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ORIGINAL ARTICLE

On aspects of damping for a vertical beam with a tuned mass damper at the top

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Abstract In this paper, the wind-induced, horizontal vibrations of a vertical Euler–Bernoulli beam will be considered. At the top of the beam, a tuned mass damper (TMD) has been installed. The horizontal vibrations can be described by an initial-boundary value problem. Perturbation methods will be applied to construct approximations of the solutions of the initial-boundary value problem, and it will be shown that the TMD uniformly damps the oscillation modes of the beam. In the analysis, it will be assumed that damping, wind-force, and gravity effects are small but not negligible.

Keywords Asymptotics · Boundary damping · Euler–Bernoulli beam · Stability · Tall building · Tuned mass damper · Two-timescales perturbation method

1 Introduction

In many mathematical models, oscillations of elastic structures are described by (non)linear wave equations, by (non)linear plate equations, or by (non)linear beam equations. Examples of wave-like or string-like problems are given in [1–3]. An example of a plate-like problem is given in [4]. In this paper, beam-like problems will be considered. Bridges [5] and tall buildings [6] can be modelled by elastic beams.

In recent years, more and more tall building were built. For tall buildings, or high-rise buildings, dampers, active or passive, are used to dissipate the energy of the vibrations of the building. Passive dampers are for instance tuned mass dampers (TMDs), tuned liquid dampers (TLDs), or tuned mass liquid dampers (TLCDs). A swimming pool or a water basin for the sprinkler installation at the top of the building already damps the vibration. A TMD is one of the most simple and economic ways to control the vibrations of a beam structure. The TMD can be modelled as a simple mass–spring–dashpot system.

In [7], a simple approach to the design of MDs is used. It is based on a 1+1 and on a 4+1 degrees of freedom (DOF) model of the system. In [6], a more complicated model is used to consider the dynamics of a tall building, where a TMD system is installed at the top. Numerical methods are used to solve this problem approximately. The damping is considered to be Coulomb damping. It has been concluded that the TMD needs much space to operate in real applications. The displacement of the mass might be much larger then that of the top floor. It has also been shown that the oscillations of the building are effectively reduced when the TMD frequency is tuned to be equal to that of the building.

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In this paper, it will be assumed that the TMD can be modelled as a simple mass–spring–dashpot system, and that the building can be modelled as a vertical Euler–Bernoulli beam. The TMD is installed at the top of the vertical beam to absorb the horizontal vibrations of the beam. The tip-mass is connected to a linear spring with spring constant $\hat{k}$, and to a dashpot with damping coefficient $\hat{c}$.

This is an example of a beam-like problem with boundary damping. Also in [1, 2, 6, 8, 9] various types of boundary damping have been considered. Furthermore, a uniform wind-flow will be considered, which causes nonlinear drag and lift forces (F_D, F_L) acting on the structure per unit length. A simple model of a vertical Euler–Bernoulli beam equation subjected to wind-forces and with a TMD at the top is given by

$$EI\eta_{XXXX} + [(gm + \rho g A(L - X))\eta_X]_X + \rho A \eta_{\tau\tau} = F_D + F_L, \quad 0 < X < L, \quad \tau > 0, \tag{1}$$

$$\eta(0,\tau) = \eta_X(0,\tau) = \eta_{XX}(L,\tau) = 0, \tau \geq 0, \tag{2}$$

$$-gm\eta_X(L,\tau) - EI\eta_{XXX}(L,\tau) + m(\eta(L,\tau) + \zeta(\tau))_{\tau\tau} = 0, \quad \tau \geq 0, \tag{3}$$

$$\hat{k}\zeta(\tau) + \hat{c}\zeta_\tau(\tau) + m(\eta(L,\tau) + \zeta(\tau))_{\tau\tau} = 0, \quad \tau \geq 0, \tag{4}$$

where E is the Young modulus, I the moment of inertia of the cross section, ρ the density, A the cross-sectional area, L the length, $\eta(X,\tau)$ the deflection of the beam in Y-direction (see Fig. 1), m the mass of the tip-mass, $\zeta(\tau)$ the displacement of the mass m relative to the top of the beam, τ the time, X the position along the beam (see Fig. 1), and g the acceleration due to gravity. In [3], it has been shown that $F_D + F_L$ can be approximated by

$$F_D + F_L = \frac{\rho_a d v_\infty^2}{2}\left(a_0 + \frac{a_1}{v_\infty}\eta_\tau + \frac{a_2}{v_\infty^2}\eta_\tau^2 + \frac{a_3}{v_\infty^3}\eta_\tau^3\right), \tag{5}$$

where ρ_a is the density of the air, d the diameter of the cross-sectional area of the beam, v_∞ the uniform wind-flow velocity, and a_0, a_1, a_2, a_3 depend on certain drag and lift coefficients, which are given explicitly in [3].

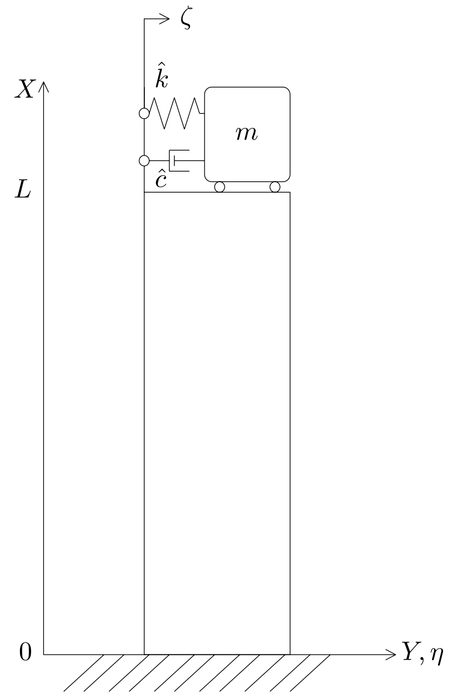

Fig. 1 A simple model for a vertical beam with a tuned mass damper at the top

To put the model in a non-dimensional form, the following substitutions $\hat{u}(x,t) = (\kappa/v_\infty)[\eta(X,\tau)/L]$, $\hat{\xi}(t) = (\kappa/v_\infty)[\zeta(\tau)/L]$, $x = X/L$ and $t = \kappa/L\tau$, where $\kappa = (1/L)\sqrt{(EI)/(A\rho)}$ will be used. In this way, the nonlinear partial differential (1) becomes $\hat{u}_{xxxx} + \epsilon_1[(\gamma + 1 - x)\hat{u}_x]_x + \hat{u}_{tt} = (\rho_a dL)/(2A\rho)(v_\infty/\kappa)(a_0 + a_1\hat{u}_t + a_2\hat{u}_t^2 + a_3\hat{u}_t^3)$, where $\gamma = m/(\rho AL)$ and where $\epsilon_1 = (g\rho AL^3)/(EI)$ is a small parameter, that is, $0 < \epsilon_1 \ll 1$. In [3], it has been shown that the right-hand side of the latter equation can be approximated by $\epsilon_2\alpha(\hat{u}_t - (b/a)\hat{u}_t^3) + \mathcal{O}(\epsilon_2^{m_1})$, with $m_1 > 1$, where a and b are specific combinations of drag and lift coefficients, which are given explicitly in [3], and are of order 1, and where $\epsilon_2\alpha = (\rho_a dL)/(2A\rho)(v_\infty/\kappa)a$, where ϵ_2 is a small parameter and $\alpha = \mathcal{O}(1)$.

Finally, the transformations $u(x,t) = \sqrt{(3b/a)}\,\hat{u}(x,t)$ and $\xi(x,t) = \sqrt{(3b/a)}\hat{\xi}(x,t)$ will be applied to obtain the following initial-boundary value problem:

$$u_{xxxx} + \epsilon_1[(\gamma + 1 - x)u_x]_x + u_{tt} = \epsilon_2\alpha\left(u_t - \frac{1}{3}u_t^3\right), \quad 0 < x < 1, \quad t > 0, \tag{6}$$

$$u(0,t)=u_x(0,t)=u_{xx}(1,t)=0, \quad t\geq 0, \tag{7}$$

$$-\epsilon_1\gamma u_x(1,t)-u_{xxx}(1,t)+\gamma(u_{tt}(1,t) + \xi_{tt}(t))=0, \quad t\geq 0, \tag{8}$$

$$k\xi(t)+\tilde{c}\xi_t(t)+\gamma(u_{tt}(1,t)+\xi_{tt}(t))=0, \quad t\geq 0, \tag{9}$$

$$u(x,0)=f(x), 0<x<1, \tag{10}$$

$$u_t(x,0)=g(x), 0<x<1, \tag{11}$$

$$\xi(0)=\xi_0 \quad \text{and} \quad \xi_t(0)=\xi_1, \tag{12}$$

where $k=\hat{k}(L^3/(EI))$, and $\tilde{c}=\hat{c}\sqrt{L^2/(EI\rho A)}$ are positive constants, where $f(x)$, $g(x)$, ξ_0, and ξ_1 are the initial displacement of the beam, the initial velocity of the beam, the initial displacement of the tip-mass, and the initial velocity of the tip-mass, respectively.

Now $\xi(t)$ will be eliminated from the coupled boundary conditions (8) and (9) to obtain an initial-boundary value problem for $u(x,t)$. This will be done in the following way. Subtract (8) from (9), and differentiate the result with respect to t, to obtain

$$-\epsilon_1\gamma u_{xt}(1,t)-u_{xxxt}(1,t)=k\xi_t(t)+\tilde{c}\xi_{tt}(t). \tag{13}$$

The boundary condition (8) gives the following expression for $\xi_{tt}(t)$

$$\xi_{tt}(t)=\epsilon_1 u_x(1,t)+\frac{1}{\gamma}u_{xxx}(1,t)-u_{tt}(1,t). \tag{14}$$

Substitution of this expression for $\xi_{tt}(t)$ into (13) yields

$$k\xi_t(t)=-\epsilon_1\gamma u_{xt}(1,t)-u_{xxxt}(1,t)-\tilde{c}\left[\epsilon_1 u_x(1,t) +\frac{1}{\gamma}u_{xxx}(1,t)-u_{tt}(1,t)\right]. \tag{15}$$

Differentiate (15) with respect to t, substitute the so-obtained expression for $\xi_{tt}(t)$ into (14), and multiply by γ, to obtain

$$\begin{aligned}\gamma u_{tt}(1,t)-\epsilon_1\gamma u_x(1,t)-u_{xxx}(1,t) &= \frac{\gamma}{k}(\epsilon_1\gamma u_x(1,t)+u_{xxx}(1,t)-\tilde{c}u_t(1,t))_{tt}\\ &\quad+\frac{\tilde{c}}{k}(\epsilon_1\gamma u_x(1,t)+u_{xxx}(1,t))_t.\end{aligned} \tag{16}$$

So, the problem (6)–(9) can be rewritten as the following initial-boundary value problem for $u(x,t)$:

$$\mathbb{L}(u)=\epsilon_2\alpha\left(u_t-\frac{1}{3}u_t^3\right), \quad 0<x<1, \quad t>0, \tag{17}$$

$$u(0,t)=u_x(0,t)=u_{xx}(1,t)=0, \quad t\geq 0, \tag{18}$$

$$\begin{aligned}\mathbb{B}_{k\gamma}(u) = &-\epsilon_1 k\gamma u_x(1,t)-\gamma(\epsilon_1\gamma u_x(1,t)\\ &-\tilde{c}u_t(1,t))_{tt}-\tilde{c}(\epsilon_1\gamma u_x(1,t)\\ &+u_{xxx}(1,t))_t, \quad t\geq 0,\end{aligned} \tag{19}$$

$$u(x,0)=f(x), \quad 0<x<1, \tag{20}$$

$$u_t(x,0)=g(x), \quad 0<x<1, \tag{21}$$

where

$$\mathbb{L}(u)\equiv u_{xxxx}+\epsilon_1[(\gamma+1-x)u_x]_x+u_{tt}, \tag{22}$$

$$\mathbb{B}_{k\gamma}(u)\equiv ku_{xxx}(1,t)+\gamma u_{xxxtt}(1,t)-k\gamma u_{tt}(1,t). \tag{23}$$

When $u(x,t)$ has been determined, also $\xi(t)$ can be obtained in the following way. Subtract (8) from (9) to obtain

$$\xi(t)=\frac{-1}{k}(\epsilon\gamma u_x(1,t)+u_{xxx}(1,t))-\frac{\tilde{c}}{k}\xi_t. \tag{24}$$

Now substitution of (15) into (24) yields $\xi(t)$ as a function of $u(x,t)$:

$$\begin{aligned}\xi(t) &= (u_{xxx}(1,t)+\epsilon_1\gamma u_x(1,t))\left(\frac{\tilde{c}^2}{\gamma k^2}-\frac{1}{k}\right)\\ &\quad+\frac{\tilde{c}}{k^2}(\epsilon_1\gamma u_x(1,t)+u_{xxx}(1,t)-\tilde{c}u_t(1,t))_t.\end{aligned} \tag{25}$$

Due to the TMD at the top of the building, the problem will have an additional degree of freedom. The displacement of the tip-mass depends on all the oscillation modes of the building. Therefore, the TMD does not have a specified frequency.

The nonlinear wind-force $\epsilon_2\alpha(u_t(x,t)-(1/3)u_t^3(x,t))$ in (17) will give a coupling between (almost) all oscillation modes. In [2, 3] also this nonlinear wind-force has been considered. It has been shown that the wind-force gives a coupling between (almost) all oscillation modes. It is also known (see

Section 4) that the nonlinear term damps the vibrations. In this paper, the linearized initial-boundary value problem will be considered because the main goal of this paper is to determine the damping. If the damper damps the vibrations due to the linearized wind-force, the damper also damps the vibration due to nonlinear wind-force because the nonlinear term in the wind-force also damps the vibrations.

