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EXTENDABILITY OF FUNCTIONS WITH PARTIALLY VANISHING TRACE

by Sebastian BECHTEL, Russell M. BROWN, Robert HALLER & Patrick TOLKSDORF (*)

Abstract. — Let $\Omega \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial\Omega$ be a closed part of its boundary. Under very mild assumptions on Ω, we construct a bounded Sobolev extension operator for the Sobolev space $\mathrm{W}^{k,p}_D(\Omega)$, $1 \leqslant p < \infty$, which consists of all functions in $\mathrm{W}^{k,p}(\Omega)$ that vanish in a suitable sense on D. In contrast to earlier work, this construction is global and does *not* use a localization argument, which allows to work with a boundary regularity that is sharp at the interface dividing D and $\partial\Omega \setminus D$. Moreover, we provide homogeneous and local estimates for the extension operator. Also, we treat the case of Lipschitz function spaces with a vanishing trace condition on D.

Résumé. — Soit $\Omega \subseteq \mathbb{R}^d$ ouvert et $D \subseteq \partial\Omega$ une partie fermée du bord. Sous des hypothèses faibles sur Ω, nous construisons un opérateur de prolongement borné pour l'espace Sobolev $\mathrm{W}^{k,p}_D(\Omega)$, $1 \leqslant p < \infty$, ce qui est le sous-espace de $\mathrm{W}^{k,p}(\Omega)$ composé des fonctions qui disparaissent sur D dans un sens approprié. Au contraire des travaux précédents, notre construction est globale et ne se sert pas d'arguments de localisation, ce qui nous permet de travailler avec une condition de régularité exacte pour l'interface entre D et $\partial\Omega \setminus D$. Aussi, nous fournissons des estimations homogènes et locales pour cet opérateur de prolongement. En plus, nous traitons le cas de fonctions Lipschitz disparaissant sur D.

1. Introduction

Sobolev spaces $\mathrm{W}^{k,p}_D(\Omega)$ that contain functions that only vanish on a portion D of the boundary of some given open set $\Omega \subseteq \mathbb{R}^d$ play an important role in the study of the mixed problem for second-order elliptic operators,

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see for example [1, 3, 7, 8, 9, 10, 11, 12, 16, 21, 22]. In the study of these spaces, an extension operator is a crucial tool.

Early contributions to the history of Sobolev extension operators include the works of Stein [20, pp. 180–192] and Calderón [4] on Lipschitz domains as well as the seminal paper of Jones [18] on (ε, δ)-domains. The latter mentioned work was later refined by Chua [5] and Rogers [19]. Even though all these constructions aim at the full Sobolev space $\mathrm{W}^{1,p}(\Omega)$, they restrict to bounded extension operators on the space with vanishing trace on D and the extensions preserve the trace condition on D if a mild regularity assumption is imposed, see [2, Lemma 3.4].

All these constructions rely on regularity assumptions for the full boundary of the underlying set. However, if we consider a (relatively) interior point of D, then it is possible to extend the function by zero around that point, so that a relaxation on the boundary regularity is feasible. This effect was exploited using localization techniques by several authors, see Brewster, Mitrea, Mitrea, and Mitrea [3] for a very mature incarnation of this idea using local (ε, δ)-charts, and [16] for a version using Lipschitz manifolds. We will present both frameworks in detail in Section 3 and show that they are included in our setup.

One drawback of this method is that the regularity assumption for the Neumann boundary part $\partial\Omega \setminus D$ has to hold not merely on this boundary portion but in a neighbourhood of it, which in particular contains interior points of D. This forbids all kinds of cusps that are arbitrarily close to the interface between the Dirichlet and the Neumann boundary part.

In this work, we will introduce an (ε, δ)-condition that is adapted to the Dirichlet condition on D. To be more precise, we also connect nearby points in Ω by ε-cigars, but these are with respect to the Neumann boundary part $\partial\Omega\setminus D$ and *not* the full boundary $\partial\Omega$, which means that ε-cigars may “leave” the domain across the Dirichlet part D to some extent that is measured by a *quasi-hyperbolic distance* condition. This allows to have certain inward and outward cusps arbitrarily close to the interface between the Dirichlet and Neumann parts, see Example 3.5 for an illustrative example. However, there are types of cusps that are particularly nasty and which are excluded from our setting by the aforementioned quasihyperbolic distance condition. In Example 3.7 we show that in these kinds of configurations there cannot exist a bounded extension operator, which emphasizes that it is indeed necessary that we have incorporated some further restriction in our setup. A detailed description of our geometric framework will be given in Assumption 2.1.

Next, we give a precise definition of what we mean by the term *extension operator*, followed by our main result.

Definition 1.1. — *Call a linear mapping E defined on $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ into the measurable functions on $\mathbb{R}^d$ an* extension operator *if it satisfies $Ef(x) = f(x)$ for almost every $x \in \Omega$ and for all $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Here, we mean by $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ the space of all measurable functions on Ω that are integrable on all bounded measurable subsets of Ω.*

Theorem 1.2. — *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $D \subseteq \partial\Omega$ be closed. Assume that Ω and D are subject to Assumption 2.1. Moreover, fix an integer $k \geqslant 0$. Then there exists an extension operator E such that for all $1 \leqslant p < \infty$ and $0 \leqslant \ell \leqslant k$ one has that E restricts to a bounded mapping from $\mathrm{W}^{\ell,p}_D(\Omega)$ to $\mathrm{W}^{\ell,p}_D(\mathbb{R}^d)$. The operator norms of E only depend on d, p, K, k, ε, δ, and λ.*

In addition, we will present some further improvements for the first-order case in Theorem 9.3 and Corollary 9.5 which include the case of Lipschitz spaces and an enlargement of admissible geometries, as well as *local* and *homogeneous* estimates in Theorem 10.2.

Outline of the article

First of all, we will present our geometric setting in Section 2 and will also give precise definitions for the relevant function spaces. A comparison with existing results and several examples and counterexamples are given in Section 3.

Then, we dive into the construction of the extension operator. Sections 4 and 5 are all about cubes. In there, we will define collections of exterior and interior cubes coming from two different Whitney decompositions, and will explain how an exterior cube can be reflected “at the Neumann boundary” to obtain an associated interior cube. In contrast to Jones, not all small cubes in the Whitney decomposition of $\overline{\Omega}$ are exterior cubes. The treatment of Whitney cubes which are “almost” exterior cubes are the central deviation from Jones construction and are thus the heart of the matter in this article. These two sections are highly technical.

Eventually, we come to the actual crafting of the extension operator for Theorem 1.2 in Section 6. This section also contains results on (adapted) polynomials which are needed to define the extension operator via “reflection”. The proof of Theorem 1.2 will be completed in Section 8. Before that, we introduce an approximation scheme that yields more regular test functions for $\mathrm{W}^{k,p}_D(\Omega)$ in Section 7. This additional regularity is crucial for Proposition 8.1.

Finally, we present some additional first-order theory in Section 9, followed by some short observations on locality and homogeneity in Section 10 which build on an observation made in Remark 6.12.

Notation

Throughout this article, the dimension $d \geqslant 2$ of the underlying Euclidean space $\mathbb{R}^d$ is fixed. Open balls around $x \in \mathbb{R}^d$ of radius $r > 0$ are denoted by $\mathrm{B}(x, r)$ and for the corresponding closed ball we write $\overline{\mathrm{B}}(x, r)$. The closure and complement of a set $A \subseteq \mathbb{R}^d$ are denoted by $\overline{A}$ and A^c. The Euclidean norm of a complex vector as well as the Lebesgue measure of a measurable set in $\mathbb{R}^d$ are denoted by $| \cdot |$. If not otherwise mentioned, cubes are closed and with sides parallel to the axes. We write $\mathcal{P}_m$ for the set of polynomials on $\mathbb{R}^d$ of degree at most m. The vector $\nabla^m f := (\partial^\beta f)_{|\beta|=m}$ is introduced for an m-times (weakly) differentiable function. The letters α and β are always supposed the mean multi-indices, possibly subject to further constraints. The distance of two sets $A, B \subseteq \mathbb{R}^d$ is denoted by $\mathrm{d}(A, B)$ and in the case $A = \{x\}$ the distance is abbreviated by $\mathrm{d}(x, B)$. For $A \subseteq \mathbb{R}^d$ and $t > 0$ put $N_t(A) := \{x \in \mathbb{R}^d \colon \mathrm{d}(x, A) < t\}$. The diameter of an arbitrary subset of $\mathbb{R}^d$ is denoted by $\operatorname{diam}(\cdot)$. Finally, we follow the standard conventions that the infimum over the empty set is ∞ and that $1/\infty = 0$.

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We would like to thank Juha Lehrbäck for drawing our attention towards the notion of quasihyperbolic distances.

2. Geometry and Function Spaces

2.1. Geometry

Let $\Xi \subseteq \mathbb{R}^d$ be open. For two points $x, y \in \Xi$ their *quasihyperbolic distance*, first introduced by Gehring and Palka [15], is given by

$$\mathrm{k}_\Xi(x, y) := \inf_\gamma \int_\gamma \frac{1}{\mathrm{d}(z, \partial \Xi)} \, |\mathrm{d}z|,$$

where the infimum is taken over all rectifiable curves γ in Ξ joining x with y. Notice that its value might be ∞. This is the case if there is no path connecting x with y in Ξ. The function k_Ξ is called the *quasihyperbolic metric*. If $\Xi' \subseteq \Xi$ define

$$\mathrm{k}_\Xi(x, \Xi') := \inf\{\mathrm{k}_\Xi(x,y) : y \in \Xi'\} \qquad (x \in \Xi).$$

To construct the Sobolev extension operator in Theorem 1.2, we will rely on the following geometric assumption.

ASSUMPTION 2.1. — *Let $\Omega \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial\Omega$ be closed, and define $\Gamma := \partial\Omega \setminus D$. We assume that there exist $\varepsilon \in (0,1]$, $\delta \in (0,\infty]$ and $K > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \delta$ there exists a rectifiable curve γ that joins x and y and takes values in $\Xi := \mathbb{R}^d \setminus \overline{\Gamma}$ and satisfies*

$$\text{(LC)} \qquad \operatorname{length}(\gamma) \leqslant \varepsilon^{-1}|x-y|,$$

$$\text{(CC-}\Gamma\text{)} \qquad \mathrm{d}(z,\Gamma) \geqslant \varepsilon\frac{|x-z||y-z|}{|x-y|} \qquad (z \in \gamma),$$

$$\text{(QHD)} \qquad \mathrm{k}_\Xi(z,\Omega) \leqslant K \qquad (z \in \gamma).$$

Furthermore, assume that there exists $\lambda > 0$ such that, for each connected component Ω_m of Ω, there holds

$$\text{(DC)} \qquad \partial\Omega_m \cap \Gamma \neq \emptyset \quad \Longrightarrow \quad \operatorname{diam}(\Omega_m) \geqslant \lambda\delta.$$

Remark 2.2. — Let $(\Xi_m)_m$ denote the connected components of Ξ. From $\mathrm{d}(z,\partial\Xi_m) = \mathrm{d}(z,\partial\Xi)$ for $z \in \Xi_m$ follows directly that $\mathrm{k}_\Xi(x,y) = \mathrm{k}_{\Xi_m}(x,y)$ holds for all $x, y \in \Xi_m$. Note that $\partial\Xi = \overline{\Gamma}$ since $\overline{\Gamma} \subseteq \partial\Omega$ contains no interior points. Moreover, for $x \in \Xi_m$ and $y \in \Xi_n$ with $m \neq n$ one has $\mathrm{k}_\Xi(x,y) = \infty$ since there is no connecting path between those points. Finally, $\mathrm{k}_{\mathbb{R}^d}(x,y) = 0$ holds for all $x, y \in \mathbb{R}^d$ by the convention $1/\infty = 0$.

Remarks 2.3.

(1) Consider the pure Dirichlet case $D = \partial\Omega$. Then the curves are allowed to take values in all of $\mathbb{R}^d$. In particular, we may connect points by a straight line, so that (LC) is clearly satisfied. Condition (CC-Γ) is void and also (QHD) is trivially fulfilled, see Remark 2.2. Moreover, the diameter condition is always fulfilled since there are no connected components that intersect Γ. Consequently, if $D = \partial\Omega$, then Assumption 2.1 is fulfilled for any open set Ω.

(2) Consider the pure Neumann case $D = \emptyset$ and fix ε, δ. The curve γ can only connect points in the same connected component of Ω. Thus, Ω is the union of at most countably many (ε,δ)-domains, whose pairwise distance is at least δ and whose diameters stay uniformly

away from zero. In particular, if $\delta = \infty$, then Ω is connected and unbounded.

(3) A similar condition on the diameter of connected components was introduced in [3, Section 2] in order to transfer Jones' construction of the Sobolev extension operator in [18] to disconnected sets. In the situation of Assumption 2.1 the positivity of the radius *only* ensures that the connected components of Ω whose boundaries have a common point with Γ do not become arbitrarily small. This is because our construction is global and not using a localization procedure. We will present a thorough comparison with the geometry from [3] in Section 3.

2.2. Function spaces

Write $\mathrm{W}^{k,p}(\Omega)$ for the vector space of all $\mathrm{L}^p(\Omega)$ functions that have weak derivatives up to the non-negative integer order k and which are again in $\mathrm{L}^p(\Omega)$. Equip $\mathrm{W}^{k,p}(\Omega)$ with the usual norm. Note that by Rademacher's theorem $\mathrm{W}^{1,\infty}(\mathbb{R}^d)$ coincides with the space $\mathrm{Lip}(\mathbb{R}^d)$ of Lipschitz continuous functions. A particular consequence is that (locally) Lipschitz continuous functions are weakly differentiable. We will exploit this fact in Section 8. Note that on domains a mild geometric assumption is needed to ensure that $\mathrm{W}^{1,\infty}(\Omega)$ coincides with $\mathrm{Lip}(\Omega)$. This can be observed by considering $\Omega = \mathrm{B}(0,1) \setminus [0,1)$ as a counterexample.

Definition 2.4. — *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $D \subseteq \overline{\Omega}$ be closed. Define the space of smooth functions on Ω which vanish in a neighborhood of D by*

$$\mathrm{C}_D^\infty(\Omega) := \big\{ f \in \mathrm{C}^\infty(\Omega) \colon \ \mathrm{d}(\mathrm{supp}(f), D) > 0 \big\}.$$

Using this space of test functions, we define Sobolev functions vanishing on D. Note that we exclude the endpoint case $p = \infty$ in that definition. However, in the case $k = 1$, we will work with a related space in Section 9.

Definition 2.5. — *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $D \subseteq \overline{\Omega}$ be closed. For a non-negative integer k and $p \in [1,\infty)$ define the Sobolev space $\mathrm{W}_D^{k,p}(\Omega)$ as the closure of $\mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$.*

In Section 7 we will see that even the space $\mathrm{C}_D^\infty(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(\Omega)$ is dense in $\mathrm{W}_D^{k,p}(\Omega)$ as long as we assume the geometry from Assumption 2.1; In fact, we will approximate by compactly supported $\mathrm{C}_D^\infty(\mathbb{R}^d)$ functions, which are therefore in particular in $\mathrm{W}^{k,p}(\Omega)$.

3. Comparison with other results and examples

This section is devoted to comparing our result with existing results. The most general geometric setup to construct a Sobolev extension operator for the spaces $\mathrm{W}^{1,p}_D(\Omega)$ was considered in the work of Brewster, Mitrea, Mitrea, and Mitrea [3, Theorem 1.3, Definition 3.4] and reads as follows.

Assumption 3.1. — *Let $\Omega \subseteq \mathbb{R}^d$ be an open, non-empty, and proper subset of $\mathbb{R}^d$, $D \subseteq \partial\Omega$ be closed, and let $\Gamma := \partial\Omega \setminus D$. Let $\varepsilon, \delta > 0$ be fixed. Assume there exist $r_0 > 0$ and an at most countable family $\{O_j\}_j$ of open subsets of $\mathbb{R}^d$ satisfying*

(1) *$\{O_j\}_j$ is locally finite and has bounded overlap,*
(2) *for all j there exists an (ε, δ)-domain $U_j \subseteq \mathbb{R}^d$ with connected components all of diameter at least r_0 and satisfying $O_j \cap \Omega = O_j \cap U_j$,*
(3) *there exists $r \in (0, \infty]$ such that for all $x \in \Gamma$ there exists j for which $\mathrm{B}(x, r) \subseteq O_j$.*

Here, an open set U_j is called an (ε, δ)-domain if there exist $\varepsilon, \delta > 0$ such that for all $x, y \in U_j$ there exists a rectifiable curve γ that joins x and y, takes its values in U_j, and satisfies (LC) and (CC-Γ) with respect to ∂U_j instead of Γ. Also note that (LC) enforces $\varepsilon \in (0, 1]$.

Proposition 3.2. — *Assumption 3.1 implies Assumption 2.1.*

Proof. — Let ε, δ, r, and r_0 be the quantities from Assumption 3.1. We have to show the quantitative connectedness condition contained in (LC), (CC-Γ), and (QHD) as well as the diameter condition (DC) for connected components touching Γ. For the rest of the proof, references to (2) and (3) refer to the respective items in Assumption 3.1.

To establish (LC), (CC-Γ), and (QHD), let $x, y \in \Omega$ and define $\kappa := r/8$ and $V_\kappa := \{x \in \mathbb{R}^d \colon \mathrm{d}(x, \Gamma) \leqslant \kappa\}$. We proceed by distinguishing two cases.

Case 1: $x, y \in V_\kappa$ with $|x - y| < \min(\delta, \varepsilon r/8)$. — Fix $x_0 \in \overline{\Gamma} \subseteq \partial\Omega$ such that $\mathrm{d}(x, \Gamma) = |x - x_0|$. Then $x, y \in \mathrm{B}(x_0, r/4)$ and by (3) we get $\mathrm{B}(x_0, r) \subseteq O_j$ for some j. Using this and (2) we furthermore get $x, y \in \mathrm{B}(x_0, r) \cap \Omega = \mathrm{B}(x_0, r) \cap U_j$. Notice that this gives in particular $x_0 \in \partial U_j$. Next, let γ denote the (ε, δ)-path subject to (LC) and (CC-Γ) that connects x and y in U_j. For $z \in \gamma$ we have by (LC)

$$|x_0 - z| \leqslant |x_0 - x| + |x - z| \leqslant \kappa + \mathrm{length}(\gamma) \leqslant \frac{r}{8} + \frac{|x - y|}{\varepsilon} < \frac{r}{4}.$$

Thus, γ takes its values in $\mathrm{B}(x_0, r/4) \cap U_j = \mathrm{B}(x_0, r/4) \cap \Omega \subseteq \Xi$. This shows that γ is an admissible path for Assumption 2.1, and of course (LC) stays valid. We also conclude

$$\mathrm{d}(z, \partial\Omega) = \mathrm{d}\big(z, \mathrm{B}(x_0, r/2) \cap \partial\Omega\big) = \mathrm{d}\big(z, \mathrm{B}(x_0, r/2) \cap \partial U_j\big) = \mathrm{d}(z, \partial U_j),$$

and thus, taking (CC-Γ) with respect to ∂U_j into account, we derive

$$\mathrm{d}(z, \Gamma) \geqslant \mathrm{d}(z, \partial\Omega) = \mathrm{d}(z, \partial U_j) \geqslant \varepsilon \frac{|x-y||y-z|}{|x-y|},$$

which is (CC-Γ) with respect to Γ. Finally, since γ takes its values in Ω, it satisfies (QHD) with $K = 0$.

