On Passivity and Power-Balance Inequalities of Nonlinear *RLC* Circuits

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***Abstract*—Arbitrary interconnections of passive (possibly nonlinear) resistors, inductors, and capacitors define passive systems, with power port variables the external source voltages and currents, and storage function the total stored energy. In this paper, we identify a class of *RLC* circuits (with convex energy function and weak electromagnetic coupling), for which it is possible to "add a differentiation" to the port terminals preserving passivity—with a new storage function that is directly related to the circuit power. To establish our results, we exploit the geometric property that voltages and currents in *RLC* circuits live in orthogonal spaces, i.e., Tellegen's theorem, and heavily rely on the seminal paper of Brayton and Moser published in the early sixties.**

***Index Terms*—Brayton–Moser equations, nonlinear circuits, passivity, Tellegen's theorem.**

I. Introduction

PASSIVITY is a fundamental property of dynamical systems that constitutes a cornerstone for many major developments in circuit and systems theory, see, e.g., [3], [9], and the references therein. It is well known that (possibly nonlinear) *RLC* circuits consisting of arbitrary interconnections of passive resistors, inductors, capacitors, and voltage and/or current sources are also passive with power port variables, the external source voltages and currents, and storage function of the total stored energy [2]. Our main contribution in this paper is the proof that for all *RL* or *RC* circuits, and a class of *RLC* circuits, it is possible to "add a differentiation" to one of the port variables (either voltage or current) preserving passivity with a storage function which is directly related to the circuit power. The new passivity property is of interest in circuit theory, but also has applications in control (see [7] for some first results regarding stabilization).

Since the supply rate (the product of the passive port variables) of the standard passivity property, as defined in, e.g., [3] and [9], is voltage $\times$ current, it is widely known that the differential form of the corresponding energy-balance establishes the *active* power-balance of the circuit. As the new supply rate is voltage $\times$ the time derivative of the current (or current $\times$ the time derivative of the voltage)—quantities which are sometimes adopted as suitable definitions of the supplied *reactive* power—our result unveils some sort of reactive power-balance.

The remainder of the paper is organized as follows. In Section II, we briefly review some fundamental results in circuits theory, like the classical definition of passivity and Tellegen's Theorem. The new passivity property for *RL* and *RC* circuits is established in Section III. In Section IV, this result is extended to a class of *RLC* circuits using the classical Brayton–Moser equations. Finally, we conclude the paper with some remarks and comments on future research.

Notation: Throughout the paper we will denote by $\nabla_x V(x,\cdot)$ the partial derivative of a vector function $V(x,\cdot)$ with respect to a n-dimensional column vector $x = \mathrm{col}(x_1,\ldots,x_n)$, i.e.,

$$\nabla_x V(x,\cdot) = \frac{\partial V}{\partial x}(x,\cdot).$$

Consequently, $\nabla_x^2 V(x,\cdot)$ denotes the second partial derivative (Hessian), i.e.,

$$\nabla_x^2 V(x,\cdot) = \frac{\partial^2 V}{\partial x^2}(x,\cdot).$$

II. Tellegen's Theorem and Passivity

Consider a circuit consisting of n_L inductors, n_C capacitors, n_R resistors, and n_S voltage and/or current sources, called the *branches* of the circuit. Let $i_\gamma = \mathrm{col}(i_{\gamma_1},\ldots,i_{\gamma_{n_\gamma}}) \in \mathbb{R}^{n_\gamma}$ and $v_\gamma = \mathrm{col}(v_{\gamma_1},\ldots,v_{\gamma_{n_\gamma}}) \in \mathbb{R}^{n_\gamma}$, with $\gamma = \{L,C,R,S\}$, denote the branch currents and voltages of the circuit, respectively. It is well known that Tellegen's Theorem [8] states that the set of branch currents (which satisfy Kirchhoff's current law), say $\mathbb{K}_i \subset \mathbb{R}^b, b = \sum_\gamma n_\gamma$, and the set of branch voltages (that satisfy Kirchhoff's voltage law), say $\mathbb{K}_v \subset \mathbb{R}^b$, are orthogonal subspaces. As an immediate consequence of this fact we have

$$\sum_\gamma i_\gamma^\top v_\gamma = 0 \tag{1}$$

which states that the total power in the circuit is preserved.