In this paper, the linearized initial-boundary value problem (17)–(21) will be considered. The damping parameter $\tilde{c}$ will be considered to be a small parameter, that is, $\tilde{c} = \epsilon_3 c$, where $0 < \epsilon_3 \ll 1$ and where $c = \mathcal{O}(1)$. Now, the following initial-boundary value problem, which describes up to $\mathcal{O}(\epsilon_2^{m_1})$, $m_1 > 1$, the horizontal, wind-induced displacement of a damped vertical beam with tip-mass at the top, can be introduced:

$$u_{xxxx} + u_{tt} = -\epsilon_1[(\gamma + 1 - x)u_x]_x + \epsilon_2 \alpha u_t, \quad 0 < x < 1, \quad t > 0, \tag{26}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \quad t \geq 0, \tag{27}$$

$$\begin{aligned} \mathbb{B}_{k\gamma}(u) = &-\epsilon_1(k\gamma u_x(1,t) + \gamma^2 u_{xtt}(1,t)) \\ &+ \epsilon_3 c(\gamma u_{ttt}(1,t) - u_{xxxt}(1,t)) \\ &- \epsilon_1 \epsilon_3 c \gamma u_{xt}(1,t), \quad t \geq 0, \end{aligned} \tag{28}$$

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{29}$$

$$u_t(x,0) = g(x), \quad 0 < x < 1. \tag{30}$$

The initial-boundary value problem (26)–(30) actually contains four small parameters $\epsilon_1, \epsilon_2, \epsilon_3$, and γ, which is the ratio of the tip-mass and the mass of the beam. In this paper, the influence of the parameters ϵ_3 and γ on the damping will be considered. The case that γ is small (but larger in order then ϵ_3), the case that γ is of order ϵ_3, and the case that γ is of order ϵ_3^2 will be studied. For each case, a different approach is needed to construct approximations of the solutions of the initial-boundary value problem (26)–(30). These three cases will be considered in this paper.

This paper is organized as follows. In Section 2, the initial-boundary value problem (26)–(30) with $c = \alpha = 0$ is considered. It will be shown that the eigenvalues of the corresponding boundary value problem are real-valued and positive. Also it will be explained why perturbation techniques are applied to solve the initial-boundary value problems. In Section 3, the vibrations of an undamped beam not subjected to wind-forces and not subjected to gravity effects, that is, the initial-boundary value problem (26)–(30) with $c = \alpha = \epsilon_1 = 0$, will be considered. This is the case of a beam equation subjected to non-classical boundary conditions. In Section 4, the energy of the beam with a TMD at the top will be considered and the boundedness of the solutions will be shown when $\alpha = 0$. Also the damping of the vibrations will be shown when $\alpha = 0$. In Section 5, approximations of the eigenvalues of the damped initial-boundary value problem (26)–(30) with $\alpha = \epsilon_1 = 0$ will be constructed by applying the method of separation of variables. By applying this method, a so-called characteristic equation is obtained. The roots of this equation will be constructed. These roots can be used to obtain the eigenvalues of the damped initial-boundary value problem (26)–(30) with $\alpha = \epsilon_1 = 0$. These eigenvalues will be used to obtain the damping rates of the oscillation modes. If ϵ_3 and γ are fixed, the roots of this equation can be found by using numerical methods. The roots can also be obtained approximately because ϵ_3 and γ are small parameters. In this section, the cases $\gamma = \mathcal{O}(1)$, $\gamma = \mathcal{O}(\epsilon_3)$, and $\gamma = \mathcal{O}(\epsilon_3^2)$ will be considered. These cases will be considered because the ratio γ can be of lower, of equal, or of higher order with respect to ϵ_3. The construction of the approximations of the roots for these cases will turn out to be different. These approximations of the eigenvalues gives a good indication what scalings are necessary to construct approximations of the solutions of the initial-boundary value problem (26)–(30) for the cases $\gamma = \mathcal{O}(1)$, $\gamma = \mathcal{O}(\epsilon_3)$, and $\gamma = \mathcal{O}(\epsilon_3^2)$. In Section 6, the multiple-timescales perturbation method will be applied to construct approximations of the solutions of the initial-boundary value problem (26)–(30). The reader is referred to the book of Nayfeh and Mook [10] for a description of this method. In this paper, only the initial-boundary value problem (26)–(30) for the case that $\gamma = \mathcal{O}(1)$ will be solved approximately. In this section also the stability of a vertical beam with a TMD at the top in a wind-field will be considered.

The constructed approximations of the solutions will be used to determine the type of damping.

Finally, some remarks will be made and some conclusions will be drawn in Section 7.

2 The undamped problem with $\alpha = 0$

In this section, the horizontal vibrations of a vertical beam with a tip-mass at the top will be studied. The wind-force and the damping are neglected. So, in this section, the initial-boundary value problem (26)–(30) with $c = \alpha = 0$ will be considered:

$$\mathbb{L}(u) = 0, \tag{31}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \tag{32}$$

$$\mathbb{B}_{k\gamma}(u) = -\epsilon_1\gamma(ku_x(1,t) + \gamma u_{xtt}(1,t)), \tag{33}$$

where $\mathbb{L}$ and $\mathbb{B}$ are given by (22) and (23), respectively. The method of separation of variables will be used to solve (31)–(33). Now look for nontrivial solutions of the partial differential Equation (31) and the boundary conditions (32)–(33) in the form $X(x)T(t)$. By substituting $u(x,t) = X(x)T(t)$ into problem (31)–(33) a boundary value problem for $X(x)$ is obtained:

$$X^{(4)}(x) + \epsilon_1[(\gamma + 1 - x)X'(x)]' = \lambda X(x), \tag{34}$$

$$X(0) = X'(0) = X''(1) = 0, \tag{35}$$

$$(\gamma\lambda - k)(\epsilon_1\gamma X'(1) + X'''(1)) = \gamma\lambda X(1), \tag{36}$$

and the following problem for $T(t)$:

$$T''(t) + \lambda T(t) = 0, \tag{37}$$

where $\lambda \in \mathbb{C}$ is a separation constant. Note that (34)–(36) is a non-standard problem. Therefore, the eigenvalues and eigenfunctions of this problem will be studied. First, it will be shown that the eigenvalues λ of problem (34)–(36) are real-valued and positive.

The case $\gamma\lambda = k$ and the case $\gamma\lambda \neq k$ will be considered. If $\gamma\lambda = k$ the eigenvalue λ is real-valued and positive because k and γ are real-valued and positive constants. Now the second case will be considered. Let the linear differential operator $\mathcal{L}$ be defined by:

$$\mathcal{L}[X] = \frac{\mathrm{d}^4X}{\mathrm{d}x^4} + \epsilon_1\frac{\mathrm{d}}{\mathrm{d}x}\left[(\gamma + 1 - x)\frac{\mathrm{d}X}{\mathrm{d}x}\right]. \tag{38}$$

Let $X_1(x)$ and $X_2(x)$ be two different solutions of the boundary value problem (34)–(36) corresponding to eigenvalues λ_1 and λ_2 respectively, then

$$\begin{aligned}\int_0^1 (\mathcal{L}[X_1]\overline{X_2} - X_1\mathcal{L}[\overline{X_2}])\mathrm{d}x &= (\epsilon_1\gamma X_1'(1) + X_1'''(1))\overline{X_2}(1)\\ &\quad - X_1(1)(\overline{\epsilon_1\gamma X_2'(1) + X_2'''(1)}),\end{aligned} \tag{39}$$

where the dependency of $X_1(x)$ and $X_2(x)$ on x has been dropped. Now substitute $\mathcal{L}[X_1] = \lambda_1 X_1$ and $\mathcal{L}[X_2] = \lambda_2 X_2$ into (39) and consider the boundary condition (36) to obtain

$$(\lambda_1 - \overline{\lambda_2})\left(\int_0^1 X_1\overline{X_2}\mathrm{d}x + \frac{(\epsilon_1\gamma X_1'(1) + X_1'''(1))(\overline{\epsilon_1\gamma X_2'(1) + X_2'''(1)})}{\lambda_1\overline{\lambda_2}}\right) = 0, \tag{40}$$

or equivalently

$$\left(\frac{\lambda_1 - \overline{\lambda_2}}{\lambda_1\overline{\lambda_2}}\right)\left(\int_0^1 \mathcal{L}[X_1]\mathcal{L}[\overline{X_2}]\mathrm{d}x + (\epsilon_1\gamma X_1'(1) + X_1'''(1))(\overline{\epsilon_1\gamma X_2'(1) + X_2'''(1)})\right) = 0. \tag{41}$$

Now introduce the following inner product on V

$$\langle u(x), v(x)\rangle = \int_0^1 \mathcal{L}[u]\mathcal{L}[\overline{v}]\mathrm{d}x + (\epsilon_1\gamma u'(1) + u'''(1))(\overline{\epsilon_1\gamma v'(1) + v'''(1)}), \tag{42}$$

where

$$\begin{aligned}V = \{v \in L^2(0,1) | v(0) = v'(0) = v'' = 0,\\ \epsilon_1\gamma v'(1) + v'''(1) \neq 0\} \cup \{v \equiv 0\}.\end{aligned} \tag{43}$$

In this notation (41) becomes

$$\left(\frac{\lambda_1 - \overline{\lambda_2}}{\lambda_1\overline{\lambda_2}}\right)\langle X_1(x), X_2(x)\rangle = 0. \tag{44}$$

Now let $\phi = X_1 = X_2$ and let $\lambda = \lambda_1 = \lambda_2$ then (44) becomes

$$\left(\frac{\lambda - \overline{\lambda}}{|\lambda|}\right)\langle \phi(x), \phi(x)\rangle = 0. \tag{45}$$

But $\langle\phi(x), \phi(x)\rangle \geq 0$ and $\phi(x)$ is not allowed to be the zero function. So, $\langle\phi(x), \phi(x)\rangle$ in Equation (44) is positive, therefore $\lambda - \overline{\lambda} = 0$, which implies that λ is real.

Since the eigenvalues λ are real, the differential Equation (34) and the boundary conditions (35) and

(36) only have real parameters (γ, ϵ_1, and λ). So, the eigenfunctions can be chosen to be real-valued. Let ϕ_i and ϕ_j be two real eigenfunctions corresponding to the eigenvalues λ_i and λ_j respectively. Now substitute $X_1 = \phi_i$, $X_2 = \phi_j$, $\lambda_1 = \lambda_i$, and $\lambda_2 = \lambda_j$ into (44), to obtain

$$\left(\frac{\lambda_i - \lambda_j}{\lambda_i \lambda_j}\right)\langle \phi_i, \phi_j \rangle = 0.$$

If $\lambda_i \neq \lambda_j$ it follows that $\langle \phi_i, \phi_j \rangle = 0$. So, eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the inner product (42).

Now it will be shown that the eigenvalues are positive. Multiply (34) by $X(x)$ and integrate the result with respect to x from 0 to 1, to obtain

$$\int_0^1 \left(X^{(4)}(x) + \epsilon_1[(\gamma + 1 - x)X'(x)]'\right)X(x)\,\mathrm{d}x = \lambda \int_0^1 X^2(x)\,\mathrm{d}x. \tag{46}$$

Integrating by parts and considering the boundary conditions (35), yields

$$I_1 + X(1)(X'''(1) + \epsilon_1 \gamma X'(1)) = \lambda I_2, \tag{47}$$

where

$$I_1 = \int_0^1 \left((X''(x))^2 - \epsilon_1(\gamma + 1 - x)(X'(x))^2\right)\mathrm{d}x, \tag{48}$$

$$I_2 = \int_0^1 (X(x))^2 \mathrm{d}x. \tag{49}$$

In [9] it has been shown for nontrivial functions $X(x)$ that $I_1 > 0$ for ϵ_1 sufficiently small, that is, $\epsilon_1(\gamma + (1/2)) < 1$. The boundary condition (36) can be rewritten in the following form

$$X(1)(X'''(1) + \epsilon_1 \gamma X'(1)) = \left(\frac{\gamma\lambda}{\gamma\lambda - k}\right)X^2(1). \tag{50}$$

By substituting (50) into (47) the following second-order polynomial in λ is obtained:

$$\gamma\lambda^2 I_2 + kI_1 = (\gamma I_1 + \gamma k X^2(1) + kI_2)\lambda. \tag{51}$$

The solutions $\lambda_{1,2}$ of (51) can be determined and are given by

$$\lambda_{1,2} = \frac{(\gamma I_1 + \gamma k X^2(1) + kI_2) \pm \sqrt{D}}{2\gamma I_2}, \tag{52}$$

where

$$\begin{aligned} D &= (\gamma I_1 + \gamma k X^2(1) + kI_2)^2 - 4k\gamma I_1 I_2 \\ &= 2\gamma k X^2(1)(\gamma I_1 + kI_2) \\ &\quad + (\gamma k X^2(1))^2 + (\gamma I_1 - kI_2)^2, \end{aligned} \tag{53}$$

and where D satisfies the following inequalities:

$$(\gamma I_1 + \gamma k X^2(1) + kI_2)^2 > D > 0.$$

These aforementioned inequalities show that the eigenvalues $\lambda_{1,2}$ are non-negative for the case $\lambda \neq (k/\gamma)$. Now by substituting $\lambda = 0$ into (51) it follows that

$$kI_1 = 0, \tag{54}$$

because $kI_1 > 0$, for ϵ_1 sufficiently small, Equation (54) does not hold, so $\lambda = 0$ is not an eigenvalue. Since for the case $\lambda \neq k/\gamma$ and the case $\lambda = k/\gamma$ the eigenvalues are not zero and non-negative it can be concluded that the eigenvalues are positive if ϵ_1 is sufficiently small. Although it can derived that the eigenvalues are real-valued and positive, the eigenvalues cannot be determined exactly because the fourth-order differential Equation (34) cannot be solved exactly. It has been assumed that $0 < \epsilon_1 \ll 1$. Then the term $\epsilon_1[(\gamma + 1 - x)X(x)']'$ in (34) is small. Now perturbation techniques can be used to solve approximately the initial-boundary value problem (35) and (36).