Case 2: $x \in V_\kappa^c$ and $|x-y| < \kappa/2$ (the case with $y \in V_\kappa^c$ works symmetrically). — Write γ for the straight line that connects x with y in $\mathbb{R}^d$. Then (LC) is clearly fulfilled and for (CC-Γ) we estimate with $z \in \gamma$ using $\max(|x-z|, |y-z|) \leqslant |x-y| \leqslant \kappa/2$ that

$$\varepsilon \frac{|x-z||y-z|}{|x-y|} < \frac{\kappa}{2} < \mathrm{d}(x, \Gamma) - \mathrm{length}(\gamma) \leqslant \mathrm{d}(z, \Gamma).$$

In particular, γ takes its values in Ξ.

To control the quasihyperbolic distance of a point $z \in \gamma$ to Ω with respect to Ξ, we estimate $\mathrm{k}_\Xi(z, x)$ using the line segment that connects $x \in \Omega$ with z. Then the integrand in the definition of the quasihyperbolic distance is bounded by $2/\kappa$ and the length of the path is at most $\kappa/2$. Hence, $\mathrm{k}_\Xi(z, \Omega) \leqslant 1$, which gives (QHD).

Therefore, conditions (LC), (CC-Γ), and (QHD) are satisfied for all $x, y \in \Omega$ as long as $|x-y| < \min(\delta, \varepsilon r/8, \kappa/2) =: \delta'$, which concludes the first part of this proof.

To show the diameter condition, let Ω_m be a connected component of Ω with $\partial\Omega_m \cap \Gamma \neq \emptyset$. Fix some x_0 in this intersection. We show $\mathrm{diam}(\Omega_m) \geqslant \min(r/2, r_0)$. This implies in particular that $\mathrm{diam}(\Omega_m) \geqslant \lambda\delta'$ for some suitable λ since δ' is finite. Suppose $\mathrm{diam}(\Omega_m) < r/2$. Then $\Omega_m \subseteq \mathrm{B}(x_0, r/2)$. According to (3), we have $\mathrm{B}(x_0, r) \subseteq O_j$ for some j. Taking also (2) into account, we get on the one hand that $\Omega_m \subseteq U_j$, and on the other hand that points in U_j close to Ω_m belong to Ω. We draw two conclusions: First, Ω_m is an open and connected subset of U_j. Second, if there were a continuous path in U_j connecting a point from $U_j \setminus \Omega_m$ with one in Ω_m, that path would eventually run in Ω and therefore Ω_m wouldn't be maximally connected in Ω, leading to a contradiction. So in total, Ω_m is a connected component of U_j and hence $\mathrm{diam}(\Omega_m) \geqslant r_0$. □

A common geometric setup which is used in many works dealing with mixed Dirichlet/Neumann boundary conditions, see for example [1, 7, 8, 9, 11, 12, 21, 22], requires Lipschitz charts around points on the closure of Γ and is presented in the following assumption.

Assumption 3.3. — *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and $D \subseteq \partial\Omega$ be closed. Assume that around each point $x \in \overline{\Gamma}$ there exists a neighborhood O_x of x and a bi-Lipschitz homeomorphism $\Phi_x : O_x \to (-1,1)^d$ such that $\Phi_x(x) = 0$, $\Phi_x(O_x \cap \Omega) = (-1,1)^{d-1} \times (0,1)$, and $\Phi_x(O_x \cap \partial\Omega) = (-1,1)^{d-1} \times \{0\}$.*

Proposition 3.4. — *Assumption 3.3 implies Assumption 3.1.*

Proof. — By [6, Lemma 2.2.20], for any $x \in \overline{\Gamma}$ the set $U_x \coloneqq O_x \cap \Omega$ is an (ε, δ)-domain. Here, ε and δ do only depend on d and the Lipschitz constant. The compactness of $\overline{\Gamma}$ implies that there exist finitely many $x_1, \dots, x_m \in \overline{\Gamma}$ such that $\overline{\Gamma} \subseteq \bigcup_{j=1}^m O_{x_j}$. Define $O_j \coloneqq O_{x_j}$ and $U_j \coloneqq U_{x_j}$ for $j = 1, \dots, m$. Due to the finiteness of the family $\{U_j\}_{j=1}^m$, the constants ε and δ can be chosen to be uniform in j. Finally, if $r > 0$ is the Lebesgue number of the covering $\{O_j\}_{j=1}^m$, then for all $x_0 \in \overline{\Gamma}$ there exists $1 \leqslant j \leqslant m$ such that $\mathrm{B}(x_0, r) \subseteq O_j$. Thus, all requirements in Assumption 3.1 are fulfilled. □

Next, we give an example of a two-dimensional domain that satisfies Assumption 2.1 but not Assumption 3.1. We further show that, within this configuration, the geometry described in Assumption 2.1 is in some sense optimal.

Example 3.5. — Let $\theta \in (0, \pi)$ and let $\mathrm{S}_\theta \subseteq \mathbb{R}^2$ denote the open sector symmetric around the positive x-axis with opening angle 2θ. Let $\Omega \subseteq \mathbb{R}^2$ be any domain satisfying

$$\Omega \cap \mathrm{S}_\theta = \{(x, y) \in \mathrm{S}_\theta : y < 0\},$$

and define

$$D \coloneqq \partial\Omega \cap \left[\mathbb{R}^2 \setminus \mathrm{S}_\theta\right] \qquad \text{and} \qquad \Gamma \coloneqq \partial\Omega \setminus D = (0, \infty) \times \{0\}.$$

Essentially, this means that inside the sector S_θ the domain Ω looks like the lower half-space and the half-space boundary that lies inside S_θ is Γ. In the complement of the sector S_θ, Ω could be any open set and the boundary of Ω in the complement of S_θ is defined to be D. See Figure 3.1 for an example of such a configuration.

To verify that such a domain fulfills the geometric setup described in Assumption 2.1, consider first the set

$$\Delta_\theta \coloneqq (\mathbb{R}^2 \setminus \overline{\mathrm{S}_\theta}) \cup \{(x, y) \in \mathbb{R}^2 : y < 0\},$$

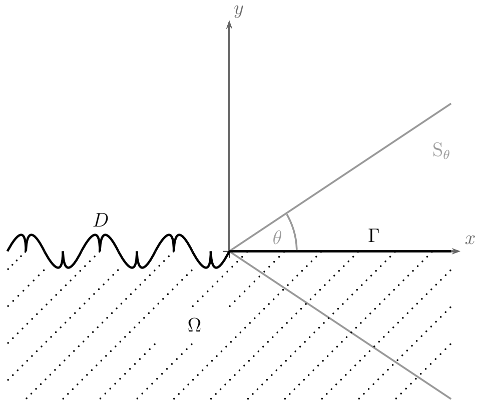

Figure 3.1. A generic picture of a domain described in Example 3.5.

which is an (ε, δ)-domain for some values $\varepsilon, \delta > 0$. Since $\Omega \subseteq \Delta_\theta$ and $\overline{\Gamma} \subseteq \partial\Delta_\theta$, the (ε, δ)-paths with respect to Δ_θ for points in Ω satisfy (LC) and (CC-Γ). Hence, to conclude the example, we only have to show that there exists $K > 0$ such that for all $z \in \Delta_\theta$ and with $\Xi = \mathbb{R}^2 \setminus \overline{\Gamma}$ it holds

$$\mathrm{k}_\Xi(z, \Omega) \leqslant K. \tag{3.1}$$

Since the paths obtained above take their values only in Δ_θ this will establish the remaining condition (QHD). Notice that since $\mathrm{S}_\theta \cap \{(x, y) \in \mathbb{R}^2 : y < 0\} \subseteq \Omega$ it suffices to show that there exists $K > 0$ such that for all $z \in \Delta_\theta$ it holds

$$\mathrm{k}_\Xi(z, \mathrm{S}_\theta \cap \{(x, y) \in \mathbb{R}^2 : y < 0\}) \leqslant K.$$

We only describe one particular case in detail, the remaining cases are similar and left to the interested reader. Assume that $\theta < \pi/2$ and pick $z = (v, w) \in \Delta_\theta$ with $v \geqslant 0$ and $w > 0$. Choose $(x, y) \in \partial\mathrm{S}_\theta$ such that $y := -w$ and let $\gamma := \gamma_1 + \gamma_2 + \gamma_3$ with

$$\begin{aligned}
\gamma_1 &: [0, 1] \longrightarrow \mathbb{R}^2, \quad t \longmapsto (x, y) + t(y - x, 0),\\
\gamma_2 &: [0, 1] \longrightarrow \mathbb{R}^2, \quad t \longmapsto (y, y) + t(0, w - y),\\
\gamma_3 &: [0, 1] \longrightarrow \mathbb{R}^2, \quad t \longmapsto (y, w) + t(v - y, 0).
\end{aligned}$$

This construction is depicted in Figure 3.2. The path γ then connects (x,y) to (v,w) and

$$\begin{aligned}\mathrm{k}_{\Xi}((x,y),(v,w)) &\leqslant \int_0^1 \frac{|y-x|}{|y|}\,\mathrm{d}t + \int_0^1 \frac{|w-y|}{|y|}\,\mathrm{d}t + \int_0^1 \frac{|v-y|}{w}\,\mathrm{d}t \\ &= 4 + \frac{x+v}{w}.\end{aligned}$$

Notice that $x = w/\tan(\theta)$ and that $v \leqslant w/\tan(\theta)$, so that

$$\mathrm{k}_{\Xi}((x,y),(v,w)) \leqslant 2\Big(2 + \frac{1}{\tan(\theta)}\Big).$$

In the remaining cases $v < 0$ and $w \geqslant 0$, $v < 0$ and $w < 0$, or $v \geqslant 0$ and $w < 0$ the quasihyperbolic distance to Ω will even be smaller. This proves the validity of (3.1) and thus, since Ω is connected and hence (DC) is void, that Ω fulfills Assumption 2.1.

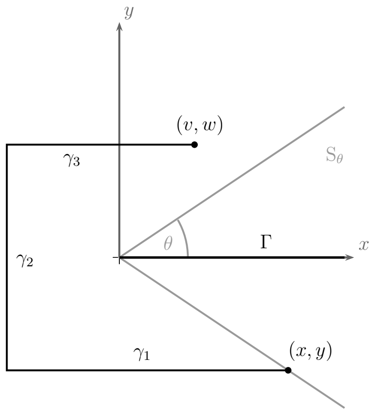

Figure 3.2. A path connecting (v,w) and (x,y) that is 'short' with respect to the quasihyperbolic distance.

Remark 3.6. — Notice that the geometric setup in Assumption 3.1 imposes boundary regularity in a neighborhood of $\overline{\Gamma}$, while in the situation described in Example 3.5 the portion D of $\partial\Omega$ can be arbitrarily irregular as long as it stays outside of S_θ.

We conclude this section by giving examples of domains where the boundary portion D fails to remain outside of a sector S_θ and show that the $\mathrm{W}^{1,p}_D$-extension property fails for these types of domains. These examples show that interior cusps that lie directly on the interface separating D and

Γ destroy the $\mathrm{W}^{1,p}_D$-extension property. The same happens with "interior cusps at infinity", that is to say, if D and Γ approach each other at infinity at a certain rate.

Example 3.7 (Interior boundary cusp at zero). — Let $\alpha \in (1,\infty)$ and consider

$$\Omega := \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \colon x \geqslant 0 \text{ and } -x^\alpha \leqslant y \leqslant 0\}.$$

Define D and Γ via

$$D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0 \text{ and } -x^\alpha = y\} \qquad \text{and} \qquad \Gamma := (0,\infty) \times \{0\}.$$

To prove that the $\mathrm{W}^{1,p}_D$-extension property fails, let $1 < p < \infty$ and $0 < r < \infty$. Let f_r be a smooth function that is supported in

$$Q_r := \{(x,y) \in \mathbb{R}^2 : r/2 \leqslant x \leqslant 2r \text{ and } 0 \leqslant y \leqslant r\},$$

satisfies $0 \leqslant f_r \leqslant 1$, and is identically 1 on

$$R_r := \{(x,y) \in \mathbb{R}^2 : 3r/4 \leqslant x \leqslant 3r/2 \text{ and } 0 \leqslant y \leqslant r/2\}.$$

Moreover, let f_r be such that $\|\nabla f_r\|_{\mathrm{L}^\infty} \lesssim r^{-1}$. In this case

$$\|f_r\|^p_{\mathrm{W}^{1,p}(\Omega)} \lesssim (r^2 + r^{2-p}). \tag{3.2}$$

Next, employ the fundamental theorem of calculus and a density argument to conclude that for all $F \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\int_{3r/4}^{3r/2} F(x,0)\,\mathrm{d}x - \int_{3r/4}^{3r/2} F(x,-x^\alpha)\,\mathrm{d}x = \int_{3r/4}^{3r/2} \int_{-x^\alpha}^{0} \partial_y F(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

If there exists a bounded extension operator $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^2)$, put $F := Ef_r$ and conclude that the second integral on the left-hand side vanishes since $Ef_r \in \mathrm{W}^{1,p}_D(\mathbb{R}^2)$. Using further that by construction the trace of Ef_r onto the set $(3r/4, 3r/2) \times \{0\}$ is identically 1, one concludes

$$\frac{3r}{4} \leqslant \int_{\frac{3r}{4}}^{\frac{3r}{2}} \int_{-x^\alpha}^{0} |\partial_y Ef_r(x,y)|\,\mathrm{d}y\,\mathrm{d}x \lesssim r^{(\alpha+1)/p'} \|Ef_r\|_{\mathrm{W}^{1,p}(\mathbb{R}^2)}.$$

Here, p' denotes the Hölder-conjugate exponent to p. Dividing by r and using that E is bounded delivers together with (3.2) the relation

$$1 \lesssim r^{(\alpha+1)/p'-1}(r^{2/p} + r^{2/p-1}), \tag{3.3}$$

which results for $r \to 0$ in the condition

$$\frac{\alpha+1}{p'} + \frac{2}{p} - 2 \leqslant 0 \quad \iff \quad \alpha \leqslant 1.$$

This is a contradiction since α is assumed to be in $(1,\infty)$. Thus, there cannot be a bounded extension operator $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^2)$.

Example 3.8 (Interior boundary cusp at infinity). — Let $\alpha \in (0,\infty)$ and consider

$$\Omega \coloneqq \{(x,y) \in \mathbb{R}^2 : \text{either } y > 0 \text{ or } x > 0 \text{ and } y < -x^{-\alpha}\}.$$

Define D and Γ via

$$D \coloneqq \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } -x^{-\alpha} = y\} \qquad \text{and} \qquad \Gamma \coloneqq \mathbb{R} \times \{0\}.$$

The proof that in this situation there does not exists a bounded extension operator E from $\mathrm{W}^{1,p}_D(\Omega)$ to $\mathrm{W}^{1,p}_D(\mathbb{R}^2)$ for any $p \in (1,\infty)$ is similar to Example 3.7 and we omit the details.

4. Whitney decompositions and the quasihyperbolic distance

In this section, we introduce the Whitney decomposition of an open subset of $\mathbb{R}^d$ and show how condition (QHD) relates to properties of Whitney cubes. A cube $Q \subseteq \mathbb{R}^d$ is always closed and is said to be *dyadic* if there exists $k \in \mathbb{Z}$ such that Q coincides with a cube of the mesh determined by the lattice $2^{-k}\mathbb{Z}^d$. Two cubes are said to *touch* if a face of one cube is completely contained in a face of the other cube, and they are said to *intersect* if their intersection is non-empty. The sidelength of a cube is denoted by $\ell(Q)$. For a number $\alpha > 0$ the dilation of Q about its center by the factor α is denoted by αQ.

Let $F \subseteq \mathbb{R}^d$ be a non-empty closed set. Then, by [20, Theorem VI.1 and following propositions] there exists a collection of cubes $\{Q_j\}_{j\in\mathbb{N}}$ with pairwise disjoint interiors such that

(i) $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^d \setminus F$,
(ii) $\operatorname{diam}(Q_j) \leqslant \mathrm{d}(Q_j, F) \leqslant 4\operatorname{diam}(Q_j)$ for all $j \in \mathbb{N}$,
(iii) the cubes $\{Q_j\}_{j\in\mathbb{N}}$ are dyadic,
(iv) $\frac{1}{4}\operatorname{diam}(Q_j) \leqslant \operatorname{diam}(Q_k) \leqslant 4\operatorname{diam}(Q_j)$ if $Q_j \cap Q_k \neq \emptyset$,
(v) each cube has at most 12^d intersecting cubes.

The collection $\{Q_j\}_{j\in\mathbb{N}}$ are called Whitney cubes and will be denoted by $\mathcal{W}(F)$. In connection with Whitney cubes, the letters (i)–(v) refer always to the above properties. We say that a collection of cubes $Q_1, \dots, Q_m \in \mathcal{W}(F)$ is a *touching chain* if Q_j and Q_{j+1} are touching cubes and that it is an *intersecting chain* if $Q_j \cap Q_{j+1} \neq \emptyset$ for all $j = 1, \dots, m-1$. The *length* of a chain is the number m.

Let us mention that for a cube $Q \in \mathcal{W}(F)$ and $x \in Q$ we have

$$\operatorname{diam}(Q) \geqslant \frac{1}{5} \operatorname{d}(x, F). \tag{4.1}$$

This follows from

$$4 \operatorname{diam}(Q) \geqslant \operatorname{d}(Q, F) \geqslant \operatorname{d}(x, F) - \operatorname{diam}(Q),$$

and will be used freely in the rest of this article.

The following lemma translates (QHD) to the existence of intersecting chains of uniformly bounded length. Notice that if $(\Xi_m)_{m \in \mathcal{I}}$ denotes the connected components of the set $\Xi = \mathbb{R}^d \setminus \bar{\Gamma}$, Gehring and Osgood [14, Lemma 1] proved that for any two points $x, y \in \Xi_m$ there exists a quasi-hyperbolic geodesic $\gamma_{x,y}$ with endpoints x and y, that is, a curve with endpoints x and y satisfying

$$\mathrm{k}_\Xi(x, y) = \int_{\gamma_{x,y}} \frac{1}{\operatorname{d}(z, \partial \Xi)} \, |\mathrm{d}z|.$$

Trivially, if $\Xi = \mathbb{R}^d$, then any path connecting x and y is a quasihyperbolic geodesic.

Lemma 4.1. — *Fix $k > 0$. There exists a constant $N = N(d, k) \in \mathbb{N}$ such that for all $x, y \in \Xi$ with $\mathrm{k}_\Xi(x, y) \leqslant k$ there exists an intersecting chain $Q_1, \dots, Q_m \in \mathcal{W}(\bar{\Gamma})$ with $x \in Q_1$ and $y \in Q_m$ and $m \leqslant N$.*

Conversely, if for $x, y \in \Xi$ there exists an intersecting chain connecting x and y of length less than $N \in \mathbb{N}$, then there exists a constant $k = k(N) > 0$ such that $\mathrm{k}_\Xi(x, y) \leqslant k$.

