Asymptotic symmetries of three dimensional Anti-de-Sitter space from a perspective of representation theory

Master Thesis

written by

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Abstract

As Wigner showed in 1939 for the Poincaré algebra, fundamental particles can be classified using symmetry algebras. In a universe including gravity, the Poincaré algebra cannot be the correct symmetry algebra, as this is the symmetry algebra for flat space. Instead, one should consider a symmetry algebra of asymptotic symmetries, which preserve only the asymptotic structure of gravity. Many of these asymptotic symmetries are not physically useful and are therefore considered 'trivial'. In this thesis we give a new, quantum, definition of trivial symmetries, namely that a symmetry is trivial if it does not change which fundamental particles are found in a classification. We then specialize to three-dimensional asymptotically Anti-de-Sitter space. To calculate which symmetries are trivial we first determine the second cohomology group of the asymptotic symmetries. Using the second cohomology group it is then found that the useful asymptotic symmetry algebra is given by $\mathfrak{w} \oplus \mathfrak{w} \oplus \mathbb{R}$, where $\mathfrak{w}$ is the Witt (centerless Virasoro) algebra, whereas the standard definition of trivial symmetry gives $\mathfrak{w} \oplus \mathfrak{w}$ as the useful symmetry algebra. The extra factor of $\mathbb{R}$ is interpreted to be a kind of center-of-mass momentum.

Acknowledgments

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Contents

Chapter 1

Introduction

Symmetries play an essential role in physics. We will show this in three different stadia of physics, starting with classical physics. This is the physical study of day-to-day objects such as billiard balls, cars, seesaws, bridges, rockets, et cetera. In classical physics, symmetry transformations such as time translations, space translations and rotations lead to energy, momentum and angular momentum conservation respectively. Let us show this in the case of the billiard balls. If a rolling billiard ball hits a stationary billiard ball, the momentum of the rolling billiard ball is transferred to the stationary billiard ball, meaning that the ball that was previously rolling will stop, and the stationary billiard ball will start rolling with the same speed as the ball that hit it. The total momentum of the two billiard balls is thus the same before and after the collision. Moreover, it is clear that space translations are a symmetry in this case; if we move both of the balls equally, the physics does not change. We can break this symmetry however, by including the walls of the billiard table. If we now move the two balls (but not the table), one of the balls could end up outside the billiard table, meaning that we no longer have space translations as a symmetry. We also no longer have momentum conservation, as the total momentum of the two balls changes whenever a ball hits a wall. Indeed, imagine that one ball is stationary and the other is rolling towards a wall. As soon as it hits the wall, the momentum flips sign and is thus not conserved. Of course, if we also translate the billiard table when performing a space translation, and if we include the momentum of the billiard table, the billiard balls and table are again symmetric under space translations, and momentum is again conserved.

The connection between symmetries and conserved quantities is made formal in Noether's theorem, which essentially tells us that every continuous symmetry corresponds to a conserved quantity [Noe18]. We will therefore call this connection between symmetries and conserved quantities the Noether perspective of symmetry.

Secondly, let us look at special relativity. This is the study of objects with a velocity on the scale of the speed of light, but without gravity being relevant. The main hypothesis of special relativity is that the speed of light is constant in every reference frame, meaning that the relative speed of a ray of light is the same when you are traveling towards this ray as when you are moving away from it. When objects travel at velocities near the speed of light, the passage of time slows down and the length of an object starts to depend on the observer. The exact way in which these phenomena occur can be derived by assuming that classical physics should be invariant under Lorentz transformations [Res68]. Lorentz transformations are symmetry transformations such as rotations, alongside additional symmetry transformations. These additional symmetry transformations are constructed in such a way that the speed of light is the same after applying a transformation, as required by the main hypothesis of special relativity. As special relativity is thus determined by these Lorentz symmetry transformations, we have found another application of symmetries.

Lastly, the Lorentz group of symmetries plays an important role in the classification of elementary particles, through Wigner's classification [Wig39]. The elementary particles are the particles that are not composed of other particles and include the particles that make up all of the matter around us, such as electrons and the quarks that make up protons and neutrons. Aside from these "matter particles", there are also "force particles", which correspond to the fundamental forces in the universe. The photon, for example, is the force particle that corresponds to the electromagnetic force. The properties of the particle tell us something about the corresponding force; e.g. the higher the mass of the particle, the shorter the effective range of the corresponding force is. We are thus interested in what kind of elementary particles can exist in our universe, and as shown by Wigner, this can be classified using the symmetry group. We will call this connection between particles and symmetries the Wigner perspective of symmetries. The Wigner perspective of symmetries is more oriented towards quantum mechanics than the Noether perspective, as the conserved quantities of the Noether perspective are only classically conserved.

Now we turn to general relativity, which is the theory that describes situations with very strong gravity, such as black holes, the big bang, or the universe itself. In a similar fashion to classical mechanics, in general relativity symmetries again lead to conserved quantities. The issue, however, is that realistic models of our universe are not symmetric. In other theories, such as special relativity or classical dynamics, there is a background space, which we can use instead to define symmetries. For example, in classical mechanics one can typically translate all bodies of interest in space with respect to the background space. If the system consists only of these bodies, the interactions only depend on the distance between the bodies and are therefore not affected by this translation, leading to a symmetry. This symmetry then gives us momentum conservation. Think of the billiard balls; when the translation symmetry was broken by the billiard table, we could simply also move the billiard table to repair this. In contrast, in general relativity this "background space", the spacetime, is the object of interest or in the billiard analogy, the spacetime is the billiard table. Now if the billiard table is the universe, what does it mean to move the billiard table? This does not make sense and therefore does not lead to a conserved quantity.

