# Characteristics of phase-averaged equations for modulated wave groups

Report submitted to RIKZ
(project RKZ-410)

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Report 09-00
October 2000

TU Delft

Delft University of Technology
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## Contents

# Abstract

The project concerns the influence of long waves on coastal morphology. The modelling of the combined motion of the long waves and short waves in the horizontal plane is done by phase-averaging over the short wave motion and using intra-wave modelling for the long waves, see e.g. Roelvink (1993).

The evolution of the long-wave motion and the short-wave energy associated with slowly-modulated wave trains can be described by applying the average Lagrangian principle of Whitham (1974). The hydrodynamic part of the present version of the model (Reniers et al. 2000) solves a reduced set of equations for the surface elevation $\zeta$, mass transport velocity $\mathbf{U}$ and short-wave energy $E$, assuming the wave angular frequency $\omega$ to be constant:

$V_m$ n.t $U_m$?

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_m}(hU_m) = 0, \tag{0.1a}$$

$$\frac{\partial U_l}{\partial t} + V_m \frac{\partial U_l}{\partial x_m} = -g\frac{\partial \zeta}{\partial x_l} - \frac{1}{\rho h}\frac{\partial S_{lm}}{\partial x_m} - \frac{\tau_{b,l}}{\rho h}, \tag{0.1b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_m}\left[(V_m + C_{g,m})E\right] = -D, \tag{0.1c}$$

where the long-wave motion is forced by the space- and time-dependent short-wave radiation stresses $S_{lm}$.

This model describes the sub-harmonic triad interactions These equations are solved with a research version of the DELFT-3D model. The short-wave energy is assumed to propagate along fixed wave rays, which are computed at the start of a computation with the HISWA model. The the long-wave dynamics, as well as the short-wave energy propagation, sediment concentrations and morphodynamics are computed with the FLOW module of DELFT-3D.

The above equations describe amplitude-modulated waves, i.e. the wave height is varying in time but the wave frequency is constant. Further the wave propagation direction is not varying in time.

In order to have frequency-modulation as well, as is the case for real sea waves, we add a fourth equation to above set (0.1) of three equations:

$$\frac{\partial k_l}{\partial t} + \frac{\partial \omega}{\partial x_l} = 0, \tag{0.2}$$

describing the evolution of the wave number vector $\mathbf{k}$. The wave direction is now also a function of time, as it is in frequency modulated and directional sea waves.

Because the wave frequency is no longer a constant, the short-wave energy equation (0.1c) has to be replaced with an equation for the short wave action $[E/(\omega - \mathbf{k}\cdot\mathbf{V})]$.

An additional advantage is that the HISWA computations are no longer needed and everything can be solved with the FLOW program in the DELFT-3D model.

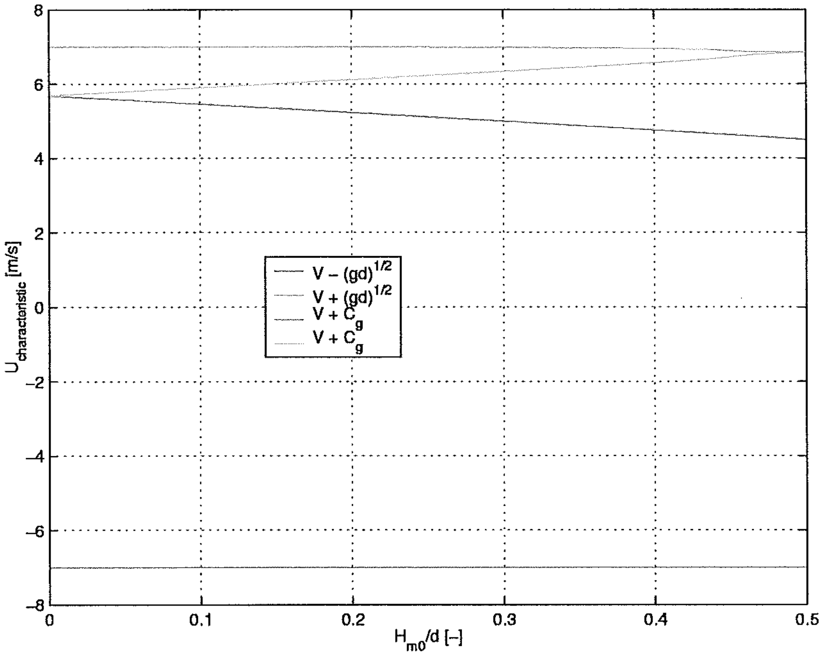
In this report we study the behaviour of the linearized version of the above long-wave and wave-propagation equations. Especially the characteristics of the one-dimensional system of equations are determined.

$C_g$-E

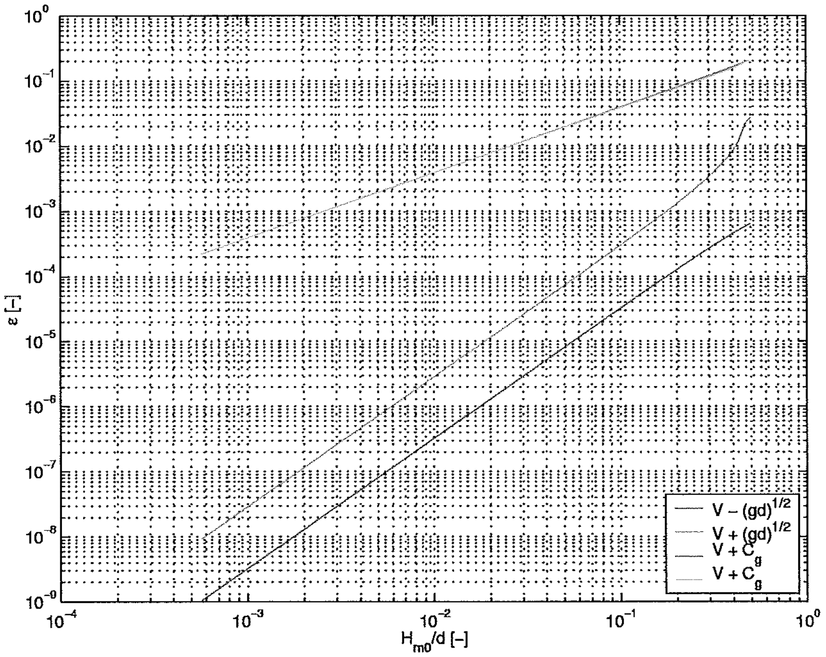
For not too high waves the 1D system is hyperbolic and has four ~~real~~ characteristic velocities. For very small wave heights and an pre-existing current with velocity $V$ these characteristic velocities are: $(V+\sqrt{gh})$, $(V-\sqrt{gh})$ and twice $(V+C_g)$. I.e. the short-wave energy $E$ and the wave number $k$ travel both with the same velocity $(V+C_g)$.

