \eta^{Kt/2} \sqrt{\lambda_{\max}(Q)}\big((1+N\ell_A)\delta + \delta_\phi\big).$$

5

As $\eta^{K/2} < 1$, the result follows by convergence of the geometric series. ■

Algorithm 8 Time-varying fully-distributed GNE seeking

Initialization: choose $\alpha > 0$ as in Theorem 5.2; for all $i \in \mathcal{I}$, set $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\hat{\boldsymbol{\sigma}}_0 = \mathbf{0}$, $\boldsymbol{z}_0 \in \mathbb{R}^m$, $\hat{\boldsymbol{\lambda}}_0 = \boldsymbol{z}_0$, $\hat{r}_{i,0} = A_i x_{i,0}$, $b_{i,0} = \mathbf{0}$, $\phi_{i,0}(\cdot) = \mathbf{0}$.

Iteration: at time $t \in \mathbb{N}_{>0}$, for each agent $i \in \mathcal{I}$,

1. Acquire $J_{i,t}$, $\phi_{i,t}$, $b_{i,t}$
2. Initialize the auxiliary variables

$$\mathring{x}^0_{i,t} = x_{i,t-1}, \tag{5.24a}$$

$$\mathring{z}^0_{i,t} = z_{i,t-1}, \tag{5.24b}$$

$$\mathring{\sigma}^0_{i,t} = \hat{\sigma}_{i,t-1} - \phi_{i,t-1}(x_{i,t-1}) + \phi_{i,t}(x_{i,t-1}), \tag{5.24c}$$

$$\mathring{r}^0_{i,t} = \hat{r}_{i,t-1} + b_{i,t} - b_{i,t-1}, \tag{5.24d}$$

$$\mathring{\lambda}^0_{i,t} = \hat{\lambda}_{i,t-1} \tag{5.24e}$$

3. For all $k \in \{0, ..., K-1\}$, for all $i \in \mathcal{I}$:
   - Local variables update:

$$\begin{aligned}
\mathring{x}^{k+1}_{i,t} &= \mathring{x}^k_{i,t} - \alpha\Big(F_{i,t}(\mathring{x}^k_{i,t}, \mathring{\sigma}^k_{i,t}) + A_i^\top \mathring{\lambda}^k_{i,t}\Big)\\
\mathring{z}^{k+1}_{i,t} &= \mathring{z}^k_{i,t} + \alpha N \mathring{r}^k_{i,t}
\end{aligned}$$

   - Estimation update: Agent i exchanges the variables $(\mathring{\sigma}^k_{i,t}, \mathring{\lambda}^k_{i,t}, \mathring{r}^t_{i,t})$ with its neighbors, and updates

$$\begin{aligned}
\mathring{\sigma}^{k+1}_{i,t} &= \sum_{j \in \mathcal{N}_i} w_{i,j} \mathring{\sigma}^k_{j,t} + \phi_{i,t}(\mathring{x}^{k+1}_{i,t}) - \phi_{i,t}(\mathring{x}^k_{i,t})\\
\mathring{r}^{k+1}_{i,t} &= \sum_{j \in \mathcal{N}_i} w_{i,j} \mathring{r}^k_{j,t} + A_i \mathring{x}^{k+1}_{i,t} - A_i \mathring{x}^k_{i,t}\\
\mathring{\lambda}^{k+1}_{i,t} &= \sum_{j \in \mathcal{N}_i} w_{i,j} \mathring{\lambda}^k_{j,t} + \mathring{z}^{k+1}_{i,t} - \mathring{z}^k_{i,t}
\end{aligned}$$

4. Set $x_{i,t} = \mathring{x}^K_{i,t}$, $z_{i,t} = \mathring{z}^K_{i,t}$, $\hat{\sigma}_{i,t} = \mathring{\sigma}^K_{i,t}$, $\hat{r}_{i,t} = \mathring{r}^K_{i,t}$, $\hat{\lambda}_{i,t} = \mathring{\lambda}^K_{i,t}$

5.6 Numerical example

We demonstrate the proposed algorithms on a market clearing problem for peer-to-peer energy market inspired by [89]. We consider $N=6$ prosumers that aim at determining their energy portfolio. At each time-step t, the agents can either purchase power from a main energy operator, produce it from a dispatchable energy source or trade it with their respective neighbors over a randomly generated undirected graph $\mathcal{G}^{\mathrm{E}}$. Furthermore, the agents can exchange information over an undirected connected communication graph $\mathcal{G}$. We denote for each agent i and each time-step t the power purchased from the main operator as $x_{i,t}^{\mathrm{mg}}$, the produced power as $x_{i,t}^{\mathrm{dg}}$ and the power that agent i purchases from agent j as $x_{i,j,t}^{\mathrm{tr}}$, $j \in \mathcal{N}_i^{\mathrm{E}}$, with $\mathcal{N}_i^{\mathrm{E}}$ the set of neighbors of agent i over $\mathcal{G}^{\mathrm{E}}$. The energy price posed by the main operator increases linearly with the aggregate power requested. Thus, by defining the aggregative value $\sigma^{\mathrm{mg}}(x^{\mathrm{mg}}) = \sum_{i\in\mathcal{I}} x_i^{\mathrm{mg}}$, the cost incurred by each agent for purchasing energy from the operator is $J^{\mathrm{mg}}(x_{i,t}^{\mathrm{mg}}, \sigma^{\mathrm{mg}}(x_t^{\mathrm{mg}})) = c^{\mathrm{mg}}\sigma^{\mathrm{mg}}(x_t^{\mathrm{mg}})x_{i,t}^{\mathrm{mg}}$, $c_{\mathrm{mg}} > 0$. We consider a quadratic cost on the power generation incurred by the agents [89, Eq. 2], with the form $J_{i,t}^{\mathrm{dg}}(x_{i,t}^{\mathrm{dg}}) = c^{\mathrm{dg}}(x_{i,t}^{\mathrm{dg}} - x_{i,t}^{\mathrm{dg,\ ref}})^2$ where $x_{i,t}^{\mathrm{dg,\ ref}}$ is the *time-varying* scheduled set point of the dispatchable generators and $c^{\mathrm{dg}} > 0$. The price (or revenue) of trading energy between peers is linear [89, Eq. 10], and the agents incur a quadratic cost on the transactions for utilizing the market; thus the total objective function related to the peer-to-peer trading is given by $J^{\mathrm{tr}}((x_{i,j,t}^{\mathrm{tr}})_{j\in\mathcal{N}_i^{\mathrm{E}}}) = \sum_{j\in\mathcal{N}_i^{\mathrm{E}}}(c^{\mathrm{tr}}x_{i,j,t}^{\mathrm{tr}} + \kappa^{\mathrm{tr}}(x_{i,j,t}^{\mathrm{tr}})^2)$. The agents cannot sell power to the main operator and, due to physical limitations, the power generated by the dispatchable units must be non-negative. As the formulation in (5.1a) does not consider inequality constraints, this is enforced by a Lipschitz continuous approximation of the logarithmic barrier function $J^{\mathrm{bar}}(x_{i,t}^{\mathrm{mg}}, x_{i,t}^{\mathrm{dg}}) = \Gamma_\gamma(x_{i,t}^{\mathrm{mg}}) + \Gamma_\gamma(x_{i,t}^{\mathrm{dg}})$, where $\Gamma_\gamma(y) := \min(-\log(y), -\gamma y + 1 - \log(1/\gamma))$ for $\gamma \gg 0$. The total cost incurred by each agent is thus

$$\begin{aligned} J_i^t(x_{i,t}, x_{-i,t}) = &J^{\mathrm{mg}}\big(x_{i,t}^{\mathrm{mg}}, \sigma^{\mathrm{mg}}(x_t^{\mathrm{mg}})\big) + J_{i,t}^{\mathrm{dg}}\big(x_{i,t}^{\mathrm{dg}}\big) \\ &+ J^{\mathrm{tr}}\big((x_{i,j,t}^{\mathrm{tr}})_{j\in\mathcal{N}_i^{\mathrm{E}}}\big) + J^{\mathrm{bar}}\big(x_{i,t}^{\mathrm{mg}}, x_{i,t}^{\mathrm{dg}}\big). \end{aligned}$$

Moreover, given a power demand $p_{i,t}^{\mathrm{d}}$, the agents need to satisfy the power balance equation [89, Eq. 1]

$$\sum_{j\in\mathcal{N}_i^{\mathrm{E}}} (x_{i,j,t}^{\mathrm{tr}}) + x_{i,t}^{\mathrm{mg}} + x_{i,t}^{\mathrm{dg}} = p_{i,t}^{\mathrm{d}}. \tag{5.28}$$

As the power balance constraints are local, we do not apply the dynamic tracking method to the associated dual variables (i.e., dual variables are managed locally, see [90, Rem. 2]). Instead, coupling constraints between the agents decisions arises via trading reciprocity constraints [89, Eq. 8]:

$$x_{i,j,t}^{\mathrm{tr}} + x_{j,i,t}^{\mathrm{tr}} = 0. \tag{5.29}$$

We first consider a time-invariant scenario and compute the day-ahead market

The code is available at `https://github.com/bemilio/Simple_peer_to_peer` clearing solution over an entire day, with time-steps of 15 minutes: namely, the cost of agent i is given by $\sum_{t=1}^{T} J_i^t$, and the constraints in (5.28)-(5.29) are imposed for all $t = 1, 2, \ldots, T$, with $T = 96$. Figure 5.7.1 shows that, as expected, Algorithm 7 exhibits a linear convergence rate with respect to the Lyapunov function in theorem 5.2.

Then, we consider a real-time scenario. In particular, the agents only have access to a prediction on their load demand and generation set point over the coming quarter of an hour; hence the cost of agent i at each time $t = 1, 2, \ldots, T$ is given by J_i^t. Note that the agents are in fact faced with a time-varying generalized game as discussed in Section 5.5, which we address via Algorithm 8. The results are shown in Figure 5.7.2. Because of the slow convergence (i.e., η is close to 1), $K = 1$ results in a significant tracking error; however, good performance is observed already for $K = 100$. Finally, in Figure 5.7.3, we show the constraint violation obtained by the proposed method over the simulation horizon. As constraints are only satisfied asymptotically, performing only a finite number of iterations per time-step leads to a constraint violation, which as expected decreases with K.

5.7 Conclusion

Strongly monotone GNE problems with full row rank equality coupling constraints can be solved with linear convergence rate, both in semi-decentralized and fully-distributed settings, via primal-dual algorithms. The contractivity properties of the iterates also allow the tracking of the solution sequence in time-varying games; in this online setting, the asymptotic tracking accuracy can be increased by increasing the update frequency.

As our results exploit the strong monotonicity of the KKT operator in a (non-diagonally) weighted space, it is not clear how to embed projections in the proposed methods, which is the main drawback of our approach. Future work should hence focus on linear convergence in generalized games with local constraints (and with inequality coupling constraints).

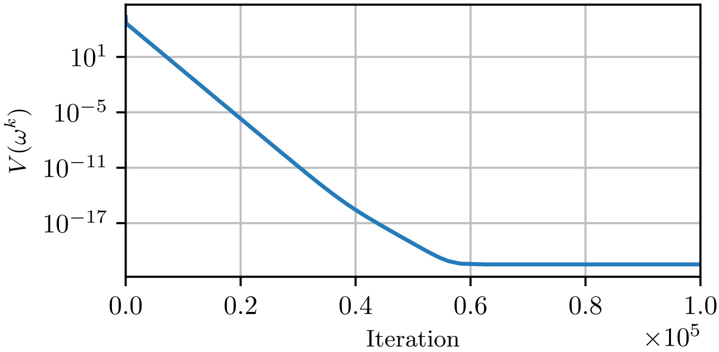


Figure 5.7.1: Convergence of Algorithm 7 for the day-ahead market clearing problem.

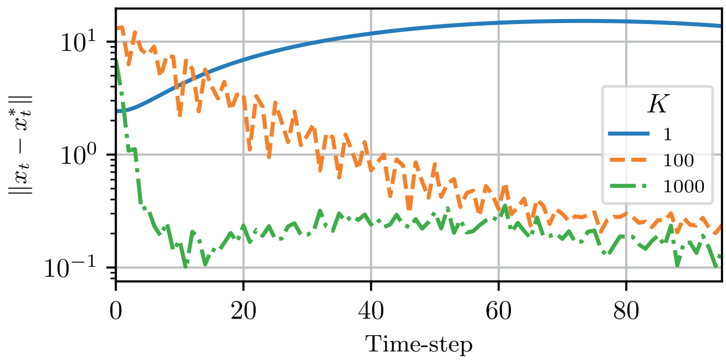


Figure 5.7.2: Tracking error of Algorithm 8 for the real-time market clearing problem with respect to the day-ahead solution computed by 10^5 iterations of Algorithm 7.

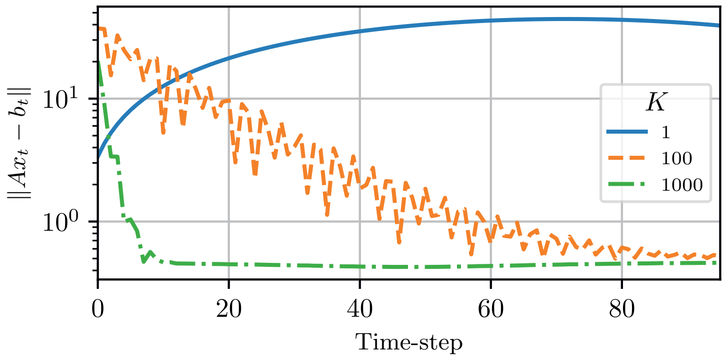


Figure 5.7.3: Constraint violation incurred by Algorithm 8 for real-time market clearing. The inequality constraints enforced via the barrier function J^{bar} are always satisfied.

II

Receding horizon control of dynamic games

6

Dynamic games as receding-horizon variational inequalities

One often gets the impression that [the Algebraic Riccati Equation] constitutes the bottleneck of all of linear system theory

Jan C. Willems, in [91]

Anything that happens, happens.
Anything that, in happening, causes something else to happen, causes something else to happen.
Anything that, in happening, causes itself to happen again, happens again.
It doesn't necessarily do it in chronological order, though.

Douglas Adams, in "Mostly harmless"

We consider dynamic games with linear dynamics and quadratic objective functions. We observe that the unconstrained open-loop Nash equilibrium (ol-NE) coincides with the linear quadratic regulator in an augmented space, thus deriving an explicit expression for the objective value achieved by the ol-NE. With such function as a terminal cost, we show asymptotic stability for the system controlled via a receding-horizon solution of the finite-horizon, constrained game. Furthermore, we show that the problem is equivalent to a non-symmetric variational inequality, which interestingly does not correspond to any Nash equilibrium problem. For unconstrained closed-loop Nash equilibria, we derive a receding-horizon controller that is equivalent to the infinite-horizon one and ensures asymptotic stability.

.

This chapter is partly based on Benenati, E. and Grammatico, S. "Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities". In: *arXiv preprint 2408.15703*, (2024)

6.1 Introduction

We consider a regulation problem for a constrained, discrete-time linear dynamics-quadratic objective (LQ) dynamic game, which emerges when the agents have interest in reaching and maintaining a known attractor while optimizing an individual objective. Dynamic games (and their continuous-time counterpart, that is, *differential* games) model a dynamical system governed by multiple inputs, each controlled by a decision maker (or agent) with a self-interested objective. Applications include robotics [2, 8, 14], robust control [92], logistics planning [93] and energy markets [5], to cite a few. An input strategy which no agent can improve via unilateral changes is called a Nash equilibrium (NE). Interestingly, depending on the information structure of the problem, dynamic games may admit different types of NE. In particular, if the agents only observe the initial state and commit to a sequence of inputs which is an optimal response to the input sequences of the remaining agents, then we have an open-loop Nash equilibrium (ol-NE); instead, if the agents are allowed to continuously observe the state, then a closed-loop Nash equilibrium (cl-NE) is an optimal feedback policy given the current state and the control policies of the other agents. It is well-known that a linear infinite-horizon cl-NE policy is linked to the solution of coupled AREs under a stabilizability and detectability condition, see e.g. [9, Proposition 6.3]. Recently, the authors of [94] develop novel sufficient conditions for a similar ol-NE characterization. In [95], the infinite-horizon continuous-time cl-NE problem was related to the problem of finding invariant linear subspaces: this observation has led to a significant development of the field [10] and solution algorithms based on a geometric approach [96]. Other algorithms, based on the iterative solution of Lyapunov or Riccati equations have been presented in [6, 97, 98]. Remarkably, some preliminary steps have been made to compute *all* linear cl-NE strategies [99] when multiple exist. The recent works [100, 101] study approximate solutions to differential games, with [102] providing a distributed computation approach. Crucially, none of these works consider the inclusion of constraints, and no established algorithmic method to compute infinite-horizon NE trajectories exists, with only [6] providing local, aposteriori convergence guarantees for the discrete-time, cl-NE case.

Computational methods for the constrained, finite-horizon case are instead available both in the ol-NE case [2] and cl-NE case [7]. This has sparked the interest for receding-horizon controllers, which compute the NE for a finite-horizon game at each time step and apply the first input of the horizon as a control action. Such approach carries multiple advantages, as the recomputation process allow the agents to react to unexpected disturbances, and operating constraints can be taken into account when solving for a finite-horizon problem [7]. Indeed, receding-horizon solutions of dynamic games were successfully employed in drone racing [14], car racing [8], logistics planning [93] and electricity market clearance [5] applications. Generally, the stability properties of the closed-loop system are not analyzed, with the exception of [5] and [103], under the restrictions of considering a *potential* game and a *stable* plant, respectively.

The receding-horizon setup is related to distributed, non-cooperative model predictive control (MPC). Crucially, traditional distributed MPC approaches postulate that all agents cooperate towards minimizing a common objective function, e.g.

[104–106] consider the local objective to be a decomposition of a shared objective function. Early attempts at fully non-cooperating MPC formulations result in the system being in general unstable, see [107, Sec. 4]. In [108–110], cooperation between agents is enforced by the introduction of additional constraints, which results in stability of the closed-loop system. In our view, the introduction of these constraints do not fully capture the non-cooperative nature of the problem.
In this chapter we take a different approach, which does not require the inclusion of additional constraints. We begin by showing that the infinite-horizon unconstrained ol-NE characterized as the solution to the asymmetric algebraic Riccati equation (ARE) in [94] can be interpreted as the input sequence resulting from a linear quadratic regulator (LQR) in an augmented state space. By leveraging this observation, we derive a novel closed-form expression for the value of the objective achieved by the ol-NE. We include this expression as terminal cost for the receding horizon controller. With this terminal cost, we show that the nominal trajectory of the receding-horizon controller coincides with the constrained infinite-horizon ol-NE - a result which generalizes the infinite-horizon optimality property of single-agent MPC [111]. The asymptotic stability to the origin of the controlled system follows then from the asymptotic convergence of the unconstrained, infinite-horizon trajectories from each initial condition. A sufficient condition for the latter to hold was derived in [94]. From a computational perspective, we show that the finite-horizon problem to be solved in receding-horizon can be cast as a variational inequality (VI) [32]. The extensive literature on VIs allows one to design efficient, decentralized algorithms with convergence guarantees under loose assumptions and even in presence of coupling constraints. Interestingly, we find that the additive terminal cost jeopardizes the structure of the problem, as the resulting VI is not associated to the Karush–Kuhn–Tucker (KKT) conditions of any static generalized Nash equilibrium problem (GNEP). In the cl-NE case, the expression of the unconstrained, infinite-horizon cost-to-go is well-known in the literature. We analyze a receding-horizon control architecture with such cost-to-go as terminal cost. Differently from the ol-NE case, we let each agent include the cl-NE feedback policy of the opponents in the prediction model. We show that the receding-horizon solution coincides with the infinite-horizon cl-NE one in the unconstrained case. Our technical contributions are summarized as follows:

- For the infinite-horizon, unconstrained ol-NE case, we propose a solution algorithm inspired by [6] based on the iterative solution of Stein equations (Section 6.3.1). We show that the ol-NE derived by the solution to the aforementioned Stein equations is also the input trajectory of a set of LQRs in an augmented space. We derive a novel expression for the objective value achieved by such ol-NE (Section 6.3.2).

- By leveraging the findings in Section 6.3.1, 6.3.2, we derive a finite-horizon constrained problem whose solution is an infinite-horizon constrained ol-NE, thus generalizing the single-agent MPC infinite-horizon optimality property [111]. We show that the system in closed loop with the receding-horizon solution of the considered finite-horizon problem is asymptotically stable and it can cast as a VI (Section 6.3.3).

- By an appropriate choice of the terminal cost and prediction model, we derive a receding-horizon controller whose trajectories are equivalent to the ones of the infinite-horizon cl-NE (Section 6.4).

In Section 6.5, we illustrate two application examples via numerical simulations on distributed automatic generation control and vehicle platooning.

6.1.1 Remarks on notation

We denote the set of T-long sequences of vectors in $\mathbb{R}^n$ as $\mathcal{S}_T^n$. For $v \in \mathcal{S}_T^n$, we denote the t-th element of the sequence as $v[t]$. We denote a sequence $v \in \mathcal{S}_T^n$ parametrized in y as $v(y)$. We then denote the t-th element of $v(y)$ as $v[t|y]$. For an invertible matrix, $A^{-\top} := (A^\top)^{-1}$.

6.2 Problem setting

Problem setting We consider the problem of regulating the state of the dynamical system

$$x[t+1] = Ax[t] + \sum_{i \in \mathcal{I}} B_i u_i[t] \tag{6.1}$$

to the origin, subject to state and coupling input constraints

$$\begin{aligned} & x[t] \in \mathbb{X} \subseteq \mathbb{R}^n, && \forall t \in \mathcal{T}; && (6.2a)\\ & u_i[t] \in \mathbb{U}_i(\boldsymbol{u}_{-i}[t]) \subseteq \mathbb{R}^m && \forall t \in \mathcal{T},\ i \in \mathcal{I}. && (6.2b) \end{aligned}$$

Each input is determined by a self-interested agent. Without loss of generality, we consider inputs with equal dimension m for each agent. We denote the state sequence of the system resulting by (6.1) with initial state x_0 and collective input sequence $\boldsymbol{u} = (u_i)_{i \in \mathcal{I}}$ as $\phi(x_0, \boldsymbol{u})$. According to the notation, we denote the state evolution at time t as $\phi[t|x_0, \boldsymbol{u}]$. We consider control problems over a horizon T, possibly infinite, with quadratic stage costs

$$\forall i \in \mathcal{I} : \ell_i(x, u_i) = \tfrac{1}{2}\|x\|_{Q_i}^2 + \tfrac{1}{2}\|u_i\|_{R_i}^2. \tag{6.3}$$

Define for each i the feasible input sequences

$$\begin{aligned} \mathcal{U}_{i,T}(x_0, \boldsymbol{u}_{-i}) := \{ u_i \in \mathcal{S}_T^m \,|\, & \\ & u_i[t] \in \mathbb{U}_i(\boldsymbol{u}_{-i}[t])\ \forall t \in \mathcal{T}; \\ & \phi[t|x_0, \boldsymbol{u}] \in \mathbb{X}\ \forall t \in \mathcal{T}^+ \} \end{aligned} \tag{6.4}$$

and the collective input sequences

$$\boldsymbol{\mathcal{U}}_T(x_0) := \{ \boldsymbol{u} \in \mathcal{S}_T^{Nm} | u_i \in \mathcal{U}_{i,T}(x_0, \boldsymbol{u}_{-i})\ \forall i \in \mathcal{I} \}. \tag{6.5}$$

Finally, we assume that the origin is strictly feasible and that the state and input weights are positive semi-definite and definite, respectively.

Assumption 6.1.

(i) $0 \in \operatorname{int}(\mathbb{X})$; $\forall i \in \mathcal{I}, 0 \in \operatorname{int}(\mathbb{U}_i(0))$.

(ii) $Q_i = C_i^\top C_i \succcurlyeq 0,\ R_i = R_i^\top \succ 0 \quad \forall i \in \mathcal{I}$.

6.3 Open-loop Nash trajectories

Depending on the information structure of the problem assumed for the agents, dynamic games can have different solutions concepts [11, Ch. 6]. In this section, we consider the infinite-horizon ol-NE trajectories, where each agent assumes that the opponents only observe the initial state and subsequently commit to the sequence of inputs computed using such observations. Define the objective

$$\forall i: \; J_i^{\infty}(u_i|x_0,\boldsymbol{u}_{-i}) := \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0,\boldsymbol{u}],u_i[t]). \tag{6.6}$$

The ol-NE trajectory is defined as follows:

Definition 6.1. *[11, Def. 6.2]: Let $x_0 \in \mathbb{X}$. The sequences $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}_\infty(x_0)$ are an open-loop Nash trajectory at x_0 if, for all $i \in \mathcal{I}$,*

$$J_i^{\infty}(u_i|x_0,\boldsymbol{u}_{-i}) \le \inf_{v_i \in \mathcal{U}_{i,\infty}(x_0,\boldsymbol{u}_{-i})} J_i^{\infty}(v_i|x_0,\boldsymbol{u}_{-i}). \tag{6.7}$$

6.3.1 The unconstrained infinite-horizon case

Let us first consider the unconstrained ol-NE problem, which is related to the solution of [112, Eq. 9]

$$\forall i \in \mathcal{I}: \begin{cases} P_i^{\mathrm{OL}} = Q_i + A^\top P_i^{\mathrm{OL}} \overline{A}^{\mathrm{OL}} & (6.8a) \\ K_i^{\mathrm{OL}} = -R_i^{-1} B_i^\top P_i^{\mathrm{OL}} \overline{A}^{\mathrm{OL}}, & (6.8b) \end{cases}$$

where

$$\overline{A}^{\mathrm{OL}} := A + \sum_{i \in \mathcal{I}} B_i K_i^{\mathrm{OL}}. \tag{6.9}$$

By leveraging [94, Theorem 4.10], one can construct an ol-NE trajectory via a solution to (6.8), as we report next:

Assumption 6.2. *[94, Assumptions 4.6, 4.7]*

(i) The matrix A is invertible.

(ii) For all $i \in \mathcal{I}$, the pairs (A,B_i) and (A,C_i) introduced in (6.1) *and Assumption 6.1 are respectively stabilizable and detectable.*

Assumption 6.3. *[94, Assumption 4.9] Denote $S_i = B_i R_i^{-1} B_i^\top$ for all i. The matrix*

$$H := \begin{bmatrix} A + \sum_{j \in \mathcal{I}} (S_j A^{-\top} Q_j) & \mathrm{row}(-S_j A^{-\top})_{j \in \mathcal{I}} \\ \mathrm{col}(-A^{-\top} Q_j)_{j \in \mathcal{I}} & I_N \otimes A^{-\top} \end{bmatrix} \tag{6.10}$$

possesses exactly n eigenvalues with modulus smaller than 1*. Moreover, an n-dimensional stable invariant subspace of H is complementary to*

$$\mathrm{Im}\left(\begin{bmatrix} \mathbf{0}_{n \times Nn} \\ I_{Nn} \end{bmatrix} \right).$$

6

Proposition 6.1. *Let Assumptions 6.1, 6.2, 6.3 hold true and let* $(K_i^{\mathrm{OL}}, P_i^{\mathrm{OL}})_{i\in\mathcal{I}}$ *satisfy* (6.8)*. Let* $\overline{A}^{\mathrm{OL}}$ *be defined as in* (6.9)*. For any* $x_0 \in \mathbb{R}^n$*, let* $\boldsymbol{u}^{\mathrm{OL}}(x_0) \in \mathcal{S}_\infty^{Nm}$*,* $x^{\mathrm{OL}}(x_0) \in \mathcal{S}_\infty^{n}$ *be defined as*

$$\forall t \in \mathbb{N}_0 : x^{\mathrm{OL}}[t|x_0] := (\overline{A}^{\mathrm{OL}})^t x_0 \tag{6.11a}$$

$$\forall i \in \mathcal{I}, t \in \mathbb{N}_0 : u_i^{\mathrm{OL}}[t|x_0] := K_i^{\mathrm{OL}} x^{\mathrm{OL}}[t|x_0]. \tag{6.11b}$$

Then, $\boldsymbol{u}^{\mathrm{OL}}(x_0)$ *is an ol-NE trajectory at the initial state* x_0 *and* $\overline{A}^{\mathrm{OL}}$ *is Schur.*

Proof. See Appendix 6.A. ■

We recall that, according to the notation table and as highlighted in Section 6.1.1, the notation $x^{\mathrm{OL}}[t|x_0]$ in (6.11) denotes the t-th element of $x^{\mathrm{OL}}(x_0)$, where $x^{\mathrm{OL}} : \mathbb{R}^n \to \mathcal{S}_\infty^N$, that is, x^{OL} is an infinite sequence parametric in x_0. A similar consideration holds for the notation $u^{\mathrm{OL}}[t|x_0]$.

If we disregard the dependence of $\overline{A}^{\mathrm{OL}}$ on $(P_j^{\mathrm{OL}})_{j\in\mathcal{I}}$, (6.8a) is a set of N Stein equations, for which off-the-shelf solvers exist [113]. One can then iteratively find $(P_i^{\mathrm{OL}})_{i\in\mathcal{I}}$ that solves Equation (6.8a) with $\overline{A}^{\mathrm{OL}}$ fixed, and then update $\overline{A}^{\mathrm{OL}}$ according to

$$\overline{A}^{\mathrm{OL}} = (I + \textstyle\sum_{j\in\mathcal{I}} S_j P_j^{\mathrm{OL}})^{-1} A. \tag{6.12}$$

The equality in (6.12) is proven in Fact 6.1 (Appendix 6.A) when A is not singular. This approach, inspired by the iteration in [6, Eq. 16] for symmetric coupled AREs, is formalized in Algorithm 9. We recall that, in Algorithm 9, $S_i = B_i R_i^{-1} B_i^\top$ for all i.

Algorithm 9 ol-NE solution via Stein recursion

Initialization: $(K_i^{(0)})_{i\in\mathcal{I}}$ such that $\overline{A}^{(0)} = A + \sum_{i\in\mathcal{I}} B_i K_i^{(0)}$ is Schur;
for $k \in \mathbb{N}$ **do**:
  for $i \in \mathcal{I}$ **do**:
    Solve $P_i^{(k+1)} = Q_i + A^\top P_i^{(k+1)} \overline{A}^{(k)}$;
  end for
  $\overline{A}^{(k+1)} \leftarrow (I + \sum_{j\in\mathcal{I}} S_j P_j^{(k+1)})^{-1} A$
  for $i \in \mathcal{I}$ **do**:
    $K_i^{(k+1)} \leftarrow -R_i^{-1} B_i^\top P_i^{(k+1)} \overline{A}^{(k+1)}$.
  end for
end for

6.3.2 The ol-NE as a linear quadratic regulator

In this section, we show that the matrices $(K_i^{\mathrm{OL}})_{i\in\mathcal{I}}$ in (6.11b) that characterize the unconstrained ol-NE are related to N LQR controllers for N dynamic systems defined in a higher-dimensional space. This result allows one to derive a novel expression for the objective value achieved by the ol-NE, which is fundamental for our main results in Section 6.3.3. For each $i \in \mathcal{I}$, let us denote by $P_i^{\mathrm{LQR}}, K_i^{\mathrm{LQR}}$ the solution to the ARE that solves the standard LQR problem for the linear time-invariant (LTI) system (A, B_i), namely:

$$P_i^{\mathrm{LQR}} = Q_i + A^\top P_i^{\mathrm{LQR}} \overline{A}_i^{\mathrm{LQR}}; \tag{6.13a}$$

$$K_i^{\mathrm{LQR}} = -R_i^{-1} B_i^\top P_i^{\mathrm{LQR}} \overline{A}_i^{\mathrm{LQR}}; \tag{6.13b}$$

$$\overline{A}_i^{\mathrm{LQR}} := A + B_i K_i^{\mathrm{LQR}}. \tag{6.13c}$$

In [94] the authors note that, for each agent i, the ol-NE is the optimal input sequence for the LTI system (A, B_i) perturbed by the actions of the other agents, which are thus treated as an affine, known disturbance signal. We observe that, along the trajectory defined by the control laws $(K_i^{\mathrm{OL}})_{i\in\mathcal{I}}$, such perturbation is fully determined by the initial state and by the dynamics of the autonomous system $\overline{A}^{\mathrm{OL}}$. Remarkably, the problem can then be cast as an augmented regulator problem by considering a system with $2n$ states that incorporates the dynamics of the perturbation. The value of the objective achieved from each initial state by the augmented regulator can be written in closed form.

