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# THE STABILITY OF THE SHORT-PERIOD MOTION OF AN AIRFRAME HAVING NON-LINEAR AERODYNAMIC CHARACTERISTICS IN PITCH AND SUBJECT TO A STEP-FUNCTION ELEVATOR DEFLECTION

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# The stability of the short-period motion of an airframe having non-linear aerodynamic characteristics in pitch and subject to a step-function elevator deflection

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## SUMMARY

The stability of the differential equation

$$\frac{d^2x}{dt^2} + B(x)\frac{dx}{dt} + C(x) = Q, \quad Q = 0, \quad t < 0,$$

with B(x) and C(x) as anti-symmetric power series, is shown to be determined by the nature of the singularity at the steady-state value given by C(x) = Q, with certain additional restrictions on the initial and final value of x. Further, the character of the transient settling down motion is directly related to the nature of this singularity and, together with the stability, is accurately predicted by criteria derived.

It is shown that the previous stability criteria can be applied to the problem of an airframe subject to a step-function elevator deflection, provided that the aerodynamic derivative $z_\eta$ is negligible. When $z_\eta$ is not small, special treatment of the stability problem is required and it is shown that the critical value of the elevator step-function which will cause instability can fairly readily be obtained from quantities taken from the elevator trim curves.

CONTENTS

## 1.0 Introduction

The differential equation

$$\frac{d^2x}{dt^2} + B'(x)\frac{dx}{dt} + C(x) = Q, \quad Q = 0, \quad t < 0 \tag{1.1}$$

describes the motion of a class of second-order systems with displacement dependent stiffness and damping and subject to a step function disturbance of magnitude Q. It is characteristic of many physical problems, for example in the stability theory of synchronous electrical motors (Refs. 1 and 2) and in the aerodynamic response of airframes having non-linear normal force and pitching moment curves (Ref. 4). The step function is, of course, a standard function for testing the transient response of systems and gives rise to many other examples whose governing equation is (1.1).

The equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Q \tag{1.2}$$

is a degenerate linear form of (1.1), where b and c are constants, and it is of interest to compare the linear with the non-linear problem. In the case of (1.2) the solution is made up of the sum of the complementary function, which is the solution of the homogeneous equation given by $Q \equiv 0$, and any particular integral, the latter necessarily involving Q. Provided that Q is of finite magnitude it follows directly from the principle of linear superposition that if the homogeneous equation, $Q \equiv 0$, can be shown to have stable solutions then (1.2) has a completely stable solution. In the linear problem, therefore, stability analysis can be restricted to the homogeneous equation.

Except in those cases where it is possible to separate the variables or the equation is exact, no explicit general solutions to non-linear differential equations are known. It is of course certain that the principle of linear superposition is invalid for such equations and as a result the stability criteria for the homogeneous, $Q \equiv 0$, and non-homogeneous, $Q \not\equiv 0$, cases are different. This means that the stability and response problems cannot be considered separately, as they are in a linear system, but involve an analysis of the stability of the response and therefore will depend on the nature and magnitude of the forcing term Q.

When the damping term is absent from (1.1) it is usually possible to obtain a first integral in the form

$$\tfrac{1}{2}v^2 + f(Q,x) = E, \tag{1.3}$$

where $v = \frac{dx}{dt}$ and E is a constant. This equation is then expressive of the energy balance in the system, which is conservative when the damping is absent. The term $\frac{1}{2}v^2$ corresponds to the kinetic energy, f(Q,x) the potential energy and E the total energy which is of course constant. A second integral is then possible by quadratures. Depending on the form of C(x) the second integral may be analytic in terms of known functions (very often elliptic integrals) or it may be necessary to resort to numerical or graphical methods to evaluate the integral.

When damping is present the system is dissipative or non-conservative and a first integral corresponding to the energy balance equation (1.3) is no longer obtainable or appropriate. If the damping is not too large the solution may be obtained by an analytic iteration procedure the starting point of which is the solution to the degenerate problem of zero damping. The success of this method will depend on the rate at which the process converges, rapid convergence being consistent with small values of the damping term $B'(x)\frac{dx}{dt}$.

The stability of the response, i.e. whether or not the system settles down to a steady value, or at least an oscillation of finite amplitude, is of importance in many applications and particularly so when dealing with systems whose damping or stiffness change sign. In the case of second or lower order systems with a step function input, it is not necessary to have an explicit expression for the response in order to determine the stability of the motion since this can be treated much more conveniently and elegantly by Poincaré's theory of singular points in the phase plane. (See Refs. 2, 3, and 4). The advantage of this method can only be fully appreciated when it is recalled that the alternative is to consider the stability within the framework of the interation procedure the conditions for convergence of which are generally not fully known.* There is in fact a second alternative method known as Lyapunov's second or direct method (see Ref. 5) but this will not be considered here.

The object of the paper is to obtain stability criteria for a specific equation of form (1.1) and compare these with numerical solutions from a digital computer. Having established the usefulness of the criteria they are then to be applied to the problem of the stability of the short-period motion of an airframe having non-linear aerodynamic characteristics in pitch and subject to a step-function deflection of the elevator.

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* This arises because it is usually impossible to state the form of the general term in the resulting series solution and therefore tests for convergence are either impossible or incomplete.

## NOTATION

| | |
|---|---|
| a, b, c, d | constants in Poincaré's theory of singular points |
| $a_1$, $a_2$, $a_3$ | constants in the expression for critical initial velocity $\dot{W}_{AC}$, equation (6.9) |
| $b_1$, $b_3$, .... | constants in the anti-symmetric function B(x) |
| $c_1$, $c_3$, .... | constants in the anti-symmetric function C(x) |
| m | airframe mass |
| q | angular velocity about axis of pitch |
| t | time |
| v = | $^{dx}/_{dt}$ |
| w | perturbation in velocity along axis of yaw |
| x | any dependent variable; often displacement. See equation (1.1). |
| $x_{c.g.}$ | distance of centre of gravity aft of reference line |
| $z_w$ = | $\frac{1}{m}\left(\frac{\partial Z}{\partial w}\right)_{w=0}$ |
| $z_q$ = | $\frac{1}{m}\left(\frac{\partial Z}{\partial q}\right)_{q=0}$ |
| $z_\eta$ = | $\frac{1}{m}\left(\frac{\partial Z}{\partial \eta}\right)_{\eta=0}$ |
| $m_w$ = | $\frac{1}{B}\left(\frac{\partial M}{\partial w}\right)_{w=0}$ |
| $m_q$ = | $\frac{1}{B}\left(\frac{\partial M}{\partial q}\right)_{q=0}$ |
| $m_\eta$ = | $\frac{1}{B}\left(\frac{\partial M}{\partial \eta}\right)_{\eta=0}$ |
| $m_{\dot{w}}$ = | $\frac{1}{B}\left(\frac{\partial M}{\partial w}\right)_{w=0}$ |
| $z_3$, $z_5$, ....... / $m_3$, $m_5$, .... | constants in the force and moment relations of equation (5.1) |
| $A_1$, = | $(U_o+z_q)m_w - m_q z_w$ |
| $A_3$, = | $(U_o+z_q)m_3 - m_q z_3$ |
| B | moment of inertia about the axis of pitch |
| $B_1$, = | $(U_o+z_q)m_w + m_q + z_w$ |
| $B_3$, = | $3z_3$ |
| B(x)= | $b_1 x + b_3 x^3 + \ldots\ldots$ |
| C(x)= | $c_1 x + c_3 x^3 + \ldots\ldots$ |
| D | the operator $^{d}/_{dt}$ |
| $\mathcal{D}$ | the discriminant $\left[B'(x)\right]^2 - 4C'(x)$ |
| F | the elevator forcing function $\left[(U_o+z_q)m_\eta - z_\eta m_q\right]H$ |
| M | moment about axis of pitch |
| P(x, v) / Q(x, v) | functions in Poincaré's theory of singular points. See equation (2.4) |
| Q | magnitude of forcing step-function, equation (1.1) |
| U | velocity tangential to flight path |
| $U_o$ | velocity along the longitudinal body axis |

NOTATION (continued)

| | |
|---|---|
| W | velocity along axis of yaw |
| Z | force along axis of yaw |
| $\alpha$ | airframe geometric incidence $\approx$ $W/U_0$, when $\alpha$ small. |
| H | total elevator angle of $\eta_T + \eta$ |
| $\eta$ | perturbation in elevator angle |
| $\theta$ | angle of pitch |
| $\vartheta$ | perturbation in angle of pitch |
| $\lambda$ | root of characteristic equation (2.7), (3.6) or (5.14). |
| $\xi$ | ordinate in x or w referred to singularity away from the origin. |

Suffixes

| | |
|---|---|
| T | refers to trimmed condition |
| S.S. | refers to steady-state condition |
| M | refers to maxima or minima on pitching moment curve. |
| P | refers to unstable singularity at point P on trim curve. |

A dot over a variable indicates differentiation with respect to time, whilst a prime indicates differentiation with respect to x or w.

## 2.0 Poincaré's Theory of Singular Points in the Phase-Plane

The stability of the solution to (1.1) can be analyzed by the phase-plane method which is given in detail in Refs. 2, 3 and 4.

Writing $v = \frac{dx}{dt}$, then $\frac{d^2x}{dx^2} = v\frac{dv}{dx}$ and (1.1) becomes

$$v\frac{dv}{dx} + B'(x)v + C(x) = Q \ . \tag{2.1}$$

The graphs of the solutions of this equation in the xv-plane, known as the phase plane, are referred to as "integral curves" and through each ordinary point in the plane there passes only one such curve. Alternatively equation (2.1) may be written as an equivalent pair of equations.

$$\left.\begin{aligned} \frac{dv}{dt} &= Q - C(x) - B'(x).\, v \\ \frac{dx}{dt} &= v \end{aligned}\right) \ , \tag{2.2}$$

which define a field vector having components $\frac{dv}{dt}$ and $\frac{dx}{dt}$; this vector is always tangential to the integral curve and indicates the direction in which t is increasing.

The stationary positions of equilibrium of (2.1) or (2.3) correspond with the singularities of the equivalent equation

$$\frac{dv}{dx} = \frac{Q - C(x) - B'(x).v}{v} \ , \tag{2.3}$$

which defines the slope of the field vector, and analysis of the character of these singularities gives considerable insight into the nature of the motion near these points. More generally, consider the singularities of the equation

$$\frac{dv}{dx} = \frac{P(x,v)}{Q(x,v)} \ , \tag{2.4}$$

which are defined by $P(x,v) = Q(x,v) = 0$. Since the origin can always be changed to correspond with the singular point, then analysis can be restricted to singularities at the origin.

When (2.4) has a singularity at the origin then it is assumed (Poincaré) that it may be written in the series form

$$\frac{dv}{dx} = \frac{ax + bv + p(x,v)}{cx + dv + q(x,v)} \ , \tag{2.5}$$

where $p(x,v)$ and $q(x,v)$ are the remaining terms of series whose lowest terms are of second degree at least. Further, if the constants obey the inequality

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0,$$

then the integral curves behave, in the neighbourhood of the singularity, as if $p(x,v)$ and $q(x,v)$ were absent.

The singularities of the reduced equation

$$\frac{dv}{dx} = \frac{ax + bv}{cx + dv} \tag{2.6}$$

are of four distinct types known as nodes, centres, spiral points and saddles respectively and each has a characteristic geometry, sometimes referred to as its "topological configuration".

NOTATION (continued)

| | |
|---|---|
| W | velocity along axis of yaw |
| Z | force along axis of yaw |
| $\alpha$ | airframe geometric incidence $\approx$ $^{W}/U_{o}$ , when $\alpha$ small. |
| H | total elevator angle of $\eta_T + \eta$ |
| $\eta$ | perturbation in elevator angle |
| $\theta$ | angle of pitch |
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| $\lambda$ | root of characteristic equation (2.7), (3.6) or (5.14). |
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$$\left.\begin{aligned} \frac{dv}{dt} &= Q - C(x) - B'(x).\,v \\ \frac{dx}{dt} &= v \end{aligned}\right) , \tag{2.2}$$

which define a field vector having components $\frac{dv}{dt}$ and $\frac{dx}{dt}$; this vector is always tangential to the integral curve and indicates the direction in which t is increasing.

The stationary positions of equilibrium of (2.1) or (2.3) correspond with the singularities of the equivalent equation

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$$\frac{dv}{dx} = \frac{P(x,v)}{Q(x,v)} \quad , \tag{2.4}$$

which are defined by $P(x,v) = Q(x,v) = 0$. Since the origin can always be changed to correspond with the singular point, then analysis can be restricted to singularities at the origin.

When (2.4) has a singularity at the origin then it is assumed (Poincaré) that it may be written in the series form

$$\frac{dv}{dx} = \frac{ax + bv + p(x,v)}{cx + dv + q(x,v)} \quad , \tag{2.5}$$

where $p(x,v)$ and $q(x,v)$ are the remaining terms of series whose lowest terms are of second degree at least. Further, if the constants obey the inequality

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0,$$

then the integral curves behave, in the neighbourhood of the singularity, as if $p(x,v)$ and $q(x,v)$ were absent.

The singularities of the reduced equation

$$\frac{dv}{dx} = \frac{ax + bv}{cx + dv} \tag{2.6}$$

are of four distinct types known as nodes, centres, spiral points and saddles respectively and each has a characteristic geometry, sometimes referred to as its "topological configuration".