Instead, one can consider *asymptotic symmetries*. These are not exact symmetries, but intuitively, they become exact symmetries for very large distances. For instance, the distance from the sun to the nearest neighboring star, Alpha Centauri, is about 268,770 AU [NAS], meaning that it is about 10,000 times further than the distance between Neptune and the sun. If we were to consider a rotation of the solar system, this would not be an

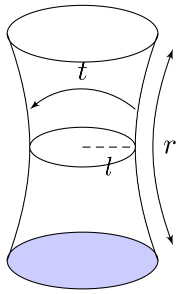

Figure 1.1: Two dimensional Anti-de Sitter space. l is the radius of the space, t is an angular coordinate and r is the radial coordinate.

exact symmetry, as spacetime is not invariant under rotations near the solar system. For very large distances from the sun, as long as we are also very far from other massive objects such as Alpha Centauri, say 10,000 AU, spacetime is almost flat and rotating it is very close to a symmetry. Intuitively, this is an approximate symmetry.

Now the solar system is well described by Newtonian gravity and thus does not benefit much from a description in terms of asymptotic symmetries, but for larger structures, such as the entire universe, these asymptotic symmetries can be valuable. For example, they can be used to define the Bondi energy, which is a measure for the total energy of the system [Wal09][BBM62]. This Bondi energy can then be used to analyze the energy carried away by gravitational waves.

Moreover, there is a great interest in asymptotic symmetries in something called the AdS/CFT correspondence. This is a duality between a string theory in an asymptotically Anti-de-Sitter (AdS) space, which is the solution of the Einstein equation for an empty universe with negative cosmological constant (see figure 1.1), and a conformal field theory (CFT) on the boundary of the AdS space. This AdS/CFT correspondence is currently one of the main avenues to explore quantum gravity [AE15] and can be used to study strongly interacting quantum field theories [Kle09]. The conserved charges of the asymptotic symmetries of AdS are used explicitly to construct the duality between the string theory and the CFT.

However, not every asymptotic symmetry is useful in the Noether point of view, as some do not lead to useful conserved charges. These symmetries are called trivial symmetries. The asymptotic symmetry group is then typically defined to be the quotient of all allowed asymptotic symmetries by the trivial asymptotic symmetries. This reduces the size of the asymptotic symmetry group and makes it easier to work with, while only losing symmetries that are trivial anyway. This Noether point of view definition of trivial symmetries is standard [BH86][Obl17].

Using this asymptotic symmetry group, the fundamental particles of AdS_3 are then typically determined by looking at the projective unitary positive energy representations, akin to Wigner's classification. We think that the Noether definition of trivial symmetries is ill-suited for this application of the asymptotic symmetries, however, as these trivial symmetries could still impact what fundamental particles are found in a classification, meaning that excluding them from the asymptotic symmetry group might not be the right choice. Indeed, there could be symmetries that do not generate a classically conserved charge but are necessary to find the correct fundamental particles, which should thus not be excluded from the asymptotic symmetry group. Furthermore, when one determines the fundamental particles of $\mathfrak{bms}_4$, which is the asymptotic symmetry algebra of four dimensional flat space in the Noether perspective, the resulting particles are not as physically expected [Obl17]. It could therefore be fruitful to consider a new definition of trivial symmetries which is better suited for determining the fundamental particles. Hence in this thesis we look at trivial symmetries from the Wigner perspective and consider a symmetry to be trivial if it is trivial in every projective unitary positive energy representation. With this new definition, taking the quotient by the trivial symmetries does not change which particles are found in a classification. Specifically, regarding this topic we will answer the following research questions:

What is the largest normal subgroup of the weak asymptotic symmetry group of AdS_3 for which every symmetry is trivial in every projective unitary positive energy representation? What is the quotient group of the weak asymptotic symmetry group by this normal subgroup and how does it differ from the standard asymptotic symmetry group?

This new definition of trivial symmetries is more oriented towards quantum mechanics, as the Noether perspective is a classical perspective of symmetries, whereas the Wigner perspective takes quantum mechanics into account.

One might wonder why we only look at projective unitary positive energy representations, or what this even means. To start answering this, we first ask ourselves what a symmetry should be in the quantum context.

In quantum theory, states are normalized complex wavefunctions on a Hilbert space. The observables are operators acting on this Hilbert space and expectation values for these observables are calculated using the inner product, for example $\langle B\rangle = \langle\psi|B|\psi\rangle = \langle\psi, B\psi\rangle$ for a wavefunction ψ and observable B. A symmetry should then act on the wavefunctions in such a way that these expectation values do not change. All the ways that a symmetry (an element of some group) can act on a Hilbert space are given by the representations of the group on the Hilbert space. The core idea is that all of these different ways of acting on a Hilbert space represent different particles.

If we thus understand all the representations, we understand all of the possible particles, which is why we consider representations. However, there are additional constraints that we should impose on the symmetries, which are given by the adjectives projective, unitary and positive energy. To start, a symmetry should preserve the expectation values, which happens precisely if the representation is unitary. Furthermore, quantum states are normalized and phases are irrelevant as multiplying a wavefunction by a phase does not change any of the expectation values. Thus a symmetry U that sends $\psi \to U\psi$ and a symmetry $\tilde{U} : \psi \to e^{i\phi}U\psi$ for some ϕ are equivalent. The projective representations are exactly the representations that take this equivalence into account. Lastly a representation is of positive energy if the energy of all the states is bounded from below. If this is not the case, a particle could keep decaying to lower energy states, whilst radiating energy. This is deemed nonphysical, leading to us only considering positive energy representations. Combining all of the above, we see that we should only consider projective unitary positive energy representations.