Proof. — Notice that $\mathrm{k}_\Xi(x, y) < \infty$ implies that x and y lie in the same connected component of Ξ. Assume first that

$$|x - y| \leqslant \frac{1}{10\sqrt{d}} \min\{\operatorname{d}(x, \Gamma), \operatorname{d}(y, \Gamma)\}. \tag{4.2}$$

Let $Q_x, Q_y \in \mathcal{W}(\bar{\Gamma})$ with $x \in Q_x$ and $y \in Q_y$, and let $\widetilde{Q}_x$ denote the region occupied by Q_x and all its intersecting Whitney cubes and similarly let $\widetilde{Q}_y$ denote its counterpart for Q_y. Then by (iv)

$$\operatorname{d}(x, \widetilde{Q}_x^c) \geqslant \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_x) \qquad \text{and} \qquad \operatorname{d}(y, \widetilde{Q}_y^c) \geqslant \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_y).$$

This combined with (4.2) yields

$$\operatorname{d}(x, \widetilde{Q}_x^c) \geqslant \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_x) \geqslant \frac{1}{2} |x - y|.$$

By symmetry, the same is valid for y instead of x. Consequently, $\widetilde{Q}_x$ and $\widetilde{Q}_y$ have a common point and thus, x and y can be connected by an intersecting chain of length at most 4.

Now, let

$$|x-y| > \frac{1}{10\sqrt{d}} \min\{\mathrm{d}(x,\Gamma), \mathrm{d}(y,\Gamma)\}.$$

Assume without loss of generality that $\mathrm{d}(x,\Gamma) \leqslant \mathrm{d}(y,\Gamma)$. Fix a quasihyperbolic geodesic $\gamma_{x,y}$ that connects x with y (see the discussion before this proof). Then Herron and Koskela [17, Proposition 2.2] ensures the existence of points $y_0 := x, y_1, \dots, y_\ell \in \mathbb{R}^d \setminus \overline{\Gamma}$ such that $\gamma_{x,y}$ is contained in the closure of $\bigcup_{i=0}^{\ell} B_i$, where $B_i := \mathrm{B}(y_i, r_i)$ with $r_i := \mathrm{d}(y_i,\Gamma)/(10\sqrt{d})$, and such that

$$\ell \leqslant 20\sqrt{d}\, \mathrm{k}_\Xi(x,y). \tag{4.3}$$

Next, we estimate the number of Whitney cubes that cover each of these balls. Denote the number of Whitney cubes that cover $\overline{B_i}$ by W_i. Let $Q \in \mathcal{W}(\overline{\Gamma})$ be such that $Q \cap \overline{B_i} \neq \emptyset$. Then,

$$\mathrm{diam}(Q) \geqslant \frac{1}{4}\mathrm{d}(Q,\Gamma) \geqslant \frac{1}{4}[\mathrm{d}(y_i,\Gamma) - r_i - \mathrm{diam}(Q)],$$

so that by definition of r_i

$$\mathrm{diam}(Q) \geqslant \frac{(10\sqrt{d}-1)\,\mathrm{d}(y_i,\Gamma)}{50\sqrt{d}}.$$

Moreover,

$$\mathrm{diam}(Q) \leqslant \mathrm{d}(Q,\Gamma) \leqslant \mathrm{d}(B_i \cap Q,\Gamma) \leqslant \mathrm{d}(y_i,\Gamma) + r_i = \left[1 + \frac{1}{10\sqrt{d}}\right] \mathrm{d}(y_i,\Gamma).$$

Consequently,

$$W_i \left[\frac{(10\sqrt{d}-1)\,\mathrm{d}(y_i,\Gamma)}{50d}\right]^d \leqslant \sum_{\substack{Q \in \mathcal{W}(\overline{\Gamma}) \\ Q \cap \overline{B_i} \neq \emptyset}} |Q| \leqslant \left| B\Big(y_i, \left[1 + \frac{1}{5\sqrt{d}}\right] \mathrm{d}(y_i,\Gamma)\Big)\right|,$$

what proves that W_i is controlled by a constant depending only on d. We conclude by (4.3) and by the bound on each W_i that there exists an intersecting chain connecting x and y of length bounded by a constant depending only on d and k.

For the other direction, let $Q_1, \dots, Q_m$ be an intersecting chain that connects x with y and with $m \leqslant N$. Thus, by definition $Q_j \cap Q_{j+1} \neq \emptyset$. Let

γ be a path connecting x and y which is constructed by linearly connecting a point in $Q_{j-1} \cap Q_j$ with a point in $Q_j \cap Q_{j+1}$. Thus, employing (ii) delivers

$$\mathrm{k}_\Xi(x,y) \leqslant \sum_{j=1}^m \int_{\gamma \cap Q_j} \frac{1}{\mathrm{d}(Q_j,\Gamma)}\,|\mathrm{d}z| \leqslant \sum_{j=1}^m \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q_j)} = m. \qquad \square$$

5. Cubes and chains

In this section, we describe how to “reflect” cubes at $\overline{\Gamma}$ if Ω is subject to Assumption 2.1 and establish some natural properties of theses “reflections”. This is an adaption of an argument of Jones presented in [18]. Throughout, assume in Sections 5 and 6 that Ω is an open set subject to Assumption 2.1 which satisfies $\overline{\Omega} \neq \mathbb{R}^d$. (When Ω is dense in $\mathbb{R}^d$, Theorem 1.2 follows in a trivial way. The details will be presented separately in the proof of the theorem). Recall that we assume $\operatorname{diam}(\Omega_m) \geqslant \lambda\delta$, where $(\Omega_m)_m$ are the connected components of Ω whose boundary hits Γ. This is in contrast to [18] where Jones assumes without loss of generality (by scaling) that the domain has radius at least 1 and that δ is at most 1. However, this has the disadvantage that homogeneous estimates will only be achievable on small scales even if $\delta = \infty$ and the domain is unbounded. We will comment on this topic later on in Remark 6.12.

Lemma 5.1. — *We have* $|\Gamma| = 0$.

Proof. — Fix $x_0 \in \Gamma$ and $y \in \Omega$ with $|x_0 - y| < \frac{\delta}{2}$. Let Q be *any* cube in $\mathbb{R}^d$ centered in x_0 with $l(Q) \leqslant \frac{1}{2}|x_0 - y|$. We will show that $[\mathbb{R}^d \setminus \Gamma] \cap Q$ has Lebesgue measure comparable to that of Q. Let $x \in \Omega$ with $|x - x_0| \leqslant \frac{1}{8} l(Q)$. Then, we have

$$(5.1) \qquad |x-y| \geqslant \frac{15}{8} l(Q) \qquad \text{and} \qquad |x-y| \leqslant \frac{17}{16}|x_0 - y|.$$

Let γ be a path connecting x and y subject to Assumption 2.1 (note that $|x-y| < \delta$ is either void if $\delta = \infty$ or otherwise it follows from the second inequality in (5.1)). By virtue of (5.1), the intermediate value theorem implies that there exists $z \in \gamma$ with $|x - z| = \frac{1}{8} l(Q)$. This point lies in $\frac{1}{2}Q$ by construction. Moreover, (CC-Γ) together with $|y-z| \geqslant |x-y| - |x-z|$ implies

$$\mathrm{d}(z,\Gamma) \geqslant \frac{\varepsilon l(Q)}{8} \frac{|x-y| - |x-z|}{|x-y|} \geqslant \frac{\varepsilon l(Q)}{8}\Big(1 - \frac{l(Q)}{8|x-y|}\Big) \geqslant \frac{7\varepsilon}{60} l(Q).$$

Thus, $\limsup_{l(Q)\to 0} \frac{|[\mathbb{R}^d \setminus \Gamma] \cap Q|}{|Q|} > 0$, where the lim sup is taken over all cubes centered at x_0. Since $\chi_{\mathbb{R}^d \setminus \Gamma}(x_0) = 0$ and $\chi_{\mathbb{R}^d \setminus \Gamma} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ Lebesgue's differentiation theorem implies $|\Gamma| = 0$. □

To proceed, we define two families of cubes. The family of interior cubes is given by

$$\mathcal{W}_i := \{Q \in \mathcal{W}(\overline{\Gamma}) : Q \cap \Omega \neq \emptyset\}.$$

These interior cubes will be the reflections of exterior cubes $\mathcal{W}_e$. To define $\mathcal{W}_e$ we use numbers $A > 0$ and $B > 2$ whose values are to be fixed during this section and define

$$\mathcal{W}_e := \{Q \in \mathcal{W}(\overline{\Omega}) : \operatorname{diam}(Q) \leqslant A\delta \text{ and } \mathrm{d}(Q, \Gamma) < B\,\mathrm{d}(Q, \partial\Omega \setminus \Gamma)\}.$$

Remark 5.2. — First, the collection $\mathcal{W}_e$ is empty if and only if $D = \partial\Omega$. Indeed, if $D = \partial\Omega$ then the second condition in the definition $\mathcal{W}_e$ can never be fulfilled. To the contrary, if Γ is non-empty, then, using the relative openness of Γ, one can fix a ball centered in Γ that does not intersect D, and small cubes inside this ball will satisfy both conditions. Second, if $D \neq \partial\Omega$, then for a cube $Q \in \mathcal{W}_e$ we have

$$\mathrm{d}(Q, \Omega) = \min\{\mathrm{d}(Q, \Gamma), \mathrm{d}(Q, \partial\Omega \setminus \Gamma)\} \geqslant B^{-1}\,\mathrm{d}(Q, \Gamma)$$

what implies that for all $Q \in \mathcal{W}_e$ it holds

$$\mathrm{d}(Q, \Omega) \leqslant \mathrm{d}(Q, \Gamma) \leqslant B\,\mathrm{d}(Q, \Omega). \tag{5.2}$$

Thus, the diameter of Q is comparable to its distance to Γ.

For the rest of this section, we assume that $\Gamma \neq \emptyset$. Before we present how to 'reflect' cubes, we prove a technical lemma that, given an exterior cube $Q \in \mathcal{W}_e$, allows us to find a connected component of Ω whose boundary intersects Γ and which is not too far away from Q.

Lemma 5.3. — *Let $Q \in \mathcal{W}_e$. Then there exists a connected component Ω_m of Ω with $\Gamma \cap \partial\Omega_m \neq \emptyset$ and $x \in \Omega_m$ with*

$$\mathrm{d}(x, Q) \leqslant 5B \operatorname{diam}(Q).$$

Proof. — By (ii) and Remark 5.2, there exists $x' \in \overline{\Gamma}$ such that $\mathrm{d}(x', Q) \leqslant 4B \operatorname{diam}(Q)$. Since $x' \in \overline{\Gamma}$ there is $x'' \in \Gamma$ with $\mathrm{d}(x'', Q) \leqslant \frac{9}{2} B \operatorname{diam}(Q)$.

Denote the at most countable family of connected components of Ω whose boundary has a non-empty intersection with Γ by $\{\Omega_m\}_m$ and the connected components whose boundary has an empty intersection with Γ by $\{\Upsilon_m\}_m$.

If there is Ω_m with $x'' \in \partial\Omega_m$, then the proof is finished. If not, pick a sequence $(x_n)_n$ in Ω that converges to $x'' \in \Gamma \subseteq \partial\Omega$. If almost all x_n are contained in the union of the Ω_m, this concludes the proof as well. Otherwise, choose a subsequence (again denoted by x_n) for which there are indices m_n such that $x_n \in \Upsilon_{m_n}$. Furthermore, $x'' \in \overline{\Upsilon}_m^c$ for all m since $\Gamma \cap \partial\Upsilon_m = \emptyset$. Now, by connecting x'' and x_n by a straight line, the intermediate value theorem implies the existence of a point $x_n' \in \partial\Upsilon_{m_n}$ with

$$|x'' - x_n'| \leqslant |x'' - x_n|.$$

Passing to the limit $n \to \infty$ yields $x'' \in D$ by the closedness of D and thus a contradiction. □

The following lemma assigns to every cube in $\mathcal{W}_e$ a "reflected" cube in $\mathcal{W}_i$. For the rest of Sections 5 and 6 we will reserve the letter N to denote the constant N appearing in Lemma 4.1 applied with $k = 2K$, where K is the number from Assumption 2.1. Notice that N solely depends on d and K. For the rest of this paper we make the following agreement.

Agreement 5.4. — *If X and Y are two quantities and if there exists a constant C depending only on d, p, K, λ, and ε such that $X \leqslant CY$ holds, then we will write $X \lesssim Y$ or $Y \gtrsim X$. If both $\frac{Y}{C} \leqslant X \leqslant CY$ holds, then we will write $X \simeq Y$.*

Lemma 5.5. — *There exist constants $C_1 = C_1(N, \varepsilon) > 0$ and $C_2 = C_2(\lambda) > 0$ such that if $AB \leqslant C_1$ and $B \geqslant C_2$, then for every $Q \in \mathcal{W}_e$ there exists a cube $R \in \mathcal{W}_i$ satisfying*

$$\operatorname{diam}(Q) \leqslant \operatorname{diam}(R) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q) \tag{5.3}$$

and

$$\mathrm{d}(R, Q) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q). \tag{5.4}$$

Proof. — Fix $Q \in \mathcal{W}_e$ and recall that $\operatorname{diam}(Q) \leqslant A\delta$ by definition of $\mathcal{W}_e$ and $B > 2$. By Lemma 5.3 there exists a connected component Ω_m of Ω with $\Gamma \cap \partial\Omega_m \neq \emptyset$ and $x \in \Omega_m$ with $\mathrm{d}(x, Q) \leqslant 5B \operatorname{diam}(Q)$. We introduce the additional lower bound $B \geqslant 3/\lambda$, which is only needed in the case $\delta < \infty$ but we choose B always that large for good measure.

So, in the case $\delta < \infty$, since $(AB)^{-1} \operatorname{diam}(Q) \leqslant \delta B^{-1} < \min(\delta, \lambda\delta/2) \leqslant \min(\delta, \operatorname{diam}(\Omega_m)/2)$ according to (DC), we find $y \in \Omega_m$ satisfying

$$|x - y| = (AB)^{-1} \operatorname{diam}(Q) \qquad \text{and} \qquad |x - y| < \delta. \tag{5.5}$$

If $\delta = \infty$, then Ω_m is unbounded, so we again find $y \in \Omega_m$ satisfying the first condition whereas the second becomes void for Assumption 2.1.

Hence, let γ be a path provided by Assumption 2.1 connecting x and y, and let $z \in \gamma$ with $|x-z| = \frac{1}{2}|x-y|$. Estimate by virtue of (CC-Γ) and (5.5)

$$\mathrm{d}(z,\Gamma) \geqslant \frac{\varepsilon}{2}|y-z| \geqslant \frac{\varepsilon}{2}\big(|x-y|-|x-z|\big) = \frac{\varepsilon}{4}(AB)^{-1}\operatorname{diam}(Q). \tag{5.6}$$

By Assumption 2.1 we have $\mathrm{k}_{\Xi}(z,\Omega) \leqslant K$, hence there exists $z' \in \Omega$ with $\mathrm{k}_{\Xi}(z,z') \leqslant 2K$. Thus, by Lemma 4.1 there exists an intersecting chain $Q_1,\ldots,Q_m \in \mathcal{W}(\bar{\Gamma})$ with $Q_m \cap \Omega \neq \emptyset$, $z \in Q_1$, and $m \leqslant N$. Choose the reflected cube as $R \coloneqq Q_m$. Using (ii) and (iv) one gets

$$\begin{aligned} 4\operatorname{diam}(R) \geqslant \mathrm{d}(R,\Gamma) &\\ &\geqslant \mathrm{d}(z,\Gamma) - \sum_{j=1}^{m}\operatorname{diam}(Q_j)\\ &\geqslant \mathrm{d}(z,\Gamma) - \sum_{j=1}^{m} 4^{m-j}\operatorname{diam}(R). \end{aligned}$$

Thus, by (5.6) and $m \leqslant N$

$$\frac{11+4^N}{3}\operatorname{diam}(R) \geqslant \frac{\varepsilon}{4}(AB)^{-1}\operatorname{diam}(Q).$$

Consequently, there exists $C = C(N,\varepsilon) > 0$ such that $AB \leqslant C$ implies $\operatorname{diam}(Q) \leqslant \operatorname{diam}(R)$.

In order to control $\operatorname{diam}(R)$ by $\operatorname{diam}(Q)$, employ (ii), (iv), and the triangle inequality to deduce

$$4^{1-m}\operatorname{diam}(R) \leqslant \operatorname{diam}(Q_1) \leqslant \mathrm{d}(z,\Gamma) \leqslant \mathrm{d}(z,Q) + \operatorname{diam}(Q) + \mathrm{d}(Q,\Gamma).$$

The right-hand side is estimated by the triangle inequality, followed by (5.2) and (ii), the choice $|x-z| = \frac{1}{2}|x-y|$ combined with (5.5), and $\mathrm{d}(x,Q) \leqslant 5B\operatorname{diam}(Q)$, yielding

$$\begin{aligned} \mathrm{d}(z,Q) + \operatorname{diam}(Q) + \mathrm{d}(Q,\Gamma) &\leqslant |z-x| + \mathrm{d}(x,Q) + \operatorname{diam}(Q) + B\,\mathrm{d}(Q,\Omega)\\ &\leqslant ((2AB)^{-1} + 1 + 9B)\operatorname{diam}(Q). \end{aligned}$$

Taking into account that $\mathrm{d}(z,R) \leqslant \operatorname{diam}(R)(4^m-1)/3$ (estimate the sizes of the cubes in the connecting chain using a geometric sum), the distance from R to Q is estimated similarly, yielding

$$\begin{aligned} \mathrm{d}(R,Q) &\leqslant |x-z| + \mathrm{d}(z,R) + \mathrm{d}(x,Q)\\ &\leqslant ((2AB)^{-1} + 5B)\operatorname{diam}(Q) + \frac{4^m-1}{3}\operatorname{diam}(R). \end{aligned}$$

Together with the previous estimate, this concludes the proof. □

For the rest of this article, we fix the notation that if $Q \in \mathcal{W}_e$ and $R \in \mathcal{W}_i$ is the cube constructed in Lemma 5.5, then R is denoted by $R = Q^*$ and Q^* is called the *reflected* cube of Q. The next lemma gives a bound on the distance of reflected cubes of two intersecting cubes. Its proof is a direct consequence of Lemma 5.5 and (iv), and is thus omitted.

Lemma 5.6. — *If $Q_1, Q_2 \in \mathcal{W}_e$ with $Q_1 \cap Q_2 \neq \emptyset$, then*

$$\mathrm{d}(Q_1^*, Q_2^*) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_1).$$

In the proof of the boundedness of the extension operator, one needs to connect Whitney cubes by appropriate touching chains. The following lemma presents a basic principle of how to build a chain out of a path γ and how the quantities $\operatorname{length}(\gamma)$ and $\mathrm{d}(\gamma, \Gamma)$ translate into the length of the chain and the distance of the cubes of the chain to Γ.