Lemma 6.1 (Nash Equilibrium as augmented LQR solution). *Let $(P_i^{\mathrm{OL}}, K_i^{\mathrm{OL}})_{i\in\mathcal{I}}$ solve* (6.8) *and let $\overline{A}^{\mathrm{OL}}$ as in* (6.9). *Let Assumptions 6.1(ii), 6.2, 6.3 hold true. For all $i \in \mathcal{I}$, define*

$$\widehat{A}_i := \begin{bmatrix} A & \sum_{j\neq i} B_j K_j^{\mathrm{OL}} \\ 0 & \overline{A}^{\mathrm{OL}} \end{bmatrix}, \quad \widehat{B}_i := \begin{bmatrix} B_i \\ 0 \end{bmatrix} \tag{6.14a}$$

$$\widehat{Q}_i := \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{R}_i := R_i. \tag{6.14b}$$

Then, the ARE

$$\widehat{P}_i = \widehat{Q}_i + \widehat{A}_i^\top \widehat{P}_i (\widehat{A}_i + \widehat{B}_i \widehat{K}_i) \tag{6.15a}$$

$$\widehat{K}_i = -(\widehat{R}_i + \widehat{B}_i^\top \widehat{P}_i \widehat{B}_i)^{-1} \widehat{B}_i^\top \widehat{P}_i \widehat{A}_i \tag{6.15b}$$

admits a unique positive semidefinite solution

$$\widehat{P}_i = \begin{bmatrix} P_i^{\mathrm{LQR}} & \widetilde{P}_i \\ \widetilde{P}_i^\top & * \end{bmatrix}; \quad \widehat{K}_i = \begin{bmatrix} K_i^{\mathrm{LQR}} & \widetilde{K}_i \end{bmatrix} \tag{6.16}$$

where $\widetilde{P}_i = P_i^{\mathrm{OL}} - P_i^{\mathrm{LQR}}$ and $\widetilde{K}_i = K_i^{\mathrm{OL}} - K_i^{\mathrm{LQR}}$. Furthermore, for all $i \in \mathcal{I}$, $\widehat{A}_i + \widehat{B}_i \widehat{K}_i$ is Schur. □

Proof. See Appendix 6.B. ■

Let us consider the lifted system $(\widehat{A}_i, \widehat{B}_i)$ for some $i \in \mathcal{I}$ with initial state $\mathrm{col}(x_0, y_0)$. Denote by $\mathrm{col}(x^*, y^*)$ and u_i^* respectively the state and input sequences of the lifted system in (6.14a) controlled with $\widehat{K}_i$ in (6.16), that is,

$$u_i^*[t] = K_i^{\mathrm{LQR}} x^*[t] + \widetilde{K}_i y^*[t]; \tag{6.17a}$$

$$x^*[t+1] = A x^*[t] + B_i u_i^*[t] + \textstyle\sum_{j\neq i} B_j K_j^{\mathrm{OL}} y^*[t]; \tag{6.17b}$$

$$y^*[t+1] = \overline{A}^{\mathrm{OL}} y^*[t]. \tag{6.17c}$$

Note that (6.17c) implies $y^*[t] = (\overline{A}^{\text{OL}})^t y_0$. Furthermore, from (6.11), $u_j^{\text{OL}}[t|y_0] = K_j^{\text{OL}}(\overline{A}^{\text{OL}})^t y_0$, for all $j \in \mathcal{I}$. Thus, substituting the latter in (6.17b),

$$x^*[t+1] = Ax^*[t] + B_i u_i^*[t] + \sum_{j \neq i} B_j u_j^{\text{OL}}[t|y_0]. \tag{6.18}$$

In other words, x^* is the sequence of states of the (non-lifted) dynamics in (6.1) resulting by the input sequences u_i^* and $\boldsymbol{u}_{-i}^{\text{OL}}(y_0)$. We write this compactly as

$$x^*[t] = \phi[t|x_0, u_i^*, \boldsymbol{u}_{-i}^{\text{OL}}(y_0)]. \tag{6.19}$$

Consider the function

$$V_i(x_0, y_0) := \frac{1}{2} \left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_{\widehat{P}_i}^2 \tag{6.20}$$

where $\widehat{P}_i$ solves (6.15). By applying a known result in the LQR literature [114, Thm. 21.1],

$$\frac{1}{2} \left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_{\widehat{P}_i}^2 = \frac{1}{2} \sum_{t=0}^{\infty} \left\| \begin{bmatrix} x^*[t] \\ y^*[t] \end{bmatrix} \right\|_{\widehat{Q}_i}^2 + \|u_i^*[t]\|_{\widehat{R}_i}^2 .$$

By substituting (6.14b) in the latter to eliminate $\widehat{Q}_i, \widehat{R}_i$, we find:

$$\begin{aligned} V_i(x_0, y_0) &= \tfrac{1}{2} \sum_{t=0}^{\infty} \|x^*[t]\|_{Q_i}^2 + \|u_i^*[t]\|_{R_i}^2 \\ &\overset{(6.19),(6.3)}{=} \tfrac{1}{2} \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0, u_i^*, \boldsymbol{u}_{-i}^{\text{OL}}(y_0)], u_i^*[t]) \\ &\overset{(6.6)}{=} J_i^{\infty}(u_i^*|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(y_0)) \\ &\leq J_i^{\infty}(u_i|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(y_0)) \quad \forall u_i \in \mathcal{S}_{\infty}^m \end{aligned} \tag{6.21}$$

where the latter inequality follows from the optimality of u_i^*. In particular, for $x_0 = y_0$, we have

$$V_i(x_0, x_0) \leq J_i^{\infty}(u_i|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(x_0)) \;\; \forall u_i \in \mathcal{S}_{\infty}^m. \tag{6.22}$$

Note that, from Definition 6.1, for all $u_i \in \mathcal{S}_{\infty}^m$ and $\boldsymbol{u}^{\text{OL}}$ defined as in (6.11),

$$J_i^{\infty}(u_i^{\text{OL}}(x_0)|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(x_0)) \leq J_i^{\infty}(u_i|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(x_0)). \tag{6.23}$$

Thus, comparing the latter with (6.22), we conclude that $V_i(x_0, x_0)$ is the infinite-horizon value of the objective achieved by the ol-NE trajectory $\boldsymbol{u}^{\text{OL}}$ from state x_0:

$$V_i(x_0, x_0) = J_i^{\infty}(u_i^{\text{OL}}(x_0)|x_0, \boldsymbol{u}_{-i}^{\text{OL}}(x_0)). \tag{6.24}$$

Finally, we observe from the Bellman optimality principle applied to the system $(\widehat{A}_i, \widehat{B}_i)$ that

$$\begin{aligned} V_i(x_0, y_0) = \min_{u_i \in \mathbb{R}^m} & \tfrac{1}{2}(\|x_0\|_{Q_i}^2 + \|u_i\|_{R_i}^2) \\ & + V_i(Ax_0 + B_i u_i + \textstyle\sum_{j \neq i} B_j K_j^{\text{OL}} y_0, \overline{A}^{\text{OL}} y_0) \end{aligned} \tag{6.25}$$

and the latter optimization problem has solution $K_i^{\text{LQR}} x_0 + \widetilde{K}_i y_0$.

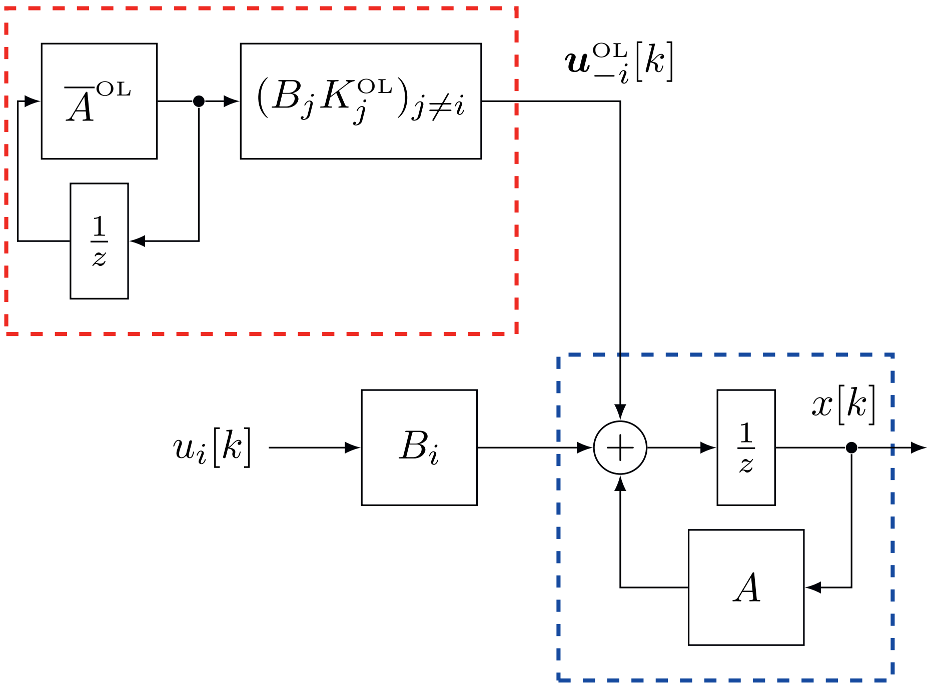


Figure 6.3.1: Block diagram of the augmented system in (6.14a) for agent i: the red dashed line highlights an uncontrollable mode which generates $\boldsymbol{u}_{-i}^{\mathrm{OL}}$. Following (6.22), u_i^{OL} is the optimal control for this system if both the subsystems highlighted in red and in blue have the same initial state.

6.3.3 Receding horizon Open-Loop Nash equilibria

In this section we construct a finite-horizon problem whose solution is a truncation of the constrained infinite-horizon ol-NE. This result, shown in Theorem 6.1, generalizes the infinite-horizon optimality property of single-agent MPC, which is known to require a careful tuning of the terminal cost function [111]. Intuitively, the finite-horizon problem in (6.26a) is defined by including V_i in (6.20) as a terminal cost: As discussed in Section 6.3.1, this function of the state captures the infinite-horizon value of the objective achieved by the ol-NE in (6.20). However, as V_i is defined in an augmented space, special care is needed. We show in Theorem 6.2 that, with this design choice, the receding-horizon control law obtained by applying the first input of the solution to the finite-horizon problem makes the origin asymptotically stable. Finally, in Proposition 6.2, we cast the finite-horizon problem as a VI. Consider the problem, parametrized in $x_0 \in \mathbb{X}$, of finding the *finite horizon solution* $\boldsymbol{u}^{\text{FH}}(x_0)$ such that

$$\forall\, i \in \mathcal{I}:\ u_i^{\text{FH}}(x_0) \in \underset{u_i \in \mathcal{U}_{T,i}(x_0, \boldsymbol{u}_{-i}^{\text{FH}})}{\operatorname{argmin}} \quad J_i(u_i | x_0, \boldsymbol{u}^{\text{FH}}), \tag{6.26a}$$

$$\begin{aligned} J_i(u_i|x_0,\boldsymbol{u}^{\text{FH}}) := {}& \textstyle\sum_{t\in\mathcal{T}} \big(\ell_i(\phi[t|x_0,u_i,\boldsymbol{u}_{-i}^{\text{FH}}],u_i[t])\big) \\ & + V_i\big(\phi[T|x_0,u_i,\boldsymbol{u}_{-i}^{\text{FH}}],\phi[T|x_0,\boldsymbol{u}^{\text{FH}}]\big), \end{aligned} \tag{6.26b}$$

where V_i is as in (6.20). Note that, for each agent i, the optimization problem in (6.26a) is parametric in the decision variables of *all* agents $\boldsymbol{u}^{\text{FH}}$. Interestingly, this structure differs from the one of a static NE problem, where the parametrization is on the decision variables of the *remaining* agents, see e.g. [3, Eq. 1]. Let $\mathbb{X}_f^{\text{OL}}$ be a constraints-admissible forward invariant set for the dynamics $x[t+1] = \overline{A}^{\text{OL}} x[t]$. Techniques for computing $\mathbb{X}_f^{\text{OL}}$ can be found in [115] and references therein. Define the set

$$\begin{aligned} \mathcal{X} := \big\{x_0 \in \mathbb{X} \mid \exists \boldsymbol{u}^{\text{FH}} &\text{ that solves (6.26a)}; \\ & \phi[T|x_0,\boldsymbol{u}^{\text{FH}}] \in \mathbb{X}_f^{\text{OL}}\big\}. \end{aligned} \tag{6.27}$$

We show next that the infinite-horizon constrained ol-NE trajectory can be recovered by the solutions to (6.26a).

Theorem 6.1. *Let Assumptions 6.1, 6.2, 6.3 hold true. Let* $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i\in\mathcal{I}}$ *solve* (6.8) *and let* $\boldsymbol{u}^{\text{OL}}(x) \in \mathcal{S}_\infty^{Nm}$ *be the unconstrained ol-NE sequence for any* $x \in \mathbb{X}$, *as defined in* (6.11). *Let* $\boldsymbol{u}^{\text{FH}} \in \mathcal{S}_T^{Nm}$ *solve* (6.26a) *for some* $x_0 \in \mathcal{X}$, *with associated state sequence* $x^{\text{FH}} := \phi(x_0, \boldsymbol{u}^{\text{FH}})$, *and define the* extended input sequence $\boldsymbol{u}^{\text{EX}} \in \mathcal{S}_\infty^{Nm}$ *as*

$$\forall i:\ u_i^{\text{EX}}[t] := \begin{cases} u_i^{\text{FH}}[t] & \text{if } t < T \\ u_i^{\text{OL}}[t-T|x^{\text{FH}}[T]] & \text{if } t \geq T. \end{cases} \tag{6.28}$$

Then, $\boldsymbol{u}^{\text{EX}}$ *is an infinite-horizon ol-NE trajectory for the system in* (6.1) *with state and input constraint sets* $\mathbb{X}, (\mathbb{U}_i)_{i\in\mathcal{I}}$, *respectively, and initial state* x_0.

Proof. See Appendix 6.C. ■

In light of Proposition 6.1, by solving (6.26a) at subsequent time instants, one expects the agents not to deviate from the previously found solution. This is because they will recover shifted truncations of the same infinite-horizon ol-NE trajectory. Indeed, in Lemma 6.2 we show that a trajectory solving the problem in (6.26a) when shifted by one time step still solves the problem in (6.26a) for the subsequent state. This is crucial, because the control action remains the same between subsequent computations of the solution, keeping the evaluated objective constant–except for the first stage cost, which does not appear in the summation. The cumulative objective of the agents decreases then at each time step and it can be used as a Lyapunov function to show the stability of the origin (modulo some technicalities, due to the fact that Q_i is only positive semidefinite for all i). Let us formalize this next.

Lemma 6.2. *Let Assumptions 6.1, 6.2, 6.3 hold. Let $x_0 \in \mathcal{X}$, with $\mathcal{X}$ defined in* (6.27). *Define the* shifted input sequence $\boldsymbol{u}^{\text{SH}} \in \mathcal{S}_T^{Nm}$ *as follows:*

$$\forall i \in \mathcal{I}: \; u_i^{\text{SH}}[t] = \begin{cases} u_i^{\text{FH}}[t+1] & \text{if } t < T-1 \\ K_i^{\text{OL}} x^{\text{FH}}[T] & \text{if } t = T-1 \end{cases} \tag{6.29}$$

where $\boldsymbol{u}^{\text{FH}}$ solves (6.26a) *with initial state x_0, $x^{\text{FH}} := \phi(x_0, \boldsymbol{u}^{\text{FH}})$ and $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i \in \mathcal{I}}$ solve* (6.8). *Then, $\boldsymbol{u}^{\text{SH}}$ is a solution for the problem in* (6.26a) *with initial state $x^{\text{FH}}[1]$.*

Proof. See Appendix 6.C. ■

In view of Lemma 6.2 and given that the problem in (6.26a) might admit multiple solutions, we assume that the shifted solution in (6.29) is actually employed when the agents solve subsequent instances of the problem in (6.26a). Assumption 6.4 is practically reasonable as, when implementing a solution algorithm for (6.26a), one can warm-start the algorithm to the shifted sequence $\boldsymbol{u}^{\text{SH}}$ defined in (6.29).

Assumption 6.4. *For any $x_0 \in \mathbb{X}$, if the problem in* (6.26a) *at the initial state x_0 admits a solution $\boldsymbol{u}^{\text{FH}}$ and if the shifted sequence $\boldsymbol{u}^{\text{SH}}$ defined in* (6.29) *is a solution of* (6.26a) *with initial state $\phi[1|x_0, \boldsymbol{u}^{\text{FH}}]$, then $\boldsymbol{u}^{\text{SH}}$ is selected by all agents when solving* (6.26a) *with initial state $\phi[1|x_0, \boldsymbol{u}^{\text{FH}}]$.*

We are now ready to conclude on asymptotic stability of the system controlled in receding horizon:

Theorem 6.2. *Let Assumptions 6.1–6.2, 6.3–6.4 hold true. Consider the system in* (6.1) *with feedback control $x \mapsto \boldsymbol{u}^{\text{FH}}[0|x]$, where $\boldsymbol{u}^{\text{FH}}(x)$ solves* (6.26a) *for the initial state x. The origin is asymptotically stable for the closed-loop system with region of attraction $\mathcal{X}$, defined in* (6.27).

Proof. See Appendix 6.C. ■

6.3.4 The open-loop Nash equilibrium as a Variational Inequality

Theorem 6.1 bridges the infinite-horizon constrained ol-NE trajectory with the solution to a finite-horizon equilibrium problem, whose solutions can be computed

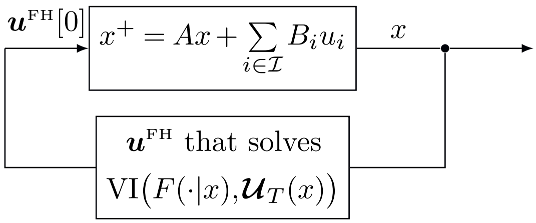


Figure 6.3.2: Block scheme of the proposed receding-horizon controller

algorithmically. In fact, we recast the problem in (6.26a) as a VI, for which a plethora of efficient solution algorithms exist under some standard monotonicity and convexity assumptions [63].

Proposition 6.2. *Assume $\boldsymbol{\mathcal{U}}_T(x_0)$ non-empty, closed and convex for all $x_0 \in \mathbb{X}$. For some $T \in \mathbb{N}$, define for all $i \in \mathcal{I}$:*

$$\begin{aligned}
\Gamma_i &:= \begin{bmatrix} B_i & 0 & \dots & 0 \\ AB_i & B_i & \dots & 0 \\ \vdots & & \ddots & \\ A^{T-1}B_i & A^{T-2}B_i & \dots & B_i \end{bmatrix}, \\
\Theta &:= \operatorname{col}(A^k)_{k\in\mathcal{T}^+}; \\
\overline{R}_i &:= I_T \otimes R_i \\
\overline{Q}_i &:= \begin{bmatrix} I_{T-1}\otimes Q_i & 0 \\ 0 & P_i^{\text{OL}} \end{bmatrix}
\end{aligned} \tag{6.30}$$

and define $F(\cdot|x_0) : \mathbb{R}^{TNm} \to \mathbb{R}^{TNm}$, parametric in $x_0 \in \mathbb{R}^n$, as

$$\begin{aligned}
F(\boldsymbol{u}|x_0) :=& \operatorname{blkdiag}(\overline{R}_i)_{i\in\mathcal{I}}\boldsymbol{u} \\
&+ \begin{bmatrix} \Gamma_1^\top \overline{Q}_1 \\ \vdots \\ \Gamma_N^\top \overline{Q}_N \end{bmatrix} \begin{bmatrix} \Gamma_1 & \dots & \Gamma_N \end{bmatrix} \boldsymbol{u} + \begin{bmatrix} \Gamma_1^\top \overline{Q}_1 \\ \vdots \\ \Gamma_N^\top \overline{Q}_N \end{bmatrix} \Theta x_0.
\end{aligned} \tag{6.31}$$

Then, any solution of $\text{VI}(F(\cdot|x_0), \boldsymbol{\mathcal{U}}_T(x_0))$ *is a solution to* (6.26a)*.*

Proof. See Appendix 6.C. ∎

It can be shown via the the generalized Gerschgorin disk theorem [76] that the VI in Proposition 6.2 is strongly monotone [63, Def. 2.3.1] if, for all i, $R_i = r_i I$, with $r_i > 0$ large enough. A strongly monotone VI admits a unique solution [63, 2.3.3], thus this design choice makes Assumption 6.4 redundant.

Remark 6.1. *It is well-known that a static convex game can be cast as a VI defined via the stacked partial gradients of the agents' cost functions [32]. We note, however, that the matrix multiplying $\boldsymbol{u}$ in* (6.31) *has non-symmetric diagonal blocks, since*

$(P_i^{\text{OL}})_{i\in\mathcal{I}}$ *are in general non-symmetric matrices. This implies that there do not exist* N *cost functions (one for each block) such that* F *corresponds to their stacked gradients. Therefore, there is no static game whose NE corresponds to the solution of* (6.26a)*. This confirms the observation made in Section 6.3.3 that the structure of* (6.26a) *differs from the one of a static NE problem.*

6.4 Closed-loop Nash equilibria

We now turn our attention to the cl-NE solution concept, where each agent assumes that the opponents can observe the state at each time step and recompute their input sequence accordingly. The game is defined over the feedback control functions $\sigma_i : \mathbb{N}\times\mathbb{R}^n \to \mathbb{R}^m$. Denote $\boldsymbol{\sigma} = (\sigma_i)_{i\in\mathcal{I}}$. We overload the notation for the state sequence of the system in (6.1) controlled by the feedback law $\boldsymbol{\sigma}$ as $\phi(x_0,\boldsymbol{\sigma})$. Furthermore, we denote as $u_i(x_0,\boldsymbol{\sigma})$ the sequence of inputs for agent $i\in\mathcal{I}$ resulting from $\boldsymbol{\sigma}$: that is, for all $t\in\mathcal{T}$,

$$\begin{aligned} u_i[t|x_0,\boldsymbol{\sigma}] &= \sigma_i(t,\phi[t|x_0,\boldsymbol{\sigma}]), \qquad \forall i\in\mathcal{I};\\ \phi[t+1|x_0,\boldsymbol{\sigma}] &= A\phi[t|x_0,\boldsymbol{\sigma}] + \textstyle\sum_{i\in\mathcal{I}} B_i u_i[t|x_0,\boldsymbol{\sigma}]. \end{aligned}$$

We also overload the notation for the objective in (6.6) as

$$\forall i\in\mathcal{I}:\ J_i^{\infty}(\sigma_i|x_0,\boldsymbol{\sigma}_{-i}) = \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0,\boldsymbol{\sigma}],u_i[t|x_0,\boldsymbol{\sigma}]). \tag{6.32}$$

We consider the unconstrained case with $\mathbb{X}=\mathbb{R}^n$, $\mathbb{U}_i=\mathbb{R}^m$, for all $i\in\mathcal{I}$.

Definition 6.2. *[11, Def. 6.3]: The feedback strategies* $\boldsymbol{\sigma}^* : \mathbb{R}^n\to\mathbb{R}^{Nm}$ *are a cl-NE if for all* $x_0\in\mathbb{R}^n$*, for all* $i\in\mathcal{I}$*,*

$$J_i^{\infty}(\sigma_i^*|x_0,\boldsymbol{\sigma}_{-i}^*) \le \inf_{\sigma_i:\mathbb{R}^n\to\mathbb{R}^m} J_i^{\infty}(\sigma_i|x_0,\boldsymbol{\sigma}_{-i}^*). \tag{6.33}$$

The unconstrained cl-NE problem admits a solution among linear feedback control laws, and a sufficient characterization of the cl-NE in terms of couple AREs is well-known in the literature [9, Prop. 6.3]. Let us report a recent result which relaxes the assumptions:

Assumption 6.5. $(A,\operatorname{row}(B_i)_{i\in\mathcal{I}})$ *is stabilizable and* $(A,\sum_{i\in\mathcal{I}}Q_i)$ *is detectable.*

Lemma 6.3. *[94, Cor. 3.3] Let Assumption 6.1(ii), 6.5 hold true. Let* $(P_i^{\text{CL}},K_i^{\text{CL}})_{i\in\mathcal{I}}$ *solve*

$$\forall i\in\mathcal{I}:\begin{cases} P_i^{\text{CL}} = Q_i + (\overline{A}_{-i}^{\text{CL}})^\top P_i^{\text{CL}}\overline{A}^{\text{CL}} & (6.34\text{a})\\ K_i^{\text{CL}} = -R_i^{-1}B_i^\top P_i^{\text{CL}}\overline{A}^{\text{CL}} & (6.34\text{b})\end{cases}$$

with $P_i^{\text{CL}} = (P_i^{\text{CL}})^\top \succcurlyeq 0$,

$$\begin{aligned} \overline{A}^{\text{CL}} &:= A + \textstyle\sum_{j\in\mathcal{I}} B_j K_j^{\text{CL}}\\ \overline{A}_{-i}^{\text{CL}} &:= A + \textstyle\sum_{j\neq i} B_j K_j^{\text{CL}}. \end{aligned} \tag{6.35}$$

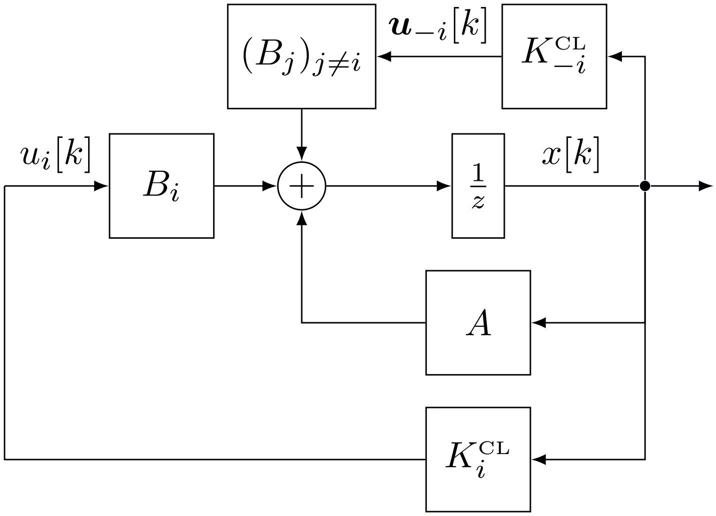


Figure 6.4.1: Block diagram of the cl-NE control problem for agent i. The linear feedback K_i^{CL} is optimal for the depicted system.

Then, the linear feedback control $\boldsymbol{\sigma} = (K_i^{\mathrm{CL}})_{i\in\mathcal{I}}$ is a cl-NE, the resulting closed-loop dynamics is asymptotically stable and $J_i(\sigma_i|x_0,\boldsymbol{\sigma}_{-i}) = \frac{1}{2}\|x_0\|^2_{P_i^{\mathrm{CL}}}$ for all $i\in\mathcal{I}$ and $x_0\in\mathbb{R}^n$.

Proof. It follows directly via algebraic calculations from [94, Cor. 3.3]. ■

If one ignores the dependence of $\overline{A}_{-i}^{\mathrm{CL}}$ on K_i^{CL} (which emerges via (6.34b)), (6.34a) is a standard Riccati equation that solves the LQR problem with state evolution matrix $\overline{A}_{-i}^{\mathrm{CL}}$ and thus one can expect its solution to be symmetric. For this reason, (6.34) is sometimes referred to in the literature as a symmetric coupled ARE, as opposed to (6.8), whose solutions are in general not symmetric. As proposed in [6], the cl-NE can either be computed by iteratively fixing $(K_j^{\mathrm{CL}})_{j\neq i}$ for each i and solving (6.34a) with a Riccati equation solver, or by rewriting (6.34a) as

$$\begin{aligned}P_i^{\mathrm{CL}} &= Q_i + (\overline{A}^{\mathrm{CL}})^\top P_i^{\mathrm{CL}}\overline{A}^{\mathrm{CL}} - (K_i^{\mathrm{CL}})^\top B_i^\top P_i^{\mathrm{CL}}\overline{A}^{\mathrm{CL}}\\ &\overset{(6.34b)}{=} \underbrace{Q_i + (P_i^{\mathrm{CL}}\overline{A}^{\mathrm{CL}})^\top S_i P_i^{\mathrm{CL}}\overline{A}^{\mathrm{CL}}}_{\widetilde{Q}_i} + (\overline{A}^{\mathrm{CL}})^\top P_i^{\mathrm{CL}}\overline{A}^{\mathrm{CL}}\end{aligned}$$

and by solving the latter via a Lyapunov equation solver, considering $\widetilde{Q}_i$ fixed at each iteration.

6.4.1 Receding horizon closed-loop Nash equilibria

In this section, we study the stability of the origin for the system (6.1) in closed-loop with the receding-horizon solution of a finite-horizon cl-NE problem. As a cl-NE is a feedback law valid on the whole state space, one needs not recomputing it at each iteration and thus the concept of a receding-horizon cl-NE is counterintuitive. However, one must consider that the state-of-the-art solution algorithm for finite-horizon cl-NE problems [7], which handles state and input constraints, only computes a single trajectory resulting from a cl-NE given an initial state. By receding-horizon cl-NE, we then mean computing at each time step a trajectory resulting from a finite-horizon cl-NE policy, and applying the first input of the sequence. For stability purposes, a natural choice for the terminal cost is $\frac{1}{2}\|\cdot\|^2_{P_i^{\mathrm{CL}}}$, where $(K_i^{\mathrm{CL}}, P_i^{\mathrm{CL}})_{i\in\mathcal{I}}$ solve (6.34). The finite-horizon cl-NE problem can be written as a NE problem with nested equilibrium constraints parametrized in the initial state [7, Theorem 2.2]:

$$\forall i: \ \sigma^*_{T,i}(x[0]) \in \arg_{u_i[0]} \min_{u_i\in\mathcal{S}_T^m} \tfrac{1}{2}\|x[T]\|^2_{P_i^{\mathrm{CL}}} + \sum_{t=0}^{T-1} \ell_i(x[t],u_i[t]) \tag{6.36a}$$

$$\text{s.t. } x[t+1] = Ax[t] + B_i u_i[t] + \sum_{j\in\mathcal{I}_{-i}} B_j \sigma^*_{T-t,j}(x[t]), \quad \forall t\in\mathcal{T}. \tag{6.36b}$$

In the latter, the notation $\arg_{u_i[0]}\min_{u_i\in\mathcal{S}_T^m}$ denotes that the minimum is taken over u_i, while only the first element $u_i[0]$ is returned. The equilibrium constraints emerge implicitly in (6.36b), as $\boldsymbol{\sigma}^*_{T-t}$ is defined by an instance of the problem in (6.36) over the shortened horizon $T-t$. The analysis of (6.36) is challenging, as the equilibrium constraints make (6.36) a set of optimal control problems over an implicitly defined, nonlinear dynamics. We then consider a surrogate problem to the one defined in (6.36), obtained by substituting $\sigma^*_{T-t,j}$ with the linear feedback K_j^{CL} for all $t\in\mathcal{T}, j\neq i$.

$$\forall i: \ u_i^*(x[0]) \in \arg \min_{u_i\in\mathcal{S}_T^m} \tfrac{1}{2}\|x[T]\|^2_{P_i^{\mathrm{CL}}} + \sum_{t=0}^{T-1} \ell_i(x[t],u_i[t]) \tag{6.37a}$$

$$\text{s.t. } x[1] = Ax[0] + B_i u_i[0] + \textstyle\sum_{j\neq i} B_j u_j[0]; \tag{6.37b}$$

$$x[t+1] = \overline{A}^{\mathrm{CL}}_{-i} x[t] + B_i u_i[t], \quad \forall t\neq 0. \tag{6.37c}$$

We remark that, given $x[0]$, (6.37) is a static NE problem (without equilibrium constraints) and it thus admits a VI reformulation via standard results [3]. Intuitively, this substitution modifies the prediction model of each agent: instead of expecting the remaining agents to apply a finite-horizon cl-NE with shrinking horizon, it is predicted that they will apply the infinite-horizon cl-NE. Next, we show that the choice of prediction model and terminal cost ensures that $(K_i^{\mathrm{CL}} x_0)_{i\in\mathcal{I}}$ is a solution to (6.37) from any initial state x_0.

Theorem 6.3. *Let Assumptions 6.1(ii), 6.5 hold. For all $T\in\mathbb{N}$, $x_0\in\mathbb{R}^n$, $\boldsymbol{u}^{\mathrm{OL}}$ is a solution to* (6.37) *for the initial state x_0, defined as*

$$\begin{aligned} x^{\mathrm{CL}}[t] &:= (\overline{A}^{\mathrm{CL}})^t x_0 && \forall t\in\mathcal{T}; \\ u_i^{\mathrm{CL}}[t] &:= K_i^{\mathrm{CL}} x^{\mathrm{CL}}[t] && \forall t\in\mathcal{T},\ \forall i\in\mathcal{I}, \end{aligned} \tag{6.38}$$

where $(K_i^{\mathrm{CL}})_{i\in\mathcal{I}}$ solve (6.34) *and $\overline{A}^{\mathrm{CL}}$ is in* (6.35).

6

Proof. See Appendix 6.D. ∎

The stability of the origin under the receding-horizon controller follows then from the one of $(K_i^{\mathrm{CL}})_{i\in\mathcal{I}}$. Note that the results of Theorem 6.3 trivially hold also when input and state constraints are included, if the initial state is in a constraints-admissible forward invariant set for the autonomous system $x[t+1]=\overline{A}^{\mathrm{CL}}x[t]$, thus still ensuring local asymptotic stability of the origin. This is because, from Theorem 6.3, the input u_i^* defined in (6.38) is the minimizer for the i-th unconstrained problem in (6.37) when the other agents apply the input $\boldsymbol{u}_{-i}^*$. Since u_i^* is constraint-admissible, it is also a minimizer of the constrained problem: note that the introduction of the constraints does not modify the model of the dynamics assumed by agent i. The same considerations do not hold for the case in (6.36), because the equilibrium constraints in (6.36b) are affected by the introduction of state and input constraints, thus $\boldsymbol{u}^*$ might not satisfy the equation in (6.36b) even when it is constraint-admissible for the state and input constraints. Intuitively speaking, the feedback $(K_i)_{i\in\mathcal{I}}$ might not be a cl-NE even when state-and-input constraint admissible, as one agent could "take advantage" of the opponents' constraints by driving the state outside the constraint-admissible region.