$C_g$-k

For higher values of the significant wave height $H_{m0}$ the characteristic velocities change, see Figure 1. Especially the group velocity splits from one value into two values. For high waves, especially in deeper water, complex-valued characteristic velocities may appear, associated with wave instabilities (see e.g. Dingemans 1997).



(a) Characteristic velocities



(b) Relative deviation of characteristic velocities from zero wave-height values

Figure 1: Example of characteristic velocities for full equations with 'exact' $k$-equation (0.2). Mean water depth $h = 5.0$ m, wave period $T = 7.0$ s.

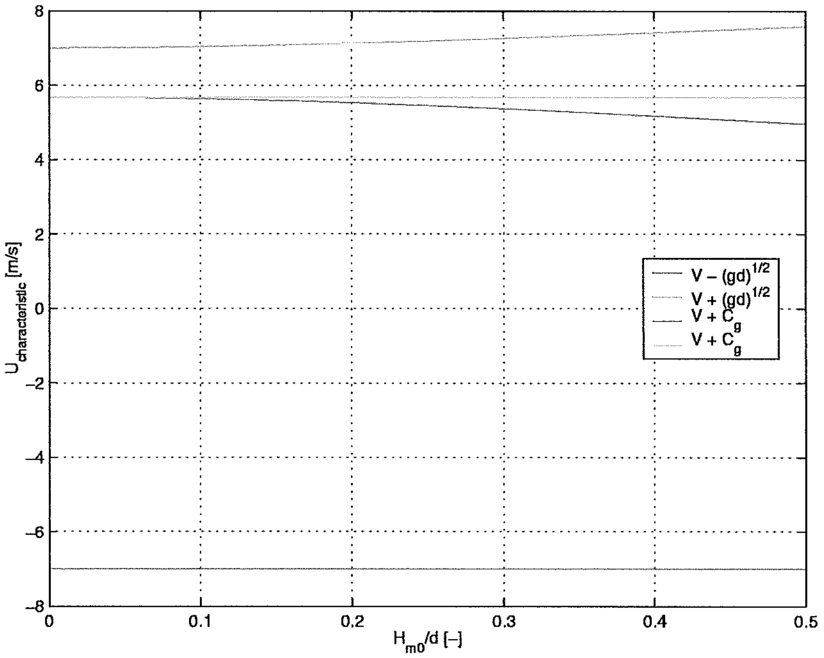
A problem with the additional equation (0.2) is, that associated with the double eigenvalue $(V + C_g)$ it also has a double eigenvector for the wave energy $E$ and wave number $k$ for very small wave heights. We do not understand how this can be the case for a realistic physic model.
In order to prevent these problems we suggest to use the following approximation of Equation (0.2), for the time being:

$$\frac{\partial k_l}{\partial t} + (V_m + C_{g,m})\,\frac{\partial k_l}{\partial x_m} = 0, \tag{0.3}$$

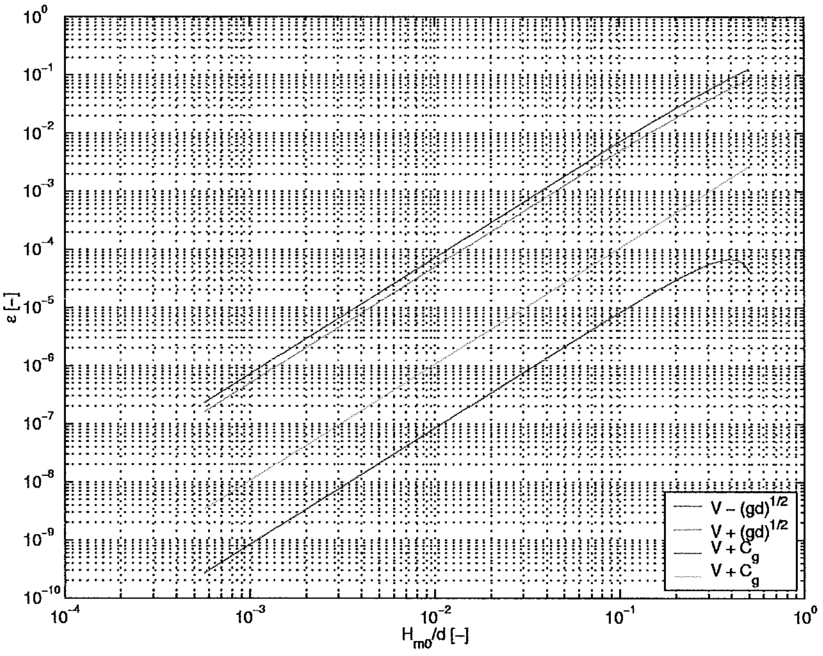
neglecting the spatial derivatives of the angular frequency $\omega$ with respect to the water depth $h$ and flow velocity $\mathbf{V}$.
The effect on the behaviour of the characteristic velocities as a function of the wave height is shown in Figure 2. As can be seen, the approximation has some influence on the characteristics for higher wave heights.