Perturbation methods can be used to solve approximately the ordinary differential Equation (26). By using this method, the approximations for the eigenvalues and the eigenfunctions will be found. These approximations can be used to construct approximations of the solution of the partial differential equation. This will be done in the next section of this paper for the initial-boundary value problem (26)–(30) with $c = \alpha = \epsilon_1 = 0$. Note that this method can be used as long as the method of separation of variables can be applied to the initial-boundary value problem.

3 The undamped problem (26)–(30) with $\alpha = \epsilon_1 = 0$

In this section, the horizontal vibrations of a beam with a tip-mass at the top will be studied. The gravity effect, the wind-force, and the damping are neglected. This problem is given by (26)–(30) with $c = \alpha = \epsilon_1 = 0$:

$$u_{xxxx} + u_{tt} = 0, \tag{55}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \tag{56}$$

$$ku_{xxx}(1,t) + \gamma u_{xxxtt}(1,t) - k\gamma u_{tt}(1,t) = 0, \tag{57}$$

$$u(x,0) = f(x), \tag{58}$$

$$u_t(x,0) = g(x). \tag{59}$$

The functions $\xi(t)$ and $u(x,t)$ are related by (25). Now also relations between the initial values $\xi(0)$ and $u(1,0)$ will be given. Substitution of $\epsilon_1 = 0, \tilde{c} = 0, t = 0$, (20), and (21) into (24) and (15) gives the following relations for the initial displacement ($f(x)$) and the initial velocity ($g(x)$) of the beam at the top and the initial displacement (ξ_0) and the initial velocity (ξ_1) of the tip-mass

$$f'''(1) = -k\xi_0, \tag{60}$$

$$g'''(1) = -k\xi_1. \tag{61}$$

The method of separation of variables will be used to solve the problem (55)–(59). Now look for nontrivial solutions of the partial differential Equation (55) and the boundary conditions (56) and (57) in the form $X(x)T(t)$. By substituting this into (55)–(57) a boundary value problem for $X(x)$ is obtained:

$$X^{(4)}(x) = \lambda X(x), \tag{62}$$

$$X(0) = X'(0) = X''(1) = 0, \tag{63}$$

$$(\gamma\lambda - k)X'''(1) = k\gamma\lambda X(1), \tag{64}$$

and the following problem for $T(t)$:

$$T''(t) + \lambda T(t) = 0, \tag{65}$$

where $\lambda \in \mathbb{C}$ is a separation constant. The boundary value problem (62)–(64) is the same problem as (34)–(36) with $\epsilon_1 = 0$. So the eigenvalues are real-valued, and positive; the eigenfunctions can be chosen to be real-valued, and two real eigenfunctions belonging to two different eigenvalues are orthogonal with respect to the inner product (42). Note that the case $X'''(1) = X(1) = 0$ and the case $X'''(1) = \lambda = 0$ only leads to trivial solutions.

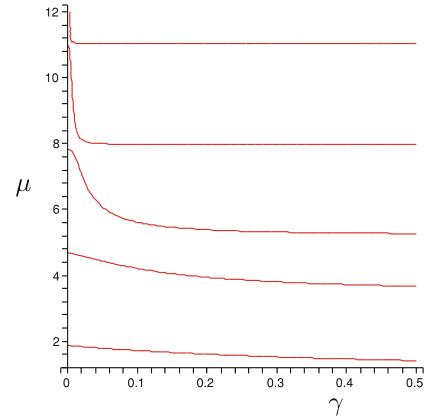

Fig. 2 The values of the first five roots μ of the characteristic Equation (66), for $k = 50$, as a function of $\gamma \in [0, \frac{1}{2}]$

The problem (62)–(64) can be solved analytically. Expressions for the eigenfunctions and the eigenvalues can be found. The eigenvalues $\lambda_n = \mu_n^4$ are implicitly given by the roots of

$$h_{k\gamma}(\mu) \equiv (\gamma\mu^4 - k)q(\mu) + k\gamma\mu s(\mu) = 0. \tag{66}$$

where

$$q(\mu) = 1 + \cosh(\mu)\cos(\mu), \tag{67}$$

$$s(\mu) = \sin(\mu)\cosh(\mu) - \cos(\mu)\sinh(\mu). \tag{68}$$

The real-valued, positive, isolated roots of $h_{k\gamma}(\mu)$ are denoted by μ_n. If μ_n is a root of (66) then also $-\mu_n$ and $\pm i\mu_n$ are roots of (66). The location of the roots depends on the value of γ. For $\gamma = 0$ the roots will be exact the roots of a cantilevered beam without tip-mass (see [8, 9]). The location of the roots of the characteristic equation (66) for $\gamma > 0$ will be close to the location of the roots of (66) for $\gamma = 0$ and of the equation $\mu^4 = k/\gamma$. In Fig. 2, the values of the first five real roots μ are shown as a function of $\gamma \in [0, (1/2)]$ for the case $k = 50$.

It follows that (for large n and γ fixed) $\mu_n \approx (n - (1/2))\pi$, but there is not a fixed $N \in \mathbb{N}$ such that $\mu_n \approx (n - (1/2))\pi$ for all $n > N$ if $\gamma \to 0$.

The eigenfunctions of problems (62)–(64) can be determined, and are given by

$$\hat{\phi}_n(x) = \sin(\mu_n x) - \sinh(\mu_n x) + \beta_n(\cosh(\mu_n x) - \cos(\mu_n x)), \quad (69)$$

where $\beta_n = [(\sin(\mu_n) + \sinh(\mu_n))]/[\cos(\mu_n) + \cosh(\mu_n)]$. In this paper, the eigenfunctions are chosen such that (see also (40))

$$\left(\int_0^1 \phi_i \phi_j \mathrm{d}x + \frac{\phi_{i_{xxx}}(1)\phi_{j_{xxx}}(1)}{\gamma \lambda_i \lambda_j}\right) = \delta_{ij}, \quad (70)$$

where δ_{ij} is the Kronecker symbol, that is, $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, and where the eigenfunctions $\phi_n(x)$ are defined by

$$\phi_n(x) = \frac{\hat{\phi}_n(x)}{\left(\int_0^1 \hat{\phi}_n^2 \mathrm{d}x + \frac{(\hat{\phi}_{n_{xxx}}(1))^2}{\gamma \lambda_n^2}\right)^{\frac{1}{2}}}. \quad (71)$$

After lengthy but elementary calculations, it can be shown that

$$\begin{aligned} \int_0^1 \hat{\phi}_n^2(x)\,\mathrm{d}x + \frac{(\hat{\phi}_{n_{xxx}}(1))^2}{\gamma \lambda_n^2} &= \left(\frac{\sinh(\mu_n) + \sin(\mu_n)}{\cosh(\mu_n) + \cos(\mu_n)}\right)^2 \\ &+ \frac{4}{\gamma \mu_n^2}\left(\frac{q(\mu_n)}{\cosh(\mu_n) + \cos(\mu_n)}\right)^2 \\ &+ \frac{3}{\mu_n}\left(\frac{q(\mu_n)s(\mu_n)}{(\cosh(\mu_n) + \cos(\mu_n))^2}\right), \end{aligned} \quad (72)$$

also it can be shown that $\int_0^1 \hat{\phi}_n^2(x)\mathrm{d}x \to 1$ if $n \to \infty$.

For each eigenvalue $T_n(t)$ can be determined. So infinitely many nontrivial solutions of the initial-boundary problem (55)–(59) have been determined. Using the superposition principle and the initial values (58) and (59), the solution of the initial-boundary value problem is obtained:

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} T_n(t)\phi_n(x) \\ &= \sum_{n=0}^{\infty} \left(A_n \cos\left(\mu_n^2 t\right) + B_n \sin\left(\mu_n^2 t\right)\right)\phi_n(x), \end{aligned} \quad (73)$$

Table 1 Numerical approximations of the first five eigenvalues μ_n, of $\phi_n(1)$, and of $-\phi_{n_{xxx}}(1)/k$ for the case $k = 1$ and $\gamma = 1$, $\gamma = 0.1$, $\gamma = 0.01$, and $\gamma = 0.001$

n	μ_n	$\phi_n(1)$	$-\phi_{n_{xxx}}(1)/k$	$(n - \frac{1}{2})\pi$
$\gamma = 1$				
0	0.9270	0.2593	0.7327	–
1	2.0177	1.9629	−2.0890	1.5708
2	4.7038	−2.0134	2.0175	4.7123
3	7.8568	2.0033	−2.0039	7.8540
4	10.996	−2.0019	2.0014	10.996
$\gamma = 0.1$				
0	1.5700	1.0591	1.6392	–
1	2.1186	1.6728	−3.3214	1.5708
2	4.7040	−2.0135	2.0555	4.7123
3	7.8568	2.0033	−2.0086	7.8540
4	10.996	−2.0013	2.0027	10.996
$\gamma = 0.01$				
0	1.8544	1.9529	0.2619	–
1	3.1881	0.3215	−10.059	1.5708
2	4.7063	−2.0141	2.5300	4.7123
3	7.8569	2.0034	−2.0574	7.8540
4	10.996	−2.0013	2.0151	10.996
$\gamma = 0.001$				
0	1.8732	1.9962	0.0249	–
1	4.6851	−1.9729	−1.8345	1.5708
2	5.6371	−0.3069	31.626	4.7123
3	7.8576	2.0040	−2.7167	7.8540
4	10.996	−2.0014	2.1483	10.996

where

$$A_n = \int_0^1 f(x)\phi_n(x)\mathrm{d}x - \frac{\phi_{n_{xxx}}(1)}{\lambda_n}(\xi_0 + f(1)), \quad (74)$$

$$\mu_n^2 B_n = \int_0^1 g(x)\phi_n(x)\mathrm{d}x - \frac{\phi_{n_{xxx}}(1)}{\lambda_n}(\xi_1 + g(1)). \quad (75)$$

Now because of (24) and (64), and because $c = \epsilon_1 = 0$, it can be deduced that the displacement $\xi(t)$ of the mass at the top of the beam with respect to the top of the beam is given by

$$\begin{aligned} \xi(t) &= \frac{-u_{xxx}(1,t)}{k} = \frac{-1}{k}\sum_{n=0}^{\infty} T_n(t)\phi_{n_{xxx}}(1) \\ &= \sum_{n=0}^{\infty} T_n(t)\left(\frac{\gamma \lambda_n \phi_n(1)}{k - \gamma \lambda_n}\right). \end{aligned} \quad (76)$$

Note that, from (64) it follows that

$$\lim_{\lambda_n \to \frac{k}{\gamma}} \left(\frac{\gamma \lambda_n \phi_n(1)}{k - \gamma \lambda_n}\right) = \frac{-\phi_{n_{xxx}}(1)}{k}.$$

In Table 1, the first five eigenvalues (μ_n) and the first five constant terms ($-\phi_{n_{xxx}}(1)/k$ and $\phi_n(1)$) of the infinite sums (73), for $x = 1$, and (76) are listed for several values of γ. From the eigenvalues (μ_n) it follows that μ_n decreases by increasing γ. Note that the case $\gamma = 1$ is not realistic for applications. The constant terms can be used to compare the direction of the displacement of the tip-mass $\xi(t)$ (i.e. (76)) and the direction of the displacement of the top of the beam $u(1,t)$ (i.e. (73) for $x = 1$) for the nth mode. It follows that these displacements have the same direction for the first oscillation modes (i.e., $\mu_n^4 < (k/\gamma)$) and have opposite directions for the higher order oscillation modes.