6

6.5 Application examples

6.5.1 Vehicle platooning

We consider the vehicle platooning scenario in [116]. The leading vehicle, indexed by 1, aims at reaching a reference speed v^{ref}, while the remaining agents $i\in\{2,\dots,N\}$ aim at matching the speed of the preceding vehicle, while maintaining a desired distance d_i, plus an additional speed-dependent term $h_i v_i$, where v_i is the speed of agent i and h_i is a design parameter. For all $i\neq 1$, the local state is

$$x_i=\begin{bmatrix} p_{i-1}-p_i-d_i-h_i v_i \\ v_{i-1}-v_i \end{bmatrix}, \tag{6.39}$$

where p_i denotes the position of agent i with respect to the one of agent 1. As the position of agent 1 with respect to itself is 0, we define $x_1=\left[0, v^{\mathrm{ref}}-v_1\right]^\top$. The dynamics is that of a single integrator and $N-1$ double integrators sampled with

Code available at `github.com/bemilio/Receding-Horizon-GNE` rate $\tau_{\mathrm{s}} = 0.1s$. With algebraic calculations, we obtain

$$
\begin{aligned}
A &= \mathrm{blkdiag}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I_{N-1} \otimes \begin{bmatrix} 1 & \tau_{\mathrm{s}} \\ 0 & 1 \end{bmatrix} \right); \\
B_1 &= \delta_2^N \otimes \begin{bmatrix} \tau_{\mathrm{s}}^2/2 \\ \tau_{\mathrm{s}} \end{bmatrix} - \delta_1^N \otimes \begin{bmatrix} 0 \\ \tau_{\mathrm{s}} \end{bmatrix}; \\
B_N &= -\delta_N^N \otimes \begin{bmatrix} h_i \tau_{\mathrm{s}} + \tau_{\mathrm{s}}^2/2 \\ \tau_{\mathrm{s}} \end{bmatrix}; \\
&\forall i \in \{2, ..., N-1\}: \\
B_i &= \delta_{i+1}^N \otimes \begin{bmatrix} \tau_{\mathrm{s}}^2/2 \\ \tau_{\mathrm{s}} \end{bmatrix} - \delta_i^N \otimes \begin{bmatrix} h_i \tau_{\mathrm{s}} + \tau_{\mathrm{s}}^2/2 \\ \tau_{\mathrm{s}} \end{bmatrix},
\end{aligned}
\tag{6.40}
$$

where $\delta_i^n \in \mathbb{R}^n$ is a vector with only non-zero element 1 at index i. We impose the following safety distance, speed and input constraints:

$$
\begin{aligned}
&p_{i-1} \geq d_i^{\min} + p_i; \\
&v_i^{\min} \leq v_i \leq v_i^{\max}; \\
&u_i^{\min} \leq u_i \leq u_i^{\max}.
\end{aligned}
\tag{6.41}
$$

As the system in (6.40) does not satisfy the stabilizability assumption 2(ii), we apply to each agent i a pre-stabilizing local controller

$$K_i^{\mathrm{stab}} = (\delta_i^N)^\top \otimes [-1, -1].$$

We then apply the ol-NE receding horizon control with state and input weights $Q_i = I$, $R_i = 1$ and horizon $T = 10$. The VI defined in Proposition 6.2 is solved via the forward-backward splitting method [63, §12.4.2], see also [4, Algorithm 1] for an implementation in the context of NE problems. This method employs dualization of the constraints which couple the decision variables of each agent (i.e. state constraints and coupling input constraints), while the local input constraints are handled via a projection step. A sample trajectory is shown in Figure 6.5.1, where we observe that the vehicles achieve the desired equilibrium state while satisfying all the constraints. We compute a suitable set $\mathbb{X}_f^{\mathrm{OL}}$ by inscribing a level set of a quadratic Lyapunov function of the autonomous system with dynamics $\overline{A}^{\mathrm{OL}}$ in the polyhedron defined by (6.41). As shown in Figure 6.5.1(c), the state enters the set $\mathcal{X}$ defined in (6.27) after $t = 11.7s$. We verify numerically that, for each subsequent time step τ, the input sequences $\boldsymbol{u}^{\mathrm{SH}}$ computed at time τ as in (6.29) are a solution for the game at time step $\tau + 1$, which is to be expected due to Lemma 6.2. Additionally, we test the robustness of the method to errors in the terminal cost function. We perturb the matrices $(P_i^{\mathrm{OL}})_{i \in \mathcal{I}}$ with an additive error matrix, generated by sampling a normal distribution. We perform 100 state realizations for each tested value of the variance. In Figure 6.5.2 we plot the difference between the resulting state trajectory and the nominal state trajectory $\hat{x}$. We observe that the problem is robust to small perturbations of the terminal cost function, as the state sequence deviates by approximately 2% when the variance is 1% of the maximum element of $(P_i^{\mathrm{OL}})_{i \in \mathcal{I}}$.

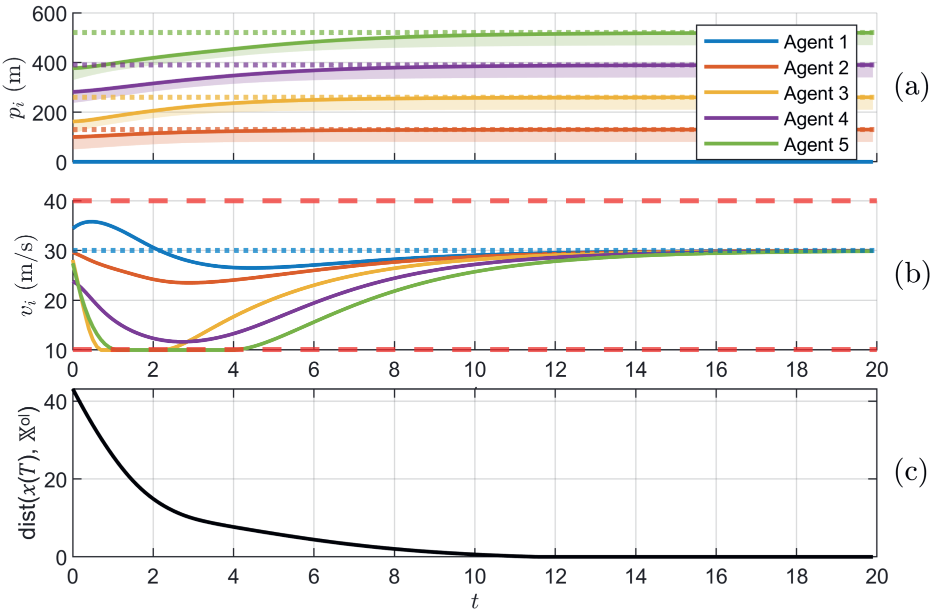


Figure 6.5.1: Vehicle platooning test case. (a) Position with respect to the leading agent. The shaded areas in the first plot represent the distance constraints for each agent. The dotted lines represent the desired values at steady state. (b) Velocity. The dotted lines represent the desired values at steady state. The dashed red lines represent the constraints. (c) Distance of the terminal state to $\mathbb{X}_f^{\mathrm{OL}}$.

The error increases significantly for increasing error values. We observe that the system remains asymptotically stable in all the tests.

6.5.2 Distributed control of interconnected generators

We test the inclusion of the proposed terminal costs on the automatic generation control problem for the power system application considered in [107]. The power system under consideration is composed of 4 generators interconnected via tie-lines arranged in a line graph. The models for the dynamics of the generators and of the tie lines linearized around a steady-state reference are the ones in [107, Eq. 17], with the model parameters specified in [117, §A.1]. This application example is chosen as it is observed in [107] that the distributed MPC control architecture (equivalent to the receding-horizon ol-NE without terminal cost in this chapter) leads to the system being unstable, and thus it is a challenging distributed control problem. The model considered has 3 states for each generator (namely, the angular velocity of the rotating element, the mechanical power applied to the rotating element and the position of the steam valve) and one for each tie line (namely, the power flow). Each agent has control authority over the reference point of their respective governor. The control objective for each agent is to regulate the deviation from the reference

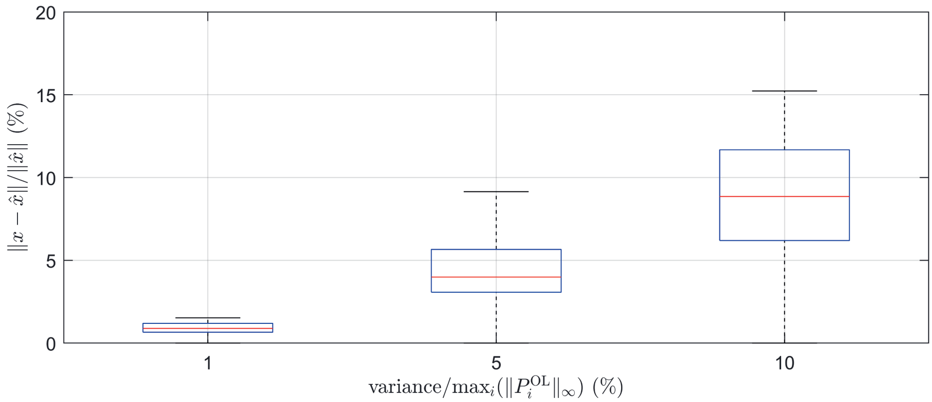


Figure 6.5.2: Deviation between the nominal state sequence $\hat{x}$ and the state sequence x resulting from the receding horizon solution of (26) when the matrices $(P_i^{\mathrm{OL}})_{i\in\mathcal{I}}$ are perturbed by an additive error sampled from a normal distribution.

angular speed of the generator rotating part and power flow at the tie-line they are connected to, with the exception of agent 1 that does not control the tie line. In numbers, we have

$$\begin{aligned}
&R_i = 1 \quad \forall i \in \{1,2,3,4\};\\
&Q_1 = \mathrm{blkdiag}(5, \mathbf{0}_{14\times 14});\\
&\forall i \in \{2,3,4\}:\\
&Q_i = \begin{bmatrix} \delta_i^4(\delta_i^4)^\top \otimes \mathrm{diag}(5,0,0) & \mathbf{0}_{12\times 3}\\ \mathbf{0}_{3\times 12} & 5\delta_{i+1}^3(\delta_{i+1}^3)^\top \end{bmatrix}.
\end{aligned} \tag{6.42}$$

We observe that the system does not satisfy Assumption 6.3, thus we could not test the performance of the ol-NE receding horizon controller. Conversely and as expected from Lemma 6.3, we find a stabilizing infinite-horizon cl-NE using the iterations in [6, Eq. 16]. We test the receding-horizon cl-NE controller which solves the problem in (6.37) from a randomly generated initial state, and we compare its stabilizing property with respect to the controller which implements (6.37) without a terminal cost in Figure 6.5.3. We estimate a constrained admissible forward invariant set of the form

$$\mathbb{X}_f^{\mathrm{CL}} = \{x \in \mathbb{X} | \|x\|_P^2 < r\} \tag{6.43}$$

where P defines a quadratic Lyapunov function for the autonomous system $\overline{A}^{\mathrm{CL}}$ defined as in (6.35) and r is determined numerically such that the controller $(K_i^{\mathrm{CL}})_{i\in\mathcal{I}}$ is feasible. We observe that the system is asymptotically stable when the initial state is in $\mathbb{X}_f^{\mathrm{CL}}$. In general, the inclusion of the terminal cost is beneficial for the asymptotic convergence of the closed-loop system.

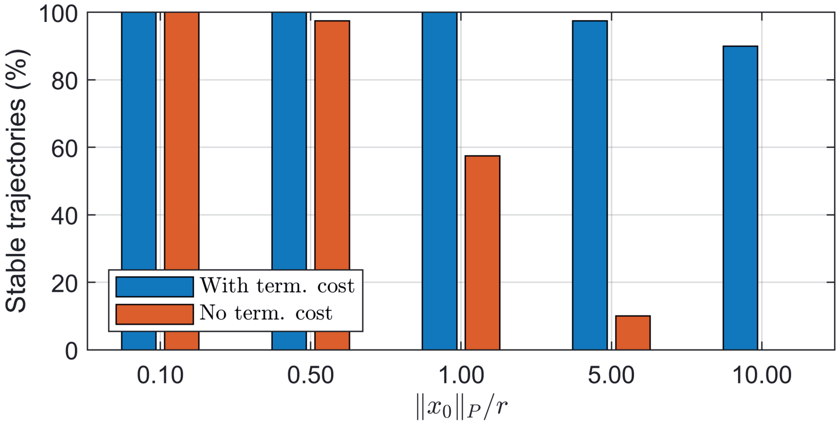


Figure 6.5.3: Percentage of asymptotically stable trajectories for the 4-zone power systems in [107] controlled by the receding-horizon cl-NE for different initial states x_0. P and r define a constraint-admissible forward invariant set as in (6.43).

Appendix

6.A Additional results and proofs to Section 6.3.1

Proof of Proposition 6.1 The proof is based on showing that (6.8) implies [94, Eq. 4.32] and then applying [94, Thm. 4.10]. We left-multiply (6.8a) by $S_i A^{-\top}$ and sum over $\mathcal{I}$ to obtain:

$$\begin{aligned}\sum_{j\in\mathcal{I}} S_j A^{-\top}(Q_j - P_j^{\mathrm{OL}}) &= \sum_{j\in\mathcal{I}} -S_j P_j^{\mathrm{OL}} \overline{A}^{\mathrm{OL}} \\ &\overset{(6.8\mathrm{b})}{=} \sum_{j\in\mathcal{I}} B_j K_j^{\mathrm{OL}}.\end{aligned} \tag{6.44}$$

Thus,

$$\overline{A}^{\mathrm{OL}} \overset{(6.9),(6.44)}{=} A + \textstyle\sum_{j\in\mathcal{I}} S_j A^{-\top}(Q_j - P_j^{\mathrm{OL}}). \tag{6.45}$$

By substituting (6.45) in (6.8a) we find

$$A^{-\top}(Q_i - P_i^{\mathrm{OL}}) = -P_i^{\mathrm{OL}}(A + \textstyle\sum_{j\in\mathcal{I}} S_j A^{-\top}(Q_j - P_j^{\mathrm{OL}})) \tag{6.46a}$$

$$K_i^{\mathrm{OL}} = -R_i^{-1} B_i^\top P_i^{\mathrm{OL}}(A + \textstyle\sum_{j\in\mathcal{I}} S_j A^{-\top}(Q_j - P_j^{\mathrm{OL}})). \tag{6.46b}$$

The latter reads as [94, Eq. 4.32]. From [94, Thm. 4.10], $\boldsymbol{u}^{\mathrm{OL}}$ defined in (6.11) is an ol-NE and $\lim_{t\to\infty} x[t|x_0, \boldsymbol{u}^{\mathrm{OL}}(x_0)] = 0$ for all x_0, thus $\overline{A}^{\mathrm{OL}}$ is Schur. ■

Fact 6.1. *Let $P_i^{\mathrm{OL}}, K_i^{\mathrm{OL}}$ solve* (6.8)*. Let A be invertible. Then,* (6.12) *holds true.*

Proof.

$$\overline{A}^{\text{OL}} \overset{(6.9)}{=} A + \textstyle\sum_{i\in\mathcal{I}} B_i K_i^{\text{OL}} \tag{6.47a}$$

$$\overset{(6.8b)}{=} A - \textstyle\sum_{i\in\mathcal{I}} S_i P_i^{\text{OL}} \overline{A}^{\text{OL}} \tag{6.47b}$$

$$\Leftrightarrow \left(I + \textstyle\sum_{i\in\mathcal{I}} S_i P_i^{\text{OL}}\right) \overline{A}^{\text{OL}} = A \tag{6.47c}$$

$$\Leftrightarrow \left(I + \textstyle\sum_{i\in\mathcal{I}} S_i P_i^{\text{OL}}\right) \overline{A}^{\text{OL}} A^{-1} = I, \tag{6.47d}$$

thus $(I + \sum_{i\in\mathcal{I}} S_i P_i^{\text{OL}})$ is invertible. Left-multiplying (6.47c) by $(I + \sum_{i\in\mathcal{I}} S_i P_i^{\text{OL}})^{-1}$ shows (6.12). ■

6.B Additional results and proofs of Section 6.3.2

Let us present a preliminary result to the proof of Lemma 6.1:

Lemma 6.4. *Let $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i\in\mathcal{I}}$ solve (6.8) and let Assumptions 6.1(ii), 6.2 hold true. Then, for each $i \in \mathcal{I}$, $P_i^{\text{OL}} = P_i^{\text{LQR}} + \widetilde{P}_i$ where $\widetilde{P}_i$ satisfies*

$$\widetilde{P}_i = (\overline{A}_i^{\text{LQR}})^\top \widetilde{P}_i \overline{A}^{\text{OL}} + (\overline{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} (\textstyle\sum_{j\neq i} B_j K_j^{\text{OL}}) \tag{6.48}$$

and $K_i^{\text{OL}} = K^{\text{LQR}} + \widetilde{K}_i$, where

$$\widetilde{K}_i = -(R_i + B_i P_i^{\text{LQR}} B_i)^{-1} B_i^\top (\widetilde{P}_i \overline{A}^{\text{OL}} + P_i^{\text{LQR}} (\textstyle\sum_{j\neq i} B_j K_j^{\text{OL}})). \tag{6.49}$$

Proof. P_i^{LQR} is symmetric [114, Thm. 22.2], thus (6.13a) can be written as

$$P_i^{\text{LQR}} = Q_i + (\overline{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} A. \tag{6.50}$$

Now note:

$$\begin{aligned} (\overline{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} B_i K_i^{\text{OL}} &\overset{(6.8b)}{=} -(\overline{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} B_i R_i^{-1} B_i^\top P_i^{\text{OL}} \overline{A}^{\text{OL}} \\ &\overset{(6.13b)}{=} (B_i K_i^{\text{LQR}})^\top P_i^{\text{OL}} \overline{A}^{\text{OL}}. \end{aligned} \tag{6.51}$$

Substituting $P_i^{\text{OL}} = P_i^{\text{LQR}} + \widetilde{P}_i$ in (6.8a):

$$\begin{aligned} P_i^{\text{LQR}} + \widetilde{P}_i &= Q_i + A^\top (P_i^{\text{LQR}} + \widetilde{P}_i) \overline{A}^{\text{OL}} \\ &\overset{(6.13c)}{=} Q_i + (\overline{A}_i^{\text{LQR}})^\top (P_i^{\text{LQR}} + \widetilde{P}_i) \overline{A}^{\text{OL}} - (B_i K_i^{\text{LQR}})^\top P_i^{\text{OL}} \overline{A}^{\text{OL}} \\ &\overset{(6.51)}{=} Q_i + (\overline{A}_i^{\text{LQR}})^\top (P_i^{\text{LQR}} + \widetilde{P}_i) \overline{A}^{\text{OL}} - (\overline{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} B_i K_i^{\text{OL}}. \end{aligned}$$

Subtract (6.50) from the latter to obtain (6.48). Let us rewrite (6.8b) and (6.13b) respectively as:

$$(R_i + B_i^\top P_i^{\text{OL}} B_i) K_i^{\text{OL}} = -B_i^\top P_i^{\text{OL}} (A + \textstyle\sum_{j\neq i} B_j K_j^{\text{OL}}); \tag{6.52}$$

$$(R_i + B_i^\top P_i^{\text{LQR}} B_i) K_i^{\text{LQR}} = -B_i^\top P_i^{\text{LQR}} A. \tag{6.53}$$

Substitute $P_i^{\text{OL}} = P_i^{\text{LQR}} + \widetilde{P}_i$ in (6.52):

$$(R_i + B_i^\top P_i^{\text{LQR}} B_i) K_i^{\text{OL}} = -B_i^\top (P_i^{\text{OL}} (A + \textstyle\sum_{j\neq i} B_j K_j^{\text{OL}}) + \widetilde{P}_i B_i K_i^{\text{OL}}),$$

which reads as (6.49) after subtracting (6.53). ■

Proof of Lemma 6.1 By the Schur property of $\overline{A}^{\text{OL}}$, it follows that any control action that stabilizes (A, B_i) is stabilizing for $(\widehat{A}_i, \widehat{B}_i)$ for all i, thus the system in (6.14a) is stabilizable. Let

$$\widehat{C}_i := \begin{bmatrix} C_i & 0 \end{bmatrix},$$

where C_i is as in Assumption 6.2. Clearly, $\widehat{Q}_i = \widehat{C}_i^\top \widehat{C}_i$. Let $\widehat{A}_i z = \lambda z$ for some $\lambda \geq 1$ and $\widehat{C}_i z = 0$, that is, let z be an unstable unobservable mode of $\widehat{A}_i$. From the stability of $\overline{A}^{\text{OL}}$, it has to be $z = [x^\top, 0^\top]^\top$ for some $x \in \mathbb{R}^n$. Then, $\widehat{A}_i z = Ax$ and it has to be $x = 0$ by the detectability of (A, C_i). Consequently, $(\widehat{A}_i, \widehat{C}_i)$ is detectable. Following [118, Cor. 13.8], (6.15a) admits a unique positive semidefinite solution and the corresponding controller is stabilizing. Consider the partition

$$\widehat{P}_i = \begin{bmatrix} \widehat{P}_i^{1,1} & \widehat{P}_i^{1,2} \\ \widehat{P}_i^{2,1} & \widehat{P}_i^{2,2} \end{bmatrix}; \quad \widehat{K}_i = [\widehat{K}_i^1, \widehat{K}_i^2].$$

By expanding (6.15) for the blocks $\widehat{P}_i^{1,1}$, $\widehat{P}_i^{2,1}$ and $\widehat{K}_i^1$, one obtains via straightforward calculations:

$$\widehat{P}_i^{1,1} = Q_i + A^\top \widehat{P}_i^{1,1}(A + B_i \widehat{K}_i^1) \tag{6.54a}$$
$$\widehat{K}_i^1 = -(R_i + B_i^\top \widehat{P}_i^{1,1} B_i)^{-1} B_i^\top \widehat{P}_i^{1,1} A \tag{6.54b}$$
$$\widehat{P}_i^{2,1} = (\textstyle\sum_{j \neq i} B_j K_j^{\text{OL}})^\top \widehat{P}_i^{1,1}(A + B_i \widehat{K}_i^1) + (\overline{A}^{\text{OL}})^\top \widehat{P}_i^{2,1}(A + B_i \widehat{K}_i^1). \tag{6.54c}$$

We note that (6.54a) and (6.54b) have the same expressions as (6.13a) and (6.13b), respectively. From $\widehat{P}_i \succcurlyeq 0$, $\widehat{P}_i^{1,1} \succcurlyeq 0$. Thus, $\widehat{P}_i^{1,1}$ is the unique positive semidefinite solution of (6.13a) and $\widehat{P}_i^{1,1} = P_i^{\text{LQR}}$, $\widehat{K}_i^1 = K_i^{\text{LQR}}$. Thus, by substituting P_i^{LQR} and K_i^{LQR} in (6.54c), one obtains that $(\widehat{P}_i^{2,1})^\top$ must satisfy (6.48). As (6.48) is a Stein equation in $\widetilde{P}_i$, its solution is unique following the Schur property of $\overline{A}_i^{\text{LQR}}$ and $\overline{A}^{\text{OL}}$ [119, Lemma 2.1]. Thus, $(\widehat{P}_i^{2,1})^\top = \widetilde{P}_i$. ■

6.C Proofs of Section 6.3.3 and 6.3.4

Proof of Theorem 6.1 By the invariance of $\mathbb{X}_f^{\text{OL}}$, $\boldsymbol{u}^{\text{EX}}$ is feasible. To prove it is an ol-NE trajectory, we proceed by contradiction. Assume there exists $v_i \in \mathcal{U}_{\infty,i}(x_0, \boldsymbol{u}_{-i}^{\text{EX}})$ such that

$$J_i^\infty(v_i | x_0, \boldsymbol{u}_{-i}^{\text{EX}}) < J_i^\infty(u_i^{\text{EX}} | x_0, \boldsymbol{u}_{-i}^{\text{EX}}) \tag{6.55}$$

where J_i^∞ is defined in (6.6). Let us substitute the definition (6.28) of $\boldsymbol{u}^{\text{EX}}$ into (6.6):

$$\begin{aligned} J_i^\infty(u_i^{\text{EX}} | x_0, \boldsymbol{u}_{-i}^{\text{EX}}) &= J_i^\infty\left(u_i^{\text{OL}}(x^{\text{FH}}[T]) | x^{\text{FH}}[T], \boldsymbol{u}_{-i}^{\text{OL}}(x^{\text{FH}}[T])\right) + \textstyle\sum_{t=0}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\overset{(6.24)}{=} V_i(x^{\text{FH}}[T], x^{\text{FH}}[T]) + \textstyle\sum_{t=0}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\overset{(6.26\text{b})}{=} J_i(u_i^{\text{FH}} | x_0, \boldsymbol{u}^{\text{FH}}). \end{aligned}$$

Denote the state sequence $x_v = \phi(x_0, v_i, \boldsymbol{u}_{-i}^{\mathrm{EX}})$: from the definition of (6.6),

$$\begin{aligned} J_i^{\infty}(v_i|x_0, \boldsymbol{u}_{-i}^{\mathrm{EX}}) &= J_i^{\infty}(v_i|x_v[T], \boldsymbol{u}_{-i}^{\mathrm{OL}}(x^{\mathrm{FH}}[T])) + \textstyle\sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\overset{(6.21)}{\geq} V_i(x_v[T], x^{\mathrm{FH}}[T]) + \textstyle\sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\overset{(6.26b)}{=} J_i(v_i|x_0, \boldsymbol{u}^{\mathrm{FH}}). \end{aligned}$$

Thus, (6.55) contradicts the definition of $\boldsymbol{u}^{\mathrm{FH}}$ in (6.26a). It follows that $\boldsymbol{u}^{\mathrm{EX}}$ is an ol-NE for all $x_0 \in \mathcal{X}$. ∎

Proof of Lemma 6.2 We first note that, for any x_0, by evaluating (6.25) at the optimal input $u_i = (K_i^{\mathrm{LQR}} + \widetilde{K}_i)x_0$ for a generic $i \in \mathcal{I}$ and substituting $K_i^{\mathrm{LQR}} + \widetilde{K}_i = K_i^{\mathrm{OL}}$ (Lemma 6.4), we obtain

$$\begin{aligned} V_i(x_0, x_0) &= \ell_i(x_0, K_i^{\mathrm{OL}} x_0) + V_i(Ax_0 + \textstyle\sum_{j\in\mathcal{I}} B_j K_j^{\mathrm{OL}} x_0, \overline{A}^{\mathrm{OL}} x_0) \\ &= \ell_i(x_0, K_i^{\mathrm{OL}} x_0) + V_i(\overline{A}^{\mathrm{OL}} x_0, \overline{A}^{\mathrm{OL}} x_0). \end{aligned} \tag{6.56}$$

We proceed by contradiction and thus assume for some $v_i \in \mathcal{U}_{i,T}(x^{\mathrm{FH}}[1], \boldsymbol{u}_{-i}^{\mathrm{SH}})$

$$J_i(v_i|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}) < J_i(u_i^{\mathrm{SH}}|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}), \tag{6.57}$$

where J_i is defined in (6.26b). Denote $x_v = \phi(x^{\mathrm{FH}}[1], v_i, \boldsymbol{u}_{-i}^{\mathrm{SH}})$. Define the auxiliary sequence

$$\hat{v}_i[t] := \begin{cases} u_i^{\mathrm{FH}}[t] & \text{if } t = 0 \\ v_i[t-1] & \text{if } t \in \{1, ..., T-1\}. \end{cases} \tag{6.58}$$

One can easily verify that

$$\phi[t+1|x_0, \hat{v}_i, \boldsymbol{u}_{-i}^{\mathrm{FH}}] = x_v[t], \ \forall t \in \mathcal{T}. \tag{6.59}$$

As $v_i \in \mathcal{U}_{i,T}(x^{\mathrm{FH}}[1], \boldsymbol{u}_{-i}^{\mathrm{SH}})$, clearly $\hat{v}_i$ is also feasible, that is, $\hat{v}_i \in \mathcal{U}_{i,T}(x_0, \boldsymbol{u}_{-i}^{\mathrm{FH}})$. By expanding (6.26b),

$$\begin{aligned} J_i(v_i|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}) &= V_i(x_v[T], \phi[T|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}]) + \textstyle\sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\overset{(6.29)}{=} V_i(x_v[T], \overline{A}^{\mathrm{OL}} x^{\mathrm{FH}}[T]) + \textstyle\sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\overset{(6.25)}{\geq} V_i(x_v[T-1], x^{\mathrm{FH}}[T]) + \textstyle\sum_{t=0}^{T-2} \ell_i(x_v[t], v_i[t]) \\ &\overset{(6.59)}{=} V_i(\phi[T|x_0, \hat{v}_i, \boldsymbol{u}_{-i}^{\mathrm{FH}}], x^{\mathrm{FH}}[T]) + \textstyle\sum_{t=1}^{T-1} \ell_i(\phi[t|x_0, \hat{v}_i, \boldsymbol{u}_{-i}^{\mathrm{FH}}], \hat{v}_i[t]) \\ &\overset{(6.26b)}{=} J_i(\hat{v}_i|x_0, \boldsymbol{u}^{\mathrm{FH}}) - \ell_i(x_0, u_i^{\mathrm{FH}}[0]). \end{aligned} \tag{6.60}$$

From (6.26b) and the definition of $\boldsymbol{u}^{\text{SH}}$ in (6.29),

$$
\begin{aligned}
J_i(u_i^{\text{SH}}|x^{\text{FH}}[1],\boldsymbol{u}^{\text{SH}}) = &\textstyle\sum_{t=1}^{T-1} \ell_i(x^{\text{FH}}[t],u_i^{\text{FH}}[t]) \\
&+\ell_i(x^{\text{FH}}[T],K_i^{\text{OL}}x^{\text{FH}}[T]) \\
&+V_i(\overline{A}^{\text{OL}}x^{\text{FH}}[T],\overline{A}^{\text{OL}}x^{\text{FH}}[T]) \\
\overset{(6.56)}{=}\ &V_i(x^{\text{FH}}[T],x^{\text{FH}}[T]) + \textstyle\sum_{t=1}^{T-1} \ell_i(x^{\text{FH}}[t],u_i^{\text{FH}}[t]) \\
\overset{(6.26\text{b})}{=}\ &J_i(u_i^{\text{FH}}|x_0,\boldsymbol{u}^{\text{FH}}) - \ell_i(x_0,u_i^{\text{FH}}[0]).
\end{aligned}
\tag{6.61}
$$

By substituting (6.60) and (6.61) in (6.57), one obtains

$$
J_i(\hat{v}_i|x_0,\boldsymbol{u}^{\text{FH}}) < J_i(u_i^{\text{FH}}|x_0,\boldsymbol{u}^{\text{FH}})
$$

which contradicts $\boldsymbol{u}^{\text{FH}}$ being a solution of (6.26a). ■

Proof of Theorem 6.2 Let $\overline{C} := \mathrm{col}(C_i)_{i\in\mathcal{I}}$, where C_i is defined in Assumption 6.2. Clearly, $(A,\overline{C})$ is detectable by the Hautus lemma and Assumption 2(*ii*). $(A,\overline{C})$ is then uniformly input/output-to-state-stable [12, Def. 2.22], [120, Prop. 3.3]. Furthermore, there exists L such that $A+L\overline{C}$ is Schur. Then, there exists P_{L} that solves the Lyapunov equation

6

$$
P_{\text{L}} - (A+L\overline{C})^\top P_{\text{L}}(A+LC) = I.
$$

For some $\gamma_{\text{x}},\gamma_{\text{y}},\gamma_{\text{u}} > 0$ and for all x, it holds that

$$
\|Ax+\textstyle\sum_{i\in\mathcal{I}} B_i u_i\|_{P_{\text{L}}}^2 - \|x\|_{P_{\text{L}}}^2 \le -\gamma_{\text{x}}\|x\|^2 + \gamma_{\text{y}}\|\overline{C}x\|^2 + \gamma_{\text{u}}\|\boldsymbol{u}\|^2. \tag{6.62}
$$

The proof of the latter inequality can be found in [120, Sec. 3.2], and it is thus omitted. From (6.62),

$$
\begin{aligned}
\|Ax+\textstyle\sum_{i\in\mathcal{I}} B_i u_i\|_{P_{\text{L}}}^2 - \|x\|_{P_{\text{L}}}^2 &\le -\gamma_{\text{x}}\|x\|^2 + \textstyle\sum_{i\in\mathcal{I}} \gamma_{\text{y}}(\|x\|_{Q_i}^2 + \gamma_{\text{u}}\|u_i\|^2) && (6.63\text{a})\\
&\le -\gamma_{\text{x}}\|x\|^2 + \overline{\gamma}\textstyle\sum_{i\in\mathcal{I}}(\|x\|_{Q_i}^2 + \|u_i\|_{R_i}^2), && (6.63\text{b})
\end{aligned}
$$

where (6.63a) follows from $\|\overline{C}x\|^2 = \sum_i \|x\|_{Q_i}^2$ and (6.63b) follows from the equivalence of norms and by taking $\overline{\gamma}$ as the maximum multiplicative constant. From the direct application of [12, Theorem B.53], there exists a continuous Λ, continuous, increasing, unbounded α_1,α_2, and a positive definite ρ such that

$$
\begin{aligned}
\alpha_1(\|x\|) \le \Lambda(x) \le \alpha_2(\|x\|) && (6.64\text{a})\\
\Lambda(Ax+\textstyle\sum_{i\in\mathcal{I}} B_i u_i) - \Lambda(x) \le -\rho(\|x\|) + \textstyle\sum_{i\in\mathcal{I}}(\|x\|_{Q_i}^2 + \|u_i\|_{R_i}^2). && (6.64\text{b})
\end{aligned}
$$

Consider the candidate Lyapunov function

$$
V(x) = \Lambda(x) + \textstyle\sum_{i\in\mathcal{I}} J_i(u_i^{\text{FH}}(x)|x,\boldsymbol{u}^{\text{FH}}(x)), \tag{6.65}
$$

where $\boldsymbol{u}^{\mathrm{FH}}(x)$ solves (6.26a) for the state x. Let $x^{\mathrm{FH}} = \phi(x, \boldsymbol{u}^{\mathrm{FH}})$. Denote the *shifted sequence* $\boldsymbol{u}^{\mathrm{SH}}$ as in (6.29), and recall that $\boldsymbol{u}^{\mathrm{SH}}$ solves (6.26a) for the state $x^{\mathrm{FH}}[1]$ following Lemma 6.2 and Assumption 6.4. Furthermore, note that

$$\phi[t|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}] = x^{\mathrm{FH}}[t+1] \quad \forall t \in \mathcal{T}. \tag{6.66}$$

Then,

$$\begin{aligned}
V(x^{\mathrm{FH}}[1]) - \Lambda(x^{\mathrm{FH}}[1]) &\overset{(6.65)}{=} \sum_{i\in\mathcal{I}} J_i(u_i^{\mathrm{SH}}|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}) \\
&\overset{(6.26\mathrm{b}),(6.29)}{=} \sum_{i\in\mathcal{I}} \Big(V_i(\overline{A}^{\mathrm{OL}} x^{\mathrm{FH}}[T], \overline{A}^{\mathrm{OL}} x^{\mathrm{FH}}[T]) \\
&\qquad + \ell_i(x^{\mathrm{FH}}[T], K_i^{\mathrm{OL}} x^{\mathrm{FH}}[T]) \\
&\qquad + \textstyle\sum_{t=0}^{T-2} \ell_i(\phi[t|x^{\mathrm{FH}}[1], \boldsymbol{u}^{\mathrm{SH}}], u_i^{\mathrm{FH}}[t+1]) \Big) \\
&\overset{(6.56),(6.66)}{=} \sum_{i\in\mathcal{I}} \Big(V_i(x^{\mathrm{FH}}[T], x^{\mathrm{FH}}[T]) + \sum_{t=1}^{T-1} \ell_i(x^{\mathrm{FH}}[t], u_i^{\mathrm{FH}}[t]) \Big) \\
&\overset{(6.26\mathrm{b})}{=} \textstyle\sum_{i\in\mathcal{I}} \Big(J_i(u_i^{\mathrm{FH}}|x, \boldsymbol{u}^{\mathrm{FH}}) - \ell_i(x, u_i^{\mathrm{FH}}[0]) \Big) \\
&\overset{(6.65)}{=} V(x) - \Lambda(x) - \textstyle\sum_{i\in\mathcal{I}} \Big(\ell_i(x, u_i^{\mathrm{FH}}[0]) \Big).
\end{aligned} \tag{6.67}$$