Lemma 5.7. — *Let $R_1, R_2 \in \mathcal{W}(\bar{\Gamma})$ with $R_1 \neq R_2$ and let $x \in R_1$, $y \in R_2$, and γ be a rectifiable path in $\mathbb{R}^d \setminus \bar{\Gamma}$ connecting x and y. Assume that there exist constants $C_1, C_2 > 0$ such that $\operatorname{length}(\gamma) \leqslant C_1 \operatorname{diam}(R_1)$ and $\mathrm{d}(z, \Gamma) \geqslant C_2 \operatorname{diam}(R_1)$ for all $z \in \gamma$. Then there exists a touching chain of cubes $R_1 = S_1, \dots, S_m = R_2$ in $\mathcal{W}(\bar{\Gamma})$, where m is bounded by a number depending only on d, C_1, and C_2. Moreover,*

$$\frac{C_2}{5} \operatorname{diam}(R_1) \leqslant \operatorname{diam}(S_i) \leqslant (5 + C_1) \operatorname{diam}(R_1) \qquad (i = 1, \dots, m).$$

Proof. — Let $\mathcal{S}$ be the *finite* set of cubes in $\mathcal{W}(\bar{\Gamma})$ intersecting γ. For $S \in \mathcal{S}$ one finds by (4.1) and by assumption that $\operatorname{diam}(S) \geqslant \frac{C_2}{5} \operatorname{diam}(R_1)$. Fix $z \in S \cap \gamma$, then

$$\mathrm{d}(z, \Gamma) \leqslant \mathrm{d}(x, \Gamma) + |x - z| \leqslant 5 \operatorname{diam}(R_1) + \operatorname{length}(\gamma) \leqslant (5 + C_1) \operatorname{diam}(R_1),$$

so that $\operatorname{diam}(S) \leqslant (5 + C_1) \operatorname{diam}(R_1)$ by (ii). This, together with $\operatorname{length}(\gamma) \leqslant C_1 \operatorname{diam}(R_1)$ implies that $S \subseteq \mathrm{B}(x, (5 + 2C_1) \operatorname{diam}(R_1))$. Because all elements of $\mathcal{S}$ are mutually disjoint one finds

$$\sharp(\mathcal{S}) \leqslant \frac{|\mathrm{B}(x, (5 + 2C_1) \operatorname{diam}(R_1))|}{(\frac{C_2}{5\sqrt{d}} \operatorname{diam}(R_1))^d} = \omega_d \Big(\frac{5\sqrt{d}(5 + 2C_1)}{C_2} \Big)^d,$$

where $\sharp(\mathcal{S})$ denotes the cardinality of $\mathcal{S}$ and $\omega_d := |\mathrm{B}(0, 1)|$. By (iii) the elements of $\mathcal{S}$ are dyadic and thus one finds a subset of $\mathcal{S}$ which is a touching chain starting at R_1 and ending at R_2. □

Lemma 5.8. — *There exist constants $C_1, C_2 > 0$ depending only on ε, d, λ, and K such that if $A \leqslant C_1$ and $B \geqslant C_2$ and if $Q_j, Q_k \in \mathcal{W}_e$ with $Q_j \cap Q_k \neq \emptyset$, then there exists a touching chain $F_{j,k} = \{Q_j^* = S_1, \dots, S_m =$*

$Q_k^*\}$ *of cubes in* $\mathcal{W}(\overline{\Gamma})$ *connecting* Q_j^* *and* Q_k^*, *where* m *can be bounded uniformly by a constant depending only on* ε, d, K, A, *and* B. *Moreover, there exist* $K_1, K_2 > 0$ *depending only on* ε, d, K, A, *and* B *such that*

$$K_1 \operatorname{diam}(Q_j) \leqslant \operatorname{diam}(S_i) \leqslant K_2 \operatorname{diam}(Q_j) \qquad (i = 1, \dots, m).$$

Proof. — If $Q_j^* = Q_k^*$ there is nothing to show. Thus, assume $Q_j^* \neq Q_k^*$ and let $x \in Q_j^*$ and $y \in Q_k^*$. We show in the following that the assumptions of Lemma 5.7 are satisfied.

Using Lemmas 5.5 and 5.6 in conjunction with (iv) gives

$$\begin{aligned} |x-y| &\leqslant \mathrm{d}(Q_j^*, Q_k^*) + \operatorname{diam}(Q_j^*) + \operatorname{diam}(Q_k^*) \\ &\lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j). \end{aligned} \tag{5.7}$$

If δ is finite we get, since $Q_j \in \mathcal{W}_e$, that

$$|x-y| \lesssim (A + AB + B^{-1})\delta,$$

so we obtain $|x-y| < \delta$ when we first choose B large enough and afterwards A sufficiently small. Let γ be a path connecting x and y according to Assumption 2.1. By (LC), (5.7), and (5.3) one finds

$$\operatorname{length}(\gamma) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j^*).$$

To estimate the distance between each $z \in \gamma$ and Γ, notice that if $|x-z| \leqslant \frac{1}{2}\operatorname{diam}(Q_j^*)$, then $\mathrm{d}(z,\Gamma) \geqslant \frac{1}{2}\operatorname{diam}(Q_j^*)$. Analogously, but by employing additionally (5.3) twice and (iv), if $|y-z| \leqslant \frac{1}{2}\operatorname{diam}(Q_k^*)$, then

$$\begin{aligned} \mathrm{d}(z,\Gamma) &\geqslant \frac{1}{2}\operatorname{diam}(Q_k^*) \\ &\geqslant \frac{1}{2}\operatorname{diam}(Q_k) \\ &\geqslant \frac{1}{8}\operatorname{diam}(Q_j) \\ &\gtrsim \big(1 + B + (AB)^{-1}\big)^{-1} \operatorname{diam}(Q_j^*). \end{aligned} \tag{5.8}$$

In the remaining case, one estimates by (CC-Γ), the calculation performed in (5.8), and (5.7) that

$$\mathrm{d}(z,\Gamma) \gtrsim \frac{\operatorname{diam}(Q_j^*)^2}{(1 + B + (AB)^{-1})|x-y|} \gtrsim \frac{\operatorname{diam}(Q_j^*)}{(1 + B + (AB)^{-1})^2}. \qquad \square$$

The following lemma provides the existence of chains that 'escape Ω' for reflections of cubes $Q \in \mathcal{W}(\overline{\Omega})$ that are close to a relatively open portion of D. These chains will be important to obtain a Poincaré inequality with a quantitative control of the constants.

Lemma 5.9. — *There exist constants $C_1, C_2 > 0$ depending only on ε, d, λ, and K such that if $A \leqslant C_1$ and $B \geqslant C_2$ and if $Q \in \mathcal{W}(\overline{\Omega}) \setminus \mathcal{W}_e$ satisfies* $\operatorname{diam}(Q) \leqslant A\delta$ *and has a non-empty intersection with a cube $Q_j \in \mathcal{W}_e$, then there exists a touching chain $F_{P,j} = \{Q_j^* = S_1, \dots, S_m\}$ of cubes in $\mathcal{W}(\overline{\Gamma})$, where m is bounded by a constant depending only on ε, d, K, A, and B and $S_m \cap Q_j$ is a dyadic cube that satisfies*

$$|S_m \cap Q_j| \gtrsim \operatorname{diam}(Q_j)^d.$$

Furthermore, all $S_i \in F_{P,j}$ satisfy

$$K_1 \operatorname{diam}(Q_j) \leqslant \operatorname{diam}(S_i) \leqslant K_2 \operatorname{diam}(Q_j) \qquad (i = 1, \dots, m).$$

The constants $K_1, K_2 > 0$ depend only on ε, d, K, A, and B.

Proof. — Let $Q_j \in \mathcal{W}_e$ be an intersecting cube of Q. Then, using properties of the Whitney cubes and $Q \notin \mathcal{W}_e$, one estimates

$$B \operatorname{d}(Q_j, \partial\Omega \setminus \Gamma) \leqslant 6B \operatorname{d}(Q, \partial\Omega \setminus \Gamma) \leqslant 6 \operatorname{d}(Q, \Gamma) \leqslant 36 \operatorname{d}(Q_j, \Gamma). \tag{5.9}$$

Let $B \geqslant 720$, then (5.9) implies $\operatorname{d}(Q_j, \partial\Omega \setminus \Gamma) = \operatorname{d}(Q_j, \Omega)$. Hence, using (5.9) again and (ii), one finds that $\operatorname{d}(Q_j, \Gamma) \geqslant \frac{B}{36} \operatorname{d}(Q_j, \Omega) \geqslant \frac{B}{36} \operatorname{diam}(Q_j)$. Let $x_0 \in D$ be such that $\operatorname{d}(x_0, Q_j) = \operatorname{d}(Q_j, \Omega) \leqslant 4 \operatorname{diam}(Q_j)$. The properties collected above then imply

$$\operatorname{d}(x_0, \Gamma) \geqslant \operatorname{d}(Q_j, \Gamma) - \operatorname{d}(x_0, Q_j) - \operatorname{diam}(Q_j) \geqslant (36^{-1}B - 5) \operatorname{diam}(Q_j),$$

and if y is any point in $\mathrm{B}(x_0, 5 \operatorname{diam}(Q_j))$ then the previous estimate delivers

$$\operatorname{d}(y, \Gamma) \geqslant \operatorname{d}(x_0, \Gamma) - 5 \operatorname{diam}(Q_j) \geqslant (36^{-1}B - 10) \operatorname{diam}(Q_j) \geqslant 10 \operatorname{diam}(Q_j).$$

Fix $y \in \mathrm{B}(x_0, 5 \operatorname{diam}(Q_j)) \cap \Omega$. Notice that the midpoint z of Q_j is contained in $\mathrm{B}(x_0, 5 \operatorname{diam}(Q_j))$. Thus, each point on the line segment γ_1 connecting y to z has a distance larger than $10 \operatorname{diam}(Q_j)$ to Γ.

For $x \in Q_j^* \cap \Omega$ Lemma 5.5 together with $\{y\} \cup Q_j \subseteq \mathrm{B}(x_0, 6 \operatorname{diam}(Q_j))$ implies

$$\begin{aligned} |x - y| &\leqslant \operatorname{d}(y, Q_j) + \operatorname{d}(Q_j, Q_j^*) + \operatorname{diam}(Q_j) + \operatorname{diam}(Q_j^*) \\ &\lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j). \end{aligned}$$

If δ is finite, we can ensure $|x - y| < \delta$ using exactly the same argument as in the proof of Lemma 5.8 and otherwise this condition is again meaningless. Let γ_2 be the path connecting x and y subject to Assumption 2.1 and let $R \in \mathcal{W}(\overline{\Gamma})$ with $y \in R$. Since $\operatorname{d}(R, \Gamma) \geqslant C \operatorname{diam}(Q_j^*)$ for some $C > 0$ depending only on ε, K, d, A, and B, one concludes as in the proof of Lemma 5.8 that the path γ_2, and hence, by the consideration above, also the path $\gamma = \gamma_1 + \gamma_2$ which connects $x \in Q_j^*$ with $z \in Q_j$ satisfies the assumptions of Lemma 5.7, where Q_j^* fulfills the role of R_1 and R_2 is some cube in $\mathcal{W}(\overline{\Gamma})$ that contains z. Note that the constants appearing in Lemma 5.7 depend only on ε, K, d, A, and B.

As in the statement of the lemma we write S_m for R_2 and distinguish cases for the relation between S_m and Q_j. Since $Q_j \cap S_m \neq \emptyset$ and since Whitney cubes are dyadic, it either holds $S_m \subseteq Q_j$ or $Q_j \subseteq S_m$. If $Q_j \subseteq S_m$ the proof is finished. If $S_m \subseteq Q_j$, then

$$4 \operatorname{diam}(S_m) \geqslant \mathrm{d}(S_m, \Gamma) \geqslant \mathrm{d}(Q_j, \Gamma) \geqslant 36^{-1} B \operatorname{diam}(Q_j),$$

so that $|S_m \cap Q_j| \gtrsim \operatorname{diam}(Q_j)^d$. □

The next lemma shows that for a fixed cube $R \in \mathcal{W}_i$, there are only finitely many cubes in $\mathcal{W}_e$, whose reflected cube is R.

Lemma 5.10. — *There is a constant $C \in \mathbb{N}$ such that for each $R \in \mathcal{W}_i$ there are at most C cubes $Q \in \mathcal{W}_e$ such that $Q^* = R$, where C solely depends on d, K, A, B, and ε.*

Proof. — Let α denote the constant from (5.4) and let $Q \in \mathcal{W}_e$ with reflected cube R, then it follows with (5.3) that $\mathrm{d}(R, Q) \leqslant \alpha \operatorname{diam}(R)$. So, if x_R denotes the center of R, every cube Q with $Q^* = R$ must be contained in $\mathrm{B}(x_R, (\alpha + \frac{3}{2}) \operatorname{diam}(R))$. Because for those cubes $\operatorname{diam}(Q)$ is controlled from below by $\operatorname{diam}(R)$ according to (5.3) and because cubes from $\mathcal{W}_e$ have disjoint interiors, the lemma follows by a counting argument. □

6. Construction of the extension operator and exterior estimates

This section is devoted to the construction of the extension operator from Theorem 1.2. We also establish estimates for the operator on $\overline{\Omega}^c$. To do so, we start with a preparatory part on (adapted) polynomials, followed by some overlap considerations. We proceed with the construction of the extension operator, in which the adapted polynomials will appear, followed by the exterior estimates, for which we will need the results on overlap.

Agreement 6.1. — *If not otherwise mentioned, the symbols k and p are supposed to refer to the parameters in Theorem 1.2. The numbers A and B which were introduced in Section 5 will be considered as fixed numbers depending only on ε, d, K, and λ such that all statements in Section 5 are valid. From now on we will use the symbols $\lesssim$ and $\gtrsim$ in a more liberal way than described in Agreement 5.4, that is, implicit constants are allowed to depend on all fixed parameters including A, B, and δ.*

Polynomial fitting and Poincaré type estimates

We record some results on polynomial approximation and Poincaré type estimates. Most of them stem from [5] and were used there for a similar purpose.

We start out with the following generic norm comparison lemma for polynomials of fixed degree, see [5, Lemma 2.3]. Recall the set of polynomials $\mathcal{P}_m$ introduced in the notation section.

Lemma 6.2. — *Let Q, R be cubes with $R \subseteq Q$ and assume that there exists a constant $\kappa > 0$ such that $|R| \geqslant \kappa |Q|$. Then for each polynomial P of degree at most m one has*

$$\|P\|_{\mathrm{L}^p(Q)} \lesssim \|P\|_{\mathrm{L}^p(R)},$$

where the implicit constant only depends on d, κ, p, and m. In particular, if S is another cube with $S \subseteq Q$ and $|S| \geqslant \kappa |Q|$, then the L^p norms over R and S are equivalent norms on $\mathcal{P}_m$ and the implicit constants only depend on d, κ, p, and m.

The following lemma provides "adapted" polynomials together with corresponding Poincaré type estimates. A proof can be found in [5, Theorem 4.5, Theorem 4.7 & Remark 4.8], note that the proof of the remark still works when replacing Whitney cubes by cubes of the same size.

Lemma 6.3. — *Let Q be a cube, R a touching cube of Q of the same size, and $k \geqslant 0$ an integer. Then there exists a projection $P : \mathrm{L}^1(Q) \to \mathcal{P}_{k-1}$ that satisfies the estimate*

$$\|\partial^\alpha P f\|_{\mathrm{L}^p(Q)} \lesssim \|\partial^\alpha f\|_{\mathrm{L}^p(Q)} \tag{6.1}$$

for $f \in \mathrm{C}_D^\infty(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(Q)$, $|\alpha| \leqslant k$ and $1 \leqslant p \leqslant \infty$. Moreover, the Poincaré type estimate

$$\|\partial^\alpha (f - Pf)\|_{\mathrm{L}^p(Q \cup R)} \lesssim \operatorname{diam}(Q)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(Q \cup R)}, \tag{6.2}$$

holds for $f \in \mathrm{C}_D^\infty(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(Q \cup R)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, and $1 \leqslant p \leqslant \infty$. The implicit constants depend only on d, k, and p. Of course, the case $Q = R$ is also permitted.

Remarks 6.4.

(1) The polynomial Pf will be denoted by $(f)_Q$. The case $|\alpha| = k$ in (6.1) was not stated in [5] but follows since the degree of Pf is at most $k-1$.
(2) That the projection is always meaningfully defined on $\mathrm{L}^1(Q)$ becomes evident from (4.2) in [5].
(3) In the case $\alpha = 0$ we can drop the intersection with $\mathrm{C}_D^\infty(\mathbb{R}^d)$ in both estimates in Lemma 6.3. This follows from a direct computation using the representation formula for the projection given in [5].

Combining these results gives a Poincaré type estimate where the polynomial is only adapted to a subcube of the domain of integration.

Corollary 6.5. — *Let Q and R be cubes with $R \subseteq Q$ such that there is $\kappa > 0$ with $|R| \geqslant \kappa |Q|$. Then with $f \in \mathrm{C}_D^\infty(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(Q)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, and $1 \leqslant p \leqslant \infty$ we obtain*

$$\|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(Q)} \lesssim \operatorname{diam}(Q)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(Q)},$$

where the implicit constant does only depend on d, k, p, and κ. We may also replace Q by the union of two touching cubes of the same size where one of them contains R as a subcube.

Proof.

Step 1. — We start with the case that Q is a single cube. Using Lemma 6.3 and Lemma 6.2 we get from the fact that $\partial^\alpha((f)_Q - (f)_R)$ is a polynomial of degree at most $k-1$ the estimate

$$\begin{aligned}\|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(Q)} &\leqslant \|\partial^\alpha (f - (f)_Q)\|_{\mathrm{L}^p(Q)} + \|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(Q)} \\ &\lesssim \operatorname{diam}(Q)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(Q)} + \|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(R)}.\end{aligned}$$

The first term satisfies the desired estimate, so we focus on the second one. Using that $R \subseteq Q$ and Lemma 6.3 twice, we estimate further

$$\begin{aligned}\|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(R)} &\leqslant \|\partial^\alpha (f - (f)_Q)\|_{\mathrm{L}^p(R)} + \|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(R)} \\ &\lesssim \|\partial^\alpha (f - (f)_Q)\|_{\mathrm{L}^p(Q)} + \operatorname{diam}(R)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(R)} \\ &\lesssim \operatorname{diam}(Q)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(Q)}.\end{aligned}$$

Step 2. — Now, assume that $Q = Q_1 \cup Q_2$ is the union of two touching cubes with $R \subseteq Q_1$ and $|R| \geqslant \kappa |Q_1|$. We reduce this case to the already shown case. Start with the triangle inequality to get

$$\|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(Q)} \leqslant \|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(Q_1)} + \|\partial^\alpha (f - (f)_R)\|_{\mathrm{L}^p(Q_2)}.$$

The first term is fine by Step 1 and for the second we continue with

$$\begin{aligned}\|\partial^\alpha(f-(f)_R)\|_{\mathrm{L}^p(Q_2)}\\ \leqslant \|\partial^\alpha(f-(f)_{Q_2})\|_{\mathrm{L}^p(Q_2)}+\|\partial^\alpha((f)_{Q_2}-(f)_R)\|_{\mathrm{L}^p(Q_2)}.\end{aligned}$$

Again, the first term is good and for the other one we exploit $Q_1 \subseteq 3Q_2$ to derive with Lemma 6.2

$$\begin{aligned}\|\partial^\alpha((f)_{Q_2}-(f)_R)\|_{\mathrm{L}^p(Q_2)} &\lesssim \|\partial^\alpha((f)_{Q_2}-(f)_R)\|_{\mathrm{L}^p(Q_1)}\\ &\leqslant \|\partial^\alpha(f-(f)_{Q_2})\|_{\mathrm{L}^p(Q)}+\|\partial^\alpha(f-(f)_R)\|_{\mathrm{L}^p(Q_1)}.\end{aligned}$$

The first term is good by Lemma 6.3 and the second by Step 1. □

Some overlap considerations

Let $Q_j \in \mathcal{W}_e$ and let $Q \in \mathcal{W}(\overline{\Omega}) \setminus \mathcal{W}_e$ be such that it intersects a cube in $\mathcal{W}_e$ and satisfies $\operatorname{diam}(Q) \leqslant A\delta$. Let $F_{j,k}$ be as in Lemma 5.8 and $F_{P,k}$ be as in Lemma 5.9. Define

$$F(Q_j) \coloneqq \bigcup_{\substack{Q_k \in \mathcal{W}_e\\ Q_j \cap Q_k \neq \emptyset}} \bigcup_{S \in F_{j,k}} 2S \qquad \text{and} \qquad F_P(Q) \coloneqq \bigcup_{\substack{Q_k \in \mathcal{W}_e\\ Q \cap Q_k \neq \emptyset}} \bigcup_{S \in F_{P,k}} 2S.$$

We count how many of these "extended" chains $F(Q_j)$ and $F_P(Q)$ can intersect a fixed point $x \in \mathbb{R}^d$. To be concise, we only present the case of $F(Q_j)$.

