RELIABILITY STUDIES OF SLOPE STABILITY

Effect of various approaches to derive distributions of soil properties in reliability studies of slope stability.

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TU Delft

Reliability studies of slope stability

Effect of various approaches to derive distributions of soil properties in reliability studies of slope stability

by

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Abstract

In the past few decades, the use of Bayes' theorem in geotechnical engineering has increased significantly. A good amount of work is done on implementing this theorem for improving the knowledge on soil properties, identification of soil strata, and quantifying different uncertainties involved in predicting soil properties. However, the results of using this approach on a geotechnical structure have not been mentioned in the past literature. Hence, in the current work, Bayes' approach is used to predict the soil property parameters and further these parameters are used in slope stability analysis of a basic slope. It is aimed to investigate if the uncertainties in such analysis can be quantified. For this, a simple Bayes' model is developed to predict the input parameters of the soil properties. This model is developed using the PyMC3 package in python which is specially developed to solve Bayes' problems. Along with Bayes' statistics, normal/frequentist statistics is also used to derive the soil properties from two different data sets (3000 data points and 24 data points). To conduct the reliability studies of a slope, a Finite element model developed at TU Delft is used. The results of slope stability analysis obtained using both approaches are further compared. Results clearly show that when uncertainty is involved Bayes' approach encapsulates its effect on calculated FOS whereas the frequentist approach doesn't. This difference in both approaches is observed due to consideration of all the possible values of input parameters by Bayes' approach. With the frequentist approach, a single input parameter is predicted neglecting all the other possible inputs. Hence, it is concluded that using the Bayes' approach better and more reliable results over the frequentist approach by providing more insight about the uncertainties involved in the analysis.

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Introduction

SOIL is complex material. Plenty of variabilities is involved in its properties. Such variability comes from various sources of uncertainties. The primary sources are inherent variability, measurement error, and transformation error [3]. Inherent variability is nothing but the actual variability of the soil which is present in the soil structure way before any project or construction is planned and is affected by various geological processes [4]. Measurement errors are generated when the soil properties are measured by some physical means. These measured values of soil properties from the physical tests are not directly used in design calculations. For these, transformation models are required to relate the measurements variables to design variables. Also, transformation models are formulated to obtain different material properties from the same set of test results. In such models, some degree of uncertainty is developed since these models are obtained by empirical data fitting [3].

These complexities or variability in the soil properties make geotechnical problems difficult to assess. It becomes necessary to carry out assessment studies of a geotechnical problem in terms of failure probability or the reliability of the structure. But often it is found that such studies provide overestimated values of soil properties. Due to the use of such overestimated values in the reliability studies, structure rarely fails than what the reliability studies assume. Zang and Cao [4] have shown that this overestimation occurs due to consideration of the total variability in the reliability studies. Total variability comprises of inherent (actual) variability of soil and knowledge uncertainties (measurement errors, statistical uncertainties, and transformation uncertainties). The inherent variability part of the total variability observed in the site characterization is relatively small compared to the total variability [4]. This makes it important to explicitly define the actual/inherent variability to increase the accuracy in reliability studies. One of the methods to do this is by implementing the Bayes' method of characterizing the soil properties.

The Bayesian technique is a method in statistics used to update the probability of a problem as more evidence/information becomes available. In geotechnical problems, this approach has been implemented to find out the actual variability of soil properties and increase the precision in the values (Zang and Cao, [4] [5] [6] [7]). A general Bayesian framework for optimizing property values consists of 3 major components: Prior knowledge, Likelihood function and Posterior knowledge [4]. The prior knowledge is obtained from desk study and site reconnaissance and the likelihood function represents the site investigation data. Using the Bayes' method, prior knowledge and likelihood function are combined to give a posterior distribution which provides the updated knowledge of the soil properties.

In this work, an attempt is made to use Bayesian statistics in reliability studies of a slope. There are different software's available which carry out reliability study of slope using limit equilibrium method (LEM) and finite element method (FEM). Herein, FEM software developed at TU Delft is used to solve the slope stability problem. To this software, a strength profile of soil is provided as input using different input parameters (trend, coefficient of variation, the scale of fluctuation, etc). Using the strength reduction method, the Factor of Safety (FOS) for a particular slope is obtained. Further, using the Random Finite Element Method (RFEM), multiple realizations of the slopes can be generated to stochastically analyze the slope. To derive these input parameters of soil properties, general statistics (frequentist approach) is used. Hence in this report, the input parameters of soil properties for a slope stability problem are derived using Bayes' statistics. The major objective is to study the effects of using the input parameters using Bayes' theorem in reliability studies of a slope and further compare those with general/frequentist statistics.

1.1. PROBLEM STATEMENT AND RESEARCH QUESTIONS

GENERALLY, to conduct reliability studies of a slope, a basic slope model is developed to which the point statistics of material property is provided as an input and the output is obtained in the form of reliability curves (pdf/CDF graphs) providing the factor of safety for the slope to fail. The main objective of this thesis is to 'study the effect of various approaches (frequentist and Bayes' approach) to derive the input statistics of soil properties on the Factor of Safety of the slope'. For this, the following questions are studied.

1. How can the total uncertainty in the slope stability analysis be quantified?
2. What is the effect of considering different data sets (in terms of data points)?

1.2. APPROACH

To tackle the above-mentioned questions, a Bayesian model for predicting the input parameters of soil properties is to be developed. This model is developed using the PyMC3 package in python. This package is specially developed to conduct Bayesian studies. The model developed is then further used to derive the input statistics for a slope stability problem.

To conduct the reliability studies on a slope, FEM software developed at TU Delft is used. Input parameters of soil properties are derived using the frequentist approach as we as using the Bayes' approach. These input parameters derived using both approaches are then used to solve the slope stability problem. The output is obtained in terms of FOS. Cumulative distribution function (cdf) and probability distribution functions (pdf) are plotted for the obtained FOS values to compare the results of both approaches. Also, different size data sets of soil properties are used to see the difference in predicting the input parameters using both approaches, ultimately showing its effect on FOS calculation.

1.3. DOCUMENT STRUCTURE

In this thesis, there is a total of 6 Chapters aiming towards answering the research questions.

- Chapter 2 is a Literature survey that discusses the use of Bayes' theorem in geotechnical uncertainties, geotechnical studies, current reliability study of slope stability problems. Further major interpretations are derived using which the problem statement of this work is derived.
- Chapter 3 discusses the basics of Bayes' theorem. An attempt is made to explain the Bayes' theorem through 2 basic examples.

- Chapter 4 the Bayes' model to be used for deriving the input parameters for slope stability problem is developed.
- Chapter 5 covers the Slope stability analysis of a basic slope model. Both approaches, frequentist and Bayes' approach are used to derive the input parameters and further the results are discussed and compared.
- Chapter 6 contains the overall conclusion of the work and covers recommended future scope.

2

Literature review

2.1. GEOTECHNICAL UNCERTAINTY

THE complexity of geotechnical variability is a result of many disparate sources of uncertainties. The three primary sources are inherent variability, measurement error and transformation error. The inherent variability is a formed due to natural geological processes that continually modifies the soil mass in situ. Measurement error is caused by equipment, procedural-operator and random testing effects. Transformation uncertainty is introduced when the field or laboratory test results are transformed into design soil properties using empirical or other correlation models. Following figure 1 illustrates the above mentioned sources of variability in soil properties.

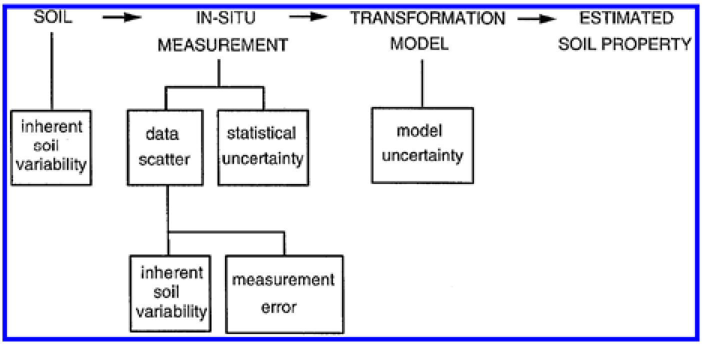

Figure 1: Uncertainty in soil property estimates.[8]

Phoon and Kulhawy conducted an extensive literature review in which the COV (coefficient of variation) and scale of fluctuation of inherent variability, measurement error [8] and transformation error [3] are evaluated in detail along with the general soil type and approximate range of mean value for which the COV is applicable. Tables with summarised data of soil type, number of tests and mean and COV of soil properties are presented. The data summaries inherent variability of strength properties (undrained shear strength, effective friction angle), index parameters (liquid limit, plastic limit, liquidity and plasticity index, relative density, etc.) and field measurements (CPT tip resistance, number of SPT blow count, etc.). Horizontal and vertical scale of fluctuation of some common geotechnical properties such as CPT tip resistance, unit weight, undrained shear strength for different soil types is summarised. Following figure 2 shows a table which summarises the COV of inherent variability for various test measurements. It also includes general soil type and the approximate range of mean value for which the COV is applicable.

Table 7. Approximate guidelines for inherent soil variability (source: Phoon et al. 1995, p. 4-49).

Test type	Property	Soil type	Mean	COV(%)
Lab strength	s_u(UC)	Clay	10–400 kN/m^2	20–55
	s_u(UU)	Clay	10–350 kN/m^2	10–30
	s_u(CIUC)	Clay	150–700 kN/m^2	20–40
	$\bar{\phi}$	Clay and sand	20–40°	5–15
CPT	q_T	Clay	0.5–2.5 MN/m^2	<20
	q_c	Clay	0.5–2.0 MN/m^2	20–40
	q_c	Sand	0.5–30.0 MN/m^2	20–60
VST	s_u(VST)	Clay	5–400 kN/m^2	10–40
SPT	N	Clay and sand	10–70 blows/ft	25–50
DMT	A	Clay	100–450 kN/m^2	10–35
	A	Sand	60–1300 kN/m^2	20–50
	B	Clay	500–880 kN/m^2	10–35
	B	Sand	350–2400 kN/m^2	20–50
	I_D	Sand	1–8	20–60
	K_D	Sand	2–30	20–60
	E_D	Sand	10–50 MN/m^2	15–65
PMT	p_L	Clay	400–2800 kN/m^2	10–35
	p_L	Sand	1600–3500 kN/m^2	20–50
	E_{PMT}	Sand	5–15 MN/m^2	15–65
Lab index	w_n	Clay and silt	13–100%	8–30
	w_L	Clay and silt	30–90%	6–30
	w_P	Clay and silt	15–25%	6–30
	PI	Clay and silt	10–40%	—[a]
	LI	Clay and silt	10%	—[a]
	γ, γ_d	Clay and silt	13–20 kN/m^3	<10
	D_r	Sand	30–70%	10–40; 50–70[b]

[a]COV = (3–12%)/mean.
[b]The first range of values gives the total variability for the direct method of determination, and the second range of values the total variability for the indirect determination using SPT values.

Figure 2: Approximate guidelines for inherent soil variability.[8]

In geotechnical design, the direct measurements of soil properties from tests performed is converted into an appropriate design property value using a transformation model. This introduces some degree of uncertainty since most of the transformation models are obtained by empirical data fitting. This uncertainty also exists for theoretical relationships due to idealizations and simplification in the theory. The transformation uncertainty can be quantified using probabilistic methods as shown in the figure 3 below.

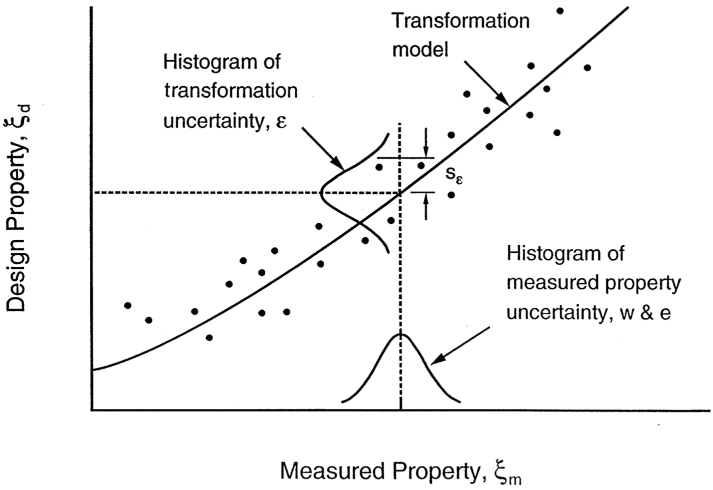

Figure 3: Probabilistic characterisation of transformation model. [3]

The transformation model typically is evaluated using regression analyses and the spread of the data around the regression curve is modelled as zero-mean random variable. The standard deviation of this zero-mean random variable is an indicator of the magnitude of transformation uncertainty. To determine the typical range of variability for some common design soil properties, a simple second-moment probabilistic approach which consistently combines the inherent variability, measurement error and transformation uncertainty is discussed. Using the statistical data on inherent variability and measurement error form the companion paper as realistic inputs, the transformation uncertainty can be evaluated. Design properties such as undrained shear strength, effective stress friction angle, in-situ horizontal stress coefficient and Young's modulus are considered in this paper. Different correlations for every design property are examined. For example, in case of undrained shear strength, the correlations with vane shear test, SPT N value, corrected cone tip resistance, plasticity index and DMT horizontal stress index is examined. Following figure 4 shows the table with summary of variability of the design properties as a function of test, correlation equation and soil type.

Design property[a]	Test[b]	Soil type	Point COV (%)	Spatial average COV[c] (%)	Correlation equation
s_u(UC)	Direct (lab)	Clay	20–55	10–40	—
s_u(UU)	Direct (lab)	Clay	10–35	7–25	—
s_u(CIUC)	Direct (lab)	Clay	20–45	10–30	—
s_u(field)	VST	Clay	15–50	15–50	14
s_u(UU)	q_T	Clay	30–40[d]	30–35[d]	18
s_u(CIUC)	q_T	Clay	35–50[d]	35–40[d]	18
s_u(UU)	N	Clay	40–60	40–55	23
s_u[e]	K_D	Clay	30–55	30–55	29
s_u(field)	PI	Clay	30–55[d]	—	32
$\bar{\phi}$	Direct (lab)	Clay, sand	7–20	6–20	—
$\bar{\phi}$ (TC)	q_T	Sand	10–15[d]	10[d]	38
$\bar{\phi}_{cv}$	PI	Clay	15–20[d]	15–20[d]	43
K_o	Direct (SBPMT)	Clay	20–45	15–45	—
K_o	Direct (SBPMT)	Sand	25–55	20–55	—
K_o	K_D	Clay	35–50[d]	35–50[d]	49
K_o	N	Clay	40–75[d]	—	54
E_{PMT}	Direct (PMT)	Sand	20–70	15–70	—
E_D	Direct (DMT)	Sand	15–70	10–70	—
E_{PMT}	N	Clay	85–95	85–95	61
E_D	N	Silt	40–60	35–55	64

[a] E_D, dilatometer modulus; E_{PMT}, pressuremeter modulus; K_o, in situ horizontal stress coefficient; s_u, undrained shear strength; s_u(field), corrected s_u from vane shear test; $\bar{\phi}$, effective stress friction angle; $\bar{\phi}_{cv}$, constant-volume $\bar{\phi}$; TC, triaxial compression; UC, unconfined compression test.
[b] K_D, dilatometer horizontal stress index; N, standard penetration test blow count; PI, plasticity index; q_T, corrected cone tip resistance.
[c] Averaging over 5 m.
[d] COV is a function of the mean; refer to COV equations in the text for details.
[e] Mixture of s_u from UU, UC, and VST.