We rearrange (6.67) and substitute (6.3) to obtain

$$\begin{aligned}
V(x^{\mathrm{FH}}[1]) - V(x) &= \Lambda(x^{\mathrm{FH}}[1]) - \Lambda(x) - \sum_{i\in\mathcal{I}} \|x\|_{Q_i}^2 + \|u_i^{\mathrm{FH}}[0]\|_{R_i}^2 \\
&\overset{(6.64\mathrm{b})}{\leq} -\rho(\|x\|).
\end{aligned} \tag{6.68}$$

From the invariance of $\mathbb{X}_f^{\mathrm{OL}}$, $\overline{A}^{\mathrm{OL}} x \in \mathbb{X}_f^{\mathrm{OL}}$. As $\overline{A}^{\mathrm{OL}} x$ is the terminal state for $\boldsymbol{u}^{\mathrm{SH}}$ with initial state $x^{\mathrm{FH}}[1]$, we obtain from the definition of $\mathcal{X}$ that $\mathcal{X}$ is an invariant set for the closed-loop system. We conclude that the closed-loop system is asymptotically stable with region of attraction $\mathcal{X}$ [12, Thm. B.13] ■

Proof of Proposition 6.2 For this proof we treat finite sequences as column vectors, that is, $u_i = \mathrm{col}(u_i[t])_{t\in\mathcal{T}}$. Let $\boldsymbol{u}^*$ solve $\mathrm{VI}(F(\cdot|x_0), \boldsymbol{\mathcal{U}}_T(x_0))$. Consider the matrix Γ_i, defined in (6.30): We can rewrite the state evolution as

$$\begin{bmatrix} \phi[1|x_0, u_i, \boldsymbol{u}^*_{-i}] \\ \vdots \\ \phi[T|x_0, u_i, \boldsymbol{u}^*_{-i}] \end{bmatrix} = \Theta x_0 + \Gamma_i u_i + \sum_{j\neq i} \Gamma_j u_j^*. \tag{6.69}$$

Denote the partitions

$$\Gamma_i = \begin{bmatrix} \overline{\Gamma}_i \\ \underline{\Gamma}_i \end{bmatrix}, \qquad \Theta = \begin{bmatrix} \overline{\Theta} \\ A^T \end{bmatrix},$$

where $\underline{\Gamma}_i$ is the last block row, that is:

$$\underline{\Gamma}_i := \begin{bmatrix} A^{T-1}B_i & A^{T-2}B_i & \dots & B_i \end{bmatrix}.$$

Using (6.69) we find the following expression for the terminal states appearing in (6.26b):

$$\begin{aligned} \phi[T|x_0, \boldsymbol{u}^*] &= A^T x_0 + \underline{\Gamma}_i u_i^* + \textstyle\sum_{j\neq i} \underline{\Gamma}_j u_j^* \\ \phi[T|x_0, u_i, \boldsymbol{u}_{-i}^*] &= A^T x_0 + \underline{\Gamma}_i u_i + \textstyle\sum_{j\neq i} \underline{\Gamma}_j u_j^*. \end{aligned} \tag{6.70}$$

Substituting (6.69) and (6.70) in (6.26b), one obtains with straightforward calculations:

$$\begin{aligned} J_i(u_i|x_0, \boldsymbol{u}^*) = & \, u_i^\top \underline{\Gamma}_i^\top (\tfrac{1}{2} P_i^{\text{LQR}} \underline{\Gamma}_i u_i + \widetilde{P}_i \underline{\Gamma}_i u_i^*) \\ & + u_i^\top \underline{\Gamma}_i^\top P_i^{\text{OL}} (A^T x_0 + \textstyle\sum_{j\neq i} \underline{\Gamma}_j u_j^*) \\ & + \tfrac{1}{2} u_i^\top (\overline{R}_i + \overline{\Gamma}_i^\top (I_{T-1} \otimes Q_i) \overline{\Gamma}_i) u_i \\ & + u_i^\top \overline{\Gamma}_i^\top (I_{T-1} \otimes Q_i)(\overline{\Theta} x_0 + \textstyle\sum_{j\neq i} \overline{\Gamma}_j u_j^*) \\ & + f(x_0, \boldsymbol{u}^*), \end{aligned} \tag{6.71}$$

where f contains the terms that do not depend on u_i. By deriving the latter with respect to u_i,

$$\begin{aligned} \nabla J_i(u_i|x_0, \boldsymbol{u}^*) = & \, \underline{\Gamma}_i^\top (P_i^{\text{LQR}} \underline{\Gamma}_i u_i + \widetilde{P}_i \underline{\Gamma}_i u_i^*) \\ & + \underline{\Gamma}_i^\top P_i^{\text{OL}} (A^T x_0 + \textstyle\sum_{j\neq i} \underline{\Gamma}_j u_j^*) \\ & + \big(\overline{R}_i + \overline{\Gamma}_i^\top (I_{T-1} \otimes Q_i) \overline{\Gamma}_i\big) u_i \\ & + \overline{\Gamma}_i^\top (I_{T-1} \otimes Q_i)(\overline{\Theta} x_0 + \textstyle\sum_{j\neq i} \overline{\Gamma}_j u_j^*). \end{aligned} \tag{6.72}$$

By evaluating the latter at u_i^* and by substituting $P_i^{\text{LQR}} + \widetilde{P}_i = P_i^{\text{OL}}$ (Lemma 6.1), it can be verified that

$$F(\boldsymbol{u}^*|x_0) = \operatorname{col}\big(\nabla J_i(u_i^*|x_0, \boldsymbol{u}^*)\big)_{i\in\mathcal{I}}.$$

Furthermore, $J_i(\cdot|x_0, \boldsymbol{u}^*)$ is convex for every x_0, because $P_i^{\text{LQR}} \succcurlyeq 0$, $Q_i \succcurlyeq 0$, $\overline{R}_i \succ 0$. Thus, by the definition of VI solution, for any $u_i \in \mathcal{U}_{i,T}(x_0, \boldsymbol{u}_{-i}^*)$:

$$\begin{aligned} 0 &\le F(\boldsymbol{u}^*|x_0)^\top \big(\operatorname{col}(u_i, \boldsymbol{u}_{-i}^*) - \boldsymbol{u}^*\big) \\ &= \nabla J_i(u_i^*|x_0, \boldsymbol{u}^*)^\top (u_i - u_i^*) \\ &\le J_i(u_i|x_0, \boldsymbol{u}^*) - J_i(u_i^*|x_0, \boldsymbol{u}^*), \end{aligned}$$

that is, $\boldsymbol{u}^*$ solves the problem in (6.26a). ■

6.D Additional results and proofs of Section 6.4

Before proving Theorem 6.3, let us show that the cl-NE satisfies an optimality principle:

Lemma 6.5. *Let* $(P_i^{\text{CL}}, K_i^{\text{CL}})_{i\in\mathcal{I}}$ *satisfy* (6.34)*. Then, for all* $i \in \mathcal{I}$,

$$\tfrac{1}{2}\|x\|^2_{P_i^{\text{CL}}} = \min_{u_i\in\mathbb{R}^m} \ \ell_i(x,u_i) + \tfrac{1}{2}\|\overline{A}_{-i}^{\text{CL}}x + B_i u_i\|^2_{P_i^{\text{CL}}} \tag{6.73}$$

and the minimum is achieved by $K_i^{\text{CL}}x$.

Proof. Let us rewrite (6.34b) as

$$K_i^{\text{CL}} = -(R_i + B_i^\top P_i^{\text{CL}} B_i)^{-1} B_i^\top P_i^{\text{CL}} \overline{A}_{-i}^{\text{CL}}.$$

By setting the gradient of (6.73) to 0 and comparing the resulting equation with the latter, one can see that the minimum is achieved by $K_i^{\text{CL}}x$. By substituting $u_i = K_i^{\text{CL}}x$ in (6.73), the minimum of (6.73) is

$$\begin{aligned}
&\|x\|^2_{Q_i} + (K_i^{\text{CL}}x)^\top R_i K_i^{\text{CL}} x + (\overline{A}^{\text{CL}}x)^\top P_i^{\text{CL}} \overline{A}^{\text{CL}} x \\
&\overset{(6.34\text{b})}{=} \|x\|^2_{Q_i} - (K_i^{\text{CL}}x)^\top B_i^\top P_i^{\text{CL}} \overline{A}^{\text{CL}} x + (\overline{A}^{\text{CL}}x)^\top P_i^{\text{CL}} \overline{A}^{\text{CL}} x \\
&= \|x\|^2_{Q_i} + x^\top (\overline{A}^{\text{CL}} - B_i K_i^{\text{CL}})^\top P_i^{\text{CL}} \overline{A}^{\text{CL}} x \\
&\overset{(6.34\text{a})}{=} \|x\|^2_{P_i^{\text{CL}}}.
\end{aligned}$$

■

Proof of Theorem 6.3 We rewrite the minimization in (6.37a) as

$$\begin{aligned}
\min_{u_i \in \mathcal{S}^m_{T-1}} \ & \tfrac{1}{2}\Big\{ \textstyle\sum_{\tau=0}^{T-2} \|x[\tau]\|^2_{Q_i} + \|u_i[\tau]\|^2_{R_i} \\
& + \min_{u_i[T-1]\in\mathbb{R}^m} (\|x[T-1]\|^2_{Q_i} + \|u_i[T-1]\|^2_{R_i} \\
& + \|x[T]\|^2_{P_i^{\text{CL}}}) \Big\}.
\end{aligned} \tag{6.74}$$

By substituting the constraint (6.37c) in (6.74) and by applying Lemma 6.5, the inner minimization in (6.74) is solved by $u_i^*[T-1] = K_i^{\text{CL}}x[T-1]$. Substituting (6.73) in the latter, we obtain

$$\min_{u_i \in \mathcal{S}^m_{T-1}} \ \tfrac{1}{2}\big(\|x[T-1]\|^2_{P_i^{\text{CL}}} + \textstyle\sum_{\tau=0}^{T-2} \|x[\tau]\|^2_{Q_i} + \|u_i[\tau]\|^2_{R_i} \big).$$

By repeating the reasoning backwards in time and substituting the constraint (6.37b), (6.37a) becomes

$$\min_{u_i[0]\in\mathbb{R}^m} \ \tfrac{1}{2}\big(\|x[0]\|^2_{Q_i} + \|u_i[0]\|^2_{R_i} + \big\|Ax[0] + \textstyle\sum_{j\in\mathcal{I}} B_j u_j[0]\big\|^2_{P_i^{\text{CL}}} \big). \tag{6.75}$$

If $u_j[0] = K_j^{\text{CL}}x[0]$ for all $j \in \mathcal{I}_{-i}$, the minimum is obtained by $u_i^*[0] = K_i^{\text{CL}}x[0]$ following Lemma 6.5. Thus, $\boldsymbol{u}^*$ is a NE. ■

6

7

Probabilistic game-theoretic traffic routing

*"The worst of the plagues which truly defame Sicily and especially Palermo in the eyes of the world…well… you already understand what I'm talking about, there's no need for me to say it… I'm even ashamed to mention this…
It's the traffic! Too many cars! It's a tentacular, swirling traffic that prevents us from living and turns us all into enemies, family against family! Too many cars!"*

From the movie "Johnny Stecchino"

"There's an infinite number of monkeys outside who want to talk to us about this script for Hamlet they've worked out."

Douglas Adams, in "The hitchhiker's guide to the galaxy"

We examine the routing problem for self-interested vehicles using stochastic decision strategies. By approximating the road latency functions and a non-linear variable transformation, we frame the problem as an aggregative game. We characterize the approximation error, and we derive a new monotonicity condition for a broad category of games that encompasses the problem under consideration. Next, we propose a semi-decentralized algorithm to calculate the routing as a variational generalized Nash equilibrium and demonstrate the solution's benefits with numerical simulations. In the particular case of potential games, which emerges for linear latency functions, we explore a receding-horizon formulation of the routing problem, showing asymptotic convergence to destinations and analyzing closed-loop performance dependence on horizon length through numerical simulations.

This chapter is partly based on Benenati, E. and Grammatico, S. "Probabilistic Game-Theoretic Traffic Routing". In: *IEEE Transactions on Intelligent Transportation Systems*, 25(10):13080–13090, (Oct. 2024).

7.1 Introduction

Traffic jams generate a heavy burden on the society [121] and, as car traffic already makes up a large share of the EU transport infrastructure costs [122], it is imperative to mitigate the road congestion without expanding the existing infrastructure. The increased availability of real-time information on the state of the road network has the potential for a more efficient traffic-aware route planning.

Related work: Previous studies have considered routing strategies which optimize a system-wide efficiency metrics [123–125]. A shortcoming of this approach is that the drivers can find a more advantageous path than the one assigned and thus they might not adhere to such an externally-imposed solution. A workaround to this issue is to limit the inconvenience caused to the users [124, 125]. However, this approach still does not fully address the inherently competitive nature of the problem, which is more properly modeled as a *game*, as noted in the seminal work[126]. Crucially, under relatively loose conditions, games admit a set of Nash equilibria (or Wardrop equilibria, if the impact of each agent on the road latency is assumed negligible), that is, a set of decision strategies from which no agent has an incentive in unilaterally deviating and thus need no external imposition. Wardrop equilibium-based routing methods have been studied first in [126], and in [127] it is shown that they exhibit bounded system-level inefficiency. The inclusion of capacity constraints was considered in [128]. The Wardrop equilibrium of the routing problem is typically found by reformulating the problem as an equivalent optimization problem [129] or variational inequality [130], and capacity constraints can be handled by Lagrangian duality [131]. However, these reformulations require a pre-computation of every route available between each origin-destination pair, which can become cumbersome for large networks. By contrast, [132, 133] propose a Markov Decision Process (MDP) model that requires no pre-computation of the paths. This idea is further elaborated in [25], where the problem is cast as a (monotone, aggregative) generalized Nash equilibrium problem (GNEP). The latter approach is particularly appealing from a computational perspective, as recent developments in algorithmic game theory has made available a plethora of efficient Nash equilibrium-seeking algorithms which allow for a decentralized computation [17].

Contributions: Inspired by [25], we cast the vehicle routing problem as a GNEP. Our contribution is threefold:

- In Section 7.2, we demonstrate that commonly used routing objective functions, e.g. [25, 132, 133] and this work, are an approximation of the expected traversing time and we characterize the approximation error.
- In Section 7.3, we establish the monotonicity of the game under a less restrictive condition than the one derived in [25, Lemma 1]. Technically, we achieve this by carefully characterizing the eigenvalues of a class of matrices whose structure emerges in the pseudogradient's Jacobian (Lemma 7.8). We then solve the game via the inertial forward-reflected backward (I-FoRB) algorithm [17], which does not require the pseudo-gradient to be cocoercive as does the algorithm adopted in [25] (namely, the preconditioned forward-backward [35]) and thus it converges without a quadratic regularization term [25, Equation 5].

- In Section 7.4, we propose a modified formulation of the problem which allows one to progressively recompute the agents' paths in a receding horizon fashion (instead of solving for the entire path in one computation) in the particular case of a *potential* game [134]. This property of the game allows one to cast the receding horizon game as an model predictive control (MPC) controller, and thus we show asymptotic convergence by a careful choice of the terminal cost. This novel approach allows one to reduce the decision horizon, thus reducing the computational burden as the vehicles move forward.

Finally, in Section 7.5, we support the theoretical results by comparative numerical simulations.

Remarks on notation We denote the set of T-long sequences of vectors in $\mathbb{R}^n$ as $\mathcal{S}_T^n$. For $v \in \mathcal{S}_T^n$, we denote the t-th element of the sequence as $v[t]$. Analogously, we denote the set of T-long sequences of $n \times m$ real matrices as $\mathcal{S}_T^{n\times m}$. Matrices are denoted with capital letters. We denote the k-th element of a vector v by a subscript, e.g. v_k. Similarly, we denote the (a,b)-th element of a matrix M as $M_{(a,b)}$. For an indexed set of vectors, e.g. $(v_i)_{i\in\{1,...,N\}}$, we denote the k-th element of the i-th vector by separating the indeces with a comma, that is, $v_{i,k}$. Similarly, for the indexed set of matrices $(M_i)_{i\in\{1,...,N\}}$, the (a,b)-th element of the i-th matrix is denoted as $M_{i,(a,b)}$.

7.2 Traffic routing as a generalized Nash equilibrium problem

Let $\mathcal{R}(\mathcal{N},\mathcal{E})$ a directed graph modelling a road network whose nodes $\mathcal{N}$ represent the junctions and each edge $(a,b)\in\mathcal{E}$ represents the road from a to b. We study the problem of routing N populations of vehicles $\mathcal{I} := \{1,...,N\}$. Denote $\mathcal{I}_{-i} := \mathcal{I}\setminus\{i\}$ for all $i \in \mathcal{I}$. Each population is made up of V vehicles, where vehicles in the same population $i \in \mathcal{I}$ share the same initial position $b_i \in \mathcal{N}$ and destination $d_i \in \mathcal{N}$.

Remark 7.1. *Each population contains the same number of vehicles without loss of generality. In fact, let each population contain $(V_i)_{i\in\mathcal{I}}$ vehicles and let $V \in \mathbb{N}$ be such that $V_i/V \in \mathbb{N}$ for all i. Then, we can split each population i into V_i/V populations of equal size V.*

Next, we ensure that each destination node can be reached:

Assumption 7.1. *$\mathcal{R}(\mathcal{N},\mathcal{E})$ is strongly connected and $(a,a)\in\mathcal{E}$ for each $a \in \mathcal{N}$.*

The vehicles aim at reaching their destinations within a time horizon T. For convenience, let us denote $\mathcal{T} := \{0,...,T-1\}$. The control action determines the probability for the controlled vehicle to drive through a certain road and it is the same for each vehicle in a population. In this setting, each population acts as a single entity, thus, we refer to each of them as an *agent*. We stress that the route of each vehicle is a realization of the probabilistic control action, thus vehicles represented by the same agent might take different routes. To formalize this, let us denote the junction visited by the v-th vehicle of agent i at time t as $s_{i,v}[t]$, which is a stochastic variable with event space $\mathcal{N}$ and probability vector $\rho_i[t] \in \Delta^{|\mathcal{N}|}$, that is, $\rho_{i,a}[t] := \mathbb{P}\{s_{i,v}[t] = a\}$, where $\rho_{i,a}[t]$ denotes the a-th element of $\rho_{i,a}[t]$. The control action is the sequence of column-stochastic matrices $\Pi_i \in \mathcal{S}_T^{|\mathcal{N}|\times|\mathcal{N}|}$, where the (a,b)-th element of the matrix at time $t \in \mathcal{T}$ is defined as

$$\Pi_{i,(a,b)}[t] = \begin{cases} \mathbb{P}\{s_{i,v}[t+1] = b | s_{i,v}[t] = a\} & \text{if } (a,b) \in \mathcal{E} \\ 0 & \text{if } (a,b) \notin \mathcal{E}. \end{cases} \tag{7.1}$$

From the law of total probability,

$$\rho_i[t+1] = (\Pi_i[t])^\top \rho_i[t] \quad \text{for all } i \in \mathcal{I}. \tag{7.2}$$

The initial state of agent i is ρ_i^{in}, with only non-zero element $\rho_{i,b_i}^{\text{in}} = 1$. In the remainder of this section we show that, under an appropriate reformulation of (7.2), the problem that arises in the proposed setting can be cast as a generalized Nash equilibrium problem (GNEP).

7.2.1 Affine formulation of the system dynamics

Similarly to the approach in [135], we reformulate the nonlinear dynamics in (7.2) in terms of the transformed variables

7

$$M_{i,(a,b)}[t] := \Pi_{i,(a,b)}[t]\rho_{i,a}[t] \tag{7.3}$$

defined for all $i \in \mathcal{I}, (a,b) \in \mathcal{E}, t \in \mathcal{T}$. By the definition of conditional probability, we have

$$M_{i,(a,b)}[t] = \mathbb{P}\{s_{i,v}[t+1] = b, s_{i,v}[t] = a\}. \tag{7.4}$$

In words, $M_{i,(a,b)}[t]$ represents the probability that, at time t, agent i traverses the road from a to b. Denoting $\mathcal{T}^+ := \{1, ..., T\}$, the decision variable of each agent is:

$$\omega_i := \begin{bmatrix} \mathrm{col}(M_{i,(a,b)}[t])_{(a,b)\in\mathcal{E}, t\in\mathcal{T}} \\ \mathrm{col}(\rho_i[t])_{t\in\mathcal{T}^+} \end{bmatrix}. \tag{7.5}$$

Without loss of generality, ω_i in (7.5) does not include any variable corresponding to $\Pi_{i,(a,b)}[t]$ with $(a,b) \notin \mathcal{E}$, since the probability of traversing a non-existing road is zero. We denote in boldface the concatenation over $\mathcal{I}$ and with boldface and indexing $-i$ the concatenation over $\mathcal{I}_{-i}$, e.g. $\boldsymbol{\omega} := \mathrm{col}(\omega_i)_{i\in\mathcal{I}}$, $\boldsymbol{\omega}_{-i} := \mathrm{col}(\omega_j)_{j\in\mathcal{I}_{-i}}$. We also define $n_\omega := T(|\mathcal{E}| + |\mathcal{N}|)$, and we note $\omega_i \in \mathbb{R}^{n_\omega}$ for all $i \in \mathcal{I}$. The following lemma states that, by imposing appropriate linear constraints on $\boldsymbol{\omega}$, the transformation in (7.3) can be inverted and the resulting matrix sequences $(\Pi_i)_{i\in\mathcal{I}}$ are coherent with the dynamics in (7.2). All proofs are provided in the Appendix.

Lemma 7.1. *For each $i \in \mathcal{I}$, let ω_i in (7.5) satisfy:*

$$\sum_{a:(a,b)\in\mathcal{E}} M_{i,(a,b)}[t] = \rho_{i,b}[t+1] \qquad \textit{for all } b \in \mathcal{N},\ t \in \mathcal{T}; \tag{7.6a}$$

$$\sum_{b:(a,b)\in\mathcal{E}} M_{i,(a,b)}[t] = \rho_{i,a}[t] \qquad \textit{for all } a \in \mathcal{N},\ t \in \{1,...,T-1\}; \tag{7.6b}$$

$$\sum_{b:(a,b)\in\mathcal{E}} M_{i,(a,b)}[0] = \rho^{\text{in}}_{i,a} \qquad \textit{for all } a \in \mathcal{N}; \tag{7.6c}$$

$$M_{i,(a,b)}[t] \geq 0 \qquad \textit{for all } (a,b) \in \mathcal{E},\ t \in \mathcal{T}. \tag{7.6d}$$

Then, $\omega_i \in (\Delta^{|\mathcal{E}|})^T \times (\Delta^{|\mathcal{N}|})^T$ and a choice of Π_i such that ρ_i follows the dynamics in (7.2) is:

$$\Pi_{i,(a,b)}[t] = \begin{cases} \frac{1}{|\mathcal{N}|} & \textit{if } \rho_{i,a}[t] = 0 \\ \frac{M_{i,(a,b)}[t]}{\rho_{i,a}[t]} & \textit{if } \rho_{i,a}[t] \neq 0 \end{cases} \tag{7.7}$$

for all $(a,b) \in \mathcal{E}, t \in \{1,...,T-1\}$, and

$$\Pi_{i,(a,b)}[0] = \begin{cases} \frac{1}{|\mathcal{N}|} & \textit{if } \rho^{\text{in}}_{i,a} = 0 \\ \frac{M_{i,(a,b)}[t]}{\rho^{\text{in}}_{i,a}} & \textit{if } \rho^{\text{in}}_{i,a}[t] \neq 0 \end{cases} \tag{7.8}$$

Note that, in (7.7) and (7.8), the a-th columns of $\Pi_i[t]$ such that $\rho_{i,a}[t] = 0$ and $\rho^{\text{in}}_{i,a} = 0$ can be chosen to be anything that sums to 1, as those values do not influence the evolution of the vehicle distribution.

7

7.2.2 Control objective and constraints

We enforce the routing of each agent by constraining the destination node d_i to be reached with high probability:

$$\rho_{i,d_i}[T] \geq 1 - \varepsilon, \qquad \forall i \in \mathcal{I}, \tag{7.9}$$

where ε is a free design parameter. Let us introduce for each $(a,b) \in \mathcal{E}$ the latency function $\ell_{(a,b)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which maps the ratio of vehicles on a road to its traversing time. A common model is the Bureau of Public Transport (BPT) function [136]:

$$\ell^{\text{BPT}}_{(a,b)}(\sigma) := \tau_{(a,b)} \left(1 + 0.15 \left(\frac{\sigma + \zeta_{(a,b)}}{c_{(a,b)}} \right)^{\xi+1} \right), \tag{7.10}$$

where $c_{(a,b)}$ and $\tau_{(a,b)}$ are the capacity and the free-flow traversing time of (a,b), respectively, $\zeta_{(a,b)} \geq 0$ is the number of uncontrolled vehicles on the road normalized by VN and $\xi \geq 0$ is a parameter often set to $\xi = 3$, e.g. [125, 136]. More generally, we consider functions that satisfy the following:

Assumption 7.2. *For each $(a,b) \in \mathcal{E}$, the latency function $\ell_{(a,b)}$ is C^2, non-negative, non-decreasing and convex.*

The number of vehicles traversing road (a,b) at time t is drawn from a Poisson's binomial distribution with NV trials, grouped into N groups of V trials with identical success probability $M_{i,(a,b)}[t]$, where $i \in \mathcal{I}$ indexes the N groups. Thus, its expected value is $\sum_{i\in\mathcal{I}} VM_{i,(a,b)}[t]$ [137, Eq. 15] and the expected ratio of vehicles on (a,b) is

$$\sigma^{\mathrm{M}}_{(a,b)}[t] := \operatorname{avg}(M_{i,(a,b)}[t])_{i\in\mathcal{I}}.$$

Let $\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]$ be the (actual) ratio of vehicles on road (a,b) at time t. The expected traversing time is $\mathbb{E}[\ell_{(a,b)}(\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t])]$. This is in general intractable to compute: let us instead consider the first-order approximation of $\ell_{(a,b)}$ around the expected value of the argument.

$$\begin{aligned}\mathbb{E}\left\{\ell_{(a,b)}(\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t])\right\} &\simeq \mathbb{E}\left\{\ell_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t]) + \ell'_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t])(\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t] - \sigma^{\mathrm{M}}_{(a,b)}[t])\right\} \\ &\overset{\{1\}}{=} \ell_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t]) + \ell'_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t])\left(\mathbb{E}\left\{\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]\right\} - \sigma^{\mathrm{M}}_{(a,b)}[t]\right) \\ &\overset{\{2\}}{=} \ell_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t])\end{aligned} \tag{7.11}$$

where in (7.11), $\{1\}$ follows from the linearity of the expected value and from the fact that $\sigma^{\mathrm{M}}_{(a,b)}[t]$ is deterministic, while $\{2\}$ follows from $\mathbb{E}\left\{\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]\right\} = \sigma^{\mathrm{M}}_{(a,b)}[t]$. Although nonlinear functions of the congestion were previously used as road traversing cost [25, 132, 133], the interpretation provided by (7.11) is novel, to the best of our knowledge. To justify the approximation in (7.11), we leverage known results on the Taylor series of stochastic functions [138, §6] in order to show that the approximation error vanishes with the number of vehicles NV:

Proposition 7.1. *Let each vehicle v in agent i draw the event $(s_{i,v}[t]=a, s_{i,v}[t+1]=b)$ with probability $M_{i,(a,b)}[t]$ independently from the remaining vehicles. Then,*

$$\left(\ell_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t]) - \ell_{(a,b)}(\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t])\right)^2 = y_{\mathrm{NV}} + z_{\mathrm{NV}},$$

where

$$\mathbb{E}\{y_{\mathrm{NV}}\} \leq \frac{1}{4NV}\ell'_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)}[t])^2$$

and, for every $\epsilon > 0$, there exists $K_\epsilon > 0$ such that

$$\sup_{N,V\in\mathbb{N}} \left(\mathbb{P}\left\{|z_{\mathrm{NV}}| \geq \frac{K_\varepsilon}{8(NV)^{3/2}}\right\}\right) \leq \epsilon.$$

We now define the cost of traversing (a,b) at time t as

$$J_{(a,b)}\left(\boldsymbol{M}_{(a,b)}[t]\right) := M_{i,(a,b)}\ell_{(a,b)}\left(\sigma^{\mathrm{M}}_{(a,b)}[t]\right). \tag{7.12}$$

The objective pursued by each agent reads then as follows:

$$J_i(\omega_i, \boldsymbol{\omega}_{-i}) := f_i(\omega_i) + \sum_{\substack{(a,b)\in\mathcal{E}\\ t\in\mathcal{T}}} J_{(a,b)}\left(\boldsymbol{M}_{(a,b)}[t]\right), \tag{7.13}$$

where $f_i : \mathbb{R}^{n_\omega} \to \mathbb{R}$ encodes a local cost for agent i. Quadratic costs are considered in [25, Eq. 5]. We consider a more general class of functions.

Assumption 7.3. *The functions* $(f_i)_{i\in\mathcal{I}}$ *in* (7.13) *are convex and belong to the class* $\mathcal{C}^2$.

Finally, we introduce the maximum capacity constraints

$$\sum_{i\in\mathcal{I}} M_{i,(a,b)}[t] \le \overline{c}_{(a,b)} \quad \text{for all } t\in\mathcal{T},(a,b)\in\mathcal{E} \tag{7.14}$$

which we recast via appropriately defined matrices $(A_i)_{i\in\mathcal{I}}$, $A_i\in\mathbb{R}^{T|\mathcal{E}|\times n_\omega}$, $b\in\mathbb{R}^{T|\mathcal{E}|}$, $A := \mathrm{row}(A_i)_{i\in\mathcal{I}}$:

$$\sum_{i\in\mathcal{I}} A_i\omega_i = A\boldsymbol{\omega} \le b. \tag{7.15}$$

7.2.3 Generalized Nash equilibrium problem

We frame the model derived in Sections 7.2.1 and 7.2.2 to the concept of GNEP introduced in Section 2.3.1. Each agent solves the local optimization problem

$$\forall i\in\mathcal{I}: \begin{cases} \min\limits_{\omega_i\in\Omega_i} & J_i(\omega_i,\boldsymbol{\omega}_{-i}) & (7.16a)\\ \text{s.t.} & A_i\omega_i \le b - \sum\limits_{j\in\mathcal{I}_{-i}} A_j\omega_j, & (7.16b)\end{cases}$$

where $\Omega_i := \{\omega\in\mathbb{R}^{n_\omega}|(7.6),(7.9) \text{ hold}\}$ for all i. The coupling between the N optimizations problems in (7.16) emerges both in the cost functions and in the constraints, thus defining a *generalized game* [3]. In particular, the game is *aggregative* because the coupling between cost functions depends only on the average of the decision variables $\sigma^{\mathrm{M}}_{(a,b)}[t]$, for all (a,b) and t. A desirable solution is the generalized Nash equilibrium (GNE) $\boldsymbol{\omega}^*$, from which no agent has an incentive to unilaterally deviate.

Definition 7.1. *A collective strategy*

$$\boldsymbol{\omega}^*\in\boldsymbol{\Omega} := \left(\bigtimes_{i\in\mathcal{I}}\Omega_i\right)\bigcap\{\boldsymbol{\omega}\in\mathbb{R}^{Nn_\omega}|A\boldsymbol{\omega}\le b\}$$

is a generalized Nash equilibrium for the game in (7.16) *if, for each* $i\in\mathcal{I}$,

$$J_i\left(\omega_i^*,\boldsymbol{\omega}_{-i}^*\right) \le J_i\left(\omega_i,\boldsymbol{\omega}_{-i}^*\right)$$

for any $\omega_i\in\Omega_i\bigcap\{y\in\mathbb{R}^{n_\omega}|A_iy\le b-\sum_{j\in\mathcal{I}_{-i}}A_j\omega_j^*\}$.

7.3 Generalized Nash equilibrium seeking

We now turn our attention to the derivation of a distributed algorithm to find a GNE of the problem in (7.16). Let us formulate the following feasibility assumption:

Assumption 7.4. *The set* $\boldsymbol{\Omega}$ *is non-empty and it satisfies Slater's constraint qualification [51, Eq. 27.50].*

7

$\boldsymbol{\Omega}$ in Definition 7.1 is compact and convex because defined by linear equations. Furthermore, the local cost functions are convex, as formalized next:

Lemma 7.2. *Let Assumptions 7.2, 7.3 hold. For each* $i \in \mathcal{I}$, $J_i(\omega_i, \boldsymbol{\omega}_{-i})$ *is convex in* $\omega_i \in \Omega_i$ *for all* $\boldsymbol{\omega}_{-i} \in \bigtimes_{j \in \mathcal{I}_{-i}} \Omega_j$.

Under Assumption 7.4 and Lemma 7.2, we conclude that a GNE exists [57, Prop. 12.11] if the game mapping

$$F(\boldsymbol{\omega}) := \operatorname{col}\big(\nabla_{\omega_i} J_i(\omega_i, \boldsymbol{\omega}_{-i})\big)_{i \in \mathcal{I}} \tag{7.17}$$

is monotone. Monotonicity of the game mapping is one of the mildest conditions under which effective GNE seeking algorithms can be derived. The authors of [25] show that monotonicity of the game defined using $\ell^{\mathrm{BPT}}_{(a,b)}$ in (7.10) for all (a,b) holds if enough non-controlled vehicles populate the roads [25, Eq. 18], with the caveat that this number must increase proportionally with the number of controlled vehicles. This might not be reasonable if the share of controlled vehicles is large. In the following, we derive a milder condition.