Three dimensional Anti-de-Sitter space is a good model to start with, for the following reasons. First of all, it is typically simpler than higher dimensional AdS spaces. It is often a good idea to start with the simplest model, as this can give you valuable intuition without getting lost in technical difficulties. Two dimensional spaces might be even simpler, but general relativity is not well defined in two dimensions, as the Einstein tensor vanishes identically in two dimensional theories. Thus AdS_3 is the simplest interesting version of AdS. From the perspective of asymptotic symmetries, AdS_3 is also different then higher dimensional spaces, as the asymptotic symmetry group of AdS_3 is infinite dimensional [BH86], whereas in higher dimensional AdS_n spaces it is the same as the normal symmetry group of AdS_n [AM84]. These facts lead to AdS_3 being an interesting space to consider for our calculation of the asymptotic symmetry group with the new definition of a trivial symmetry. Three dimensional AdS can essentially be pictured as in figure 1.1, except each point on the hyperboloid is actually a circle of points.

We will start this thesis by quickly introducing the core concepts of Lie theory and locally convex vector spaces in chapter 2. Lie theory is essential, as this is used to describe continuous symmetry groups. A continuous symmetry is a symmetry transformation that depends on a parameter, say x, which takes values in $\mathbb{R}$, so for each $x \in \mathbb{R}$ we have a symmetry transformation. An example is given by the translation $x_0 \to x_0 + x$. These continuous symmetries are easier to handle than non-continuous (discrete) symmetries, as we can differentiate continuous symmetries in their parameter x. These differentials, also called infinitesimal symmetries, turn out to be easier to handle while still giving us plenty of information about the continuous symmetries. They form an object called a Lie algebra, and these are the objects we will focus on in the thesis, thus we spend some time on their basic properties.

We then turn to representation theory in chapter 3. As mentioned earlier, symmetries in a quantum context are given by representations, thus we must define them rigorously, and the same holds for projective, unitary and positive energy representations. Combining all of these adjectives, we get projective unitary positive energy representations, and it is shown in section 3.3 that these can be understood through something called Lie algebra cohomology. We define this cohomology in section 3.4 and we lastly give a nontrivial example of Lie algebra cohomology in section 3.5 by looking at symmetries of the circle, which form the Witt (or centerless Virasoro) algebra.

We are then ready to introduce the actual asymptotic symmetries, which is done in chapter 4. We first look at what asymptotic symmetries should be in general in section 4.1 and we then specify to AdS_3 in section 4.2. In the next two sections we take care to define the boundary conditions in a mathematically rigorous fashion. We can then finally compute what the asymptotic symmetries are, which is done in section 4.5.

Next we look at the Lie algebra cohomology of the asymptotic symmetries to gain an understanding of the projective unitary positive energy representations in chapter 5. A start to this is made in section 5.1. We then need some more powerful tools, which are developed next in section 5.2. Using these tools, we can compute the rest of the cohomology in sections 5.3, 5.4.

Before continuing with the calculation of the trivial symmetries in the Wigner perspective, we first shortly discuss how the trivial symmetries are determined in the Noether perspective and what the result is. Afterwards, having computed the cohomology, we are ready to compute the trivial symmetries, which is done in section 6.2. The result is nontrivial, so we give some ideas on how the result is to be interpreted in section 6.3. We finish by giving a summary of the results and a conclusion in chapter 7.

Chapter 2

Lie theory

The ultimate goal of this thesis is to describe asymptotic symmetries. These are continuous symmetries and to understand those we need Lie theory. Symmetry groups are often Lie groups, which are groups where the multiplication and inverse are smooth operations. To understand what it means for these operations to be smooth, we need a bit of differential geometry. We will discuss both in section 2.1, in an informal manner as we will only use them to motivate other definitions. For a more complete introduction to differential geometry and Lie groups, we refer to Ref. [Lee12].

The symmetry group we will consider will consist of asymptotic symmetries. As we only restrict the behavior of these symmetries at infinity, they are very free in finite space, leading to the group or algebra of asymptotic symmetries being infinite dimensional. To handle infinite dimensional Lie groups and algebras we will need the theory of locally convex vector spaces, which will be treated in section 2.2. This section is a bit technical and if your goal is only to understand the main ideas of this thesis, this section can safely be skipped.

Instead of working with symmetries themselves, it is typically a lot easier to work with the generators of symmetries, as these form a vector space. In fact, this space of generators forms a Lie algebra, which we will define in section 2.3, on a rigorous level as this is the structure we will mainly work with. We will then also discuss some of the properties of Lie algebras with a focus on quotient and complexification constructions.

2.1 Differential geometry and Lie groups

Lie groups are inspired by groups of continuous symmetries, meaning that they should be both a group and be able to vary continuously in some sense. For the last property, the correct setting is a *manifold*:

- A finite dimensional manifold is a topological space that locally looks like $\mathbb{R}^n$ for some dimension n.
- A finite dimensional Lie group is both a manifold and a group, in such a way that group multiplication and inverses are smooth

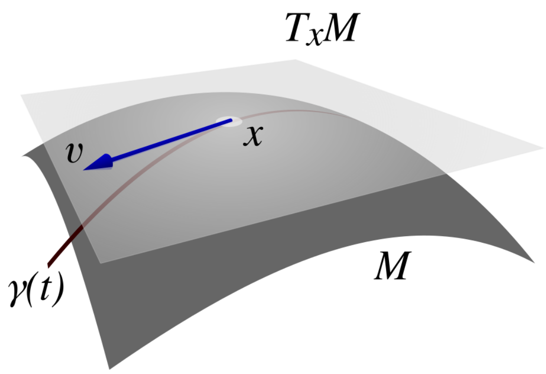

Figure 2.1: Tangent space T_xM of a manifold (or Lie group) M at the point x. A tangent vector v with corresponding curve $\gamma(t)$ is also shown. Figure was taken from Wikipedia [TN].

Now because a manifold M locally looks like $\mathbb{R}^n$, at each point x in M we can consider the tangent space T_xM of that point, see figure 2.1. This tangent space describes all the derivatives of curves $\gamma(t)$ going through x. These derivatives are vectors, denoted by v in the figure. For a Lie group, using the multiplicative structure, one can identify all the tangent spaces T_x with the tangent space at the identity T_e. This tangent space has the structure of a Lie algebra, which we will describe in section 2.3.