Criteria for distinguishing the type of singularity are obtained from the characteristic equation

$$\lambda^2 - \lambda(b + c) - (ad - bc) = 0 \tag{2.7}$$

which has the roots

$$\lambda_{1,2} = \tfrac{1}{2}\left\{ (b + c) \pm \left[ (b+c)^2 + 4(ad - bc)\right]^{\frac{1}{2}} \right\}$$

Three important cases are distinguished, as follows.

Roots real and unequal. These arise when the discriminant $(b + c)^2 + 4(ad - bc)$ is positive. The sub-cases are: if $\Delta > 0$ then $\lambda_2/\lambda_1$ is negative and corresponds to a saddle point, and if $\Delta < 0$ then $\lambda_2/\lambda_1$ is positive and corresponds to a node. In the latter case, if $\lambda_1$ and $\lambda_2$ are both negative the node is stable, whereas if $\lambda_1$ and $\lambda_2$ are both positive the node is unstable.

Roots complex conjugate. Complex roots occur when the discriminant is negative. If the real part of the root is negative the integral curves are stable spirals, whereas if the real part is positive the curves are unstable spirals. When the roots are purely imaginary the singularity is a centre; however, under these conditions the singularity of (2.6) is not necessarily that of (2.5) and the higher order terms in $p(x, v)$ and $q(x, v)$ have to be considered.

Roots real and equal. The singularity is a node and its stability is governed the same way as if the roots were unequal.

## 3.0 Stability Criteria for a Particular Second-Order Equation.

The singularities of (2.3) are defined by $Q - C(x) = 0$, $v = 0$ and therefore the number of singular points will depend on the form of $C(x)$. It was shown in Ref. 4 that the anti-symmetric form for $c(x)$ is particularly valuable, i.e.

$$C(x) = c_1x + c_3x^3 + c_5x^5 + \ldots\ldots\ldots\ldots,$$

and can be used to represent a wide class of practical non linearities. For the remainder of this paper attention will be restricted to equations in which $B(x)$ and $C(x)$ are capable of representation by power series in odd powers of x. The number of singular points of (2.3) is then equal to the number of unequal roots of the equation

$$c_1x + c_3x^3 + c_5x^5 + \ldots\ldots\ldots\ldots = Q, \tag{3.1}$$

and therefore dependent on the number of terms, n, used to represent $C(x)$. For simplicity only the first two terms are used in the remainder of the analysis, although this can readily be extended to any other reasonable number of terms.

For a given value of Q the singular points of (3.1) are in fact the equilibrium or steady-state positions of the system and can readily be evaluated. Several cases exist depending of the signs of $c_1$ and $c_3$. Those of engineering interest are:

(a) $c_1 > 0$, $c_3 > 0$, corresponding to a "hard" system in which the stiffness is initially positive,
(b) $c_1 > 0$, $c_3 < 0$, corresponding to a "soft" system in which the stiffness is initially positive and
(c) $c_1 < 0$, $c_3 > 0$, corresponding to a "hard" system in which the stiffness is initially negative.

In addition if B is taken in the form

$$B(x) = b_1x + b_3x^3$$

then

$$B'(x) = \frac{dB}{dx} = b_1 + 3b_3x^2, \tag{3.2}$$

in which the cases of practical interest are $b_1>0$, $b_3>0$; $b_1>0$, $b_3<0$; and $b_1<0$, $b_3>0$.

Typical curves of the steady state values of x versus Q for the cases considered are shown in Fig. 1. It can be seen that for a given positive value of Q there exists one singular point for case (a) and three each for cases (b) and (c) when Q is relatively small, reducing to one each when Q is greater than the maximum or less than the minimum in the curve. Restricting attention to positive values of Q only, the singular point in (a) has a value, $x_{s.s.}>0$; in (b) two of the values are positive and one negative, whilst in (c) one value is positive and two are negative. Only in the degenerate case $Q\equiv 0$ is the singular point at the origin. In addition, on Fig. 1, curves of $B'(x)$ are shown.

Since the singular points of (2.3) are in general away from the origin then it may be written

$$\frac{dv}{d\xi} = \frac{Q - B'(x_{s.s.}+\xi)v - C(x_{s.s.}+\xi)}{v} , \tag{3.3}$$

where $x_{s.s.}$, 0 are the co-ordinates of the singular point and $\xi$ is the displacement co-ordinate referred to the singular point. Substituting for $B'(x)$ and $C(x)$ then gives

$$\frac{dv}{d\xi} = \frac{Q - C_1(x_{s.s.}+\xi) - C_3(x_{s.s.}+\xi)^3 - \left[b_1 + 3b_3(x_{s.s.}+\xi)^2\right] v}{v}$$

Expanding, remembering that for the present case

$$Q = C(x_{s.s.}) = c_1 x_{s.s.} + c_3 x^3_{s.s.} , \tag{3.4}$$

the equation for the slope of the field vector becomes

$$\frac{dv}{d\xi} = \frac{-c_1\xi - c_3(3x^2_{s.s.}\xi + 3x_{s.s.}\xi^2 + \xi^3) - \left[b_1 + 3b_3(x_{s.s.}+\xi)^2\right] v}{v} \tag{3.5a}$$

which in the first approximation reduces to

$$\frac{dv}{d\xi} = \frac{-(c_1 + 3c_3 x^2_{s.s.})\xi - (b_1 + 3b_3 x^2_{s.s.})v}{v} \tag{3.5b}$$

Comparing with the standard form, equation (2.6),

$a\equiv -(c_1 + 3c_3x^2_{s.s.})$, $b\equiv -(b_1 + 3b_3x^2_{s.s.})$, $c\equiv 0$ and $d\equiv 1$.

The roots of the characteristic equation are

$$\begin{aligned}\lambda_{1,2} &= \tfrac{1}{2}\left\{ -(b_1 + 3b_3x^2_{s.s.}) \pm\left[(b_1 + 3b_3x^2_{s.s.})^2 - 4(c_1 + 3c_3x^2_{s.s.})\right]^{\frac{1}{2}}\right\}\\ &= \tfrac{1}{2}\left\{ B'(x)\pm\mathcal{D}^{\frac{1}{2}}\right\} ,\end{aligned} \tag{3.6}$$

where $\mathcal{D} = [B'(x)]^2 - 4C'(x)$, is the discriminant.

The stability of the motion near the steady-state value of the system, as expressed by (3.6) and discussed in Section 2, can conveniently be summarized on a diagram of the type shown in Fig. 2. Starting with $Q = 0$, the variation of $x_{s.s.}$ and thereby of $B'(x)$ and $C'(x)$ may be obtained and the appropriate curves superimposed on Fig. 2. This then permits a ready assessment of the nature of the singularity at $x_{s.s.}$, 0. Typical curves for the various cases are shown in Figs. 3 and 4. Consider each of these in turn:

$b_1>0$, $c_1>0$, $c_3>0$

In Fig. 3(a), A corresponds to the origin at which $Q = 0$, $B'(x) = b_1$ and $C'(x) = c_1$.

The value of $b_1$ has been taken relatively small and the singularity is a stable spiral; alternatively with higher damping the singularity at the origin could be a stable node, point E. With $b_3>0$, increase or decrease of Q now gives rise to an increase of $B'(x)$ and $C'(x)$ along curve AB, until at B the character of the singularity changes from a stable spiral to a stable node; the value of $x_{s.s.}$ at which this change takes place corresponds to the disappearance of the discriminant and is given by

$$\mathcal{D} = (b_1 + 3b_3x^2_{s.s.})^2 - 4(c_1 + 3c_3x^2_{s.s.}) = 0$$

or

$$9b_3^2(x^2_{s.s.})^2 + (6b_1b_3 - 12c_3)x^2_{s.s.} + (b_1^2 - 4c_1) = 0$$

which has the solution

$$x_{s.s.} = \left\{ \frac{1}{3b_3^2} \left[2c_3 - b_1b_3 \pm 2(c_3^2 + c_1b_3^2 - b_1b_3c_3)^{\frac{1}{2}}\right] \right\}^{\frac{1}{2}} \tag{3.7}$$

in which only real values are appropriate. In the degenerate case when $b_3 = 0$, the value of $x_{s.s.}$ corresponding to (3.7) becomes

$$x_{s.s.} = \left\{ \frac{1}{12c_3} \quad (b_1^2 - 4c_1) \right\}^{\frac{1}{2}} \tag{3.8}$$

which can only have real values when $b_1^2>4c_1$, i.e. the initial point A on Fig. 3(a) must be above the curve $\mathcal{D} = 0$. Such a point is marked E, and increase or decrease of Q takes place along the curve EF, F corresponding to the value $x_{s.s.}$ given by (3.8) at which the singularity changes from a stable node to a stable spiral.

When $b_3<0$ the damping decreases as Q increases or decreases until at C, Fig. 3(a), the curve meets the damping boundary $B'(x) = 0$. At this point the singularity at $x_{s.s.}$, 0 becomes an unstable spiral, thereby demonstrating that transients having values of Q which cause $B'(x)$ to become negative are unstable since the motion in the neighbourhood of the steady-state value, predicted by stiffness considerations alone, Fig. 1, is unstable. The critical value of $x_{s.s.}$ for which the system goes unstable is given by

$$B'(x) = b_1 + 3b_3x^2_{s.s.} = 0$$

or

$$x_{s.s.} = \left(-\tfrac{1}{3}\frac{b_1}{b_3}\right)^{\frac{1}{2}} \tag{3.9}$$

<u>$b_1>0$, $c_1>0$, $c_3<0$</u>

Again A corresponds to the origin Q = 0. With increase or decrease of Q the stiffness $C'(x)$ decreases and provided $b_3$ is not too large and negative, the nature of the change in the singularities at $x_{s.s.}$0 are typified by curve ABC or AEF. The first change in the character of the singularities occurs at B or E where the change to nodal point takes place. The steady-state value for $\mathcal{D} = 0$ are again given by (3.7) or (3.8).

With further increase or decrease of Q a point is reached, C or F, where the stiffness changes sign i.e.

$$C'(x) = c_1 + 3c_3x^2_{s.s.} = 0$$

or

$$x_{s.s.} = \left(-\tfrac{1}{3}\frac{c_1}{c_3}\right)^{\frac{1}{2}}, \tag{3.10}$$

corresponding to the maxima or minima on the Q, $x_{s.s.}$ curves of Fig 1(b). It follows that transients having values of Q which would make $x_{s.s.}$, on the basis of the curves of Fig. 1(b), lie outside the region between the maxima and minima, are unstable.

With $b_3 << 0$ it is possible for the system to reach the damping boundary, G, before the stiffness boundary. The value of $x_{s.s.}$ for which this occurs is given by (3.9)

$b_1 > 0, \; c_1 < 0, \; c_3 > 0$

The singularity at the origin is a saddle point and with $b_3$ not too large and negative the singularities change along typical curves ACEF or AGHJ. At the stiffness boundary, $C'(x) = 0$, corresponding to the minima and maxima on the Q, $x_{s.s.}$ curves of Fig. 1(c), the singularities become stable nodes. In a practical system this means that the system will never settle down to steady-state values such as C or H, Fig. 1(c), but will depart to one or other of the alternative stable singularities A or B, or E or F. The direction in which the system moves will depend on the initial acceleration.

$$x = Q, \; t = 0$$

i.e. the stable singularity at which the system settles will have a displacement of the same sign as Q. In Fig. 1(c), B and E will be the appropriate settling points, rather than A or F, when subject to transients of magnitudes numerically less than those corresponding to the maxima or minima. This means that at Q = 0 the system will rest at values given by

$$Q = c_1 x_{s.s.} + c_3 x^3_{s.s.} = 0, \; x_{s.s.} \neq 0,$$

or

$$x_{s.s.} = \left[ - \frac{c_1}{c_3} \right]^{\frac{1}{2}} \; ; \qquad (3.11)$$

the values of stiffness corresponding to (3.11) are

$$C'(x) = c_1 + 3c_3 \left[ - \frac{c_1}{c_3} \right] = -2c_1 \qquad (3.12)$$

Taking E, Fig.3, as a typical point given by (3.11), then application of step-function disturbances Q of the same sign as the displacement will cause the singularity at $x_{s.s.}$, 0 to move along EF; alternatively if the sign of the disturbance, Q, is of opposite sign to the displacement then the singularity moves along EC becoming unstable at the stiffness boundary. In this latter case the instability at the stiffness boundary is not indicative of unbounded displacement, since further increase in the numerical value of Q will cause the system to jump to the other stable singularity, this being of opposite sign in displacement.

$b_1 < 0, \; b_3 > 0, \; c_1 > 0, \; c_3 > 0$

All the cases where $b_1 < 0$, $b_3 > 0$ are characterized by the possible existence of 'limit cycles", see Ref. 6. In Fig. 4(a), A is an unstable spiral point about which oscillations of increasing amplitude will develop. With increasing displacement from the origin the damping, $B'(x)$, increases until it reaches zero. At this condition there is established a stable oscillation known as a limit cycle whose amplitude corresponds to the displacement OL in Fig. 1(f) i.e. the limit cycle amplitude is given by $B'(x) = 0$, or

$$a_{LC} = \left( -\frac{1}{3} \frac{b_1}{b_3} \right)^{\frac{1}{2}} \qquad (3.13)$$

With vanishingly small values of Q the amplitude of the limit cycle is that of (3.13). Increasing values of Q, corresponding to moving along curve A to B in Fig. 4(a), causes a shift in the point about which the limit cycle oscillation occurs and a reduction in the amplitude.