4 The energy of the beam with a TMD device

The energy of the vertical beam with a TMD device at the top and not subjected to wind-forces, is defined to be

$$E(t) = \int_0^1 \frac{1}{2}(u_t^2(x,t) + u_{xx}^2(x,t) - \epsilon_1(\gamma + 1 - x)u_x^2(x,t))\mathrm{d}x + \frac{\gamma}{2}(u_t(1,t) + \xi_t(t))^2 + \frac{k}{2}\xi^2(t). \tag{77}$$

The time derivative of the energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -c\epsilon_3\xi_t^2(1,t). \tag{78}$$

So, the energy is bounded if the initial energy is bounded. Substituting (15) into (78) gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\epsilon_3 c}{k^2}\Bigg(-\epsilon_1\gamma u_{xt}(1,t) - u_{xxxt}(1,t) - \epsilon_3 c\left[\epsilon_1 u_x(1,t) + \frac{1}{\gamma}u_{xxx}(1,t) - u_{tt}(1,t)\right]\Bigg)^2. \tag{79}$$

So, not only the damping parameter c does have significant influence on the damping, but also the spring constant k and the mass of the tip-mass γ. The existence of a solution of $u(x,t)$ is assumed, where $u(x,t)$ is a twice continuously differentiable function with respect to t and a four times continuously differentiable function with respect to x. What can be shown for the boundedness of $u(x,t)$ and $\xi(t)$? Since $u_x(x,t)$ and $u_{xx}(x,t)$ are continuous it follows that

$$u(x,t) = \int_0^x u_s(s,t)\,\mathrm{d}s, \tag{80}$$

and

$$u_x(x,t) = \int_0^x u_{ss}(s,t)\mathrm{d}s, \tag{81}$$

respectively. It then follows, using the Cauchy–Schwarz inequality

$$|u_x(x,t)| \le \int_0^1 |u_{xx}(x,t)|\mathrm{d}x \le \sqrt{\int_0^1 u_{xx}^2(x,t)\,\mathrm{d}x}. \tag{82}$$

From the first and the second inequality of (82) it follows that

$$u_x^2(x,t) \le \int_0^1 u_{xx}^2(x,t)\,\mathrm{d}x. \tag{83}$$

By using (83) the following inequality is obtained

$$\int_0^1 (u_{xx}^2(x,t) - \epsilon_1(\gamma + 1 - x)u_x^2(x,t))\,\mathrm{d}x \ge \int_0^1 \left(1 - \epsilon_1\left(\gamma + \frac{1}{2}\right)\right)u_{xx}^2(x,t)\,\mathrm{d}x. \tag{84}$$

Now by substituting (84) into (82) it follows that

$$|u_x(x,t)| \le \sqrt{\frac{2E(t)}{1 - \epsilon_1\left(\gamma + \frac{1}{2}\right)}} \le \sqrt{\frac{2E(0)}{1 - \epsilon_1\left(\gamma + \frac{1}{2}\right)}}. \tag{85}$$

It then follows from (85) and (80) that

$$|u(x,t)| \le \int_0^1 |u_x(x,t)|\,\mathrm{d}x \le \int_0^1 \sqrt{\frac{2E(0)}{1 - \epsilon_1\left(\gamma + \frac{1}{2}\right)}}\,\mathrm{d}x = \sqrt{\frac{2E(0)}{1 - \epsilon_1\left(\gamma + \frac{1}{2}\right)}}. \tag{86}$$

So also $u(x, t)$ is bounded if the initial energy is bounded. The displacement of the mass with respect to the top of the beam is also bounded,

$$|\xi(t)| \leq \sqrt{|\xi^2(t)|} \leq \sqrt{\frac{2}{k}E(t)} \leq \sqrt{\frac{2E(0)}{k}}. \tag{87}$$

Note that $\xi(t)$ should not be bigger then the width of the top floor because otherwise the mass will not be at the floor. We find that larger values of k give smaller values of $\xi(t)$ but smaller values of $\xi(t)$ will give less damping(see also (79)).

The time derivative of the energy of the damped beam with tip-mass, subjected to nonlinear wind-forces (see also (17)), is

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -c\epsilon_3\xi_t^2(1, t)\\ &\quad + \epsilon_2\alpha \int_0^1 \left(u_t^2(x, t) - \frac{1}{3}u_t^4(x, t)\right)\mathrm{d}x. \end{aligned} \tag{88}$$

Since $\int_0^1 u_t^4(x, t)\,\mathrm{d}x$ is positive, the nonlinear term in the wind-force is a damping term.

5 The problem (26)–(30) with $\alpha = \epsilon_1 = 0$

In this section, the horizontal vibrations of a beam with a TMD device at the top will be studied. The gravity effect and the wind-force are neglected. So, in this section the problems (26)–(30) with $\alpha = \epsilon_1 = 0$ will be considered:

$$u_{xxxx} + u_{tt} = 0, \tag{89}$$

$$u(0, t) = u_x(0, t) = u_{xx}(1, t) = 0, \tag{90}$$

$$\mathbb{B}_{k\gamma}(u) = \epsilon c(\gamma u_{ttt}(1, t) - u_{xxxt}(1, t)), \tag{91}$$

$$u(x, 0) = f(x), \tag{92}$$

$$u_t(x, 0) = g(x), \tag{93}$$

where $\epsilon = \epsilon_3$ with $0 < \epsilon \ll 1$. The ratio $\gamma = m/(\rho AL)$ is also a small parameter. The ratio can be large with respect to ϵ, can be of the order ϵ, and can be small with respect to ϵ. Therefore, the cases $\gamma = \mathcal{O}(1)$, $\gamma = \mathcal{O}(\epsilon)$, and $\gamma = \mathcal{O}(\epsilon^2)$ will be considered in this section. The method of separation of variables will be used to solve the problem (89)–(93) and to obtain the so-called characteristic equation. At first the location of the roots of the characteristic equation will be considered. Secondly, in Section 5.1, numerical methods will be used to obtain the roots of the characteristic equation. Finally, in Sections 5.2–5.4 perturbation techniques will be used to obtain approximations of the roots of the characteristic equation for the cases $\gamma = \mathcal{O}(1)$, $\gamma = \mathcal{O}(\epsilon^2)$, and $\gamma = \mathcal{O}(\epsilon)$, respectively. The obtained approximations can be used to obtain the damping rates. The approximations can also be used to obtain a good indication what scalings are necessary to construct approximations of the solutions of the initial-boundary value problems (26)–(30) for the cases $\gamma = \mathcal{O}(1)$, $\gamma = \mathcal{O}(\epsilon_3)$, and $\gamma = \mathcal{O}(\epsilon_3^2)$.

Now look for nontrivial solutions of the partial differential equation (89) and the boundary conditions (90) and (91) in the form $X(x)T(t)$. By substituting this into (89)–(91), a boundary value problem for $X(x)$ is obtained:

$$X^{(4)}(x) = \lambda X(x), \tag{94}$$

$$X(0) = X'(0) = X''(1) = 0, \tag{95}$$

$$\begin{aligned}&\gamma\lambda X'''(1) - k(X'''(1) + \gamma\lambda X(1))\\ &\quad = \frac{\epsilon c T'(t)}{T(t)}(X'''(1) + \gamma\lambda X(1)),\end{aligned} \tag{96}$$

and the following problem for $T(t)$:

$$T''(t) + \lambda T(t) = 0, \tag{97}$$

where $\lambda \in \mathbb{C}$ is the separation constant. The case $\lambda = 0$ only leads to trivial solutions. From $(X'''(1) + \gamma\lambda X(1)) = 0$ follows that $\lambda = X'''(1) = 0$ or that $X'''(1) = X(1) = 0$. Both cases only lead to trivial solutions. So the case $(X'''(1) + \gamma\lambda X(1)) = 0$ only leads to trivial solutions.

Now set $\lambda = \mu^4$ where $\mu = \mu_1 + \mu_2 i$ with $\mu_1, \mu_2 \in \mathbb{R}$. Then because of (94) and (95) and because $\lambda = 0$ is not an eigenvalue it follows that:

$$X(x) = A\phi(x), \tag{98}$$

where A is an arbitrary constant and where

$$\begin{aligned}\phi(x) &= (\cos(\mu) + \cosh(\mu))(\sin(\mu x) - \sinh(\mu x))\\ &\quad + (\sin(\mu) + \sinh(\mu))(\cosh(\mu x) - \cos(\mu x)).\end{aligned} \tag{99}$$

By substituting (99) into (96) and because $\mu = 0$ does not correspond to an eigenvalue it follows that

$$(\gamma\mu^4 q(\mu) + k(\gamma\mu s(\mu) - q(\mu)))T(t) \\ = \epsilon c T'(t)(q(\mu) - \gamma\mu s(\mu)), \tag{100}$$

where

$$q(\tau) = 1 + \cos(\tau)\cosh(\tau), \tag{101}$$

$$s(\tau) = \sin(\tau)\cosh(\tau) - \cos(\tau)\sinh(\tau). \tag{102}$$

Since the case $(X'''(1) + \gamma\lambda X(1)) = 0$ only leads to trivial solutions also the case $(kq(\mu) - \gamma\mu s(\mu)) = 0$ only leads to trivial solutions. Then (100) can be written as

$$T'(t) = \theta T(t), \tag{103}$$

where $\theta = \theta_1 + \theta_2 i$, with $\theta_1, \theta_2 \in \mathbb{R}$, is defined by

$$\theta = \frac{(\gamma\mu^4 q(\mu) + k(\gamma\mu s(\mu) - q(\mu)))}{\epsilon c(q(\mu) - \gamma\mu s(\mu))}.$$

The solution of (103) is given by

$$T(t) = c_0 e^{(\theta_1 + i\theta_2)t}, \tag{104}$$

where $c_0 \in \mathbb{C}$. Now the oscillation mode with frequency θ_2 will be damped if $\theta_1 < 0$. The constant θ_1 will be called the damping coefficient or damping rate corresponding to the oscillation mode. The main goal of this section is to determine this damping coefficient.

Because of (97) and (103) the following relation between θ and λ is obtained: $\lambda = -\theta^2$. Now substitution of

$$\theta = \frac{(\gamma\mu^4 q(\mu) + k(\gamma\mu s(\mu) - q(\mu)))}{\epsilon c(q(\mu) - \gamma\mu s(\mu))}$$

and $\lambda = \mu^4$ into $\lambda = \theta^2$ yields:

$$\mu^4 = -\frac{(\gamma\mu^4 q(\mu) + k(\gamma\mu s(\mu) - q(\mu)))^2}{\epsilon^2 c^2 (q(\mu) - \gamma\mu s(\mu))^2}. \tag{105}$$

Equation (105) can be written as:

$$\pm\ i\epsilon c\mu^2(q(\mu) - \gamma\mu s(\mu)) \\ = \gamma\mu^4 q(\mu) + k(\gamma\mu s(\mu) - q(\mu)), \tag{106}$$

where $\theta = \pm i\mu^2$. Now, only consider the case $\theta = +i\mu^2$(the case $\theta = -i\mu^2$ will lead to the same θ). Then the so-called characteristic equation is obtained, given by

$$h_{k\gamma c}(\mu) \equiv (\gamma\mu^4 - k)q(\mu) + \gamma k\mu s(\mu) \\ -i\epsilon c(\mu^2 q(\mu) - \gamma\mu^3 s(\mu)) \tag{107}$$

$$\equiv (\gamma\mu^4 - k - i\epsilon c\mu^2)q(\mu) \\ +\gamma\mu(k + i\epsilon c\mu^2)s(\mu) = 0. \tag{108}$$

If a root μ is found θ can be determined by considering the relation $\theta = i\mu^2$. So, the damping coefficient is given by $\theta_1 = -2\mu_1\mu_2$. Taking apart the real and imaginary parts in the characteristic Equation (107) a system of two nonlinear equations for μ_1 and μ_2 is obtained. Note that (107) can be expressed as a function depending on θ. This is an entire function of order $1/2$. Since an entire function of nonintegral order have infinitely many zeros, also $h_{k\gamma c}(\mu)$ has infinitely many zeros (see [11]).

The roots of $h_{k\gamma c}(\mu)$ are such that if $\mu_1 + \mu_2 i$ is a solution then also $\mu_2 + \mu_1 i$, $-\mu_1 - \mu_2 i$, and $-\mu_2 - \mu_1 i$ are solutions. Since $\mu_1 + \mu_2 i$ and $\mu_2 + \mu_1 i$ are both solutions, θ occurs in complex conjugate pairs. Before approximations of the roots are constructed the location of the roots in the complex plane will be considered. The roots of $h_{k\gamma c}(\mu)$ will be compared to the roots of a more simple function. Rouché's theorem will be applied to show that the roots of $h_{k\gamma c}(\mu)$ are close to the roots of the more simple function. The function $h_{k\gamma c}(\mu)$ will be compared to two simple functions.

The zeros of $h_{k\gamma c}(\mu)$ for $c = 0$ have been considered in Section 3. The roots of this equation are purely imaginary or real. Now it will be shown that there exist a sequence $R_k \in \mathbb{R}$ such that $R_k \to \infty$ as $k \to \infty$ and such that the number of roots of $h_{k\gamma}(\mu) = 0$ and $h_{k\gamma c}(\mu) = 0$ is the same, counting multiplicities, in $B(0, R_k)$, where $B(0, R) = \{\tau \in \mathbb{C} \mid |\tau| \le R\}$. Then the roots of $h_{k\gamma c}(\mu) = 0$ can be enumerated in a similar way for the controlled case $c > 0$ and for the uncontrolled case $c = 0$.

Let $R > 0$ be given. Now, by Rouché's theorem, $h_{k\gamma}(\mu)$ and $h_{k\gamma c}(\mu)$ have the same number of roots, counting multiplicities, in $B(0, R)$ if

$$\left|\frac{\epsilon c(\mu^2 q(\mu) - \gamma\mu^3 s(\mu))}{(\gamma\mu^4 - k)q(\mu) + \gamma k\mu s(\mu)}\right| < 1, \tag{109}$$

for $|\mu| = R$. Now, it will be shown that there exist a sequence $R_k \in \mathbb{R}$ such that $R_k \to \infty$ as $k \to \infty$ and such that (109) is true for $|\mu| = R_k$. To show that such a sequence exist it will be shown that the following inequality is true for sufficiently large values of R:

$$\left|\frac{s(\mu)}{\mu q(\mu)}\right| < \frac{1}{\epsilon c + \frac{k}{|\mu^2|}} - \frac{1}{\gamma |\mu^2|}. \tag{110}$$

For $\mu = Re^{i\varsigma}$, $R = 2n\pi$, and $0 \leq \varsigma \leq 2\pi$ it has been shown that

$$\lim_{n\to\infty}\left|\frac{s(\mu)}{\mu q(\mu)}\right| = 0$$

(see Appendix A of [8]). It can also be shown that

$$\left(\frac{1}{\epsilon c + \frac{k}{|\mu^2|}} - \frac{1}{\gamma |\mu^2|}\right) \to \frac{1}{\epsilon c}$$

if $|\mu| \to \infty$. Hence, there exists a sequence $R_k = 2k\pi$, $k \in \mathbb{N}$ and $k \to \infty$ such that inequality (110) is valid for $|\mu| = R_k$. Then by using the triangle inequality, it follows that

$$\left|\frac{s(\mu)}{\mu q(\mu)} - \frac{1}{\gamma \mu^2}\right| = \left|\frac{q(\mu) - \gamma \mu s(\mu)}{\gamma \mu^2 q(\mu)}\right| < \frac{1}{\epsilon c + \frac{k}{|\mu^2|}}. \tag{111}$$

But then by using (111) it follows that

$$\left|\frac{\gamma \mu^4 q(\mu)}{\mu^2 q(\mu) - \gamma \mu^3 s(\mu)} - \frac{k}{\mu^2}\right| \geq \left|\frac{\gamma \mu^4 q(\mu)}{\mu^2 q(\mu) - \gamma \mu^3 s(\mu)}\right| - \frac{k}{|\mu^2|} > \epsilon c. \tag{112}$$

So, finally it is obtained that (109) is true.