Figure 4: Summarised approximate guidelines for design soil properties.[3]

This extensive literature review conducted by Phoon and Kulhway [3],[8] to estimate the statistics of inherent variability, measurement error and Transformation error concluded that, the COV of inherent variability for sand is higher than that for clay when the soil type is considered. The highest COV of inherent variability is associated with measurements in horizontal direction and measurements of soil modulus. Scale of fluctuation is an important parameter in identifying the inherent variability but the information on this is limited. The studies show that the vertical and horizontal scales of fluctuation are highest for index parameters. Statistical information on measurement error is also limited. In case of transformation error, the method of calculating values of parameters (direct/indirect) can affect the range of COV.

2.2. USE OF BAYES THEOREM IN CURRENT GEOTECHNICAL STUDIES

In today's geotechnical analysis, probability of failure is predicted using the total variability of soil (actual variability + measurement errors and statistical uncertainties) arising two important questions 'why are failure less frequent than our reliability studies?' and 'what is the actual variability of soil?'. To figure out the answers for these questions Yu Wang, et. al. [4] developed a Bayesian inverse analysis framework for direct quantification of inherent variability of soil and rock properties. This robust framework is developed such that it streamlines the formulation of likelihood functions for various soil and rock properties when measured using different in-situ and laboratory tests.

The Bayesian framework consists of 3 key components: prior distribution, likelihood function and solving Bayesian equation to get posterior information. Prior knowledge is obtained by doing desk study and site reconnaissance whereas the likelihood function is formulated using the site observation data obtained from in-situ/laboratory testing. Following figure 5 shows the Bayesian framework for geotechnical characterization of a project site. X_M represents the observation data from site, X_D represent the design property value which is calculated using a transformation model which is function of X_M and ϵ_t which is a random variable representing error in transformation model M_T. M_P is a probabilistic model (random variables or fields) for inherent variability of X_D. Both M_T and M_P together form a likelihood model M_L.

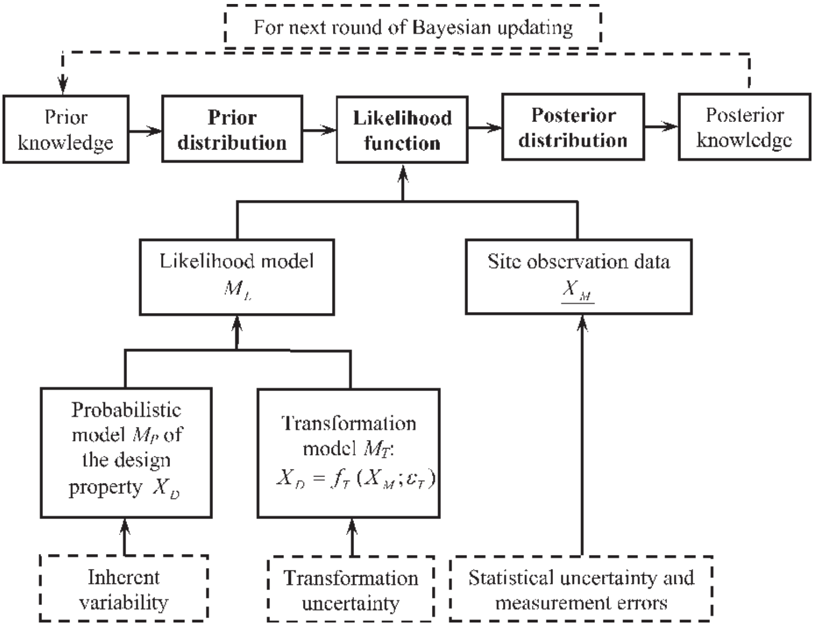

Figure 5: Bayesian framework for geotechnical characterization of a project site.[4]

The most critical components in this framework are formulating the prior distribution and likelihood function. As shown in the figure, the likelihood model needed to derive the likelihood function depends on the probabilistic model and transformation model. Probabilistic models like random fields or random variables can be adopted. Implementation procedure of this approach and two ways of solving the Bayesian equation is further discussed in brief. One of the method is Bayesian equivalent sample method in which the actual variability of design property in a soil layer is modelled by a random variable and its model parameters. In this method MCMCS process is used to overcome the difficulty in expressing the complicated pdf of design propert because of sophisticated likelihood function and prior knowledge. MCMCS (Markov Chain Monte Carlo Simulation) is a numerical process which simulates a sequence of samples of random variable. It is a feasible way to generate samples from an arbitrary pdf[9]. A large number of samples are generated by using MCMCS and are used to plot pdf, cdf of design property to estimate its statistics. This method solves the problem of estimating the statistics of soil and rock properties from limited amount of site-specific observation data.

The second method is Bayesian system identification and model class selection method. In this method, for a soil profile with multiple layers, mutually independent random fields are used to characterize the inherent spatial variability of design property of each layer. The obtained model parameters of the posterior distribution are then maximized to get the most probable value (MPVs) of model parameters using Laplace asymptotic approximation. Along with the MPV's, the Bayesian system identification method gives the posterior standard deviations of random field model parameters.

Following are few examples in which Bayes' technique is applied with brief discussion on it's implementation procedure:

- Probabilistic characterization of Young's modulus of soil (Yu Wang, Zijun Cao [9]): A MCMCS based approach is developed to probabilistically characterize the undrained Young's modulus (Eu) of clay using SPT tests. This approach uses the site information available prior to the project i.e. 'prior knowledge' and project specific information from the laboratory and in-situ tests performed. First in the Bayesian framework, the probabilistic modelling of the inherent variability of Eu and the transformation uncertainty associated with regression between Eu and SPT (N) values is done. Further a PDF of Eu based on prior knowledge and project specific SPT data is derived. Then, using MCMCS and derived PDF, 30,000 equivalent samples of Eu are generated and with the help of conventional statistical analysis of these samples, the statistics of Eu and its characteristic value both are determined. MCMCS is integrated with the Bayesian approach to effectively tackle the difficulty in generating samples form realistic prior distributions which are generally complex posterior pdf's. To explicitly model the inherent variability, Eu is represented by a lognormal random variable with a mean and standard deviation. The Eu value of the soil is obtained by the regression between the Eu and the N values measured during the SPT test using the eq. provided by Kullaway and Phoon. Metropolis-Hastings (MH) algorithm is used in MCMCS to generate equivalent samples of Eu. These samples are equivalent to those measured physically in laboratory or in-situ tests from statistical point of view. The MCMCS samples are generated form the pdf of Eu, hence contain the integrated information of both site-specific observation data and prior knowledge. The implementation procedure of the MCMCS based Bayesian approach is applied on the sample obtained from the clay site of the NGES at Texas A&M University. It was found that the proposed approach provides quite similar results when compared to the results obtained from large laboratory and in-situ test data (42 pressuremeter tests in this case). Following figure 6 shows the comparison of cdf's of E¬u estimated using 30,000 equivalent samples and 42 pressuremeter tests.

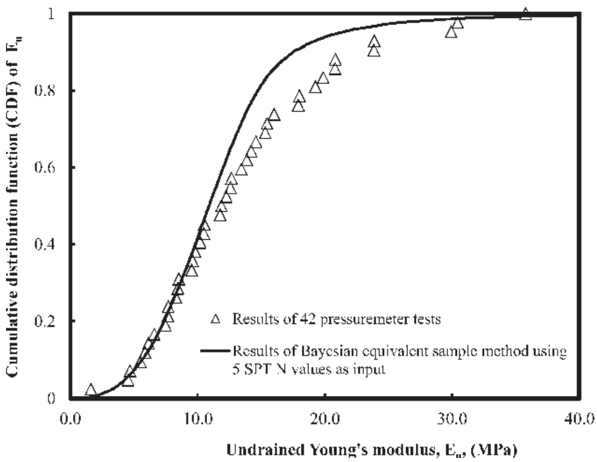

Figure 6: Validation of the probability distribution for the undrained Young's modulus estimated from equivalent samples.[9]

It can be clearly seen that data provided by 30,000 equivalent samples is consistent with the data obtained from 42 pressuremeter tests performed. Sensitivity studies (to study the effect of quantity of project specific test data) were conducted using 10 SPT data sets with different no. of SPT (N) values in every set. Results showed that in case of estimating the mean value, the equivalent samples are better estimations than those directly estimated from the regression model without prior knowledge. By incorporating relatively limited available prior knowledge, the equivalent sample approach improves the mean value estimation significantly. Similar results are achieved in case of standard deviation estimations. Following figure 7 shows the estimates of the mean and standard deviation.

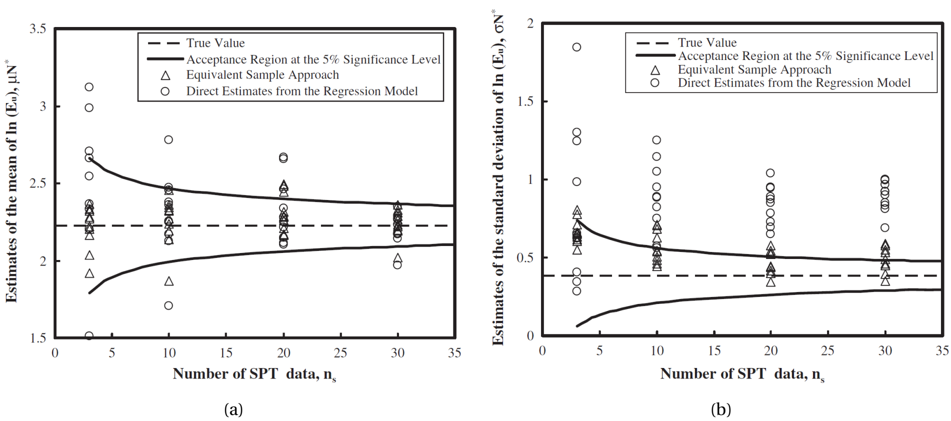

Figure 7: Effects of project specific test data: Estimates of the mean of ln(Eu) (a) and Estimates of the standard deviation of ln(Eu) (b).[9]

Sensitivity study on prior knowledge is also carried out using 4 different sets of prior knowledge. Each set have different range of mean and standard deviation values. 2 sets of prior knowledge follow a uniform distribution, 1st set is used as baseline case and in 2nd the range of mean and standard deviation is much smaller than 1st set. Sets 3 and 4 follow an arbitrary histogram type of distribution. 3rd set has similar range of mean and standard deviation but with relatively high PDF values allocated close to true values of mean and standard deviation. 4th set is the truncated Gaussian best fit of set 3. In case of sets following uniform distribution, both the mean and the standard deviation values were improved with consistent and informative set of the prior knowledge which is obvious result. Sets following an arbitrary histogram type of distribution are slightly more informative than those following a uniform distribution. The proposed approach effectively tackles the difficulty in generating meaningful statistics from very limited soil data obtained from the site investigation. Also, the sensitivity analysis shows that the approach is general and applicable to different type of project specific test data and prior knowledge.

- Probabilistic characterization of sand friction angles (Yu Wang, et. al. [5]): A Bayesian framework is developed in conjunction with limited number of cone penetration tests for estimating the effective friction angle of sand and model the inherent spatial variability with random fields. First the random field modelling of the effective friction angle of sand and regression between the cone tip resistance and effective friction angle is carried out followed by the development of the Bayesian framework. To model the inherent spatial variability of the effective friction angle, random field theory by Vanmarcke [10] is applied. A semi-log regression equation between the cone tip resistance and sand friction angle provided by Kulhaway and Mayne [11] is used. In the Bayesian framework, the updated knowledge (mean, standard deviation and correlation length) about the model parameters is reflected through their joint posterior distribution given the observation data and prior information. The proposed Bayesian framework was applied to get probabilistic characterisation of the effective friction angle at the sand site of NGES at Texas A&M University. It has been shown analytically that uncertainty about effective friction angle stems the inherent spatial variability, and uncertainty regarding the estimates of spatial variability and the mean effective friction angle. The spatial variability can only be improved in accuracy, but cannot be eliminated, by collecting more data. On the other hand, uncertainty regarding the estimates of spatial variability and the mean effective friction angle can be reduced as the amount of independent data increases, providing a way to determine whether the amount of project-specific information (e.g., in-situ and/or laboratory tests) is sufficient in site characterization. It also concluded that if uncertainty regarding estimates of spatial variability and the mean effective friction angle is relatively small compared to inherent spatial variability, more tests (laboratory/in-situ) will provide very less reduction in overall uncertainty whereas if the difference is large, additional tests will effectively reduce the overall uncertainty. To study the effect of different sources of information in Bayesian technique, analytical solutions on two asymptotic cases of posterior mean is derived which can be used to check the results of using Bayesian technique. The application example with real data showed that the Bayesian approach properly accounts for the effects of prior information and data in the updating process.

- Underground soil stratification using CPT's (Yu Wang, et. al.[7]): The uncertainty in CPT-based soil classification using Robertson chart is explicitly modelled using Bayesian technique. Maximum entropy principle is used to figure out that to which soil type in Robertson chart does the sample belong. Previous approaches of soil classification both deterministic and probabilistic focus mainly on the soil type from a particular data point leading to a question of how to stratify the underground soil profile and identify different soil layers from a large number of continuous CPT data. The main issue of this problem is the uncertainty in the CPT based soil classification and the spatial distribution of the CPT data. Identification of the underground soil layers based on CPT data is to determine the layer thickness in a soil profile with N soil layers. This number of layers is considered deterministic but unknown and are further determined using a Bayesian model class selection approach presented in the paper. In this study for the probabilistic soil classification based on the Robertson chart, the pdf (probability density function) is taken as the commonly used Gaussian distribution which is consistent with the results obtained by using the principle of maximum entropy, a theoretical basis of the Bayesian School to quantify uncertainty and assign a pdf for a given set of information. Maximum entropy principle states that the proper pdf for a given set of information is the one with the maximum entropy while satisfying all the constraints given by the information. In the context of CPT-based probabilistic soil classification, the information available from the CPT test is the measured values of $[\ln(F_R i), \ln(Q_t i)]$. Such values can be reasoned as the centre or expected values of the pdf that spreads over the Robertson chart. $F_R i$ is the normalised friction ratio and $Q_t i$ is normalised cone resistance.

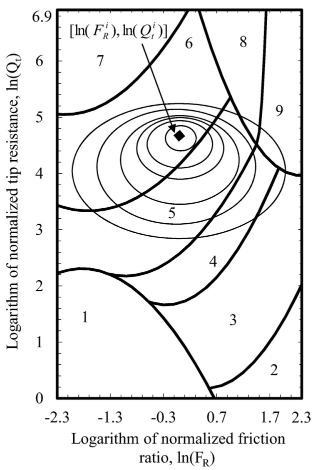

Figure 8: Illustration of the probability density contour and two dimensional joint pdf for soil classification based on a given CPT data point. [7]

The parameters of interest are thickness of the soil layers. The Bayesian framework used in this study contain two major components: Bayesian model class selection approach to identify the most probable number of underground soil layers and Bayesian system identification approach to simultaneously estimate the most probable layer thickness and classify the soil type. Following figure 9 shows the implementation procedure provided.