By Lemma 5.8, we know that a chain $F_{j,k}$ has length less than a constant M which only depends on d, K, λ, and ε. If $x \in F(Q_j)$, then there exist $k \in \mathbb{N}$ and $S \in F_{j,k}$ with $x \in 2S$. Assume $R \in \mathcal{W}(\overline{\Gamma})$ is any cube such that also $x \in 2R$. By (ii) and an elementary geometric consideration one infers for $z \in S$ that

$$4\operatorname{diam}(R) \geqslant \mathrm{d}(R,\Gamma) \geqslant \mathrm{d}(z,\Gamma) - |x-z| - \frac{3}{2}\operatorname{diam}(R).$$

Pick some z that satisfies $|x-z| \leqslant \operatorname{diam}(S)/2$. Then

$$4\operatorname{diam}(R) \geqslant \mathrm{d}(S,\Gamma) - \frac{1}{2}\operatorname{diam}(S) - \frac{3}{2}\operatorname{diam}(R) \geqslant \frac{1}{2}\operatorname{diam}(S) - \frac{3}{2}\operatorname{diam}(R).$$

By symmetry (interchange S and R) this implies that

$$\frac{1}{11}\operatorname{diam}(S) \leqslant \operatorname{diam}(R) \leqslant 11\operatorname{diam}(S). \tag{6.3}$$

Now let $F_{\alpha,\beta}$ be another chain such that $x \in \bigcup_{S \in F_{\alpha,\beta}} 2S$. This means that there is a cube in $F_{\alpha,\beta}$ that fulfills the role of R above. Since Q_α^* and R as well as Q_j^* and S are connected by touching chains of Whitney cubes each of length at most M, we deduce from (6.3) that $\operatorname{diam}(Q_\alpha^*) \approx \operatorname{diam}(Q_j^*)$ and conclude $\mathrm{d}(Q_j^*, Q_\alpha^*) \lesssim \operatorname{diam}(Q_j^*)$. Then the usual counting argument yields a bound on such reflected cubes Q_α^*. Finally, Lemma 5.10 implies that there exists a constant $C > 0$ that depends only on d, K, λ, and ε such that

$$\sum_{Q_j \in \mathcal{W}_e} \chi_{F(Q_j)}(x) \leqslant C. \tag{6.4}$$

Construction of the extension operator

Fix an enumeration $(Q_j)_j$ of $\mathcal{W}_e$ and take a partition of unity $(\varphi_j)_j$ on $\bigcup_{Q_j \in \mathcal{W}_e} Q_j$ valued in $[0,1]$ on $\mathbb{R}^d$ and satisfying $\operatorname{supp}(\varphi_j) \subseteq \frac{17}{16} Q_j$ as well as $\|\partial^\alpha \varphi_j\|_{\mathrm{L}^\infty} \lesssim \operatorname{diam}(Q_j)^{-|\alpha|}$ for $|\alpha| \leqslant k$ and with an implicit constant only depending on k.

Let f be a measurable function on Ω and $A \subseteq \mathbb{R}^d$ closed. Write $E_A f$ for the zero extension of f to A. Clearly, E_A is isometric from $\mathrm{L}^p(A \cap \Omega)$ to $\mathrm{L}^p(A)$ for all $1 \leqslant p \leqslant \infty$. Moreover, if $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ and $A \cap \overline{\Gamma} = \emptyset$, then $E_A f$ is again in $\mathrm{C}_D^\infty(A) \cap \mathrm{W}^{k,p}(A)$. A relevant example is $A = Q \in \mathcal{W}_i$. Note that then $\|\partial^\alpha E_A f\|_{\mathrm{L}^p(A)} = \|\partial^\alpha f\|_{\mathrm{L}^p(A \cap \Omega)}$ holds for any $|\alpha| \leqslant k$.

Recall the notation introduced in Remark 6.4. Define the extension operator E on some locally integrable f by

$$Ef(x) \coloneqq \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in D, \\ \sum_{Q_j \in \mathcal{W}_e} (E_{Q_j^*} f)_{Q_j^*}(x) \varphi_j(x), & x \in \overline{\Omega}^c. \end{cases}$$

If $\mathcal{W}_e$ is empty (which is the case if $D = \partial\Omega$ according to Remark 5.2) then the sum is empty and its value is considered to be zero.

Remark 6.6. — Note that due to the properties of the Whitney cubes, $\varphi_j(x) \neq 0$ only for finitely many j. If $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ in the sense of Definition 1.1, then Ef is defined almost everywhere on $\mathbb{R}^d$ according to Lemma 5.1. Moreover, Ef is smooth on $\overline{\Omega}^c$ by construction. Due to Remark 6.4, E restricts to a bounded operator from $\mathrm{L}^p(\Omega)$ to $\mathrm{L}^p(\mathbb{R}^d)$ for all $1 \leqslant p \leqslant \infty$. If $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ then Ef vanishes almost everywhere around D. Indeed, this follows from the support assumption on f and the fact that Q_j^* is close to D if x is close to D, see (5.4).

Estimates for the extension operator

We show estimates for the extension operator on different types of cubes. The overlap considerations from before will permit us to sum them up in Proposition 6.11 to arrive at exterior estimates for the extension operator.

Lemma 6.7. — *Let $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, and $1 \leqslant p \leqslant \infty$. If $S_1, \dots, S_m$ is a touching chain of Whitney cubes with respect to $\overline{\Gamma}$ whose length is bounded by a constant M, then*

$$\|\partial^\alpha((E_{S_1} f)_{S_1} - (E_{S_m} f)_{S_m})\|_{\mathrm{L}^p(S_1)} \lesssim \operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(\bigcup_{r=1}^m (2S_r) \cap \Omega)},$$

where the implicit constant does only depend on d, k, p, and M. The assertion remains true if the chain consists of cubes in Ξ of fixed size (not necessarily Whitney cubes). In that case, the set $\bigcup_{r=1}^m (2S_r) \cap \Omega$ in the L^p-norm on the right-hand side can be replaced by $\bigcup_{r=1}^m S_r \cap \Omega$.

Proof. — We focus on the case of Whitney cubes; the other case is even simpler.

Note first that the sizes of cubes from the chain are pairwise comparable due to the bound on the chain length. Using Lemma 6.2 (observe that the whole chain is contained in a comparably larger cube) we get

$$\begin{aligned}
&\|\partial^\alpha((E_{S_1} f)_{S_1} - (E_{S_m} f)_{S_m})\|_{\mathrm{L}^p(S_1)} \\
&\qquad \lesssim \sum_{r=1}^{m-1} \|\partial^\alpha((E_{S_r} f)_{S_r} - (E_{S_{r+1}} f)_{S_{r+1}})\|_{\mathrm{L}^p(S_{r+1})} \\
&\qquad = \sum_{r=1}^{m-1} \|\partial^\alpha((E_{S_r \cup S_{r+1}} f)_{S_r}) - (E_{S_{r+1}} f)_{S_{r+1}})\|_{\mathrm{L}^p(S_{r+1})} \\
&\qquad \leqslant \sum_{r=1}^{m-1} \|\partial^\alpha((E_{S_r \cup S_{r+1}} f)_{S_r} - E_{S_{r+1}} f)\|_{\mathrm{L}^p(S_{r+1})} \\
&\qquad\qquad + \|\partial^\alpha(E_{S_{r+1}} f - (E_{S_{r+1}} f)_{S_{r+1}})\|_{\mathrm{L}^p(S_{r+1})} \\
&\qquad \leqslant \sum_{r=1}^{m-1} \|\partial^\alpha(E_{S_r \cup S_{r+1}} f - (E_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r \cup S_{r+1})} \\
&\qquad\qquad + \|\partial^\alpha((E_{S_{r+1}} f)_{S_{r+1}} - E_{S_{r+1}} f)\|_{\mathrm{L}^p(S_{r+1})}.
\end{aligned}$$

By virtue of Lemma 6.3, the second term in the sum on the right-hand side is controlled by $\operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_{r+1} \cap \Omega)}$.

If S_r and S_{r+1} are of the same size, the first term can be controlled using Lemma 6.3. Otherwise, assume without loss of generality that $\operatorname{diam}(S_{r+1}) <$

$\operatorname{diam}(S_r)$. Since the cubes are dyadic, it follows that $S_r \cup S_{r+1} \subseteq 2S_r$. Moreover,

$$\mathrm{d}(2S_r, \Gamma) \geqslant \mathrm{d}(S_r, \Gamma) - \frac{1}{2}\operatorname{diam}(S_r) \geqslant \frac{1}{2}\operatorname{diam}(S_r).$$

So, $E_{2S_r} f$ is a smooth extension of $E_{S_r \cup S_{r+1}} f$ to $2S_r$, in particular one has $(E_{S_r \cup S_{r+1}} f)_{S_r} = (E_{2S_r} f)_{S_r}$. Invoking Corollary 6.5 yields

$$\begin{aligned} \|\partial^\alpha (E_{S_r \cup S_{r+1}} f - (E_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r \cup S_{r+1})} & \\ &\leqslant \|\partial^\alpha (E_{2S_r} f - (E_{2S_r} f)_{S_r})\|_{\mathrm{L}^p(2S_r)} \\ &\lesssim \operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p((2S_r)\cap\Omega)}. \end{aligned}$$ □

Lemma 6.8. — *Let $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, and $1 \leqslant p \leqslant \infty$. If $Q_j \in \mathcal{W}_e$, then*

$$\|\partial^\alpha E f\|_{\mathrm{L}^p(Q_j)} \lesssim \operatorname{diam}(Q_j)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F(Q_j)\cap\Omega)} + \|\partial^\alpha f\|_{\mathrm{L}^p(Q_j^* \cap \Omega)}.$$

Proof. — Observe that φ_k vanishes on Q_j if $Q_k \cap Q_j = \emptyset$. Hence, by definition it holds $Ef = \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} (E_{Q_k^*} f)_{Q_k^*} \varphi_k$ and $\sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \varphi_k \equiv 1$ on Q_j. Consequently, using the Leibniz rule we get

$$\begin{aligned} &\|\partial^\alpha E f\|_{\mathrm{L}^p(Q_j)} \\ &\quad\leqslant \Bigg\| \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \sum_{\beta \leqslant \alpha} c_{\alpha,\beta} \partial^{\alpha-\beta} \big[(E_{Q_k^*} f)_{Q_k^*} - (E_{Q_j^*} f)_{Q_j^*}\big] \partial^\beta \varphi_k \Bigg\|_{\mathrm{L}^p(Q_j)} \\ &\qquad\qquad + \|\partial^\alpha (E_{Q_j^*} f)_{Q_j^*}\|_{\mathrm{L}^p(Q_j)} \\ &\quad =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We employ the estimate for $\partial^\beta \varphi_k$ and Lemma 6.2 (taking Lemma 5.5 into account), followed by Lemma 6.7 and (v) to derive

$$\begin{aligned} \mathrm{I} &\lesssim \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \sum_{\beta \leqslant \alpha} \operatorname{diam}(Q_k)^{-|\beta|} \|\partial^{\alpha-\beta} \big[(E_{Q_k^*} f)_{Q_k^*} - (E_{Q_j^*} f)_{Q_j^*}\big]\|_{\mathrm{L}^p(Q_j^*)} \\ &\lesssim \operatorname{diam}(Q_j)^{\ell-|\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F(Q_j)\cap\Omega)}. \end{aligned}$$

The term II is controlled by $\|\partial^\alpha f\|_{\mathrm{L}^p(Q_j^* \cap \Omega)}$ using (6.1) from Lemma 6.3; Note that we can switch to Q_j^* using Lemma 6.2 as in the estimate for term I. □

Lemma 6.9. — *Let* $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, *and* $1 \leqslant p \leqslant \infty$. *If* $Q \in \mathcal{W}(\overline{\Omega}) \setminus \mathcal{W}_e$ *intersects a cube in* $\mathcal{W}_e$ *and satisfies* $\operatorname{diam}(Q) \leqslant A\delta$, *then*

$$\|\partial^\alpha Ef\|_{\mathrm{L}^p(Q)} \lesssim \operatorname{diam}(Q)^{\ell-|\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F_P(Q)\cap\Omega)}.$$

Proof. — Note that Q satisfies the assumptions of Lemma 5.9. For $Q_j \in \mathcal{W}_e$ an intersecting cube of Q let $Q_j^* = S_1, \dots, S_{m_j}$ be the corresponding touching chain. Then

$$\begin{aligned}
&\|\partial^\alpha Ef\|_{\mathrm{L}^p(Q)} \\
&\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leqslant \alpha} \operatorname{diam}(Q_j^*)^{-|\beta|} \big\|\partial^{\alpha-\beta}\big[(E_{Q_j^*} f)_{Q_j^*}\big]\big\|_{\mathrm{L}^p(Q_j^*)} \\
&\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leqslant \alpha} \operatorname{diam}(Q_j^*)^{-|\beta|} \Big[\big\|\partial^{\alpha-\beta}\big[(E_{S_1} f)_{S_1} - (E_{S_{m_j}} f)_{S_{m_j}}\big]\big\|_{\mathrm{L}^p(S_1)} \\
&\qquad\qquad\qquad + \|\partial^{\alpha-\beta}(E_{S_{m_j}} f)_{S_{m_j}}\|_{\mathrm{L}^p(S_{m_j})}\Big].
\end{aligned}$$

By virtue of Lemma 6.7 and (v) the first term inside the double sum can be controlled by $\operatorname{diam}(Q)^{\ell-|\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(\bigcup_{r=1}^{m_j}(2S_r)\cap\Omega)}$. For the second term in the sum, note that $E_{S_{m_j}} f \equiv 0$ on the cube $S_{m_j} \cap Q_j$ and that $|S_{m_j} \cap Q_j| \gtrsim \operatorname{diam}(Q_j)^d$ by Lemma 5.9. Estimate using Lemma 6.2 and the fact that $(E_{S_{m_j}} f)_{S_{m_j} \cap Q_j}$ vanishes that

$$\begin{aligned}
&\big\|\partial^{\alpha-\beta}(E_{S_{m_j}} f)_{S_{m_j}}\big\|_{\mathrm{L}^p(S_{m_j})} \\
&\qquad\qquad \lesssim \big\|\partial^{\alpha-\beta}\big[(E_{S_{m_j}} f)_{S_{m_j}} - (E_{S_{m_j}} f)_{S_{m_j} \cap Q_j}\big]\big\|_{\mathrm{L}^p(S_{m_j} \cap Q_j)}.
\end{aligned}$$

Using Lemma 6.3 and $\operatorname{diam}(S_{m_j}) \approx \operatorname{diam}(Q_j)$ we further estimate

$$\begin{aligned}
&\leqslant \big\|\partial^{\alpha-\beta}\big[E_{S_{m_j}} f - (E_{S_{m_j}} f)_{S_{m_j}}\big]\big\|_{\mathrm{L}^p(S_{m_j})} \\
&\qquad\qquad + \big\|\partial^{\alpha-\beta}\big[E_{S_{m_j} \cap Q_j} f - (E_{S_{m_j}} f)_{S_{m_j} \cap Q_j}\big]\big\|_{\mathrm{L}^p(S_{m_j} \cap Q_j)} \\
&\lesssim \operatorname{diam}(S_{m_j})^{\ell-|\alpha|+|\beta|}\|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap \Omega)} \\
&\qquad\qquad + \operatorname{diam}(S_{m_j} \cap Q_j)^{\ell-|\alpha|+|\beta|}\|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap Q_j \cap \Omega)} \\
&\lesssim \operatorname{diam}(Q_j)^{\ell-|\alpha|+|\beta|}\|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap \Omega)}.
\end{aligned}$$

With (v) and $\operatorname{diam}(Q_j) \approx \operatorname{diam}(Q)$ this concludes the proof. □

Lemma 6.10. — *Let* $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$, $0 \leqslant \ell \leqslant k$, $|\alpha| \leqslant \ell$, *and* $1 \leqslant p \leqslant \infty$. *If* $Q \in \mathcal{W}(\overline{\Omega}) \setminus \mathcal{W}_e$ *intersects a cube in* $\mathcal{W}_e$ *and satisfies*

$\operatorname{diam}(Q) > A\delta$, *then*

$$\|\partial^\alpha Ef\|_{\mathrm{L}^p(Q)} \lesssim \max(1,\delta^{-\ell})\|f\|_{\mathrm{W}^{\ell,p}\left(\bigcup_{\substack{Q_j\in\mathcal{W}_e\\ Q\cap Q_j\neq\emptyset}} Q_j^*\cap\Omega\right)}.$$

Proof. — Note that in fact $\operatorname{diam}(Q)\approx\delta$ because Q intersects $\mathcal{W}_e$. The same is true for its intersecting Whitney cubes. Hence, with a similar calculation as in Lemma 6.8 we derive

$$\begin{aligned}\|\partial^\alpha Ef\|_{\mathrm{L}^p(Q)} &\lesssim \sum_{\substack{Q_j\in\mathcal{W}_e\\ Q\cap Q_j\neq\emptyset}}\sum_{\beta\leqslant\alpha}\delta^{-|\beta|}\|\partial^{\alpha-\beta}(E_{Q_j^*}f)_{Q_j^*}\|_{\mathrm{L}^p(Q_j^*)}\\ &\lesssim \sum_{\substack{Q_j\in\mathcal{W}_e\\ Q\cap Q_j\neq\emptyset}}\sum_{\beta\leqslant\alpha}\delta^{-|\beta|}\|\partial^{\alpha-\beta}f\|_{\mathrm{L}^p(Q_j^*\cap\Omega)}\\ &\lesssim \max(1,\delta^{-\ell})\|f\|_{\mathrm{W}^{\ell,p}\left(\bigcup_{\substack{Q_j\in\mathcal{W}_e\\ Q\cap Q_j\neq\emptyset}} Q_j^*\cap\Omega\right)}.\end{aligned}$$ □