Hence, there exists a sequence $R_k = 2k\pi$, $k \in \mathbb{N}$ and $k \to \infty$ such that (109) is valid for $|\mu| = R_k$. Therefore, by Rouché's theorem, the number of roots of $h_{k\gamma c}(\mu)$ for $c = 0$ and $h_{k\gamma c}(\mu)$ for $c > 0$ is the same in $B(0, R_k)$, counting multiplicities.

In a similar way, the roots of $h_{k\gamma c}(\mu)$ can be compared to the roots of $(\gamma \mu^4 - k - i\epsilon c\mu^2)q(\mu)$ and it can be shown for γ fixed that the number of roots of these functions is the same in $B(0, R_k)$, counting multiplicities.

5.1 Numerical approximations of the roots of the characteristic equation

Now consider the characteristic Equation (107), where ϵ and γ are small parameters. In applications, these small parameters and the parameters c and k will be fixed. Now Maple can be used to construct the roots of Equation (107) numerically. First approximations of the eigenvalues will be given for k, c, ϵ, and γ fixed and n sufficiently large. Consider the characteristic Equation (107), multiplying this equation by $(2e^{\mu})/(\gamma \mu^4)$ yields

$$\cos(\mu) = \frac{ic}{\mu}(\sin(\mu) - \cos(\mu)) + \mathcal{O}\left(\frac{1}{|\mu|^2}\right), \tag{113}$$

or

$$\cos(\mu) = \mathcal{O}\left(\frac{1}{|\mu|}\right), \tag{114}$$

which is valid for values of μ in a small neighborhood of $(n - (1/2))\pi$ where $n \in \mathbb{N}$. In [12], it has been shown that these equations give the following asymptotic solutions for θ_n and μ_n

$$\theta_n = -2\epsilon c + \mathcal{O}\left(\frac{1}{n}\right) + i\left((m\pi)^2 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{115}$$

$$\mu_n = \frac{\epsilon c}{m\pi} + \mathcal{O}\left(\frac{1}{n^2}\right) + i\left(m\pi + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{116}$$

which are valid for sufficiently large $n \in \mathbb{N}$, and where $m = (n - \frac{1}{2})$. Note that the obtained approximations of the damping coefficient are similar to the approximations of the damping coefficients of a weakly damped beam, that is, a beam where the damping at the top is proportional to the velocity of the top (see [8, 9]). The expressions (115) and (116) show that the damping coefficient of the eigenvalues with large index n are dependent on ϵc. Now, it can be concluded that the oscillations are damped uniformly, because (78) holds. The asymptotic approximations of the damping rates are only valid for sufficiently large $n \in \mathbb{N}$. The damping rates for the lower order modes can be obtained numerically by using Maple. The first five roots μ_n and the first five θ_n for several values of ϵc, k, and γ are listed in Table 2. For the cases considered in Table 2, it has been found that the damping rates $\theta_{1,0}$ of the first oscillation mode are small and that the damping rates $\theta_{1,1}$ of the second oscillation modes are large with respect to

Table 2 Numerical approximations of the eigenvalues θ_n and the solutions μ_n of the characteristic equation (107) for the case $k=1$, $\epsilon c=0.1$, $\gamma=0.1$; the case $k=1, \epsilon c=0.1, \gamma=0.08$; the case $k=1, \epsilon c=0.1$, $\gamma=0.05$; the case $k=1$, $\epsilon c=0.1, \gamma=0.01$; and the case $k=1, \epsilon c=0.01$, $\gamma=0.1$

n	$\mu_{1,n}$	$\mu_{2,n}$	$\theta_{1,n}$	$\theta_{2,n}$	$(n-\frac{1}{2})\pi$
$k=1, \epsilon c=0.1,$	$\gamma=0.1$				
0	0.04218	1.5779	−0.13313	2.4880	–
1	0.13161	2.1039	−0.55380	4.4089	1.5708
2	0.02238	4.7026	−0.21050	22.114	4.7124
3	0.01283	7.8564	−0.20159	61.724	7.8540
4	0.00912	10.996	−0.20049	120.92	10.996
$k=1, \epsilon c=0.1,$	$\gamma=0.08$				
0	0.03762	1.6404	−0.12343	2.6894	–
1	0.16048	2.1383	−0.68630	4.5467	1.5708
2	0.02258	4.7024	−0.21239	22.112	4.7124
3	0.01284	7.8564	−0.20181	61.723	7.8540
4	0.00912	10.996	−0.20054	120.91	10.996
$k=1, \epsilon c=0.1,$	$\gamma=0.05$				
0	0.02125	1.7418	−0.07402	3.0334	–
1	0.24445	2.2589	−1.10439	5.0430	1.5708
2	0.02317	4.7016	−0.21783	22.105	4.7124
3	0.01288	7.8563	−0.20243	61.721	7.8540
4	0.00913	10.996	−0.20069	120.91	10.996
$k=1, \epsilon c=0.1,$	$\gamma=0.01$				
0	0.00081	1.8547	−0.00299	3.4399	–
1	0.83382	3.0885	−5.15050	8.8435	1.5708
2	0.02546	4.6913	−0.23892	22.007	4.7124
3	0.01306	7.8546	−0.20522	61.694	7.8540
4	0.00916	10.995	−0.20136	120.92	10.996
$k=1, \epsilon c=0.01,$	$\gamma=0.1$				
0	0.00428	1.5700	−0.01343	2.4650	–
1	0.01302	2.1185	−0.05516	4.4878	1.5708
2	0.00225	4.7040	−0.02112	22.127	4.7124
3	0.00128	7.8568	−0.02017	61.730	7.8540
4	0.00091	10.996	−0.02005	120.92	10.996

the damping rates of the other oscillation modes. Now numerical values for μ_n and θ_n have been obtained. Then $T(t)$ can be approximated by

$$T_n(t)=e^{\theta_{1,n}t}(A_n\cos(\theta_{2,n}t)+B_n\sin(\theta_{1,n}t)). \tag{117}$$

By using the superposition principle the general solution of (26)–(30) with $\alpha=\epsilon_1=0$ is given by

$$\begin{aligned} u(x,t)=\sum_{n=0}^{\infty}e^{\theta_{1,n}t}&(A_n\cos(\theta_{2,n}t)\\ &+B_n\sin(\theta_{1,n}t))\phi_n(x), \end{aligned} \tag{118}$$

where

$$\begin{aligned} \phi_n(x)=&(\cos(\mu_n)+\cosh(\mu_n))(\sin(\mu_n x)-\sinh(\mu_n x))\\ &+(\sin(\mu_n)+\sinh(\mu_n))(\cosh(\mu_n x)\\ &-\cos(\mu_n x)), \end{aligned} \tag{119}$$

and where the constants A_n and B_n can be determined by the initial conditions (92) and (93). Substitution of (118) into (25) yields

$$\begin{aligned} \xi(t)=&\left(\frac{\epsilon^2c^2}{k^2\gamma}-\frac{1}{k}\right)u_{xxx}(1,t)\\ &+\frac{\epsilon c}{k^2}(u_{xxxt}(1,t)-\epsilon c u_{tt}(1,t))\\ =&\sum_{n=0}^{\infty}\left(\frac{\epsilon^2c^2}{k^2\gamma}-\frac{1}{k}\right)\phi_{n_{xxx}}(1)T_{n_t}(t)\\ &+\frac{\epsilon c}{k^2}(\phi_{n_{xxx}}(1,t)T_{n_t}(t)-\epsilon c\phi_n(1)T_{n_{tt}}(t)). \end{aligned} \tag{120}$$

So, $u(x,t)$ and $\xi(t)$ will be damped in a completely similar way.

5.2 Construction of the approximations of the roots of (107) for the case $\gamma = \mathcal{O}(1)$

In this section, only order ϵ approximation of the roots of the characteristic equation will be considered. We are not interested in the higher order approximations. The approximations are such that these are approximations for $\epsilon \downarrow 0$, but also such that these are valid for all oscillation modes (i.e., $\forall n \in \mathbb{N} \cup \{0\}$). The roots of the following equation will be considered

$$(\gamma\mu^4 - k)q(\mu) + \gamma k\mu s(\mu) \\ -i\epsilon c(\mu^2 q(\mu) - \gamma\mu^3 s(\mu)) = 0, \tag{121}$$

where $q(\tau)$ and $s(\tau)$ are given by (101) and (102) respectively. The roots of this equation are close to the roots of the uncontrolled case (that is, the roots of $h_{k\gamma}(\mu)$ as considered in Section 3). Now, it is assumed that a root $\mu_n = \mu_{1,n} + i\mu_{2,n}$ of (121) can be expressed in a power series in ϵ, that is,

$$\mu_{1,n} = \mu_{1,0,n} + \epsilon\mu_{1,1,n} + \cdots, \tag{122}$$

$$\mu_{2,n} = \mu_{2,0,n} + \epsilon\mu_{2,1,n} + \cdots, \tag{123}$$

where $\mu_{i,j,n} \in \mathbb{R}$ for $i = 1, 2$ and $j, n \in \mathbb{N} \cup \{0\}$. To approximate μ_n also $q(\mu)$ and $s(\mu)$ are expanded in power series in ϵ. For the case $(\gamma\mu^4 - k)q(\mu) + \gamma k\mu s(\mu) = 0 + \mathcal{O}(\epsilon)$ it follows that $\mu_n = \mu_{1,0,n} + i\mu_{2,0,n} + \mathcal{O}(\epsilon) = \mu_{0,n} + \mathcal{O}(\epsilon)$, where $\mu_{0,n}$ is the $(n+1)$th positive root of (66). Now by substituting (122) and (123) into (121) and by equating the coefficients of equal powers of ϵ for $n \in \{0, 1, 2, \ldots\}$ it follows (after lengthy but elementary calculations) that

$$\mu_{1,1,n} = 0, \tag{124}$$

and that

$$\mu_{2,1,n} \\ = \frac{c\mu_{0,n}^2(q(\mu_{0,n}) - \gamma\mu_{0,n}s(\mu_{0,n}))}{2k\gamma p(\mu_{0,n}) + 4\gamma\mu_{0,n}^3 q(\mu_{0,n}) + (k\gamma + k - \gamma\mu_{0,n}^4)s(\mu_{0,n})}, \tag{125}$$

where $p(\mu_{0,n}) = \sin(\mu_{0,n})\sinh(\mu_{0,n})$ and where $q(\mu_{0,n})$ and $s(\mu_{0,n})$ are given by (101) and (102) respectively. Now approximations of the damping coefficients $\theta_{1,n}$ up to order ϵ can be found and are given by

$$\theta_{1,n} = \frac{-2\epsilon c\mu_{0,n}^3(q(\mu_{0,n}) - \gamma\mu_{0,n}s(\mu_{0,n}))}{2k\gamma p(\mu_{0,n}) + 4\gamma\mu_{0,n}^3 q(\mu_{0,n}) + (k\gamma + k - \gamma\mu_{0,n}^4)s(\mu_{0,n})}, \tag{126}$$

where $\mu_{0,n}$ is the $(n+1)$th positive root of $h_{k\gamma}(\mu) = 0$, and where $\theta_{1,n}$ is negative for all $n \in \mathbb{N} \cup \{0\}$. So, the damping coefficients can be calculated if the positive roots $\mu_{0,n}$ of $h_{k\gamma}(\mu) = 0$ are known. In Table 3, the first eight values of the damping coefficient are listed for $k = 1$ and $\gamma = 1$, $\gamma = 0.1$, $\gamma = 0.01$, and $\gamma = 0.001$. Now compare the values of Tables 2 and 3. In this section, roots of (107) have been constructed for the case $\gamma = \mathcal{O}(1)$. So only the values of Table 2 for the case $k = 1$, $\epsilon c = 0.01$, and $\gamma = 0.1$ can be compared to the values of Table 3.

Table 3 Numerical approximations of the damping coefficient $\theta_{1,n}$ for $k = 1$ and $\gamma = 1$, $\gamma = 0.1$, $\gamma = 0.01$, and $\gamma = 0.001$

n	$\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$	$\gamma = 0.001$
0	$-0.2684\epsilon c$	$-1.3435\epsilon c$	$-0.0344\epsilon c$	$-0.000310\epsilon c$
1	$-2.1819\epsilon c$	$-5.5157\epsilon c$	$-50.595\epsilon c$	$-1.6826\epsilon c$
2	$-2.0352\epsilon c$	$-2.1125\epsilon c$	$-3.1998\epsilon c$	$-500.10\epsilon c$
3	$-2.0077\epsilon c$	$-2.0173\epsilon c$	$-2.1164\epsilon c$	$-3.6902\epsilon c$
4	$-2.0029\epsilon c$	$-2.0053\epsilon c$	$-2.0303\epsilon c$	$-2.3076\epsilon c$
5	$-2.0014\epsilon c$	$-2.0023\epsilon c$	$-2.0113\epsilon c$	$-2.1054\epsilon c$
6	$-2.0008\epsilon c$	$-2.0012\epsilon c$	$-2.0052\epsilon c$	$-2.0463\epsilon c$
7	$-2.0005\epsilon c$	$-2.0007\epsilon c$	$-2.0027\epsilon c$	$-2.0236\epsilon c$

Since $\mu_n \to (n - \frac{1}{2})\pi$ for $n \to \infty$ it follows that

$$\theta_{1,n} \to -2\epsilon c, \tag{127}$$

for n sufficiently large. So, the oscillation modes will be damped uniformly. Using a multiple-timescales perturbation method an approximation of the solution of (26)–(30) can be constructed. It now follows that the following timescales are necessary: x, t and $\tau = \epsilon t$. In Section 6, such an approximation of the solution will be constructed.