At the moment the above extensions to the hydrodynamic part of the morphodynamic model are implemented into the program.

initial condition
+ boundary condition
K at the boundary is
obtained from linear
dispersion relation



(a) Characteristic velocities



(b) Relative deviation of characteristic velocities from zero wave-height values

Figure 2: Example of characteristic velocities for full equations with approximate $k$-equation (0.3). Mean water depth $h = 5.0$ m, wave period $T = 7.0$ s.

# 1 Introduction

The evolution of wave groups and their effects on the low-frequency waves can be modeled by using the average Lagrangian principle. These equations describe the evolution as a function of time $t$ and horizontal coordinate $\mathbf{x}$ of the free surface level $\zeta$, depth-averaged mass-transport velocity $\mathbf{U}$, wave-number vector $\mathbf{k}$ and short-wave energy $E$.
The quantities $\zeta$, $\mathbf{U}$, $\mathbf{k}$ and $E$ are phase-averaged over the short wave period, but still describe the variations with respect to the wave group time and length scales.
Applied to water waves, the result correct to order $\mathcal{O}(E^2)$ is (Whitham 1974, Paragraph 16.8; Phillips 1977, Paragraph 3.6):

is that correct

eulerian

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_m}(hU_m) = 0, \tag{1.4a}$$

$$\frac{\partial U_l}{\partial t} + V_m \frac{\partial U_l}{\partial x_m} = -g\frac{\partial \zeta}{\partial x_l} - \frac{1}{\rho h}\frac{\partial S_{lm}}{\partial x_m} - \frac{\tau_{b,l}}{\rho h}, \tag{1.4b}$$

4

$E_S \cdot E_S$ , $E^2 \rightarrow 4$

$$\frac{\partial k_l}{\partial t} + \frac{\partial \omega}{\partial x_l} = 0, \tag{1.4c}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_m}[(V_m + C_{g,m})E] = -\beta\frac{1}{2}\left(\frac{\partial V_l}{\partial x_m} + \frac{\partial V_m}{\partial x_l}\right)S_{lm} - D, \tag{1.4d}$$

where $h = d + \zeta$ is the mean water-depth, $z = -d$ is the sea bottom level, $\zeta$ is the mean free-surface elevation, $x_m$, $U_m$, $V_m$ and $k_m$ ($m = 1, 2$) are the components of the horizontal coordinate $\mathbf{x}$, mean mass-transport velocity $\mathbf{U}$, mean Eulerian velocity $\mathbf{V}$ and wave number vector $\mathbf{k}$ respectively, $g$ is the gravitational acceleration, $\rho$ is the mass density of the water, $S_{lm}$ ($l, m = 1, 2$) are the components of the wave radiation-stress tensor, $\tau_{b,l}$ ($l = 1, 2$) are the components of the bottom shear stress, $\omega$ is the angular frequency of the short waves, $C_{g,m}$ ($m = 1, 2$) are the components of the short wave group velocity and $D$ is the short-wave energy dissipation.

The mean mass-transport velocity $\mathbf{U}$ is obtained by integrating the horizontal velocities first from the bed to the instantaneous free surface elevation and there after average over the short wave phase. Therefor $\mathbf{U}$ also includes the mass transport due to the waves in the region between the wave trough and crest. This wave mass transport is due to the positive correlation between the instantaneous free surface elevation and the near-surface horizontal wave-related velocities.
The relation between the mean mass-transport velocity $\mathbf{U}$ and the mean Eulerian velocity $\mathbf{V}$, obtained by averaging over wave phase in fixed points below wave trough level, is:

$$\mathbf{U} = \mathbf{V} + \frac{\mathbf{k}}{k}\frac{E}{\rho C h}, \tag{1.5}$$

with $k = |\mathbf{k}|$ the length of vector $\mathbf{k}$. This relationship is correct to second order in wave amplitude and with $E/(\rho C)$ the mass flux of the waves in the splash zone.

The summation rule for indices is used, i.e. when indices appear more than once in a term of a equation summation over all possible values is meant, for example:

$$\frac{\partial}{\partial x_m}(hU_m) \equiv \sum_{l=1}^{2} \frac{\partial}{\partial x_l}(hU_l). \tag{1.6}$$

The factor $\beta$ is either 0 or 1, and is used to switch on the term describing the change in $E$ due to the horizontal variations in $\mathbf{U}$.

The angular frequency $\omega$ is approximately related to the absolute value $k \equiv \sqrt{k_m k_m}$ of the wave number vector $\mathbf{k}$ by the dispersion relation according to linear wave theory:

$$\begin{aligned} \omega &= \sigma + k_m V_m = \\ &= \sigma + k_m U_m - \frac{kE}{\rho C h}, \end{aligned} \tag{1.7a}$$

$$\sigma^2 = gkT, \tag{1.7b}$$

$$T = \tanh(kh), \tag{1.7c}$$

where $\sigma$ is the intrinsic or relative angular wave frequency as seen by an observer moving with the current velocity $\mathbf{V}$.
A third-order (in wave amplitude) improvement of the dispersion relation is given by:

$$\sigma^2 = gkT \left( 1 + \frac{9T^4 - 10T^2 + 9}{4T^2} \frac{k^2 E}{\rho g} \right) \tag{1.8}$$

but we will not use this any further for the time being.
The components $C_m$ of the wave propagation velocity $\mathbf{C}$ and $C_{g,m}$ of the group velocity $\mathbf{C}_g$, as observed in a frame of reference moving with the current $\mathbf{U}$, are according to linear wave theory:

$$C_m = \frac{k_m}{k} C, \qquad C = \frac{\sigma}{k}, \tag{1.9a}$$

$$C_{g,m} = \frac{k_m}{k} C_g, \qquad C_g = \frac{\partial \sigma}{\partial k}, \tag{1.9b}$$

where $C$ and $C_g$ are the lengths of the vectors $\mathbf{C}$ and $\mathbf{C}_g$ respectively.
The components $S_{lm}$ (for $l, m = 1, 2$) of the wave radiation-stress tensor are related to the short wave energy $E$ and the ratio $n \equiv C_g / C$ by:

$$S_{lm} = \left[ n \, \frac{k_l k_m}{k^2} + \left( n - \frac{1}{2} \right) \delta_{lm} \right] E, \tag{1.10}$$

where $\delta_{lm}$ is the Kronecker delta function, i.e.:

$$\delta_{lm} = \begin{cases} 1 \text{ if } l = m, \\ 0 \text{ if } l \neq m. \end{cases} \tag{1.11}$$