7.3.1 Monotonicity of the game

To study the monotonicity of the proposed formulation, let us define for each $(a,b) \in \mathcal{E}$ the operators $F_{(a,b)} : \mathbb{R}^N \to \mathbb{R}^N$ as

$$F_{(a,b)}(\boldsymbol{y}) := \operatorname{col}\left(\nabla_{y_i} J_{(a,b)}(\boldsymbol{y})\right)_{i \in \mathcal{I}} \tag{7.18}$$

where we compute

$$\nabla_{y_i} J_{(a,b)}(\boldsymbol{y}) = \ell_{(a,b)}\big(\operatorname{avg}(\boldsymbol{y})_{i \in \mathcal{I}}\big) + \tfrac{1}{N} y_i \ell'_{(a,b)}\big(\operatorname{avg}(\boldsymbol{y})_{i \in \mathcal{I}}\big). \tag{7.19}$$

We now link the monotonicity of F to that of each $F_{(a,b)}$.

Lemma 7.3. *The operator* F *in* (7.17) *is monotone if* $F_{(a,b)}$ *in* (7.18) *is monotone for each* $(a,b) \in \mathcal{E}$.

We find a monotonicity condition for the particular class of $\ell_{(a,b)}$ described via the following assumption. Note that this includes $\ell^{\mathrm{BPT}}_{(a,b)}$ in (7.10) for large enough $\zeta_{(a,b)}$, and that Assumption 7.2 is implied by Assumption 7.5.

Assumption 7.5. *For all* $(a,b) \in \mathcal{E}$, $\ell_{(a,b)}$ *in* (7.12) *is in the form*

$$\ell_{(a,b)}(\sigma) = \tau_{(a,b)} + \tfrac{k_{(a,b)}}{\xi+1}(\sigma + \zeta_{(a,b)})^{\xi+1}$$

where ξ, $\tau_{(a,b)}$, $k_{(a,b)}, \zeta_{(a,b)} \in \mathbb{R}_{\geq 0}$ *and* $\zeta_{(a,b)}$ *satisfies*

$$\zeta_{(a,b)} \geq \max\left(\tfrac{\xi^2-8}{8N}, \tfrac{\xi-2}{2N}\right). \tag{7.20}$$

Lemma 7.4. *Let* $\ell_{(a,b)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *satisfy Assumption 7.5. Then,* $F_{(a,b)}$ *defined in* (7.18) *is monotone on* $[0,1]^N$.

Remark 7.2. (7.20) *is satisfied for any* $\zeta_{(a,b)} \geq 0$ *whenever* $\xi \leq 2$.

Remark 7.3. *Let us compare the condition in* (7.20) *with the previously known monotonicity condition derived in [25] for the latency function* $\ell^{\mathrm{BPT}}_{(a,b)}$ *with* $\xi = 3$. *Following* (7.20) *the game is monotone if* $\zeta_{(a,b)} \geq \frac{1}{2N}$ *for all* (a,b), *which is satisfied if at least* $\frac{V}{2}$ *uncontrolled vehicles traverse each road. By contrast, taking into account that [25] considered* $V = 1$, *the bound in [25, Eq. 18] requires* $\frac{3NV}{8}$ *uncontrolled vehicles on each road when translated to our setting. By applying Lemma 7.4, one can then find more general conditions for the convergence of [25, Alg. 1] than the ones specified in [25, Thm. 1].*

As $\boldsymbol{\Omega} \subset [0,1]^{Nn_\omega}$ by Lemma 7.1, the following result is immediate by Lemmas 7.3 and 7.4:

Lemma 7.5. *Under Assumption 7.5,* F *in* (7.17) *is monotone on* $\boldsymbol{\Omega}$.

Lemma 7.5 is fundamental for guaranteeing the convergence of the GNE-seeking algorithm proposed in Section 7.3.2.

7.3.2 Semi-decentralized equilibrium seeking

To solve the game in (7.16), we focus on the computation of a variational GNE (v-GNE) [3, Def. 3.10], that is, the subset of GNEs which satisfy the Karush–Kuhn–Tucker (KKT) conditions

$$\begin{bmatrix} \boldsymbol{\omega} \\ \lambda \end{bmatrix} \in \mathrm{zer}\left(\begin{bmatrix} \times_{i\in\mathcal{I}} \mathrm{N}_{\Omega_i}(\omega_i) + F(\boldsymbol{\omega}) + \mathrm{col}(A_i^\top \lambda)_i \\ \mathrm{N}_{\mathbb{R}^{|\mathcal{E}|}_{\geq 0}}(\lambda) - A\boldsymbol{\omega} + b \end{bmatrix} \right) \tag{7.21}$$

where $\lambda \in \mathbb{R}^{|\mathcal{E}|}_{\geq 0}$ is the dual variable associated to the shared constraints in (7.14). The v-GNEs have desirable characteristics of fairness between agents and there exist several efficient algorithms for their computation. In particular, we adopt the Inertial Forward-Reflected-Backward (inertial forward-reflected backward (I-FoRB)) algorithm [17], for its convergence speed and low computational complexity. The I-FoRB algorithm converges in the general class of (non-strictly) monotone games. On the contrary, the algorithm proposed in [25] converges only if the game is strongly monotone, thus an additive quadratic cost is necessary, which is not needed in our model. The agents perform a *reflected* projected-gradient descent of the Lagrangian function with an inertial term (7.22b). Then, the agents communicate the primal variable and auxiliary variables $\boldsymbol{d}$ to the aggregator. In turn, the aggregator updates the aggregate variable and the dual variable via a reflected dual ascent with inertia (7.23b) and communicates them to the agents. We now state the main result of this section, where we denote by L_i^f the Lipschitz constant of f_i for all $i \in \mathcal{I}$ (which exist following Assumption 7.3 and the compactness of $\boldsymbol{\Omega}$):

Algorithm 10 I-FoRB GNE seeking for traffic routing

Initialization. $\forall i \in \mathcal{I}$ set $\omega_i^{(0)}, \omega_i^{(1)} \in \mathbb{R}^{n_\omega}$; $\lambda^{(1)}, \lambda^{(0)} \in \mathbb{R}^{|\mathcal{E}|}_{\geq 0}$.
For $k \in \mathbb{N}$:

1. Each agent $i \in \mathcal{I}$ receives $\sigma^{(k)}$, $\lambda^{(k)}$ and computes:

$$r_i^{(k)} = 2\nabla_{\omega_i} J_i(\omega_i^{(k)}, \boldsymbol{\omega}_{-i}^{(k)}) - \nabla_{\omega_i} J_i(\omega_i^{(k-1)}, \boldsymbol{\omega}_{-i}^{(k-1)}) \tag{7.22a}$$

$$\omega_i^{(k+1)} = \mathrm{proj}_{\Omega_i}\left(\omega_i^{(k)} - \alpha_i(r_i^{(k)} + A_i^\top \lambda^{(k)}) + \theta(\omega_i^{(k)} - \omega_i^{(k-1)})\right) \tag{7.22b}$$

$$d_i^{(k+1)} = 2A_i\omega_i^{(k+1)} - A_i\omega_i^{(k)} - b_i. \tag{7.22c}$$

2. The aggregator receives $(\boldsymbol{\omega}^{(k+1)}, \boldsymbol{d}^{(k+1)})$ and computes:

$$\sigma^{(k+1)} = \mathrm{avg}(\omega_i^{(k+1)})_{i \in \mathcal{I}} \tag{7.23a}$$

$$\lambda^{(k+1)} = \mathrm{proj}_{\mathbb{R}^{|\mathcal{E}|}_{\geq 0}}\left(\lambda^{(k)} + \beta \mathrm{avg}(d_i^{(k+1)})_{i \in \mathcal{I}} + \theta(\lambda^{(k)} - \lambda^{(k-1)})\right) \tag{7.23b}$$

Proposition 7.2. *Let Assumptions 7.3, 7.4, 7.5 hold and let*

$$\begin{aligned}
L_{(a,b)} &\geq \tfrac{k_{(a,b)}}{N}\left((1+\zeta_{(a,b)})^\xi + \xi(1+\zeta_{(a,b)})^{\xi-1}\right) \\
L &\geq \max_{i \in \mathcal{I}}(L_i^f) + \max_{(a,b)\in\mathcal{E}}(L_{(a,b)}) \\
\theta &\in [0, \tfrac{1}{3}), \\
\delta &> 2L/(1-3\theta).
\end{aligned}$$

Then, $(\boldsymbol{\omega}^{(k)}, \lambda^{(k)})_{k \in \mathbb{N}}$ *generated by Algorithm 10 with stepsizes*

$$\begin{aligned}
&0 < \alpha_i \leq \left(\|A_i\| + \delta\right)^{-1} && \textit{for all } i \in \mathcal{I} \\
&0 < \beta \leq N\left(\textstyle\sum_{i=1}^N \|A_i\| + \delta\right)^{-1}
\end{aligned}$$

converges to $\mathrm{col}(\boldsymbol{\omega}^*, \lambda^*)$ *where* $\boldsymbol{\omega}^*$ *is a v-GNE of* (7.16).

7.4 Receding horizon formulation

Due to the constraint in (7.9), the problem in (7.16) admits a solution only if, for all i, the destination d_i is reachable from the starting node b_i in T steps. In common applications, choosing a large enough T that guarantees feasibility might lead to a heavy computational burden. In this section, we propose an alternative formulation of the problem in (7.16) without the constraint in (7.9) as N coupled Finite Horizon Optimal Control Problems (FHOCPs), which we label Finite Horizon Dynamic Game (FHDG). Inspired by the Model Predictive Control (model predictive control (MPC)) literature, we propose to repeatedly solve the FHDG in receding horizon and to iteratively apply the first input of the computed sequence. The agents are directed to their destinations by means of a terminal cost which penalizes the distance from their destination. Remarkably, the introduction of a terminal cost is reminiscent of the classic stability requirements of MPC [12, Sec. 2].

7.4.1 Equivalent finite horizon optimal control problem

Let us formalize the game under consideration, parametrized in the initial distributions $(\rho_i^{\text{in}})_{i\in\mathcal{I}}$ through (7.6c):

$$\text{for all } i \in \mathcal{I}: \min_{\omega_i \in \mathcal{Y}_i(\rho_i^{\text{in}})} J_i(\omega_i, \boldsymbol{\omega}_{-i}) \tag{7.24}$$

where $\mathcal{Y}_i(\rho_i^{\text{in}}) := \{\omega_i \in \mathbb{R}^{n_\omega} | (7.6)\}$. We emphasize that we do not include the constraint in (7.14): due to the probabilistic control action, an unlucky realization might lead the constraint (7.14) to be unfeasible at the successive time steps. Instead, $\mathcal{Y}_i(\rho_i^{\text{in}})$ is non-empty for any ρ_i^{in}. In this section, we show the equivalence of the problem in (7.24) to a FHOCP. As a first step, we rewrite the equations defining $\mathcal{Y}_i$ as the state-space representation of a constrained linear system. We define the desired distribution ρ_i^{eq} as the one where every vehicle is at their destination, that is, $[\rho_i^{\text{eq}}]_a = \delta_{d_i}(a)$, where δ_{d_i} is a Kronecker delta centered in d_i, and the associated equilibrium input $u_i^{\text{eq}} := \text{col}(\delta_{d_i}(a)\delta_{d_i}(b))_{(a,b)\in\mathcal{E}}$, that is, the vector of edge transitions associated to remaining in the destination node d_i with probability 1. Let the state dimension $n := |\mathcal{N}|$ and the input dimension $m := |\mathcal{E}|$. We define the state and input sequences $x_i \in \mathcal{S}^n_{T+1}$ and $u_i \in \mathcal{S}^m_T$, respectively, as follows:

$$x_i[0] := \rho_i^{\text{in}} - \rho_i^{\text{eq}}, \tag{7.25a}$$

$$x_i[t] := \rho_i[t] - \rho_i^{\text{eq}}, \qquad \forall t \in \mathcal{T}^+; \tag{7.25b}$$

$$u_i[t] := \text{col}(M_{i,(a,b)}[t])_{(a,b)\in\mathcal{E}} - u_i^{\text{eq}}, \qquad \forall t \in \mathcal{T}. \tag{7.25c}$$

This definition ensures that the origin coincides with the desired distribution and equilibrium input. We define the selection vectors $S^{\text{edge}}_{(a,b)} \in \mathbb{R}^m$ for all $(a,b) \in \mathcal{E}$ such that $(S^{\text{edge}}_{(a,b)})^\top (u_i[t] + u_i^{\text{eq}}) = M_{i,(a,b)}[t]$, as well as

$$B := \text{col}\left(\sum_{a:(a,b)\in\mathcal{E}} (S^{\text{edge}}_{(a,b)})^\top\right)_{b\in\mathcal{N}}$$

$$P := \text{col}\left(\sum_{b:(a,b)\in\mathcal{E}} (S^{\text{edge}}_{(a,b)})^\top\right)_{a\in\mathcal{N}}.$$

It can be verified that $Bu_i^{\text{eq}} = Pu_i^{\text{eq}} = \rho_i^{\text{eq}}$ and thus, by substituting the definitions of B and P in (7.6a) and (7.6b):

$$(7.6\text{a}) \Leftrightarrow x_i[t+1] = Bu_i[t] \qquad \forall t \in \mathcal{T}; \tag{7.26a}$$

$$(7.6\text{b}),(7.6\text{c}) \Leftrightarrow x_i[t] = Pu_i[t] \qquad \forall t \in \{0,...,T\}. \tag{7.26b}$$

The desired state (that is, the origin) is an equilibrium under the input $u_i[t]=0$ which, following (7.25c) and the definition of u_i^{eq}, corresponds to $M_{i,(d_i,d_i)}[t]=1$. We ensure that such action is cost-free for all agents (and thus, trivially, a Nash equilibrium (NE) strategy) when the initial state is the origin by assuming that the self-loops have no traversing cost. Another restriction on the cost functions is supported by the following lemma, which states that $\ell_{(a,b)}$ must be affine for the game in (7.16) to be potential [134, Sec. 2]. This property is crucial to rewrite (7.24) as a single optimization problem.

Lemma 7.6. *Let Assumption 7.2 hold. The game in* (7.24) *is potential if and only if* $\ell_{(a,b)}$ *in* (7.12) *is affine for each* $(a,b)\in\mathcal{E}$*.*

In view of these requirements, we formulate the following assumption (which implies Assumption 7.5):

Assumption 7.6. *For each* $(a,b)\in\mathcal{E}$*, the congestion function* $\ell_{(a,b)}$ *is of the form*

$$\ell_{(a,b)}(\sigma)=\tau_{(a,b)}+k_{(a,b)}\sigma,$$

with $\tau_{(a,a)}=0$*,* $k_{(a,a)}=0$ *and* $\tau_{(a,b)}>0$ *for all* $a\in\mathcal{N}$*,* $b\in\mathcal{N}\setminus\{a\}$*.*

Linear(ized) edge traversing costs in traffic routing problems are considered in [125, 132] and in the numerical analysis of [25]. Naturally, Assumption 7.6 implies that an optimal policy for each agent is to remain at their initial state. We then impose that the agents cannot choose to remain still in a node, unless that node is their destination:

7

$$(S^{\mathrm{edge}}_{(a,a)})^\top u_i[t]=0 \quad \forall\, a\in\mathcal{N}\setminus\{d_i\},\ i\in\mathcal{I},\ t\in\mathcal{T}. \tag{7.27}$$

We can compactly rewrite the set $\mathcal{Y}_i(\rho_i^{\mathrm{in}})$ in (7.24) with the additional constraint (7.27) as the dynamics in (7.26a) with constraint set $\mathbb{Z}_i$ defined as

$$\mathbb{X}_i := \Delta^{|\mathcal{N}|}-\{\rho_i^{\mathrm{eq}}\}, \tag{7.28a}$$

$$\mathbb{U}_i := \big\{u\in\mathbb{R}^{|\mathcal{E}|}_{\geq 0}-\{u_i^{\mathrm{eq}}\} | (S^{\mathrm{edge}}_{(a,a)})^\top u=0\ \forall a\in\mathcal{N}\setminus\{d_i\}\big\}, \tag{7.28b}$$

$$\mathbb{Z}_i := \big\{(x,u)\in\mathbb{X}_i\times\mathbb{U}_i | u\in\mathrm{null}(P)+\{P^\dagger x\}\big\}. \tag{7.28c}$$

As a next step towards the formulation of the multi-stage decision problem, we rewrite the objective functions $(J_i)_{i\in\mathcal{I}}$ in terms of stage cost, that is, as a sum of terms which only depends on the variables at time step $t\in\mathcal{T}$. In the remainder of this section, we overload the notation $(J_i)_{i\in\mathcal{I}}$ and $(f_i)_{i\in\mathcal{I}}$ to accept $\boldsymbol{x}[0]$ and $\boldsymbol{u}$ as arguments (oppositely to $\boldsymbol{\omega}$ as in section 7.2). Note that this redefinition only entails computing the solution of the dynamics in (7.26a) and translating the state and inputs, as for (7.25b), (7.25c). We make the following technical assumption, which is typical of multi-stage decision problems:

Assumption 7.7. *The local cost* f_i *is separable in* t*, that is,*

$$f_i(x_i[0],u_i)=\sum_{t\in\mathcal{T}} f_i^{\mathrm{S}}(x_i[t],u_i[t])+f_i^{\mathrm{T}}(x_i[T]).$$

Furthermore, the stage cost f_i^{S} *and the terminal cost* f_i^{T} *are non-negative and* $f_i^{\mathrm{S}}(0,0) = f_i^{\mathrm{T}}(0) = 0$*.*

Let us collect the network parameters in

$$\overline{\tau} := \mathrm{col}(\tau_{(a,b)})_{(a,b)\in\mathcal{E}}$$

and in the matrix C with (a,b)-th element:

$$C_{(a,b)} := \begin{cases} k_{(a,b)} & \text{if } (a,b) \in \mathcal{E} \\ 0 & \text{if } (a,b) \notin \mathcal{E}. \end{cases}$$

It can then be shown (see Appendix 7.C) that under Assumptions 7.6, 7.7, J_i in (7.24) can be written in terms of stage costs as

$$J_i(x_i[0],\boldsymbol{u}) = f_i^{\mathrm{T}}(x_i[T]) + \sum_{t\in\mathcal{T}} f_i^{\mathrm{S}}(x_i[t],u_i[t]) + \overline{\tau}^\top u_i[t] + \mathrm{avg}(\boldsymbol{u}[t])_{i\in\mathcal{I}}^\top C u_i[t]. \tag{7.29}$$

We then formulate the FHDG $\mathcal{G}(\boldsymbol{x}[0])$ for any initial state $\boldsymbol{x}[0] \in \bigtimes_{i\in\mathcal{I}} \mathbb{X}_i$:

$$\forall i: \begin{cases} \min\limits_{u_i\in\mathcal{S}_T^m} J_i\big(x_i[0],\boldsymbol{u}\big) & \text{(7.30a)} \\ \text{s.t.} \quad (x_i[t],u_i[t]) \in \mathbb{Z}_i \qquad \forall t \in \mathcal{T}, & \text{(7.30b)} \end{cases}$$

which is equivalent to (7.24) with the additional constraint in (7.27). Note that, in (7.30), the optimization problems are only defined over $\boldsymbol{u}$ as the variables $(\boldsymbol{x}[t])_{t\in\mathcal{T}^+}$ are eliminated by computing the solution of the dynamical system in (7.26a).

The collective state variable $\boldsymbol{x}[t]$ evolves according to the collective dynamics

$$\boldsymbol{x}[t+1] = (I_N \otimes B)\boldsymbol{u}[t].$$

Define the constraint sets for the collective states and inputs:

$$\begin{aligned} \mathbb{X} &:= \bigtimes_{i\in\mathcal{I}} \mathbb{X}_i; \\ \mathbb{U} &:= \bigtimes_{i\in\mathcal{I}} \mathbb{U}_i; \\ \mathbb{Z} &:= \big\{(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{X}\times\mathbb{U} | (x_i,u_i) \in \mathbb{Z}_i \ \text{ for all } i\big\}. \end{aligned}$$

In the following Lemma 7.7 we find a common control objective p that the agents unknowingly, but willingly, aim at minimizing when solving the game $\mathcal{G}$:

Lemma 7.7. *For any* $\boldsymbol{x}[0] \in \mathbb{X}$*, the pseudogradient of* $\mathcal{G}(\boldsymbol{x}[0])$ *in* (7.30) *is equal to* $\nabla_{\boldsymbol{u}} p$*, where*

$$p^{\mathrm{S}}(\boldsymbol{x}[t],\boldsymbol{u}[t]) := \tfrac{1}{2N}\|\boldsymbol{u}[t]\|^2_{(I_N+\mathbf{1}\mathbf{1}^\top)\otimes C} + \sum_{i\in\mathcal{I}} f_i^{\mathrm{S}}(x_i[t],u_i[t]) + \overline{\tau}^\top u_i[t] \tag{7.31a}$$

$$p^{\mathrm{T}}(\boldsymbol{x}[T]) := \sum_{i\in\mathcal{I}} f_i^{\mathrm{T}}(x_i[T]) \tag{7.31b}$$

$$p(\boldsymbol{x}[0],\boldsymbol{u}) := p^{\mathrm{T}}(\boldsymbol{x}[T]) + \sum_{t\in\mathcal{T}} p^{\mathrm{S}}(\boldsymbol{x}[t],\boldsymbol{u}[t]).$$

7

The potential function p allows us to conclude that, by Lemma 7.7 and [139, Theorem 2], a NE of $\mathcal{G}(\boldsymbol{x}[0])$ is a solution to the FHOCP $\mathcal{O}(\boldsymbol{x}[0])$, defined as

$$\begin{cases} \min\limits_{\boldsymbol{u}\in\mathcal{S}_T^{Nm}} & p\big(\boldsymbol{x}[0],\boldsymbol{u}\big) \qquad (7.32a) \\ \quad \text{s.t.} & (\boldsymbol{x}[t],\boldsymbol{u}[t]) \in \mathbb{Z} \quad \forall t \in \mathcal{T}. \qquad (7.32b) \end{cases}$$

We now derive conditions for the asymptotic stability of the receding horizon solution of (7.32), and in turn of (7.30), via standard MPC results.

7.4.2 Stability of receding horizon Nash equilibrium control

At every time-step, the agents apply the first input corresponding to a NE of the game in (7.30). This is formalized via the following control actions:

$$\kappa_i : y \mapsto u_i^*[0], \text{ where } \boldsymbol{u}^* \in \mathcal{S}_T^{Nm} \text{ is a NE of } \mathcal{G}(y). \tag{7.33}$$

Intuitively, κ_i leads the i-th agent to the desired equilibrium if the agents have a high enough incentive to approach their destinations. For this purpose, let us assume that each agent knows a path to its destination, formalized by the mappings $(\mathrm{KP}_i)_{i\in\mathcal{I}} : \mathcal{N} \to \mathcal{N}$ with the following characteristics:

$$\begin{aligned} &\mathrm{KP}_i(d_i) = d_i; \\ &(a, \mathrm{KP}_i(a)) \in \mathcal{E}, \forall a \in \mathcal{N}; \\ &\exists\ T_i^P \in \mathbb{N} \text{ such that } \mathrm{KP}_i^t(a) := \underbrace{\mathrm{KP}_i \circ \ldots \circ \mathrm{KP}_i}_{t \text{ times}}(a) = d_i, \quad \text{for all } a \in \mathcal{N},\ t \geq T_i^P. \end{aligned}$$

An example is the shortest path computed with edge weights $\overline{\tau}$. We collect the traversing time of the next edge along the known path in the vector τ_i^{kp}, and for the whole path starting from each node in the vector χ_i^{kp}:

$$\begin{aligned} \tau_i^{\mathrm{kp}} \in \mathbb{R}_{\geq 0}^{|\mathcal{N}|};\ \tau_{i,a}^{\mathrm{kp}} &:= \tau_{(a,\mathrm{KP}_i(a))}, \\ \chi_i^{\mathrm{kp}} \in \mathbb{R}_{\geq 0}^{|\mathcal{N}|};\ \chi_{i,a}^{\mathrm{kp}} &:= \sum_{t=0}^{\infty} \tau_{(\mathrm{KP}_i^t(a),\mathrm{KP}_i^{t+1}(a))}, \end{aligned}$$

and the following auxiliary input, designed such that every vehicle takes the next edge of the known path:

$$\begin{aligned} &u_i^{\mathrm{kp}} : \mathbb{X}_i \to \mathbb{U}_i \text{ for all } i \text{ such that} \\ &(S_{(a,b)}^{\mathrm{edge}})^\top u_i^{\mathrm{kp}}(x_i) = \begin{cases} x_{i,a} - \delta_{d_i}(a) & \text{if } b = \mathrm{KP}_i(a) \\ 0 & \text{if } b \neq \mathrm{KP}_i(a). \end{cases} \end{aligned} \tag{7.34}$$

We then postulate the following technical assumption, which encodes the fact that each agent evaluates the distance of the final state from the destination by means of the known path:

Assumption 7.8. *The local costs satisfy Assumption 7.7 with*

$$f_i^{\mathrm{T}}(x) = \xi_i^{\mathrm{T}}\left(x^\top \chi_i^{\mathrm{kp}}\right)$$
$$f_i^{\mathrm{S}}\left(x, u_i^{\mathrm{kp}}(x)\right) \le \xi_i^{\mathrm{S}}\left(x^\top \tau_i^{\mathrm{kp}}\right),$$

where ξ_i^{T} *is* μ^{T}*-strongly monotone and* ξ_i^{S} *is an* L_S*-Lipschitz continuous function for all* i*, with* $\xi_i^{\mathrm{T}}(0) = \xi_i^{\mathrm{S}}(0) = 0$.

For example, Assumption 7.8 is satisfied by $f_i^{\mathrm{T}}(x) = \gamma_1 x^\top \chi_i^{\mathrm{kp}}$, with $\gamma_1 > 0$ and $f_i^{\mathrm{S}}(x,u) = \gamma_2 (\tau_i^{\mathrm{kp}})^\top x$, with $\gamma_2 \ge 0$. In the main result of this section we show that, if the agents have a high enough incentive to reach the destination (encoded by the strong monotonicity constant of the terminal cost μ^{T}), then the system in (7.26a) controlled by the receding-horizon NE control action defined in (7.33) asymptotically reaches the origin (that is, the state at which every vehicle is at its destination with probability 1). The proof follows from the equivalence between (7.30) and (7.32). Specifically, we show that p^{T} is a control Lyapunov function under control action u^{kp} for the collective system whose map from input to state is $I_N \otimes B$ (cf. (7.26a)). We then apply a known result in MPC theory [12, Theorem 2.19] to conclude the asymptotic stability.

Theorem 7.1. *Denote* $\overline{k} := \max_{(a,b)}(k_{(a,b)})$ *and* $\tau^{\min} := \min_{(a,b),a\neq b} \tau_{(a,b)}$. *Under Assumptions 7.1,7.3,7.6–7.8 and if*

$$\mu^{\mathrm{T}} \ge 1 + L_S + \tfrac{\overline{k}(N+1)}{2N\tau_{\min}}, \tag{7.35}$$

then the origin is asymptotically stable for the systems $x_i[t+1] = B\kappa_i(\boldsymbol{x}_t)$ *for all* $i \in \mathcal{I}$*, with* κ_i *as in* (7.33).

Let us present the resulting approach in Algorithm 11.

7

Algorithm 11 Receding horizon NE seeking for traffic routing

Initialization. Set $x_i[0]$ as in (7.25a) for each $i \in \mathcal{I}$.
For $\tau \in \mathbb{N}$:

1. **Agents control computation**:

   (a) A NE $\boldsymbol{u}^*$ of $\mathcal{G}(\boldsymbol{x}[\tau])$ is computed using Algorithm 10, where each $(\Omega_i)_{i\in\mathcal{I}}$ in (7.22b) is substituted with $\{\omega_i \in \mathcal{Y}_i | (7.27) \text{ holds}\}$, $\lambda^{(0)} = \mathbf{0}$ and the dual update (7.23b) is ignored.

   (b) Each agent i computes $(M_{i,(a,b)}[0])_{(a,b)\in\mathcal{E}}$ according to (7.25c) and $\Pi_i^*[0]$ as in (7.7).

2. **Vehicles node update**:
   For all $v \in \{1,...,V\}$, $i \in \mathcal{I}$ draw $s_{i,v}[\tau+1] \in \mathcal{N}$ from the probability distribution $\mathrm{col}(\Pi_i^*[1]_{(s_{i,v}[\tau],b)})_{b\in\mathcal{N}}$

3. **Agents state update**:
   Each agent updates the empirical distribution and the state:

$$
\begin{aligned}
&\pi_{n,i} = |\{v \in \{1,...,V\} \text{ s.t. } s_{i,v}[\tau+1] = n\}| \text{ for all } n \in \mathcal{N} \\
&\rho_i[\tau+1] = \mathrm{col}(\pi_{n,i}/V)_{n\in\mathcal{N}} \\
&x_i[\tau+1] = \rho_i[\tau+1] - \rho_i^{\mathrm{eq}}.
\end{aligned}
$$

7

7.5 Numerical study

We study the behavior of Algorithm 10 on multiple randomly generated simple scenarios, in order to better observe the characteristics of the solution. We implement a randomly generated directed graph with 12 nodes and 27 edges for $N = 8$ agents. We consider the case where the agents only take the traversing time into account when choosing the road, and thus we set $f_i \equiv 0$ for all i. We set every road to have the same length and capacity, by considering $\ell_{(a,b)}$ to the BPT latency function in (7.10) with $\tau_{(a,b)} = 0.1$, $\zeta_{(a,b)} = \frac{1}{N}$, $c_{(a,b)} = 0.1$ for all $(a,b) \in \mathcal{E}$ and $\xi = 3$. The road limit in (7.14) is defined as $\overline{c}_{(a,b)} = 0.2$. We solve the problem in (7.16) for 100 random initial states and destinations of the agents. The solution to the game in (7.16) is then compared to the routing obtained by the shortest path with no traffic information, that is, with edge weight $\tau_{(a,b)}$ for all $(a,b) \in \mathcal{E}$. We can conclude from Figure 7.5.1 that the baseline solution tends to overcrowd some roads (cf. edge 6) and underutilize others (cf. edge 3), while the proposed GNE routing, which exploits traffic information, obtains a more uniform usage of the network. In Figure 7.5.2, we show the normalized approximation error for the travel time computed using (7.11). The approximation error for each link $(a,b) \in \mathcal{E}$ and instant t is computed as $|\ell_{(a,b)}(\sigma^{\mathrm{M}}(a,b),t) - \ell_{(a,b)}(\hat{\sigma}^{\mathrm{M}}(a,b),t)|$, where $\sigma^{\mathrm{M}}(a,b),t$ is the expected

Code available at `https://github.com/bemilio/MDP_traffic_nonlinear`

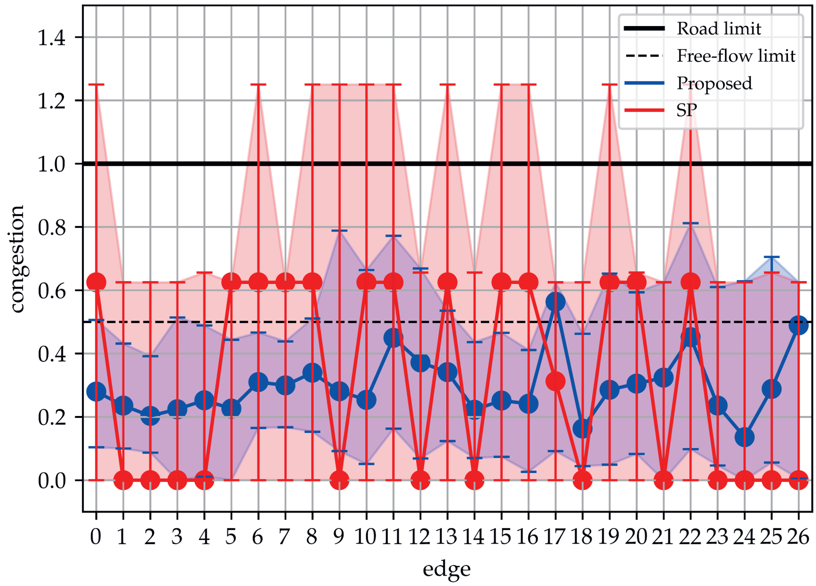


Figure 7.5.1: $\max_t \sigma^t_{(a,b)}/\bar{c}_{(a,b)}$, compared to the congestion obtained by the shortest path routing. The dotted line denotes $c_{(a,b)}/\bar{c}_{(a,b)}$. The dots show the median values. The shaded area highlights the 95% confidence interval. We show in red the performance of the shortest path solution (SP).

road occupation defined in §7.2.2 and $\hat{\sigma}^{\mathrm{M}}(a,b),t$ is the realized number of vehicles on (a,b) at time t divided by NV. Evidently, increasing the population size reduces the approximation error. We then apply Algorithm 11 for $\tau \in \{1,...,10\}$ with the terminal cost $f_i^{\mathrm{T}}(x) = \gamma(\chi_i^{\mathrm{kp}})^\top x$ and γ as in the right-hand side of (7.35), which ensures that the assumptions of Theorem 7.1 are satisfied. The results are compared to the pre-computed open-loop solution of problem (7.16) without the constraint in (7.14), denoted by the ∞ horizon, in terms of the relative total traversing time reduction with respect to the shortest-path solution without traffic information. Figure 7.5.3 shows that the traversing time experienced is reduced with respect to the shortest path solution, and this advantage increases with the time horizon. In a practical sense, the results in Figures 7.5.1, 7.5.3 show that the availability of real-time traffic information allows to reduce the congestion on heavily utilized links and the total traversing time experienced by the drivers.

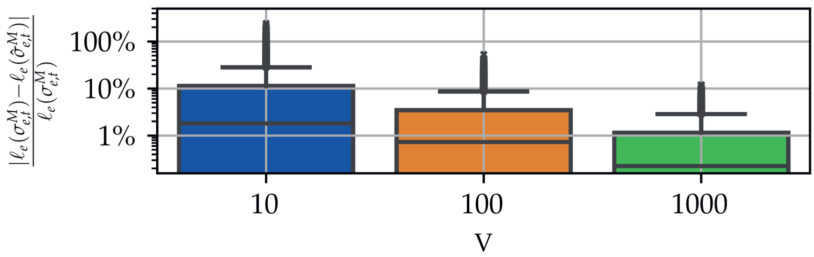


Figure 7.5.2: Difference between approximated and empirical travel time with respect to V, the number of vehicles per population.