5.3 Construction of the approximations of the roots of (107) for the case $\gamma = \mathcal{O}(\epsilon^2)$

In this section, the first two terms of the approximation of the roots of the characteristic Equation (107) for $\gamma = \mathcal{O}(\epsilon^2)$ will be considered. We are not interested in the higher order approximations. The approximations are such that these are approximations for $\epsilon \downarrow 0$ but also such that these are valid for all oscillation modes (i.e., $\forall n \in \mathbb{N} \cup \{0\}$). The characteristic Equation (107) for $\gamma = \mathcal{O}(\epsilon^2)$ is given by

$$(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2)q(\mu) = -\epsilon^2\gamma_2\mu(k + i\epsilon c\mu^2)s(\mu), \tag{128}$$

where $\gamma = \epsilon^2\gamma_2$ and where γ_2 is ϵ-independent. The roots can be expressed in series in ϵ. Now it will be studied how these expansions can be chosen. By substituting $\mu = \tilde{\mu}\epsilon^\beta = (\tilde{\mu}_{re} + i\tilde{\mu}_{im})\epsilon^\beta$, where $\beta, \tilde{\mu}_{re}, \tilde{\mu}_{im} \in \mathbb{R}$ and where $\tilde{\mu}_{re}, \tilde{\mu}_{im} = \mathcal{O}(1)$, into (128) yields

$$(\gamma_2\tilde{\mu}^4\epsilon^{2+4\beta} - k - ic\tilde{\mu}^2\epsilon^{1+2\beta})q(\tilde{\mu}\epsilon^\beta) = -(\gamma_2 k\tilde{\mu}\epsilon^{2+\beta} + i\gamma_2 c\tilde{\mu}^3\epsilon^{3+3\beta})s(\tilde{\mu}\epsilon^\beta). \tag{129}$$

A significant degeneration (see also [13]) of (129) arises if $\beta = -1/2$, which yields

$$(\gamma_2\tilde{\mu}^4 - k - ic\tilde{\mu}^2)q\left(\frac{\tilde{\mu}}{\sqrt{\epsilon}}\right) = -\epsilon^{\frac{3}{2}}(\gamma_2 k\tilde{\mu} + i\gamma_2 c\tilde{\mu}^3)s\left(\frac{\tilde{\mu}}{\sqrt{\epsilon}}\right). \tag{130}$$

Since $\frac{s(\frac{\tilde{\mu}}{\sqrt{\epsilon}})}{q(\frac{\tilde{\mu}}{\sqrt{\epsilon}})} \to -\frac{\tilde{\mu}_{re}}{|\tilde{\mu}_{re}|} + i\frac{\tilde{\mu}_{im}}{|\tilde{\mu}_{im}|}$ for $\epsilon \downarrow 0$, $\tilde{\mu}_{re} \neq 0$, and for $\tilde{\mu}_{im} \neq 0$ the case $(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2) = 0 + \mathcal{O}(\epsilon^{\frac{3}{2}})$ will be considered. For this case the first-order approximation of μ is proportional to $\frac{1}{\sqrt{\epsilon}}$. This case will be studied further in Section 5.3.2.

Now consider the case $(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2) \neq 0 + \mathcal{O}(\epsilon^{\frac{3}{2}})$. For this case it can be shown that

$$\left(\frac{\gamma_2 k\tilde{\mu}\epsilon^{2+\beta} + i\gamma_2 c\tilde{\mu}^3\epsilon^{3+3\beta}}{\gamma_2\tilde{\mu}^4\epsilon^{2+4\beta} - k - ic\tilde{\mu}^2\epsilon^{1+2\beta}}\right) = \mathcal{O}(\epsilon)$$

for all values of $\tilde{\mu}$ and for $\epsilon \downarrow 0$. Then (128) is given by $q(\mu) = 0 + \mathcal{O}(\epsilon)$. Therefore, also the case $q(\mu) = 0 + \mathcal{O}(\epsilon)$ will be considered. This case will be studied in Section 5.3.1.

5.3.1 *The case $q(\mu) = 0 + \mathcal{O}(\epsilon)$*

Now (128) can be written as

$$q(\mu) = -\epsilon\left(\frac{\gamma_2\epsilon\mu(k + i\epsilon c\mu^2)}{\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2}\right)s(\mu). \tag{131}$$

The order in ϵ of

$$\left(\frac{\gamma_2\epsilon\mu(k + i\epsilon c\mu^2)}{\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2}\right)$$

depends not only on ϵ but also on the order in ϵ of μ. For each order of μ the order of

$$\left(\frac{\gamma_2\epsilon\mu(k + i\epsilon c\mu^2)}{\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2}\right)$$

will be different. But it can be shown that

$$\left(\frac{\gamma_2\epsilon\mu(k + i\epsilon c\mu^2)}{\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2}\right) = \mathcal{O}(1)$$

for all values of μ except for the case that $(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2) = 0 + \mathcal{O}(\epsilon^{\frac{3}{2}})$. Now, the following ϵ-dependent constants are introduced: $G_1(\epsilon) = \epsilon^2\gamma_2$, $G_2(\epsilon) = \epsilon\gamma_2$, and $C(\epsilon) = \epsilon c$. By using these constants an expansion for the roots of (131) can be obtained which is valid for all these roots. By using these constants (131) becomes

$$q(\mu) = -\epsilon\left(\frac{G_2(\epsilon)\mu(k + iC(\epsilon)\mu^2)}{G_1(\epsilon)\mu^4 - k - iC(\epsilon)\mu^2}\right)s(\mu). \tag{132}$$

Now, it is assumed that a root $\mu_n = \mu_{1,n} + i\mu_{2,n}$ of (132) can be expressed in a series in ϵ, that is,

$$\mu_{1,n} = \mu_{1,0,n} + \epsilon\mu_{1,1,n}(\epsilon) + \cdots, \tag{133}$$

$$\mu_{2,n} = \mu_{2,0,n} + \epsilon\mu_{2,1,n}(\epsilon) + \cdots, \tag{134}$$

where $\mu_{i,0,n} \in \mathbb{R}$, $\mu_{i,j,n}(\epsilon) \in \mathbb{R}$, and $\mu_{i,j,n}(\epsilon) = \mathcal{O}(1)$ for $i = 1, 2$ and $j, n \in \mathbb{N}$. To approximate μ_n $q(\mu)$ and $s(\mu)$ will also be expanded in power series in ϵ. For the case $q(\mu) = 0 + \mathcal{O}(\epsilon)$ it follows that $\mu_n = \mu_{1,0,n} + i\mu_{2,0,n} + \mathcal{O}(\epsilon) = \mu_{0,n} + \mathcal{O}(\epsilon)$, where $\mu_{0,n}$ is the nth positive root of $q(\mu) = 1 + \cos(\mu)\cosh(\mu) = 0$ and where $\mu_{0,n} \to (n - \frac{1}{2})\pi$ if $n \to \infty$ (see also [8, 9]). Now by substituting (133) and (134) into (131) and by equating the coefficients of equal powers of ϵ for $n \in \{1, 2, \ldots\}$ it follows(after lengthy but elementary calculations) that

$$\mu_{1,1,n}(\epsilon) = \frac{G_1(\epsilon)G_2(\epsilon)C(\epsilon)\mu_{0,n}^7}{\left(G_1(\epsilon)\mu_{0,n}^4 - k\right)^2 + C^2(\epsilon)\mu_{0,n}^4}, \tag{135}$$

$$\mu_{2,1,n}(\epsilon) = \frac{G_2(\epsilon)\mu_{0,n}\left(k(G_2(\epsilon)\mu_{0,n}^4 - k) - C^2(\epsilon)\mu_{0,n}^4\right)}{\left(G_1(\epsilon)\mu_{0,n}^4 - k\right)^2 + C^2(\epsilon)\mu_{0,n}^4}. \tag{136}$$

Now approximations of μ_n for the roots of (132) have been found. Now, also an approximation for the damping coefficients $\theta_{1,n} = -2\mu_{1,n}\mu_{2,n}$ have been found:

$$\theta_{1,n} = \frac{-2\epsilon G_1(\epsilon)G_2(\epsilon)C(\epsilon)\mu_{0,n}^8}{\left(G_1(\epsilon)\mu_{0,n}^4 - k\right)^2 + C^2(\epsilon)\mu_{0,n}^4}. \tag{137}$$

Now substitute $G_1(\epsilon) = \epsilon^2\gamma_2, G_2(\epsilon) = \epsilon\gamma_2$, and $C(\epsilon) = \epsilon c$ into (137) to obtain the damping coefficient for the nth oscillation mode

$$\theta_{1,n} = \frac{-2\epsilon^5\gamma_2^2 c\mu_{0,n}^8}{\left(\epsilon^2\gamma_2\mu_{0,n}^4 - k\right)^2 + \epsilon^2 c^2\mu_{0,n}^4}. \tag{138}$$

So, it follows for the higher order modes (i.e., for n sufficiently large) that

$$\theta_{1,n} \approx -2\epsilon c. \tag{139}$$

So, the higher order modes are damped weakly, but the damping for the first oscillation modes is very small, that is, $\theta_{1,n} = \mathcal{O}(\epsilon^5)$. Since in applications the first oscillation modes are important the parameter γ should not be small with respect to the damping parameter ϵc to obtain damping of order ϵ.

5.3.2 The case $(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2) = 0 + \mathcal{O}(\epsilon^{\frac{3}{2}})$

Now (128) can be written in the following way

$$\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2 = -\epsilon^2(\gamma_2 k\mu + i\epsilon c\gamma_2\mu^3)\left(\frac{s(\mu)}{q(\mu)}\right). \tag{140}$$

The roots of (140) will be denoted by μ_0. Now approximations of μ_0 will be considered. It was observed that in this case the first-order approximation of μ_0 is proportional to $1/\sqrt{\epsilon}$. It should also be observed that the small parameter in (130) is $\epsilon\sqrt{\epsilon}$. For these reasons the root μ_0 will be expanded in

$$\mu_0 = \frac{1}{\sqrt{\epsilon}}\left(\mu_{0,0} + \epsilon\sqrt{\epsilon}\mu_{1,0} + \cdots\right). \tag{141}$$

Note that both the real part $\mu_{0_{re}}$ and the imaginary part $\mu_{0_{im}}$ of μ_0 are both $\mathcal{O}(1/\sqrt{\epsilon})$. Then it can be shown that

$$\frac{s(\mu_0)}{q(\mu_0)} \to -\frac{\mu_{0_{re}}}{|\mu_{0_{re}}|} + i\frac{\mu_{0_{im}}}{|\mu_{0_{im}}|}$$

if $\epsilon \downarrow 0$. Now by substituting (141) into (140) and by equating equal powers of ϵ it is obtained that $\mu_{0,0}$ is the root of the following equation

$$\gamma_2\mu_{0,0}^4 - k - ic\mu_{0,0}^2 = 0. \tag{142}$$

The roots are such that if $\mu_{0,0_{re}} + \mu_{0,0_{im}}i$ is a solution then also $\mu_{0,0_{im}} + \mu_{0,0_{re}}i$, $-\mu_{0,0_{re}} - \mu_{0,0_{im}}i$, and $-\mu_{0,0_{re}} - \mu_{0,0_{im}}i$ are solutions. Now it is obtained that

$$\mu_{0,0} = \frac{\pm 1}{\sqrt{2\gamma_2}}\sqrt{ic \pm \sqrt{4k\gamma_2 - c^2}}. \tag{143}$$

If a root μ_0 of (140) is found the oscillation mode $\theta_0 = \theta_{0_{re}} + i\theta_{0_{im}}$, where $\theta_{0_{re}}, \theta_{0_{im}} \in \mathbb{R}$, can be determined by considering the relation: $\theta_0 = i\mu_0^2$. Note that $\theta_{0_{re}}$ is the damping coefficient of the mode θ_0. Hence, an approximation for θ_0 has been found, given by

$$\theta_0 = \frac{1}{2\epsilon\gamma_2}\left(-c \pm \sqrt{c^2 - 4k\gamma_2}\right). \tag{144}$$

Now also an approximation of the solution of (103) can be obtained. Depending on the sign of $4k\gamma_2 - c^2$ three cases have to be considered. The mode will be damped critically for $c^2 = 4k\gamma_2$, and the mode will be overdamped for $c^2 > 4k\gamma_2$. If c^2 is large with respect to $4k\gamma$ the damping coefficients $\theta_{0_{re}}$ will be close to 0 and $\frac{-c}{\gamma_2}$. So, the damping parameter c of the TMD should not be chosen too large, that is, $c^2 < 4k\gamma$. Therefore, in this paper these cases will not be considered and it is assumed that $c^2 < 4k\gamma$.

Now, by assuming that $c^2 < 4k\gamma_2$, it is found, after lengthy but elementary calculations, that

$$\mu_{1,0} = \left(-\frac{\mu_{0,0_{re}}}{|\mu_{0,0_{re}}|} + i\frac{\mu_{0,0_{im}}}{|\mu_{0,0_{im}}|} \right) \times \left(\frac{2\gamma_2 k - c^2 + ic\sqrt{4\gamma_2 k - c^2}}{4\sqrt{4\gamma_2 k - c^2}} \right). \quad (145)$$

Since the damping coefficients (138) and the real part of (144) are negative and do not tend to zero for n large the oscillation modes will be damped uniformly.