However, Roelvink (1993) and Reniers (1992) solve not Equation (1.4c) for the conservation of wave phase, but instead assume that the angular frequency $\omega$ is a constant. We will study both the effects of this assumption regarding constant $\omega$ and the neglect of the work $\frac{1}{2}(\partial U_l / \partial x_m + \partial U_m / \partial x_l) S_{lm}$ by the velocity gradient in the wave energy equation (1.4d).

## 2 Linearized equations

The evolution equations (1.4) for the long waves and the short-wave energy are linearized around a mean position. We introduce the mean quantities:

$$\overline{h} = d, \quad \overline{\omega} = \Omega, \quad \overline{k_l} = K_l, \text{ and } \overline{\zeta} = 0. \tag{2.12}$$

The mean frequency $\Omega$ is assumed to be a constant. Further we introduce the perturbations of quantities around the mean value:

$$h = d + \zeta, \tag{2.13a}$$
$$U_l = \overline{U}_l + u_l, \tag{2.13b}$$
$$V_l = \overline{V}_l + v_l, \tag{2.13c}$$
$$E = \overline{E} + e, \tag{2.13d}$$
$$S_{lm} = \overline{S}_{lm} + s_{lm}, \tag{2.13e}$$
$$\omega = \Omega + \varpi \text{ and} \tag{2.13f}$$
$$k_l = K_l + \kappa_l. \tag{2.13g}$$

Neglecting the bottom shear stress $\tau_{b,l}$ and wave energy dissipation $D$, the linearized evolution equations (1.4) for the perturbations around the mean values become:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_m}(d u_m + \overline{U}_m \zeta) = 0, \tag{2.14a}$$

$$\frac{\partial u_l}{\partial t} + \overline{V}_m \frac{\partial u_l}{\partial x_m} + v_m \frac{\partial \overline{U}_l}{\partial x_m} = -g \frac{\partial \zeta}{\partial x_l} - \frac{1}{\rho d} \frac{\partial s_{lm}}{\partial x_m} + \frac{1}{\rho d^2} \frac{\partial \overline{S}_{lm}}{\partial x_m} \zeta, \tag{2.14b}$$

$$\frac{\partial \kappa_l}{\partial t} + \frac{\partial \varpi}{\partial x_l} = 0, \tag{2.14c}$$

$$\begin{aligned} \frac{\partial e}{\partial t} + \frac{\partial}{\partial x_m} \left[ (\overline{V}_m + \overline{C}_{g,m}) e + \overline{E} v_m + \frac{\overline{\partial C_g}}{\partial k} \overline{E} \kappa_m + \frac{\overline{\partial C_g}}{\partial h} \overline{E} \zeta \right] &= \\ = -\beta \, \frac{1}{2} \left( \frac{\partial \overline{V}_l}{\partial x_m} + \frac{\partial \overline{V}_m}{\partial x_l} \right) s_{lm} - \beta \, \frac{1}{2} \overline{S}_{lm} \left( \frac{\partial v_l}{\partial x_m} + \frac{\partial v_m}{\partial x_l} \right). \end{aligned} \tag{2.14d}$$

From Equations (1.10) & (1.7) we have for the radiation stress oscillations $s_{lm}$:

$$s_{lm} = \left[ \frac{\overline{C}_g}{\overline{C}} \frac{K_l K_m}{K^2} + \left( \frac{\overline{C}_g}{\overline{C}} - \frac{1}{2} \right) \delta_{lm} \right] e \; + \frac{\overline{\partial S_{lm}}}{\partial k} \frac{K_j}{K} \, \kappa_j \; + \frac{\overline{\partial S_{lm}}}{\partial h} \, \zeta, \tag{2.15}$$

for the mass-transport velocity oscillations $u_l$:

$$u_l = v_l \; + \frac{\overline{E}}{\rho \overline{\sigma} d} \, \kappa_l \; + \frac{K_l}{\rho \overline{\sigma} d} \, e \; - \frac{K_l \overline{E}}{\rho \overline{\sigma} d^2} \left( 1 + \frac{d}{\overline{\sigma}} \frac{\overline{\partial \sigma}}{\partial h} \right) \zeta \; - \frac{K_l \overline{E}}{\rho \overline{\sigma}^2 d} \, \frac{\overline{\partial \sigma}}{\partial k} \, \frac{K_m}{K} \, \kappa_m \tag{2.16}$$

and for the angular frequency oscillations $\varpi$:

$$\varpi = \frac{\overline{\partial \omega}}{\partial h} \, \zeta \; + \frac{\overline{\partial \omega}}{\partial U} \, \frac{\overline{U}_m}{\overline{U}} \, u_m \; + \frac{\overline{\partial \omega}}{\partial k} \, \frac{K_m}{K} \, \kappa_m \; + \frac{\overline{\partial \omega}}{\partial E} \, e. \tag{2.17}$$

In case the assumption is made that $\omega$ is a constant, as done by Roelvink (1993) and Reniers (1992), than $\varpi = 0$ and then Equation (2.17) gives a relation between the variations in $\zeta$, $u_m$

and $\kappa_m$.