7

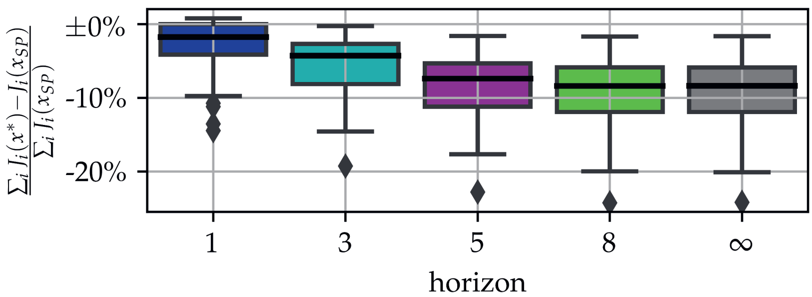


Figure 7.5.3: Comparison of the total cost incurred by the agents, with respect to the shortest path without traffic information.

7.6 Conclusion

Traffic routing of multiple vehicles can be modelled as an aggregative game with mixed strategies using a first-order approximation of the latency function. The approximation error decreases as the number of controlled vehicles increases. The particular structure of the road latency function guarantees the monotonicity of the game under mild conditions, allowing for solution via existing equilibrium-seeking algorithms. If the latency function is linear, then the game can be solved in receding horizon whenever the local objective functions satisfy a set of conditions inherited from the MPC literature. Numerical simulations show that the proposed solution reduces the overall network congestion and traversing time, compared to the optimal routing computed without traffic information.

Appendix

Lemma 7.8. *The only nonzero eigenvalues of a matrix*

$$A(\boldsymbol{y}) := 2(\sigma+\zeta)\mathbf{1}_N\mathbf{1}_N^\top + \tfrac{\xi}{N}(\boldsymbol{y}\mathbf{1}_N^\top + \mathbf{1}_N\boldsymbol{y}^\top) \tag{7.36}$$

where $\boldsymbol{y} \in \mathbb{R}^N_{\geq 0}$, $\sigma := \mathrm{avg}(y_i)_{i\in\mathcal{I}}$, $\zeta \geq 0$, *are* $\lambda_- := \xi\sigma + \gamma_-$ *and* $\lambda_+ := \xi\sigma + \gamma_+$, *where*

$$\gamma_\pm := N(\sigma+\zeta) \pm \sqrt{N^2(\sigma+\zeta)^2 + 2N\xi(\sigma+\zeta)\sigma + \tfrac{\xi^2\|\boldsymbol{y}\|^2}{N}}. \tag{7.37}$$

Proof (sketch): As $A(\boldsymbol{y})$ is a sum of 3 rank-1 matrices, it is at most rank 3. We verify that $\lambda_\pm$ are eigenvalues with eigenvectors $\xi\boldsymbol{y} + \gamma_\pm\mathbf{1}_N$. There is no third non-zero eigenvalue as $\mathrm{trace}\big(A(\boldsymbol{y})\big) = \lambda_- + \lambda_+$. ■

7.A Proofs of Section 7.2

Proof of Lemma 7.1: We prove that (7.2) and (7.3) hold true for the matrices computed as in (7.7). First, note $\rho_i[0] = \rho_i^{\text{in}}$ for all $i \in \mathcal{I}$. If $\rho_{i,\overline{a}}[t] = 0$ for some $\overline{a} \in \mathcal{N}$ and $t \in \mathcal{T}$, then from (7.6b) and (7.6c) we obtain

$$\sum_{b:(\overline{a},b)\in\mathcal{E}} M_{i,(\overline{a},b)}[t] = 0$$

and, together with (7.6d), the latter implies $M_{i,(\overline{a},b)}[t] = 0$ for all $b \in \mathcal{N}$. Substituting in (7.7) and , we obtain (7.3):

$$\Pi_{i,(a,b)}[t]\rho_{i,a}[t] = \begin{cases} 0 & \text{if } \rho_{i,a}[t] = 0 \\ M_{i,(a,b)}[t] & \text{if } \rho_{i,a}[t] \neq 0 \end{cases} = M_{i,(a,b)}[t] \quad \forall (a,b) \in \mathcal{E},\ t \in \mathcal{T}.$$

By expanding the product $(\Pi_i[t])^\top \rho_i[t]$ and by substituting the latter and (7.6a), one obtains (7.2). Finally, we sum both sides of (7.6a) and (7.6b) for all $b \in \mathcal{N}$ and $a \in \mathcal{N}$, respectively, to obtain:

$$\sum_{b\in\mathcal{N}} \rho_{i,b}[t+1] = \sum_{(a,b)\in\mathcal{E}} M_{i,(a,b)}[t] = \sum_{a\in\mathcal{N}} \rho_{i,a}[t].$$

Since $\rho_i^{\text{in}} \in \Delta^{|\mathcal{N}|}$, we conclude by induction that $\rho_i[t] \in \Delta^{|\mathcal{N}|}$ and $\mathrm{col}(M_{i,(a,b)}[t])_{(a,b)\in\mathcal{E}} \in \Delta^{|\mathcal{E}|}$. ■

Proof of Proposition 7.1: As $\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]$ is drawn from a Poisson binomial distribution scaled by NV, from [137, Eq. 15] and from $M_{i,(a,b)}[t] \in [0,1]$,

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]) &= \tfrac{1}{(NV)^2} \sum_{\substack{i\in\mathcal{I},\\ v\in\{1,\dots,V\}}} (1-M_{i,(a,b)}[t])M_{i,(a,b)}[t] \\ &\leq \frac{1}{4NV}.\end{aligned}$$

By the Chebyschev's inequality, for any $\epsilon > 0$ and $K_\epsilon = \frac{1}{\sqrt{\epsilon}}$,

$$\mathbb{P}\left\{(\sigma^{\mathrm{M}}_{(a,b)}[t] - \hat{\sigma}^{\mathrm{M}}_{(a,b)}[t]) \geq \tfrac{K_\epsilon}{2\sqrt{NV}}\right\} \leq \epsilon.$$

The result then follows from [138, Theorem 6.2.3] by substituting $r_n = \frac{1}{2\sqrt{NV}}$ and $\boldsymbol{a} = \sigma^{\mathrm{M}}_{(a,b)}[t]$ (in the reference notation). ■

7.B Proofs of Section 7.3

Proof of Lemma 7.2 (sketch): Compute $J''_{(a,b)}(\cdot, \boldsymbol{M}_{-i,(a,b)}[t])$ for a generic $(a,b), t, i$ and note that it is non-negative, following Assumption 7.2. The result then follows from [51, Prop. 8.14, 8,17], and Assumption. 7.3. ■

Proof of Lemma 7.3: Let us compute F:

$$F(\boldsymbol{\omega}) = \mathrm{col}\left(\nabla f_i(\omega_i) + \begin{bmatrix}\mathrm{col}\left(\nabla_{M_{i,(a,b)}[t]} J_{(a,b)}(\boldsymbol{M}_{(a,b)}[t])\right)_{(a,b)\in\mathcal{E},t\in\mathcal{T}} \\ \mathbf{0}_{|\mathcal{N}|(T+1)}\end{bmatrix}\right)_{i\in\mathcal{I}}, \tag{7.38}$$

where the zero vector appears because the latency functions do not depend on $\boldsymbol{\rho}$. From Assumption 7.3 and [51, Example 20.3], ∇f_i is monotone for each i. Then, $\mathrm{col}(\nabla f_i)_{i\in\mathcal{I}}$ is monotone by [51, Prop. 20.23]. Let us denote the second addend in (7.38) as $T(\boldsymbol{\omega})$. From [51, Prop. 20.10], F is monotone if T is monotone. Let us define the permutation matrix P such that

$$P\boldsymbol{\omega} = \begin{bmatrix}\mathrm{col}(\boldsymbol{M}_{(a,b)}[t])_{(a,b)\in\mathcal{E},t\in\mathcal{T}} \\ \mathrm{col}(\boldsymbol{\rho}[t])_{t\in\mathcal{T}^+}\end{bmatrix}.$$

It holds, from the definition of $F_{(a,b)}$,

$$\begin{aligned}PF(\boldsymbol{\omega}) &\overset{(7.38)}{=} \begin{bmatrix}\mathrm{col}\left(\mathrm{col}\left(\nabla_{M_{i,(a,b)}[t]} J_{(a,b)}(\boldsymbol{M}_{(a,b)}[t])\right)_{i\in\mathcal{I}}\right)_{(a,b)\in\mathcal{E},t\in\mathcal{T}} \\ \mathbf{0}_{NT|\mathcal{N}|}\end{bmatrix} \\ &\overset{(7.18)}{=} \begin{bmatrix}\mathrm{col}\left(F_{(a,b)}(\boldsymbol{M}_{(a,b)}[t])\right)_{(a,b)\in\mathcal{E},t\in\mathcal{T}} \\ \mathbf{0}_{NT|\mathcal{N}|}\end{bmatrix}.\end{aligned} \tag{7.39}$$

As $PP^\top = I$, for all $\boldsymbol{\omega},\boldsymbol{\omega}'$:

$$\begin{aligned}\langle T(\boldsymbol{\omega})-T(\boldsymbol{\omega}'),\boldsymbol{\omega}-\boldsymbol{\omega}'\rangle &= \langle PT(\boldsymbol{\omega})-PT(\boldsymbol{\omega}'),P\boldsymbol{\omega}-P\boldsymbol{\omega}'\rangle\\ &= \sum_{\substack{(a,b)\in\mathcal{E}\\ t\in\mathcal{T}}}\left\langle F_{(a,b)}(\boldsymbol{M}_{(a,b)}[t])-F_{(a,b)}(\boldsymbol{M}'_{(a,b)}[t]),\boldsymbol{M}_{(a,b)}[t]-\boldsymbol{M}'_{(a,b)}[t]\right\rangle\end{aligned}$$

which is non-negative if $F_{(a,b)}$ is monotone for each $(a,b),t$. ■

Proof of Lemma 7.4: For compactness of notation, we suppress the subscript of $\zeta_{(a,b)}$ and $\xi_{(a,b)}$. Following [140, Prop. 12.3], $F_{(a,b)}$ in (7.18) is monotone if

$$DF_{(a,b)}(\boldsymbol{y})+DF_{(a,b)}(\boldsymbol{y})^\top \succcurlyeq 0 \quad \forall \boldsymbol{y}=\mathrm{col}(y_i)_{i\in\mathcal{I}},\ y_i\in[0,1].$$

Denote $\sigma=\mathrm{avg}(\boldsymbol{y})_{i\in\mathcal{I}}$.

$$DF_{(a,b)}(\boldsymbol{y}) = \tfrac{1}{N}\ell'_{(a,b)}(\sigma)(I_N+\mathbf{1}\mathbf{1}^\top)+\tfrac{1}{N^2}\ell''_{(a,b)}(\sigma)(\boldsymbol{y}\mathbf{1}^\top). \tag{7.40}$$

As $\ell'_{(a,b)}(\sigma)=k(\sigma+\zeta)^\xi$, $\ell''_{(a,b)}(\sigma)=k\xi(\sigma+\zeta)^{\xi-1}$, we compute

$$\begin{aligned}DF_{(a,b)}(\boldsymbol{y})+DF^\top_{(a,b)}(\boldsymbol{y}) =& \tfrac{2k}{N}(\sigma+\zeta)^\xi I_N\\ &+\tfrac{k}{N}(\sigma+\zeta)^{\xi-1}\big(2(\sigma+\zeta)\mathbf{1}\mathbf{1}^\top+\tfrac{\xi}{N}(\boldsymbol{y}\mathbf{1}^\top+\mathbf{1}\boldsymbol{y}^\top)\big).\end{aligned} \tag{7.41}$$

By Lemma 7.8, $DF_{(a,b)}(\boldsymbol{y})+DF^\top_{(a,b)}(\boldsymbol{y})\succcurlyeq 0$ if

$$\tfrac{2k}{N}(\sigma+\zeta)^\xi+\tfrac{k}{N}\big(\sigma+\zeta)^{\xi-1}(\xi\sigma+\gamma_-(\boldsymbol{y})\big)\geq 0, \tag{7.42}$$

where γ_- is defined in (7.37). Excluding the trivial case $\boldsymbol{y}=0,\zeta=0$, we divide by $\frac{k}{N}(\sigma+\zeta)^\xi$ to obtain

$$\begin{aligned}(7.42)\Leftrightarrow\quad & 2+\tfrac{\xi\sigma}{\sigma+\zeta}+\tfrac{\gamma_-(\boldsymbol{y})}{\sigma+\zeta}\geq 0\\ \Leftrightarrow\quad & 2+\tfrac{\xi\sigma}{\sigma+\zeta}+N\geq\sqrt{N^2+2\tfrac{N\xi\sigma}{\sigma+\zeta}+\tfrac{\xi^2\|\boldsymbol{y}\|^2}{N(\sigma+\zeta)^2}}\\ \Leftrightarrow\quad & 4+\tfrac{\xi^2\sigma^2}{(\sigma+\zeta)^2}+\cancel{N^2}+\tfrac{4\xi\sigma}{\sigma+\zeta}+4N+\cancel{\tfrac{2N\xi\sigma}{\sigma+\zeta}}\geq\cancel{N^2}+\cancel{\tfrac{2N\xi\sigma}{\sigma+\zeta}}+\tfrac{\xi^2\|\boldsymbol{y}\|^2}{N(\sigma+\zeta)^2}\\ \Leftrightarrow\quad & f(\boldsymbol{y}):=4(N+1)+\tfrac{\xi^2\sigma^2}{(\sigma+\zeta)^2}+\tfrac{4\xi\sigma}{\sigma+\zeta}-\tfrac{\xi^2\|\boldsymbol{y}\|^2}{N(\sigma+\zeta)^2}\geq 0.\end{aligned} \tag{7.43}$$

We look for the minimum of the left-hand side of the latter inequality. Notice that $\nabla_{\boldsymbol{y}}\sigma=\frac{1}{N}\mathbf{1}_N$. Then,

$$\begin{aligned}\nabla f(\boldsymbol{y}) =& \tfrac{2\xi^2}{N}\tfrac{\sigma(\sigma+\zeta)^2-\sigma^2(\sigma+\zeta)}{(\sigma+\zeta)^4}\mathbf{1}_N-\tfrac{4\xi}{N}\tfrac{\zeta}{(\sigma+\zeta)^2}\mathbf{1}_N\\ &-\xi^2\tfrac{2N\boldsymbol{y}(\sigma+\zeta)^2-2(\sigma+\zeta)\|\boldsymbol{y}\|^2\mathbf{1}_N}{N^2(\sigma+\zeta)^4}.\end{aligned}$$

Since $\nabla f(\boldsymbol{y})$ contain either terms that multiply $\mathbf{1}_N$ or $\boldsymbol{y}$, it must be $\boldsymbol{y}=\alpha\mathbf{1}_N$ for some $\alpha\in(0,1]$ for $\boldsymbol{y}$ to be a stationary point. Therefore, the minimum of $f(\boldsymbol{y})$ is

either obtained for $\boldsymbol{y}=\alpha\mathbf{1}_N$ or at an extreme point of $[0,1]^N$, that is, $\boldsymbol{y}=\sum_{i\in\mathcal{Q}}\boldsymbol{e}_i$, where $\boldsymbol{e}_i\in\mathbb{R}^N$ with only non-zero element $[\boldsymbol{e}_i]_i=1$ and $\mathcal{Q}\subset\{1,....,N\}$. Let us study the two cases separately:
Case $\boldsymbol{y}=\alpha\mathbf{1}_N$*:* In this case, $\sigma=\alpha$ and $\|\boldsymbol{y}\|^2=\alpha^2N$. We substitute these values in (7.43) to find that it is always satisfied and, thus, $DF_{(a,b)}(\boldsymbol{y})+DF_{(a,b)}(\boldsymbol{y})^\top\succcurlyeq 0$:

$$f(\boldsymbol{y})=4(N+1)+\cancel{\tfrac{\xi^2\alpha^2}{(\alpha+\zeta)^2}}+\tfrac{4\xi\alpha}{\alpha+\zeta}-\cancel{\tfrac{\xi^2\alpha^2N}{N(\alpha+\zeta)^2}}\geq 0.$$

Case $\boldsymbol{y}=\sum_{i\in\mathcal{Q}}\boldsymbol{e}_i$*:* In this case, define $q:=|\mathcal{Q}|$, we compute $\sigma=\frac{q}{N}$ and $\|\boldsymbol{y}\|^2=q$. We then substitute in (7.43) to find

$$f(\boldsymbol{y})=4N+4+\tfrac{\xi^2q^2}{(q+N\zeta)^2}+\tfrac{4\xi q}{q+N\zeta}-\tfrac{\xi^2qN}{(q+N\zeta)^2}\geq 0.$$

A sufficient condition for the latter is that the first addend is greater than the negative one, which is true if

$$g(q):=4(q+\zeta N)^2-q\xi^2\geq 0. \tag{7.44}$$

Let us study the first derivative of g:

$$g'(q)=8\,(q+\zeta N)-\xi^2=0\Leftrightarrow q=\tfrac{\xi^2}{8}-\zeta N.$$

Note that $q\in\{1,...,N\}$. Therefore, $g(q)$ has the minimum in $q=1$ if $\zeta\geq\frac{\xi^2-8}{8N}$. We then note from (7.44) that $g(1)\geq 0$ if $\zeta\geq\frac{\xi-2}{2N}$. Therefore, $g(q)\geq 0$ for all $q\in\{1,...,N\}$ if (7.20) holds true, which in turn guarantees that (7.42) holds true for all $\boldsymbol{y}\in[0,1]^N$. ■

Proof of Proposition 7.2: $DF_{(a,b)}$ is computed in (7.40). Denote $\sigma=\mathrm{avg}(\boldsymbol{y})$. For $\boldsymbol{y}\in[0,1]^N$, $\sigma\leq 1$ and $\|\boldsymbol{y}\|^2\leq N$. Thus,

$$\|\boldsymbol{y}\mathbf{1}_N^\top\|=\sqrt{\lambda_{\max}(\boldsymbol{y}\mathbf{1}_N^\top\mathbf{1}_N\boldsymbol{y})}=\sqrt{N\|\boldsymbol{y}\|^2}\leq N$$

From subadditivity, $\|\mathbf{1}_N\mathbf{1}_N^\top\|=N$ and the latter,

$$\begin{aligned}
\max_{y\in[0,1]^N}\|DF_{(a,b),t}(\boldsymbol{y})\| &\leq \max_{y\in[0,1]^N}\tfrac{1}{N}\ell'_{(a,b)}(\sigma)\|I+\mathbf{1}\mathbf{1}^\top\|+\tfrac{1}{N^2}\ell''_{(a,b)}(\sigma)\|\boldsymbol{y}\mathbf{1}^\top\|\\
&\leq \max_{y\in[0,1]^N}\tfrac{1}{N}\ell'_{(a,b)}(\sigma)(1+N)+\tfrac{1}{N^2}\ell''_{(a,b)}(\sigma)N\\
&\overset{(\sigma\leq 1)}{\leq} \tfrac{k_{(a,b)}}{N}\left((1+\zeta_{(a,b)})^\xi+\xi(1+\zeta_{(a,b)})^{\xi-1}\right)\\
&=L_{(a,b)}.
\end{aligned}$$

$F_{(a,b)}$ is therefore $L_{(a,b)}$-Lipschitz continuous on $[0,1]^N$ following [140, Thm. 9.2, 9.7]. As $\mathbf{\Omega}\subset[0,1]^{Nn_\omega}$, it can be shown that the second addend of (7.38) is Lipschitz continuous with constant $\max_{(a,b)\in\mathcal{E}}L_{(a,b)}$ and thus F is $L-$Lipschitz continuous. By applying Lemma 7.5 and [51, Cor. 20.28], F is maximally monotone. Finally, by applying Lemma 7.2, all the assumptions of [17, Thm. 1] are satisfied and the claim follows. ■

7.C Proofs of Section 7.4

Proof of Lemma 7.6: Denote by F the pseudogradient of (7.24). We compute

$$\begin{aligned} DF(\boldsymbol{\omega}) =& \operatorname{diag}\big(D\nabla f_i(\omega_i)\big)_{i\in\mathcal{I}} \\ &+ \frac{1}{N}(I_N + \mathbf{1}_N\mathbf{1}_N^\top)\otimes \begin{bmatrix} \operatorname{diag}\left(\ell'_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)})\right)_{t\in\mathcal{T},(a,b)\in\mathcal{E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} \\ &+ \frac{1}{N^2}\operatorname{col}\left(\mathbf{1}_N^\top \otimes \begin{bmatrix} \operatorname{diag}\left(M_{i,(a,b)}[t]\ell''_{(a,b)}(\sigma^{\mathrm{M}}_{(a,b)})\right)_{t\in\mathcal{T},(a,b)\in\mathcal{E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right)_{i\in\mathcal{I}}. \end{aligned}$$

The arguments of col in the second addend are in general different for each $i \in \mathcal{I}$. Thus, $DF(\boldsymbol{\omega})$ is symmetric only if $\ell''_{(a,b)} \equiv 0$. The thesis then follows from [134, Theorem 4.5]. ■

Derivation of (7.29): By leveraging Assumption 7.6 and $(S^{\text{edge}}_{(a,b)})^\top(u_i[t]+u_i^{\text{eq}}) = M_{i,(a,b)}[t]$, we write $J_{(a,b)}$ in (7.12) for a generic $(a,b)\in\mathcal{E}$ as

$$J_{(a,b)}(\boldsymbol{u}[t]) = \Big(\tau_{(a,b)} + \tfrac{k_{(a,b)}}{N}\sum_{j\in\mathcal{I}}\big[(S^{\text{edge}}_{(a,b)})^\top(u_j[t]+u_j^{\text{eq}})\big]\Big)(S^{\text{edge}}_{(a,b)})^\top(u_i[t]+u_i^{\text{eq}})$$

By substituting the latter and by leveraging Assumption 7.7, we then rewrite J_i in (7.13) as

$$\begin{aligned} J_i(x_i[0],\boldsymbol{u}) =& f_i^{\mathrm{T}}(x_i[T]) + \sum_{t\in\mathcal{T}}\Big\{ f_i^{\mathrm{S}}(x_i[t],u_i[t]) \\ &+ \sum_{(a,b)\in\mathcal{E}}\Big[\Big(\tau_{(a,b)} + \tfrac{k_{(a,b)}}{N}\sum_{j\in\mathcal{I}}\big[(S^{\text{edge}}_{(a,b)})^\top(u_j[t]+u_j^{\text{eq}})\big]\Big)(S^{\text{edge}}_{(a,b)})^\top(u_i[t]+u_i^{\text{eq}})\Big]\Big\}. \end{aligned}$$

By using the definitions of C and $\overline{\tau}$ and rearranging,

$$\begin{aligned} J_i(x_i[0],\boldsymbol{u}) =& f_i^{\mathrm{T}}(x_i[T]) + \sum_{t\in\mathcal{T}} f_i^{\mathrm{S}}(x_i[t],u_i[t]) \\ &+ (\overline{\tau}^\top + \operatorname{avg}(\boldsymbol{u}[t]+\boldsymbol{u}_{\text{eq}})^\top C)(u_i[t]+u_i^{\text{eq}}). \end{aligned}$$

From Assumption 7.6 and the definition of C and $\overline{\tau}$, $Cu_i^{\text{eq}} = \mathbf{0}$, $\overline{\tau}^\top u_i^{\text{eq}} = 0$ for any $i\in\mathcal{I}$, thus (7.29) follows. ■

Proof of Lemma 7.7: In this proof, we treat time sequences as column vectors, that is, $u_i = \operatorname{col}(u_i[t])_{t\in\mathcal{T}}$; $\boldsymbol{u} = \operatorname{col}(u_i)_{i\in\mathcal{I}}$. We can rewrite the agent cost in (7.29) as

$$J_i(x_i[0],\boldsymbol{u}) = f_i^{\mathrm{T}}(x_i[T]) + \sum_{t\in\mathcal{T}}\Big\{ f_i^{\mathrm{S}}(x_i[t],u_i[t]) + \overline{\tau}^\top u_i[t])\Big\} + \sum_{j\in\mathcal{I}}\Big\{\tfrac{1}{N}u_j^\top(I_T\otimes C)u_i\Big\}$$

The pseudo-gradient of (7.30) reads then as [17, Eq. 32]

$$\begin{aligned} F(\boldsymbol{x}[0],\boldsymbol{u}) =& \operatorname{col}\Big(\nabla_{u_i}\Big\{f_i^{\mathrm{T}}(x_i[T]) + \textstyle\sum_{t\in\mathcal{T}}\left\{f_i^{\mathrm{S}}(x_i[t],u_i[t]) + \overline{\tau}^\top u_i[t]\right\}\Big\}\Big)_{i\in\mathcal{I}} \\ &+ \tfrac{1}{N}(I_N + \mathbf{1}_N\mathbf{1}_N^\top)\otimes(I_T\otimes C)\boldsymbol{u}. \end{aligned}$$

Note that the first addend term in the latter only depends on the local variables of each agent, due to the decoupling of the dynamics, while the second addend is a coupling term. It can be verified by expanding the quadratic forms that

$$\sum_{t\in\mathcal{T}} \|\boldsymbol{u}[t]\|^2_{\mathbf{1}_N\mathbf{1}_N^\top\otimes C} = \|\boldsymbol{u}\|^2_{\mathbf{1}_N\mathbf{1}_N^\top\otimes(I_T\otimes C)}$$
$$\sum_{t\in\mathcal{T}} \|\boldsymbol{u}[t]\|^2_{I_N\otimes C} = \|\boldsymbol{u}\|^2_{I_{TN}\otimes C}$$

By substituting the latter in (7.31), one obtains

$$p(\boldsymbol{x}[0],\boldsymbol{u}) = \sum_{i\in\mathcal{I}} \Big\{ f_i^{\mathrm{T}}(x_i[T]) + \sum_{t\in\mathcal{T}} \{ f_i^{\mathrm{S}}(x_i[t],u_i[t]) + \overline{\tau}^\top u_i[t] \} \Big\} + $$
$$+ \tfrac{1}{2N}\|\boldsymbol{u}\|^2_{(I_N+\mathbf{1}_N\mathbf{1}_N^\top)\otimes(I_T\otimes C)}.$$

One can then compute $\nabla_{\boldsymbol{u}} p$ to verify that it reads as F. ■

Proof of Theorem 7.1: Theorem 7.1 follows by verifying the hypothesis of [12, Thm. 2.19]. Namely, we prove a lower bound for the stage cost (Lemma 7.10) and that the terminal cost is a control Lyapunov function for the collective system $\boldsymbol{x}[t+1] = (I_N \otimes B)\boldsymbol{u}[t]$ (Lemma 7.11). We first show some technical relations in Lemma 7.9:

Lemma 7.9. *The following hold for all* $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{Z}, i \in \mathcal{I}$*:*

$$\sum_{a\neq b} (S^{\text{edge}}_{(a,b)})^\top u_i = -(S^{\text{edge}}_{(d_i,d_i)})^\top u_i; \tag{7.45a}$$

$$x_{i,d_i} \geq (S^{\text{edge}}_{(d_i,d_i)})^\top u_i \quad ; \tag{7.45b}$$

$$-x_{i,d_i} \geq \max_{a\in\mathcal{N}} x_{i,a}. \tag{7.45c}$$

Proof. (7.45a): From the definition of $\mathbb{Z}$, $Pu_i = x_i$. Substituting the definition of P and summing each row,

$$\sum_{a\in\mathcal{N}} \sum_{b:(a,b)\in\mathcal{E}} (S^{\text{edge}}_{(a,b)})^\top u_i = \sum_{a\in\mathcal{N}} x_{i,a} = 0 \tag{7.46}$$

where we used the definition of $\mathbb{X}_i$ in (7.28a) and $\sum_{a\in\mathcal{N}} \rho_i^{\text{eq}} = 1$. Using the definition of $\mathbb{U}$, (7.45a) follows by noting

$$\sum_{a\neq b} (S^{\text{edge}}_{(a,b)})^\top u_i = \sum_{(a,b)\in\mathcal{E}} \{(S^{\text{edge}}_{(a,b)})^\top u_i\} - (S^{\text{edge}}_{(d_i,d_i)})^\top u_i.$$

(7.45b)*:* From $u_i \in \mathbb{R}^{|\mathcal{E}|}_{\geq 0} - \{u_i^{\text{eq}}\}$ in (7.28b) and $u_{i,a}^{\text{eq}} = 0$ for each a-th element not associated to the edge (d_i,d_i), it follows $(S^{\text{edge}}_{(d_i,b)})^\top u_i \geq 0$ for all $b \in \mathcal{N}\setminus\{d_i\}$. As $x_i = Pu_i$, from the definition of P:

$$x_{i,d_i} = \sum_{b:(d_i,b)\in\mathcal{E}} (S^{\text{edge}}_{(d_i,b)})^\top u_i \geq (S^{\text{edge}}_{(d_i,d_i)})^\top u_i.$$

7

(7.45c)*:* From $x_i + \rho_i^{\text{eq}} \in \Delta^{|\mathcal{N}|}$ and the definition of ρ_i^{eq}, it follows $x_{i,a} \geq 0 \ \forall \ a \neq d_i$ and $x_{i,d_i} \leq 0$. Thus,

$$-x_{i,d_i} \overset{(7.46)}{=} \sum_{b \in \mathcal{N} \setminus \{d_i\}} x_{i,b} \geq x_{i,a} \quad \forall \ a \in \mathcal{N}. \qquad \blacksquare$$

Lemma 7.10. *For all* $(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{Z}$*, the stage cost in* (7.31a) *satisfies*

$$p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) \geq \frac{\tau^{\min} \|\boldsymbol{x}\|}{Nn}. \tag{7.47}$$

Proof. From $C \succcurlyeq 0$ and Assumption 7.7, $p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) \geq \sum_i \overline{\tau}^\top u_i$. Thus,

$$\begin{aligned} p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) &\geq \sum_{i \in \mathcal{I}} \sum_{(a,b) \in \mathcal{E}} \tau_{(a,b)} (S^{\text{edge}}_{(a,b)})^\top u_i \\ &\overset{\text{Ass. 7.6}}{\geq} \tau^{\min} \sum_{i \in \mathcal{I}} \sum_{a \neq b} (S^{\text{edge}}_{(a,b)})^\top u_i. \end{aligned} \tag{7.48}$$

We then note

$$\sum_{a \neq b} (S^{\text{edge}}_{(a,b)})^\top u_i \overset{(7.45\text{a})}{=} -(S^{\text{edge}}_{(d_i,d_i)})^\top u_i \overset{(7.45\text{b})}{\geq} -x_{i,d_i} = |x_{i,d_i}|.$$

Substituting in (7.48),

$$p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) \geq \tau^{\min} \sum_{i \in \mathcal{I}} |x_{i,d_i}|. \tag{7.49}$$

From (7.45c), $|x_{i,d_i}| = \|x_i\|_\infty$. Substituting in (7.49),

$$p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) \geq \tau^{\min} \sum_{i \in \mathcal{I}} \|x_i\|_\infty.$$

We recall $\boldsymbol{x} = \text{col}_{i \in \mathcal{I}}(x_i)$, thus

$$\|\boldsymbol{x}\|_\infty = \max_{i,a} |x_{i,a}| = \max_{i \in \mathcal{I}} \|x_i\|_\infty \leq \sum_{i \in \mathcal{I}} \|x_i\|_\infty.$$

We then obtain $p^{\text{S}}(\boldsymbol{x}, \boldsymbol{u}) \geq \tau^{\min} \|\boldsymbol{x}\|_\infty$. As for any $y \in \mathbb{R}^n$, $n\|y\|_\infty \geq \|y\|_2$ and since $\boldsymbol{x} \in \mathbb{R}^{Nn}$, we obtain (7.47). $\blacksquare$

Lemma 7.11. *Let* p^{T} *be as in* (7.31b) *and let Assumption 7.8 and Equation* (7.35) *hold true. For all* $\boldsymbol{x} \in \mathbb{X}$, $(\boldsymbol{x}, \boldsymbol{u}^{\text{kp}}(x)) \in \mathbb{Z}$ *and*

$$p^{\text{T}}\big((I_N \otimes B)\boldsymbol{u}^{\text{kp}}(\boldsymbol{x})\big) - p^{\text{T}}(\boldsymbol{x}) \leq -p^{\text{S}}\big(\boldsymbol{x}, \boldsymbol{u}^{\text{kp}}(\boldsymbol{x})\big). \tag{7.50}$$

Proof. For compactness of notation, we drop the dependencies of $\boldsymbol{u}^{\text{kp}}$ on $\boldsymbol{x}$, and we let $\boldsymbol{x}^+ = (I_N \otimes B)\boldsymbol{u}^{\text{kp}}$. Define the mapping from each node to all parent nodes along the known path KP_i:

$$\text{KP}_i^{-1} : \mathcal{N} \rightrightarrows \mathcal{N}, \quad \text{KP}_i^{-1} : b \mapsto \{a : b = \text{KP}_i(a)\}.$$

Then, from the definition of B and (7.34),

$$\begin{aligned} x^+_{i,b} &= \sum_{a:(a,b)\in\mathcal{E}} (S^{\text{edge}}_{(a,b)})^\top u^{\text{kp}}_i \\ &= \sum_{a\in\mathrm{KP}_i^{-1}(b)} x_{i,a} - \delta_{d_i}(a). \end{aligned} \tag{7.51}$$

From the definition of χ^{kp}_i, its element with index $\mathrm{KP}_i(a)$ satisfies the following for each $a\in\mathcal{N}$:

$$\chi^{\text{kp}}_{i,\mathrm{KP}_i(a)} = \chi^{\text{kp}}_{i,a} - \tau^{\text{kp}}_{i,a}. \quad \text{for all } a\in\mathcal{N}. \tag{7.52}$$