The symmetry groups that we will consider will be infinite dimensional however. In that case, instead of saying that a Lie group locally looks like $\mathbb{R}^n$, we say it locally looks like V where V is some infinite dimensional vector space. Furthermore, to construct the tangent space T_e, we need to differentiate the curves $\gamma(t)$ as in figure 2.1. This requires V to have a topology, as differentials are defined via limits. For finite dimensional vector spaces there is only one Hausdorff topology such that addition and scalar multiplication are continuous, which is the standard topology on $\mathbb{R}^n$. In contrast, for infinite dimensional vector spaces there are multiple topologies and not every one of them can be used to define derivatives. In fact, we need the vector space V to be locally convex [Nee06], which we will define in the next section. We will say that a Lie group G is locally convex if the vector space V is locally convex.

Remark 2.1. We will mainly be concerned with infinite dimensional Lie groups, which are not as well behaved as finite dimensional Lie groups, for example the exponential map from the corresponding Lie algebra might not be surjective on a neighborhood of the identity [Mil85]. Translating information between the Lie group and the Lie algebra therefore becomes harder, which is why we will mainly stay on the Lie algebra level. For a survey on infinite dimensional Lie groups, see Ref. [Nee06].

Now the symmetries we are interested in are symmetries on a spacetime. These are *diffeomorphisms*, which are smooth bijective maps $\phi : M \to N$ between manifolds M, N such that the inverse is also smooth. If a diffeomorphism exists between M, N, for the intents and purposes of differential geometry they are the same or isomorphic in mathematical language. The reason for the smoothness requirements on the diffeomorphism becomes

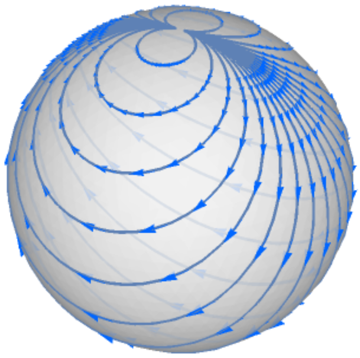

Figure 2.2: A vector field on the sphere. Figure taken from Wikipedia [BTo]

clear from this point of view; we want to be able to translate a smooth function $f : M \to \mathbb{R}$ to a smooth function $\tilde{f} : N \to \mathbb{R}$ if M, N are isomorphic, and the smoothness requirements on the diffeomorphism ensure that $\tilde{f}$ is again smooth. A symmetry should moreover be an automorphism, meaning that it is a diffeomorphism $\phi : M \to M$. Lastly, spacetime is not just a manifold. It is a manifold with a *metric* (also called a Lorentzian manifold). This metric specifies the distance between two points on the manifold. On 4 dimensional flat space, the metric is for example typically written as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, where we use natural units with $c = 1$, as we will use all throughout the thesis. We could however, specify a different metric on $\mathbb{R}^4$ such that we get a hyperbolic space. In such a space, parallel lines do not stay parallel and the sum of the angles in a triangle do not add up to 180 degrees, thus spacetime would look very different. A true symmetry should therefore not change the metric. An *isometry* is a diffeomorphism $\phi : M \to M$ such that the metric stays the same when transformed using ϕ and is thus the exact symmetry we are interested in. Asymptotic symmetries are diffeomorphisms that become isometries infinitely far away from all matter in the spacetime.

The symmetry groups that we will look at are thus groups of diffeomorphisms. These are however not always Lie groups. Regardless, we will see that we can associate a Lie algebra with them, by looking at curves of diffeomorphisms and differentiating them, just as before. To see what kind of objects we get by differentiating these curves, let us look at a special kind of curves, namely one-parameter groups of curves. These are curves $\gamma : \mathbb{R} \to \mathrm{Diff}(M)$ such that $\gamma(t)\gamma(s) = \gamma(t + s)$ and $\gamma(\cdot)(\cdot) : \mathbb{R} \times M \to M$ is continuous. They are also called global flows. Now note that for $x \in M$, $\gamma(t)(x)$ is a curve in M going through x. The derivative in time is thus a vector in T_xM. We conclude that the derivative of a global flow specifies a vector in T_xM for each $x \in M$. This is called a vector field and an example is given in figure 2.2. Conversely, a vector field on M also generates a flow, although this need not be a global flow. Instead, for each $x \in M$, the map $\gamma(\cdot)(x)$ (which now represents a local flow) is only defined for some some $t \in (-t_x, t_x)$ and satisfies the additive property on this domain. On the Lie algebra level, the symmetry algebra that we are looking for thus consists of vector fields on the manifold M.

2.2 Locally convex vector spaces

A normed vector space is a vector space together with a norm. In some cases a norm is too strict however. For example, for the smooth functions on the circle $C^\infty(\mathbb{S}^1)$ the standard supremum norm makes differentiating discontinuous. Instead of a single norm, one can also consider a family of *seminorms*.

Definition 2.2 (Seminorm)**.** Let X be a vector space over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. A seminorm p is a map $X \to \mathbb{R}$ such that:

1. p is nonnegative: $p(x) \geq 0$ for all $x \in X$
2. p is absolutely homogeneous: $p(sx) = |s|p(x)$ for all $s \in \mathbb{K}, x \in X$
3. p satisfies the triangle inequality: $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

Note that the nonnegativity actually follows from

$$0 = 0p(x-x) = p(0(x-x)) = p(x-x) \leq p(x) + p(-x) = 2p(x) \tag{2.1}$$

where we only used the homogeneity and the triangle inequality.

The topology generated by this family of seminorms then forms a locally convex space.