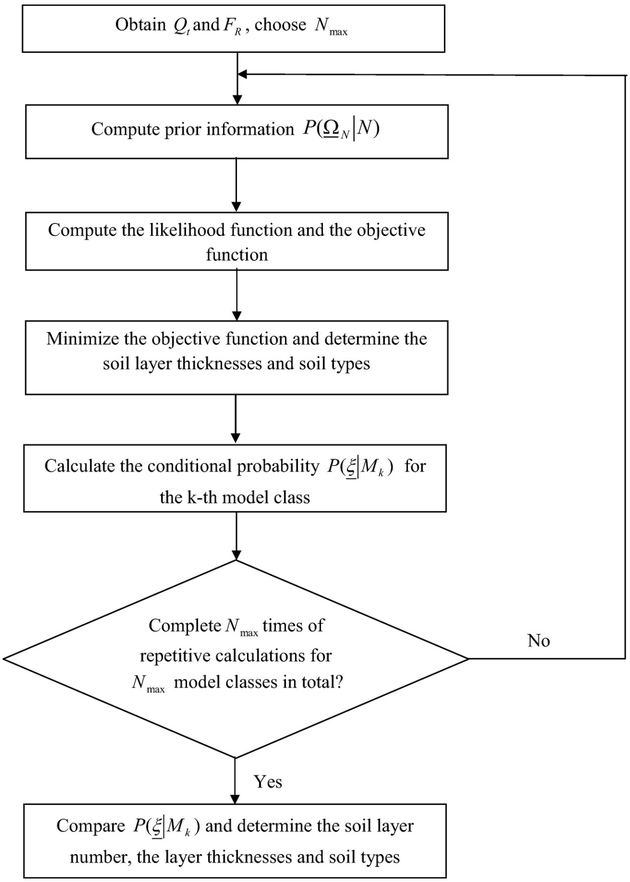

Figure 9: Flowchart for the proposed Bayesian process to identify the soil layers.[7]

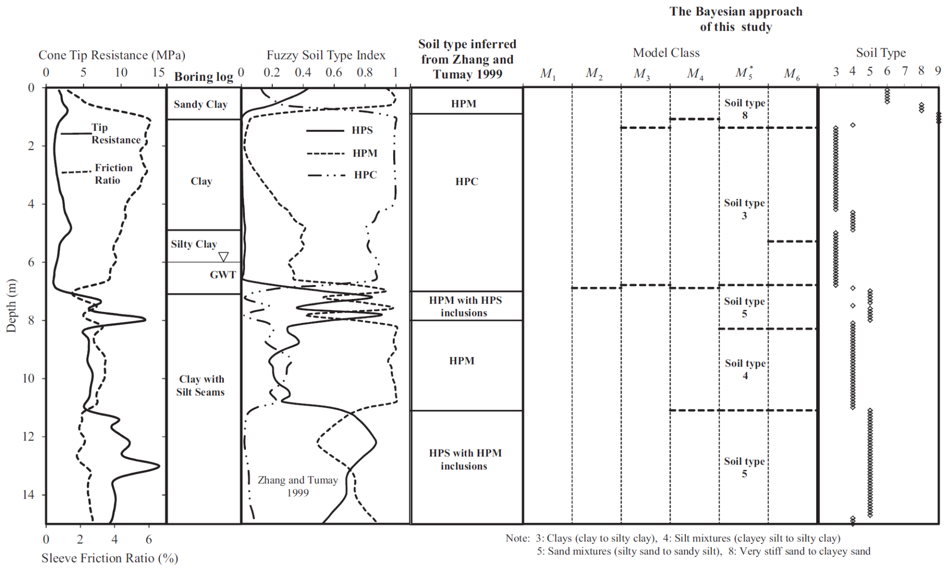

Figure 10: Comparison among the result of the boundaries determined by different approaches in the illustrative example.[7]

The proposed approach is applied on a case study involving sample CPT test performed at the NGES site of Texas A&M University. Figure 10 compares the results of proposed Bayesian approach with other approaches. It has been shown that the proposed approaches properly identify the underground soil stratification and classify the soil type of each layer. In addition, as the number of model classes increases, the Bayesian model class selection approach identifies the soil layers progressively, starting from the statistically most significant boundary and gradually zooming into less significant ones with improved resolution. It is also found that, although the most probable model class (the most probable number of soil layers) and its corresponding most probable layer thicknesses are of primary interest in the identification of the underground soil stratification, the evolution of the identified soil strata as the model class increases also provides valuable information for assisting in the interpretation of CPT data in a rational and transparent manner.

- Identification of soil strata using water content data (Yu Wang, et.al. [6]): LCF (London Clay Formation) is generally determined based on their lithological characters and fossil contents with which geotechnical engineers are less familiar. In identification of the soil strata in LCF, attempts have been made to correlate results of conventional geotechnical tests like SPT's and index tests such as water content and liquid limit. However, it is found out that the water content in the soil profile is inevitably scattered and contains various uncertainties. Following figure 11 shows an example of water content profile at St James's Park, London.

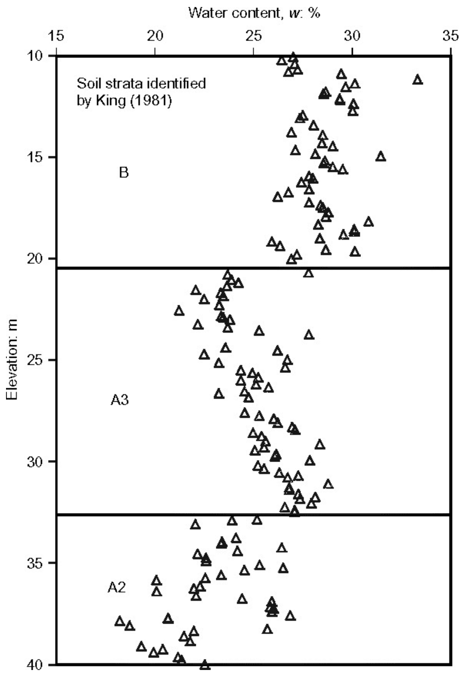

Figure 11: Water content profile and soil strata identified in London clay at St James's Park, London.[6]

The division of a water content profile into multiple layers is, therefore, somehow obscured, subjective and may be inconsistent among different engineers. To assist in division of a water content profile into multiple layers in an objective manner and to facilitate identification of soil strata in LCF using water content data, a Bayesian approach is developed. The approach contains a Bayesian system identification method that is used to identify the thickness of soil layer and Bayesian model class selection to select the most probable number of soil strata. Based on likelihood function and the prior distribution, Bayesian analysis is carried out to provide the posterior distribution (pdf) for the model parameters and the most probable thickness of soil layer. The number of soil layers in Bayesian system identification is considered deterministic but unknown. Using Bayesian model class selection approach, the most probable value of number of layers among a pool of candidate model class (family of stratification models that share the same number of soil layers in LCF but have different model parameters) can be selected. LFC water content profile at St James's Park, London is used to illustrate the proposed approach. Following figure 12 shows the results of the proposed approach for identifying the soil strata.

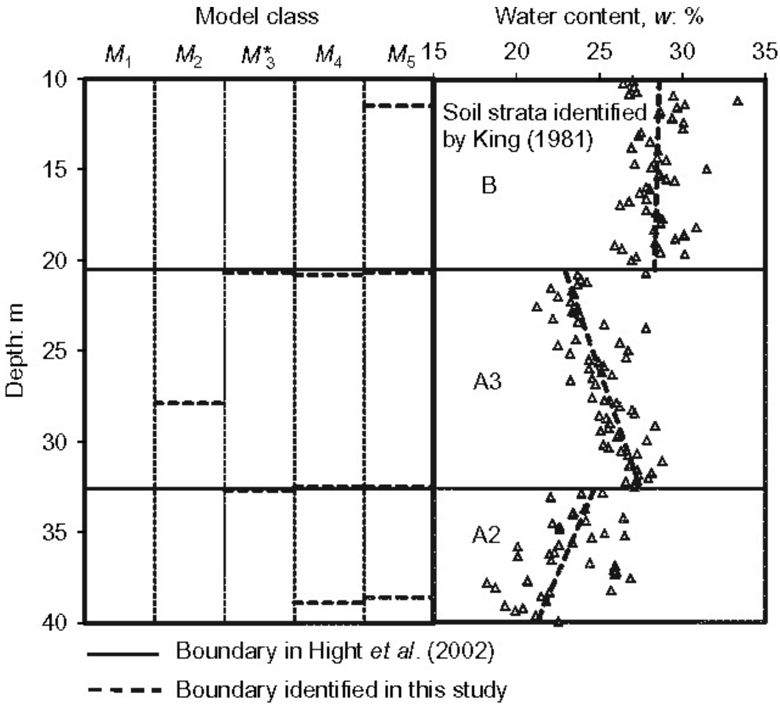

Figure 12: The most probable soil strata identified from the proposed approach.[6]

Further, sensitivity analysis is performed to study the effect of data quantity (measurement interval and the number of measurements at same depth in water content profile). It concluded that relatively small measurement intervals or repeated measurements at same depth are needed to properly identify LCF soil strata. Also using the information form other sources such as Atterberg limits and lithological characters reduces the quantity of water content data required in the proposed Bayesian approach.

In this section, few application of Bayes' theorem in geotechnical engineering are discussed. It is intended to use this theorem in slope stability analysis to answer the research questions mentioned in section 1.1. For this it is important to understand current practice on slope stability analysis. Hence, in next section, a background study on probabilistic approach in slope stability analysis is provided.

2.3. PROBABILISTIC APPROACH IN THE SLOPE STABILITY ANALYSIS

A case study on assessment and re-design of an existing dyke (section of dyke ring at the Starnmeer polder situated in province of North Holland) founded on a layered soil is presented by Michael A. Hicks, et. al. [12]. A stability assessment carried out by Hoogheermraadschap Hollands Noorderkwartier (HHNK) using the limit equilibrium software D-Geo stability showed that the sections do not comply to current safety requirements and return the factor of safety (FOS) values as low as 0.5 even though the dyke has stayed stable for hundreds of years.[12]

Re-analysis of a dyke section, which showed FOS of 0.59 using D-Geo stability software [13], is carried out by the authors using an in-house finite element software developed at TU Delft. Similar to the assessment carried out by HHNK, the strength parameters used in this analysis were cohesion and tangent of friction angle. Both deterministic and stochastic analysis were performed and the FOS values for using mean, 5-percentile and design property values were obtained. The following figure 13 shows the table that summarises the results.[12]

Deterministic analysis		Stochastic analysis		
			F corresponding to CDF of 0.05	
Property values	F	θ_h (m)	without partial factors	with partial factors
Mean	1.31	0.5	1.10	–
5-percentile	0.66	6.0	0.98	0.85
Design	0.54	12.0	0.98	–

Figure 13: Results of re-analysis carried out using in-house finite element software.[12]

In stochastic analysis, to account for the spatial variability, random finite element method (RFEM) involving 500 realisations is used. This process uses same point statistics as used for deterministic analysis but additional vertical and horizontal scales of fluctuation are specified to quantify the distance over which the property values are corelated. Since insufficient data are available for the cross-section, the vertical scale of fluctuation is assumed as 0.5 (conservative estimate) and for horizontal scale of fluctuation, initially 3 values are assumed: 0.5, 6 and 12. Following figure 14 shows the cdf of FOS computed using deterministic as well as the stochastic analysis.[12]

FOS of 0.98 is obtained for reliability of 95% when value of horizontal scale of fluctuation is used as 6 in stochastic approach. The value FOS for corresponding design property values, determined by scaling down the property distribution of cohesion and tangent of friction angle is obtained as 0.85 for reliability of 95%. This value is less than safety requirement FOS of 0.95 but there is a significant increase of 57% in FOS when the spatial variability is accounted.[12]

Redesigning of the dyke led to increase in FOS from 0.59 to 1.33 for deterministic approach using D-Geo stability software and 1.21 using inhouse software. Using RFEM approach gave FOS of 1.531 showing an increase of 27% with respect to deterministic approach. Characteristic soil property values consistent with the Eurocode 7 were back-calculated and found to be 34% for all material zones. This shows increase in strength capacity when compared to the point-statistics based interpretations of Eurocode 7.[12]

2.4. SUMMARY

- The complexity in geotechnical variability is due to the uncertainties of inherent variability, measurement errors and transformation uncertainties. A thorough review was carried out to estimate the statistics on inherent variability, measurement errors [8] and transformation uncertainties [3]. This work provided guide-

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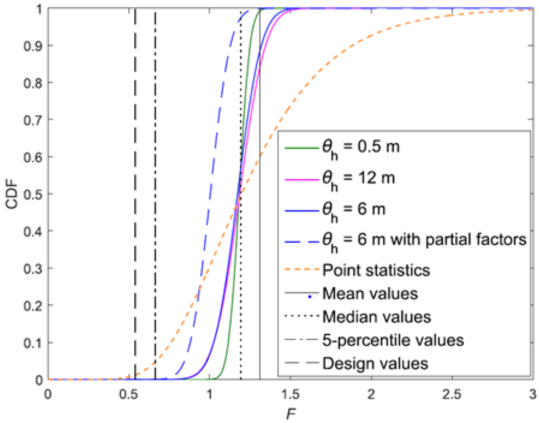


Figure 14: Comparison of deterministic and stochastic solutions for factor of safety.[12]

lines on the probable range of COV of soil properties which can prove to be useful first order approximations. These ranges are derived from the reported statistics on soil properties in geotechnical literature. Due to this data being limited, it is not suitable for statistical analysis. To overcome this problem and to have better insight on variability, the site-specific data and the prior knowledge from generalised guidelines can be combined using Bayesian updating techniques.

- Y. Wang, et. al [4] have combined recent studies of using Bayesian technique in soil characterisation and have developed a Bayesian inverse analysis to directly quantify the actual (inherent) variability of soil and rock properties. The study provides two different ways for solving the Bayesian problem. First method is Bayesian equivalent sample method which uses MCMCS to create multiple samples from limited samples for analysing the statistics of soil properties. Second method is Bayesian system and model class selection method in which, for multi layered soil profile, mutually independent random fields are used to characterise design property for each layer and further using Laplace asymptotic approximation, MPV is obtained.

- This Bayesian framework has been applied to characterise different soil properties such as Young's Modulus (Wang and Cao [9]), sand friction angles (Wang and Cao [5]), identification of soil strata in London clay using water content data (Wang and Cao [6]), soil strata identification using CPT (Wang and Cao [7]). These studies show that actual variability is the major component which affects the actual response of geotechnical system. The measurement errors and transformation uncertainties do not affect the actual response but they do affect the reliability analysis since we use the total variability in reliability studies. This overestimates the reliability estimates (overestimation of FOS of structures).

- This Bayesian approach provides promising results for quantifying the inherent variability by reducing the measurement errors and transformation errors in the characteristic values.

- Current Dutch stability assessments of rural dykes using statistical methods are based the guidelines provided in clause (11) of Eurocode 7. Hicks, et. al [12] investigated a case study involving reliability-based assessment and redesign of an existing dyke. Both, deterministic and random finite element methods (RFEM) were used and when compared showed significant increase in FOS when spatial variability is considered. The characteristic property values used in the approach were consistent with requirements of Eurocode 7. In this study, no correlation was assumed between the design soil properties (cohesion and tangent of friction). In RFEM for each soil property, due to limited data available, values of both the vertical as well as horizontal scale of fluctuations are assumed based on past knowledge.

As discussed in above section, previous studies have reported the use of Bayes' statistics in geotechnical engineering mainly to calculate the property values of soil (eg. Young's Modulus, Shear strength, Water content, etc.), soil stratification, etc. The attempt of such approach was made to quantify the transformation, inherent and measurement uncertainties. However, little attention has been paid to implementing Bayes' statistics in the reliability studies of slope. As the problem statement suggests, the present work focuses on implementing this method and quantifying the uncertainty involved in the current reliability studies of slope.

3

Bayes' Equation

BAYES THEOREM:
Bayes theorem in statistics and probability theory describes the probability of an event based on prior knowledge of conditions that might be related to the event (Joyce and James [14]). It is a tool that can be used to quantify the uncertainty of the variable value. Bayes' Theorem is formulated as,

$$f(y|x) = \frac{f(x|y)f(y)}{f(x)} \tag{3.1}$$

Due to the generality of Bayes' Theorem, it has a wide range of applications in everyday life. To understand how the above equation 3.1, a classic probability example of disease-test is considered. Assume that 'y' is the hypothesis that a person has a disease and 'x' is the event that the test carried out to confirm the disease is positive. Every parameter in equation 3.1 for this example is defined as follows.

f(y): It is the probability that a person has a disease without conducting any test to confirm the disease. It is also known as prior distribution as it shows knowledge on disease prior to performing the test.

f(x): It is the total probability of testing positive for the disease. It is also treated as the normalizing constant as there is no dependency of f(x) on f(y). It is the summation of the probability of having disease and correctly testing positive and probability of not having disease and falsely being identified. Mathematically, it can be written as,

$$f(x) = f(y) \cdot f(x|y) + f(-y) \cdot f(x|-y) \tag{3.2}$$

$$f(x) = \sum f(x|y) \cdot f(y) \tag{3.3}$$

$$f(x) = \sum \int f(x|y) \cdot f(y) dy \tag{3.4}$$

Equation 3.3 is used in case of discrete parameter values and equation 3.4 is used when the parameter values are continuous.

f(x|y): It shows the probability of the test being positive given that the person has the disease. This function is also known as the likelihood function.

f(y|x): This function represents the probability of having the disease given the probability of the test being positive. It is called posterior distribution.