Corollary 1: Voltages and currents in a (possibly nonlinear) *RLC* circuit satisfy

$$\sum_\gamma v_\gamma^\top \frac{di_\gamma}{dt} = 0 \tag{2}$$

as well as

$$\sum_\gamma i_\gamma^\top \frac{dv_\gamma}{dt} = 0. \tag{3}$$

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The proof of this corollary is easily established noting that, if $i_\gamma \in \mathbb{K}_i$ (respectively, $v_\gamma \in \mathbb{K}_v$), then, clearly also $(di_\gamma)/(dt) \in \mathbb{K}_i$ (respectively, $(dv_\gamma)/(dt) \in \mathbb{K}_v$), and then, invoking orthogonality of $\mathbb{K}_i$ and $\mathbb{K}_v$, see also [1], [8] and the references therein.

Another immediate consequence of Tellegen's theorem is the following, slight variation of the classical result in circuit theory, see, e.g., [2, Sec. 19.3.3], whose proof is provided for the sake of completeness.

Proposition 1: Arbitrary interconnections of inductors and capacitors with passive resistors verify the energy-balance inequality

$$\int_0^t i_S^\top(t') v_S(t')\, dt' \geq E[\varphi_L(t), q_C(t)] - E[\varphi_L(0), q_C(0)] \tag{4}$$

where we have defined the total stored energy $E(\varphi_L, q_C) = E_L(\varphi_L) + E_C(q_C)$ with $\varphi_L \in \mathbb{R}^{n_L}$ and $q_C \in \mathbb{R}^{n_C}$ the inductor fluxes and the capacitor charges, respectively. If, furthermore, the inductors and capacitors are also passive, then, the network defines a passive system with power port variables $i_S, v_S \in \mathbb{R}^{n_S}$ and storage function of the total energy.

Proof: First, notice that $(dE)/(dt) = i_L^\top v_L + i_C^\top v_C$, where we have used the fact that $i_L = \nabla_{\varphi_L} E_L(\varphi_L)$ and $v_C = \nabla_{q_C} E_L(q_C)$, and the relations $v_L = (d\varphi_L)/(dt)$ and $i_C = (dq_C)/(dt)$. Then, by (1) we have that $i_L^\top v_L + i_C^\top v_C = i_S^\top v_S - i_R^\top v_R$ (notice that we have adopted the standard sign convention for the supplied power). Hence, noting that $i_R^\top v_R \geq 0$ for passive resistors, and integrating the latter equations form 0 to t, we obtain (4). Passivity follows from positivity of $E(\varphi_L, q_C)$ for passive inductors and capacitors. ■

III. New Passivity Property for *RL* and *RC* Circuits

In this section, we first consider circuits consisting solely of inductors and current-controlled resistors and sources, denoted by Σ_L, and circuits consisting solely of capacitors and voltage-controlled resistors and sources, denoted by Σ_C. Furthermore, to present the new passivity property, we need to define some additional concepts that are well known in circuit theory [6], [8], and will be instrumental to formulate our results.

Definition 1: The *content* of a current-controlled resistor is defined as

$$F_k\left(i_{R_k}\right) = \int_0^{i_{R_k}} \hat{v}_{R_k}\left(i'_{R_k}\right) di'_{R_k} \tag{5}$$

while, for a voltage-controlled resistors, the function

$$G_k\left(v_{R_k}\right) = \int_0^{v_{R_k}} \hat{i}_{R_k}\left(v'_{R_k}\right) dv'_{R_k} \tag{6}$$

is called the resistors *co-content*.

Proposition 2: Arbitrary interconnections of passive inductors with convex energy function $E_L(\varphi_L)$, current-controlled resistors and sources, satisfy the power-balance inequality

$$\int_0^t v_S^\top(t') \frac{di_S}{dt'}(t')\, dt' \geq F[i_R(t)] - F[i_R(0)] \tag{7}$$

where $F(i_R) = \sum_{k=1}^{n_R} F_k(i_{R_k})$. If the resistors are passive, the circuit Σ_L defines a passive system with power port variables $(v_S, (di_S)/(dt))$ and storage function of the total resistors content.

Similarly, arbitrary interconnections of passive capacitors with the convex energy function $E_C(q_C)$, voltage-controlled resistors, and sources, satisfy the power-balance inequality

$$\int_0^t i_S^\top(t') \frac{dv_S}{dt'}(t')\, dt' \geq G[v_R(t)] - G[v_R(0)] \tag{8}$$

where $G(v_R) = \sum_{k=1}^{n_R} G_k(v_{R_k})$. If the resistors are passive, the circuit Σ_C defines a passive system with power port variables $(i_S, (dv_S)/(dt))$ and storage function the total resistors co-content.