The new origin of the limit cycle motion will be at the value of $x_{s.s.}$ given by stiffness considerations i.e. at solutions of

$$Q = c_1 x_{s.s.} + c_3 x^3_{s.s.}$$

and the amplitude will be

$$a_{LC.} = \left| \left( -\frac{1}{3}\frac{b_1}{b_3} \right)^{\frac{1}{2}} \right| - \left| x_{s.s.} \right| \tag{3.14}$$

When Q is of sufficient magnitude to cause $B'(x_{s.s.})$ to be zero, corresponding to B, then the amplitude of the limit cycle becomes zero and the transient response becomes very similar to that obtained if the singular point at $x_{s.s.}$, 0 were a stable node.

For larger values of Q the singular point at $x_{s.s.}$, 0 is a stable spiral, or eventually a node. These are similar to the cases of Fig. 3(a), but the nature of the transient motion will differ as a result of the negative damping experienced in the earlier portion of the motion.

$b_1<0, \; b_3>0, \; c_1>0, \; c_3<0$

In this case two possibilities exist, Fig. 4(b). With a relatively small amount of negative damping initially, increase or decrease of Q is associated with a limit cycle whose amplitude decreases in a similar way to the previous case. The nature of the singularity from B through C to E is then similar to the second case, Fig. 3(b), $b_3>0$. Alternatively the variation may follow the curve FG in which the singularities are always unstable.

$b_1<0, \; b_3>0, \; c_1<0, \; c_3>0$

The initial point F, Fig. 4(c), is a saddle point. Small values of Q cause the system to diverge, however, with increase of displacement the stiffness changes sign and the motion changes, during the transient, to a limit cycle whose origin is the value of $x_{s.s.}$ at the stiffness boundary and amplitude corresponds to the difference of $x_{s.s.}$, as given by stiffness considerations alone, at the points G and H. i.e. the origin of the limit cycle is given by

$$C'(x) = c_1 x_{s.s.} + c_3 x^2_{s.s.} = 0$$

or

$$x_{s.s.} = \left( \frac{-c_1}{c_3} \right)^{\frac{1}{2}} \tag{3.15}$$

and its amplitude is

$$a_{LC.} = \left| \left( -\frac{1}{3}\frac{b_1}{b_3} \right)^{\frac{1}{2}} \right| - \left| \left( -\frac{c_1}{c_3} \right)^{\frac{1}{2}} \right| \tag{3.16}$$

With increase of Q the amplitude of the limit cycle decreases in a similar way to that discussed in the fourth case, Fig. 4(a).

Alternatively, following the curve ABCE, the damping boundary may be reached prior to the stiffness boundary. In this case no limit cycle develops and the singular point variation is similar to that of the third case, Fig. 3(c), $b_3>0$.

These then are the six cases of engineering interest and describe the nature of the singular points at the steady-state values given by $C(x) = 0$. They are not in themselves sufficient to determine the stability of the transient motion.

Returning to equation (3.5a), it can be seen that the slope of the field vector in the phase plane depends on $x_{s.s.}$, $\xi$ and v. This means that if the system is initially at rest at a singularity $x_{s.s.}$, v = 0 in the phase plane, then the stability can be decided by an instantaneous change in the orientation of the singular points, corresponding to the changed values of Q. On this basis the initial position of rest becomes an ordinary point in the phase plane and the system will either move to a stable singularity, not necessarily the nearest one, diverge indefinitely or limit cycle. The transient motion of the system is described by the integral curve through this ordinary initial point and the stability or otherwise. This movement can only be completely determined by reference to the form of the integral curves.

It is convenient to use the curves of Fig. 10, where for the present purpose x and v are identified with W and $\dot{W}$ respectively, $x_A$ is the starting point, $x_E$ the position of the nearest stable singularity and $x_p$ the nearest saddle point. Taking the cases in turn

$b_1 > 0,\ c_1 > 0,\ c_3 > 0$

There is only one singular point for a given value of Q, fig. 10(a), and since this is a stable spiral or node all integral curves move into it.

$b_1 > 0,\ c_1 > 0,\ c_3 < 0$

A stable singularity can only exist if Q lies between the points of maxima and minima on the Q, x curve. For Q between these limits there are three singularities consisting of a stable spiral or node lying between two saddle points, Fig. 10(b). In order that the system will settle at the stable singularity the initial point $x_A$ must lie between the saddle points, i.e. the initial and final positions must not be separated by an integral curve which passes through a saddle.

$b_1 > 0,\ c_1 < 0,\ c_3 > 0$

When Q lies between the minima and maxima of the Q, x curve three singularities exist, these consisting of a saddle point lying between two stable spirals or nodes, Fig. 10(c). If A lies between a stable singularity and a saddle it will always settle at that stable singularity. When $x_A$ is numerically greater than the x co-ordinate of the stable singularity the system will continue to settle at this singularity until an initial displacement is reached at which A is separated from E by an integral curve passing through the saddle. The integral curve through A now moves, not into E, but to the stable singularity on the other side of the saddle.

The complete criteria for stability of the response of a step function are therefore

(1) The singular point at the steady state value given by C(x) = Q must be stable.

(2) On the phase plane diagram associated with the final steady state values, the final and initial positions of the system must not be separated by an integral curve passing through a saddle point.

## 4.0 A Numerical Example.

As a check on the validity and accuracy of the stability criteria obtained in Section 3.0, a limited number of solutions of the equation

$$x + bx + c_1 x + c_3 x^3 = Q, \quad Q = 0, \ t < 0$$

have been obtained on the Ferranti "Pegasus" digital computer at The College of Aeronautics. In these examples the damping was taken constant and positive, thereby excluding limit cycling from the solutions. The values of the coefficients used were as follows:

(a) (i) $b = 0.6,\ c_1 = 1.0,\ c_3 = 1.0,\ Q = 0$ to $2.1$

(ii) $b = 0.6,\ c_1 = 0.05,\ c_3 = 0.04,\ Q = 0$ to $0.6$

(b) $b = 0.6,\ c_1 = 1.0,\ c_3 = -1.0,\ Q = 0$ to $0.5$

(c) $b = 0.6,\ c_1 = 1.0,\ c_3 = 1.0,\ Q = 0$ to $1.0$.

The transient responses obtained are shown plotted in Figs. 5,6,7 and 8 in which the displacement co-ordinate has been normalized by division by the appropriate steady state value. Taking the cases in turn:

(a) (i)

Here the initial stiffness is positive and $\mathcal{D} < 0$ for all values of Q. As expected the settling down motion is oscillatory for all values of Q. The response curves tend to the linear result as $Q \to 0$, increase of Q produces an increase of frequency of the oscillation about the steady-state. It is also evident that the amount of "overshoot" and "undershoot" is dependent on Q. This feature of the curves is outside the scope of the present stability investigation and to analyse it, in terms of b, $c_1$, $c_3$ and Q, would require an analytic solution to the equation.

(a) (ii)

Again the initial stiffness is positive, but $\mathcal{D} > 0$. At large values of Q the discriminant $\mathcal{D}$ becomes negative, corresponding to the transient settling down motion becoming oscillatory. With increase of Q the motion becomes more damped until at $\mathcal{D} = 0$ boundary the transient becomes non-oscillatory in character. This occurs when $x_{s.s.}$ has the value given by (3.8), i.e.

$$x_{s.s.} = (\tfrac{1}{3})^{\frac{1}{2}} = 0.580$$

and

$$Q = 0.05 \times 0.580 + 0.04 \times 0.580^3 = 0.037.$$

(b)

With Q small, $\mathcal{D} < 0$, and the singularity at $x_{s.s.}$ is a stable spiral. Increase of Q corresponds to increase of $\mathcal{D}$, until $\mathcal{D} = 0$ when

$$x_{s.s.} = \left\{\frac{-1}{12 \times 1}\ (0.6^2 - 4 \times 1)\right\}^{\frac{1}{2}} = 0.551$$

and

$$Q = 0.551 - 0.551^3 = 0.384.$$

Further increase of Q causes a reduction in stiffness, the zero stiffness boundary being reached when

$$x_{s.s.} = (-\frac{1}{3} - \frac{1}{-1}\ )^{\frac{1}{2}} = 0.580$$

and

$$Q = 0.580 - 0.580^3 = 0.385$$

From this it can be seen that in terms of Q the $\mathcal{D}=0$ and $C'(x) = 0$ boundaries are very close together. The computer results, Fig. 7, indicate that instability occurs at the lower value $Q = 0.36 - 0.37$. This difference is probably due to instability in the Runge-Kutta iteration procedure used to perform the machine integration.

(c)

In this case the singularity at the origin is a saddle and any positive initial acceleration will cause the system to move to the stable spiral point at a positive value of $x_{s.s.}$ The magnitude of Q determines $x_{s.s.}$ and has a marked influence on the "rise time" and frequency of the settling down motion, Fig. 8. The results of Fig. 8 do not have very much engineering significance since if such a system were employed any transient would normally commence from a stable condition of equilibrium. If for instance the initial condition corresponded to a stable spiral point at a positive value of x, then positive step-functions of Q would produce response curves similar to Fig. 5. Small negative step-functions of Q would produce curve similar to Fig. 7. Large negative step-functions of Q would cause the system to jump to negative values of $x_{s.s.}$ and would presumably have a transients similar to that sketched in Fig. 9.

Summarizing, the computer solutions show that the character of the transient settling down motion is directly related to the nature of the singularity at the steady-state condition and is accurately predicted by the criteria of Section 3.0.

## 5.0 Stability of the Short-Period Motion of an Airframe Subject to a Step-Function Elevator Disturbance.

In Ref. 4 the author has shown how to introduce non-linear normal force and pitching moment characteristics into the equations of motion of an airframe whose dominant mode of oscillation is the "short-period" motion. For this purpose the characteristics are taken to be of anti-symmetric form and are expressed analytically as

$$\left.\begin{aligned} \frac{Z(w)}{m} &= z_w W + z_3 W^3 + z_5 W^5 + \ldots\ldots \\ &\text{and} \\ \frac{M(w)}{B} &= m_w W + m_3 W^3 + m_5 W^5 + \ldots\ldots \end{aligned}\right\} \quad (5.1)$$

Upon substituting these expressions into the equations of motion and eliminating $\dot{\vartheta}$ between them, the equation of motion in the vertical velocity, W, becomes

$$\ddot{W} - \left[(U_o + z_q)m_{\dot{w}} + m_q + z_w + 3z_3W^2 \ \ldots\right]\dot{W} - \left[(U_o + z_q)m_w - m_q z_w\right]W$$

$$- \left[(U_o + z_q)m_3 - m_q z_3\right]W^3 = \left[z_\eta D + (U_o + z_q)m_\eta - z_\eta m_q\right]H, \quad (5.2)$$

where

$$W = w_T + w, \quad (5.3)$$

$$H = \eta_T + \eta \quad (5.4)$$

and suffix $_T$ refers to initial trimmed conditions. In the present problem the increment in vertical velocity, w, arises from the application of a step-function disturbance of the elevator, $\eta$. Restricting (5.1) to two terms, then (5.2) becomes

$$\ddot{w} - \left[B_1 + B_3(w_T + w)^2\right]\dot{w} - \left[A_1(w_T + w) + A_3(w_T + w)^3\right]$$

$$= \left[z_\eta D + (U_o + z_q)m_\eta - z_\eta m_q\right](\eta_T + \eta), \qquad (5.5)$$

where

$$\left.\begin{aligned} A_1 &= (U_o + z_q)m_w - m_q z_w \\ A_3 &= (U_o + z_q)m_3 - m_q z_3 \\ B_1 &= (U_o + z_q)m_{\dot{w}} + m_q + z_w \\ B_3 &= 3z_3 \end{aligned}\right) \qquad (5.6)$$

It is now convenient to refer the w co-ordinate to the final steady-state value $w_{s.s.}$ Write

$$w = w_{s.s.} + \xi \quad , \qquad (5.7)$$

$$w_{s.s.} = w_T + w_{s.s.} \qquad (5.8)$$

and (5.5) becomes

$$\ddot{w} - \left[B_1 + B_3(W_{s.s.} + \xi)^2\right]\dot{w} - \left[A_1(W_{s.s.} + \xi) + A_3(W_{s.s.} + \xi)^3\right]$$

$$= \left[z_\eta D + (U_o + z_o)m_\eta - z_\eta m_q\right](\eta_T + \eta) \qquad (5.9)$$

The steady-state condition is defined by

$$-A_1 W_{s.s.} - A_3 W^3_{s.s.} = \left[(U_o + z_q)m_\eta - z_\eta m_q\right](\eta_T + \eta), \qquad (5.10)$$

which upon substitution into (5.9) gives

$$\ddot{w} - \left[B_1 + B_3(W_{s.s.} + \xi)^2\right]\dot{w} - \left\{A_1\xi + A_3\left[3W^2_{s.s.}\xi + 3W_{s.s.}\xi^2 + \xi^3\right]\right\}$$

$$= z_\eta \dot{\eta} \qquad (5.11)$$

Following Poincaré, an expression is sought for the slope of the field vector in the $\xi, \dot{\xi}$ plane in the region of the singularity $W_{s.s.}, 0$. For $t>0$, $\dot{\eta}$, which is a Dirac delta function, is zero, therefore in the first approximation (5.11) becomes

$$\ddot{\xi} - (B_1 + B_3 W^2_{s.s.})\dot{\xi} - (A_1 + 3A_3 W^2_{s.s.})\xi = 0 \qquad (5.12)$$

The problem described by (5.11) differs from that of Section 3 because of the term $z_\eta \dot{\eta}$, which gives rise to an initial velocity

$$|\dot{W}_{t=0}| = |z_\eta| , \qquad (5.13)$$

and can have an important influence on the system stability. (See Ref. 7 for a discussion on the determination of initial conditions). For many airframes with small elevators situated well away from the centre of gravity the value of $z_\eta$ is small and the effect of initial velocity may in many circumstances be neglected. For moving wing configurations the effect is dominant and requires special treatment. In the present discussion attention will be focussed on the former, whilst certain aspects of the effect of initial velocity on the stability will be discussed in Section 6.0

Neglecting then the initial velocity, the stability of the response to a step-function elevator deflection may be treated in a similar way to the problem of Section 3. The roots of the characteristic equation become

$$\lambda_{1,2} = \tfrac{1}{2}\left\{(B_1 + B_3W^2_{s.s.}) \pm\left[(B_1 + B_3W^2_{s.s.})^2 + 4(A_1 + 3A_3W^2_{s.s.})\right]^{\frac{1}{2}}\right\} \qquad (5.14)$$

Write the equation of trim, (5.10), in the form

$$F = \left[(U_o + z_q)m_\eta - z_\eta m_q\right](\eta_T + \eta) = -A_1W_{s.s.} - A_3W^3_{s.s.};$$

identify F, W, $-A_1$, $-A_3$, $-B_1$ and $-B_3$ with Q, x, $c_1$, $c_3$, $b_1$ and $3b_3$ respectively in Section 3 and the discussion of airframe stability can then proceed on the basis of Figs. 1 and 3.