Definition 2.3 (Locally convex topological vector space)**.**

- A *topological vector space* X is a vector space over a topological field $\mathbb{K}$, which is a field with a topology such that all operations of the field are continuous, together with a topology on the vector space such that addition of vectors and scalar multiplication are continuous.
- A *locally convex topological vector space* (LCTVS) is a vector space V together with a family $\mathcal{P}$ of seminorms. The topology on V is the coarsest topology such that all seminorms $p \in \mathcal{P}$ are continuous.

Equivalently, a LCTVS is a topological vector space where the neighborhood basis at the origin consists of convex and balanced sets. These are sets S such that:

1. For each point $x, y \in S$ the line between x, y is contained in S (Convex)
2. For each point $x \in S$ and scalar $|a| \leq 1$, $ax \in S$ (Balanced)

This is where the name *locally convex* comes from. For a more complete introduction, see Ref. [NB10]. The similarity in terminology between convex sets and convex functions is no coincidence. Indeed, a function is convex precisely if the set of points above and including the graph is convex.

In a normed vector space the norm generates the whole topology and in a LCTVS all of the seminorms are needed to generate the topology. The norm is thus replaced by a typically infinite set of seminorms. Furthermore, every LCTVS is a topological vector space [NB10].

An example of a LCTVS is $C^\infty(\mathbb{S}^1)$ with the seminorms:

$$p_k(f) = \sup_{x\in\mathbb{S}^1} |\partial_x^k f(x)| \tag{2.2}$$

The absolute homogeneity is immediate and the triangle inequality holds as the triangle inequality holds for both the absolute value and the supremum. Aside from the seminorms in the generating family, there can be other continuous seminorms. We show that these are all bounded by a finite linear combination of seminorms in the generator family.

Lemma 2.4. *Let q be a seminorm on a locally convex vector space V with family of seminorms $\mathcal{P}$. Then q is continuous if and only if there exist $p_1, \dots, p_n \in \mathcal{P}$ such that $q \leq C_1p_1 + \cdots + C_np_n$ where $C_1, \dots, C_n$ are positive constants.*

Proof. First assume that q is continuous, then $q^{-1}([0,1))$ is an open neighborhood of zero in V. Now the family of preimages $p^{-1}([0,r))$ for $p \in \mathcal{P}, r > 0$ form a neighborhood subbasis of 0 [NB10], so there exist $p_1, \dots, p_n \in \mathcal{P}, r_1, \dots, r_n > 0$ such that $\bigcap_{i=1}^n p_i^{-1}([0,r_i)) \subseteq q^{-1}([0,1))$. Now let $x \in V$, and assume that $p_i(x) = 0$ for all $1 \leq i \leq n$. Then for all $l > 1$, $p_i(lx) = 0 < r_i$, so $q(lx) < 1$ meaning that $q(x) < 1/l$ for all $l > 1$. Thus $q(x) = 0$ and the conclusion is trivially true. Now if there exists i such that $p_i(x) \neq 0$, define $y = \frac{x}{\sum_{i=1}^n p_i(x)2/r_i}$. Now either $p_j(y) = 0 < r_j$ or $p_j(y) = \frac{p_j(x)}{\sum_{i=1}^n p_i(x)2/r_i} \leq \frac{p_j(x)}{p_j(x)2/r_j} = r_j/2 < r_j$, and thus also $q(y) = \frac{q(x)}{\sum_{i=1}^n p_i(x)2/r_i} < 1$, so $q(x) < \sum_{i=1}^n p_i(x)2/r_i$.

Now for the other direction, assume there exist $p_1, \dots, p_n \in \mathcal{P}$ such that $q \leq C_1p_1 + \cdots + C_np_n$. Note that $p = C_1p_1 + \cdots + C_np_n$ is a continuous seminorm as scalar multiplication and addition are continuous in topological vector sp aces, and $q \leq p$ implies that q is continuous whenever p is continuous, which is shown in Ref. [NB10]. □

Two families of seminorms P, Q can induce the same topology on a vector space V. We will here give a condition for this to be the case:

Lemma 2.5. *Let P, Q be two families of seminorms on the vector space V. Suppose that for all $p \in P, q \in Q$ there exist $p_1, \dots, p_n \in P, q_1, \dots, q_k \in Q$ such that*

$$\begin{aligned} p(x) &\leq q_1(x) + \cdots + q_k(x) \\ q(x) &\leq p_1(x) + \cdots + p_n(x) \end{aligned} \tag{2.3}$$

Then the topologies induced by P, Q are equivalent.

Proof. Let $(V, P), (V, Q)$ be the locally convex vector spaces with topologies induced by P, Q respectively and let $\iota : (V, P) \to (V, Q)$ be the identity mapping. We will show that this is a homeomorphism. It is clearly a bijection. For continuity, note that by assumption for all $q \in Q$ there exist $p_1, \ldots, p_n \in P$ such that $q(\iota(x)) \leq p_1(x) + \cdots + p_n(x)$, meaning that ι is continuous [NB10]. The proof that the inverse is continuous is the same. $\square$

We will regularly encounter quotient and product spaces. The next two lemmas from Ref. [NB10] show that these are again LCTVSs and describe the corresponding family of seminorms.