The derivatives with respect to $k$, $h$ and $U$ appearing in the above equations can be evaluated as:

$$\frac{\partial C_g}{\partial k} = -\frac{1}{2}\frac{\sigma}{k^2}\left[1 + (kh)^2\frac{1-T^4}{T^2}\right] + \frac{C_g^2}{\sigma}, \tag{2.18a}$$

$$\frac{\partial C_g}{\partial h} = \frac{1}{2}(\sigma + kC_g)\frac{1-T^2}{T} - \frac{1}{2}\sigma kh\frac{1-T^4}{T^2}, \tag{2.18b}$$

$$\frac{\partial \sigma}{\partial k} = C_g = \frac{1}{2}C\left(1 + kh\frac{1-T^2}{T}\right), \tag{2.18c}$$

$$\frac{\partial \sigma}{\partial h} = \frac{1}{2}\sigma k\frac{1-T^2}{T} = \frac{k}{h}C_g - \frac{1}{2}\frac{\sigma}{h}, \tag{2.18d}$$

$$\frac{\partial S}{\partial k} = h\left(\frac{1-T^2}{T} - kh\frac{1-T^4}{T^2}\right)E, \tag{2.18e}$$

$$\frac{\partial S}{\partial h} = k\left(\frac{1-T^2}{T} - kh\frac{1-T^4}{T^2}\right)E, \tag{2.18f}$$

$$\frac{\partial \omega}{\partial k} = U + C_g + \left(\frac{C_g}{C} - 2\right)\frac{E}{\rho C h}, \tag{2.18g}$$

$$\frac{\partial \omega}{\partial U} = k, \tag{2.18h}$$

$$\frac{\partial \omega}{\partial h} = \frac{1}{2}\sigma k\frac{1-T^2}{T} + \frac{k^2 E}{\rho\sigma h^2}, \tag{2.18i}$$

$$\frac{\partial \omega}{\partial E} = -\frac{k^2}{\rho\sigma h}. \tag{2.18j}$$

Next we will consider harmonic solutions for the one-dimensional case with a horizontal bed and constant mean quantities, first for the full equations. There after we will consider the consequences of the assumptions that the angular frequency $\omega$ is a constant and that we can neglect the source term $\frac{1}{2}(\partial U_l/\partial x_m + \partial U_m/\partial x_l)S_{lm}$ in the wave energy equation.

# 3 One-dimensional harmonic solutions for the full equations

## 3.1 The eigenvalue problem formulation

In the one-dimensional case for a uniform depth $d$ and mean current $\overline{U}$ the linearized equations (2.14) can be written as:

$$\frac{\partial \zeta}{\partial t} + d\frac{\partial u}{\partial x} + \overline{U}\frac{\partial \zeta}{\partial x} = 0, \tag{3.19a}$$

$$\frac{\partial u}{\partial t} + \overline{V}\frac{\partial u}{\partial x} = -g\frac{\partial \zeta}{\partial x} - \frac{1}{\rho d}\frac{\partial s}{\partial x}, \tag{3.19b}$$

$$\frac{\partial \kappa}{\partial t} + \frac{\partial \varpi}{\partial x} = 0, \tag{3.19c}$$

$$\frac{\partial e}{\partial t} + (\overline{V} + \overline{C}_g)\,\frac{\partial e}{\partial x} + (\overline{E} + \beta\overline{S})\,\frac{\partial v}{\partial x} + \overline{\frac{\partial C_g}{\partial k}}\,\overline{E}\,\frac{\partial \kappa}{\partial x} + \overline{\frac{\partial C_g}{\partial h}}\,\overline{E}\,\frac{\partial \zeta}{\partial x} = 0, \tag{3.19d}$$

with the mean quantities like $d$, $\overline{U}$, $\overline{V}$, $\Omega$, $K$ and $\overline{E}$ being constants. With the mean value of a derivative is meant evaluating the derivative at the mean value of the relevant quantities. The perturbations in the radiation stress $s$, mass-transport velocity $u$ and angular frequency $\varpi$ become:

$$s = \left(2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}\right)\, e + \overline{\frac{\partial S}{\partial k}}\,\kappa + \overline{\frac{\partial S}{\partial h}}\,\zeta, \tag{3.20}$$

$$u = v + \frac{\overline{E}}{\rho\overline{\sigma}d}\left(1 - \frac{\overline{C}_g}{\overline{C}}\right)\,\kappa + \frac{K}{\rho\overline{\sigma}d}\,e - \frac{K\overline{E}}{\rho\overline{\sigma}d^2}\left(1 + \frac{1}{\overline{\sigma}}\,\overline{\frac{\partial \sigma}{\partial h}}\right)\,\zeta \tag{3.21}$$

and

$$\varpi = \alpha\,\overline{\frac{\partial \omega}{\partial h}}\,\zeta + \alpha\,\overline{\frac{\partial \omega}{\partial U}}\,u + \overline{\frac{\partial \omega}{\partial k}}\,\kappa + \alpha\,\overline{\frac{\partial \omega}{\partial E}}\,e, \tag{3.22}$$

where $\alpha$ is a coefficient to turn off ($\alpha = 0$) or on ($\alpha = 1$) the dependence of $\omega$ on $\zeta$, $U$ and $E$. Next we consider harmonic oscillations with long-wave number $\mu$ and long-wave angular frequency $\nu$:

$$\phi(x,t) = \Re\left\{\hat{\phi}\,\mathrm{e}^{i(\mu x - \nu t)}\right\}, \tag{3.23}$$

with $i$ the imaginary unit and $\phi$ denoting perturbed quantities like $\zeta$, $u$, $\varpi$, $\kappa$ or $e$. A hat like $\hat{\phi}$ denotes the complex-valued amplitude of quantity $\phi$.
The linearized perturbation equations (3.19) become:

$$-i\nu\hat{\zeta} + i\mu d\hat{u} + i\mu\overline{U}\hat{\zeta} = 0, \tag{3.24a}$$

$$-i\nu\hat{u} + i\mu\overline{V}\hat{u} = -i\mu g\hat{\zeta} - i\mu\frac{1}{\rho d}\hat{s}, \tag{3.24b}$$