For a normal airframe (aeroplane or missile) $B_1$ is negative, thereby excluding the possibility of a change from negative to positive damping and the associated limit cycling. The cases to be considered are therefore similar to the first three discussed in 3.0. Taking these in turn:

$B_1<0$, $A_1<0$, $A_3<0$

The dominant term in $A_1$ is $U_0m_W$. Since $m_W$ is proportional to the centre of gravity margin then $A_1<0$ implies that the airframe is statically stable at low incidence. The term $U_0m_3$ is dominant in $A_3$ and with $A_3<0$ the airframe increases its static stability with increase of incidence. Alternatively this may be described by saying that the aerodynamic stiffness increases with incidence, thereby constituting a "hard" system.

$B_3$ can be of either sign. Many airframes having wings of low aspect-ratio of axi-symmetric body configurations exhibit W, $C_Z$ characteristics whose slope increases over the whole of the useful incidence range, corresponding to $z_3$ and $B_3$ being negative. Others have wings of higher aspect-ratio which stall at relatively small incidences, an effect which can be represented approximately by taking $z_3>0$. The approximation involved is satisfactory provided the transient motion does not cause the incidence to increase very much above the stall. When oscillating through the stall it is fairly certain that aerodynamic hysteresis will occur, (i.e. the W, $C_Z$ curve followed during the nose-up swing of the airframe will not be re-traced during the subsequent nose-down swing) and the form used in (5.1) to represent the forces and moments will be inadequate.

The point A, in Fig. 3(a), now corresponds to the origin of the F, W curve and the problem resolves itself into deciding the stability of transients when moving from any typical singular point, corresponding to the final trimmed condition, along curves such as AB or AC. With $B_3<0$ the character of the settling down motion can change from oscillatory to heavily damped and the boundary between these conditions is given by

$$(B_1 + B_3W^2_{s.s.})^2 + 4(A_1 + 3A_3W^2_{s.s.}) = 0$$

or

$$W_{s.s.} = \left\{\frac{1}{B_3^2}\left[-6A_3 - B_1B_3 \pm 6\left[A_3^2 + \tfrac{1}{3}A_3B_1B_3 - \frac{1}{9}A_1B_3^2\right]^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \qquad (5.15)$$

With $B_3 > 0$ the curve AC will meet the damping boundary when

$$-B_1 - B_3 W^2_{s.s.} = 0,$$

or

$$W_{s.s.} = \left(-\frac{B_1}{B_3}\right)^{\frac{1}{2}} = \left[\frac{(U_o + z_q)m_{\dot{w}} + m_q + z_w}{-3z_3}\right]^{\frac{1}{2}} \tag{5.16}$$

Now from (5.1)

$$\frac{Z(w)}{m} = z_w W + z_3 W^3$$

which upon differentiation with respect to W gives

$$\frac{1}{m}\frac{dZ(w)}{dW} = z_w + 3z_3 W^2 .$$

The stall, which occurs at symmetrical values about zero incidence, is defined by

$$\frac{dZ(w)}{dW} = 0,$$

or

$$W_{stall} = \left(-\frac{z_w}{3z_3}\right)^{\frac{1}{2}} \tag{5.17}$$

The values of $z_w$, $m_q$ and $m_{\dot{w}}$ will normally be negative and for this case $z_3$ will have to be positive. It follows that the steady-state incidence for which the damping becomes zero is numerically greater than the stalling incidence by an amount

$$\frac{1}{U_\infty}\left\{\left|\left(\frac{-B_1}{B_3}\right)^{\frac{1}{2}}\right| - \left|\left(\frac{z_w}{3z_3}\right)^{\frac{1}{2}}\right|\right\},$$

where the incidence has been taken to be $W/U_\infty$. This result must, of course, be treated with some reservation since it is unlikely that the "damping in pitch" term

$$(U_o + z_q)m_{\dot{w}} + m_q$$

will be constant for an oscillation which includes the stall. However, it is equally unlikely that this term will change sign and therefore it may be concluded that the previous statement regarding the damping boundary is qualitatively true, but leaves uncertainty as to the amount the damping boundary exceeds the stalling incidence.

<u>$B_1 < 0$, $A_1 < 0$, $A_3 > 0$</u>

Again the airframe is statically stable at low incidence, but now the static stability decreases with incidence, thereby constituting a "soft" system. A notable example of such an airframe is the canard missile configuration which experiences considerable non-linear body lift and, as a result of the centre of pressure of the non-linear body lift being ahead of the centre of gravity, develops a nose-up pitching moment. This configuration would normally have $A_3 > 0$ associated with $B_3 < 0$.

This case will have boundaries similar to those expressed by (5.15) and (5.16), but in addition will have a zero stiffness boundary given by

$$-A_1 - 3A_3 W^2_{s.s.} = 0, \; W_{s.s.} \neq 0,$$

or

$$W_{s.s.} = \left(-\tfrac{1}{3}.\frac{A_1}{A_3}\right)^{\frac{1}{2}} = \left[-\tfrac{1}{3}.\frac{(U_o + z_q)m_w - m_q z_w}{(U_o + z_q)m_3 - m_q z_3}\right]^{\frac{1}{2}} \tag{5.18}$$

From (5.1)

$$\frac{M(w)}{m} = m_w W + m_3 W^3$$

and the symmetrically disposed points of maxima and minima on the pitching moment versus incidence curve will occur at

$$W_M = \left(-\tfrac{1}{3}.\frac{m_w}{m}\right)^{\frac{1}{2}} \tag{5.19}$$

Now $z_w$ and $m_q$ will normally be negative and it follows that if $z_3<0$ then the steady-state incidence for which the stiffness becomes zero is numerically greater than $\frac{W_M}{U}$. When $z_3>0$, the zero stiffness boundary may be at small or greater values of incidence than $\frac{W_M}{U}$ depending on the relative magnitude of $z_w$ and $z_3$.

<u>$B_1<0$, $A_1>0$, $A_3<0$</u>

Here the airframe is statically unstable at low incidence. With $F = 0$ the airframe will trim out at values of $W_{s.s.}$, given by

$$F = -A_1 W_{s.s.} - A_3 W^3_{s.s.} = 0, \; W_{s.s.} \neq 0,$$

or

$$W_{s.s.} = \left(\frac{-A_1}{A_3}\right)^{\frac{1}{2}} \tag{5.20}$$

and the corresponding value of the stiffness will be

$$-A_1 - 3A_3\left(\frac{-A_1}{A_3}\right) = -2A_1. \tag{5.21}$$

The line $C'(x) = -2c_1$, in Fig. 3(c), is now interpreted as a line of constant stiffness $-2A_1$. Increasing values of F now cause the singularity to move along EF or HJ. If F is negative then the movement is along EC or HG, becoming unstable at the stiffness boundary as given by (5.18). Since $m_w>0$ and $m_3<0$, the value of W corresponding to the stiffness boundary can be either greater or smaller than $W_M$ and in particular when $z_w<0$, $z_3<0$ then W at zero stiffness will be $>W_M$.

The instability at the stiffness boundary consists of a jump to the remaining stable singularity, corresponding to a change in sign and increase of magnitude of the trimmed incidence. Obviously this sort of behaviour is out of the question for an aeroplane, but possible on a missile having no automatic control system. As indicated in Ref. 8, when dealing with a rear-controlled missile a useful increase in aerodynamic gain, $(W/\eta)_{s.s.}$, can be obtained by making the airframe statically unstable at small incidence. This will be offset by having a region around $W = 0$ for which a stable trim condition cannot be obtained, at least not without the use of a closed-loop control system.

It was shown in Section 3.0 that in addition to having a stable singularity at $x_{s.s.}$, the conditions for complete stability imposed limitations on the initial and final values of trim, i.e. limited the magnitude of the step function imposed. This condition has not been included in the present section since it will be seen to constitute a special case of the criteria developed in Section 6.0.

## 6.0 Stability of Short-Period Motion When $z_\eta$ is Not Small.

As indicated in Section 5.0, when $z_\eta$ is not small, the initial velocity $|\dot{W}_{t=0}| = |z_\eta|$ can under certain circumstances have an important influence on the stability. Again the stability is treated by assuming an instantaneous change in the orientation of the singular points, corresponding to a step-function change in F. When $z_\eta$ is not small there will be an appreciable value of $\dot{W}_{t=0}$ and thereby the initial position of rest is transformed to an ordinary point in the phase-plane with a known displacement from the new singularities and an initial velocity $\dot{W}_{t=0}$. Consider each case in turn:

$B_1<0,\ A_1<0,\ A_3<0$

For each value of F there exists only one singularity, which will be stable if the damping is positive. In this case the system will always settle to the singularity regardless of the sign of $\dot{W}_{t=0}$, although the settling time will be less for negative values of $\dot{W}$, shown as $\dot{W}_1$ in Fig. 10(a).

$B_1<0,\ A_1<0, A_3>0$

For small positive values of F three singular points exist, Fig. 10(b), one stable spiral at positive W, one saddle at larger positive W and another saddle at negative W. Now when $z_\eta$ and hence $\dot{W}_1$ or $\dot{W}_2$ are zero then a sufficient condition for stability is that the dashed line, representing the initial value of W, shall lie between the saddle points, bearing in mind that the orientation changes with F. Obviously when $\dot{W}_1$ and $\dot{W}_2$ are small the same criterion is approximately true. With increasing values of $\dot{W}_{t=0}$ a point is reached at which the integral curves no longer spiral in to the stable singularity. When $\dot{W}_1$ is numerically greater than a critical value, $(\dot{W}_1)_C$, which lies on the integral curve passing through the saddle point at negative W, then the motion diverges indefinitely in the negative sense. A similar divergence occurs if $\dot{W}_2>(\dot{W}_2)_C$.

The interesting problem here is to determine the critical values of $\dot{W}_{t=0}$ or the corresponding value of $z_\eta$ for a given step change of F or $\eta$. Alternatively if $z_\eta$ is fixed the problem is to determine the magnitude of the step change in F or $\eta$ for which $\dot{W}_{t=0}$ reaches a critical value. As will be seen from Ref. 2, pp.61-80, this problem is very similar to that of determining the critical disturbance of a damped pendulum or the "pull-out torque" of a synchronous motor.

Assuming that the airframe is trimmed at a small positive incidence and $z_\eta$ is negative (this is always true for the convention adopted), then for a rear controlled missile an increase of incidence is obtained by making $\eta$ more negative and the associated value of $\dot{W}_{t=0} = -z_\eta$ will be positive. For a canard or moving wing arrangement an increase of incidence is achieved by making $\eta$ more positive and therefore $\dot{W}_{t=0} = -z_\eta$ will be negative.

Consider the case of a canard airframe, for which the conditions $A_1<0$, $A_3>0$ are characteristic (a feature arising from the centre of gravity being behind the centre of pressure of the non-linear body lift) and assume that it is trimmed at a positive incidence corresponding to the point A on Fig. 10(b) and below the maximum on the W,F curve. A negative step-function of $\eta$ is applied to the elevator and the new trim value will be at the centre, E, of the stable spiral shown. However, in this case $\dot{W}_{t=0} = -z_\eta$ is positive and will oppose the motion toward the stable condition. If $\dot{W}_{t=0} < \dot{W}_{A_C}$ the critical value, the response will finally settle at the stable singularity, whereas if $\dot{W}_{t=0} > \dot{W}_{A_C}$ the airframe will become unstable.