Lemma 2.6 (From Ref. [NB10])**.** *Let X be a LCTVS and let $\mathcal{P}$ be the corresponding family of seminorms. Suppose Y is a subspace of X, then the quotient X/Y is again a LCTVS with seminorms $\bar{\mathcal{P}}$, where $\bar{p} \in \bar{\mathcal{P}}$ is given by $\bar{p}(\bar{x}) = \inf_{y \in Y} p(x + y)$ for $\bar{x} \in X/Y$.*

Lemma 2.7 (From Ref. [NB10])**.** *Let $(X_s)_{s \in S}$ be a family of LCTVSs and for each $s \in S$, let $\mathcal{P}_s$ be the corresponding family of seminorms on X_s. Then the product $\prod_{s \in S} X_s$ with the product topology is again a LCTVS and the seminorms on $\prod_{s \in S} X_s$ are given by $p \circ \pi_s$ where π_s is the projection on the subspace X_s and $p \in \mathcal{P}_s$.*

The following lemma will help us prove when a multilinear map to a LCTVS is continuous:

Lemma 2.8 (From Ref. [NB10])**.** *Let A be a multilinear map $A : X_1 \times \cdots \times X_n \to Y$ where Y is a locally convex vector space determined by the family of seminorms $\mathcal{Q}$ and all X_i are topological vector spaces. Then A is continuous if and only if for each seminorm $q \in \mathcal{Q}$ there are continuous seminorms $p_1, \ldots, p_n$ on $X_1, \ldots, X_n$ respectively and $C > 0$ such that $q(A(x_1, \ldots, x_n)) \leq C p_1(x_1) \ldots p_n(x_n)$ for all $x_1, \ldots, x_n \in X_1 \times \cdots \times X_n$.*

For example, the derivative $\partial_x : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ is continuous as $p_k(\partial_x f) = p_{k+1}(\partial x)$.

Lastly, we give a lemma that shows when a LCTVS is metrizable, in which case it is sequential, so we also characterize when a sequence converges:

Lemma 2.9 (Paraphrased from Ref. [NB10])**.** *Suppose X is a LCTVS with a countable family of seminorms $\mathcal{P}$. Then X is pseudometrizable. If furthermore for all $x, y \in X$ there exists $p \in \mathcal{P}$ such that $p(x - y) \neq 0$, X is metrizable and therefore sequential.*

In any case, a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ converges to $x \in X$ if and only if $p(x_n - x) \to 0$ for each $p \in \mathcal{P}$.

2.3 Lie algebra

We can now define what a Lie algebra is. As described above, this structure is inspired from the tangent space structure of a Lie group.

Definition 2.10 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ together with a *Lie bracket* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

1. The bracket $[\cdot,\cdot]$ is linear in both arguments

2. The bracket is antisymmetric: $[a,b] = -[b,a]$ for all $a,b \in \mathfrak{g}$

3. The Jacobi identity holds: $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for all $a,b,c \in \mathfrak{g}$

The prototypical example of a Lie algebra is $\mathfrak{gl}(n)$, which is the set of $n \times n$ matrices. The bracket is then given by the commutator $[A,B] = AB - BA$. This is the tangent space of the Lie group $\mathrm{GL}(n)$, which consists of the matrices with nonzero determinant and standard matrix multiplication. The commutator $[A,B]$ then describes how the elements $\exp(A), \exp(B)$ commute in the Lie group.

The Lie algebra of a locally convex Lie group will again be locally convex. We therefore define locally convex Lie algebras abstractly as well.

Definition 2.11 (Locally convex Lie algebra)**.**

- A *locally convex Lie algebra* is a Lie algebra which is locally convex as a vector space and has a continuous Lie bracket.

- A *Lie algebra homomorphism* is a linear map that preserves the Lie bracket.

We saw previously that the smooth functions on the circle, $C^\infty(\mathbb{S}^1)$, form a locally convex vector space. With the Lie bracket $[f,g] = fg' - gf'$, where the prime denotes differentiation, $C^\infty(\mathbb{S}^1)$ becomes a locally convex Lie algebra, which will be proved in section 3.5.

A perfect Lie algebra is essentially a Lie algebra in which each element can be written as a finite sum of commutators of other elements. This property can come in handy in certain computations.

Definition 2.12 (Perfectness)**.** A Lie algebra is called perfect if it is equal to its own commutator ideal:

$$\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] := \mathrm{span}\{[a,b] | a,b \in \mathfrak{g}\} \tag{2.4}$$

Remark 2.13. For a locally convex Lie algebra, one can also consider the closed linear span of the commutator. Our definition is slightly stricter and gives more information about the algebraic structure of the Lie algebra.

2.3.1 Complexification

Often it is easier to work in the complexification of a Lie algebra, for the same reason that it is typically easier to work with complex exponentials than with sines and cosines. We therefore define this complexified structure and show that we can use it to draw conclusions about the real Lie algebra.

Definition 2.14 (Complexification of a Lie algebra)**.** The complexification of a locally convex Lie algebra $\mathfrak{g}$ with bracket $[\cdot,\cdot]$, written as $\mathfrak{g}_\mathbb{C}$, has the following vector space structure:

$$\mathfrak{g} \oplus i\mathfrak{g} \tag{2.5}$$

together with the bracket $[\cdot,\cdot]' : \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$:

$$[a+ib, c+id]' = ([a,c]-[b,d]) + i([b,c]+[a,d]) \tag{2.6}$$

The topology is given by the product topology of $\mathfrak{g} \times \mathfrak{g}$.

That the bracket $[\cdot,\cdot]'$ is a continuous Lie bracket will now be explained. First of all, it is clearly antisymmetric. Furthermore, $\mathbb{C}$-linearity of $[\cdot,\cdot]'$ follows from the $\mathbb{R}$-linearity of $[\cdot,\cdot]$ is. The Jacobi identity clearly holds for $[a,c]', [ib,c]', [a,id]', [ib,id]'$ and by linearity it holds in general. The Lie bracket is continuous as it is a linear combination of continuous functions.

We also define an involution on the complexification, namely $\overline{a+bi} = a - bi$. It is immediate that for $z \in \mathfrak{g}_\mathbb{C}$, if $\overline{z} = z$, $z \in \mathfrak{g}$. Now for any subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\mathbb{C}$, we define the real part to be $\mathrm{Re}(\mathfrak{h}) = \{z \in \mathfrak{h} | \overline{z} = z\}$. Clearly $\mathrm{Re}(\mathfrak{g}_\mathbb{C}) = \mathfrak{g}$.