$$-i\nu\hat{\kappa} + i\mu\hat{\varpi} = 0, \tag{3.24c}$$

$$-i\nu\hat{e} + i\mu(\overline{V} + \overline{C}_g)\,\hat{e} + i\mu(\overline{E} + \beta\overline{S})\,\hat{v} + i\mu\overline{\frac{\partial C_g}{\partial k}}\,\overline{E}\,\hat{\kappa} + i\mu\overline{\frac{\partial C_g}{\partial h}}\,\overline{E}\,\hat{\zeta} = 0, \tag{3.24d}$$

where the radiation-stress amplitude $\hat{s}$, velocity amplitude $\hat{v}$ and angular-frequency amplitude $\hat{\varpi}$ are given by:

$$\hat{s} = \left(2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}\right)\,\hat{e} + \overline{\frac{\partial S}{\partial k}}\,\hat{\kappa} + \overline{\frac{\partial S}{\partial h}}\,\hat{\zeta}, \tag{3.25a}$$

$$\hat{\upsilon} = \hat{u} - \frac{\overline{E}}{\rho\overline{\sigma}d}\left(1 - \frac{\overline{C}_g}{\overline{C}}\right)\hat{\kappa} - \frac{K}{\rho\overline{\sigma}d}\,\hat{e} + \frac{K\overline{E}}{\rho\overline{\sigma}d^2}\left(1 + \frac{d}{\overline{\sigma}}\overline{\frac{\partial\sigma}{\partial h}}\right)\hat{\zeta}, \tag{3.25b}$$

$$\hat{\varpi} = \alpha\overline{\frac{\partial\omega}{\partial h}}\,\hat{\zeta} + \alpha\overline{\frac{\partial\omega}{\partial U}}\,\hat{u} + \overline{\frac{\partial\omega}{\partial k}}\,\hat{\kappa} + \alpha\overline{\frac{\partial\omega}{\partial E}}\,\hat{e}. \tag{3.25c}$$

Dividing by $i\mu$ and filling in $\hat{s}$, $\hat{\upsilon}$ and $\hat{\varpi}$ gives the following equations for the complex-valued amplitudes $\hat{\zeta}$, $\hat{u}$, $\hat{\kappa}$ and $\hat{e}$:

$$-\vartheta\hat{\zeta} + d\hat{u} + \overline{U}\hat{\zeta} = 0, \tag{3.26a}$$

$$-\vartheta\hat{u} + \overline{V}\hat{u} = -g\hat{\zeta} - \frac{1}{\rho d}\left[\left(2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}\right)\hat{e} + \overline{\frac{\partial S}{\partial k}}\,\hat{\kappa} + \overline{\frac{\partial S}{\partial h}}\,\hat{\zeta}\right], \tag{3.26b}$$

$$-\vartheta\hat{\kappa} + \alpha\overline{\frac{\partial\omega}{\partial h}}\,\hat{\zeta} + \alpha\overline{\frac{\partial\omega}{\partial U}}\,\hat{u} + \overline{\frac{\partial\omega}{\partial k}}\,\hat{\kappa} + \alpha\overline{\frac{\partial\omega}{\partial E}}\,\hat{e} = 0, \tag{3.26c}$$

$$\begin{aligned} -\vartheta\hat{e} + \left[\overline{V} + \overline{C}_g - \frac{K(\overline{E} + \beta\overline{S})}{\rho\overline{\sigma}d}\right]\hat{e} + (\overline{E} + \beta\overline{S})\,\hat{u} + {} \\ + \left[\overline{\frac{\partial C_g}{\partial k}} - \frac{\overline{E} + \beta\overline{S}}{\rho\overline{\sigma}d}\left(1 - \frac{\overline{C}_g}{\overline{C}}\right)\right]\overline{E}\,\hat{\kappa} + {} \\ + \left[\overline{\frac{\partial C_g}{\partial h}} + \frac{K(\overline{E} + \beta\overline{S})}{\rho\overline{\sigma}d^2}\left(1 + \frac{d}{\overline{\sigma}}\overline{\frac{\partial\sigma}{\partial h}}\right)\right]\overline{E}\,\hat{\zeta} = 0, \end{aligned} \tag{3.26d}$$

with $\vartheta = \nu/\mu$ the characteristic velocity.
This is an eigenvalue problem, with $\vartheta$ the eigenvalues. In matrix notation the equations can be written in the following familiar eigenproblem form:

$$(A - \vartheta I)\,\mathbf{p} = \mathbf{0}, \tag{3.27}$$

where $I$ is the identity matrix, $\mathbf{p}$ is the vector:

$$\mathbf{p} = \begin{pmatrix} \hat{\zeta} \\ \hat{u} \\ \hat{\kappa} \\ \left(\dfrac{\hat{e}}{\rho g}\right) \end{pmatrix}, \tag{3.28}$$

and the matrix $A$ is given by:

$$A = \begin{pmatrix} \overline{U} & d & 0 & 0 \\ g + \dfrac{1}{\rho g}\overline{\dfrac{\partial S}{\partial h}} & \overline{U} - \dfrac{\overline{E}}{\rho\overline{C}d} & \dfrac{1}{\rho g}\overline{\dfrac{\partial S}{\partial k}} & \dfrac{g}{d}\left(2\dfrac{\overline{C}_g}{\overline{C}} - \dfrac{1}{2}\right) \\ \alpha\overline{\dfrac{\partial\omega}{\partial h}} & \alpha\overline{\dfrac{\partial\omega}{\partial U}} & \overline{\dfrac{\partial\omega}{\partial k}} & \alpha\rho g\overline{\dfrac{\partial\omega}{\partial E}} \\ \left(\overline{\dfrac{\partial C_g}{\partial h}} + \gamma_{41}\right)\dfrac{\overline{E}}{\rho g} & \dfrac{\overline{E} + \beta\overline{S}}{\rho g} & \left(\overline{\dfrac{\partial C_g}{\partial k}} + \gamma_{43}\right)\dfrac{\overline{E}}{\rho g} & \left(\overline{U} + \overline{C}_g + \gamma_{44}\dfrac{\overline{E}}{\rho g}\right) \end{pmatrix}, \tag{3.29}$$

with:

$$\gamma_{41} = \frac{K(\overline{E}+\beta\overline{S})}{\rho\overline{\sigma}d^2}\left(1+\frac{d}{\overline{\sigma}}\frac{\overline{\partial\sigma}}{\partial h}\right), \tag{3.30a}$$

$$\gamma_{43} = -\frac{\overline{E}+\beta\overline{S}}{\rho\overline{\sigma}d}\left(1-\frac{\overline{C}_g}{\overline{C}}\right) \qquad \text{and} \tag{3.30b}$$

$$\gamma_{44} = -\frac{gK(2+\beta\frac{\overline{S}}{\overline{E}})}{\overline{\sigma}d}. \tag{3.30c}$$