Proof: The proof of the new passivity property for *RL* circuits is established as follows. First, differentiate the resistors content

$$\frac{dF}{dt}(i_R) = v_R^\top \frac{di_R}{dt}. \tag{9}$$

Then, by using the fact that

$$\frac{di_L}{dt} = \frac{d}{dt}\left[\nabla_{\varphi_L} E_L(\varphi_L)\right] = \left(\frac{d\varphi_L}{dt}\right)^\top \nabla^2_{\varphi_L} E_L(\varphi_L)$$

and by invoking Faraday's law, i.e., $v_L = (d\varphi_L)/(dt)$, we obtain

$$v_L^\top \frac{di_L}{dt} = v_L^\top \nabla^2_{\varphi_L} E_L(\varphi_L) v_L \geq 0 \tag{10}$$

where the nonnegativity stems from the convexity assumption. Finally, by substituting (9) and (10) into (2) of Corollary 1, with $v_C = 0$ and $i_C = (dq_C)/(dt) = 0$, and integrating form 0 to t yields the result.

The proof for *RC* circuits follows verbatim, but now using (3) of Corollary 1 instead of (2), the relation $(dv_C)/(dt) = \nabla^2_{q_C} E_C(q_C) i_C$ and the definition of the co-content. ■

Remark 1: In some cases it is also possible to apply Proposition 2 to *RL* (respectively, *RC*) circuits containing voltage-controlled resistors in Σ_L (respectively, current-controlled resistor in Σ_C) under the condition that the (i_R, v_R) curves are invertible. If, for example, Σ_L contains a voltage-controlled resistor, say R_k, and its constitutive relation $i_{R_k} = \hat{i}_{R_k}(v_{R_k})$ is invertible, it should then be possible to rewrite the characteristic equation in terms of the current, i.e., $v_{R_k} = \hat{v}_{R_k}(i_{R_k})$. In the linear case, this means that instead of writing $i_{R_k} = (1/R_k)v_{R_k}$ (or in terms of the resistors co-content: $G_k(v_{R_k}) = (1/(2R_k))v_{R_k}^2$), we may write $v_{R_k} = R_k i_{R_k}$ (Ohm's law in its conventional form), and hence its content reads $F_k(i_{R_k}) = (1/2)R_k i_{R_k}^2$ and the new passivity property (7) can be established; see also, Fig. 1. □

Remark 2: The new passivity properties of Proposition 2 differ from the standard result of Proposition 1 in the following respects. First, while Proposition 1 holds for general *RLC* circuits, the new properties are valid only for *RL* or *RC* systems. Using the fact that passivity is invariant with respect to negative feedback interconnections it is, of course, possible to combine *RL* and *RC* circuits and establish the new passivity property for some *RLC* circuits. A class of *RLC* circuits for which a similar

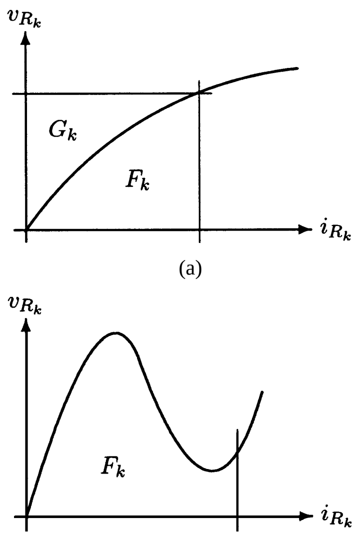

Fig. 1. (a) Resistor characteristic with invertible (i_{R_k}, v_{R_k}) curve. (b) Noninvertible resistor characteristic.

property holds will be identified in Section IV. Second, the condition of convexity of the energy functions required for Proposition 2 is sufficient, but not necessary for passivity of the dynamic *L* and *C* elements. Hence, the class of admissible dynamic elements is more restrictive. □

Remark 3: It is interesting to remark that the supply rate of the new passive systems defined by either the product $v_S^\top (di_S)/(dt)$ or $i_S^\top (dv_S)/(dt)$, relates with an alternative definition of *reactive* power. The interested reader is referred to, e.g., [5] and [11] for more details on this subject. □