Proposition 2.15. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then the complexified commutator can be calculated from commutator of the complexification:*

$$[\mathfrak{g},\mathfrak{h}]_\mathbb{C} = [\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}] \tag{2.7}$$

and in particular, the normal commutator is the real part:

$$[\mathfrak{g},\mathfrak{h}] = \mathrm{Re}([\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}]) \tag{2.8}$$

Proof. First let $a \in [\mathfrak{g},\mathfrak{h}]_\mathbb{C}$, then $a = b + ci$ with $b = \sum_k [u_k, v_k], c = \sum_j [x_j, y_j]$ and $u_k, x_j \in \mathfrak{g}, v_k, y_j \in \mathfrak{h}$. Now $u_k, ix_j \in \mathfrak{g}_\mathbb{C}, v_k, y_j \in \mathfrak{h}_\mathbb{C}$ and $a = \sum_k [u_k, v_k] + \sum_j [ix_j, y_j]$ so $a \in [\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}]$.

For the converse, let $a \in [\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}]$, so $a = \sum_k [u_k, v_k]$ with $u_k = b_k + ic_k, v_k = x_k + iy_k$, then $a = \sum_k ([b_k, x_k] - [c_k, y_k]) + i([c_k, x_k] + [b_k, y_k]) \in [\mathfrak{g},\mathfrak{h}]_\mathbb{C}$

Now we have

$$[\mathfrak{g},\mathfrak{h}] = \mathrm{Re}([\mathfrak{g},\mathfrak{h}]_\mathbb{C}) = \mathrm{Re}([\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}]) \tag{2.9}$$

□

2.3.2 Quotient algebra

Next, in a lot of situations it can be fruitful to consider the quotient of a Lie algebra by an ideal, which is a subspace $\mathfrak{i}$ such that $[\mathfrak{g},\mathfrak{i}] \subseteq \mathfrak{i}$. We now show that the Lie algebra structure of the quotient vector space is unique.

Lemma 2.16 (Lemma 17.10 in [Ban10])**.** *Let $\mathfrak{i}$ be an ideal of the locally convex Lie algebra $\mathfrak{g}$. The linear quotient $\mathfrak{g}/\mathfrak{i}$ has a unique Lie algebra structure such that the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ is a continuous and open Lie algebra homomorphism.*

Proof. As π is surjective, for $[a],[b] \in \mathfrak{g}/\mathfrak{i}$ we can always find $a, b \in \mathfrak{g}$ such that $\pi(a) = [a], \pi(b) = [b]$. If we want π to be a Lie algebra homomorphism, we need $[[a],[b]] =$

$[\pi(a), \pi(b)] = \pi([a,b])$. We will show that this is indeed a Lie bracket on $\mathfrak{g}/\mathfrak{i}$ and in that case it is clearly unique. First, note that it is well defined as for $x, y \in \mathfrak{i}$, $[[a+x],[b+y]] = [\pi(a+x), \pi(b+y)] = [\pi(a), \pi(b)] = [[a],[b]]$. Next, linearity and antisymmetry of the bracket follow immediately from the linearity of π. Lastly, the Jacobi identity also follows:

$$\Big[[[a],[b]],[c]\Big] + \Big[[[b],[c]],[a]\Big] + \Big[[[c],[a]],[b]\Big] = \pi\left([[a,b],c] + [[b,c],a] + [[c,a],b]\right) = 0 \tag{2.10}$$

Thus the bracket on $\mathfrak{g}/\mathfrak{i}$ is indeed a Lie bracket and unique. With the quotient topology, π is continuous and open by definition. □

The quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ is therefore also defined with this bracket. It is Hausdorff if and only if $\mathfrak{i}$ is closed, which follows from lemma 2.6. The quotient "commutes" with the commutator, as the projection is a Lie algebra homomorphism, which gives the following corollary:

Corollary 2.17. *Suppose $\mathfrak{i}$ is an ideal of the Lie algebra $\mathfrak{g}$ and $\mathfrak{i} \subseteq [\mathfrak{g}, \mathfrak{g}]$.*

$$[\mathfrak{g}, \mathfrak{g}]/\mathfrak{i} = [\mathfrak{g}/\mathfrak{i}, \mathfrak{g}/\mathfrak{i}] \tag{2.11}$$

Proof. Let $[a] \in [\mathfrak{g}, \mathfrak{g}]/\mathfrak{i}$. As the projection $\pi : \mathfrak{g} \to i$ is surjective, there exists $a \in [\mathfrak{g}, \mathfrak{g}]$ such that $\pi(a) = [a]$ and $a = \sum_i [b_i, c_i]$ where $b_i, c_i \in \mathfrak{g}$ for all i and the sum is finite. Now as π is a Lie algebra homomorphism, $\pi(a) = \sum_i [\pi(b_i), \pi(c_i)]$, thus $a \in [\mathfrak{g}/\mathfrak{i}, \mathfrak{g}/\mathfrak{i}]$.