One of the critical and difficult parts of this equation is to figure out the prior data. Sometimes it is no better than a guess. To understand how the above equation 3.1 equation works, assume a sample size of 1000 people. It is assumed that the frequency of the disease in the population, f(x)=0.001. Also, it is assumed that 1% of the total population would be falsely identified as having the disease (f(x|-y) = 0.1). Using these values, equation 3.1 can be solved as;

$$f(y|x) = \frac{(0.99) \cdot (0.001)}{(0.001) \cdot (0.99) + (0.999) \cdot (0.01)} \tag{3.5}$$

$$f(y|x) = 0.09 \tag{3.6}$$

This solution shows that the probability that a person has the disease after testing positive for it, in reality, is very low. Statistically, this solution is more logical. The person who actually has the disease belongs to a group of 11 people who tested positive. One person in this group actually has the disease. So chance that a person has the disease after testing positive for it is 1 in 11 i.e. 9%

In this example, the assumptions made allow computing the normalizing constant quite easily. But most of the time the parameters (y and x) have discrete or continuous values. For these type of parameters, equations 3.3 and 3.4 are used to compute fx. When a parameter is continuous, it gets very difficult to solve equation 3.4 analytically as well as numerically. To bypass this issue, Markov Chain Monte Carlo (MCMC) can be used. The starting point for MCMC is the parameter space. Each parameter in the defined Bayes' model has a possible range of values. Parameter space is the space/area that covers all possible combinations for all possible values of the model parameters. For each parameter value in that space, the likelihood function can be computed. Also, the Prior probability for any possible value is known (assumed Prior probabilities in the model). Knowing the likelihood and the respective prior, the numerator in Bayes' eq. is calculated for any given point in the parameter space. MCMC algorithm starts with a random point in this parameter space, computes the multiplication of likelihood and prior for that particular point, and then moves to a new point to carry out a similar operation. Thousands of such samples which are also correlated are drawn, forming a distribution that is nothing but a posterior distribution. This way by using MCMC, the difficulty in computing the normalizing constant is bypassed. The most common MCMC algorithms used are Metropolis-Hastings, Hamiltonian Monte Carlo, Gibbs, No-U-Turn Sampler (NUTS), etc. In this work to carry out the Bayes' statistics, the PyMC3 package in python [15] is used.

Following is an example of implementing Bayes' statistics for estimating the values of undrained shear strength (s_u). For this, 20 random samples of s_u for clay soil are assumed to be obtained by laboratory/Field tests. A normal distribution is fitted over these 20 samples and the mean and standard deviation of this distribution are derived as shown in the figure 15. Hence, this observed data of s_u is represented by a normal random variable with a mean μ and standard deviation σ and is defined as:

$$s_u = \mu + \sigma \cdot z \tag{3.7}$$

in which z is a standard Gaussian random variable. Since these parameters (mean and standard deviation of Su) are derived from a relatively small sample set, their values may not be perfectly correct and might have a lot of uncertainties involved. In order to predict better values and see the uncertainties involved in the observed values, Bayes' statistics are used. For this, a simple model is created in the PyMC3 environment in python [15].

Predicting/improving the values in this case essentially means getting better insight on the mean (μ) and standard deviation (σ) of s_u. To do so, following Bayes' equation is to be solved.

$$f(\mu,\sigma|x) = k \cdot f(x|\mu,\sigma) \cdot f(\mu,\sigma) \tag{3.8}$$

Where k is normalising constant given as,

$$k = \left[\int_{\mu,\sigma} f(x|\mu,\sigma) \cdot f(\mu,\sigma)\, d\mu\, d\sigma\right]^{-1} \tag{3.9}$$

To solve this equation, from the normal distribution of the observed data given by equation 3.7, n_s independent values of s_u are considered. the likelihood function given by $f(x|\mu,\sigma)$ in equation 3.8 is expressed as:

$$f(x|\mu,\sigma) = \prod_{i=1}^{n_s} f(\mu,\sigma|x) \tag{3.10}$$

$$f(x|\mu,\sigma) = \prod_{i=1}^{n_s} \frac{1}{\sqrt{2\pi\sigma}} exp\left\{-\frac{1}{2}\left[\frac{x-\mu}{\sqrt{\sigma}}\right]^2\right\} \tag{3.11}$$

Since there are two model parameters (μ and σ) assumed, as prior data ($f(\mu,\sigma)$) a joint distribution between μ and σ. For this joint distribution, the following distributions are assumed.

$$\mu \sim \mathcal{N}(\mu_a, \sigma_a)$$
$$\sigma \sim \mathcal{N}(\mu_b, \sigma_b)$$

It is known that for the clay sample, the mean and Coefficient of variation of Su values lie in the range of 10-400 kPa and 0.2% to 0.55%. Hence, as prior data on the mean and standard deviation of Su, these values summarised by Phoon and Kulhawy [8] are used. Now, to get the posterior distribution ($f(\mu,\sigma|x)$) 1000 samples are drawn using No-U-Turn sampler. NUTS is an extension to Hamiltonian Monte Carlo (HMC). This sampler uses dual averaging, making it possible to get similar or sometimes more efficiency is obtained compared to HMC without wasting time and effort in hand-tuning HMC. Using NUTS the Bayesian inference can be performed efficiently (more information on NUTS can be found here [16]). Following figure 16 shows the posterior distribution of the mean and standard deviation obtained by 1000 samples along with their prior distributions. The figure shows a smooth posterior distribution as a normal distribution is fitted over the samples just to compare the values. The actual posterior distribution is not smooth.

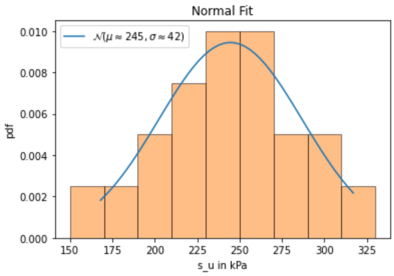
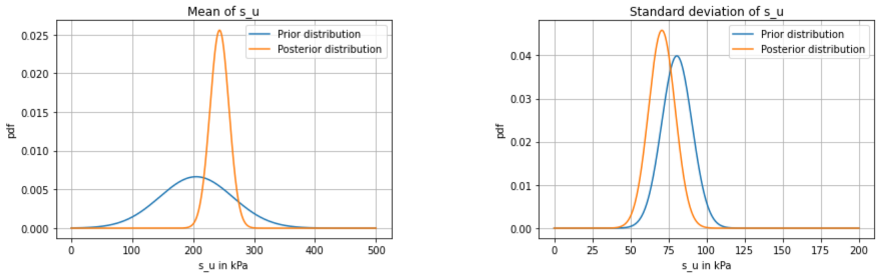


Figure 15: Observation distribution



(a) Mean of s_u

(b) Mean of s_u

Figure 16: Prior and posterior distributions

Following are the details of prior and posterior distributions.

$$\begin{gathered}\text{Prior Distributions :}\\ \text{mean of } s_u \sim \mathcal{N}(205, 60)\\ \text{standard deviation of } s_u \sim \mathcal{N}(80.5, 10)\end{gathered}$$

$$\begin{gathered}\text{Posterior distribution in terms of } (\mu,\sigma)\\ \text{mean of } s_u = (243.336, 15.593)\\ \text{standard deviation of } s_u = (70.706, 8.707)\end{gathered}$$

It can be clearly seen that the posterior distributions are narrower than the prior distributions. This is expected because when new/more information is available, the confidence level in the data increases. By using the Bayes' method, two major interpretations are obtained;

- Rather than getting a single value of the parameters, it provides the distribution of the values.
- By combining the prior knowledge and the observed knowledge, the uncertainties in the parameter values can be seen.

When general statistics (frequentist approach) is applied to the 20 data points, a single value of mean and standard deviation is obtained, neglecting all the other possible values. With Bayes' approach, all the possible values are obtained given the data set and assumed prior data showing the uncertainty involved.

The above examples explain how exactly the Bayes' approach works in improving the understanding of particular parameters using the past knowledge and the observations made. As discussed in section 1.1 the major aim of this study is to implement Bayes' statistical approach to derive the input statistics of soil properties and compare the results with the frequentist approach. In the next chapter, a very basic Bayes' model is developed to predict these input parameters.

4

Basic Bayes' Model

In this chapter, the Bayes' model is introduced step by step over some basic examples below. In every example, bits and pieces are added to scale up the model to reach the actual target problem of slope stability. As discussed earlier, in this work a finite element model which is developed at TU Delft is used to analyze a basic slope problem. This model uses a strength profile to calculate the FOS of the slope. To the finite element model, this profile is provided as input in terms of Intercept (c) and Slope (m) of the linear trend as shown in the figure, Coefficient of Variation (CoV) of the data around the trend, and vertical scale of fluctuation (θ_v) (It is assumed that there is no horizontal scale of fluctuation in this work). In the examples below, a model to predict the linear trend (m and c) and CoV is developed. To start with, a basic linear data set is generated in example 4.1 and a simple Bayes' model is tried to fit over it.

4.1. EXAMPLE 1.

In this example, a data set is generated around a linear line using the following linear equation (eq. 4.1). It is assumed that the value slope (m) is 0.5 and that of intercept (c) is 1 and is represented by β_1 and β_0 respectively in vector form. The parameter ϵ used for generating the data around this line is assumed to be normally distributed with a mean of 0 and standard deviation of 2. 15 data points are generated around the assumed linear line

$$Y_i = \beta_1 \cdot X_i + \beta_0 + \epsilon_i \tag{4.1}$$

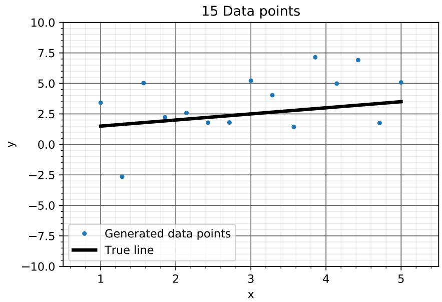
Figure 17a shows the linear line and the data generated around it. Now it is assumed that the values of β_1, β_0 and ϵ_i are unknown. Using the Bayes' approach these values are to be predicted. A major part in modeling the Bayes' equation is assuming the type of prior data (prior knowledge on β_1, β_0 and ϵ_i) and likelihood function. For this example following assumptions are made for prior data.

$$\begin{aligned} \beta_1 &\sim \mathcal{N}(0, 100) \\ \beta_0 &\sim \mathcal{N}(0, 100) \\ \epsilon_i &\sim \mathcal{N}(0, 10) \end{aligned}$$

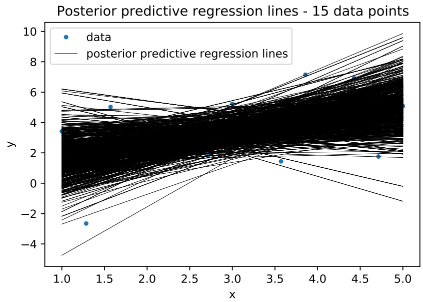
The likelihood function is also assumed to be normally distributed with a mean as $(\beta_1 \cdot X_i + \beta_0)$ and standard deviation as ϵ_i. It can be written as:

$$Y_i \sim \mathcal{N}[\beta_1 \cdot X_i + \beta_0, \epsilon_i]$$

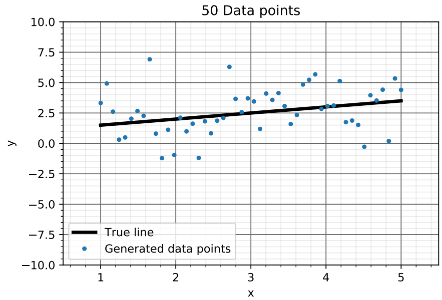
To plot the posterior, 1000 samples are drawn using MCMC (as discussed in chapter 3). Following figure 17b shows the results of this example.



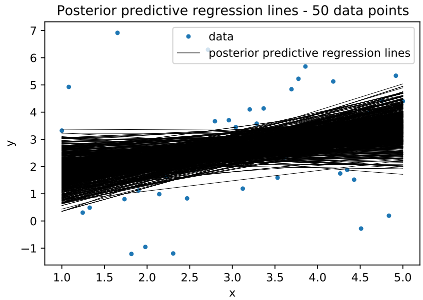
(a) Generated Data - 15 points



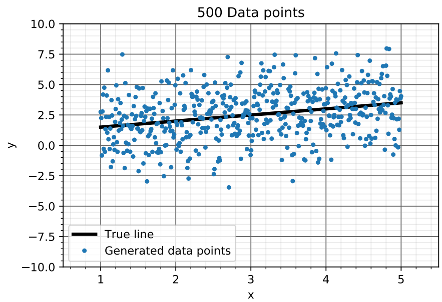
(b) Posterior predictive regression lines - 15 points



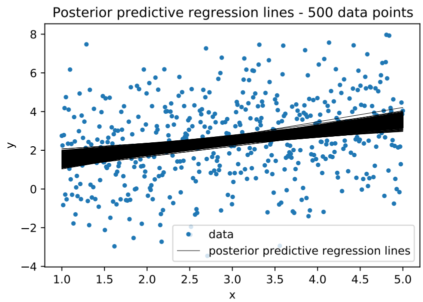
(c) Generated Data - 50 points



(d) Posterior predictive regression lines - 50 points



(e) Generated Data - 500 points



(f) Posterior predictive regression lines - 500 points

Figure 17: Different data sets with respective prediction of possible regression lines

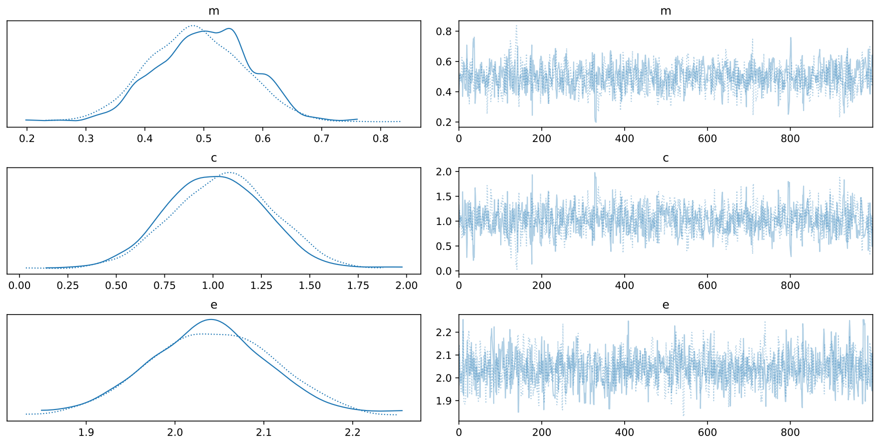
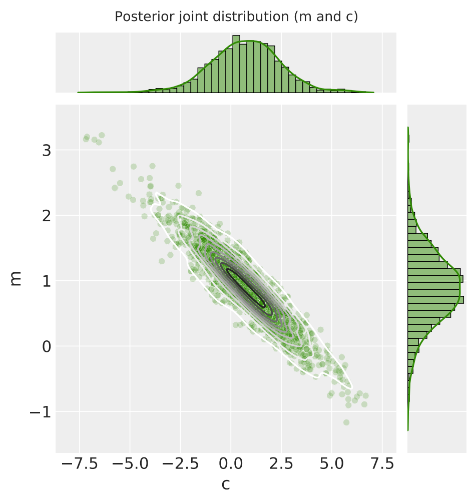
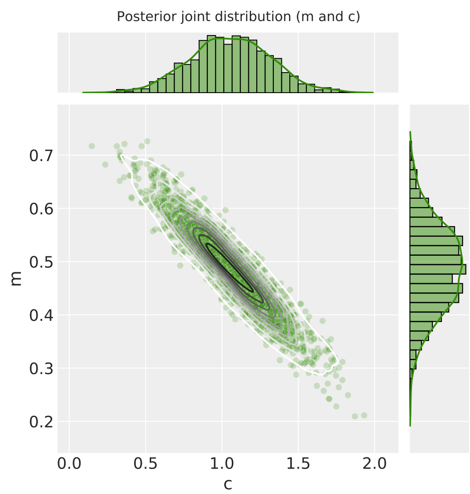


Figure 18: Posterior distribution of model parameters (left) with samples of Markov chain (right) for data set with 500 points



(a) 15 data points



(b) 500 data points

Figure 19: Joint Distribution between slope (m) and intercept (c)

As seen in figure 17b, the Bayes' approach provides a wide range of possible linear lines that can fit over the provided set of data. These are 1000 different lines since 1000 different samples were drawn to get the posterior distribution. The widespread of lines indicate the uncertainty involved in predicting the linear line. To see the effect of different data sizes, sample data sets with 50 and 500 data points were generated using the same linear line and error as used for generating data set with 15 points. Figures 17d and 17f shows the predicted lines for data set with 50 points and 500 points respectively. It can be seen that as the number of data points increases, the uncertainty decreases in the prediction of linear lines.