Then,

$$\begin{aligned} (\chi^{\text{kp}}_i)^\top x_i &= \sum_{a\in\mathcal{N}} \chi^{\text{kp}}_{i,a} x_{i,a} \\ &\overset{(7.52)}{=} \sum_{a\in\mathcal{N}} (\chi^{\text{kp}}_{i,\mathrm{KP}_i(a)} + \tau^{\text{kp}}_{i,a}) x_{i,a} \\ &= (\tau^{\text{kp}}_i)^\top x_i + \sum_{a\in\mathcal{N}} \chi^{\text{kp}}_{i,\mathrm{KP}_i(a)} x_{i,a} \\ &= (\tau^{\text{kp}}_i)^\top x_i + \sum_{b\in\mathcal{N}} \chi^{\text{kp}}_{i,b} \sum_{a\in\mathrm{KP}_i^{-1}(b)} x_{i,a} \\ &\overset{(7.51)}{=} (\tau^{\text{kp}}_i)^\top x_i + \sum_{b\in\mathcal{N}} \chi^{\text{kp}}_{i,b}\big(x^+_{i,b} + \delta_{d_i}(b)\big). \end{aligned}$$

We note that $\mathrm{KP}_i(d_i) = d_i$, thus $\chi^{\text{kp}}_{i,d_i} = 0$ from Assumption 7.6, therefore $\chi^{\text{kp}}_{i,b}\delta_{d_i}(b) = 0$ for all b. Thus,

$$(\chi^{\text{kp}}_i)^\top x_i = (\tau^{\text{kp}}_i)^\top x_i + (\chi^{\text{kp}}_i)^\top x^+_i \tag{7.53}$$

From Assumption 7.8, the monotonicity of ξ_i implies for each i:

$$\begin{aligned} &\Big(\xi^{\mathrm{T}}_i(x_i^\top \chi^{\text{kp}}_i) - \xi^{\mathrm{T}}_i\big((x^+_i)^\top \chi^{\text{kp}}_i\big)\Big)\big(x_i^\top \chi^{\text{kp}}_i - (x^+_i)^\top \chi^{\text{kp}}_i\big) \ge \mu^{\mathrm{T}}\big(x_i^\top \chi^{\text{kp}}_i - (x^+_i)^\top \chi^{\text{kp}}_i\big)^2 \\ &\overset{(7.53)}{\Rightarrow} \Big(\xi^{\mathrm{T}}_i(x_i^\top \chi^{\text{kp}}_i) - \xi^{\mathrm{T}}_i\big((x^+_i)^\top \chi^{\text{kp}}_i\big)\Big)\big(x_i^\top \tau^{\text{kp}}_i\big) \ge \mu^{\mathrm{T}}\big(x_i^\top \tau^{\text{kp}}_i\big)^2 \\ &\Rightarrow \Big(\xi^{\mathrm{T}}_i(x_i^\top \chi^{\text{kp}}_i) - \xi^{\mathrm{T}}_i\big((x^+_i)^\top \chi^{\text{kp}}_i\big)\Big) \ge \mu^{\mathrm{T}}\big(x_i^\top \tau^{\text{kp}}_i\big) \end{aligned} \tag{7.54}$$

where the latter follows from $x_i^\top \tau^{\text{kp}}_i \ge 0$, which in turn follows from $x_{i,a}\ge 0$, $\tau^{\text{kp}}_{i,a}\ge 0$ for all $a\ne d_i$, and $\tau^{\text{kp}}_{i,d_i} = 0$. Thus,

$$\begin{aligned} p^{\mathrm{T}}(\boldsymbol{x}) - p^{\mathrm{T}}(\boldsymbol{x}^+) &\overset{\text{As. 7.8}}{=} \sum_{i\in\mathcal{I}} \xi^{\mathrm{T}}_i(x_i^\top \chi^{\text{kp}}_i) - \xi^{\mathrm{T}}_i\big((x^+_i)^\top \chi^{\text{kp}}_i\big) \\ &\overset{(7.54)}{\ge} \mu^{\mathrm{T}}\big((\boldsymbol{\chi}^{\text{kp}})^\top \boldsymbol{x} - (\boldsymbol{\chi}^{\text{kp}})^\top \boldsymbol{x}^+\big) \\ &\overset{(7.53)}{=} \mu^{\mathrm{T}}(\boldsymbol{\tau}^{\text{kp}})^\top \boldsymbol{x}. \end{aligned} \tag{7.55}$$

From the definition of $\overline{\tau}$ and from $\tau_{(d_i,d_i)} = 0$, $\forall \boldsymbol{x} \in \mathbb{X}$,

$$\begin{aligned}\overline{\tau}^\top u_i^{\mathrm{kp}} &= \sum_{(a,b)\in\mathcal{E}} \tau_{(a,b)} (S^{\mathrm{edge}}_{(a,b)})^\top u_i^{\mathrm{kp}} \\ &\overset{(7.34)}{=} \sum_{a\in\mathcal{N}} \tau_{(a,\mathrm{KP}_i(a))}(x_{i,a} - \delta_{d_i}(a)) \\ &= (\tau_i^{\mathrm{kp}})^\top x_i.\end{aligned}$$

By Assumption 7.8 and denoting $\overline{C} = (I_N + \mathbf{1}\mathbf{1}^\top) \otimes C$,

$$\begin{aligned}p^{\mathrm{S}}(\boldsymbol{x},\boldsymbol{u}^{\mathrm{kp}}) &= \tfrac{1}{2N} \|\boldsymbol{u}^{\mathrm{kp}}\|^2_{\overline{C}} + \sum_{i\in\mathcal{I}} f_i^{\mathrm{S}}(x_i, u_i^{\mathrm{kp}}) + x_i^\top \tau_i^{\mathrm{kp}} \\ &\le \tfrac{1}{2N} \|\boldsymbol{u}^{\mathrm{kp}}\|^2_{\overline{C}} + \sum_{i\in\mathcal{I}} (L_S+1) x_i^\top \tau_i^{\mathrm{kp}}.\end{aligned} \tag{7.56}$$

From (7.55) and (7.56), then (7.50) holds if

$$(\mu^{\mathrm{T}} - 1 - L_{\mathrm{S}})(\boldsymbol{\tau}^{\mathrm{kp}})^\top \boldsymbol{x} \ge \tfrac{1}{2N} \|\boldsymbol{u}^{\mathrm{kp}}\|^2_{\overline{C}}. \tag{7.57}$$

Let us find a lower bound for the LHS of (7.57).

$$\begin{aligned}(\boldsymbol{\tau}^{\mathrm{kp}})^\top \boldsymbol{x} &= \sum_{i,a} \tau^{\mathrm{kp}}_{i,a} x_{i,a} \\ &\overset{\text{As. 7.6}}{=} \sum_i \sum_{a\neq d_i} \tau^{\mathrm{kp}}_{i,a} x_{i,a} \\ &\ge \tau^{\min} \sum_i \sum_{a\neq d_i} x_{i,a} \\ &\overset{(7.46)}{=} \tau^{\min} \sum_i (-x_{i,d_i}).\end{aligned} \tag{7.58}$$

We now rewrite the RHS of (7.57):

$$\|\boldsymbol{u}^{\mathrm{kp}}\|^2_{\overline{C}} = \sum_{i\in\mathcal{I}} \Big\{ (u_i^{\mathrm{kp}})^\top C u_i^{\mathrm{kp}} + \sum_{j\in\mathcal{I}} \{ (u_j^{\mathrm{kp}})^\top C u_i^{\mathrm{kp}} \} \Big\}. \tag{7.59}$$

We then note that for all $i,j \in \mathcal{I}$, from the definition of C:

$$(u_j^{\mathrm{kp}})^\top C u_i^{\mathrm{kp}} = \sum_{(a,b)\in\mathcal{E}} k_{(a,b)} (u_j^{\mathrm{kp}})^\top S^{\mathrm{edge}}_{(a,b)} (S^{\mathrm{edge}}_{(a,b)})^\top u_i^{\mathrm{kp}}. \tag{7.60}$$

From (7.34), $(u_j^{\mathrm{kp}})^\top S^{\mathrm{edge}}_{(a,b)} \le 1$ for all (a,b) and $(S^{\mathrm{edge}}_{(a,b)})^\top u_i^{\mathrm{kp}} = 0$ if $b \neq \mathrm{KP}_i(a)$. We continue from (7.60):

$$
\begin{aligned}
&\leq \sum_{a\in\mathcal{N}} k_{(a,\mathrm{KP}_i(a))} S^{\mathrm{edge}}_{(a,\mathrm{KP}_i(a))} u_i^{\mathrm{kp}} \\
&= \sum_{a\in\mathcal{N}} k_{(a,\mathrm{KP}_i(a))} (x_{i,a} - \delta_{d_i}(a)) \\
&= \sum_{a\neq d_i} k_{(a,\mathrm{KP}_i(a))} x_{i,a} \\
&\leq \overline{k} \sum_{a\neq d_i} x_{i,a} \\
&\overset{(7.46)}{=} -\overline{k} x_{i,d_i}
\end{aligned}
$$

where we noted $k_{(d_i,\mathrm{KP}_i(d_i))} = k_{(d_i,d_i)} = 0$ from Assumption 7.6. Substituting the latter in (7.59),

$$\|\boldsymbol{u}^{\mathrm{kp}}\|^2_{\overline{C}} \leq (N+1)\overline{k} \sum_{i\in\mathcal{I}} (-x_{i,d_i}). \tag{7.61}$$

From (7.61) and (7.58), (7.57) holds true under (7.35). ■

We are now ready to present the proof of Theorem 7.1:

Proof. By [139, Thm. 2], for any $\boldsymbol{x}\in\mathbb{X}$, a solution of $\mathcal{G}(\boldsymbol{x})$ solves $\mathcal{O}(\boldsymbol{x})$. Then, $\mathrm{col}(\kappa_i(\boldsymbol{x}))_i$ is the first input of a sequence which solves (7.32) with initial state $\boldsymbol{x}$. Problem (7.32) satisfies [12, Assm. 2.2, 2.3] under Assumptions 7.3 and 7.6. [12, Assm. 2.14a] follows from Lemma 7.11. By Assumption 7.3, p^{T} is Lipschitz continuous. Thus, [12, Assm. 2.14b] is satisfied by Lemma 7.10. $\mathbb{X}$ is control invariant for $\boldsymbol{u}^{\mathrm{kp}}(\cdot)$, as verified by computing $(I\otimes B)\boldsymbol{u}^{\mathrm{kp}}(\boldsymbol{x})$ for a generic $\boldsymbol{x}\in\mathbb{X}$. [12, Assm. 2.17] is then satisfied by applying [12, Prop. 2.16]. The claim follows from [12, Thm 2.19]. ■

7

8

Concluding remarks

Zen is the "spirit of the valley", not the mountaintop.
The only Zen you find on the tops of mountains is the Zen you bring up there.

Robert M. Pirsig, in "Zen and the art of motorcycle maintenance"

In this thesis, we have addressed the optimal generalized Nash equilibrium selection problem, and we have contributed to the field of (optimal) generalized Nash equilibrium seeking in time-varying and dynamic games. In this chapter, we provide a summary of the main contributions, we address the research questions formulated in Chapter 1, and we explore possible directions for future developments.

This thesis provides contributions to the field of game theory and game-theoretic control of dynamical systems by developing algorithms that are able of computing an optimal generalized Nash equilibrium (GNE) according to a generic preference metric, of tracking a (possibly selected) GNE in a dynamic environment and by providing design guidelines and stability guarantees for receding-horizon game-theoretic controllers. The main contributions of this thesis are as follows:

- **Optimal GNE selection**
  In the first part of the thesis, we identify a shortcoming of current state-of-the art GNE-seeking algorithms, which compute an unspecified, initialization-dependent GNE when multiple solutions are present. We design the first semi-decentralized algorithms that can then solve the problem of selecting the variational GNE (v-GNE) that minimizes an arbitrary performance metric. We compare two design methods, based either on the hybrid steepest-descent method (HSDM), which relies on a reformulation of the v-GNE selection problem as a fixed-point selection problem, or on the Tikhonov regularization method, where the problem is solved as a sequence of variational inequality (VI)s regularized with the selection objective. We design ad-hoc algorithms for the particular cases of aggregative and cocoercive games. We remark that the semi-decentralized algorithms proposed in Chapter 2 can be computed in a fully decentralized fashion, if the selection objective can be computed with local information only.

- **Tracking of a time-varying (optimal) GNE**
  In Chapter 2, we examine the performance of a v-GNE selection algorithm inspired by the HSDM paired with the forward-backward-forward (FBF) splitting method. Compared to the standard HSDM, we apply a non-vanishing step size to the gradient step of the selection function and a finite number of iterations, accounting for the limited computation time allotted between subsequent time steps. In Chapter 5, we derive a GNE-seeking algorithm with a linear convergence rate for a strongly monotone game subject to linear equality constraints in a partial-information setting. This result represents the first linear convergence rate result for a GNE-seeking algorithm in the presence of coupling constraints. This fast convergence rate, which implies a input-to-state stability (ISS) property, is particularly relevant for the online GNE tracking problem. We leverage these properties to derive an explicit tracking error bound.

- **Receding-horizon game-theoretic control**
  In the second part of this thesis, we derive a novel game-theoretic control methodology based on the receding-horizon solution of a finite-horizon equilibrium seeking problem. We design two alternative finite-horizon formulations whose solutions match either the open-loop Nash equilibrium (ol-NE) or the closed-loop Nash equilibrium (cl-NE) infinite-horizon control input. Compared to other results in the literature, which introduce additional constraints to limit the variation of the agents' input trajectories between time steps [110],

or assume the stability of the plant [103], we provide the first stability result for a receding-horizon non-cooperative game-theoretic control without either of these requirements. In deriving this result, we also establish a novel characterization of the value of the infinite-horizon achieved by the ol-NE for a linear-quadratic dynamic game, by casting each agents' optimization problem as a linear quadratic regulator (LQR) problem in an augmented space. For the particular case of a potential game, we illustrate a design methodology that guarantees asymptotic stability to a known steady state by adding a carefully designed terminal cost to the agents' objectives. This design is applied on a practical traffic routing setup.

8.1 Addressing the research questions

Q1 How to deterministically compute a specific GNE, optimally selected according to a design criterion?
As highlighted in Chapter 2, a generalized Nash equilibrium problem (GNEP) admits, in general, an infinite number of GNEs. The problem of computing a generic GNE is an unsolved, challenging problem, even without taking its optimal selection problem into account. The current state-of-the-art GNE-seeking algorithms only compute an unspecified element of the set of v-GNEs of a monotone game, that is, a subset of GNEs such that the dual variables associated to the shared constraints are equal for each agent. We then restricted our focus to the problem of selecting the v-GNE that optimizes a generic, convex metric. In the case of a monotone game, the set of v-GNEs is closed and convex, thus the v-GNE selection problem is a convex optimization problem, which can be solved by first-order gradient methods–with the catch that one cannot derive a projection operator on the set of v-GNEs. One can instead resort to the HSDM method [15], which pairs a first-order gradient descent of the selection function with a quasi-shrinking or attracting fixed-point operator (in place of the projection operator typically found in a projected-gradient method). We found that the FBF, preconditioned forward-backward (pFB) and preconditioned proximal-point (PPP) v-GNE seeking algorithms all possess the characteristics required for the convergence of the HSDM. An alternative algorithm design is based on the Tikhonov regularization method proposed in [21] for the solution of a VI-constrained VI. This method involves defining a sequence of auxiliary, regularized games, where the selection function is included in the agents' objectives as an additive term with a vanishing weight and an inertial term. One then obtains a bi-level algorithm, where each iteration requires the approximate solution of a strongly monotone VI. The HSDM and the Tikhonov-regularization methods exhibit similar convergence speed and, although the Tikhonov regularization method requires a more careful tuning process, it is technically more flexible. In fact, the HSDM only works when paired with quasi-shrinking algorithms that solve merely monotone games, while the Tikhonov regularization method can be paired to any GNE-seeking method for strongly monotone games.

Q2 How to track the (possibly selected) time-varying GNE of a game, by means of an algorithm which only employs a finite number of iterations?
In Chapter 5, we show that, in the particular case of a strongly monotone GNEP that only presents linear equality constraints, we are able to derive a fully distributed algorithm that achieves a linear convergence rate, even in a partial information scenario. As we show in Section 5.5, this convergence regime is linked to a degree of robustness of the algorithm to bounded variations of the equilibrium point, which allows one to derive a tight upper bound on the tracking error of a time-varying GNE. In the general case of a merely monotone GNEP, the problem admits multiple solutions and one cannot expect a linear convergence rate of a solution algorithm. However, in this case, one can augment the GNEP problem with a strongly monotone selection function, thus obtaining a GNE selection problem. The optimal GNE selection problem admits a unique v-GNE solution and, following [15, Lemma 4(a)], the HSDM method applied to its solution with a constant step size is contractive. However, we note that the HSDM requires a vanishing step size to converge to the exact solution, meaning that using a constant step size introduces an additional tracking error. This error can be controlled by reducing the step size and increasing the number of algorithm iterations per time step. Determining the necessary algorithm parameters in advance is challenging, as that would require prior knowledge of the shrinkage function of the solution algorithm, which is typically not available.

Q3 Is a multiagent dynamical system controlled by the receding-horizon solution of a GNEP asymptotically stable?
In general, no. In the particular case when the finite-horizon problem defines a potential game, one can introduce an additive terminal cost and/or constraint such that the potential function is a Lyapunov function for the closed-loop system. In practice, the potential function behaves analogously to the objective function of a (single-agent) model predictive control (MPC), which the agents unknowingly, but willingly, contribute to minimize. The stability of the origin can then be established using standard arguments from the MPC literature, as shown in Chapter 7 with a practical traffic routing application. These findings, however, are not valid in the general non-potential case, which we studied in Chapter 6. Specifically, in the non-potential case, the GNE of the finite-horizon game is only optimal for each agent given a fixed input sequence from the remaining agents. When the solution of the finite-horizon problem is recomputed at each time step, the previously found solution may no longer be optimal for some agents, potentially leading to increases in the agents' objective functions across subsequent time steps. Consequently, one cannot use (a combination of) the agents' objective functions as Lyapunov functions, as is typically done in single-agent MPC. In cases where an infinite-horizon Nash equilibrium (NE) control is both computable and stabilizing to the origin, then an effective design strategy for a non-cooperative receding-horizon controller with stability guarantees of the origin for the closed-loop system is that of designing the finite-horizon problem such that its solution coincides with the

infinite-horizon control input.

Q4 Can the solution to a finite-horizon GNEP problem involving a multiagent linear time-invariant (LTI) dynamical system match the one of an infinite-horizon GNEP?
Yes, in a multi-agent LTI system with quadratic objective functions such that the infinite-horizon solution is asymptotically stable to the origin. In Chapter 6 we studied two solution concepts, namely the ol-NE and the cl-NE. The infinite-horizon cl-NE renders the origin asymptotically stable under easily verifiable detectability and stabilizability assumptions. Conversely, the asymptotic stability properties of the origin under the ol-NE control action cannot be established by the problem's priors at the present stage. In both cases, one can include a carefully tuned terminal cost to the finite-horizon equilibrium problem, such that the resulting finite-horizon solution coincides with the infinite-horizon solution, provided that the initial condition belongs to a certain region of the space. Computationally, the cl-NE case results in a finite-horizon GNE with nested equilibrium constraints, which must be solved by an ad-hoc algorithm (namely, [7]). Instead, the ol-NE translates into a finite-horizon VI, for which a wider literature of solution algorithms is available.

8.2 Future research directions

The development of this thesis has raised several further research questions. We present the main ones, highlighting the potential opportunities and challenges each poses.

- **Computation and stability guarantees for infinite-horizon NEs**
  In Chapter 6, we design game-theoretic receding-horizon controllers that inherit the stabilizing properties of the infinite-horizon, unconstrained GNE. However, there is still no established algorithm for solving the coupled Riccati equations that characterize an ol-NE or a cl-NE, nor any stability guarantee for an ol-NE. The recent work [6] partially addresses this problem by proposing algorithms for the computation of a cl-NE. However, the convergence of the algorithms is not directly linked to the characteristics of either the dynamics or the objective functions (e.g. controllability, observability, dynamic coupling, objective coupling), which in practice limits the possibility of designing a multiagent system that meets the algorithm's convergence assumptions. A promising, alternative direction is to reformulate the dynamic game as an equilibrium problem in the cone of positive semidefinite matrices, thus generalizing a well-known set of results in the field of optimal control [141]. We emphasize that this research direction requires some significant technical development, as the known results on monotone games (typically derived in the space of real vectors) should be extended to the cone of positive-semidefinite matrices.

- **Monotonicity conditions for finite-horizon games**
  The finite-horizon ol-NE formulation in Section 6.3.3 and the surrogate finite-horizon cl-NE formulation in Section 6.4 can both be solved via a standard

8

VI solution algorithm, provided that the finite-horizon problem is monotone. It is then valuable to explore the assumptions required on the system priors that result in a monotone equilibrium-seeking problem. This fundamental issue in dynamic games, which is trivial in the single-agent MPC case, is surprisingly challenging in a game-theoretic setting. Previous results in the field of monotone games [142] suggest that this property is linked to the objective coupling between the agents which, however, in the dynamic case depends also on the system dynamics. Moreover, from our numerical experience, it appears that the monotonicity of the problem is dependent on the horizon length, thus highlighting another fundamental difference with the single-agent MPC case. We find this a valuable and insightful research direction which would extend the practical applicability of the control designs proposed in this thesis, as well as providing greater insights on the whole field of dynamic games.

- **Robust game-theoretic control**
The control design method proposed in Chapter 6 requires each agent's terminal cost to match the objective value achieved by the infinite-horizon NE *exactly*. In practice, however, one cannot hope to have access to the agents' exact objectives, and the dynamics is often subject to unknown external disturbances or model inaccuracies. If the system is nonlinear, the necessary terminal cost expression is generally not even computable in closed form. Notably, single-agent MPC enjoys several robustness guarantees, and it can be designed to be provably stable even when employed on a nonlinear plant [12, 143]. This suggests that the controllers proposed in Chapter 6 may still be stabilizing in a non-nominal setting, making it worthwhile to study the sensitivity of the control action to modelling errors. We also envision possible extensions of the proposed design: for example, introducing additional robustness constraints or relaxing the terminal cost requirements.

- **Zero-order and partial-information Nash equilibrium selection**
In Chapters 2, 3 and 4, we proposed semi-decentralized algorithms for solving the optimal GNE selection problem. The proposed algorithms are also effective in a decentralized setting if the gradient of the coupling objective function and the selection function can be computed with only the local information and the one available at the neighbor nodes. This may be an unpractical assumption if some agent's objective depends on agents they cannot communicate with, or if the closed-form expression of the gradients is not available. We refer to these settings as partial-information and zero-order GNE optimal selection, respectively. Some GNE-seeking algorithms were recently proposed, where the missing information is compensated via peer-to-peer communication in the partial-information case [81] and some gradient estimation technique in the zero-order case, e.g. [144]. Either scenario introduces additional technical challenges, as the partial-information scenario typically leads to a zero-seeking problem for a non-monotone operator, while the zero-order information case requires the study of a perturbed dynamical system. Deriving GNE selection

algorithms in these scenarios entails then exploring how the non-monotonicity of the operator and gradient perturbations influence the convergence of the HSDM.

- **Applications of game-theoretic control to highly competitive scenarios**
  We envision that a game-theoretic control system such as the one presented in Chapter 6 does not necessarily have to be implemented on multiple agents acting independently, but it could also be internally solved by a single agent performing in a competitive setting. The game-theoretic model endows the agent with an adversarial model of the others, thus rendering it able to counteract the "best move" of the adversaries. This implicitly defined internal model of the adversaries may provide an advantage in highly competitive scenarios, such as autonomous drone and car racing, as already partially explored in [8] and [14], and evasive maneuvering [145]. We believe that this application field needs further exploration, and that the insights provided in Chapters 6 and 7 is a solid starting point to the development of more sophisticated receding-horizon control algorithms.

Bibliography

References

[1] Annaswamy, A.M., Johansson, K.H. and Pappas, G. "Control for Societal-Scale Challenges: Road Map 2030". In: *IEEE Control Systems*, 44(3):30–32, (June 2024).

[2] Le Cleac'h, S., Schwager, M. and Manchester, Z. "ALGAMES: A fast solver for constrained dynamic game". In: *Robotics: Science and Systems*, (2020).

[3] Facchinei, F. and Kanzow, C. "Generalized Nash equilibrium problems". In: *Annals of Operations Research*, 175(1):177–211, (2010).

[4] Belgioioso, G. et al. "Distributed Generalized Nash Equilibrium Seeking: An Operator-Theoretic Perspective". In: *IEEE Control Systems Magazine*, 42(4):87–102, (Aug. 2022).

[5] Hall, S. et al. "Receding Horizon Games with Coupling Constraints for Demand-Side Management". (June 2022).

[6] Nortmann, B. et al. "Nash Equilibria for Linear Quadratic Discrete-time Dynamic Games via Iterative and Data-driven Algorithms". In: *IEEE Transactions on Automatic Control*, pages 1–15, (2024).

[7] Laine, F. et al. "The Computation of Approximate Generalized Feedback Nash Equilibria". In: *SIAM Journal on Optimization*, 33(1):294–318, (Mar. 2023).

[8] Wang, M. et al. "Game-Theoretic Planning for Self-Driving Cars in Multivehicle Competitive Scenarios". In: *IEEE Transactions on Robotics*, 37(4):1313–1325, (Aug. 2021).

[9] Başar, T. and Olsder, G.J. *Dynamic Noncooperative Game Theory*. Number 23 in Classics in Applied Mathematics. SIAM, Philadelphia, 2nd ed edition, (1999).

[10] Engwerda, J. *LQ Dynamic Optimization and Differential Games*. J. Wiley & Sons, Chicester, West Sussex, England Hoboken, NJ, (2005).

[11] Haurie, A., Krawczyk, J.B. and Zaccour, G. *Games and Dynamic Games*. World Scientific Publishing Company, (2012).

[12] Rawlings, J.B., Mayne, D.Q. and Diehl, M. *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, Madison, Wisconsin, 2nd edition edition, (2017).

[13] Ananduta, W. and Grammatico, S. “Equilibrium Seeking and Optimal Selection Algorithms in Peer-to-Peer Energy Markets”. In: *Games*, 13(5):66, (Oct. 2022).

[14] Spica, R. et al. “A Real-Time Game Theoretic Planner for Autonomous Two-Player Drone Racing”. In: *IEEE Transactions on Robotics*, 36(5):1389–1403, (Oct. 2020).

[15] Yamada, I. and Ogura, N. “Hybrid Steepest Descent Method for Variational Inequality Problem over the Fixed Point Set of Certain Quasi-nonexpansive Mappings”. In: *Numerical Functional Analysis and Optimization*, 25(7-8):619–655, (2005).

[16] Franci, B., Staudigl, M. and Grammatico, S. “Distributed forward-backward (half) forward algorithms for generalized Nash equilibrium seeking”. In: *Proceedings of the 2020 European Control Conference (ECC)*, pages 1274–1279. IEEE, (2020).

[17] Belgioioso, G. and Grammatico, S. “Semi-Decentralized Generalized Nash Equilibrium Seeking in Monotone Aggregative Games”. In: *IEEE Transactions on Automatic Control*, 68(1):140–155, (2023).

[18] Benenati, E., Ananduta, W. and Grammatico, S. “Optimal Selection and Tracking Of Generalized Nash Equilibria in Monotone Games”. In: *IEEE Transactions on Automatic Control*, 68(12):7644–7659, (Dec. 2023).

[19] Belgioioso, G. and Grammatico, S. “A distributed proximal-point algorithm for Nash equilibrium seeking in generalized potential games with linearly coupled cost functions”. In: *2019 18th European Control Conference (ECC)*, pages 1–6, Naples, Italy, (June 2019). IEEE.

[20] Benenati, E., Ananduta, W. and Grammatico, S. “On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games”. In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 6389–6394, Cancun, Mexico, (Dec. 2022). IEEE.

[21] Facchinei, F. et al. “VI-constrained hemivariational inequalities: Distributed algorithms and power control in ad-hoc networks”. In: *Mathematical Programming*, 145(1-2):59–96, (June 2014).

[22] Benenati, E., Ananduta, W. and Grammatico, S. “A Semi-Decentralized Tikhonov-Based Algorithm for Optimal Generalized Nash Equilibrium Selection”. In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 4243–4248, Singapore, Singapore, (Dec. 2023). IEEE.

[23] Bianchi, M., Benenati, E. and Grammatico, S. “Linear Convergence in Time-Varying Generalized Nash Equilibrium Problems”. In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 7220–7226, Singapore, Singapore, (Dec. 2023). IEEE.

[24] Benenati, E. and Grammatico, S. "Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities". In: *arXiv preprint 2408.15703*, (2024).

[25] Bakhshayesh, B.G. and Kebriaei, H. "Decentralized Equilibrium Seeking of Joint Routing and Destination Planning of Electric Vehicles: A Constrained Aggregative Game Approach". In: *IEEE Transactions on Intelligent Transportation Systems*, pages 1–10, (2021).

[26] Benenati, E. and Grammatico, S. "Probabilistic Game-Theoretic Traffic Routing". In: *IEEE Transactions on Intelligent Transportation Systems*, 25(10):13080–13090, (Oct. 2024).

[27] Shilov, I., Le Cadre, H. and Busic, A. "Privacy impact on generalized Nash equilibrium in peer-to-peer electricity market". In: *Operations Research Letters*, 49(5):759–766, (2021).

[28] Belgioioso, G. et al. "Operationally-Safe Peer-to-Peer Energy Trading in Distribution Grids: A Game-Theoretic Market-Clearing Mechanism". In: *IEEE Transactions on Smart Grid*, (2022).

[29] Pang, J.S. et al. "Distributed power allocation with rate constraints in Gaussian parallel interference channels". In: *IEEE Transactions on Information Theory*, 54(8):3471–3489, (2008).

[30] Scutari, G. et al. "Monotone Games for Cognitive Radio Systems". In: *Distributed Decision Making and Control*, pages 83–112. Springer London, (2012).

[31] Wang, J. et al. "A Generalized Nash Equilibrium Approach for Robust Cognitive Radio Networks via Generalized Variational Inequalities". In: *IEEE Transactions on Wireless Communications*, 13(7):3701–3714, (2014).

[32] Facchinei, F., Fischer, A. and Piccialli, V. "On generalized Nash games and variational inequalities". In: *Operations Research Letters*, 35(2):159–164, (2007).

[33] Kulkarni, A.A. and Shanbhag, U.V. "On the variational equilibrium as a refinement of the generalized Nash equilibrium". In: *Automatica*, 48(1):45–55, (2012).

[34] Paccagnan, D. et al. "Nash and Wardrop equilibria in aggregative games with coupling constraints". In: *IEEE Transactions on Automatic Control*, 64(4):1373–1388, (2018).

[35] Belgioioso, G. and Grammatico, S. "Projected-gradient algorithms for Generalized Equilibrium seeking in Aggregative Games are preconditioned Forward-Backward methods". In: *Proceedings of the 2018 European Control Conference (ECC)*, pages 2188–2193, (2018).

[36] Belgioioso, G. and Grammatico, S. “Semi-decentralized Nash equilibrium seeking in aggregative games with separable coupling constraints and non-differentiable cost functions”. In: *IEEE Control Systems Letters*, 1(2):400–405, (2017).

[37] Yi, P. and Pavel, L. “An operator splitting approach for distributed generalized Nash equilibria computation”. In: *Automatica*, 102:111–121, (2019).

[38] Yi, P. and Pavel, L. “Distributed Generalized Nash Equilibria Computation of Monotone Games via Double-Layer Preconditioned Proximal-Point Algorithms”. In: *IEEE Transactions on Control of Network Systems*, 6(1):299–311, (2019).

[39] Belgioioso, G., Nedić, A. and Grammatico, S. “Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks”. In: *IEEE Transactions on Automatic Control*, 66(5):2061–2075, (2020).

[40] Bianchi, M., Belgioioso, G. and Grammatico, S. “Fast generalized Nash equilibrium seeking under partial-decision information”. In: *Automatica*, 136(110080), (2022).

[41] Marden, J.R. and Roughgarden, T. “Generalized Efficiency Bounds in Distributed Resource Allocation”. In: *IEEE Transactions on Automatic Control*, 59(3):571–584, (2014).

[42] Yin, H., Shanbhag, U.V. and Mehta, P.G. “Nash equilibrium problems with scaled congestion costs and shared constraints”. In: *IEEE Transactions on Automatic Control*, 56(7):1702–1708, (2011).

[43] Scutari, G. et al. “Equilibrium selection in power control games on the interference channel”. In: *2012 Proceedings IEEE INFOCOM*, pages 675–683, Orlando, FL, USA, (Mar. 2012). IEEE.

[44] Scutari, G. et al. “Real and Complex Monotone Communication Games”. In: *IEEE Transactions on Information Theory*, 60(7):4197–4231, (July 2014).

[45] Dreves, A. “How to select a solution in generalized Nash equilibrium problems”. In: *Journal of Optimization Theory and Applications*, 178(3):973–997, (2018).

[46] Dreves, A. “An algorithm for equilibrium selection in generalized Nash equilibrium problems”. In: *Computational Optimization and Applications*, 73(3):821–837, (2019).

[47] Kaushik, H.D. and Yousefian, F. “A Method with Convergence Rates for Optimization Problems with Variational Inequality Constraints”. In: *SIAM Journal on Optimization*, 31(3):2171–2198, (2021).

[48] Simonetto, A. et al. “Time-Varying Convex Optimization: Time-Structured Algorithms and Applications”. In: *Proceedings of the IEEE*, 108(11):2032–2048, (2020).

[49] Lu, K., Li, G. and Wang, L. “Online distributed algorithms for seeking generalized Nash equilibria in dynamic environments”. In: *IEEE Transactions on Automatic Control*, 66(5):2289–2296, (2021).

[50] Meng, M. et al. “Decentralized Online Learning for Noncooperative Games in Dynamic Environments”. In: *arXiv preprint, https://arxiv.org/abs/2105.06200*, (2021).

[51] Bauschke, H.H. and Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, (2017).

[52] Xu, HK. and Kim, TH. “Convergence of hybrid steepest-descent methods for variational inequalities”. In: *Journal of Optimization Theory and Applications*, 119(1):185–201, (2003).

[53] Cegielski, A. and Zalas, R. “Properties of a class of approximately shrinking operators and their applications”. In: *Fixed Point Theory*, 15(2):399–426, (2014).