Proposition 6.11. — *For all $1\leqslant p\leqslant\infty$ and $0\leqslant\ell\leqslant k$ there exists a constant $C>0$ depending only on d, ε, δ, k, p, λ, and K such that for all $f\in\mathrm{C}_D^\infty(\Omega)\cap\mathrm{W}^{k,p}(\Omega)$ and $|\alpha|\leqslant\ell$ one has*

$$\|\partial^\alpha Ef\|_{\mathrm{L}^p(\overline{\Omega}^c)}\leqslant C\|f\|_{\mathrm{W}^{\ell,p}(\Omega)}.$$

Proof. — The estimates for the derivatives in the case $p<\infty$ are deduced by the following calculation based on Lemmas 6.8, 6.9, and 6.10

$$\begin{aligned}&\|\partial^\alpha Ef\|^p_{\mathrm{L}^p(\overline{\Omega}^c)}\\ &\quad=\sum_{Q_j\in\mathcal{W}_e}\|\partial^\alpha Ef\|^p_{\mathrm{L}^p(Q_j)}+\sum_{\substack{Q\in\mathcal{W}(\overline{\Omega})\setminus\mathcal{W}_e\\ Q\cap\mathcal{W}_e\neq\emptyset}}\|\partial^\alpha Ef\|^p_{\mathrm{L}^p(Q)}\\ &\quad\lesssim\sum_{Q_j\in\mathcal{W}_e}\left(\operatorname{diam}(Q_j)^{(\ell-|\alpha|)p}\|\nabla^\ell f\|^p_{\mathrm{L}^p(F(Q_j)\cap\Omega)}+\|\partial^\alpha f\|^p_{\mathrm{L}^p(Q_j^*\cap\Omega)}\right)\\ &\qquad\qquad+\sum_{\substack{Q\in\mathcal{W}(\overline{\Omega})\setminus\mathcal{W}_e\\ Q\cap\mathcal{W}_e\neq\emptyset\\ \operatorname{diam}(Q)>A\delta}}\max(1,\delta^{-\ell p})\,\|f\|^p_{\mathrm{W}^{\ell,p}\left(\bigcup_{\substack{Q_j\in\mathcal{W}_e\\ Q\cap Q_j\neq\emptyset}}Q_j^*\cap\Omega\right)}\\ &\qquad\qquad+\sum_{\substack{Q\in\mathcal{W}(\overline{\Omega})\setminus\mathcal{W}_e\\ Q\cap\mathcal{W}_e\neq\emptyset\\ \operatorname{diam}(Q)\leqslant A\delta}}\operatorname{diam}(Q_j)^{(\ell-|\alpha|)p}\|\nabla^\ell f\|^p_{\mathrm{L}^p(F_P(Q)\cap\Omega)}.\end{aligned}$$

Since $\ell-|\alpha|\geqslant 0$ and $\operatorname{diam}(Q_j)$ is comparably smaller than δ, we can get rid of the factors in front of the norm terms to the cost of an implicit constant depending only on δ and k. The estimate then follows from Lemma 5.10

and from (6.4), which holds also true for the chains $F_P(Q)$ as mentioned in the discussion previous to (6.4).

The estimate in the case $p = \infty$ is even simpler because we can use the same estimates but can omit the overlap argument. □

Remark 6.12. — Assume that G and H are subsets of $\mathbb{R}^d$ such that the following version of (6.4) holds true:

$$\sum_{\substack{Q_j \in \mathcal{W}_e \\ Q_j \cap G \neq \emptyset}} \chi_{F(Q_j)}(x) \leqslant C\chi_H(x).$$

Then we may replace the $\mathrm{L}^p(\overline{\Omega}^c)$ norm on the left-hand side of the estimate in Proposition 6.11 by an $\mathrm{L}^p(G \cap \overline{\Omega}^c)$ norm and the $\mathrm{W}^{\ell,p}(\Omega)$ norm on the right-hand side by an $\mathrm{W}^{\ell,p}(H \cap \Omega)$ norm. Moreover, if $|\alpha| = \ell$ and G is contained in $N_{A\delta}(\Omega)$, then it suffices to estimate against $\|\nabla^\ell f\|_{\mathrm{L}^p(H\cap\Omega)}$. Indeed, in this case the second term in the final estimate in Proposition 6.11 vanishes. We will benefit from these observations in Section 10.

7. Approximation with smooth functions on $\mathbb{R}^d$

In this section, we show that smooth and compactly supported functions on $\mathbb{R}^d$ whose support stays away from D are dense in $\mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$. In particular, both classes of functions have the same $\mathrm{W}^{k,p}(\Omega)$-closure. We will benefit from this fact in Section 8. To do so, we use an approximation scheme similar to that introduced in [18, Section 4]. The arguments rely on techniques similar to what we have used in the construction of the extension operator.

To begin with, let $f \in \mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$, $1 \leqslant p < \infty$, and put $\kappa := \mathrm{d}(\mathrm{supp}(f), D) > 0$. Furthermore, let $\eta > 0$ quantify the approximation error. We need parameters A, B, s, t, and ρ for which we will collect several constraints in the course of this section (similar to what we have done in Section 5). Some parameters depend on each other, but there is a non-cyclic order in which they can be picked. This will enable us to prove the following proposition.

Proposition 7.1. — *Let f, η, and κ be as above. Then there exists a function g which is smooth on $\mathbb{R}^d$ and satisfies $\mathrm{d}(\mathrm{supp}(g), D) > \kappa/2$ as well as $\|f - g\|_{\mathrm{W}^{k,p}(\Omega)} \lesssim \eta$. In particular, smooth and compactly supported functions on $\mathbb{R}^d$ whose support has positive distance to D are dense in $\mathrm{C}_D^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ with respect to the $\mathrm{W}^{k,p}(\Omega)$ topology.*

For brevity, put $\widetilde{B}_t \coloneqq N_t(\partial\Omega)$ for the tubular neighborhood of size t around $\partial\Omega$ and choose $0 < s < \min(1, \kappa/2)$ in such a way that we have the estimate

$$\|f\|_{\mathrm{W}^{k,p}(\tilde{B}_{3s}\cap\Omega)} \leqslant \eta. \tag{7.1}$$

Furthermore, we define a region near Γ that stays away from D and is adapted to the support of f, namely

$$B_t \coloneqq \left\{x \in \mathbb{R}^d :\ \mathrm{d}(x,\Gamma) < t \text{ and } \mathrm{d}(x,D) > \frac{\kappa}{2}\right\}.$$

Later on, we will only deal with $t \in (0, 3s)$, so that (7.1) will in particular be applicable on $B_t \cap \Omega$.

Denote the zero extension of f to

$$\Omega_0 \coloneqq \Omega \cup \bigcup_{x\in D} \mathrm{B}(x, 3\kappa/4)$$

by $E_0 f$. Note that this function is again smooth since

$$\mathrm{d}(\mathrm{B}(x,3\kappa/4), \operatorname{supp}(f)) \geqslant \frac{\kappa}{4} \text{ for } x \in D.$$

Lemma 7.2. — *Let $x \in \Omega \setminus B_s$, then $\mathrm{B}(x,t) \subseteq \Omega_0$ for all $0 < t < s/2$.*

Proof. — Recall $s < \kappa/2$. We distinguish two cases.

Case 1: $\mathrm{d}(x,D) \leqslant \kappa/2$. — Let $z \in D$ with $|x - z| = \mathrm{d}(x,D)$. For $y \in \mathrm{B}(x,\kappa/4)$ we derive

$$|y - z| \leqslant |x - y| + |x - z| < \kappa/4 + \kappa/2 = 3\kappa/4,$$

so by choice of s we see

$$\mathrm{B}(x,t) \subseteq \mathrm{B}(x,\kappa/4) \subseteq \mathrm{B}(z,3\kappa/4) \subseteq \Omega_0.$$

Case 2: $\mathrm{d}(x,D) > \kappa/2$ and consequently $\mathrm{d}(x,\Gamma) \geqslant s$. — Then $\mathrm{d}(x,\partial\Omega) \geqslant \min(\kappa/2, s) = s > t$, therefore $\mathrm{B}(x,t) \subseteq \Omega \subseteq \Omega_0$ (keep in mind $x \in \Omega$). ☐

A family of interior cubes

Assume that ρ is a dyadic number and $\mathcal{G}$ is the collection of dyadic cubes of sidelength ρ. Recall $\Xi = \mathbb{R}^d \setminus \overline{\Gamma}$. As before, we write $(\Omega_m)_m$ for the connected components of Ω whose boundary intersects Γ and $(\Upsilon_m)_m$ for the remaining ones. Define

$$\Sigma' \coloneqq \{R \in \mathcal{G} : R \subseteq \Xi\}.$$

Moreover, we introduce the collection of cubes

$$\Sigma := \left\{ R \in \mathcal{G} : \text{there exist } S \in \mathcal{W}(\Gamma) \text{ and } m\text{:} \begin{array}{c} \operatorname{diam}(S) \geqslant A\rho, \\ R \subseteq S \;\&\; R \cap \Omega_m \neq \emptyset \end{array} \right\}.$$

These cubes take the role of $\mathcal{W}_i$ in the upcoming approximation construction. Note that $\Sigma \subseteq \Sigma'$. For $R \in \Sigma$ define enlarged cubes

$$\widehat{R} := B\,R \qquad \text{and} \qquad \widehat{\widehat{R}} := 2B\,R.$$

We claim that if we choose $\rho \leqslant \frac{\kappa}{2\sqrt{d}}$, then $R \subseteq \Omega_0$. Indeed, if $R \cap D = \emptyset$, then R is properly contained in Ω since it has a non-trivial intersection with Ω and avoids its boundary. Otherwise, let $z \in R \cap D$, then $R \subseteq \overline{\mathrm{B}}(z, \operatorname{diam}(R)) \subseteq \mathrm{B}(z, 3\kappa/4)$.

Lemma 7.3. — *There exist positive constants $C_1 = C_1(d) > 0$ and $C_2 = C_2(A, s) > 0$ such that*

$$\bigcup_m \Omega_m \setminus N_s(\Gamma) \subseteq \bigcup_{R \in \Sigma} R,$$

provided $A \geqslant C_1$ and $\rho \leqslant C_2$.

Proof. — Let $x \in \Omega_m \setminus N_s(\Gamma)$. In particular, $x \in \Xi$ and hence there exists $S \in \mathcal{W}(\overline{\Gamma})$ that contains x. Since $\mathrm{d}(x, \Gamma) \geqslant s$ by choice of x we conclude $\operatorname{diam}(S) \geqslant \frac{1}{5}\,\mathrm{d}(x, \Gamma) \geqslant \frac{s}{5}$. Hence, if we choose $\rho \leqslant \frac{s}{5A}$, then $\operatorname{diam}(S) \geqslant A\rho$. Let R be some cube in $\mathcal{G}$ that contains x. If we demand $A \geqslant \sqrt{d}$, then $R \subseteq S$ because both are dyadic cubes and they have a common point. Finally, $R \cap \Omega_m \neq \emptyset$ since $x \in \Omega_m$, so $R \in \Sigma$. □

The following lemma provides a covering of the support of f close to the Neumann boundary region by the enlarged cubes $\widehat{R}$.

Lemma 7.4. — *There exist constants $C_1 = C_1(A, \varepsilon) > 0$ and $C_2 = C_2(A, \delta, \varepsilon, \kappa, \lambda) > 0$ such that*

$$B_{2s} \cap \bigcup_m \Omega_m \subseteq \bigcup_{R \in \Sigma} \widehat{R},$$

provided $B \geqslant C_1$ and $\rho \leqslant C_2$.

Proof. — Let $x \in B_{2s} \cap \Omega_m$. Choose $\rho \leqslant \frac{\varepsilon}{80A} \min(\delta, \lambda\delta)$. Then $\frac{20A}{\varepsilon}\rho < \lambda\delta/2 \leqslant \operatorname{diam}(\Omega_m)/2$ by (DC), hence there exists some $y \in \Omega_m$ satisfying $|x - y| = \frac{20A}{\varepsilon}\rho$. Moreover, since $|x - y| < \delta$, there is a curve γ subject to Assumption 2.1 that connects x and y. Let $z \in \gamma$ with $|x - z| = \frac{1}{2}|x - y|$. Then

$$\mathrm{d}(z, \Gamma) \geqslant \frac{\varepsilon}{2}|y - z| \geqslant \frac{\varepsilon}{4}|x - y| = 5A\rho.$$

Since γ takes its values in Ξ, there exists a cube $S \in \mathcal{W}(\overline{\Gamma})$ with $z \in S$. We deduce $\operatorname{diam}(S) \geqslant \frac{1}{5}\,\mathrm{d}(z,\Gamma) \geqslant A\rho$. As in the previous lemma, there is some cube $R \in \mathcal{G}$ that contains z and consequently is a subcube of S. To conclude that $R \in \Sigma$ we must ensure that γ cannot escape Ω_m. To this end, let us assume that $z \notin \Omega_m$. Since $x \in \Omega_m$, there would be some $\widetilde{z} \in \gamma$ with $\widetilde{z} \in \partial\Omega_m$. Since $\widetilde{z} \notin \Gamma$ by definition of γ, we must have $\widetilde{z} \in D$. Now recall that by definition of B_{2s} it holds $\mathrm{d}(x,D) > \frac{\kappa}{2}$. On imposing the constraint $\rho \leqslant \frac{\varepsilon^2\kappa}{40A}$ we then get the contradiction

$$\mathrm{d}(x,D) \leqslant |x-\widetilde{z}| \leqslant \operatorname{length}(\gamma) \leqslant \frac{20A}{\varepsilon^2}\rho \leqslant \frac{\kappa}{2} < \mathrm{d}(x,D).$$

So, indeed, $z \in \Omega_m$ and therefore $R \in \Sigma$. Denote the center of R by x_R and estimate

$$|x-x_R|_\infty \leqslant |x-z| + |x_R - z|_\infty \leqslant \left(\frac{10A}{\varepsilon} + \frac{1}{2}\right)\rho.$$

So, if we choose $B \geqslant \frac{20A}{\varepsilon} + 1$, then $x \in \widehat{R}$. ☐

We have already mentioned that the collection Σ is a substitute for $\mathcal{W}_i$, so it is not surprising that we want to connect nearby cubes in Σ by a touching chain of cubes (which we allow to be in Σ') of bounded length.

Lemma 7.5. — *There are constants $C_1 = C_1(B,d,\varepsilon) > 0$, $C_2 = C_2(d,\varepsilon) > 0$, and $C_3 = C_3(B,d,\delta) > 0$ such that any pair of cubes $R,S \in \Sigma$ with $\widehat{\widehat{R}} \cap \widehat{\widehat{S}} \neq \emptyset$ can be connected by a touching chain of cubes in Σ' whose length is controlled by C_1, provided that $A \geqslant C_2$ and $\rho \leqslant C_3$.*

Proof. — By definition of Σ we can pick $x \in R \cap \Omega_m$ and $y \in S \cap \Omega_\ell$. By assumption we moreover fix $\widehat{\widehat{z}} \in \widehat{\widehat{R}} \cap \widehat{\widehat{S}}$. Let x_R, y_S denote the centers of R and S, then

$$\begin{aligned} |x-y| &\leqslant \sqrt{d}\big(|x-x_R|_\infty + |x_R - \widehat{\widehat{z}}|_\infty + |y_S - \widehat{\widehat{z}}|_\infty + |y-y_S|_\infty\big) \\ &\leqslant \sqrt{d}(1+2B)\rho. \end{aligned} \tag{7.2}$$

If we choose $\rho \leqslant \frac{\delta}{2\sqrt{d}(1+B)}$, then $|x-y| < \delta$ and we can connect x and y by a curve γ subject to Assumption 2.1. Let $z \in \gamma$ and pick $Q \in \mathcal{G}$ such that $z \in Q$. By symmetry we assume without loss of generality that $|x-z| \leqslant |y-z|$. This implies, in particular, that $|x-y| \leqslant 2|y-z|$.

Case 1: $|x-z| \leqslant \frac{4\sqrt{d}}{\varepsilon}\rho$. — Then, since $R \in \Sigma$, we find $\widetilde{Q} \in \mathcal{W}(\overline{\Gamma})$ with $R \subseteq \widetilde{Q}$ and $\operatorname{diam}(\widetilde{Q}) \geqslant A\rho$. Using $x \in R \subseteq \widetilde{Q}$, it follows

$$\mathrm{d}(x,\Gamma) \geqslant \mathrm{d}(\widetilde{Q},\Gamma) \geqslant \operatorname{diam}(\widetilde{Q}) \geqslant A\rho,$$

consequently

$$\mathrm{d}(Q,\Gamma) \geqslant \mathrm{d}(x,\Gamma) - |x-z| - \mathrm{diam}(Q) \geqslant \Big(A - \frac{4\sqrt{d}}{\varepsilon} - \sqrt{d}\Big)\rho.$$

We choose $A \geqslant \sqrt{d}(4/\varepsilon + 2)$ to conclude $\mathrm{d}(Q,\Gamma) \geqslant \mathrm{diam}(Q)$, in particular $Q \in \Sigma'$.

Case 2: $|x-z| > \frac{4\sqrt{d}}{\varepsilon}\rho$. — We calculate using (CC-Γ)

$$\mathrm{d}(Q,\Gamma) \geqslant \mathrm{d}(z,\Gamma) - \mathrm{diam}(Q) \geqslant \frac{\varepsilon}{2}|x-z| - \sqrt{d}\rho > \mathrm{diam}(Q).$$

So, as before, $Q \in \Sigma'$.

Taking (LC) and (7.2) into account, we get $\mathrm{length}(\gamma) + \mathrm{diam}(Q) \leqslant \sqrt{d}\big(\frac{2B+1}{\varepsilon} + 1\big)\rho$ and $Q \subseteq \overline{\mathrm{B}}(x, \mathrm{length}(\gamma) + \mathrm{diam}(Q))$. By the usual counting argument that we have already used in Lemma 5.7 it follows that the number of such cubes Q can be bounded by a constant depending only on B, d, and ε. We select a touching chain out of that collection of cubes to conclude the proof. □

Remark 7.6. — There is a constant $C = C(B,d,\varepsilon,s)$ such that for $R,S \in \Sigma$ as in the foregoing lemma with $R \cap B_{2s} \neq \emptyset$ we have that the connecting chain stays in $\widetilde{B}_{3s}$ provided $\rho \leqslant C$. Indeed, let $\widetilde{C}$ be the constant C_1 from that lemma with dependence on B, d, and ε. If x is contained in some cube from the connecting chain between R and S and $y \in R \cap B_{2s}$, then $\mathrm{d}(x,\partial\Omega) \leqslant \mathrm{d}(y,\Gamma) + \widetilde{C}\sqrt{d}\rho < 2s + \widetilde{C}\sqrt{d}\rho$, so the claim follows if we choose $\rho \leqslant s(\widetilde{C}\sqrt{d})^{-1}$.

So far, we have seen that near Γ and away from D we can reasonably cover the components Ω_m. The next two lemmas show that we will not have to bother with the components Υ_m.