Figure 18 shows the sampling results for the data set with 500 data points. The left part of the figure shows the posterior distributions of the 3 model parameters (m (β_1), c (β_0) and ϵ_i) and the right side of the figure shows the samples of the Markov chain plotted in sequential order. The two posterior distributions (dotted and solid) represent a sampling from 2 chains. Using multiple chains helps to check the convergence of the distribution. PyMC3 uses Gelman-Rubin diagnostics to check the convergence of the posterior distribution. The maximum Gelman–Rubin diagnostic across all model parameters is labeled as Max Gelman–Rubin Rc (or $\hat{R}$). If the diagnostic $\hat{R}$ values are less than 1.1 for all the model parameters, the convergence can be declared. For all the 3 data sets, the value of $\hat{R}$ is 1 suggesting convergence of the posterior distribution. Also, the model parameters of the samples drawn are correlated to each other. Figure 19 shows the joint distribution between slope (m) and intercept (c). The following table 1 shows the summary of the posterior distributions of the model parameters.

Table 1: Summary of posteriors for example 1

Observation Points	**Intercept (c)(β_0)**		**Slope (m)(β_1)**		ϵ_i	
	Mean	**S.D.**	**Mean**	**S.D.**	**Mean**	**S.D.**
15	0.723	1.778	0.892	0.552	2.536	0.562
50	1.362	0.764	0.423	0.233	1.905	0.196
500	1.056	0.263	0.487	0.081	2.041	0.066

The results in the above table show that, as the number of samples increases, the uncertainty in the prediction decreases. With the increase in the number of samples, the standard deviation of the model parameter decreases, and the mean value moves more close to the true values. It is expected to get such results since more information is available, more confidence/certainty is present. Hence, it is safe to say that the assumptions made in this model on prior data and likelihood function provide reliable results. In this example, the data points are not correlated to each other and are randomly generated.

The sample set generated in the above example does not represent the trend of the general soil strength profile. Usually, as the depth increases, the values of S_u scatters farther away from the mean. To capture such behavior, a heteroskedastic type of data set is tried in the next example.

4.2. EXAMPLE 2.

In Heteroskedatic data, the deviation in the values of y from the trend line increases with an increase in values of x. To create such a data set following equation is used.

$$Y_i = \beta_1 \cdot X_i + \beta_0 + \epsilon_i \cdot X_i \tag{4.2}$$

Now, similar to example 1 the values of β_1, β_0 are assumed as 0.5, 1 and $\epsilon_i \sim \mathcal{N}(0,1)$

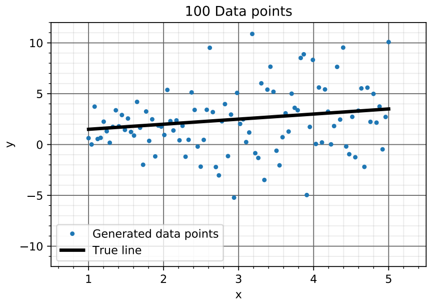
respectively. Figure 20a shows data set generated using the above formula 4.2. For Bayes' model, a model used in example 1 is also used here with a change in the likelihood function. Change is made in the standard deviation of the likelihood function where the ϵ_i is multiplied with the depth X_i. The likelihood function is given as:

$$Y_i \sim \mathcal{N}[(\beta_1 \cdot X_i + \beta_0), (\epsilon_i \cdot X_i)]$$

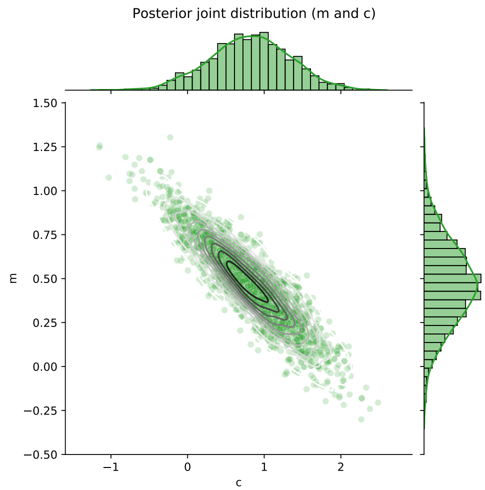
Following set of prior data is assumed:

$$\beta_1 \sim \mathcal{N}(0, 100)$$
$$\beta_0 \sim \mathcal{N}(0, 100)$$
$$\epsilon_i \sim \mathcal{N}(0, 10)$$

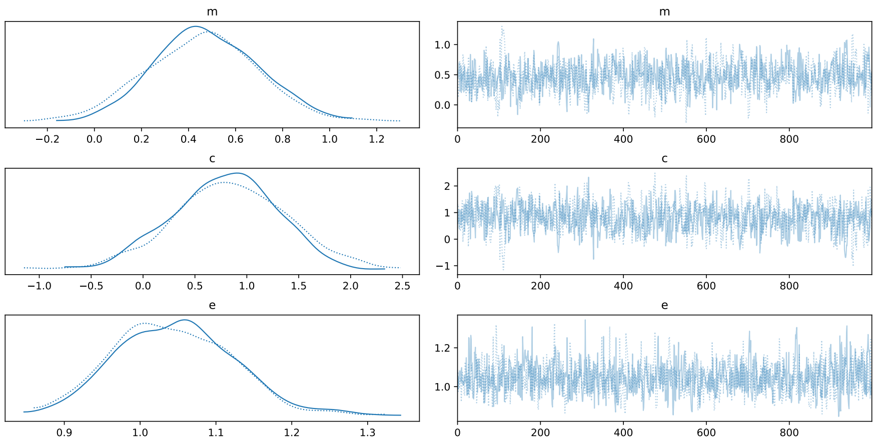
Following figure 20 shows the results of implementing this model.



(a) Generated Data



(b) Joint Distribution between slope (m) and intercept (c)



(c) Sampling results

Figure 20: Results for Heteroskedastic data

For this example, data set with 100 data points was generated. Similar to example 1, this heteroskedastic model also provide promising results. The mean values of m (β_1), c (β_0) and ϵ_i are 0.468, 0.812 and 1.045 respectively which are very close to the true values (m (β_1) = 0.5, c (β_0) = 1 and $\epsilon_i \sim \mathcal{N}(0,1)$). Also, the model parameters (m (β_1), c (β_0) and ϵ_i) exhibit correlation with each other. Figure 20b shows the joint graph of m and c. It can be seen that with increasing value of c, the value of m decreases; showing negative correlation.

But does the heteroskedastic data correctly represents the CPT data? Consider figure 21 below. In this figure, actual CPT data is plotted along with sample heteroskedastic data. This heteroskedastic data is generated using values of m, c and ϵ as 3, 15, $\epsilon \sim \mathcal{N}(0,1)$) respectively. Looking at the heteroskedastic data set, the data points keep on moving away from the mean trend. This suggests that in the case of CPT's, at larger depths the deviation of the data point from the mean trend is very large. In most of the conditions, this is not true. The increase of deviation of the data point is more gradual in actual CPT's compared to the heteroskedastic data. Also, at a shallower depth, the values of CPT relatively show larger deviations compared to the heteroskedastic data. Hence, it can be said that the heteroskedastic model is not the perfect fit for working with CPT data sets. In the next example, a different equation is used in order to create a more suitable data set for CPT data sets.

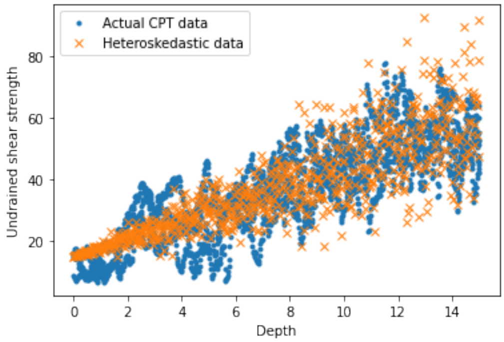


Figure 21: Comparison between heteroskedastic and CPT data

4.3. EXAMPLE 3.

In this example, the following equation is used to generate the data set. This equation is taken from the coursework documents of the Risk and Variability course (CIE4395) at TU Delft [17].

$$Y_i = (\beta_1 \cdot X_i + \beta_0) \cdot (1 + \beta_2 \cdot \zeta_i) \tag{4.3}$$

In the above equation, the mean trend $(\beta_1 \cdot X_i + \beta_0)$ is multiplied by term $(1 + \beta_2 \cdot \zeta_i)$. β_2 is CoV (coefficient of variation) and ζ_i is random number generating parameter following a normal distribution with mean as 0 and standard deviation as 1. Figure 22a shows the data generated using equation 4.3. The assumed value of m (β_1), c (β_0) and CoV (β_2) to generate this data set is 0.5, 1 and 0.25 respectively.

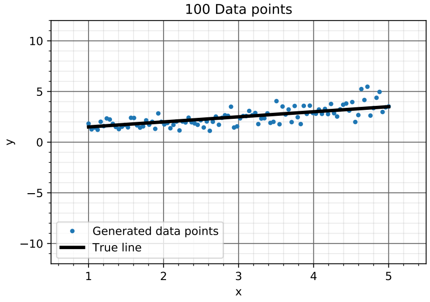
In this example to solve Bayes's equation, a different, more uninformative prior data set is assumed. For model parameter m (β_1) and c (β_0), a uniform distribution with values ranging from 0 to 100 is assumed. Since CoV (β_2) is often expressed in percentage, it is assumed that its value will lie between 0 and 1. Hence, for CoV (β_2) a uniform prior is assumed with values ranging from 0 to 1. Following are the assumed prior distribution:

$$\beta_1 \sim \mathrm{U}(0,100)$$
$$\beta_0 \sim \mathrm{U}(0,100)$$
$$\epsilon_2 \sim \mathrm{U}(0,1)$$

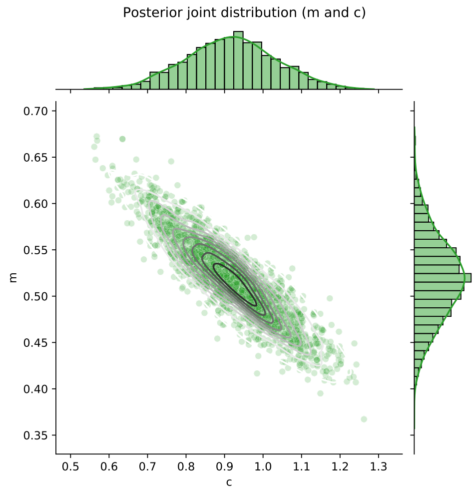
For the likelihood function, the following distribution is assumed.

$$Y_i \sim \mathcal{N}[(\beta_1 \cdot X_i + \beta_0), \beta_2 \cdot \zeta_i(\beta_1 \cdot X_i + \beta_0)]$$

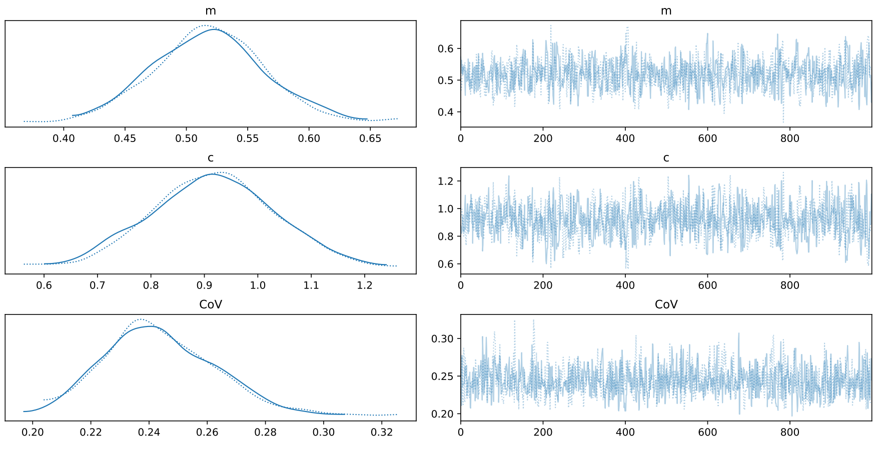
Following figure 22 shows the results obtained by sampling 1000 posterior samples.



(a) Generated Data



(b) Joint Distribution between slope (m) and intercept (c)



(c) Sampling results

Figure 22: Results for data generated using equation 4.3

Again, looking at the figure 22, it can be concluded that the assumed Bayes' model provides very good results. The mean value of the model parameters (m (β_1) = 0.519 , c (β_0) = 0.916 , CoV (β_2) = 0.244) drawn from their respective posteriors are very close to the true values (m=0.5 , c=1 , CoV=0.25). Also, the standard deviation of these posteriors is very small. This is because of the high number of available data points. As the number of data points will decreases, the confidence in the observed data set will decrease. Due to this, the posterior will move towards the assumed prior value of the model parameter.

Following figure 23 provides comparison between the data created using actual equation 4.3 and actual CPT. It can be clearly seen that relative to the heteroskedastic model, the data created using equation 4.3 provides a better fit over the CPT data. Hence, this model (example 3) is used to derive the input statistics for the slope stability analysis presented in the next chapter.

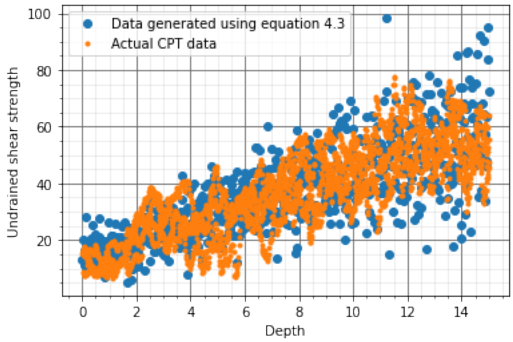


Figure 23: Comparison between data created using equation 4.3 and CPT data

Here in this chapter, 3 different types of data sets were modeled. Comparatively, the data set in example 3 shows more similarity with the actual CPT data set. In the next chapter, the Bayes model used in example 3 is implemented on a basic slope stability problem modeled in FEM. Also, the input statistics are derived using the frequentist approach and further, the FOS results are compared.

5

SLOPE STABILITY STUDY

In this chapter, slope stability analysis is performed on a basic slope modeled in FEM. Along with the basic statistical approach (Frequentist approach), the Bayes' approach is also used to derive input statistics and further the results of FOS are compared.