This eigenvalue problem cannot be solved analytically in general since the determinant of $(A-\vartheta I)$ results in a fourth-order polynomial equation which cannot be solved analytically. Some numerical results are shown in Figure 1.

Next, we will try to find an approximate solution by using perturbation series around $\overline{E}=0$.

## 3.2 Perturbation-series solution for the orders zero and one

We expand the matrix $A$ in a series $A = A^{(0)} + \varepsilon A^{(1)} + \varepsilon^2 A^{(2)}$, with $\varepsilon = K^2\overline{E}/(\rho g)$. Here we will only consider the first two terms $A^{(0)}$ and $A^{(1)}$. The part $A^{(0)}$ is independent of $\overline{E}$ and $A^{(1)}$ being the part proportional to $\varepsilon$:

$$A^{(0)} = \begin{pmatrix} \overline{U} & d & 0 & 0 \\ g & \overline{U} & 0 & \frac{g}{d}\left(2\frac{\overline{C}_g}{\overline{C}}-\frac{1}{2}\right) \\ \alpha\frac{\overline{\partial\sigma}}{\partial h} & \alpha K & (\overline{U}+\overline{C}_g) & \alpha\left(-\frac{gK^2}{\overline{\sigma}d}\right) \\ 0 & 0 & 0 & (\overline{U}+\overline{C}_g) \end{pmatrix} \tag{3.31}$$

and

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{K^2\overline{E}}\frac{\overline{\partial S}}{\partial h} & -\frac{g}{K^2\overline{C}d} & \frac{1}{K^2\overline{E}}\frac{\overline{\partial S}}{\partial k} & 0 \\ \alpha\left(\frac{g}{\overline{\sigma}d^2}\right) & 0 & \left(\frac{\overline{C}_g}{\overline{C}}-2\right)\frac{g}{\overline{\sigma}Kd} & 0 \\ \frac{1}{K^2}\frac{\overline{\partial C_g}}{\partial h} & \frac{1}{K^2}\left(1+\beta\frac{\overline{S}}{\overline{E}}\right) & \frac{1}{K^2}\frac{\overline{\partial C_g}}{\partial k} & -\frac{g\,(2+\beta\frac{\overline{S}}{\overline{E}})}{K\overline{\sigma}d} \end{pmatrix}. \tag{3.32}$$

We write the eigenvalues $\vartheta$ and eigenvectors $\mathbf{p}$ as a perturbation series in $\varepsilon$:

$$\vartheta = \vartheta^{(0)} + \varepsilon\,\vartheta^{(1)} + \varepsilon^2\,\vartheta^{(2)} + \cdots, \tag{3.33a}$$

$$\mathbf{p} = \mathbf{p}^{(0)} + \varepsilon\,\mathbf{p}^{(1)} + \varepsilon^2\,\mathbf{p}^{(2)} + \cdots. \tag{3.33b}$$

Using these expansions the eigenvalue problem (3.27) can be expanded as:

$$(A^{(0)}-\vartheta^{(0)}I)\,\mathbf{p}^{(0)} + \varepsilon\,\left[(A^{(0)}-\vartheta^{(0)}I)\,\mathbf{p}^{(1)} + (A^{(1)}-\vartheta^{(1)}I)\,\mathbf{p}^{(0)}\right] + \mathcal{O}(\varepsilon^2) = 0, \tag{3.34}$$

which can be solved sequentially for each order of $\varepsilon$.
For $\alpha = 1$ the $0^{\text{th}}$-order eigenproblem $(A^{(0)} - \vartheta^{(0)} I)\ \mathbf{p}^{(0)} = 0$ has the solution:

$$\boldsymbol{\vartheta}^{(0)} = \begin{pmatrix} \overline{U} + \sqrt{gd} & \overline{U} - \sqrt{gd} & \overline{U} + \overline{C}_g & \overline{U} + \overline{C}_g \end{pmatrix}^T, \tag{3.35a}$$

$$P^{(0)} = \begin{pmatrix} \sqrt{\frac{d}{g}} & -\sqrt{\frac{d}{g}} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \dfrac{\overline{\frac{\partial \omega}{\partial h}}\sqrt{\frac{d}{g}} + K}{\overline{C}_g - \sqrt{gd}} & \dfrac{\overline{\frac{\partial \omega}{\partial h}}\sqrt{\frac{d}{g}} - K}{\overline{C}_g + \sqrt{gd}} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.35b}$$

Here the elements of $\boldsymbol{\vartheta}^{(0)}$ are the eigenvalues, and the columns of matrix $P^{(0)}$ are the corresponding eigenvectors. The superscript $(\cdot)^T$ denotes the matrix transpose.
The system has a double eigenvalue and double eigenvector at $\vartheta^{(0)}_{3,4} = \overline{U} + \overline{C}_g$. The short waves are only traveling in one direction due to the assumptions made in deriving the modulation equations using the average Lagrangian principle (Whitham 1974, Chapter 16).
Next we want to solve the first-order system:

$$\left(A^{(0)} - \vartheta_j^{(0)} I\right)\ \mathbf{p}_j^{(1)} + \left(A^{(1)} - \vartheta_j^{(1)} I\right)\ \mathbf{p}_j^{(0)} = 0, \qquad \text{for } j = 1 \cdots 4. \tag{3.36}$$