Lemma 7.7. — *There is a constant $C = C(B,d,\delta,\varepsilon,\kappa) > 0$ such that for any $R \in \Sigma$ with $\widehat{\widehat{R}} \cap B_{2s} \neq \emptyset$ it follows $\widehat{\widehat{R}} \cap \bigcup_m \Upsilon_m = \emptyset$ provided that $\rho \leqslant C$.*

Proof. — Assume there exists $y \in \widehat{\widehat{R}} \cap \Upsilon_m$. Furthermore, let $x \in R \cap \Omega_\ell$. It holds $|x-y| \leqslant 2B\sqrt{d}\rho$, so x and y can be connected by a path in Ξ subject to Assumption 2.1 if we ensure $\rho \leqslant (4\sqrt{d}B)^{-1}\delta$, and its length can be controlled by $\mathrm{length}(\gamma) \leqslant \varepsilon^{-1}|x-y|$ according to (LC). Since x and y are in different connected components by assumption, there must be a point $z \in \gamma$ which satisfies $z \in D$. By assumption we may pick some $\widetilde{z} \in \widehat{\widehat{R}} \cap B_{2s}$. Then

$$\mathrm{d}(x,D) \geqslant \mathrm{d}(\widehat{\widehat{R}},D) \geqslant \mathrm{d}(\widetilde{z},D) - \mathrm{diam}(\widehat{\widehat{R}}) > \kappa/2 - 2B\sqrt{d}\rho.$$

On the other hand,

$$|x-z| \leqslant \operatorname{length}(\gamma) \leqslant \frac{2B\sqrt{d}}{\varepsilon}\rho.$$

If we choose $\rho \leqslant \frac{\varepsilon\kappa}{16\sqrt{d}B}$ as well as $\rho \leqslant \frac{\kappa}{8B\sqrt{d}}$, then we arrive at the contradiction

$$\mathrm{d}(x,D) \leqslant |x-z| \leqslant \frac{\kappa}{8} < \frac{\kappa}{4} \leqslant \mathrm{d}(x,D). \qquad \square$$

Lemma 7.8. — *Let $x \in B_{2s} \cap \bigcup_m \Upsilon_m$, then $x \notin \operatorname{supp}(f)$.*

Proof. — Let $x \in B_{2s} \cap \Upsilon_m$, then there is $y \in \Gamma$ such that $|x-y| < 2s$. Since $y \notin \overline{\Upsilon_m}$, there is $z \in \partial\Upsilon_m \subseteq D$ on the connecting line between x and y. Thus,

$$\mathrm{d}(x,D) \leqslant |x-z| \leqslant |x-y| < 2s < \kappa = \mathrm{d}(\operatorname{supp}(f),D).$$

Consequently, $x \notin \operatorname{supp}(f)$. $\square$

Construction of the approximation and estimates

Let ψ be a cutoff function valued in $[0,1]$ which is 1 on $\overline{B_s}$, supported in $N_s(\overline{B_s})$, and satisfies $|\partial^\alpha\psi| \lesssim s^{-|\alpha|}$ for $|\alpha| \leqslant k$. Moreover, fix an enumeration $(R_j)_j$ of Σ and let φ_j be a partition of unity on $\bigcup_j \widehat{R}_j$ with $\operatorname{supp}(\varphi_j) \subseteq \widehat{\widehat{R}}_j$ and $|\partial^\alpha\varphi_j| \lesssim \rho^{-|\alpha|}$. The implicit constants depend on α, d, and B. Note that according to Lemma 7.4 this partition of unity exists in particular on $B_{2s} \cap \bigcup_m \Omega_m$.

Now we may construct the approximation g of f for Proposition 7.1. With Lemma 7.2 in mind, choose $t \in (0, s/2)$ small enough that

$$\|f - E_0 f * \Phi_t\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{B_s})} \leqslant \eta s^k, \tag{7.3}$$

where Φ_t is a mollifier function supported in $\mathrm{B}(0,t)$. Recall the notation for adapted polynomials introduced in Remark 6.4 and put

$$g_1 \coloneqq \sum_j (E_0 f)_{R_j}\varphi_j, \quad g_2 \coloneqq E_0 f * \Phi_t, \quad \text{and} \quad g \coloneqq \psi g_1 + (1-\psi)g_2.$$

With a further constraint on ρ we see that g_1 vanishes near D.

Lemma 7.9. — *There exists a constant $C = C(d,\kappa) > 0$ such that $\mathrm{d}(\operatorname{supp}(g_1),D) \geqslant 3\kappa/4$, provided $\rho \leqslant C$.*

Proof. — Let $x \in \mathbb{R}^d$ with $\mathrm{d}(x, D) \leqslant \frac{3\kappa}{4}$. If $x \in \operatorname{supp}(\varphi_j)$, then fix some $y \in R_j$. We estimate (with z the center of R_j)

$$\mathrm{d}(y, D) \leqslant |y - z| + |x - z| + \mathrm{d}(x, D) \leqslant \frac{1}{2}\sqrt{d}\rho + B\sqrt{d}\rho + \frac{3\kappa}{4}.$$

Choose $\rho \leqslant \frac{\kappa}{4(1+2B)\sqrt{d}}$, then $\mathrm{d}(y, D) \leqslant \frac{7}{8}\kappa < \mathrm{d}(\operatorname{supp}(f), D)$, so $y \notin \operatorname{supp}(f)$ and $(f)_{R_j} = 0$ by linearity of the projection. But this means $g_1(x) = 0$. □

Proof of Proposition 7.1. — Assume that all constraints on the parameters collected in this section are fulfilled. We split the proof into several steps.

Step 1: g is well-defined and smooth. — We have already noticed after the definition of Σ that we can ensure that all its cubes are contained in Ω_0, so the usage of polynomial approximations is justified and yields the smooth function g_1 on $\mathbb{R}^d$. By definition of the mollification, g_2 is a smooth function in $\Omega \backslash \overline{B_s}$. If $x \in \Omega$ with $\mathrm{d}(x, D) \leqslant \kappa/2$, then we get as in Lemma 7.2 that $\mathrm{B}(x, \kappa/4) \subseteq \mathrm{B}(z, 3\kappa/4) \subseteq \Omega_0$ for some $z \in D$ and $E_0 f$ vanishes on this ball, so by definition of the mollification, g_2 vanishes in that neighborhood of D. Together with the knowledge on the support of $1 - \psi$ we infer that $(1 - \psi)g_2$ can be extend by zero to a smooth function on $\mathbb{R}^d$.

Step 2: $\mathrm{d}(\operatorname{supp}(g), D) \geqslant \kappa/2$. — First, we have $\mathrm{d}(\operatorname{supp}(g_1), D) \geqslant 3\kappa/4$ by Lemma 7.9. On the other hand, we have already noticed in Step 1 that $\mathrm{d}(\operatorname{supp}(g_2), D) \geqslant \kappa/2$, which in total gives a distance of at least $\kappa/2$ to D.

Step 3: Split up the terms for estimation. — Let α be some multi-index with $|\alpha| \leqslant k$. Then

$$\begin{aligned}\partial^\alpha(f - g) &= \partial^\alpha(\psi(f - g_1)) + \partial^\alpha((1 - \psi)(f - g_2)) \\ &= \sum_{\beta \leqslant \alpha} c_{\alpha,\beta} \left(\partial^{\alpha-\beta}\psi\partial^\beta(f - g_1) + \partial^{\alpha-\beta}(1 - \psi)\partial^\beta(f - g_2)\right) \\ &=: \sum_{\beta \leqslant \alpha} c_{\alpha,\beta}(\mathrm{I}_{\alpha,\beta} + \mathrm{II}_{\alpha,\beta}).\end{aligned}$$

Clearly, it suffices to estimate for fixed α and β the terms $\mathrm{I}_{\alpha,\beta}$ and $\mathrm{II}_{\alpha,\beta}$ in the $\mathrm{L}^p(\Omega)$-norm. The estimate for $\mathrm{II}_{\alpha,\beta}$ is possible in a uniform manner whereas for $\mathrm{I}_{\alpha,\beta}$ we will have to carefully consider different relations between $|\alpha|$, $|\beta|$, and k.

Step 4: Estimate of $\mathrm{II}_{\alpha,\beta}$. — Owing to (7.3), this term is under control on keeping $|\partial^{\alpha-\beta}(1 - \psi)| \lesssim s^{-|\alpha-\beta|} \leqslant s^{-k}$ in mind (recall $s < 1$).

Step 5: Reduction of the area of integration in $\mathrm{I}_{\alpha,\beta}$. — Since the support of ψ is contained in $N_s(\overline{B_s})$, we only have to consider this region. Assume $x \in N_s(\overline{B_s}) \setminus B_{2s}$. Then we must have $\mathrm{d}(x, D) \leqslant \kappa/2$. But in this region f

and g_1 vanish according to the definition of κ and Step 2. So we only have to deal with B_{2s}. Furthermore, f vanishes on $B_{2s} \cap \bigcup_m \Upsilon_m$ according to Lemma 7.8 and the same is true for g_1 owing to Lemma 7.7. So in summary, we only need to estimate the term $\mathrm{I}_{\alpha,\beta}$ on $B_{2s} \cap \bigcup_m \Omega_m$.

Step 6: Estimate of $\mathrm{I}_{\alpha,\beta}$ *if* $|\beta| < |\alpha|$. — Since $\psi = 1$ on B_s and $|\alpha - \beta| \neq 0$, we even only have to estimate the L^p norm over $(B_{2s} \setminus B_s) \cap \bigcup_m \Omega_m$. Write M for this set. The fact $(B_{2s} \setminus B_s) \cap N_s(\Gamma) = \emptyset$ allows us to use Lemma 7.3 to cover M by cubes from Σ to calculate

$$\|\partial^{\alpha-\beta}\psi\partial^\beta(f - g_1)\|^p_{\mathrm{L}^p(M)} \leqslant \sum_{\substack{R_j \in \Sigma \\ R_j \cap B_{2s} \neq \emptyset}} s^{p(|\beta|-|\alpha|)} \Bigg\| \partial^\beta \Bigg(f - \sum_{\substack{R_k \in \Sigma \\ \hat{\hat{R}}_k \cap R_j \neq \emptyset}} (E_0 f)_{R_k} \varphi_k \Bigg) \Bigg\|^p_{\mathrm{L}^p(R_j)}.$$

Using that $(\varphi_k)_k$ is a partition of unity on R_j, we derive using the Leibniz rule that on R_j we have

$$\partial^\beta \sum_{\substack{R_k \in \Sigma \\ \hat{\hat{R}}_k \cap R_j \neq \emptyset}} (E_0 f)_{R_k} \varphi_k = \partial^\beta (E_0 f)_{R_j} + \partial^\beta \sum_{\substack{R_k \in \Sigma \\ \hat{\hat{R}}_k \cap R_j \neq \emptyset}} \big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big] \varphi_k.$$

Using Lemma 6.3 we can estimate the norm of $\partial^\beta\big[f - (E_0 f)_{R_j}\big]$ against $\rho^{k-|\beta|} \|\nabla^k f\|_{\mathrm{L}^p(R_j)}$. From $\rho \leqslant s \leqslant 1$ we obtain $s^{|\beta|-|\alpha|}\rho^{k-|\beta|} \leqslant 1$, so we infer with (7.1) that

$$\sum_{\substack{R_j \in \Sigma \\ R_j \cap B_{2s} \neq \emptyset}} s^{p(|\beta|-|\alpha|)} \|\partial^\beta\big[f - (E_0 f)_{R_j}\big]\|^p_{\mathrm{L}^p(R_j)} \lesssim \|\nabla^k f\|^p_{\mathrm{L}^p(\tilde{B}_{3s} \cap \Omega)} \leqslant \eta^p.$$

For the second term, we first expand using the Leibniz rule to obtain

$$\partial^\beta \sum_{\substack{R_k \in \Sigma \\ \hat{\hat{R}}_k \cap R_j \neq \emptyset}} \big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big] \varphi_k = \sum_{\substack{R_k \in \Sigma \\ \hat{\hat{R}}_k \cap R_j \neq \emptyset}} \sum_{\gamma \leqslant \beta} c_{\beta,\gamma} \partial^{\beta-\gamma}\big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big] \partial^\gamma \varphi_k.$$

According to Lemma 7.5 we can apply Lemma 6.7 to the effect that

$$\|\partial^{\beta-\gamma}\big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big]\|_{\mathrm{L}^p(R_j)} \lesssim \rho^{k-|\beta|+|\gamma|} \|\nabla^k f\|_{\mathrm{L}^p(G_{j,k})},$$

where $G_{j,k}$ denotes the connecting chain from Lemma 7.5 between R_j and R_k. The ρ factor compensates for $s^{|\beta|-|\alpha|}$ and $|\partial^\gamma \varphi_k|$ as before. The sums in k and j add up by similar (but simpler) overlap considerations as already seen in Section 6 for $F_{j,k}$. Finally, since $G_{j,k}$ stays in $\widetilde{B}_{3s}$ by Remark 7.6, we get an estimate against η as was the case for the first term.

Step 7: Estimate of $\mathrm{I}_{\alpha,\beta}$ *if* $|\beta| = |\alpha|$. — The estimate follows the same ideas as in Step 6, so we only mention which modifications are needed.

First of all, we have to estimate over the whole $B_{2s} \cap \bigcup_m \Omega_m$. According to Lemma 7.4, this set can be covered by the enlarged cubes $\widehat{R}_j$. As there are no derivatives on ψ, this term can be ignored. For the $\mathrm{L}^p(\widehat{R}_j)$ norm of

$$\partial^\beta \sum_{\substack{R_k \in \Sigma \\ \hat{R}_k \cap R_j \neq \emptyset}} \big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big]\varphi_k$$

we use Lemma 6.2 to estimate

$$\|\partial^{\beta-\gamma}\big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big]\|_{\mathrm{L}^p(\hat{R}_j)} \lesssim \|\partial^{\beta-\gamma}\big[(E_0 f)_{R_k} - (E_0 f)_{R_j}\big]\|_{\mathrm{L}^p(R_j)},$$

where the implicit constant introduces a dependence on B (which determines κ in that lemma). Then this term can be handled as in Step 6.

For the term $\partial^\beta\big[f - (E_0 f)_{R_j}\big]$ we crudely apply the triangle inequality. Then we can estimate $\partial^\beta f$ directly with (7.1), and for $\partial^\beta (E_0 f)_{R_j}$ we estimate with Lemma 6.2 and Lemma 6.3 that

$$\|\partial^\beta (E_0 f)_{R_j}\|_{\mathrm{L}^p(\hat{R}_j)} \lesssim \|\partial^\beta (E_0 f)_{R_j}\|_{\mathrm{L}^p(R_j)} \lesssim \|\nabla^k f\|_{\mathrm{L}^p(R_j)}.$$

Step 8: Approximation by compactly supported functions. — As we have seen in the previous steps, g is an approximation to f that satisfies all properties but the compact support. But if we multiply g with a cutoff ψ_n from $\mathrm{B}(0,n)$ to $\mathrm{B}(0,2n)$ then this sequence does the job. □

8. Conclusion of the proof of Theorem 1.2

First, we show that the extension of a compactly supported function in $\mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(\Omega)$ constructed in Section 6 is weakly differentiable up to order k. More precisely, we show this for the larger class $\mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,\infty}(\Omega)$, which makes this result also applicable for Section 9. Clearly, compactly supported functions in $\mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(\Omega)$ belong to this class, though the inclusion is not topological. Combined with the exterior estimates from Proposition 6.11 and the density result from Section 7, this allows us to conclude Theorem 1.2.

Proposition 8.1. — *Let* $f \in \mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,\infty}(\Omega)$ *and* $|\alpha| \leqslant k-1$, *then* $\partial^\alpha E f$ *exists on* $\mathbb{R}^d$ *in the weak sense and has a Lipschitz continuous representative* g_α *which satisfies* $\mathrm{d}(\mathrm{supp}(g_\alpha), D) > 0$.

Proof. — Fix an extension $F \in \mathrm{C}^\infty_D(\mathbb{R}^d)$ of f. We show the claim by induction over $|\alpha|$. By Proposition 6.11, Ef is well-defined and bounded. Now assume that $|\alpha| < k$ and $\partial^\alpha Ef$ is well-defined and bounded. It suffices to show that $\partial^\alpha Ef$ is given by a Lipschitz function. To this end, define g_α to equal $\partial^\alpha F$ on $\overline{\Omega}$ and $\partial^\alpha Ef$ otherwise. We proceed in two steps.

Step 1: g_α is a representative of $\partial^\alpha Ef$. — That g_α and $\partial^\alpha Ef$ coincide on $\Omega \cup \overline{\Omega}^c$ is by definition. It follows from Remark 6.6 that $\partial^\alpha Ef$ vanishes on D. The same is true for F by assumption. Consequently, Lemma 5.1 reveals that g_α is a representative of $\partial^\alpha Ef$.

Step 2: g_α is Lipschitz continuous. — By assumption, g_α is Lipschitz on $\overline{\Omega}$. Furthermore, g_α is smooth on $\overline{\Omega}^c$ and its gradient is bounded according to Proposition 6.11. Hence, g_α is Lipschitz on any line segment contained in the exterior of Ω. The claim follows if we show that g_α is continuous on $\partial\Omega$. This is already established around D, so it only remains to show continuity in $x \in \Gamma$ with $\mathrm{d}(x, D) > 0$.

Clearly, it suffices to consider $y \in \overline{\Omega}^c$ close to x to show continuity. Moreover, using the positive distance of x to D, we may assume using Lemma 5.5 that $y \in Q_j$ for some cube $Q_j \in \mathcal{W}_e$ and that $Q_j^* \subseteq \Omega$. Write y^j for the center of Q_j. Fix some cube R which contains Q_j and Q_j^* with size comparable to Q_j^*. Also, note that $Ef(z) = (E_{Q_j^*} f)_{Q_j^*}(z)$ in a neighborhood of y^j by choice of the partition of unity used in the construction of E, and that $E_{Q_j^*} f = F$ on Q_j^* since Q_j^* is properly contained in Ω. Then

$$
\begin{aligned}
&|g_\alpha(x) - g_\alpha(y)| \\
&\qquad \leqslant |\partial^\alpha F(x) - \partial^\alpha F(y^j)| + |\partial^\alpha F(y^j) - \partial^\alpha (E_{Q_j^*} f)_{Q_j^*}(y^j)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad + |\partial^\alpha Ef(y^j) - \partial^\alpha Ef(y)| \\
&\qquad \leqslant \|\partial^\alpha F\|_{\mathrm{Lip}(\mathbb{R}^d)} |x - y^j| + \|\partial^\alpha \big(F - (F)_{Q_j^*}\big)\|_{\mathrm{L}^\infty(R)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \|\partial^\alpha Ef\|_{\mathrm{Lip}(Q_j)} \operatorname{diam}(Q_j).
\end{aligned}
$$

Clearly, the first and the last term tend to zero when y approaches x. Finally, we estimate the second term using Corollary 6.5 to get decay of order $\operatorname{diam}(R) \approx \operatorname{diam}(Q_j)$. Hence, g_α is indeed continuous in x. □

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. — Let $f \in \mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(\Omega)$ with compact support. First, we treat the trivial case $\overline{\Omega} = \mathbb{R}^d$. In this situation, we extend f to D by zero. This is a representative according to Lemma 5.1, it is weakly differentiable of all orders by assumption on f, and the extension is isometric with respect to the norm of $\mathrm{W}^{k,p}(\Omega)$. Hence, this case can be completed by continuity, compare with the conclusion of the general case below.