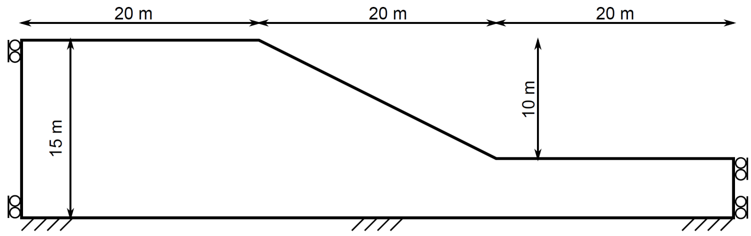


Figure 24: Slope geometry and boundary condition

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Figure 24 shows the modeled geometry of the slope being investigated. The height of the slope is 10m excavated in a spatially variable clay layer of 15m. It is assumed that there is no horizontal scale of fluctuation. For this spatially variable soil, 4 undrained shear strength profiles were generated (using CIE4395_CW1.exe [18]) as shown in figure 25 (**Note: This is Example 1**). Each profile has 750 data points equally distributed throughout the depth. These profiles are used to derive the input statistics of undrained shear strength (Intercept, slope, coefficient of variation (CoV), vertical scale of fluctuation (θ_v)). These statistics are ultimately provided to the FEM model as input to analyze the stability of the slope. Different methods such as the graphical method, regression method are used to derive these parameters. In this chapter, in the first section, the statistics of strength profile are derived and in the second section, the analysis of slope using the derived statistics is presented.

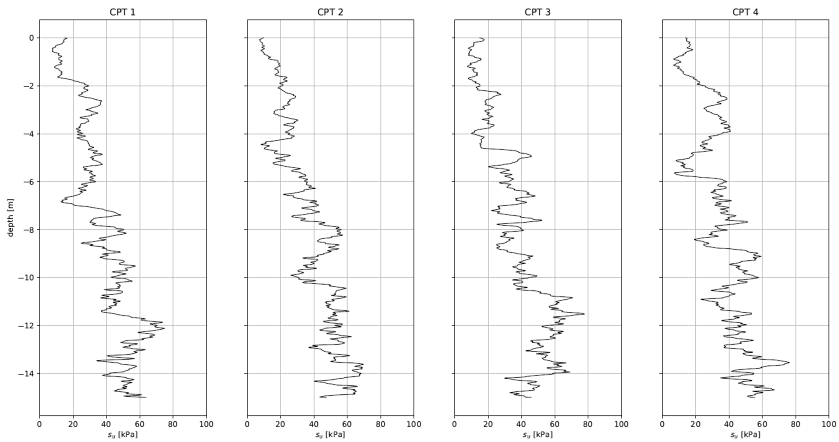


Figure 25: Undrained shear strength (s_u) profiles

As mentioned in section 1.1 one of the research questions in this thesis is to look at the effect of using different data sets. In sections 5.1.1 - 5.1.3 below, a sample set of CPT's shown in figure 25 is used to calculate the input statistics (using frequentist approach) for slope stability analysis. In these sections, the analysis is performed for data set with 3000 data points (750 data points from 4 CPT's each). In section 5.1.4 results of similar analysis on smaller data set with 24 data points is presented. To create this small data set, from each CPT 6 data points are selected spread equally throughout the depth (6x4 = 24 data points).

5.1. ESTIMATION OF INPUT STATISTICS (FREQUENTIST APPROACH):

In this work, it is considered that the undrained shear strength follows a log-normal distribution with a linear trend $mz + c$ over the depth z with a constant coefficient of variation (CoV). Following equation 5.1 represents the undrained shear strength profile.

$$s_u(z) = (c + mz)\epsilon(z) \tag{5.1}$$

where,

$$\epsilon(z) = (1 + CoV \cdot z)$$

Here, $\epsilon(z)$ is the error around the linear trend. It is assumed that this error follows a log-normal distribution with mean as 1 and coefficient of variation CoV. This log-normal trend of s_u profile can be transformed to a standard normal distribution using the following equation.

$$U(z) = \frac{X(z) - \mu_x}{\sigma_x} \tag{5.2}$$

where,

$$X(z) = ln(\epsilon(z))$$

$$\mu_x = ln\left(\frac{m}{\sqrt{1 + CoV^2}}\right)$$

$$\sigma_x = \sqrt{ln(1 + CoV^2)}$$

For describing the correlation between two points at a relative distance of Δz, the Markov correlation function $\rho_M(\Delta z)$ is used since it is one of the common correlation functions considered for the soil variability.

$$\rho_M(\Delta z) = exp\left(-2\frac{|\Delta z|}{\theta}\right) \tag{5.3}$$

In the above equation, θ is nothing but scale of fluctuation. This scale of fluctuation is defined as the area under the correlation function as,

$$\theta = 2\int_0^{\infty} \rho(\Delta z)\, d\Delta z \tag{5.4}$$

For this way of estimating the scale of fluctuation, the correlation function has to be estimated from the available data and should be non-trivial. There are several methods to estimate θ which are further discussed below in section 5.1.3

5.1.1. ESTIMATING THE TREND (INTERCEPT, SLOPE) AND COV USING GRAPHICAL METHOD:

In this method, a median trend line (P_{50}) is drawn through 4 combined strength profiles such that at each depth, 50% data falls below the trend line. Two additional trend lines P_5 and P_{95} are drawn such that 5% and 95% strength data fall below these lines respectively. Following are the equations of these trend lines.

$$P_{50}(z) = b_{50} + a_{50} \cdot z \tag{5.5}$$

$$P_5(z) = \alpha_5 \cdot P_{50}(z) \tag{5.6}$$

$$P_{95}(z) = \alpha_{95} \cdot P_{50}(z) \tag{5.7}$$

To estimate the mean trend given as $M(z) = b + a \cdot z$ draw a line which is slightly different from the median trend $P_{50}(z)$. Due to skew in lognormal distribution the mean trend and median trend are slightly different. To estimate the CoV, following formula 5.8 is used.

$$1.645 \cdot \sigma_x = \frac{1}{2} \cdot ln\left(\frac{\alpha_{95}}{\alpha_5}\right) \tag{5.8}$$

Figure 26 shows the trend lines and mean line drawn over the combined CPT's. Using this method, following values of the trend and CoV were estimated.

Intercept (b) = 13.5 kPa
slope (a) = 2.9 kPa
CoV = 0.25

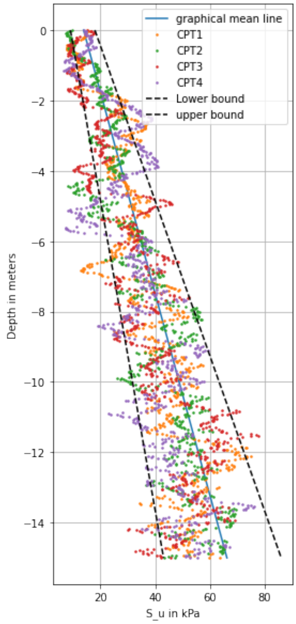


Figure 26: Trend line and mean line over combined CPT's

5.1.2. ESTIMATING THE TREND (INTERCEPT, SLOPE) AND COV USING REGRESSION METHOD:

In this method, the numerical data of the strength profile is used for estimating the trend and CoV. The trend $M(z) = b + a \cdot z$ is estimated using polyfit function from Python's NumPy package. For estimating the CoV the trend in the strength profile is removed and the variability component ϵ(z) is obtained. Using the mean and standard deviation of this component, CoV can be calculated. Following figure 27 shows the detrended strength profile.

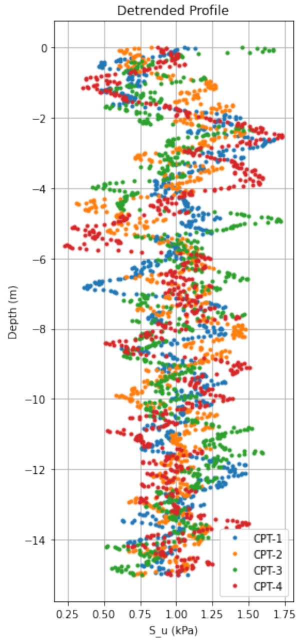


Figure 27: De-trended profile

Following are the estimates made using the regression method.

Intercept (b) = 13.076 kPa
slope (a) = 3.059 kPa
CoV = 0.2556

5.1.3. ESTIMATING VERTICAL SCALE OF FLUCTUATION (θ_v)

There are different methods using which the scale of fluctuation can be predicted. Following are two methods used here to predict θ_v.

GRAPHICAL METHOD:

In this method, the P_{50} line drawn for combined profiles is drawn separately on individual profiles and the number of times the profile crossing the trend line is calculated (N_c). a minimum interval is considered between two crossings (δz). Following formula 5.9 is used to calculate vertical scale of fluctuation. (For details refer [8]

$$\hat{\theta} = \frac{1}{\delta z}\left(\frac{2L}{\pi N_c}\right)^2 \tag{5.9}$$

Where L is the cumulative length of the 4 profiles. The value of θ_v calculated using this method is **2.154 meters**.

CORRELATION FUNCTION FIT METHOD:

In this method after estimating the linear trend over all the data, data is detrended to get the standard normal equivalent data profile. After removing the trend, the experimental correlation function is derived as,

$$\hat{\rho}(\Delta z) = \frac{1}{n_{\Delta z}} \sum_{i=1}^{n_{\Delta z}} U(z_i)U(z_i + \Delta z) \tag{5.10}$$

where, $U(z_i)$ and $U(z_i + \Delta z)$ are two data points in detrended random field spaced at a distance of Δz. $n_{\Delta z}$ is the number of such pairs of data points in the whole data set. This correlation function is plotted as shown in figure 28 and $\hat{\theta}$ is estimated by fitting a theoretical Markov correlation function (green curve) to the experimental result. The Markov correlation function is given as,

$$\rho_M = exp\left(-2\frac{|\Delta z|}{\theta}\right) \tag{5.11}$$

5

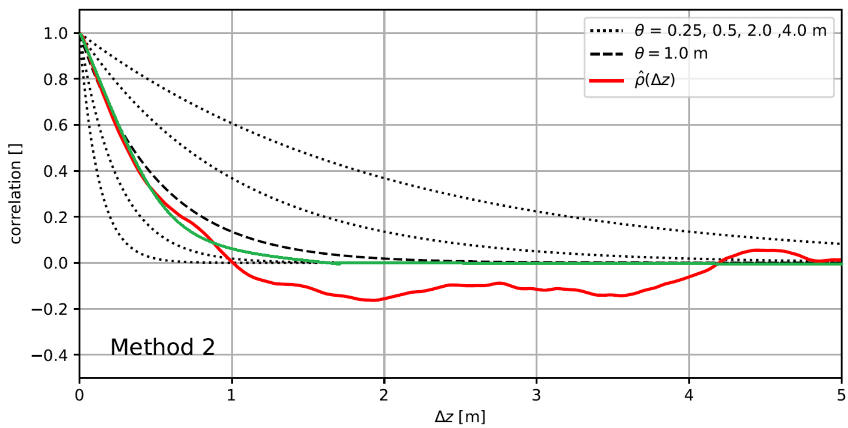


Figure 28: Likelihood Plot

From the above graph the value of θ_v is approximately estimated to be **0.85 meters**. (For details refer [19])

VARIANCE FUNCTION METHOD:

In this method, variance function given by equation 5.12 below is used. With increasing Δz the inner integral tends towards scale of fluctuation (θ) since the integral over $\rho(\Delta z)$ is θ. Hence, θ is equal to the limit given in equation 5.13

$$\Gamma^2(\Delta z) = \frac{1}{\Delta z^2} \int_0^{\Delta z} \int_0^{\Delta z} \rho(z_a - z_b) dz_a dz_b \tag{5.12}$$

$$\theta = \lim_{\Delta z \to \infty} \Gamma^2(\Delta z)\Delta z \tag{5.13}$$

From the variance of moving average profiles of CPT data, the variance function can be estimated. After plotting $\Gamma^2(\Delta z).\Delta z$ against Δz, the peak of $\Gamma^2(\Delta z).\Delta z$ can be considered as good estimate of θ. (For details refer [20])

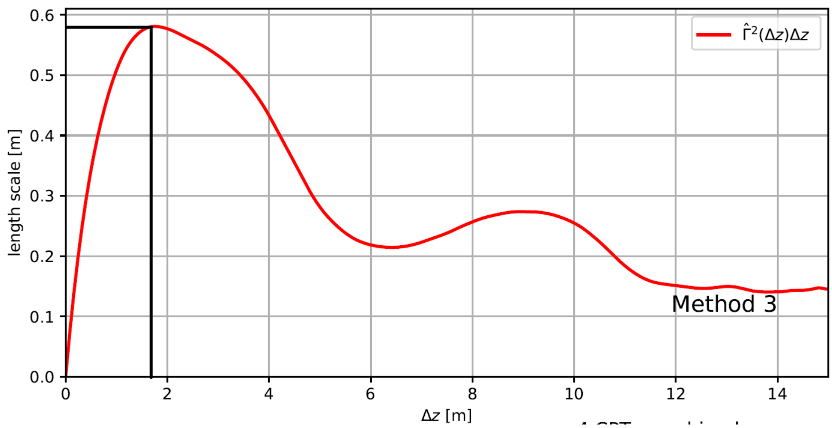


Figure 29: Likelihood Plot

From above figure 29, it can be said that the value of θ is approximately **0.58 meters**

MOST LIKELIHOOD METHOD:

A log-likelihood is calculated for a series of possible scales of fluctuation. Following equation 5.14 is used to calculate the log-likelihood. (For details refer [19])

$$L(\theta_i|s_u(z)) = -\frac{n}{2}log(2\pi) - \frac{1}{2}log|R_i| - \frac{1}{2}U^T {R_i}^{-1} U \tag{5.14}$$

where, R_i is the correlation matrix between data $U(z)$ for θ_i and the normalised measurement data U. This likelihood is plotted against θ_i as shown in figure 30. The peak of this plot is taken as the best estimate of $\hat{\theta}$ which in this case is **1.15 meters**.

From above methods, best estimates of all the parameters is considers for finite element analysis. In case of the above example (CPT's shown in figure 25) following are the best estimates considered.