The problem then is to develop an expression describing the integral curves through the unstable singularity P with the intention of using the expression to determine $\dot{W}_{A_C}$. If $\xi$ is the W co-ordinate relative to P, then

$$W = W_p + \xi$$

and the integral curves in the region of $W_p$ are described by

$$\ddot{\xi} - [B_1 + B_3(W_p + \xi)^2]\dot{\xi} - \left\{A_1\xi + A_3[3W_p^2.\xi + 3W_p\xi + \xi^3]\right\} = 0 \qquad (6.1)$$

Now $\ddot{\xi} = \dot{\xi}\dfrac{d\dot{\xi}}{d\xi}$ and (6.1) may be re-written in the form

$$\dot{\xi}\left\{\frac{d\dot{\xi}}{d\xi} - [B_1 + B_3(W_p + \xi)^2]\right\} = (A_1 + 3A_3W_p^2)\,\xi + 3A_3W_p\xi^2 + \xi^3 \qquad (6.2)$$

At $\xi = 0$, $\dot{W} = \dot{\xi} = 0$ and $\dot{\xi}$ may be developed as a power series in the form

$$\dot{\xi} = a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots\ldots\ldots, \qquad (6.3)$$

which upon differentiation with respect to $\xi$, gives

$$\frac{d\dot{\xi}}{d\xi} = a_1 + 2a_2\xi + 2a_2\xi + 3a_3\xi^2 + \ldots\ldots\ldots, \qquad (6.4)$$

Substitution from (6.3) and (6.4) into (6.2) then gives

$$(a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots\ldots.)\,[(a_1 - B_1 - B_3W_p^2) + 2(a_2 - B_3W_p)\xi$$

$$+ (3a_3 - B_3)\xi^2 + \ldots\ldots.] = (A_1 + 3A_3W_p^2)\xi + 3A_3W_p\xi^2 + A_3\xi^3 \qquad (6.5)$$

Since (6.5) is to be true for all values of $\xi$, then the coefficients of like powers of $\xi$ may be equated giving the indicial equations:

1) $$a_1^2 - (B_1 + B_3W_p^2)a_1 - (A_1 + 3A_3W_p^2) = 0$$

or

$$a_1 = \tfrac{1}{2}\left\{(B_1 + B_3W_p^2) \pm [(B_1 + B_3W_p^2)^2 + 4(A_1 + 3A_3W_p^2)]^{\frac{1}{2}}\right\} \qquad (6.6)$$

2) $$a_1(2a_2 - 2B_3W_p) + a_2(a_1 - B_1 - B_3W_p^2) = 3A_3W_p$$

or

$$a_2 = \frac{(3a_3 + 2a_1B_3)W_p}{3a_1 - B_1 - B_3W_p^2} \qquad (6.7)$$

3) $$a_1(3a_3 - B_3) + a_2(2a_2 - 2B_3W_p) + a_3(a_1 - B_1 - B_3W_p^2) = A_3$$

or

$$a_3 = \frac{A_3 - 2a_2(a_2 - B_3W_p) + a_1B_3}{4a_1 - B_1 - B_3W_p^2} \qquad (6.8)$$

The value of $\dot{W}_{A_C}$ will then be

$$\dot{W}_{A_C} = a_1(w_{T_A} - W_p) + a_2(w_{T_A} - W_p)^2 + a_3(w_{T_A} - W_p)^3 \qquad (6.9)$$

From (6.6), $a_1$ has two values. Since P is a saddle point then $(A_1 + 3A_3W_p^2) > 0$ and the values of $a_1$ must be real, positive and negative respectively. These two values correspond to the initial slopes of the two integral curves passing through the singularity. In the present problem the negative value of $a_1$ is appropriate.

With known aerodynamic characteristics, $A_1$, $A_3$, ... etc., it is now possible to determine the critical value of $\dot{W}_{t=0}$ or $z_\eta$ for a range of various initial and final trimmed incidences.

$B_1<0,\ A_1>0,\ A_3<0$

When flying trimmed in the lower range of positive incidence three singular points exists, Fig. 10(c). This case would be typical of the rear controlled airframe if it were statically unstable at low incidence.

A positive step-function of $\eta$ is applied in order to reduce the trimmed incidence to a point corresponding to E. When $z_\eta = 0$, it is a sufficient condition for the motion to settle at E, that the starting point A shall not lie outside the integral curve passing through the saddle point complementary to the stable spiral at E. This means that if the magnitude of the step-function is too great the motion will not settle at E, but will pass over to the other stable spiral at negative incidence. The limiting value can only be obtained by constructing the integral curve through the unstable singularity at P.

When $z_\eta \neq 0$, the value of $\dot{W}_{t=0} = z_\eta$, which is negative, and if this value is numerically greater than $\dot{W}_{A_C}$ the airframe will not trim at E but will depart to the other stable singularity.

From these three cases it can be seen that the term $z_\eta$ does not alter the stability criteria of Section 3.0, provided these are interpreted in a slightly more general manner. As before the final steady state condition must be a stable singularity, and the initial and final points on the phase plane must not be separated by an integral curve passing through a saddle point which is complementary to the stable singularity at the steady state value. Whereas previously the initial value was at a point $x_A$, $v = 0$, when $z_\eta \neq 0$ the initial value will be $x_A$, $v \neq 0$. The cases described in Section 3.0 and 5.0 are special cases for which $v = 0$.

A Numerical Example

In order to demonstrate the calculation of $\dot{W}_{A_C}$ and show its influence on the effective stability boundaries, an example has been chosen of a rear controlled missile conforming to the conditions $B_1<0$, $B_3<0$, $A_1>0$, $A_3<0$. The missile is a cruciform, air-to-air type having a useful speed range of 1,500 to 3,500 f.p.s. Its operation and aerodynamic characteristics are given in detail in Ref. 8.

Taking a flight altitude of 60,000 feet at a speed of 2,000 f.p.s. and a centre of gravity position $x_{c.g.} = 0.5$ feet, the aerodynamic derivatives become

$z_w = -0.227$ sec.$^{-1}$, $z_3 = -1.33 \times 10^{-7}$ft.$^{-2}$ sec.,

$m_w = 0.00566$ ft.$^{-1}$sec.$^{-1}$, $m_3 = -0.354 \times 10^{-7}$ft.$^{-3}$ sec.,

$z_\eta = -86.7$ ft. sec.$^{-2}$, $m_\eta = -37.91$ sec.$^{-2}$,

$z_q = -0.259$ ft. sec.$^{-1}$, $m_q = -0.1134$ sec.$^{-1}$,

$m_{\dot{w}} = -0.0340 \times 10^{-3}$ ft.$^{-1}$,

giving

$A_1 = 11.30$ sec.$^{-2}$, $A_3 = -0.708 \times 10^{-4}$ ft.$^{-2}$,

$B_1 = -0.408$ sec.$^{-1}$, $B_3 = -3.399 \times 10^{-7}$ ft.$^{-2}$ sec.

The equation of trim is

$$-2{,}000\, A_1 \alpha_T - 2{,}000^3 A_3 \alpha_T^3 = \left[(2{,}000 + z_q)\, m_\eta - z_\eta m_q\right] \eta_T ,$$

or

$$-22.60 \times 10^3 \alpha_T + 5.664 \times 10^5 \alpha_T^3 = -75{,}829 \eta_T, \tag{6.10}$$

where $\alpha_T$ and $\eta_T$ are in radians and $\alpha_T = {}^{w_T}/_U$ . The trim curve is shown in Fig. 11.

When the airframe is statically stable at low incidence (corresponding, in this airframe, to $x_{c.g.} < 0.3$ feet) the values of $\eta_T$ will all be negative. In the present case, with static instability at low incidence, a region exists between M and N, Fig. 11, where a stable trimmed positive incidence is achieved with $\eta_T$ positive. The minimum positive value of $\alpha_T$ for which a stable trimmed condition can exist will be given by

$$(\alpha_T)_M = \left(\frac{22.60 \times 10^3}{3 \times 5.664 \times 10^5}\right)^{\frac{1}{2}} = 0.1154 \text{ radian},$$

where only the positive value is relevant in the present problem. The corresponding value of $\eta_T$ is $+0.0229$ radian. When $\alpha_T$ is greater than that corresponding to the point S, Fig. 11, only one stable trimmed condition can exist, the limit being given by the maximum positive root of

$$-22.60 \times 10^3 \alpha_T + 5.664 \times 10^5 \alpha_T^3 = -75{,}829 \times (-0.0229)$$

or

$$(\alpha_T)_S = 0.232 \text{ radian}.$$

Let E be the point corresponding to the final trim condition; then if it lies between M and S there will always be a complementary saddle singularity at the point P. This implies that a critical value of $\dot{W}_{t=0}$ or $\dot{\alpha}_{t=0} = (\dot{W}_{t=0})/U$ will exist only if E lies between M and S. A range of values of $(\alpha_T)_E$ , between $(\alpha_T)_M$ and $(\alpha_T)_S$ , can now be selected and the associated values of $(\eta_T)_{E,P}$ , $(\alpha_T)_P = {}^{W_P}/_U$ , $a_1$, $a_2$ and $a_3$ calculated. If now a range of values of the initial trimmed incidence, $(\alpha_T)_A$ , is associated with each value of $(\alpha_T)_E$, then the critical integral curves, i.e. the integral curves through $(\alpha_T)_p$ , may be evaluated and are shown plotted in Fig. 12 in the form $\dot{W}_{A_C}$ versus $\alpha_T$.

Taking first cases in which $(\alpha_T)_A < (\alpha_T)_E$ , for which the minimum positive value is 0.1154 radian and $\dot{W}_{A_C} = -z_\eta = 86.7$ ft. sec.$^{-2}$ . It can be seen from Fig. 12 that the critical integral curve can never separate the initial and final positions, implying that the airframe is stable when subjected to negative step functions of $\eta$ of any magnitude within the limits of the elevator mechanical stops.

When $(\alpha_T)_A < (\alpha_T)_E$, $\dot{W}_{A_C} = z_\eta = 86.7$ ft. sec.$^{-2}$ and values of $(\alpha_T)_A$ may be reached in which the system will not trim finally at $(\alpha_T)_E$. The critical values of $(\alpha_T)_A$ correspond to the points of intersection of the line $\dot{W}_{A_C} = z_\eta$ and the critical integral curves, only the higher of the two values given being relevant. These values have been plotted in Fig. 13. For a given value of initial trimmed incidence the airframe can only be re-trimmed, by means of a single step function elevator deflection, to final values of incidence greater than that of the boundary shown. If the elevator step deflection is greater than the value of the $\Delta\eta$ boundary, with the intention of re-trimming at $\alpha_T$ less than the incidence boundary, the airframe will shoot past the desired trim position and trim out at negative incidence.

The minimum value of $(\alpha_T)_E$ for which any re-trimming, by means of elevator step deflections, is possible at all corresponds to the condition when the minimum on a critical integral curve just touches the line $\dot{W}_{A_C} = z_\eta$. This value is $(\alpha_T)_E = 0.13$ radian and corresponds to a limiting value of $(\alpha_T)_A = 0.14$ radian. The difference between these values arises from the initial value of $\dot{W}$ produced by $z_\eta$; when $z_\eta = 0$ the minimum values of $(\alpha_T)_E = (\alpha_T)_A = 0.1154$ radian.

## 7.0 Conclusions

The conclusions which may be drawn from this study are as follows:

(1) That the stability of the equation (1.1) with B(x) and C(x) as anti-symmetric functions is determined by the nature of the singularity at the steady-state value given by C(x) = Q, with the additional condition that on the phase plane diagram associated with the final steady state values, the initial and final points must not be separated by an integral curve passing through a saddle point. Further, the character of transient settling down motion is directly related to the nature of the singularity at the steady state (e.g. the settling down is oscillatory if the singularity is a stable spiral) and, together with the stability, is accurately predicted by the criteria of Section 3.0.

(2) The short-period motion of an airframe subject to a step-function elevator deflection is governed by an equation similar to (1.1), but differing as a result of the term $z_\eta . \eta$ on the right hand side. When $z_\eta$ is small, which it usually is for rear controlled and canard arrangements, it is found that the stability criteria for (1.1) are applicable to the airframe problem. When $z_\eta$ is not small special treatment is required in order to determine the critical value of the elevator step-function which will cause instability. It is found that the critical value may fairly readily be obtained from quantities taken from the elevator trim curves.