This is itself not an eigenvalue problem, but denotes the first-order changes to be made to the eigenvalues and eigenvectors.
To solve this we first introduce the eigenvectors $\mathbf{q}_j^{(0)}$, $j = 1 \cdots 4$ of the transpose of $A^{(0)}$:

$$\left((A^{(0)})^T - \vartheta_j^{(0)} I\right)\ \mathbf{q}_j^{(0)} = 0, \qquad \text{for } j = 1 \cdots 4. \tag{3.37}$$

The eigenvalues of the transposed matrix $(A^{(0)})^T$ are the same as those of $A^{(0)}$. For the eigenvectors of $(A^{(0)})^T$ we have:

$$Q^{(0)} = \begin{pmatrix} \left(\sqrt{\frac{d}{g}}\ \frac{\sqrt{gd} - \overline{C}_g}{2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}}\right) & \left(\sqrt{\frac{d}{g}}\ \frac{\sqrt{gd} + \overline{C}_g}{2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}}\right) & 0 & 0 \\ \left(+\frac{d}{g}\ \frac{\sqrt{gd} - \overline{C}_g}{2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}}\right) & \left(-\frac{d}{g}\ \frac{\sqrt{gd} + \overline{C}_g}{2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{3.38}$$

with the columns of matrix $Q^{(0)}$ denoting the eigenvectors $\mathbf{q}_j^{(0)}$ for the corresponding eigenvalues $\vartheta_j^{(0)}$, $j = 1 \cdots 4$.

# 4 One-dimensional harmonic solutions for the reduced equations

## 4.1 Introduction of the constant frequency assumption

Roelvink (1993) and Reniers (1992) assume a constant angular frequency $\omega$ in the whole domain, instead of the conservation equation for the wave phase, Eq. (1.4c). So the set of equations becomes:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_m}(hU_m) = 0, \tag{4.39a}$$

$$\frac{\partial U_l}{\partial t} + V_m \frac{\partial U_l}{\partial x_m} = -g\frac{\partial \zeta}{\partial x_l} - \frac{1}{\rho h}\frac{\partial S_{lm}}{\partial x_m} - \frac{\tau_{b,l}}{\rho h}, \tag{4.39b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_m}\left[(V_m + C_{g,m})E\right] = -\beta\,\frac{1}{2}\left(\frac{\partial V_l}{\partial x_m} + \frac{\partial V_m}{\partial x_l}\right) S_{lm} - D. \tag{4.39c}$$

The wave number $k$ is to be solved from the dispersion relationship Equation (1.7) as a function of the local water depth $h$, flow velocity $\mathbf{U}$, angular frequency $\omega$ and short-wave energy $E$.

## 4.2 The eigenvalue problem formulation

The same assumptions as used for the full equations will be used, i.e. no dissipation, harmonic solutions with long-wave number $\mu$, long-wave angular frequency $\nu$, waves and current in the same direction, a uniform water depth $h$ and a constant mean current velocity $\overline{U}$.

The resulting set of equations is:

$$-i\nu\hat{\zeta} + i\mu d\hat{u} + i\mu\overline{U}\hat{\zeta} = 0, \tag{4.40a}$$

$$-i\nu\hat{u} + i\mu\overline{V}\hat{u} = -i\mu g\hat{\zeta} - i\mu\frac{1}{\rho d}\hat{s}, \tag{4.40b}$$

$$-i\nu\hat{e} + i\mu(\overline{V} + \overline{C}_g)\ \hat{e} + i\mu(\overline{E} + \beta\overline{S})\ \hat{v} + i\mu\overline{\frac{\partial C_g}{\partial k}}\overline{E}\ \hat{\kappa} + i\mu\overline{\frac{\partial C_g}{\partial h}}\overline{E}\ \hat{\zeta} = 0, \tag{4.40c}$$

where the radiation-stress amplitude $\hat{s}$ and velocity amplitude $\hat{v}$ are given by:

$$\hat{s} = \left(2\frac{\overline{C}_g}{\overline{C}} - \frac{1}{2}\right)\ \hat{e}\ + \overline{\frac{\partial S}{\partial k}}\ \hat{\kappa}\ + \overline{\frac{\partial S}{\partial h}}\ \hat{\zeta}, \tag{4.41a}$$

$$\hat{v} = \hat{u} - \frac{\overline{E}}{\rho\overline{\sigma}d}\ \left(1 - \frac{\overline{C}_g}{\overline{C}}\right)\ \hat{\kappa}\ - \frac{K}{\rho\overline{\sigma}d}\ \hat{e}\ + \frac{K\overline{E}}{\rho\overline{\sigma}d^2}\ \left(1 + \frac{d}{\overline{\sigma}}\ \overline{\frac{\partial \sigma}{\partial h}}\right)\ \hat{\zeta}. \tag{4.41b}$$

The wave number amplitude $\hat{\kappa}$ can be determined from the requirement that the angular frequency $\omega = \omega(h, k, V)$ is a constant, so also the $x$-derivative of $\omega$ should be equal to zero. Using the chain rule regarding differentiation and the previously introduced harmonic solutions, we get:

$$\overline{\frac{\partial \omega}{\partial k}}\ \hat{\kappa}\ +\ \overline{\frac{\partial \omega}{\partial h}}\ \hat{\zeta}\ +\ \overline{\frac{\partial \omega}{\partial V}}\ \hat{v}\ =\ 0, \tag{4.42}$$

from which the following equation for the wave-number amplitude $\hat{\kappa}$ is obtained:

$$\hat{\kappa}\ =\ -\frac{1}{\overline{C}_g}\ \left(\overline{\frac{\partial \omega}{\partial h}}\ \hat{\zeta}\ +\ K\ \hat{v}\right). \tag{4.43}$$

The next step will be to construct the equations for the eigenproblem and to study the behaviour of the reduced system.

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