Intercept (b) = 13.076 kPa

Slope (a) = 3.059 kPa

CoV = 0.2556

Vertical scale of Fluctuation (θ_v) = 1.15 meters

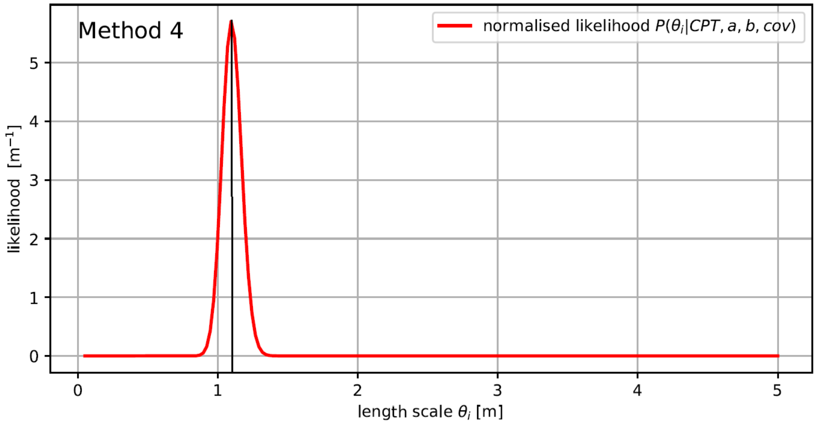
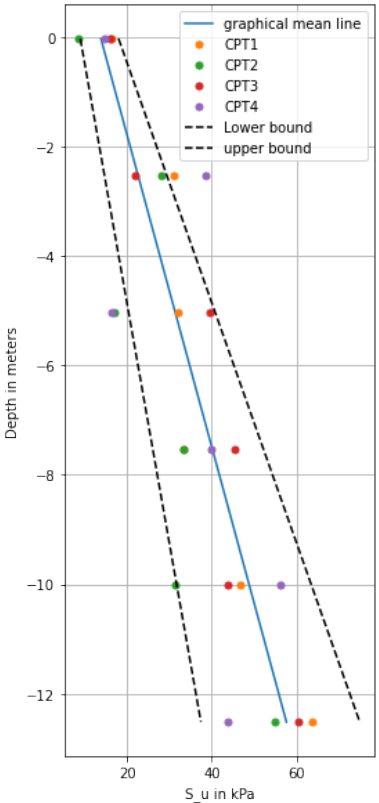


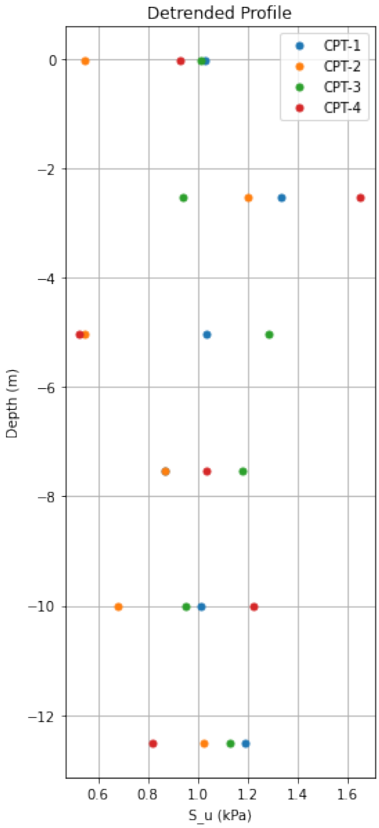
Figure 30: Likelihood Plot

5.1.4. RESULTS OF FREQUENTIST METHOD ON SMALLER DATA SET (24 DATA POINTS):

All the methods used for bigger data set were used to estimate the input parameters using a smaller data set. Figure 31a shows the strength profile and figure 31b shows the detrended profile with 24 data points.



(a) Profile used to calculate parameter using graphical approach



(b) De-trended profile

Figure 31: CPT Profile and de-trended profile with 24 data points

The following table 2 shows the results obtained for trend and CoV.

Table 2: Results of input parameters for small data for first example

	Intercept(b) in kPa	Slope (a) in kPa	CoV
Graphical Method	13.8	3.5	0.213
Regression Method	15.9	3	0.258

To calculate the vertical scale of fluctuation, the most likelihood method discussed in section 5.1.3 is used. Following figure 32 show the result for the same. As seen from the graph, the value of θ_v is nearly zero. The reason behind such a result might be that the data points are so far apart that they don't show any correlation between them. Hence, for the sake of performing slope stability analysis, it is considered that the value of the vertical scale of fluctuation is known. A similar value of θ_v as calculated for bigger data set (1.15 meters) is further considered for the slope stable is analysis. However, this assumption does undermine the results for FOS and hence, a furthermore better investigation is needed for the scale of fluctuation.

Hence, based on above results the best estimate of the input parameters derived from smaller data set is:

Intercept (b) = 15.9 kPa

Slope (a) = 3 kPa

CoV = 0.2556

Vertical scale of Fluctuation (θ_v) = 1.15 meters

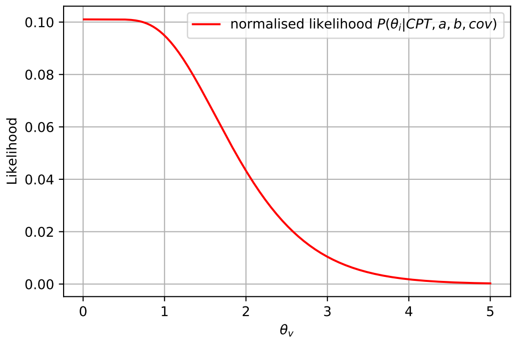


Figure 32: Likelihood Plot for θ_v estimated from 24 data points

5.2. ESTIMATION OF STATISTICS USING THE BAYES' APPROACH:

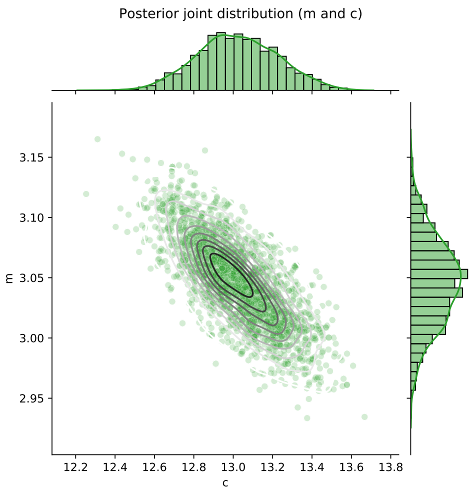
In this section, Bayes' approach is used to predict the statistical parameters (slope(m), intercept(c), and CoV). **Note that the Bayes' model to predict the scale of fluctuation (vertical and horizontal scale of fluctuation is not developed in this work. Hence for further reliability studies performed in 5.3, the value of the vertical scale of fluctuation considered is derived from the frequentist approach**. In this section, the Bayes' model presented in example 4.3 is used. As observed data, a total of 3000 data points are available from 4 strength profiles($4 \times 750 = 3000$). The following set of prior data is considered.

$$\text{m} \sim U(0, 15^2)$$
$$\text{c} \sim U(0, 50^2)$$
$$\text{CoV} \sim U(0.1, 1)$$

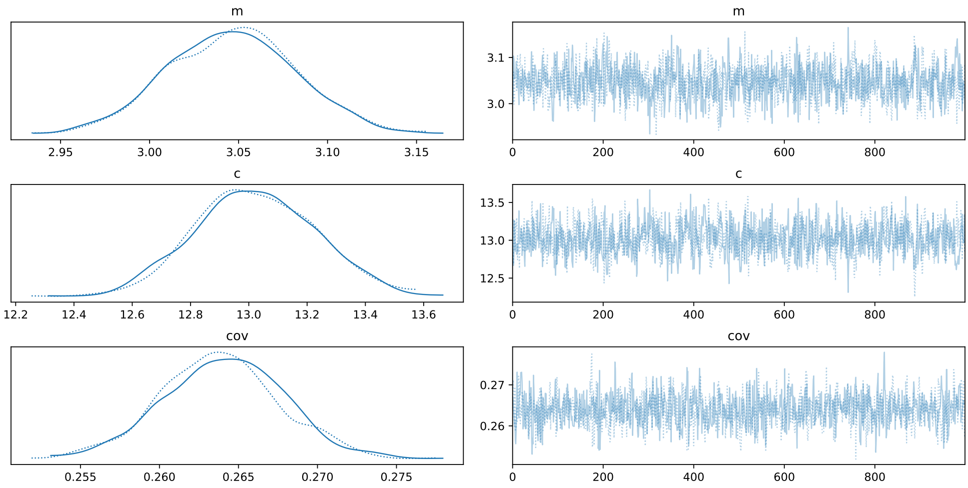
This set of prior data is randomly assumed. As likelihood Function, following distribution is considered.

$$\text{y} \sim \mathcal{N}[(m \cdot x + c), CoV \cdot \zeta(m \cdot x + c)]$$

To get the posterior distribution on the statistical parameters, 1000 samples (2000 from 2 chains) were drawn. Following figure 33 shows the results of this sampling.



(a) Joint Distribution between slope (m) and intercept (c)



(b) Sampling results

Figure 33: Results for 4 strength profile data

The details of the posterior distribution in terms of (μ,σ) are,

m = (3.046, 0.036)
c = (13.022, 0.205)
CoV = (0.264, 0.004)

Very small standard deviation in slope and CoV values clearly suggest that there is very little uncertainty involved. This is because the amount of data points considered here is very large. All the points are highly co-related to each other. Now, the same simulation is performed on a data set with 24 data points. Following results of posterior distribution were obtained after using the Bayes' approach.

m = (2.903, 0.470)
c = (16.479, 2.281)
CoV = (0.285, 0.047)

It can be seen that the standard deviation of the parameters is very high. Also compared to results with the 3000 data points, the mean values have also changed significantly. These wide distributions of the parameters show nothing but the uncertainty involved in estimating statistics with fewer data points.

Using these parameters derived by Frequentist and Bayesian statistics, in the next section reliability analysis is performed on the slope using FEM software and further, the results obtained are compared.

5.3. RELIABILITY ANALYSIS:

To perform the reliability studies on the slope shown in figure 24, a Random Finite Element code CIE4395-CW2.exe [21] is used. All the boundary conditions, slope geometry, spatial discretization, etc. are implemented in the program. Using the statistics derived in the above section, the analysis is performed using the strength reduction method.

First, the input parameters derived from the frequentist approach are used. Using such a method, a single value of all the input parameters is obtained. Analysis on both data sets (3000 and 24 data points) is performed. For each data set, RFEM analysis involving 1000 realizations is conducted. Figure 34 shows the mechanism of input parameters and realizations. As seen in the figure for a single input parameter, different realizations are generated. For each realization, a random field is generated using the derived input parameters and the FOS is calculated using the strength reduction method. Further using these 1000 FOS values cdf is plotted as shown in the figure 36

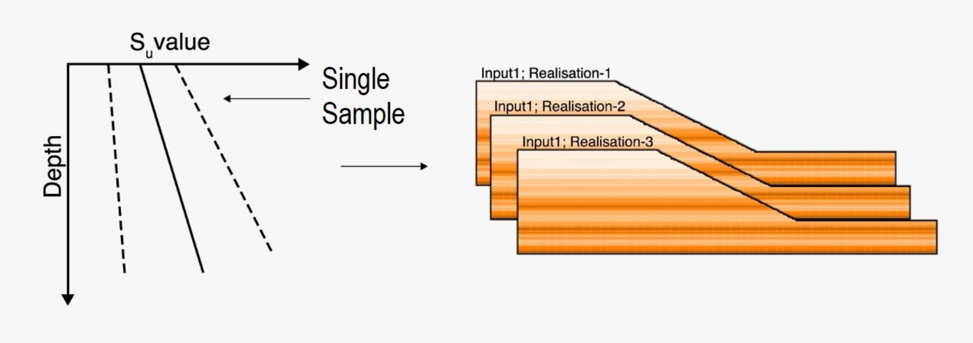


Figure 34: Input mechanism for frequentist approach

To use the input parameter values derived using Bayes' approach a different way of input is adopted. Using Bayes' statistics, a total of 2000 posterior samples (1000 samples from 2 chains each) of each input parameter (except the vertical scale of fluctuation) is obtained. Also, these parameters are correlated with each other. Hence, rather than using multiple random fields for 1 sample set of input parameters, multiple correlated samples are used. Figure 41 shows the mechanism of input parameters and realizations for Bayes' approach. For each sample input, RFEM analysis involving 1 realization is performed and a single value of FOS is obtained. This is done for 1000 such sample inputs and 1000 different FOS's are obtained. This analysis is performed on both the data sets and the cdf's of Factor of safety (FOS) obtained are plotted in the figure 36

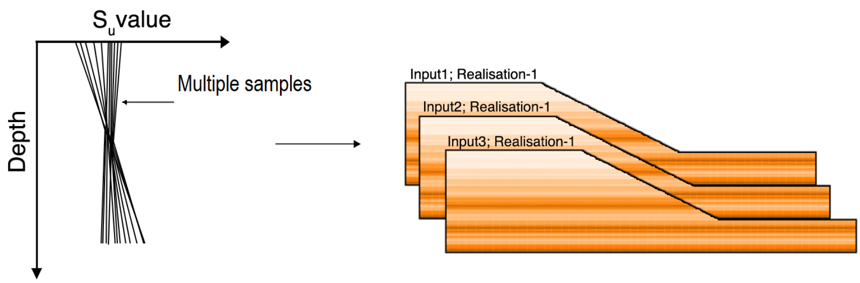


Figure 35: Input mechanism for Bayes' approach

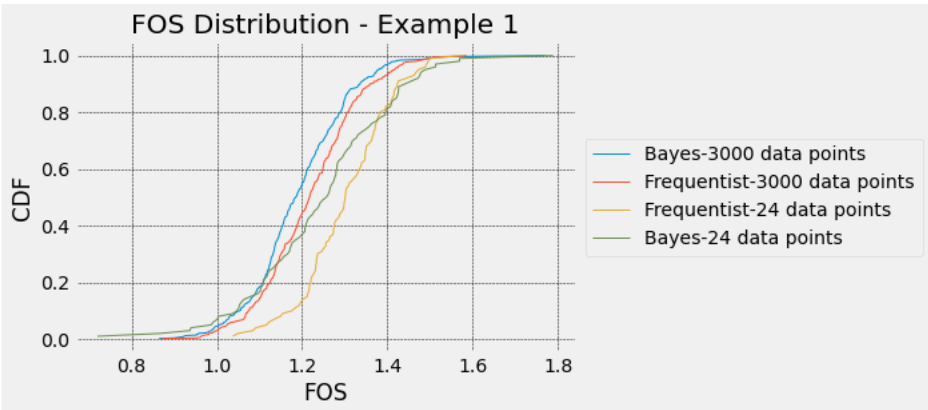


Figure 36: CDF's of Factor of Safety

In the above figure, in the legend, 'Bayes' represents 'Method of deriving statistics' and '3000 data points' represents the number of data points in the data set. Following table 3 summarises the values of FOS for 5%, 50% and 95% slope reliability for all cdf's obtained as shown in figure 36

Table 3: Results of first slope example

	Data Points	95% Reliability	50% Reliability	5% Reliability
Frequentist Approach	**3000**	1.0237	1.2171	1.4167
	24	1.1167	1.3012	1.4725
Bayes' Approach	**3000**	1.0063	1.1841	1.3777
	24	0.9862	1.2562	1.4879

For bigger data set, there is no significant difference seen in FOS values for all three reliabilities (5%, 50%, and 95%) obtained by both frequentist and Bayes' approaches. This is not the case for smaller data set. The FOS value for 5% reliability is nearly similar for both approaches but the FOS value for 95% reliability obtained using the Bayes approach is significantly lower than the one obtained using the frequentist approach. The FOS value for 95% reliability for smaller data set obtained using Bayes' approach is very close to one obtained for bigger data set.

Assuming that the slope will fail at FOS of 1, all three cdf's except the one obtained using frequentist approach on smaller data set show some failure probability. Cdf obtained using frequentist approach on smaller data show negligible failure probability, which is not true. This clearly shows overestimation of FOS values when frequentist approach is used on limited data.

The FOS values for 5% reliability: When Bayes' approach is used on smaller data set, 8% increase in FOS value is observed when compared to the bigger data set. For the frequentist approach, an increment of 3.7% is observed.

The FOS values for 95% reliability: The FOS value computed using Bayes' approach for smaller data set is less compared to the FOS calculated for bigger data set, but both the values are very close to each other. In the case of the frequentist approach similar trend of increase in FOS is observed for 95% reliability as seen in the FOS for 5% reliability.

It can be said that in the frequentist approach there is a shift in the cdf curve from lower FOS computed for bigger data set to higher FOS computed for smaller data set. The main reason behind this shift is the change in the input value of the intercept of the trend. All the input parameters (slope, CoV, scale of fluctuation) are the same for both smaller and bigger data set except the intercept which is higher in the case of smaller data set. The Bayesian approach provides the distribution of input parameters rather than a single value. Also, all the samples used to form these distributions are correlated with each other. Considering such a wide range of possible input values captures a better picture of the FOS values compared to the frequentist approach in which single input is used.