## Acknowledgement

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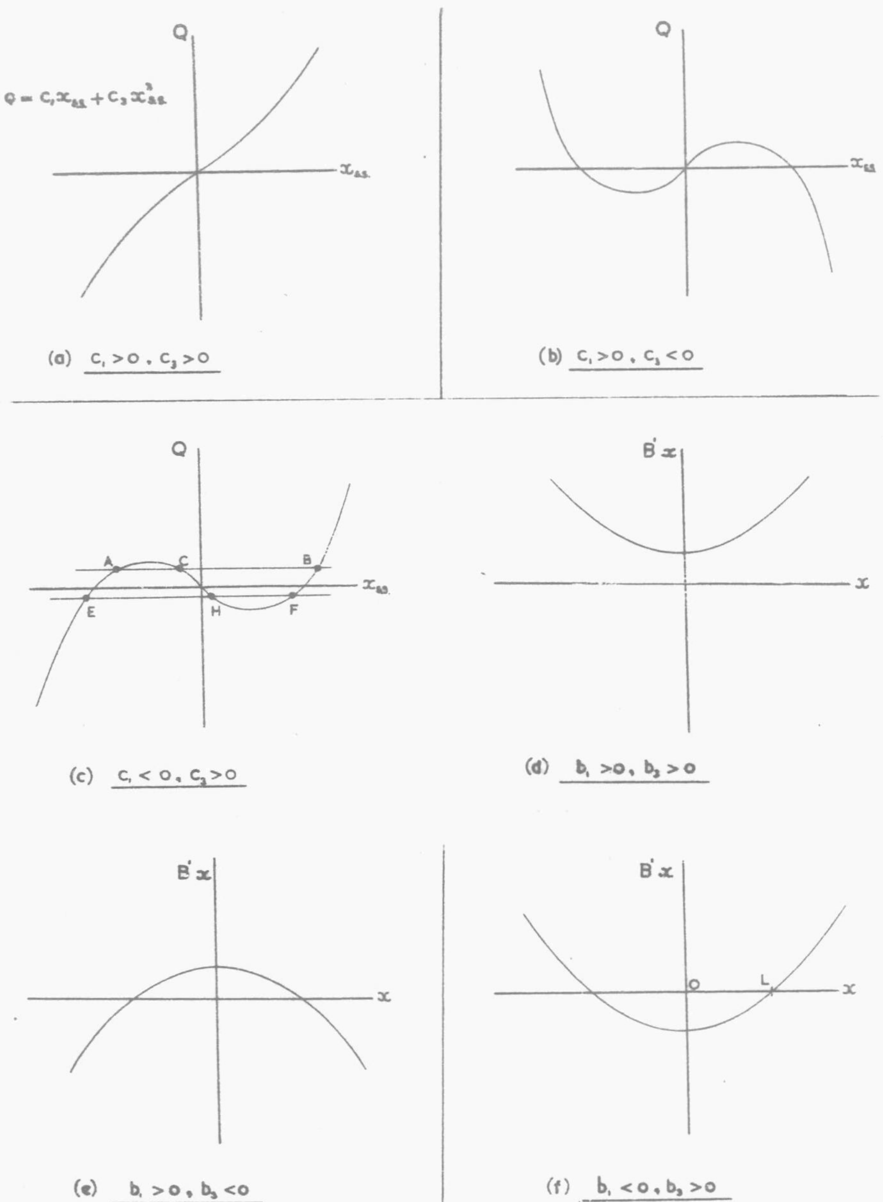



FIG. 1 VARIATION OF Q WITH $x_{s.s.}$ and $B'(x)$ with x

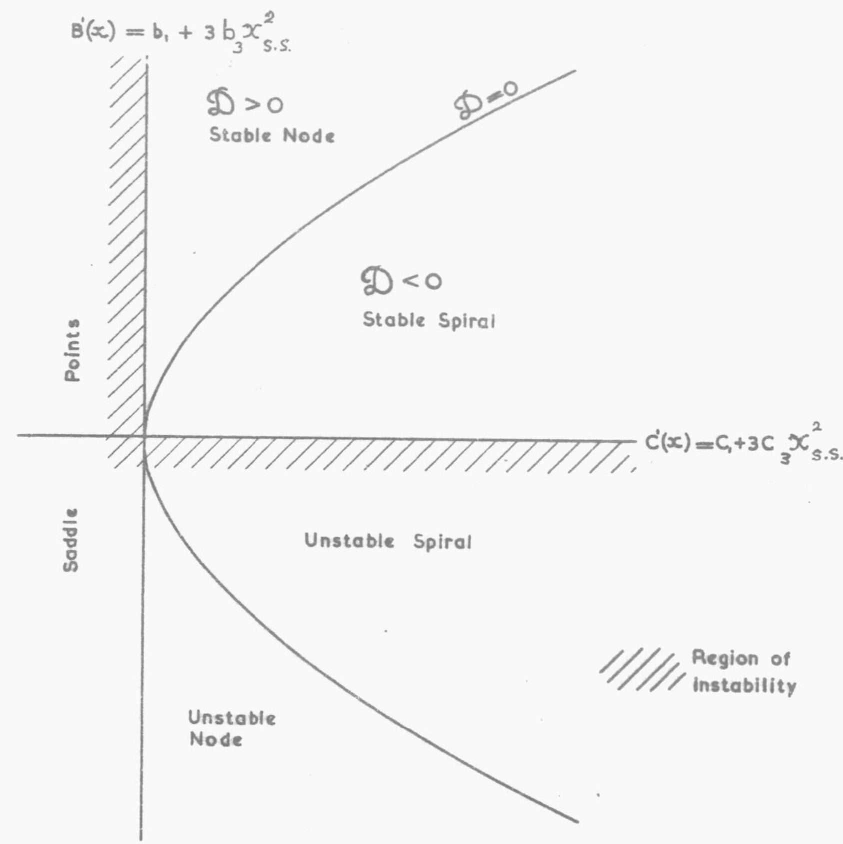



FIG. 2 STABLE AND UNSTABLE SINGULARITIES DEFINED BY EQUATION (3.6)

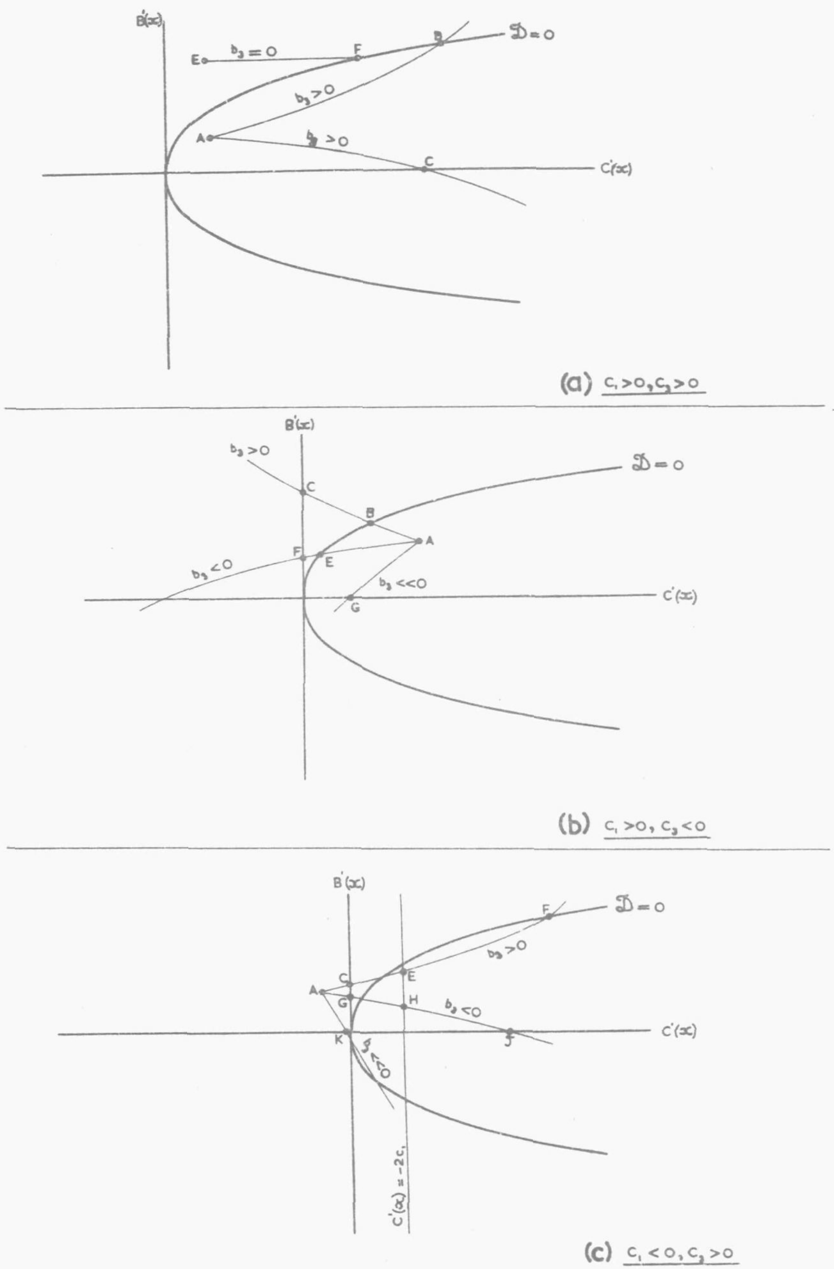



FIG. 3 SINGULAR POINTS DESCRIBED BY EQUATION (3.6) $b_1 > 0$

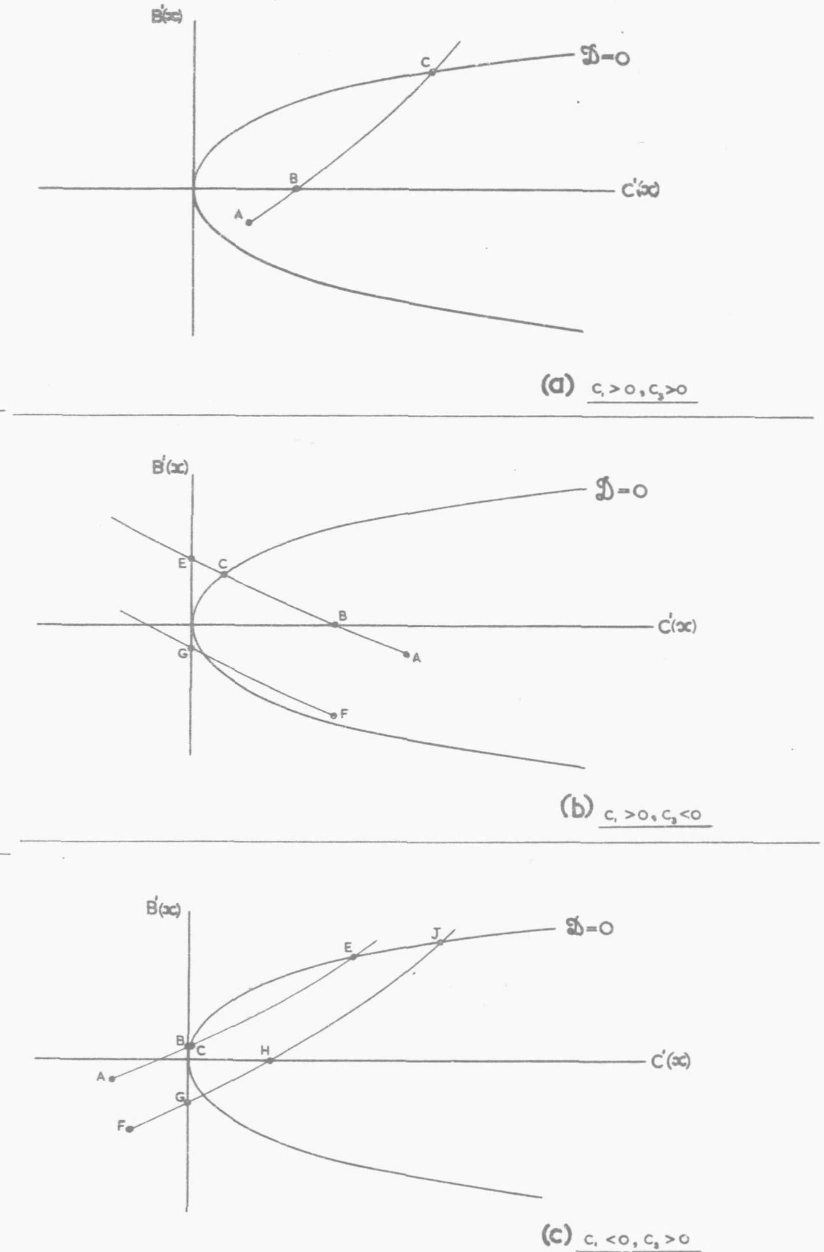
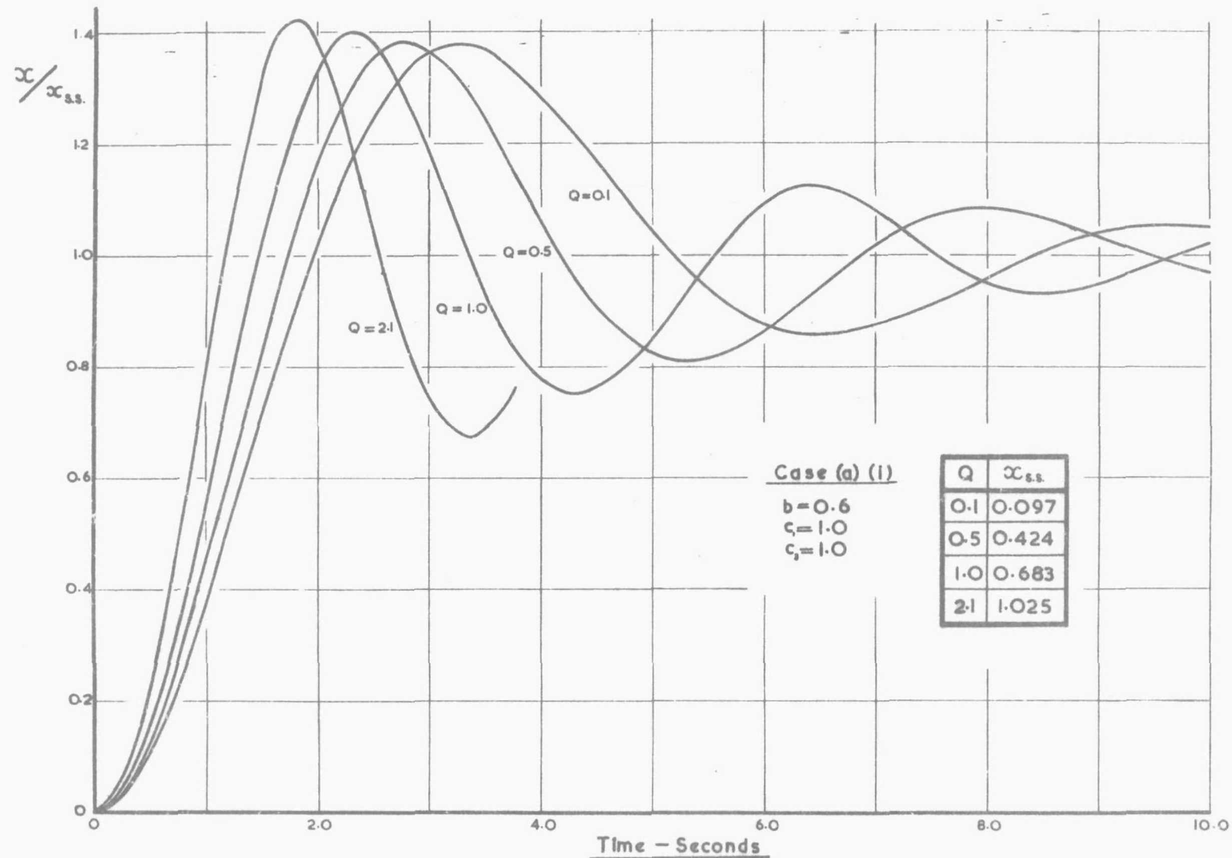
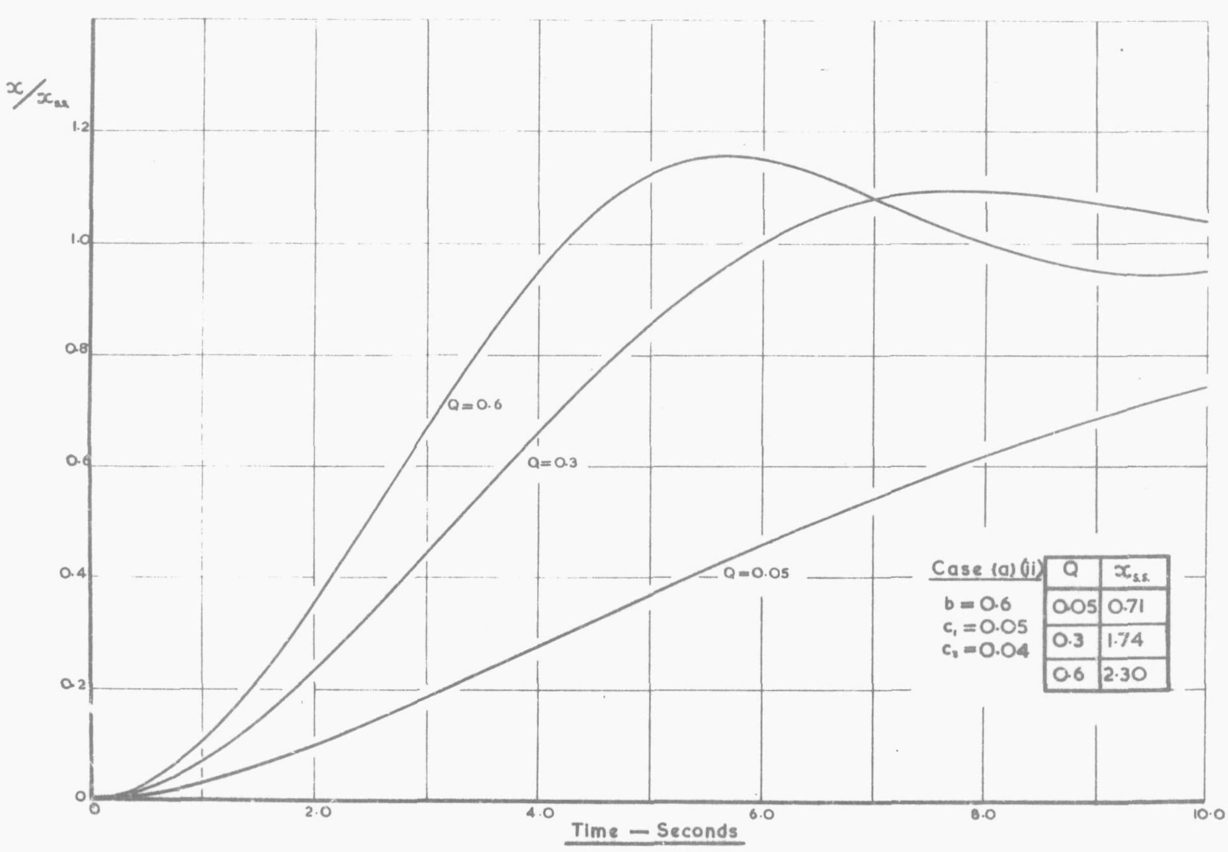