IV. Passivity of Brayton–Moser Circuits

The previous developments show that, using the content and co-content as storage functions and the reactive power as supply rate, we can identify new passivity properties of *RL* and *RC* circuits. In this section, we will establish similar properties for *RLC* circuits. Toward this end, we strongly rely on some fundamental results reported in [1]. Furthermore, we assume that the n_{R_L} current-controlled resistors $R_{L_k} \in R_L$, with $R_L \in \mathbb{R}^{n_{R_L} \times n_{R_L}}$, are contained in Σ_L and the n_{R_C} voltage-controlled resistors $R_{R_k} \in R_C$, with $R_C \in \mathbb{R}^{n_{R_C} \times n_{R_C}}$, are contained in Σ_C. The class of *RLC* circuits considered here is then composed by an interconnection of Σ_L and Σ_C.

A. Brayton and Moser's Equations

In the early 1960s, Brayton and Moser [1] have shown that the dynamic behavior of a topologically complete[1] circuit (without external sources) is governed by the following differential equations:

$$-L(i_L)\frac{di_L}{dt} = \nabla_{i_L}\tilde{P}(i_L, v_C)$$
$$C(v_C)\frac{dv_C}{dt} = \nabla_{v_C}\tilde{P}(i_L, v_C) \qquad (11)$$

where $L(i_L) = \nabla_{i_L}\hat{\varphi}_L(i_L) \in \mathbb{R}^{n_L \times n_L}$ is the inductance matrix, $C(v_C) = \nabla_{v_C}\hat{q}_C(v_C) \in \mathbb{R}^{n_C \times n_C}$ is the capacitance matrix, $\tilde{P} : \mathbb{R}^{n_L + n_C} \to \mathbb{R}$ is called the mixed-potential and is given by

$$\tilde{P}(i_L, v_C) = i_L^\top \Gamma v_C + F(i_L) - G(v_C) \qquad (12)$$

where $\Gamma \in \mathbb{R}^{n_L \times n_C}$ is a (full-rank) matrix that captures the interconnection structure between the inductors and capacitors.

If we add external sources[2], (11) can be written as

$$Q(x)\dot{x} = \nabla_x \tilde{P}(x) - Bv_S \qquad (13)$$

where $x = \mathrm{col}(i_L, v_C), \dot{x} = (dx)/(dt), Q(x) = \mathrm{diag}(-L(i_L), C(v_C))$, and $B = \mathrm{col}(B_S, 0)$ with $B_S \in \mathbb{R}^{n_L \times n_S}$.

Remark 4: Notice that the mixed-potential function contains both the content and co-content which are, due to the topological completeness assumption, described in terms of the inductor currents and capacitor voltages, respectively. In other words, for topologically complete circuits there exist a matrix $\Gamma_L \in \mathbb{R}^{n_{R_L} \times n_L}$ such that, for the resistors contained in Σ_L, we have that $i_{R_L} = \Gamma_L i_L$, while for the resistors contained in Σ_C we have $v_{R_C} = \Gamma_C v_C$, with $\Gamma_C \in \mathbb{R}^{n_{R_C} \times n_C}$. □

B. Generation of New Storage Function Candidates

Let us next see how the Brayton–Moser equations (13), can be used to generate storage functions for *RLC* circuits. Suppose we multiply (13) by $\dot{x}^\top$, i.e.,

$$\dot{x}^\top Q(x)\dot{x} = \underbrace{\dot{x}^\top \nabla_x \tilde{P}(x)}_{\frac{d}{dt}\tilde{P}(x)} - \dot{x}^\top B v_S$$

which, after reorganizing the terms, yields the following equation:

$$\frac{d\tilde{P}}{dt}(x) = \dot{x}^\top Q(x)\dot{x} + \dot{x}^\top B v_S. \qquad (14)$$

That is, $(d\tilde{P})/(dt)(x)$ consists of the sum of a quadratic term plus the inner product of the source port variables in the desired form $\dot{x}^\top B v_S = v_S^\top (di_S)/(dt)$ (compare with the left-hand side of (7) of Proposition 2). Unfortunately, even under the reasonable assumption that the inductor and capacitor have convex energy functions, the presence of the negative sign in the first main diagonal block of $Q(x)$ makes the quadratic form sign-indefinite, and not negative (semi-)definite as desired. Hence, we cannot establish a power-balance inequality from (14). Moreover, to obtain the passivity property an additional difficulty stems from the fact that $\tilde{P}(x)$ is also not sign-definite.