[54] Bastianello, N., Simonetto, A. and Carli, R. “Primal and Dual Prediction-Correction Methods for Time-Varying Convex Optimization”. In: *arXiv preprint, arXiv:2004.11709*, (2020).

[55] Simonetto, A. “Time Varying Convex Optimization via Time-Varying Averaged Operators”. In: *arXiv preprint, arXiv:1704.07338*, (2017).

[56] Cegielski, A. et al. “An algorithm for solving the variational inequality problem over the fixed point set of a quasi-nonexpansive operator in Euclidean space”. In: *Numerical Functional Analysis and Optimization*, 34(10):1067–1096, (2013).

[57] Palomar, D.P. and Eldar, Y.C. *Convex Optimization in Signal Processing and Communications.* Cambridge university press, (2010).

[58] Auslender, A. and Teboulle, M. “Lagrangian duality and related multiplier methods for variational inequality problems”. In: *SIAM Journal on Optimization*, 10(4):1097–1115, (2000).

[59] Moret, F. et al. “Loss Allocation in Joint Transmission and Distribution Peer-to-Peer Markets”. In: *IEEE Transactions on Power Systems*, 36(3):1833–1842, (May 2021).

[60] Malitsky, Y. and Tam, M.K. “A Forward-Backward Splitting Method for Monotone Inclusions Without Cocoercivity”. In: *SIAM Journal on Optimization*, 30(2):1451–1472, (Jan. 2020).

[61] Tseng, P. “A modified forward-backward splitting method for maximal monotone mappings”. In: *SIAM Journal on Control and Optimization*, 38(2):431–446, (2000).

[62] Dall'Anese, E. and Simonetto, A. “Optimal Power Flow Pursuit”. In: *IEEE Transactions on Smart Grid*, (2016).

[63] Facchinei, F. and Pang, J.S. *Finite-Dimensional Variational Inequalities and Complementarity Problems.* Springer Series in Operations Research. Springer, New York, (2007).

[64] Dontchev, A.L. and Rockafellar, R.T. *Implicit Functions and Solution Mappings.* Springer Science & Business Media, (2014).

[65] Zinkevich, M. “Online convex programming and generalized infinitesimal gradient ascent”. In: *Proceedings of the 20th international conference on machine learning*, (2003).

[66] Ogura, N. and Yamada, I. “Nonstrictly Convex Minimization over the Bounded Fixed Point Set of a Nonexpansive Mapping”. In: *Numerical Functional Analysis and Optimization*, 24(1-2):129–135, (2003).

[67] Zhou, P. et al. “Private and Truthful Aggregative Game for Large-Scale Spectrum Sharing”. In: *IEEE Journal on Selected Areas in Communications*, 35(2):463–477, (Feb. 2017).

[68] Liu, Z. et al. “Optimal Day-Ahead Charging Scheduling of Electric Vehicles Through an Aggregative Game Model”. In: *IEEE Transactions on Smart Grid*, 9(5):5173–5184, (Sep. 2018).

[69] Koshal, J., Nedić, A. and Shanbhag, U.V. “Distributed Algorithms for Aggregative Games on Graphs”. In: *Operations Research*, 64(3):680–704, (June 2016).

[70] Liang, S., Yi, P. and Hong, Y. “Distributed Nash equilibrium seeking for aggregative games with coupled constraints”. In: *Automatica*, 85:179–185, (Nov. 2017).

[71] Gadjov, D. and Pavel, L. “Single-Timescale Distributed GNE Seeking for Aggregative Games Over Networks via Forward–Backward Operator Splitting”. In: *IEEE Transactions on Automatic Control*, 66(7):3259–3266, (July 2021).

[72] Lei, J., Shanbhag, U.V. and Chen, J. “Distributed Computation of Nash Equilibria for Monotone Aggregative Games via Iterative Regularization”. In: *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 2285–2290, Jeju, Korea (South), (Dec. 2020). IEEE.

[73] Chen, X. et al. “Aggregate Power Flexibility in Unbalanced Distribution Systems”. In: *IEEE Transactions on Smart Grid*, 11(1):258–269, (Jan. 2020).

[74] Yamada, I. “The Hybrid Steepest Descent Method for the Variational Inequality Problem Over the Intersection of Fixed Point Sets of Nonexpansive Mappings”. In: Butnariu, D., Censor, Y. and Reich, S., editors, *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*,

volume 8 of *Studies in Computational Mathematics*, pages 473–504. Elsevier, (2001).

[75] Hirstoaga, S. “Iterative selection method for common fixed point problems”. In: *Journal of Mathematical Analysis and Applications*, 324:1020–1035, (Dec. 2006).

[76] Feingold, D.G. and Varga, R. “Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem”. In: *Pacific Journal of Mathematics*, 12(4):1241–1250, (Dec. 1962).

[77] Kannan, A. and Shanbhag, U.V. “Distributed Computation of Equilibria in Monotone Nash Games via Iterative Regularization Techniques”. In: *SIAM Journal on Optimization*, 22(4):1177–1205, (Jan. 2012).

[78] Lin, W. et al. “Distributed formation control with open-loop Nash strategy”. In: *Automatica*, 106:266–273, (Aug. 2019).

[79] Pavel, L. “Distributed GNE Seeking Under Partial-Decision Information Over Networks via a Doubly-Augmented Operator Splitting Approach”. In: *IEEE Transactions on Automatic Control*, 65(4):1584–1597, (Apr. 2020).

[80] Deng, Z. and Nian, X. “Distributed Generalized Nash Equilibrium Seeking Algorithm Design for Aggregative Games Over Weight-Balanced Digraphs”. In: *IEEE Transactions on Neural Networks and Learning Systems*, 30(3):695–706, (Mar. 2019).

[81] Bianchi, M. and Grammatico, S. “Fully Distributed Nash Equilibrium Seeking Over Time-Varying Communication Networks With Linear Convergence Rate”. In: *IEEE Control Systems Letters*, 5(2):499–504, (Apr. 2021).

[82] Dall’Anese, E. et al. “Optimization and Learning with Information Streams: Time-varying Algorithms and Applications”. In: *arXiv preprint arXiv:1910.08123v2*, (2020).

[83] Li, T. et al. “The Confluence of Networks, Games, and Learning: A Game-Theoretic Framework for Multiagent Decision Making Over Networks”. In: *IEEE Control Systems*, 42(4):35–67, (Aug. 2022).

[84] Su, Y. et al. “Online distributed tracking of generalized Nash equilibrium on physical networks: Closing the Loop via Measurement Feedback”. In: *Autonomous Intelligent Systems*, 1(1):6, (Dec. 2021).

[85] Stein, O. and Sudermann-Merx, N. “The noncooperative transportation problem and linear generalized Nash games”. In: *European Journal of Operational Research*, 266(2):543–553, (Apr. 2018).

[86] Parise, F., Gentile, B. and Lygeros, J. “A Distributed Algorithm For Almost-Nash Equilibria of Average Aggregative Games With Coupling Constraints”. In: *IEEE Transactions on Control of Network Systems*, 7(2):770–782, (June 2020).

[87] Grammatico, S. “Dynamic Control of Agents Playing Aggregative Games With Coupling Constraints”. In: *IEEE Transactions on Automatic Control*, 62(9):4537–4548, (Sep. 2017).

[88] Bullo, F. *Lectures on Network Systems*. Kindle Direct Publishing, 1.7 edition, (2024).

[89] Belgioioso, G. et al. “Energy Management and Peer-to-peer Trading in Future Smart Grids: A Distributed Game-Theoretic Approach”. In: *2020 European Control Conference (ECC)*, pages 1324–1329, Saint Petersburg, Russia, (May 2020). IEEE.

[90] Bianchi, M. and Grammatico, S. “Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents”. In: *Automatica*, 129:109660, (July 2021).

[91] Willems, J. “Least squares stationary optimal control and the algebraic Riccati equation”. In: *IEEE Transactions on Automatic Control*, 16(6):621–634, (Dec. 1971).

[92] Theodor, Y. and Shaked, U. “Output-feedback mixed H-infinity and H-2 control – a dynamic game approach”. In: *International Journal of Control*, 64(2):263–279, (May 1996).

[93] Hall, S. et al. “Game-theoretic Model Predictive Control for Modelling Competitive Supply Chains”. (Jan. 2024).

[94] Monti, A. et al. “Feedback and Open-Loop Nash Equilibria for LQ Infinite-Horizon Discrete-Time Dynamic Games”. In: *SIAM Journal on Control and Optimization*, 62(3):1417–1436, (June 2024).

[95] Papavassilopoulos, G.P. and Olsder, G.J. “On the linear-quadratic, closed-loop, no-memory Nash game”. In: *Journal of Optimization Theory and Applications*, 42(4):551–560, (Apr. 1984).

[96] Engwerda, J. “Algorithms for computing Nash equilibria in deterministic LQ games”. In: *Computational Management Science*, 4(2):113–140, (Apr. 2007).

[97] Li, T.Y. and Gajic, Z. “Lyapunov Iterations for Solving Coupled Algebraic Riccati Equations of Nash Differential Games and Algebraic Riccati Equations of Zero-Sum Games”. In: *New Trends in Dynamic Games and Applications*, pages 333–351, (1995).

[98] Freiling, G., Jank, G. and Abou-Kandil, H. “On global existence of solutions to coupled matrix Riccati equations in closed-loop Nash games”. In: *IEEE Transactions on Automatic Control*, 41(2):264–269, (Feb. 1996).

[99] Possieri, C. and Sassano, M. “An algebraic geometry approach for the computation of all linear feedback Nash equilibria in LQ differential games”. In: *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 5197–5202, Osaka, (Dec. 2015). IEEE.

[100] Mylvaganam, T., Sassano, M. and Astolfi, A. “Constructive ε-Nash equilibria for nonzero-sum differential games”. In: *IEEE Transactions on Automatic Control*, 60(4):950–965, (Apr. 2015).

[101] Cappello, D. and Mylvaganam, T. “Approximate Nash Equilibrium Solutions of Linear Quadratic Differential Games”. In: *21st IFAC World Congress*, 53(2):6685–6690, (2020).

[102] Cappello, D. and Mylvaganam, T. “Distributed Differential Games for Control of Multi-Agent Systems”. In: *IEEE Transactions on Control of Network Systems*, 9(2):635–646, (June 2022).

[103] Hall, S. et al. “Stability Certificates for Receding Horizon Games”. (Apr. 2024).

[104] Alessio, A. and Bemporad, A. “Decentralized model predictive control of constrained linear systems”. In: *2007 European Control Conference (ECC)*, pages 2813–2818, Kos, (July 2007). IEEE.

[105] Alessio, A., Barcelli, D. and Bemporad, A. “Decentralized model predictive control of dynamically coupled linear systems”. In: *Journal of Process Control*, 21(5):705–714, (June 2011).

[106] Keviczky, T., Borrelli, F. and Balas, G.J. “Decentralized receding horizon control for large scale dynamically decoupled systems”. In: *Automatica*, 42(12):2105–2115, (Dec. 2006).

[107] Venkat, A. et al. “Distributed MPC Strategies With Application to Power System Automatic Generation Control”. In: *IEEE Transactions on Control Systems Technology*, 16(6):1192–1206, (Nov. 2008).

[108] Camponogara, E. et al. “Distributed Model Predictive Control”. In: *IEEE Control Systems Magazine*, (2002).

[109] Magni, L. et al, editors. *Nonlinear Model Predictive Control: Towards New Challenging Applications*, volume 384 of *Lecture Notes in Control and Information Sciences*. Springer Berlin Heidelberg, Berlin, Heidelberg, (2009).

[110] Dunbar, W.B. “Distributed Receding Horizon Control of Dynamically Coupled Nonlinear Systems”. In: *IEEE Transactions on Automatic Control*, 52(7):1249–1263, (July 2007).

[111] Bei Hu and Linnemann, A. “Toward infinite-horizon optimality in nonlinear model predictive control”. In: *IEEE Transactions on Automatic Control*, 47(4):679–682, (Apr. 2002).

[112] Freiling, G., Jank, G. and Abou-Kandil, H. “Discrete-Time Riccati Equations in Open-Loop Nash and Stackelberg Games”. In: *European Journal of Control*, (1999).

[113] Van Huffel, S. et al. "High-performance numerical software for control". In: *IEEE Control Systems Magazine*, 24(1):60–76, (2004).

[114] Hespanha, J.P. *Linear Systems Theory.* Princeton University Press, Princeton Oxford, second edition edition, (2018).

[115] Darup, M.S. and Mönnigmann, M. "Computation of the Largest Constraint Admissible Set for Linear Continuous-Time Systems with State and Input Constraints". In: *IFAC Proceedings Volumes*, 47(3):5574–5579, (2014).

[116] Shi, S. and Lazar, M. "On distributed model predictive control for vehicle platooning with a recursive feasibility guarantee". In: *IFAC-PapersOnLine*, 50(1):7193–7198, (July 2017).

[117] Venkat, A.N. *Distributed Model Predictive Control: Theory and Applications.* (2012).

[118] Zhou, K., Doyle, J. and Glover, K. *Robust and Optimal Control.* Prentice Hall, (1996).

[119] Jiang, T. and Wei, M. "On solutions of the matrix equations X-AXB=C and X-AXB=C". In: *Linear Algebra and its Applications*, 367:225–233, (July 2003).

[120] Cai, C. and Teel, A.R. "Input–output-to-state stability for discrete-time systems". In: *Automatica*, 44(2):326–336, (Feb. 2008).

[121] Barth, M. and Boriboonsomsin, K. "Traffic congestion and greenhouse gases". In: *Access Magazine*, 1(35):2–9, (2009).

[122] European Commission and Directorate-General for Mobility and Transport. *Sustainable Transport Infrastructure Charging and Internalisation of Transport Externalities: Executive Summary.* (2019).

[123] Jahn, O., Möhring, R.H. and Schulz, A.S. "Optimal Routing of Traffic Flows with Length Restrictions in Networks with Congestion". In: *Operations Research Proceedings*, pages 437–442. Springer, (1999).

[124] Jahn, O. et al. "System-Optimal Routing of Traffic Flows with User Constraints in Networks with Congestion". In: *Operations Research*, 53(4), (2005).

[125] Angelelli, E. et al. "System optimal routing of traffic flows with user constraints using linear programming". In: *European Journal of Operational Research*, 293(3):863–879, (Sep. 2021).

[126] Wardrop, J.G. "Some theoretical aspects of road traffic research". In: *Proceedings of the Institution of Civil Engineers*, (3), (1952).

[127] Roughgarden, T. and Tardos, E.V. "How bad is selfish routing?". In: *Journal of the Association for Computing Machinery*, 49(2), (2002).

[128] Correa, R. and Stier-Moses, N.E. "Selfish Routing in Capacitated Networks". In: *Mathematics of Operations Research*, 29(4), (2004).

[129] Dafermos, S.C. "The Traffic Assignment Problem for Multiclass-User Transportation Networks". In: *Transportation Science*, 6(1), (1972).

[130] Dafermos, S. "Traffic Equilibrium and Variational Inequalities". In: *Transportation Science*, 14(1):42–54, (Feb. 1980).

[131] Larsson, T. and Patriksson, M. "An augmented lagrangean dual algorithm for link capacity side constrained traffic assignment problems". In: *Transportation Research Part B: Methodological*, 29(6), (1995).

[132] Calderone, D. and Sastry, S.S. "Markov decision process routing games". In: *8th International Conference on Cyber-Physical Systems*, (Apr. 2017).

[133] Li, S.H.Q. et al. "Tolling for Constraint Satisfaction in Markov Decision Process Congestion Games". In: *American Control Conference*. IEEE, (July 2019).

[134] Monderer, D. and Shapley, L. "Potential Games". In: *Games and Economic Behavior*, 14:124–143, (1996).

[135] Benenati, E., Colombino, M. and Dall'Anese, E. "A tractable formulation for multi-period linearized optimal power flow in presence of thermostatically controlled loads". In: *Conference on Decision and Control*, (2019).

[136] U. S. B. of Public Roads,. "Traffic assignment manual for application with a large, high speed computer". In: *US Department of Commerce, Bureau of Public Roads*, (1964).

[137] Wang, Y.H. "On the number of successes in independent trials". In: *Statistica Sinica*, 3:295–312, (1993).

[138] Wolter, K.M. *Introduction to Variance Estimation*. Statistics for Social and Behavioral Sciences. Springer, New York, 2nd ed edition, (2007).

[139] Scutari, G., Barbarossa, S. and Palomar, D. "Potential Games: A Framework for Vector Power Control Problems With Coupled Constraints". In: *International Conference on Acoustics Speed and Signal Processing Proceedings*. IEEE, (2006).

[140] Rockafellar, R., Wets, M. and Wets, R. *Variational Analysis*. Grundlehren Der Mathematischen Wissenschaften. Springer Berlin Heidelberg, (2009).

[141] Balakrishnan, V. and Vandenberghe, L. "Semidefinite programming duality and linear time-invariant systems". In: *IEEE Transactions on Automatic Control*, 48(1):30–41, (Jan. 2003).

[142] Belgioioso, G. and Grammatico, S. "On convexity and monotonicity in generalized aggregative games". In: *IFAC-PapersOnLine*, 50(1):14338–14343, (July 2017).

[143] Grimm, G. et al. "Nominally Robust Model Predictive Control With State Constraints". In: *IEEE Transactions on Automatic Control*, 52(10):1856–1870, (Oct. 2007).

[144] Krilašević, S. and Grammatico, S. "Learning generalized Nash equilibria in monotone games: A hybrid adaptive extremum seeking control approach". In: *Automatica*, 151:110931, (May 2023).

[145] Li, A. et al. "Receding Horizon Control Based Real-time Strategy for Missile Pursuit-evasion Game". In: *2024 IEEE 25th China Conference on System Simulation Technology and Its Application (CCSSTA)*, pages 408–415, Tianjin, China, (July 2024). IEEE.

[146] Ciuffo, B. et al. "Robotic Competitions to Design Future Transport Systems: The Case of JRC AUTOTRAC 2020". In: *Transportation Research Record: Journal of the Transportation Research Board*, 2677(2):1165–1178, (Feb. 2023).

[147] L'Erario, G. et al. "Modeling, Identification and Control of Model Jet Engines for Jet Powered Robotics". In: *IEEE Robotics and Automation Letters*, 5(2):2070–2077, (Apr. 2020).

[148] Benenati, E., Colombino, M. and Dall'Anese, E. "A tractable formulation for multi-period linearized optimal power flow in presence of thermostatically controlled loads". In: *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 4189–4194, Nice, France, (Dec. 2019). IEEE.

[149] Benenati, E. and Grammatico, S. "An Adaptive Convex Combination between Prediction and Correction in Online Quadratic optimization". In: *2021 25th International Conference on System Theory, Control and Computing (ICSTCC)*, pages 138–143, Iasi, Romania, (Oct. 2021). IEEE.

Acronyms

ARE	algebraic Riccati equation
cl-NE	closed-loop Nash equilibrium
FB	forward-backward
FBF	forward-backward-forward
FoRB	forward-reflected backward
GNE	generalized Nash equilibrium
GNEP	generalized Nash equilibrium problem
HSDM	hybrid steepest-descent method
I-FoRB	inertial forward-reflected backward
ISS	input-to-state stability
KKT	Karush–Kuhn–Tucker
LQ	linear dynamics-quadratic objective
LQR	linear quadratic regulator
LTI	linear time-invariant
MPC	model predictive control
NE	Nash equilibrium
OCP	optimal control problem
ol-NE	open-loop Nash equilibrium
OPF	optimal power flow
pFB	preconditioned forward-backward
PPP	preconditioned proximal-point
v-GNE	variational GNE
VI	variational inequality

About the author

Emilio Benenati was born in Catania, Italy, in 1994.

He received the Bachelor's degree in Electrical Engineering (with special distinction) from the University of Catania, Italy, in July 2016, with a thesis on the control of a robotic arm developed under the supervision of Prof. G. Muscato.

He obtained the Master's degree in Robotics, Systems and Control from the Federal Institute of Technology (ETH) Zürich, Switzerland, in April 2019. He wrote his Master thesis on demand-side control of an electric grid with Dr. Ing. M. Colombino, Prof. E. Dall'Anese and Prof. F. Dörfler during a research visit at the University of Colorado Boulder and the National Renewable Energies Lab (NREL), USA, in 2018.

In 2019-2020, he held a research position at the Italian Institute of Technology (IIT), Genova, Italy, in the Artificial and Mechanical Intelligence Lab (AMI), under the supervision of Dr. Ing. D. Pucci.

Since 2020 until the present day he has been a PhD student at the Delft Center for Systems and Control (DCSC) at the Technical University of Delft, the Netherlands, under the supervision of Prof. S. Grammatico. His research focuses on game-theoretic control of multi-agent systems and algorithms for optimal Nash equilibrium selection. During his studies, he visited as a research scholar the University of California in Santa Barbara, USA, in 2024, working with Prof. F. Bullo.

List of Publications

Journal articles and preprints

1. Benenati, E., Ananduta, W. and Grammatico, S. “Optimal Selection and Tracking Of Generalized Nash Equilibria in Monotone Games”. In: *IEEE Transactions on Automatic Control*, 68(12):7644–7659, (Dec. 2023)

2. Benenati, E. and Grammatico, S. “Probabilistic Game-Theoretic Traffic Routing”. In: *IEEE Transactions on Intelligent Transportation Systems*, 25(10):13080–13090, (Oct. 2024)

3. Benenati, E. and Grammatico, S. “Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities”. In: *arXiv preprint 2408.15703*, (2024)

4. Ciuffo, B. et al. “Robotic Competitions to Design Future Transport Systems: The Case of JRC AUTOTRAC 2020”. In: *Transportation Research Record: Journal of the Transportation Research Board*, 2677(2):1165–1178, (Feb. 2023)

5. L’Erario, G. et al. “Modeling, Identification and Control of Model Jet Engines for Jet Powered Robotics”. In: *IEEE Robotics and Automation Letters*, 5(2):2070–2077, (Apr. 2020)

Conference papers

1. Benenati, E., Ananduta, W. and Grammatico, S. “On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games”. In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 6389–6394, Cancun, Mexico, (Dec. 2022). IEEE

2. Benenati, E., Ananduta, W. and Grammatico, S. “A Semi-Decentralized Tikhonov-Based Algorithm for Optimal Generalized Nash Equilibrium Selection”. In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 4243–4248, Singapore, Singapore, (Dec. 2023). IEEE

3. Bianchi, M., Benenati, E. and Grammatico, S. “Linear Convergence in Time-Varying Generalized Nash Equilibrium Problems”. In: *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 7220–7226, Singapore, Singapore, (Dec. 2023). IEEE

4. Benenati, E., Colombino, M. and Dall’Anese, E. “A tractable formulation for multi-period linearized optimal power flow in presence of thermostatically controlled loads”. In: *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 4189–4194, Nice, France, (Dec. 2019). IEEE

5. Benenati, E. and Grammatico, S. “An Adaptive Convex Combination between Prediction and Correction in Online Quadratic optimization”. In: *2021 25th International*

Conference on System Theory, Control and Computing (ICSTCC), pages 138–143, Iasi, Romania, (Oct. 2021). IEEE

Included in this thesis.
Won 1^{st} place at the JRC AUTOTRAC 2020 competition.

Afterword

The Buddha, the Godhead, resides quite as comfortably in the circuits of a digital computer or the gears of a cycle transmission as he does at the top of the mountain, or in the petals of a flower. To think otherwise is to demean the Buddha - which is to demean oneself.

Robert M. Pirsig, in "The Zen and the art of motorcycle maintenance"

There's the math of music and there's the emotional part. Anybody can do math. The emotional part, the «why am I attracted to this as opposed to that» is the delineating point of a songwriter.

Billy Corghan

The journey that has led me to write this thesis has been long and complex. The beauty and the crux of working in research is that it combines the technical and scientific challenges with the puzzling experience of venturing into the uncharted territory of the creative process. There has been intense study and work involved, yet the most fierce challenges I faced came from the fact that I was absolutely unaware of what the profession of being creative implies. In this section, I humbly share with you my perspective on the matter.

The creative process in a scientific field is in essence not much different from the one involved in other, say, artistic ones. I noticed that ideas usually spur from moments of freedom and playfulness, during which the mind is able to roam freely and juggle anything that comes at hand. It is in this state that a mathematician makes a key breakthrough in a proof, that an engineer gets an unconventional design idea, that a musician performs an improvisation or comes up with a new song idea. One can find himself in this fluid state of mind at the work desk just as much while taking a shower: famously, Hamilton was thunderstruck by an idea on quaternions while strolling with his wife, and he ended up carving their fundamental formula on a bridge in Dublin. In an interview, Billy Corgan from The Smashing Pumpkins remembers that, while writing his hit song *Today*: "one day, out of the blue, I heard the opening lick note for note in my head". I believe that anyone who worked in a creative sector has had a similar experience. The first lesson I had to learn in my journey is that this state of mind is difficult to reach and delicate to maintain. Its most ruthless killer is the fear of making a mistake: there is nothing that impedes

your mind to roam freely like the paralyzing fear of tripping over. On the other hand, a necessary condition for it is a solid intuition over your subject. Without it, you will be weighing every step taken, which is not a good premise for a successful exploration. Intuition is a consequence of mastery: a skillful creator is able to take sharp decisions without much thought–or, to be more precise, with an *interiorized* stream of thoughts. In the words of Robert M. Pirsig:
"The craftsman isn't ever following a single line of instruction. He's making decisions as he goes along. For that reason he'll be absorbed and attentive to what he's doing even though he doesn't deliberately contrive this. His motions and the machine are in a kind of harmony. He isn't following any set of written instructions because the nature of the material at hand determines his thoughts and motions, which simultaneously change the nature of the material at hand. The material and his thoughts are changing together in a progression of changes until his mind's at rest at the same time the material's right."
A piece of improvised music comes from a musician who does not need to fit every single note as in a puzzle, but manages to smoothly connect their soul to the hands. A sculptor must exert a precise amount of pressure on the chisel, without letting this technical necessity become a friction point in their connection with the artistic creation. Similarly, a control engineer needs to master the necessary mathematical tools at an intuitive level, using them as a smooth connection between the concept in their mind and the technical creation. The need for intuition as a solid ground on which to build a creative playground is often overlooked in engineering at an academic level, sacrificed on the altar of pedantry. On this note, I firmly believe that a fundamental task for a teacher is to help their students develop a strong intuition of their subject, and that one of the roles of a researcher is to make their research clear and accessible, in order to make others build an intuition on the research output.

A second lesson comes from the realization that in such an unstructured state of mind, one cannot expect to conclude anything of scientific value. As John Cleese said quoting Alan Watts, "you can't be spontaneous within reason". Spontaneity is a building block for creativity but eventually, and especially in the scientific realm, reason must step in. One needs the discipline of restraining the playfulness of exploration when necessary, and proceed into a structured, controlled state of mind, that is fundamental for studying and conducting a scientific analysis. It is in this restricted state of mind that a scientist writes and reads pieces of scientific literature, runs experiments, polishes proofs. Just as is in the same state of mind a musician studies the theory, analyses music sheets, practices the technique, finalizes a composition. Alternating between these two states of mind is surprisingly hard, as I have come to realize.

These two states of mind are both necessary and codependent. Without discipline, one will not have the mastery in the tools needed to step into a playful state without fear of mistakes and friction between the mind and the act. Without letting the mind roam free, one will never find a diamond in the rough to polish and shape. The third lesson that I learned is that, ultimately, the result of this virtuous cycle of alternations is but a reflection of oneself. The Zen, this experience of fluid flowing of one's mind with the surrounding in a continuous shaping of amorphous ideas, imprints the object of creation with quality and beauty–if the beauty is in one's mind in the first place. As a quality piece of art is a crystallization of the artist's soul, so is a quality piece of mathematics or a technical creation. And to imprint a theorem, or a design, with quality, one must be at peace with themselves. This inner peace is also the source of the fuel that runs this two-cycle engine of alternating between playfulness and rigidity. One usually refers to such fuel as “motivation”, but I prefer the term used by Robert Pirsig: The *gumption*. This gumption behaves exactly like the fuel in a tank, that runs low over time–especially when the path gets rough, and one must periodically refill it by experiencing beauty. Simply put, connecting with friends, enjoying a sunny afternoon, listening to the right music, enjoying one's hobbies. Anything beautiful in life that refills your gumption and gets your flooded Zen engine running again.

I don't have the presumption to say that this work is an example of beauty–far from it, although I experienced that this creative free-flow has led to the bits that I believe are most valid. The key message I want to convey here is that this work has really been a collective construction. Without my academic mentors I would not have found the discipline to study with perseverance and I would not have had bright examples to follow and to motivate me. Without everyone else in my life that I hold dear, I would not have had the gumption to fuel me to the end of this journey. To you all–thank you, from the bottom of my heart. In the next chapter, I included a list with more specific acknowledgments, which I try to make as exhaustive as possible. Please forgive me if I missed you.

Acknowledgments

There are places I'll remember
All my life, though some have changed
Some forever, not for better
Some have gone and some remain

All these places have their moments
With lovers and friends, I still can recall
Some are dead and some are living
In my life, I've loved them all.

The Beatles, "In my life"

Perhaps the ache of homesickness was a fair price to pay
for having so many good people in her life.

Becky Chambers, in "The long way to a small, angry planet"

First and foremost, a warm thank you to my supervisor, dr. Sergio Grammatico, for everything he has taught me. He has been patient in the first steps of my academic path, perseverant in motivating me to produce quality research and supportive in my professional growth. His guidance has been invaluable.

Thank you to my co-promotor prof. Bart De Schutter, for providing reliable advice whenever asked. I deeply appreciate his dedication as the head of the department, always approachable and open to feedback.

Thank you to prof. Francesco Bullo and his research group, for hosting me during my visit at UCSB. His unwavering passion for research, his pursuit of clarity, intuition, and dedication to understanding the "core" of the problems have been an inspiration. It was an honor working with him, given that one of his books (and Prof. Florian Dörfler teaching its content) has led me to this academic path.

Thank you to all the academic mentors that I had the honor to work with. My gratitude goes to prof. Emiliano Dall'Anese and dr. Marcello Colombino, for hosting me at CU Boulder and co-authoring my first paper. I am grateful for this opportunity. Thank you to dr. Daniele Pucci and the colleagues at IIT, for a fantastic year of research. It was thrilling to work at the Italian JPL. Thank you to prof. Giovanni Muscato, for making me fall in love with control theory in the first place.

Thank you to my fantastic friends and research group fellows Suad, Mattia, Barbara, Wicak, Alitazaz, Georgios, Luyao, Reza, Silvia, Shije, Arghya, Nico, Lorenzo. It was an honor and a pleasure to work with brilliant scholars like you. Sharing the office made every work day brighter.

Thank you to all the remaining researchers at DCSC, for creating a fantastic workplace that I loved being a part of. I would mention you all, but there are just too many of you. Special thanks go to Alex, Claudia, Léonore, Frederik, Leila, Luca, Lotfi, Alessandro, Sam, Gianpietro, Francesco, Athina, David, Ilias, Maarten.

Thank you to all the DCSC staff, especially our beloved secretaries Francy, Helen, Bo, Sandra, Renate, Erica, Marieke, that have always made sorting things out a piece of cake.

Thanks to my students Vishwanath and Anbang, for their outstanding job, and for everything they taught me. I wish you the best in your careers.

Thanks to the AUTOTRAC team members Suad, Cas, And Vishwanath, and to Wim for supporting our project. Not even the sun could stop us.

The warmest thank you goes to my family, for always loving, supporting and motivating me in this long journey. In particular, thanks to dad, for being a bright example of passion and curiosity that have led me to become an engineer, for showing me the importance of hobbies in one's life, and for continuously inspiring and challenging me. Grazie a mia mamma, per avermi fatto sentire sempre amato e supportato nelle mie scelte di vita, nonostante mi abbiano portato lontano da casa. Thanks to Elena, my strongest ally at home. I know we will always rely on each other, and I am proud of you. Grazie a Nonna Lina, per aver infiammato la mia ambizione fin da bambino. Grazie a Nonna Elena, per la sua dolcezza, pazienza, sincerità, per tutte le storie, per tutti gli insegnamenti. Thanks to zia Sabrina and zio Franco, for showing me that there is a way off the beaten path. I truly wish I could have grown up closer to you. Thanks to zia Simona, zio Gianni, Andrea e Gabriele, for being my second family. Un caro ricordo a Nonno Tano e Nonno Emilio, venuti a mancare troppo presto, che conservo nel cuore. Vi voglio bene tutti.

Thank you Dominika, my lover and closest friend. You are a cornerstone of my life. Thank you for your unwavering trust in our relationship despite its unconventionality through all these years. Home is really wherever we are together.

Thank you, Rosario, for this friendship that has run parallel to each and every university exam and beyond. I couldn't have wished for a better companion.

Thank you, Francesco, Davide, Rachele, Mara, for making Bruxelles a tranquil cove where to refill my gumption.

Thank you, Bruno, Giulio, Gianluca, Andrea, Pietro, Federico, Lou, Ilario, Luca, Andrea, Matteo, Giacomo, Nausicaa, Maria Chiara, Alice, Luigi, Amalia, Cristina, for making Catania a place where it's always beautiful to come back, and for all the good memories.

Thank you, Chiara, for trying to convince me that the beauty that lies in the stars does not lie in the study of them. I still disagree.

Thank you, Marta and Elena, for our beautiful and long-lasting distance friendship.

Thank you, Charlotte, Elena, and Marco, for making Santa Barbara feel like home, and to the KCSB radio folks for all the music we discovered together.

Thank you, Kasia, for making Genova feel like home.

Thank you, Susan, Daniel, Stephanie, and Magda, for making Boulder feel like home.

Thank you, to all the colleagues at ABB and the folks at EESTEC, for all the fun we had in Zürich and for making it feel like home.

Thank you, Rolijne, Marleen, Anna, Cristina, Giulia, Claire, Riccardo, Giovanni, Luka, Phil, Paula, Kavya, Francesca, Aga, KD, for making Delft feel like home.

Thank you, Adriano, for the cover illustration of this thesis.

Thanks to music, “bigger than words and wider than pictures”, and to all the artists that with it have colored my life. Especially Kevin Shields. Mr. Shields, in the unlikely case that you are reading this, I’d be thrilled to exchange autographs.

Yours truly,
Emilio Benenati