FIG. 4 SINGULAR POINTS DESCRIBED BY EQUATION (3.6), $b_1 < 0$ $b_3 > 0$



$x/x_{s.s.}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2.0
4.0
6.0
8.0
10.0
Time — Seconds
Q = 0.1
Q = 0.5
Q = 1.0
Q = 2.1
Case (a) (i)
b = 0.6
$c_1$ = 1.0
$c_2$ = 1.0
Q | $x_{s.s.}$
0.1 | 0.097
0.5 | 0.424
1.0 | 0.683
2.1 | 1.025


FIG. 5



$x/x_{s.s.}$
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2.0
4.0
6.0
8.0
10.0
Time — Seconds
Q = 0.6
Q = 0.3
Q = 0.05
Case (a) (ii)
b = 0.6
$c_1$ = 0.05
$c_2$ = 0.04
Q | $x_{s.s.}$
0.05 | 0.71
0.3 | 1.74
0.6 | 2.30


FIG. 6

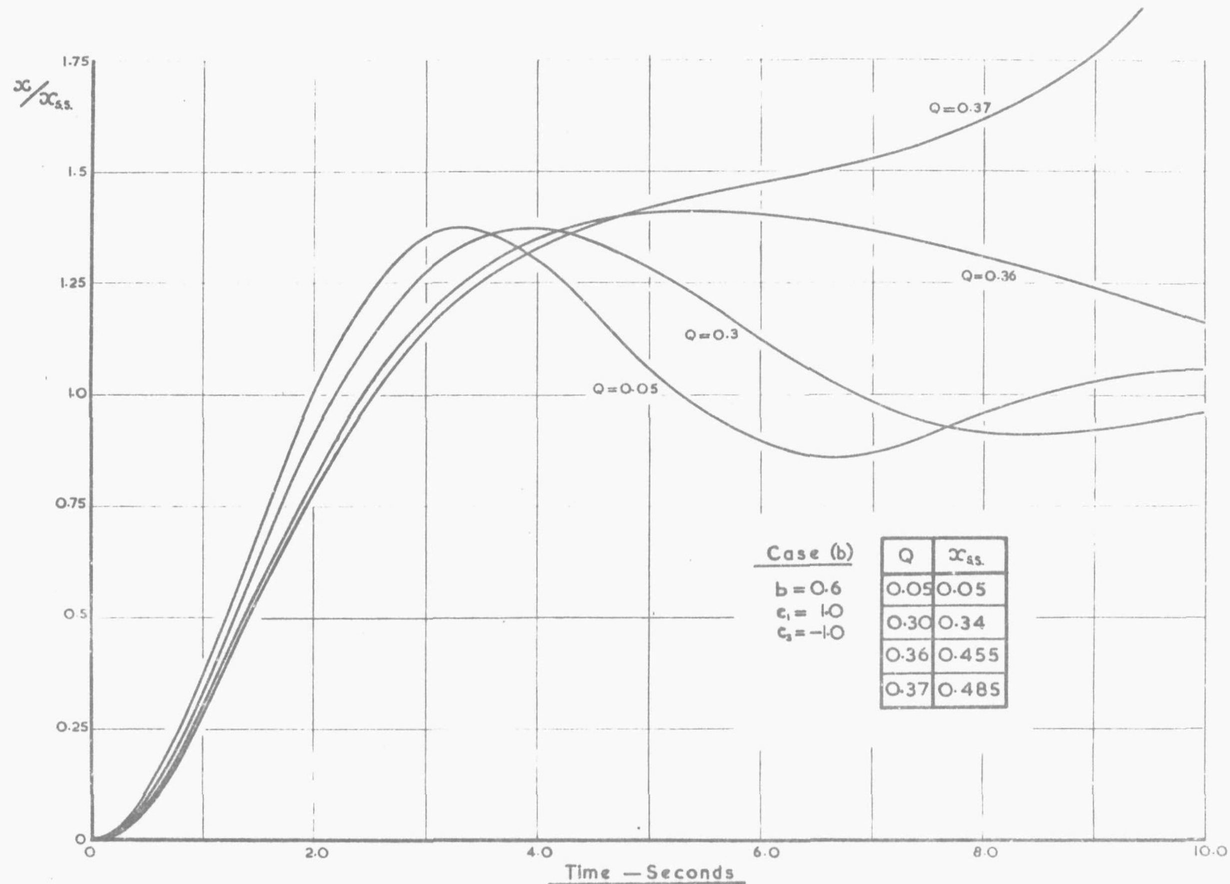

$x/x_{s.s.}$
Q=0.37
Q=0.36
Q=0.3
Q=0.05
Case (b)
b=0.6
$c_1$= 1.0
$c_3$= -1.0
Q | $x_{s.s.}$
0.05 | 0.05
0.30 | 0.34
0.36 | 0.455
0.37 | 0.485
Time — Seconds

FIG. 7

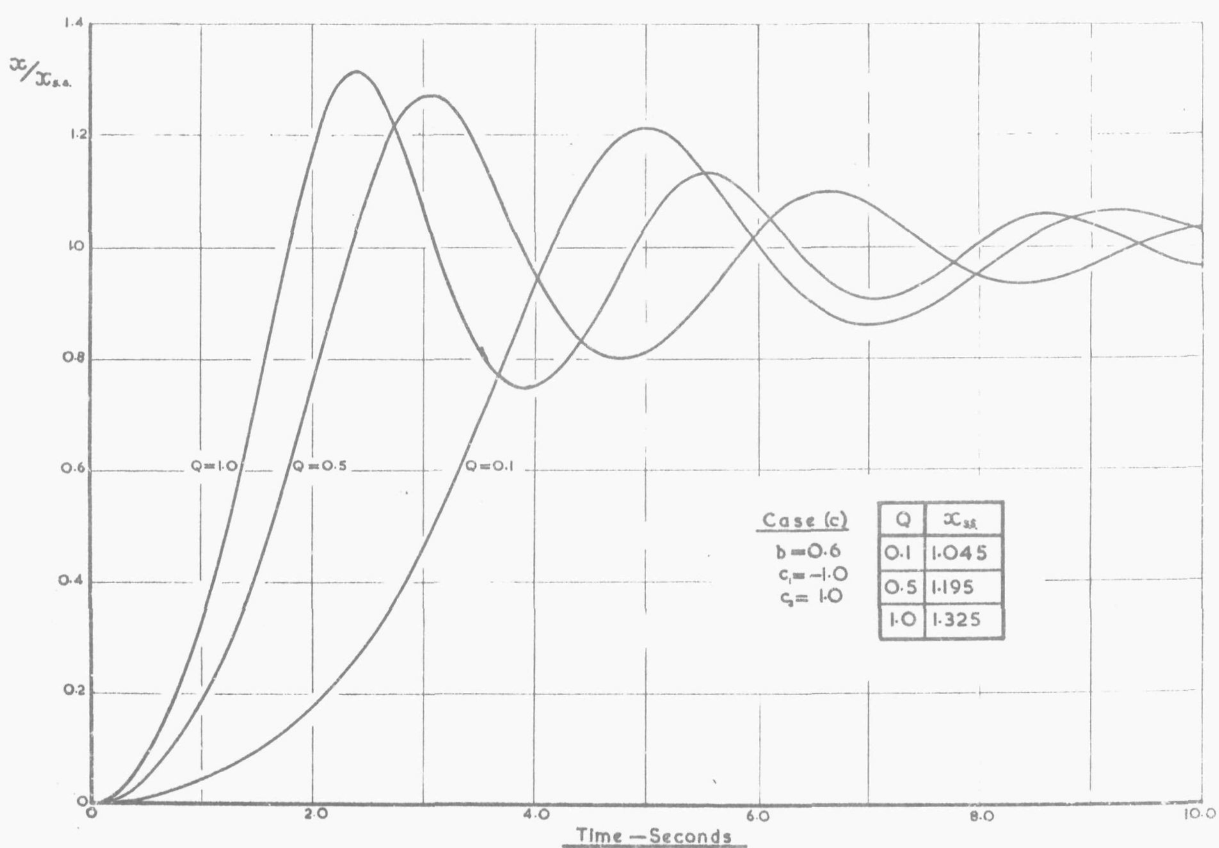

$x/x_{s.s.}$
Q=1.0
Q=0.5
Q=0.1
Case (c)
b=0.6
$c_1$=-1.0
$c_3$= 1.0
Q | $x_{s.s.}$
0.1 | 1.045
0.5 | 1.195
1.0 | 1.325
Time — Seconds