To overcome these difficulties we borrow inspiration from [1] and look for other suitable pairs, say $Q_A(x)$ and $\tilde{P}_A(x)$, which we call *admissible*, that preserve the form of (13). More precisely, we want to find matrix functions $Q_A(x) \in \mathbb{R}^{n \times n}$, with $n = n_L + n_C$, verifying

$$Q_A^\top(x) + Q_A(x) \leq 0 \qquad (15)$$

[1]A circuit is called "topologically complete" if it can be described by an independent set of inductor currents and capacitor voltages such that Kirchhoff's laws are satisfied. For a detailed treatment on topologically completeness, see [10].

[2]Restricting, for simplicity, to circuits having only voltage sources in series with the inductors.

and scalar functions $\tilde{P}_A : \mathbb{R}^n \to \mathbb{R}$ (if possible, positive semi-definite), such that the circuit dynamics (13) can be (re)written as

$$Q_A(x)\dot{x} = \nabla_x \tilde{P}_A(x) - Bv_S. \tag{16}$$

If we multiply (16) by $\dot{x}^\top$ like before, we have that

$$\begin{aligned}\dot{x}^\top Q_A(x)\dot{x} &= \dot{x}^\top \nabla_x \tilde{P}_A(x) - \dot{x}^\top Bv_S\\ &= \frac{d\tilde{P}_A}{dt}(x) - \dot{x}^\top Bv_S.\end{aligned}$$

Hence, if the symmetric part of $Q_A(x)$ is negative semi-definite, that is, if (15) is satisfied and thus $\dot{x}^\top Q_A(x)\dot{x} \leq 0$, we may state (noting that $i_S = B_S^\top i_L$) that

$$\frac{d\tilde{P}_A}{dt}(x) \leq v_S^\top \frac{di_S}{dt}$$

from which we obtain a power-balance inequality with the desired port variables. Furthermore, if $\tilde{P}_A(x)$ is positive semi-definite we are able to establish the required passivity property.

In the proposition below, we will provide a complete characterization of the admissible pairs $Q_A(x)$ and $\tilde{P}_A(x)$. For that, we find it convenient to use the general form (11), i.e., $Q(x)\dot{x} = \nabla_x P(x)$, where for the case considered here $P(x) = \tilde{P}(x) - x^\top Bv_S$.

Proposition 3: For any $\lambda \in \mathbb{R}$ and any constant symmetric matrix $M \in \mathbb{R}^{n\times n}$

$$Q_A(x) = \lambda Q(x) + \nabla_x^2 P(x) M Q(x) \tag{17}$$

$$P_A(x) = \lambda P(x) + \frac{1}{2}\nabla_x^\top P(x) M \nabla_x P(x). \tag{18}$$

Proof: A detailed proof of (17) and (18) can be found in [1, p.19]. ■

An important observation regarding Proposition 3 is that, for suitable choices of λ and M, we can now try to generate a matrix $Q_A(x)$ with the required negativity property, i.e., $Q_A^\top(x) + Q_A(x) \leq 0$.

Remark 5: Since $P(x)$ has the units of power and $P_A(x) = \lambda P(x) +$ *a quadratic term in the gradient of* $P(x)$ (see (18)), $P_A(x)$ also has the units of power. A similar discussion holds for $\tilde{P}_A(x)$, which is the mixed-potential without the sources. The difference between $P_A(x)$ and the original mixed-potential $P(x)$ is that we have "swapped" the resistive terms. However, the solution of the differential equation (13) precisely coincides with the solution of (16), i.e.,

$$Q^{-1}(x)\nabla_x P(x) = Q_A^{-1}(x)\nabla_x P_A(x) \quad (=\dot{x}).$$

□

Remark 6: Some simple calculations show that a change of coordinate $z = \Phi(i_L, v_C), \Phi : \mathbb{R}^n \to \mathbb{R}^n$, on the dynamical system (11) acts as a similarity transformation on $Q(x)$. Therefore, this kind of transformation is of no use for our purposes where we want to change the sign of $Q(x)$ to render the quadratic form sign-definite. □

C. Power-Balance Inequality and the New Passivity Property

Before we present our main result, we first remark that in order to preserve the port variables $(v_S, (di_s)/(dt))$, we must ensure that the transformed dynamics (16) can be expressed in the form (13), which is equivalent to requiring that $P(x) = \tilde{P}(x) - x^\top Bv_S$. This naturally restricts the freedom in the choices for λ and M in Proposition 3.