Otherwise, derive from Proposition 8.1 that Ef has weak derivates up to order k and satisfies $\mathrm{d}(\mathrm{supp}(Ef), D) > 0$. From the latter follows in particular that $\partial^\alpha Ef$ vanishes in D. Proposition 6.11 yields the desired estimate on $\mathbb{R}^d \setminus \overline{\Omega}$. Taking Lemma 5.1 into account, these estimates sum up to an estimate that holds almost everywhere on $\mathbb{R}^d \setminus \Omega$, which completes the boundedness assertion.

Because we have the positive distance of the support of Ef to D, a convolution argument shows that moreover $Ef \in \mathrm{W}^{k,p}_D(\mathbb{R}^d)$. Finally, we can extend E by density to $\mathrm{W}^{k,p}_D(\Omega)$ using the definition of that space and the density of $\mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{k,p}(\Omega)$ shown in Section 7. □

9. Some additional first-order results

9.1. Extension of Lipschitz functions vanishing on *D*

Definition 9.1. — *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $D \subseteq \overline{\Omega}$ be closed. The space of Lipschitz continuous functions that vanish on D is given by*

$$\mathrm{Lip}_D(\Omega) \coloneqq \{u : \overline{\Omega} \longrightarrow \mathbb{R} : u \text{ Lipschitz and } u = 0 \text{ on } D\}$$

with norm

$$\|u\|_{\mathrm{Lip}_D(\Omega)} \coloneqq \max(\|u\|_{\mathrm{L}^\infty(\Omega)}, |u|_{\mathrm{Lip}(\Omega)}).$$

Here, $|u|_{\mathrm{Lip}(\Omega)}$ is defined as

$$|u|_{\mathrm{Lip}(\Omega)} \coloneqq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|}.$$

The following approximation lemma for functions in $\mathrm{Lip}_D(\Omega)$ is a modified version of an argument of Stein [20, p. 188] and is used as a substitute for the result from Section 7 in the case $p < \infty$.

Lemma 9.2. — *Let $f \in \mathrm{Lip}_D(\Omega)$. Then there exists a bounded sequence $(f_n)_n \subseteq \mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\Omega)$ that converges to f in $\mathrm{L}^\infty(\Omega)$ and satisfies the estimate $\|f_n\|_{\mathrm{Lip}(\Omega)} \lesssim \|f\|_{\mathrm{Lip}(\Omega)}$, where the implicit constant only depends on d.*

Proof. — It suffices to show the claim for functions defined on $\mathbb{R}^d$ since by Whitney's extension theorem [13, Theorem 3.1.1] there exists an extension $F \in \mathrm{Lip}_D(\mathbb{R}^d)$ of f that satisfies $\|F\|_{\mathrm{Lip}(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{Lip}(\Omega)}$, where the implicit constant depends only on the dimension d. For convenience, we drop $\mathbb{R}^d$ in the notation of function spaces for the rest of this proof.

Pick a family of functions $\varphi_n : [0,\infty) \to [0,1]$ satisfying for $y \geqslant x > 0$

(i) $\varphi_n = 0$ on $[0, 1/n)$
(ii) $\varphi_n = 1$ on $(2/n, \infty)$
(iii) $|\varphi_n(x) - \varphi_n(y)| \lesssim \frac{1}{x}|x-y|$,

for an explicit construction see [16, Theorem 3.7]. Put $\psi_n(x) \coloneqq \varphi_n(\mathrm{d}_D(x))$. By construction, ψ_n vanishes around D and, by Lipschitz continuity of the distance function, (iii) yields for $x, y \in \mathbb{R}^d$ with $\mathrm{d}_D(x) \leqslant \mathrm{d}_D(y)$

$$|\psi_n(x) - \psi_n(y)| \lesssim \mathrm{d}_D(x)^{-1}|x-y|. \tag{9.1}$$

It suffices to show that there is a sequence of Lipschitz functions whose supports have positive distance to D which fulfill all claims but smoothness, since then we can conclude using mollification. Note that the mollified sequence converges in L^∞ because we have Lipschitz continuity.

In this light, define the sequence of functions $f_n \coloneqq \psi_n f$. Clearly, these functions are Lipschitz, and their supports stay away from D because ψ_n has this property. Next, we show that f_n converges to f in L^∞. To this end, let $x \in \mathbb{R}^d$ and pick $z \in D$ satisfying $|x-z| = \mathrm{d}_D(x)$. Since $f(z) = 0$, we get

$$|f(x) - f_n(x)| = (1-\psi_n(x))|f(x) - f(z)| \leqslant \|f\|_{\mathrm{Lip}}(1-\psi_n(x))\,\mathrm{d}_D(x).$$

By definition of ψ_n, $(1-\psi_n(x))\,\mathrm{d}_D(x) \leqslant 2/n$. Consequently, it follows $|f(x) - f_n(x)| \to 0$ uniformly in x.

It remains to show that the Lipschitz seminorms of f_n can be estimated against $\|f\|_{\mathrm{Lip}}$. The argument uses the same trick using an element from D as we have just seen. So, let $x, y \in \mathbb{R}^d \setminus D$. Assume without loss of generality that $\mathrm{d}_D(x) \leqslant \mathrm{d}_D(y)$ and let z realize the distance from x to D. Using (9.1), we obtain

$$\begin{aligned}|f_n(x) - f_n(y)| &\leqslant |f(x) - f(y)|\psi_n(y) + |f(x)||\psi_n(x) - \psi_n(y)| \\ &\lesssim \|f\|_{\mathrm{Lip}}|x-y| + |f(x) - f(z)|\,\mathrm{d}_D(x)^{-1}|x-y|.\end{aligned}$$

The first term is fine and for the second we notice that

$$|f(x) - f(z)| \leqslant \|f\|_{\mathrm{Lip}}|x-z| = \|f\|_{\mathrm{Lip}}\,\mathrm{d}_D(x). \qquad \square$$

Theorem 9.3. — *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $D \subseteq \overline{\Omega}$ be closed such that Ω and D are subject to Assumption 2.1. Then there exists an extension operator E which is bounded from $\mathrm{W}^{1,p}_D(\Omega)$ to $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$ for all $1 \leqslant p < \infty$ and from $\mathrm{Lip}_D(\Omega)$ to $\mathrm{Lip}_D(\mathbb{R}^d)$.*

Proof. — Write E for the extension operator constructed in Section 6 for the case $k = 1$. The boundedness on $\mathrm{W}^{1,p}_D(\Omega)$ is the content of Theorem 1.2. Now, let $f \in \mathrm{Lip}_D(\Omega)$ and let $(\varphi_n)_n$ be the approximation in $\mathrm{C}^\infty_D(\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\Omega)$ constructed in Lemma 9.2. According to Proposition 6.11 we have L^∞ bounds for E on φ_n. In particular, this shows the $\mathrm{L}^\infty(\mathbb{R}^d)$ bound for E on f. Moreover, this permits us to calculate for almost every $x, y \in \mathbb{R}^d$ that

$$|Ef(x) - Ef(y)| = \lim_{n\to\infty} |E\varphi_n(x) - E\varphi_n(y)|.$$

By Proposition 8.1, $E\varphi_n$ is Lipschitz and hence

$$\lim_{n\to\infty} |E\varphi_n(x) - E\varphi_n(y)| \leqslant \liminf_{n\to\infty} \|\nabla E\varphi_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} |x-y|.$$

Proceeding by Proposition 6.11 and Lemma 9.2, we obtain

$$\liminf_{n\to\infty} \|\nabla E\varphi_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \liminf_{n\to\infty} \|\varphi_n\|_{\mathrm{W}^{1,\infty}(\Omega)} \lesssim \|f\|_{\mathrm{Lip}(\Omega)}.$$

So, Ef satisfies a Lipschitz estimate against $\|f\|_{\mathrm{Lip}(\Omega)}$ almost everywhere. Hence, Ef possesses a representative which is Lipschitz on $\mathbb{R}^d$ and satisfies the boundedness estimate. □

Remark 9.4. — If one is merely interested in extending functions in $\mathrm{Lip}_D(\Omega)$ to $\mathbb{R}^d$, this is possible without any geometric quality using for example an extension operator of Whitney type [20, p. 174]. However, this operator is not an extension operator in the sense of Definition 1.1 and in particular does not extend to a consistent extension operator on the spaces $\mathrm{W}^{1,p}_D(\Omega)$ in this general situation.

9.2. An extension using reference geometries

As a corollary, we obtain the existence of an extension operator on even more general but very inexplicit geometries.

Corollary 9.5. — *Let $\Omega \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial\Omega$ be closed, and define $\Gamma := \partial\Omega \setminus D$. Further, assume that there exists a proper open superset $\Omega_\Gamma \supset \Omega$ such that Γ is contained in $\partial\Omega_\Gamma$ and is relatively open with respect to $\partial\Omega_\Gamma$. Finally, assume that Ω_Γ and $D' := \partial\Omega_\Gamma \setminus \Gamma$ satisfy Assumption 2.1. Then there exists an extension operator E that restricts to a bounded operator from $\mathrm{L}^p(\Omega)$ to $\mathrm{L}^p(\mathbb{R}^d)$ as well as from $\mathrm{W}^{1,p}_D(\Omega)$ to $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$ in the case $1 \leqslant p < \infty$, and which restricts to a bounded operator from $\mathrm{L}^\infty(\Omega)$ to $\mathrm{L}^\infty(\mathbb{R}^d)$ as well as from $\mathrm{Lip}_D(\Omega)$ to $\mathrm{Lip}_D(\mathbb{R}^d)$. The operator norms of E only depend on d, p, K, ε, δ, and λ. Here, the quantities K, ε, δ, and λ are measured with respect to Ω_Γ.*

Proof. — Throughout this proof, let $\varepsilon, \delta > 0$ be the parameters from Assumption 2.1 with respect to Ω_Γ and D'.

Let E_0 be the operator that extends functions by zero from Ω to Ω_Γ, and let E_Γ denote the extension operator constructed for Ω_Γ in Theorem 1.2 for $k = 1$. We claim that $E \coloneqq E_\Gamma \circ E_0$ is the desired extension operator. We proceed in several steps.

Step 1: E is L^p bounded for $1 \leqslant p \leqslant \infty$. — The respective estimate for E_0 is clear by construction. The same is true for E_Γ by Theorem 1.2 in the case $p < \infty$. Owing to Remark 6.6, the L^p-estimates for E_Γ also hold in the case $p = \infty$. Hence, the claim follows by composition.

Step 2: E_0 maps $\mathrm{Lip}_D(\Omega)$ boundedly into $\mathrm{Lip}_{D'}(\Omega_\Gamma)$. Let $f \in \mathrm{Lip}_D(\Omega)$.

Claim 1: $E_0 f$ is Lipschitz continuous on Ω_Γ. — It suffices to consider $x \in \Omega$ and $y \in \Omega_\Gamma \setminus \Omega$ with $|x - y| < \delta$. Let γ be the path connecting x with y subject to Assumption 2.1. By virtue of (CC-Γ) and the intermediate value theorem there exists $z \in \gamma \cap D$. Now, by Lipschitz continuity of f, the fact that f vanishes on D, and by (LC) one estimates

$$
\begin{aligned}
(9.2) \qquad |E_0 f(x) - E_0 f(y)| &= |f(x) - f(z)| \leqslant |f|_{\mathrm{Lip}(\Omega)} |x - z| \\
&\leqslant \frac{|f|_{\mathrm{Lip}(\Omega)}}{\varepsilon} |x - y|.
\end{aligned}
$$

Claim 2: $E_0 f$ vanishes on D'. Let $x \in D'$. — If $x \in \partial\Omega$, then $x \in D$ since $x \notin \Gamma$ by definition of D'. Hence, $E_0 f(x) = f(x) = 0$ by choice of f. Otherwise, there is a ball B around x that avoids Ω. Choose a sequence x_n in $B \cap \Omega_\Gamma$ that approaches x, by construction of $E_0 f$ and continuity shown in Step 1 we conclude that $E_0 f$ vanishes in x.

Claim 3: E_0 is bounded. — The crucial estimate was shown in (9.2).

Step 3: E_0 maps $\mathrm{W}^{1,p}_D(\Omega)$ boundedly into $\mathrm{W}^{1,p}_{D'}(\Omega_\Gamma)$ for $1 \leqslant p < \infty$. — Let $f \in \mathrm{W}^{1,p}_D(\Omega)$ and pick an approximating sequence $(f_n)_n \subseteq \mathrm{C}^\infty_D(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$, which exists by definition of the space. Since f vanishes around D, $E_0 f_n$ is weakly differentiable on Ω_Γ with $\nabla E_0 f_n = E_0 \nabla f_n$ almost everywhere. Therefore,

$$
\|E_0 f_n\|_{\mathrm{W}^{1,p}(\Omega_\Gamma)} = \|f_n\|_{\mathrm{W}^{1,p}(\Omega)},
$$

which yields that there exists $g \in \mathrm{W}^{1,p}(\Omega_\Gamma)$ such that a subsequence $E_0 f_{n_j}$ converges weakly to g in $\mathrm{W}^{1,p}(\Omega_\Gamma)$. By the L^p-continuity of E_0 we conclude that $E_0 f$ coincides with g. Finally, $E_0 f$ belongs to $\mathrm{W}^{1,p}_{D'}(\Omega_\Gamma)$ by construction.

Step 4: E is bounded in $\mathrm{W}^{1,p}_D(\Omega)$ for $1 \leqslant p < \infty$ and $\mathrm{Lip}_D(\Omega)$. — This follows by composition using Steps 2 or 3 together with Theorem 9.3 or Theorem 1.2. □

Remarks 9.6.

(1) Notice that Assumption 2.1 is an explicit assumption that uses only information on points in Ω. To the contrary of that, the geometry described in Corollary 9.5 has an inexplicit nature, as it is a priori not clear how to construct such a set Ω_Γ. However, there are important examples where this condition can be checked in the "blink of an eye", see Example 9.7 below.

(2) We suggest that a similar result holds in the higher-order case using an induction similar to that in Proposition 8.1. Moreover, a more involved approximation procedure than our truncation method employed at the end of Step 3 would be needed.

Example 9.7 (Exterior boundary cusps at zero or at infinity). — Let Ω be a domain that has an exterior boundary cusp either at zero or at infinity, as it is informally depicted in Figure 9.1. In this case, Ω is an $\mathrm{W}^{1,p}_D$-extension domain as a simple reflection argument shows. However, it is not so clear if it satisfies Assumption 2.1. Nevertheless, it is simple to verify the validity of the geometric setting stated in Corollary 9.5. Indeed, simply take as Ω_Γ the lower half-space and notice that the parameter K in Assumption 2.1 can be set to zero because it is already an (ε, δ)-domain.

We can even go further and extend the geometric setting from Example 3.5 to the following one (see Figure 9.1).

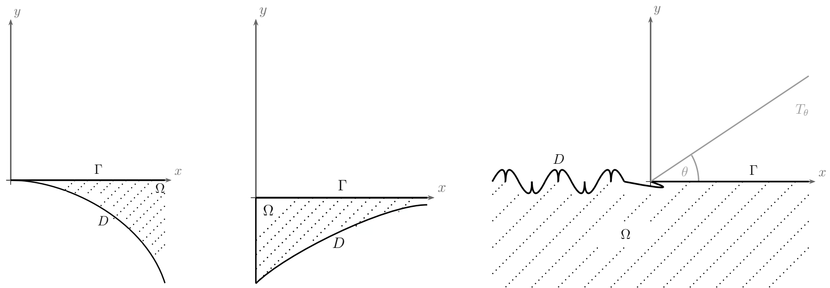

Figure 9.1. Situations in Example 9.7.

Let $\theta \in (0, \pi)$ and let $\mathrm{S}_\theta \subseteq \mathbb{R}^2$ denote the open sector symmetric about the positive x-axis with opening angle 2θ. Define $T_\theta \coloneqq \mathrm{S}_\theta \cap \{(x, y) \in \mathbb{R}^2 :$

$y > 0\}$. Let $\Omega \subseteq \mathbb{R}^2$ be a domain satisfying $T_\theta \subseteq \Omega^c$ and define

$$\Gamma \coloneqq (0,\infty) \times \{0\} \qquad \text{and} \qquad D \coloneqq \partial\Omega \setminus \Gamma.$$

Assume further, that Ω is such that D is closed (this avoids that D touches Γ from below). To apply Corollary 9.5 take $\Omega_\Gamma \coloneqq (\overline{T_\theta})^c$. As this is an (ε,δ)-domain, it satisfies Assumption 2.1 with $K = 0$.

10. Homogeneous estimates

We provide further estimates for the extension operator from Theorem 1.2 which concern homogeneous estimates and locality (see Definition 10.1 for a proper definition). These results build on the observations made in Remark 6.12.

DEFINITION 10.1. — *An extension operator E on $\mathrm{W}^{k,p}_D(\Omega)$ is called* local *if there exist constants $r_0, \kappa > 0$ such that*

$$\|\nabla^\ell Ef\|_{\mathrm{L}^p(\mathrm{B}(x,r))} \lesssim \|f\|_{\mathrm{W}^{k,p}(\Omega\cap\mathrm{B}(x,\kappa r))}$$

for all $x \in \partial\Omega$, $r \in (0, r_0)$, and $\ell \leqslant k$. Moreover, call E homogeneous *if one can replace the right-hand side of that estimate by $\|\nabla^\ell f\|_{\mathrm{L}^p(\Omega\cap\mathrm{B}(x,\kappa r))}$.*

To verify that E is local, we choose $G = \mathrm{B}(x,r)$ in Remark 6.12 and let $Q_j \in \mathcal{W}_e$ with $Q_j \cap \mathrm{B}(x,r) \neq \emptyset$. On using (5.3), (5.4), the bound on the chain length from Lemmas 5.8 as well as the properties of Whitney cubes, we see that $F(Q_j)$ is contained in the ball $\mathrm{B}(x,\kappa r)$ for some κ depending only on ε, d, K, and λ (as before, an analogous version for $F_P(Q)$ holds on using Lemma 5.9 instead of Lemma 5.8 and a similar reasoning). So, with $H = \mathrm{B}(x,\kappa r)$ we derive locality from Remark 6.12 with $r_0 = \infty$. If we restrict to $r_0 = A\delta$, the same remark also yields that E is homogeneous. Note that in the case of $\delta = \infty$ this restriction is void. We summarize this result in the following theorem.

THEOREM 10.2. — *Let $\Omega \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial\Omega$ be closed such that Ω and D are subject to Assumption 2.1, and fix some integer $k \geqslant 0$. Then there exist $A, \kappa > 0$ and an extension operator E such that for all $1 \leqslant p < \infty$ one has that E restricts to a bounded mapping from $\mathrm{W}^{k,p}_D(\Omega)$ to $\mathrm{W}^{k,p}_D(\mathbb{R}^d)$ and which is moreover homogeneous and local, that is, the estimate*

$$\|\nabla^\ell Ef\|_{\mathrm{L}^p(\mathrm{B}(x,r))} \lesssim \|\nabla^\ell f\|_{\mathrm{L}^p(\mathrm{B}(x,\kappa r)\cap\Omega)}$$

holds for $f \in \mathrm{W}^{k,p}_D(\Omega)$, $\ell \leqslant k$, $x \in \partial\Omega$, and $r \in (0, A\delta)$. The implicit constant in that estimate depends on d, p, K, k, ε, δ, and λ.

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