In Section 5, it has been shown that there exist a $R_k \in \mathbb{R}$ such that the number of roots of (128) and $(\epsilon^2\gamma_2\mu^4 - k - i\epsilon c\mu^2)q(\mu) = 0$ is the same, counting multiplicities, in $B(0, R_k)$. Therefore, approximations of all the roots of the so-called characteristic equation for the case $\gamma = \mathcal{O}(\epsilon^2)$ have been constructed. It also has been shown that the oscillation modes will be damped uniformly. Using a multiple-timescales perturbation method an approximation of the solution of (26)–(30) for the case $\gamma = \mathcal{O}(\epsilon^2)$ can be constructed. From (104) and (144) it follows that the timescale $\bar{t} = t/\epsilon$ is necessary. Substitution of (141) into (119) leads to the timescale $\bar{x} = x/\sqrt{\epsilon}$. It now follows that the following timescales are necessary: x, t, $\bar{t} = t/\epsilon$, $\bar{x} = x/\sqrt{\epsilon}$, and $\tau = \epsilon t$. This case will not be studied in this paper.

5.4 Construction of the approximation of the first roots of (107) for the case $\gamma = \mathcal{O}(\epsilon)$

In the previous section, it has been shown that the damping coefficient of the first oscillation mode is relatively small with respect to the other damping coefficients. Therefore only the first roots of (107) for the case $\gamma = \mathcal{O}(\epsilon)$ will be considered in this section. The obtained approximation is only valid for roots μ such that $\epsilon|\mu|^4 \ll 1$. The roots for the case $\epsilon|\mu|^4 \approx 1$ and the case $\epsilon|\mu|^4 \gg 1$ can be obtained by using numerical methods. The characteristic Equation (107) for $\gamma = \mathcal{O}(\epsilon)$ is given by

$$q(\mu) = \frac{\epsilon}{k}(\gamma_1\mu^4 q(\mu) + \gamma_1 k\mu s(\mu) - ic(\mu^2 q(\mu) - \epsilon\gamma_1\mu^3 s(\mu))), \quad (146)$$

where $\gamma = \epsilon\gamma_1$ and where γ_1 is ϵ-independent. Now, it is assumed that a root $\mu_n = \mu_{1,n} + i\mu_{2,n}$ of (146) can be expressed in a power series in ϵ, that is,

$$\mu_{1,n} = \mu_{1,0,n} + \epsilon\mu_{1,1,n} + \cdots, \quad (147)$$

$$\mu_{2,n} = \mu_{2,0,n} + \epsilon\mu_{2,1,n} + \cdots, \quad (148)$$

where $\mu_{i,j,n} \in \mathbb{R}$ for $i = 1, 2$ and $j, n \in \mathbb{N} \cup \{0\}$. To approximate μ_n also $q(\mu)$ and $s(\mu)$ are expressed in power series in ϵ. For the case $q(\mu) = 0 + \mathcal{O}(\epsilon)$ it follows that $\mu_n = \mu_{1,0,n} + i\mu_{2,0,n} + \mathcal{O}(\epsilon) = \mu_{0,n} + \mathcal{O}(\epsilon)$, where $\mu_{0,n}$ is the $(n+1)$th positive root of $q(\mu) = 1 + \cos(\mu)\cosh(\mu) = 0$, and where $\mu_{0,n} \to (n + (1/2))\pi$ if $n \to \infty$. Now by substituting (147) and (148) into (146) and by equating the coefficients of equal powers of ϵ for $n \in \{0, 1, 2, \ldots\}$ it follows that

$$\mu_{1,1,n} = -\gamma_1\mu_{0,n}, \quad (149)$$

$$\mu_{1,2,n} = -\gamma_1^2\mu_{0,n}\left(\mu_{0,n}^4 - k - \mu_{0,n}k \times \left(\frac{\sin(\mu_{0,n})\sinh(\mu_{0,n})\cosh(\mu_{0,n})}{\sinh(\mu_{0,n}) + \sin(\mu_{0,n})\cosh^2(\mu_{0,n})}\right)\right), \quad (150)$$

and that

$$\mu_{2,1,n} = 0, \quad \mu_{2,2,n} = 0, \quad \mu_{2,3,n} = \frac{c\gamma_1^2\mu_{0,n}^7}{k^2}. \quad (151)$$

Now it is found that an approximation of the damping coefficient($\theta_{1,n} = -2i\mu_{1,n}\mu_{2,n}$) up to order ϵ^3 is given by:

$$\theta_{1,n} = \frac{-2\epsilon^3 c\gamma_1^2\mu_{0,n}^8}{k^2}. \quad (152)$$

So, the first damping coefficients are small with respect to the damping parameter ϵc and the ratio $\epsilon\gamma_1$. Also it has been found that (152) has the smallest value for $n = 0$ with respect to the other oscillation modes such that $\epsilon|\mu_n|^4 \ll 1$.

6 Formal approximations

In Section 5.2 problems (26)–(30) with $\alpha = \epsilon_1 = 0$ has been considered. It has also been mentioned that a slow timescale like $\tau = \epsilon t$ is needed to solve the problems (26)–(30) with $\alpha = \epsilon_1 = 0$ approximately, by using a two-timescales perturbation method. In this section, an approximation of the solution of the initial-boundary value problem (26)–(30) with $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3$ will be constructed. This is the case of a vertical beam with a TMD at the top in a wind-field. In this section, conditions like $t > 0$, $t \geq 0$, and $0 < x < 1$ will be dropped, for abbreviation.

It is assumed that the solution can be expanded in a Taylor series with respect to ϵ in the following way

$$u(x, t; \epsilon) = \hat{u}_0(x, t) + \epsilon \hat{u}_1(x, t) + \epsilon^2 \hat{u}_2(x, t) + \cdots . \tag{153}$$

It is assumed that the functions $\hat{u}_i(x, t)$ are $\mathcal{O}(1)$. The approximation of the solution will contain secular terms. Since the $\hat{u}_i(x, t)$ are assumed to be $\mathcal{O}(1)$, and since the solutions are bounded, secular terms should be avoided when approximations are constructed on a time-scale of $\mathcal{O}(\epsilon^{-1})$. That is why a two-timescales perturbation method will be applied. Using such a two-timescales perturbation method the function $u(x, t)$ is supposed to be a function of x, t and $\tau = \epsilon t$. So put

$$u(x, t) = w(x, t, \tau; \epsilon). \tag{154}$$

A result of this is

$$\begin{aligned} u_t &= w_t + \epsilon w_\tau, \\ u_{tt} &= w_{tt} + 2\epsilon w_{t\tau} + \epsilon^2 w_{\tau\tau}, \\ u_{ttt} &= w_{ttt} + 3\epsilon w_{tt\tau} + 3\epsilon^2 w_{t\tau\tau} + \epsilon^3 w_{\tau\tau\tau}. \end{aligned} \tag{155}$$

Substitution of (154) and (155) into the problems (26)–(30) yields

$$\begin{aligned} w_{xxxx} + w_{tt} &= -\epsilon[(\gamma + 1 - x)w_x]_x - 2\epsilon w_{t\tau} \\ &\quad -\epsilon^2 w_{\tau\tau} + \epsilon\alpha w_t + \epsilon^2 \alpha w_\tau, \end{aligned} \tag{156}$$

$$w(0, t, \tau) = w_x(0, t, \tau) = w_{xx}(1, t, \tau) = 0, \tag{157}$$

$$\begin{aligned} &k w_{xxx}(1, t, \tau) \\ &\quad = -\gamma(w_{xxxtt}(1, t, \tau) + 2\epsilon w_{xxxt\tau}(1, t, \tau) \\ &\qquad + \epsilon^2 w_{xxx\tau\tau}(1, t, \tau)) + k\gamma(w_{tt}(1, t, \tau) \\ &\qquad + 2\epsilon w_{t\tau}(1, t, \tau) + \epsilon^2 w_{\tau\tau}(1, t, \tau)) \\ &\qquad - \epsilon\gamma^2(w_{xtt}(1, t, \tau) + 2\epsilon w_{xt\tau}(1, t, \tau) \\ &\qquad + \epsilon^2 w_{x\tau\tau}(1, t, \tau)) - \epsilon\gamma k w_x(1, t, \tau) \\ &\qquad - \epsilon^2 c\gamma(w_{xt}(1, t, \tau) + \epsilon w_{x\tau}(1, t, \tau)) \\ &\qquad + \epsilon c\gamma(3\epsilon w_{tt\tau}(1, t, \tau) + 3\epsilon^2 w_{t\tau\tau}(1, t, \tau) \\ &\qquad + \epsilon^3 w_{\tau\tau\tau}(1, t, \tau)) - \epsilon c(w_{xxxt}(1, t, \tau) \\ &\qquad + w_{xxx\tau}(1, t, \tau)) + \epsilon c\gamma w_{ttt}(1, t, \tau), \end{aligned} \tag{158}$$

$$w(x, 0, 0) = f(x), \tag{159}$$

$$w_t(x, 0, 0) = g(x) - \epsilon w_\tau(x, 0, 0). \tag{160}$$

Assuming that

$$\begin{aligned} w(x, t, \tau) &= u_0(x, t, \tau) + \epsilon u_1(x, t, \tau) \\ &\quad + \epsilon^2 u_2(x, t, \tau) + \cdots , \end{aligned} \tag{161}$$

then by collecting terms of equal powers in ϵ, it follows from (156)–(160) that the $\mathcal{O}(1)$ problem is:

$$u_{0_{xxxx}} + u_{0_{tt}} = 0, \tag{162}$$

$$u_0(0, t, \tau) = u_{0_x}(0, t, \tau) = u_{0_{xx}}(1, t, \tau) = 0, \tag{163}$$

$$\mathbb{B}_{k\gamma}(u_0) = 0, \tag{164}$$

$$u_0(x, 0, 0) = f(x), \tag{165}$$

$$u_{0_t}(x, 0, 0) = g(x), \tag{166}$$

and that the $\mathcal{O}(\epsilon)$ problem is:

$$u_{1_{xxxx}} + u_{1_{tt}} = -[(\gamma + 1 - x)u_{0_x}]_x - 2u_{0_{t\tau}} + \alpha u_{0_t}, \tag{167}$$

$$u_1(0, t, \tau) = u_{1_x}(0, t, \tau) = u_{1_{xx}}(1, t, \tau) = 0, \tag{168}$$

$$\begin{aligned} \mathbb{B}_{k\gamma}(u_1) &= c\left(\gamma u_{0_{ttt}}(1, t, \tau) - u_{0xxxt}(1, t, \tau)\right) \\ &\quad -2\gamma u_{0_{xxxt\tau}}(1, t, \tau) + 2k\gamma u_{0_{t\tau}}(1, t, \tau) \\ &\quad -\gamma^2 u_{0_{xtt}}(1, t, \tau) - k\gamma u_{0_x}(1, t, \tau), \end{aligned} \tag{169}$$

$$u_1(x, 0, 0) = 0, \tag{170}$$

$$u_{1_t}(x,0,0) = -u_{0_\tau}(x,0,0), \tag{171}$$

where (see also (23))

$$\mathbb{B}_{k\gamma}(\psi) \equiv k\psi_{xxx}(1,t,\tau) + \gamma\psi_{xxxtt}(1,t,\tau) - k\gamma\psi_{tt}(1,t,\tau). \tag{172}$$

The solution of the $\mathcal{O}(1)$-problem (162)–(166) has been determined in Section 3 and is given by

$$u_0(x,t,\tau) = \sum_{n=0}^{\infty} T_{0n}(t,\tau)\phi_n(x), \tag{173}$$

where $\phi_n(x)$ is an eigenfunction, corresponding to λ_n and

$$\left(\int_0^1 \phi_i(x)\phi_j(x)\mathrm{d}x + \frac{\phi_{i_{xxx}}(1)\phi_{j_{xxx}}(1)}{\gamma\lambda_i\lambda_j}\right) = \delta_{ij}, \tag{174}$$

where δ_{ij} is the Kronecker symbol and where

$$T_{0n}(t,\tau) = A_{0n}(\tau)\cos(\mu_n^2 t) + B_{0n}(\tau)\sin(\mu_n^2 t), \tag{175}$$

where $A_{0n}(0)$ and $B_{0n}(0)$ are defined by (74) and (75), respectively.

Now the solution of the $\mathcal{O}(\epsilon)$-problem will be determined. The problems (167)–(171) have an inhomogeneous boundary condition.

For classical inhomogeneous boundary conditions the inhomogeneous boundary conditions are made homogeneous. However, for inhomogeneous non-classical boundary conditions such as (169) a different procedure has to be followed. In fact, a transformation will be used such that the partial differential equation and the inhomogeneous boundary condition, after the transformation, match; if a solution which is expanded in eigenfunctions $\phi_n(x)$, satisfies the transformed partial differential equation it immediately satisfies the transformed inhomogeneous boundary condition. A similar matching for a string-like problem and a beam-like problem has been used in [1] and in [9] respectively.