Theorem 1: Consider a (possibly nonlinear) *RLC* circuit satisfying (13). Assume the following.

- A.1. The inductors and capacitors are passive and have strictly convex energy functions.
- A.2. The voltage-controlled resistors R_C in Σ_C are passive, linear, and time-invariant. Also, $\det(R_C) \neq 0$, and thus by taking the sum of (6) we have that $G(v_C) = (1/2)v_C^\top R_C^{-1} v_C \geq 0$ for all v_C.
- A.3. Uniformly in i_L, v_C, we have

$$\|C^{1/2}(v_C) R_C \Gamma^\top L^{-1/2}(i_L)\| < 1$$

where $\|\cdot\|$ denotes the spectral norm of a matrix.

Under these conditions, we have the following power-balance inequality:

$$\int_0^t v_S^\top(t')\frac{di_S}{dt'}(t')\,dt' \geq \tilde{P}_A[i_L(t), v_C(t)] - \tilde{P}_A[i_L(0), v_C(0)] \tag{19}$$

where the transformed mixed-potential function is defined as

$$\begin{aligned}\tilde{P}_A(i_L, v_C) &= F(i_L) + \frac{1}{2} i_L \Gamma R_C \Gamma^\top i_L\\ &\quad + \frac{1}{2}\left(\Gamma^\top i_L - R_C^{-1}v_C\right)^\top R_C \left(\Gamma^\top i_L - R_C^{-1}v_C\right).\end{aligned}$$

If, furthermore

- A.4. The current-controlled resistors are passive, i.e., $F(i_L) \geq 0$.

Then, the circuit defines a passive system with power port variables $(v_S, (di_S)/(dt))$ and storage function the transformed mixed-potential $\tilde{P}_A(i_L, v_C)$.

Proof: The proof consists in first defining the parameters λ and M of Proposition 3 so that, under the conditions A.1–A.4 of the theorem, the resulting Q_A satisfies (15) and $\tilde{P}_A$ is a positive semi-definite function.

First, notice that under assumption A.2 the co-content is linear and quadratic. To ensure that $P_A(x)$ is linear in v_S, as is required to preserve the desired port variables, we may select $\lambda = 1$ and $M = \mathrm{diag}(0, 2R_C)$. Now, using (17) we obtain after some straightforward calculations

$$Q_A(i_L, v_C) = \begin{bmatrix} -L(i_L) & 2\Gamma R_C C(v_C) \\ 0 & -C(v_C)\end{bmatrix}.$$

Assumption A.1 ensures that $L(i_L)$ and $C(v_C)$ are positive definite. Hence, a Schur complement analysis [4] proves that, under Assumption A.3, (19) holds. This proves the power-balance inequality. Passivity follows from the fact that, under Assumption A.2 and A.4, the mixed-potential function $\tilde{P}_A(i_L, v_C)$ is positive semi-definite for all i_L and v_C. This completes the proof. ■

Remark 7: Assumption A.3 is satisfied if the voltage-controlled resistances $R_{C_k} \in R_C$ are "small." Recalling that these resistors are contained in Σ_C, this means that the coupling between Σ_L and Σ_C, that is, the coupling between the inductors and capacitors, is weak. □

Remark 8: We have considered here only voltage sources. Some preliminary calculations suggest that current sources can be treated analogously using an alternative definition of the mixed potential. Furthermore, it is interesting to underscore that from (14) we can obtain, as a particular case with $\tilde{P}(i_L) = F(i_L)$, the new passivity property for *RL* circuits of Proposition 2, namely

$$\frac{dF}{dt}(i_L) = v_S^\top \frac{di_S}{dt} - v_L^\top \nabla^2_{\varphi_L} E_L(\varphi_L) v_L.$$

However, the corresponding property for *RC* circuits

$$\frac{dG}{dt}(v_C) = i_S^\top \frac{dv_S}{dt} - i_C^\top \nabla^2_{q_C} E_C(q_C) i_C$$

does not follow directly from (14), as it requires the utilization of (3) instead of (2), as done above. □