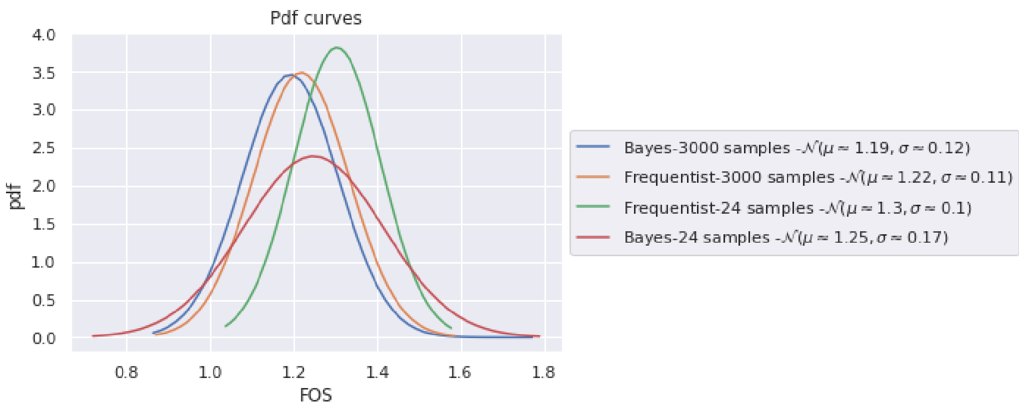


Figure 37: Pdf's of Factor of Safety

Above figure 40 shows pdf's plotted for all 4 results. For smaller data set, a significant difference between the pdf's obtained using both approaches for smaller data set is observed. The pdf obtained using the Bayes approach shows the wider distribution for smaller data set indicating the uncertainty in calculating the FOS values. With the frequentist approach, the pdf is relatively narrower and predicts a higher FOS value. Looking at the tails of the pdf, Bayes' approach provides more information about the FOS compared to the frequentist approach.

Overall, it can be said that using the frequentist approach (especially on limited data) overestimates the FOS results. Also, the uncertainty involved can not be well showcased using this approach whereas the Bayes approach showcases the uncertainty involved and provides relatively better and more reliable FOS results depending upon data size.

The above interpretations are very specific to this particular slope and strength profile. To see if similar trends are observed in examples with different CPT profiles, a similar analysis is performed on a different set of data in the next section.

5.4. SLOPE STABILITY ANALYSIS: EXAMPLE 2

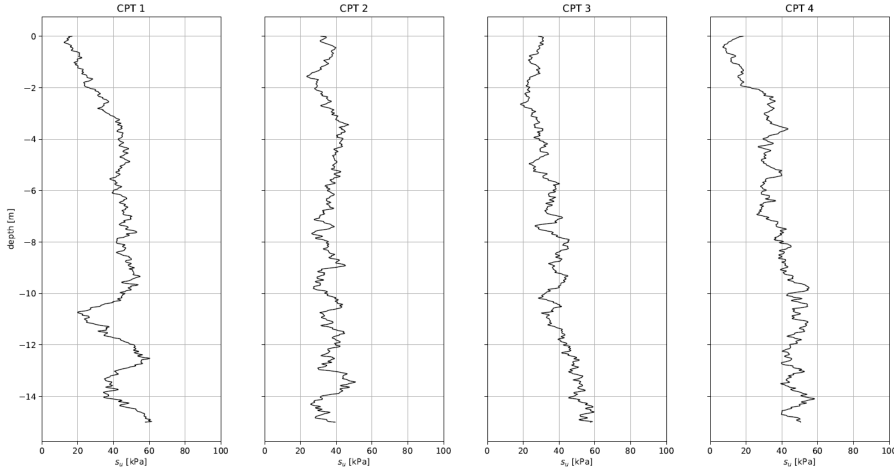


Figure 38: Undrained shear strength (s_u) profiles : Example 2

The above figure 38 shows 4 different CPT profiles compared to the previous example. Similar to the previous example, two data sets are considered; one with 3000 data points and one with 24 data points. Using Frequentist approach as shown in section 5.1 and Bayes' approach shown in section 5.2, input statistics were calculated. Following table 4 shows the results obtained.

Table 4: Results of input parameters for second example

	Data Points	**Intercept (b) in kPa**	**Slope (a) in kPa**	**CoV**	**Vertical scale of fluctuation in meters**
Frequentist Approach	**3000**	26.78	1.4	0.214	2.9
	24	26.15	1.59	0.168	2.9
Bayes' Approach	**3000**	(27.145, 0.259)	(1.349, 0.034)	(0.214, 0.003)	2.9
	24	(27.252, 2.618)	(1.421, 0.407)	(0.246, 0.041)	2.9

Once these input statistics are derived, slope stability analysis is performed. In this analysis same slope is assumed as in the previous example. The following figure 39 shows the results for the same.

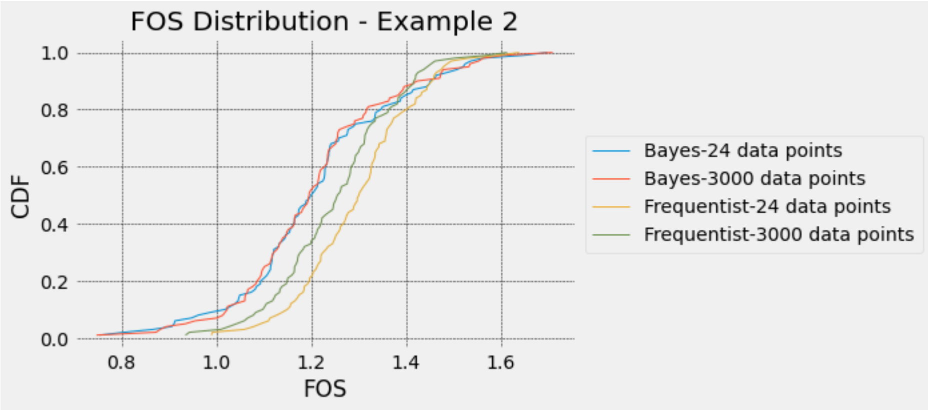


Figure 39: CDF's of Factor of Safety

5

Table 5: Results of second slope example

	Data Points	95% Reliability	50% Reliability	5% Reliability
Frequentist Approach	**3000**	1.0429	1.2555	1.4794
	24	1.0939	1.3009	1.4454
Bayes' Approach	**3000**	0.9353	1.1960	1.5333
	24	0.9111	1.1996	1.5211

Above table 5 summarises the values of FOS for example 2. Compared to the previous example (example 1), the trend of cdf's of FOS obtained in this example is significantly different. In this example, the cdf's obtained using Bayes' approach for both the data sets are nearly identical, showing little to no difference in the FOS values for all three reliability percentages. With the frequentist approach, the FOS values computed for smaller data set are higher than those computed for bigger data set showing a shift in the cdf.

Assuming that the structure fails at FOS of 1, about 9% failure probability is obtained using the Bayes' approach. With the frequentist approach, about 2.5% failure probability is obtained. This shows that the frequentist approach predicts higher reliability of slope compared to the prediction using Bayes' approach. The following figure shows the pdf's curves of the FOS obtained in this example.

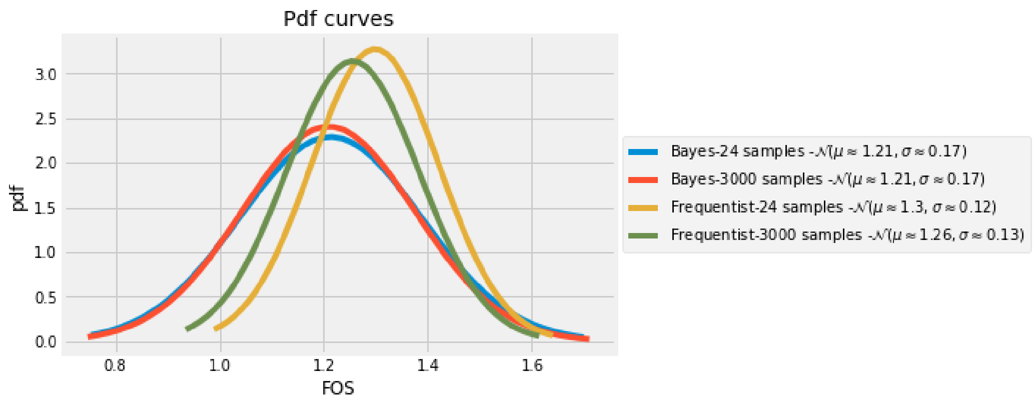


Figure 40: Pdf's of Factor of Safety

The major difference between the results of these two examples is the different trends of the cdf's obtained using Bayes' approach. The cdf's obtained using Bayes' approach for smaller and bigger data sets is different in example 1 whereas it is the same in the case of example 2. The primary cause for this might be the value of the vertical scale of fluctuation considered.

As mentioned previously in the case of the smaller data set (24 data points), the distance between two consecutive data points in one strength profile is 2.5 meters. The input value of the vertical scale of prediction used in the analysis of example 1 is 1.15 meters. Since for the smaller data set the distance between two consecutive data points is larger than the assumed vertical scale of fluctuation, there is no correlation present between the data points. In the case of the bigger data set, two consecutive data points lie within the range of the assumed vertical scale of fluctuation suggesting that the data points are correlated. The absence of correlation in smaller data ultimately adds up more uncertainty in calculating the FOS. Hence, when compared to the cdf of FOS drawn using results of the bigger data set, the cdf drawn using results of smaller data set is wider. This wide cure shows the uncertainty involved in the calculations. For example 2, the assumed vertical scale of fluctuation is 2.9 meters. Hence, for both the data sets it can be said that correlation is present. This provides the reason for getting similar cdf curves for both data sets in example 2.

To summarize the results of both the examples, Bayes' approach provides wider cdf curves compared to the frequentist approach. Wide cdf is the indicator of uncertainty involved in the calculation of FOS. Wider is the cdf more is the uncertainty involved. Also, with change in vertical scale of fluctuation, with Bayes' approach, significant change is observed in cdf's of FOS obtained by smaller and bigger data sets. With the frequentist approach, such difference is not observed since for both the examples, a similar trend of FOS is present.

The fundamental reason behind such difference in results of both approaches: In Bayes' approach, using the prior knowledge and available test data; all the possible values of input parameters are predicted. Every parameter is correlated with each other.

Each trend line has its specific CoV. An individual set of input parameters will provide different FOS values. Using all the possible sets of input parameters provides a better picture of FOS values. In the case of the frequentist approach, different general statistical approaches are used to predict the input parameters and the best estimate is selected to run the FEM analysis. It neglects the other possible values of input parameters which ultimately affect the estimation of FOS values. Hence, it can be said that Bayes' approach should be used in calculating the parameter values over the frequentist approach since it provides better overall insight on the possible FOS values.

However, in this work, the vertical scale of fluctuation is not predicted using the Bayes' approach. Hence, the same value of the vertical scale of fluctuation is used in both approaches. Hence, the uncertainty specifically generated by the vertical scale of fluctuation is not considered in this work. Considering all the possible values of the scale of fluctuation will affect the results. The next chapter provides conclusions on the work presented in this report.

6

CONCLUSION

6.1. CONCLUSIONS

In this thesis 'Effect of various approaches to derive distributions of soil properties in reliability studies of slope stability, two major research questions were aimed to investigate: 'How can the total uncertainty in the slope stability analysis be quantified?' and 'What is the effect of considering different data sets?'

As seen from the literature studies, one of the best methods for such investigation is Bayes' Theorem. To conduct the reliability studies in this work, the Finite Element Method is used (FEM model developed at TU Delft) which also considers spatial variability using random fields. To this model, as input the shear strength profile of the soil is to be provided in terms of trend, the scale of fluctuation, coefficient of variation of the data around mean (input parameters). These parameters are derived using Frequentist as well as Bayes statistics. Different data sets (3000 data points and 24 data points) were considered to derive these parameters. These input parameters were further used to analyze the sample slope stability problem and the results were compared to answer the research questions.

- **How can the total uncertainty in the slope stability analysis be quantified?**
  Yes, Using Bayes' theorem the total uncertainty in slope stability analysis can be clearly observed. The cdf curve of Factor of safety observed using the Bayes' approach is relatively wider than those obtained using the frequentist approach indicating uncertainty.

- **What is the effect of considering different data sets?**
  It is clearly observed that with smaller data set, there is more uncertainty involved compared to bigger data set when Bayes' approach is used. With the frequentist approach, this uncertainty can't be seen since the cdf curves are nearly similar for both data sets with a shift in FOS values.

Looking at the results of both the slope stability problem, the frequentist approach can provide distinct results of FOS values. It might overestimate or underestimate the FOS results (as it overestimates the results in example 2). It follows the same trend in cdf for different data sets for both examples. With smaller data set, it predicts higher FOS compared to the bigger data set. This shows the overestimation in the value of FOS in the case of smaller data set.

Bayes' approach provides completely different results compared to the frequentist approach. It provides wider cdf curves compared to the frequentist curves showing the uncertainty in FOS values. Also for both the data sets, the cdf's might vary. As seen in example 1, the cdf of smaller data set is wider than bigger data set. This result is reasonable assuming that with limited data, more uncertainty is involved. In example 2, cdf obtained for both the data sets using Bayes' approach is similar to each other but wider when compared to the curves obtained using the frequentist approach showing uncertainty. The primary cause of this change in cdf for different data sets in the Bayes' approach might be the consideration of the vertical scale of fluctuation. In example 2 the data points from both the data sets lie within the assumed scale of fluctuation, providing the same cdf's. In example 1 the data points for smaller data set lie beyond the assumed scale of fluctuation, providing different cdf's for both data sets.

In this work, results obtained using Bayes' approach neglect the uncertainty generated due to the vertical scale of fluctuation. The vertical scale of fluctuation is not predicted using Bayes' approach. The value predicted using the most likelihood method is considered for all the approaches. Hence, more investigation involving the use of the vertical scale of fluctuation predicted using Bayes' approach is needed.

The major reason behind such difference between the two approaches: With the Bayes' approach by considering the past knowledge and available test data all the possible values of input parameters are calculated. With the frequentist approach, using the available test data, the single best estimate of the input parameter is derived. By using all the possible input parameters derived by Bayes' approach, a more complete and better picture of FOS values is captured. Such a picture of FOS values is not captured by using the frequentist approach since it just uses one best estimate of input parameters and totally neglects other possible values. Hence it can be concluded that overall, by using the input parameters predicted using the Bayes' approach provides better and reliable results over the frequentist approach by providing more insight into the uncertainties involved in the analysis.

6.2. FUTURE WORK

Following are a few recommendations for future work on implementing Bayes' Statistics in Reliability studies of a slope.

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- Deriving scale of fluctuations using Bayes' statistics: In current work, the scale of fluctuation (vertical) used is derived only from the frequentist method. Using Bayes' approach in determining this parameter will significantly affect the results and provide better insight into uncertainty.

- Use of actual variability assumptions: In this work, uninformative priors are assumed to predict the input parameters of soil property. Hence only total variability can be quantified. By using the informative priors (actual variability ranges provided by Phoon n Kulhawy) more detailed information on total and inherent variability and its effect can be obtained.

- Testing this approach on actual data: In this work, the very basic slope problem (no horizontal scale of fluctuation, homogeneous soil profile, CPT data without noise) is investigated. It will be interesting to study how this approach works on a problem having heterogeneous soil profile, CPT data with noise, etc.

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APPENDIX

PyMC3 syntax

```
obs_su = np.random.normal(245, 42, 20)
with pm.Model() as basic_su:
    su_mu = pm.Normal('su_mu', mu=205, sd=60)
    su_sd = pm.Normal('su_sd', mu=81.5, sd=10)

    y = pm.Normal('y', mu=su_mu, sd=su_sd, observed=obs_su)

    trace = pm.sample(1000)
```

Observed Data

Prior Data

Likelihood Function

Posterior Sample

```
Sequential sampling (2 chains in 1 job)
NUTS: [su_sd, su_mu]
100.00% [2000/2000 00:01<00:00 Sampling chain 0, 0 divergences]
100.00% [2000/2000 00:01<00:00 Sampling chain 1, 0 divergences]
Sampling 2 chains for 1_000 tune and 1_000 draw iterations (2_000 + 2_000 draws total) took 3 seconds.
```

Figure 41: Input mechanism for Bayes' approach