To solve this problem, the following transformation will be used

$$u_1(x,t,\tau) = v(x,t,\tau) + \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h(t,\tau). \tag{176}$$

Substitution of (176) into (167)–(171) yields the following problem for $v(x,t,\tau)$

$$v_{xxxx} + v_{tt} = -[(\gamma+1-x)u_{0_x}]_x - 2u_{0_{t\tau}} - \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h_{tt}(t,\tau) + \alpha u_{0_t}, \tag{177}$$

$$v(0,t,\tau) = v_x(0,t,\tau) = v_{xx}(1,t,\tau) = 0, \tag{178}$$

$$\begin{aligned}\mathbb{B}_{k\gamma}(v) = {} & c(\gamma u_{0_{ttt}}(1,t,\tau) - u_{0_{xxxt}}(1,t,\tau)) \\ & -kh(t,\tau) - \gamma h_{tt}(t,\tau) - \frac{k\gamma}{3}h_{tt}(t,\tau) \\ & -2\gamma u_{0_{xxxt\tau}}(1,t,\tau) + 2k\gamma u_{0_{t\tau}}(1,t,\tau) \\ & -k\gamma u_{0_x}(1,t,\tau) - \gamma^2 u_{0_{xtt}}(1,t,\tau),\end{aligned} \tag{179}$$

$$v(x,0,0) = \left(\frac{x^2}{2} - \frac{x^3}{6}\right)h(0,0), \tag{180}$$

$$v_t(x,0,0) = \left(\frac{x^2}{2} - \frac{x^3}{6}\right)h_t(0,0) - u_{0_\tau}(x,0,0). \tag{181}$$

It is assumed that $v(x,t,\tau)$ can be expressed in series of eigenfunctions,

$$v(x,t,\tau) = \sum_{m=0}^{\infty} v_n(t,\tau)\phi_n(x). \tag{182}$$

Substitute (182) into the partial differential Equation (177) and the boundary condition (179) to obtain

$$\sum_{n=0}^{\infty}(v_{n_{tt}} + \lambda_n v_n)\phi_n(x) = -[(\gamma+1-x)u_{0_x}]_x - 2u_{0_{t\tau}} + \alpha u_{0_t} - \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h_{tt}(t,\tau), \tag{183}$$

$$\begin{aligned}\sum_{n=0}^{\infty}(v_{n_{tt}} + \lambda_n v_n)\left(\frac{k\phi_{n_{xxx}}(1)}{\lambda_n}\right) = {} & c(\gamma u_{0_{ttt}}(1,t,\tau) - u_{0_{xxxt}}(1,t,\tau)) \\ & -kh(t,\tau) - \gamma h_{tt}(t,\tau) - \frac{k\gamma}{3}h_{tt}(t,\tau) \\ & -2\gamma u_{0_{xxxt\tau}}(1,t,\tau) + 2k\gamma u_{0_{t\tau}}(1,t,\tau) \\ & -k\gamma u_{0_x}(1,t,\tau) - \gamma^2 u_{0_{xtt}}(1,t,\tau),\end{aligned} \tag{184}$$

respectively. Now the function $h(t,\tau)$ will be derived. By differentiating (183) with respect to x thrice, by multiplying by γ, and by taking the limit to $x=1$ in the so-obtained equation, yields

$$\sum_{n=0}^{\infty}(v_{n_{tt}}+\lambda_n v_n)\gamma\phi_{n_{xxx}}(1)$$
$$=-2\gamma u_{0_{xxxt\tau}}(1,t,\tau)+\alpha\gamma u_{0_{xxxt}}(1,t,\tau)$$
$$-\gamma h_{tt}(t,\tau)-\gamma^2 u_{0_{xtt}}(1,t,\tau)+4\gamma\lambda_n u_{0_{tt}}(1,t,\tau). \quad (185)$$

By taking the limit $x=1$ in (183) and by multiplying to so-obtained result by $k\gamma$, yields

$$\sum_{n=0}^{\infty}(v_{n_{tt}}+\lambda_n v_n)k\gamma\phi_n(1)$$
$$=-2k\gamma u_{0_{t\tau}}(1,t,\tau)+\alpha k\gamma u_{0_t}(1,t,\tau)$$
$$+\frac{k\gamma}{3}h_{tt}(t,\tau)+k\gamma u_{0_x}(1,t,\tau). \quad (186)$$

Now by subtracting (184) and (186) from (185) and by using the second boundary condition in $x=1$ (i.e., $(k-\gamma\lambda)X'''(1)+k\gamma\lambda X(1)=0$) it follows that

$$kh(t,\tau)$$
$$=c(\gamma u_{0_{ttt}}(1,t,\tau)-u_{0_{xxxt}}(1,t,\tau))$$
$$-\alpha\gamma(u_{0_{xxxt}}(1,t,\tau)-ku_{0_t}(1,t,\tau))-4u_{0_{tt}}(1,t,\tau). \quad (187)$$

The initial-boundary value problems (167)–(171) can be solved after expanding

$$\left(\frac{-x^2}{2}+\frac{x^3}{6}\right)$$

in a series of the orthonormal eigenfunctions $\phi_n(x)$:

$$\frac{-x^2}{2}+\frac{x^3}{6}=\sum_{n=0}^{\infty}C_n\phi_n(x), \quad (188)$$

where

$$C_n=\int_0^1\left(\frac{-x^2}{2}+\frac{x^3}{6}\right)\phi_n(x)\mathrm{d}x$$
$$=-\left(\frac{\phi_{n_{xxx}}(1)+3\phi_n(1)}{3\lambda_n}\right). \quad (189)$$

Now the solution $v(x,t,\tau)$ will be derived. Multiply Equation (183) by $\phi_m(x)$ and integrate with respect to x from 0 to 1 to obtain

$$\sum_{n=0}^{\infty}\left(v_{n_{tt}}+\lambda_n v_n\right)\int_0^1\phi_n\phi_m\mathrm{d}x$$
$$=-\int_0^1\left([(\gamma+1-x)u_{0_x}]_x+2u_{0_{t\tau}}-\alpha u_{0_t}\right)\phi_m\mathrm{d}x$$
$$+\left(\frac{\phi_{n_{xxx}}(1)+3\phi_n(1)}{3\lambda_n}\right)h_{tt}(t,\tau). \quad (190)$$

Now by multiplying Equation (184) by

$$\left(\frac{\phi_{m_{xxx}}(1)}{\gamma k\lambda_m}\right),$$

adding Equation (190), and by using (174) the differential equation for $v_n(t,\tau)$ it follows that

$$v_{n_{tt}}+\lambda_n v_n=-2T_{0n_{t\tau}}-\left(\frac{\phi_{n_{xxx}}(1)-k\phi_n(1)}{k\lambda_n}\right)h_{tt}(t,\tau)$$
$$+\alpha T_{0n_t}(t,\tau)+\sum_{j=0}^{\infty}T_{0j}(t,\tau)(\Theta_{jn}$$
$$-\gamma\phi_{n_x}(1)\phi_j(1))$$
$$-\left(\frac{\phi_{n_{xxx}}(1)}{k\gamma\lambda_n}\right)(\gamma^2 u_{0_{xtt}}(1,t,\tau)$$
$$+k\gamma u_{0_x}(1,t,\tau)-4u_{0_{tt}}(1,t,\tau)), \quad (191)$$

where

$$\Theta_{mn}=\int_0^1(\gamma+1-x)\phi_{m_x}(x)\phi_{n_x}(x)\,\mathrm{d}x. \quad (192)$$

To avoid secular terms it then follows that

$$-2T_{0n_{t\tau}}-(\alpha\gamma(k\phi_n(1)-\phi_{n_{xxx}}(1))-c(\gamma\lambda\phi_n(1)$$
$$+\phi_{n_{xxx}}(1)))\left(\frac{\phi_{n_{xxx}}(1)-k\phi_n(1)}{k\lambda_n}\right)\frac{T_{0n_{ttt}}}{k}$$
$$+\alpha T_{0n_t}+T_{0n}\Theta_{nn}=0, \quad (193)$$

where Θ_{nn} is given by (192). Since $T_{0n}(t,\tau)=A_{0n}(\tau)\cos(\mu_n^2 t)+B_{0n}(\tau)\sin(\mu_n^2 t)$ and because of the boundary condition (64) (i.e., $(\gamma\lambda-k)X'''(1)=k\gamma\lambda X(1)$) Equation (193) gives the following coupled differential equations for $A_{0n}(\tau)$ and $B_{0n}(\tau)$:

$$\frac{\mathrm{d}A_{0n}}{\mathrm{d}\tau}+\left(\left(\frac{c}{2}+\frac{\alpha k^2}{\gamma\lambda_n^2}\right)\left(\frac{\phi_{n_{xxx}}(1)}{k}\right)^2-\frac{\alpha}{2}\right)A_{0n} + \left(\frac{\Theta_{nn}}{2\mu_n^2}\right)B_{0n}=0, \tag{194}$$

$$\frac{\mathrm{d}B_{0n}}{\mathrm{d}\tau}+\left(\left(\frac{c}{2}+\frac{\alpha k^2}{\gamma\lambda_n^2}\right)\left(\frac{\phi_{n_{xxx}}(1)}{k}\right)^2-\frac{\alpha}{2}\right)B_{0n} - \left(\frac{\Theta_{nn}}{2\mu_n^2}\right)A_{0n}=0. \tag{195}$$

Define the following constants

$$k_{1n}=\left(\left(\frac{c}{2}+\frac{\alpha k^2}{\gamma\lambda_n^2}\right)\left(\frac{\phi_{n_{xxx}}(1)}{k}\right)^2-\frac{\alpha}{2}\right) = \left(\left(\frac{c}{2}+\frac{\alpha k^2}{\gamma\lambda_n^2}\right)\left(\frac{\gamma\lambda_n\phi_n(1)}{\gamma\lambda_n-k}\right)^2-\frac{\alpha}{2}\right), \tag{196}$$

$$k_{2n}=\frac{\Theta_{nn}}{2\mu_n^2}. \tag{197}$$

From (194) and (195) $A_{0n}(\tau)$ and $B_{0n}(\tau)$ can be determined, yielding

$$A_{0n}(\tau)=e^{-k_{1n}\tau}\left(A_{0n}(0)\cos(k_{2n}\tau)-B_{0n}(0)\sin(k_{2n}\tau)\right),$$
$$B_{0n}(\tau)=e^{-k_{1n}\tau}\left(B_{0n}(0)\cos(k_{2n}\tau)+A_{0n}(0)\sin(k_{2n}\tau)\right),$$

Consider (196), if the wind-force is not included (i.e., $\alpha=0$) then $k_{1n}>0$. Since $\phi_n^2(1)\to 4$ for $n\to\infty$ it follows that $k_{1n}\to 2c$ for $n\to\infty$. So, the oscillations will be damped uniformly for every positive value of c.

In applications only the first oscillation modes are important. In Table 4, the quotient

$$\frac{1}{2}\left(\frac{\gamma\lambda_n\phi_n(1)}{\gamma\lambda_n-k}\right)^2$$

of the first eight oscillation modes is listed for several values of γ. Note that also the case that γ is small, but not $\mathcal{O}(\epsilon)$, has been considered. Since the quotient is small for the first oscillation mode, c has to be large to suppress the wind-force.

Note that the values of the parameters in Table 4 are similar to the values in Table 3.

Table 4 Numerical approximations of $(1/2)([\gamma\lambda_n\phi_n(1)]/[\gamma\lambda_n-k])^2$ for $k=1$ and $\gamma=1$, $\gamma=0.1$, $\gamma=0.01$, and $\gamma=0.001$

n	$\gamma=1$	$\gamma=0.1$	$\gamma=0.01$	$\gamma=0.001$
0	0.2684	1.3435	0.0344	0.000310
1	2.1819	5.5157	50.595	1.6826
2	2.0352	2.1125	3.1998	500.10
3	2.0077	2.0173	2.1164	3.6902
4	2.0029	2.0053	2.0303	2.3076
5	2.0014	2.0023	2.0113	2.1054
6	2.0008	2.0012	2.0052	2.0463
7	2.0005	2.0007	2.0027	2.0236

The damping coefficient is equal to $-(\epsilon c/2)([\gamma\lambda_n\phi_n(1)]/[\gamma\lambda_n-k])^2$

The functions $A_{0n}(\tau)$ and $B_{0n}(\tau)$ have been obtained. Now the expression for $v_n(t,\tau)$, $u_0(x,t,\tau)$, and $u_1(x,t,\tau)$ can be derived, and also an order ϵ approximation of $\xi(t,\tau)$ can be obtained from (25). It is beyond the scope of this paper to prove that the $\mathcal{O}(\epsilon)$-approximations are indeed valid on timescales of $\mathcal{O}(\epsilon^{-1})$.

7 Conclusions

In this paper, a beam subjected to wind-forces and with a TMD at the top as a model for a tall building in a wind-field has been considered. The TMD is modelled as a simple mass-spring-dashpot system. The oscillations of this beam are described by an initial-boundary value problem. For this problem the nonlinear terms in the beam model have been omitted. The problem has been solved approximately by using perturbation techniques and by using the method of separation of variables. All the calculations in this paper are formal. The well-posedness of the problem has been assumed, and a proof of this is beyond the scope of this paper. Note that the well-posedness of the problem is not an easy question. The method of separation of variables cannot always be applied to find the solution of a linear partial differential equation. A typical example is

$$y_{tt}(x,t)-y_{xx}(x,t)=0, \tag{198}$$

$$y(0,t)=0,\ y_x(1,t)=-y_t(1,t). \tag{199}$$

For this problem, the method of separation of variables cannot be used to find non-trivial solutions. Also it is possible that the method of separation of variables does not mean anything for a problem (see [14]). For the problems considered in this paper the method of separation of variables works fine.

In this paper the stability of the system has been considered. The energy integral has been used to show that the system (not subjected to wind-forces) is damped. Also the influence of the ratio (γ) of the mass of the TMD (the tip-mass) with respect to the mass of the beam, and of the damping parameter of the dashpot ($\epsilon_3 c$, where $0 < \epsilon_3 \ll 1$) on the damping rates of the system has been considered. It has been found (see Table 2 and formula (138)) that the ratio (γ) should not be small with respect to the damping parameter ($\epsilon_3 c$) to obtain appropriate damping rates for the first oscillation modes. For the case that γ and $\epsilon_3 c$ are of equal order it has been shown (see formula (152)) that the first damping rates will become small with respect to the damping rates of the higher order modes if ϵ_3 tends to 0. Furthermore, it has been shown that the TMD can be used efficiently to damp the higher order modes.

One of the boundary conditions contains a small parameter. A multiple-timescales perturbation method has been used to construct approximations of the solution. It has been shown how the timescales should be chosen.

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