V. Example

Consider the *RLC* circuit depicted in Fig. 2. For simplicity, assume that all the circuit elements are linear and time-invariant, except for the resistor R_{L_1}. The voltage-current relation of R_{L_1} is described by $v_{R_{L_1}} = f_{R_{L_1}}(i_{L_1})$. The interconnection matrix Γ, the content $F(i_{L_1})$ and the co-content $G(v_{C_1})$ are readily found to be $\Gamma = [1, -1]^\top$, $F(i_{L_1}) = \int_0^{i_{L_1}} f_{R_{L_1}}(i'_{L_1})\, di'_{L_1}$, and $G(v_{C_1}) = (1/(2R_{C_1}))v_{C_1}^2$, respectively, and thus, the mixed-potential for the circuit is

$$\tilde{P}(i_{L_1}, i_{L_2}, v_{C_1}) = \int_0^{i_{L_1}} f_{R_{L_1}}(i'_{L_1})\, di'_{L_1} - \frac{1}{2R_{C_1}} v_{C_1}^2 + i_{L_1} v_{C_1} - i_{L_2} v_{C_1}.$$

Hence, the differential equations describing the dynamics of the circuit are given by

$$\begin{aligned} -L_1 \frac{di_{L_1}}{dt} &= f_{R_{L_1}}(i_{L_1}) - v_{S_1} + v_{C_1} \\ -L_2 \frac{di_{L_2}}{dt} &= -v_{C_1} \\ C_1 \frac{dv_{C_1}}{dt} &= i_{L_1} - \frac{v_{C_1}}{R_{C_1}} - i_{L_2}. \end{aligned}$$

The new passivity property is obtained by selecting $\lambda = 1$ and $M = \mathrm{diag}(0, 0, 2R_{C_1})$, yielding that $Q_A^\top + Q_A \leq 0$ if and only if

$$R_{C_1} < \sqrt{\frac{L_1 L_2}{C_1(L_1 + L_2)}}. \tag{20}$$

Under the condition that $F(i_{L_1}) \geq 0$ and $R_{C_1} > 0$, positivity of $\tilde{P}_A$ is easily checked by calculating (18), i.e.,

$$\begin{aligned} \tilde{P}_A(i_{L_1}, i_{L_2}, v_{C_1}) = &\int_0^{i_{L_1}} f_{R_{L_1}}(i'_{L_1})\, di'_{L_1} \\ &+ \frac{R_{C_1}}{2}\left(i_{L_1}^2 + i_{L_2}^2\right) + \frac{R_{C_1}}{2} \\ &\times \left(i_{L_1} - i_{L_2} - \frac{v_{C_1}}{R_{C_1}}\right)^2. \end{aligned}$$

In conclusion, if (20) is satisfied, then the circuit of Fig. 2 defines a passive system with power port variables $(v_{S_1}, (di_{L_1})/(dt))$ and storage function $\tilde{P}_A(i_{L_1}, i_{L_2}, v_{C_1}) \geq 0$.

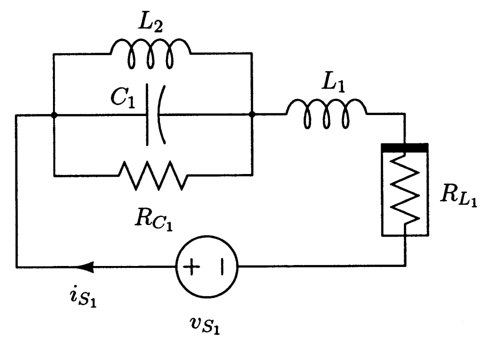

Fig. 2. Simple *RLC* circuit with nonlinear current-controlled resistor.

VI. Concluding Remarks

Our main motivation in this paper was to establish a new passivity property for *RL*, *RC*, and a class of *RLC* circuits. We have proven that for this class of circuits it is possible to "add a differentiation" to the port variables preserving passivity with respect to a storage function which is directly related to the circuit's power. The new supply rate naturally coincides with the definition of reactive power.

Instrumental for our developments was the exploitation of Tellegen's theorem. Dirac structures, as proposed in [9], provide a natural generalization to this theorem, characterizing in an elegant geometrical language the key notion of power preserving interconnections. It seems that this is the right notion to try to extend our results beyond the realm of *RLC* circuits, e.g., to mechanical or electromechanical systems. A related question is whether we can find Brayton–Moser like models for this class of systems.

There are close connections of our result and the shrinking dissipation Theorem of [12], which is extensively used in analog very large-scale integration circuit design. Exploring the ramifications of our research in that direction is a question of significant practical interest.

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