FIG. 8

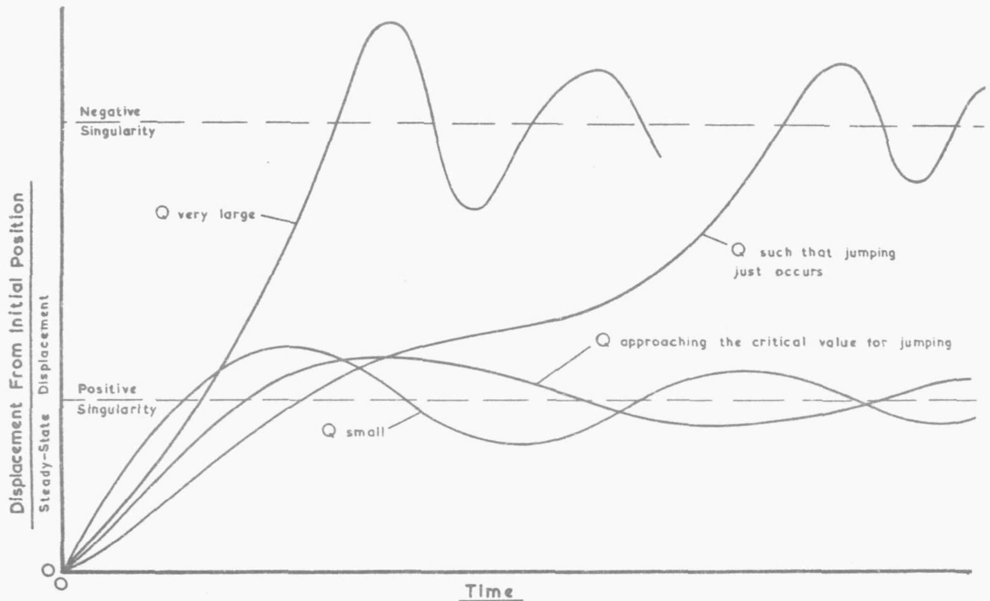



FIG. 9 SKETCH OF TRANSIENTS DESCRIBED IN CASE (C) SECTION 4.0

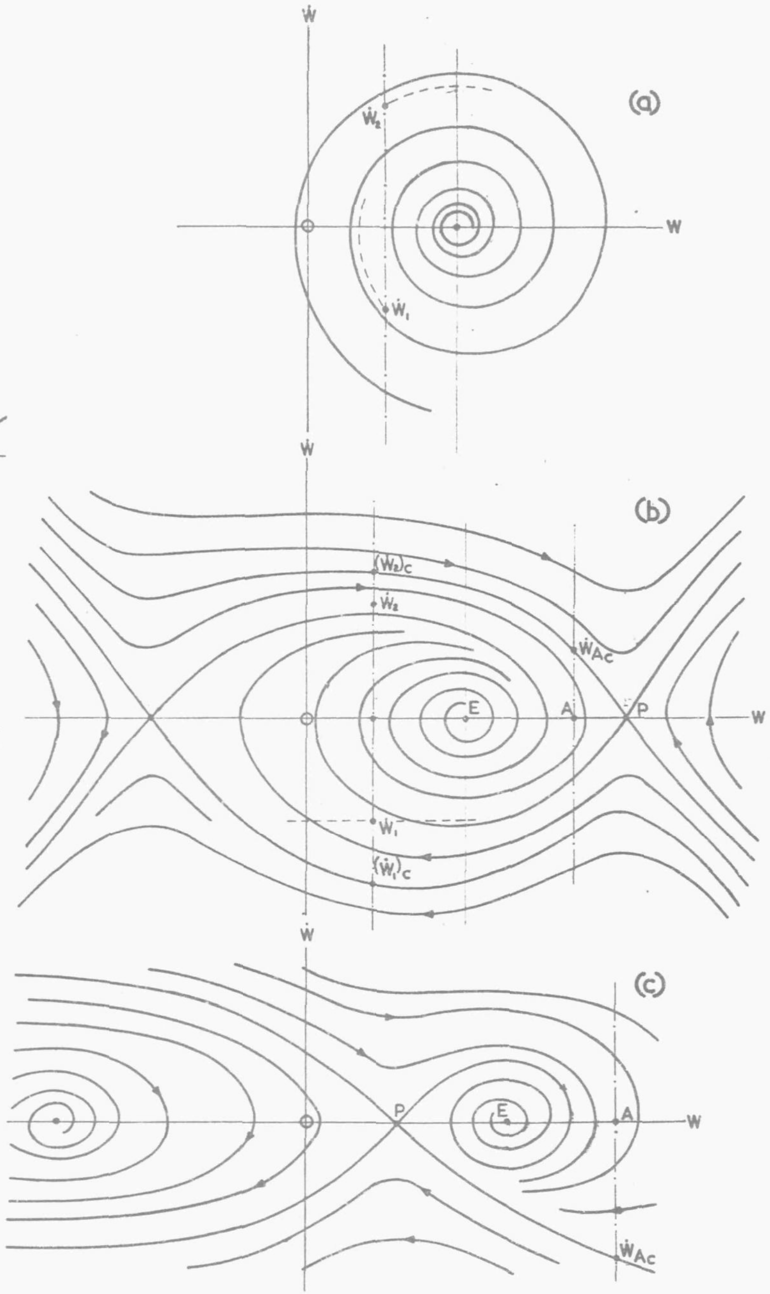



FIG. 10

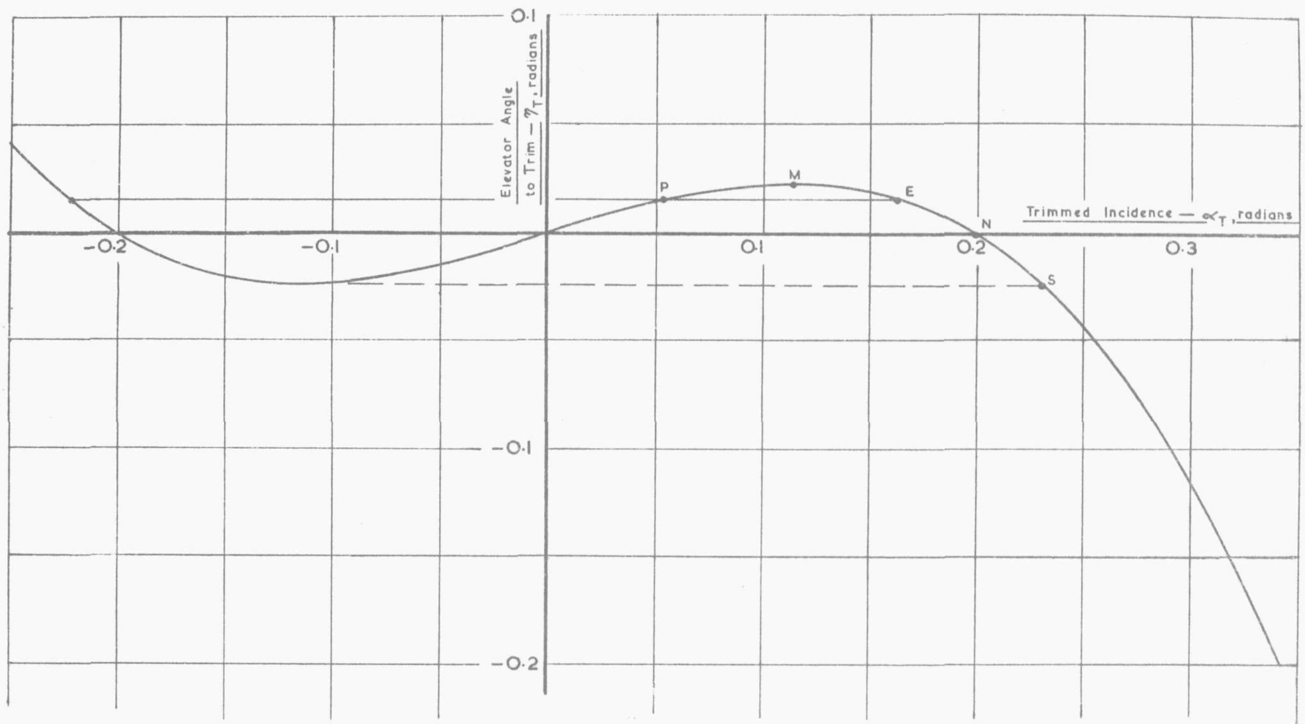



FIG. 11 TRIM CURVE ($x_{c.g.}$ = 0.5 feet)

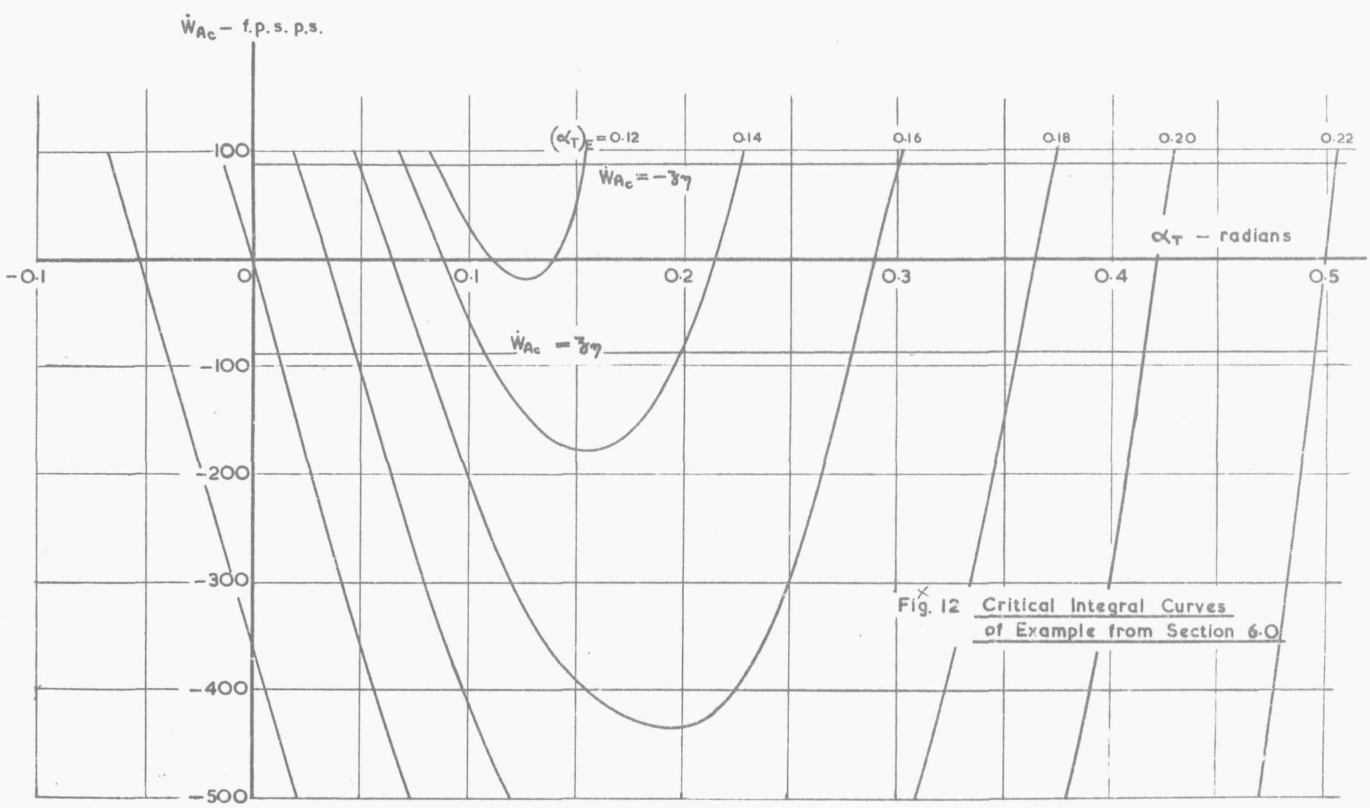



FIG. 12 CRITICAL INTEGRAL CURVES OF EXAMPLE FROM SECTION 6.0

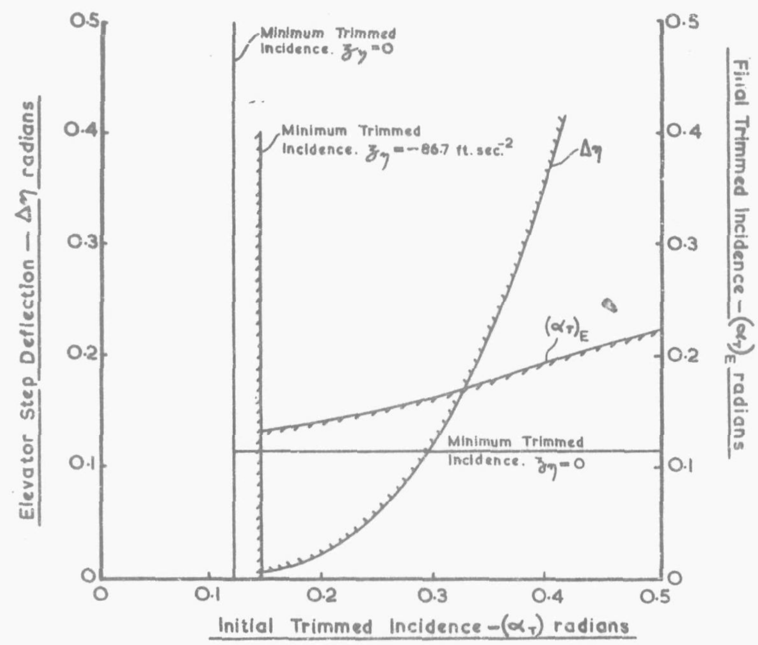



FIG. 13 INFLUENCE OF $z_\eta$ ON THE STABILITY WHEN RE-TRIMMING BY MEANS OF ELEVATOR STEP DEFLECTIONS