The reverse inclusion follows from the above argument but in reverse, again using that π is surjective. □

The complexification and quotient structures behave nicely with respect to each other; the complexification of the quotient is the quotient of the complexifications:

Lemma 2.18. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ an ideal, then*

$$(\mathfrak{g}/\mathfrak{i})_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{i}_{\mathbb{C}} \tag{2.12}$$

Proof. Let $\phi : (\mathfrak{g}/\mathfrak{i})_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{i}_{\mathbb{C}}$ be given by $\phi([a] + i[b]) = [a + ib]$, which is clearly well-defined and linear. Now if $\phi([a] + i[b]) = [0]$, $a + ib \in \mathfrak{i}_{\mathbb{C}}$, meaning that $[a] = [b] = 0$, so ϕ is injective and surjectivity is trivial. Lastly, ϕ also preserves the bracket:

$$\begin{gathered}\phi(\big[[a] + i[b], [c] + i[d]\big]) = \phi\Big(\big[[a],[c]\big] - \big[[b],[d]\big] + i\big[[b],[c]\big] + i\big[[a],[d]\big]\Big) = \\ \phi\Big(\big[[a,c]\big] - \big[[b,d]\big] + i\big[[b,c]\big] + i\big[[a,d]\big]\Big) = \big[[a,c] - [b,d] + i[b,c] + i[a,d]\big] = \big[[a+bi, c+di]\big] = \\ \big[[a+bi],[c+di]\big] = \big[\phi([a]+i[b]), \phi([c]+i[d])\big]\end{gathered} \tag{2.13}$$

□

We will also need to know when the sequence of locally convex Lie algebras

$$0 \to \mathfrak{i} \xrightarrow{\iota_{\mathfrak{i}}} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{i} \to 0 \tag{2.14}$$

with $\mathfrak{i}$ an ideal, *splits*, which means that there exists an $s : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}$ such that it is a left-inverse of π. For vector spaces, such a s always exists, namely $s([x]) = x - \tilde{\pi}(x)$ where $\tilde{\pi}$ is the linear projection onto $\mathfrak{i}$. For topological vector spaces this already becomes a bit more difficult, as $\tilde{\pi}$ need not be continuous. The following lemma will show when the sequence splits for locally convex Lie algebras.

Lemma 2.19. *Suppose we have a short exact sequence of locally convex Lie algebras, where* $\mathfrak{i}$ *is an ideal :*

$$0 \to \mathfrak{i} \xrightarrow{\iota_{\mathfrak{i}}} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{i} \to 0 \tag{2.15}$$

Further assume that there exists a linear projection $\tilde{\pi} : \mathfrak{g} \to \mathfrak{i}$ *that is continuous. Then this sequence splits if* $\mathfrak{g}/\mathfrak{i}$ *is isomorphic to a subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$.

Proof. Suppose that $\mathfrak{g}/\mathfrak{i}$ is isomorphic to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and suppose $\tilde{\pi} : \mathfrak{g} \to \mathfrak{i}$ is the continuous linear projection. Then as topological vector spaces $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{g}/\mathfrak{i} \cong \mathfrak{i} \oplus \mathfrak{h}$. For $x, y \in \mathfrak{g}$, we can write $x = \tilde{\pi}(x) + (x - \tilde{\pi}(x))$ with $(x - \tilde{\pi}(x)) \in \mathfrak{h}$. Now define $s : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}$ by $s([x]) = x - \tilde{\pi}(x)$. This is well defined, as if $a \in \mathfrak{i}$, $s([a]) = a - \tilde{\pi}(a) = 0$. Furthermore s is linear and we will show that it preserves the bracket:

$$\begin{aligned} &s\big(\big[[x],[y]\big]\big) = s\big(\big[[x,y]\big]\big) = [x,y] - \tilde{\pi}([x,y]) = \\ &[x,y] - \tilde{\pi}\big(\big[\tilde{\pi}(x) + (x-\tilde{\pi}(x)), \tilde{\pi}(y) + (y - \tilde{\pi}(y))\big]\big) = \\ &[x,y] - \big[\tilde{\pi}(x), \tilde{\pi}(y)\big] - \big[x - \tilde{\pi}(x), \tilde{\pi}(y)\big] - \big[\tilde{\pi}(x), y - \tilde{\pi}(y)\big] \end{aligned} \tag{2.16}$$

where the last equality follows as $\big[x - \tilde{\pi}(x), y - \tilde{\pi}(y)\big] \in \mathfrak{h}$ as it is a subalgebra. Continuing the calculation:

$$\begin{aligned} &[x,y] - \big[\tilde{\pi}(x), \tilde{\pi}(y)\big] - \big[x - \tilde{\pi}(x), \tilde{\pi}(y)\big] - \big[\tilde{\pi}(x), y - \tilde{\pi}(y)\big] = \\ &[x,y] + \big[\tilde{\pi}(x), \tilde{\pi}(y)\big] - \big[x, \tilde{\pi}(y)\big] - \big[\tilde{\pi}(x), y\big] = [x - \tilde{\pi}(x), y - \tilde{\pi}(y)] = \big[s([x]), s([y])\big] \end{aligned} \tag{2.17}$$

Thus s is a continuous Lie algebra homomorphism. Lastly $\pi \circ s = 1$ is immediate. □

The lifting or splitting operator s is defined to be the operator s in the above lemma. If $\mathfrak{g}/\mathfrak{i}$ is not isomorphic to a subalgebra of $\mathfrak{g}$, a lifting operator is any linear splitting, which always exists (although it need not be continuous).

We further define the notion of the unitary Lie algebra, which is the Lie algebra of unitary operators. This will be used later to define the unitary representations.

Definition 2.20 (Unitary Lie algebra [JN25])**.** Let $\mathcal{D}$ be a complex vector space with inner product $\langle \cdot, \cdot \rangle$. The space of linear operators from $\mathcal{D} \to \mathcal{D}$ is denoted by $\mathcal{L}(\mathcal{D})$. We then define the space of linear operators with adjoint:

$$\mathcal{L}^{\dagger}(\mathcal{D}) = \Big\{ X \in \mathcal{L}(\mathcal{D}) | \exists X^{\dagger} \in \mathcal{L}(\mathcal{D}) : \langle X^{\dagger}\psi, \eta \rangle = \langle \psi, X\eta \rangle \Big\} \tag{2.18}$$

We can now define the unitary Lie algebra:

$$\mathfrak{u}(\mathcal{D}) = \left\{ X \in \mathcal{L}^\dagger(\mathcal{D}) | X^\dagger + X = 0 \right\} \tag{2.19}$$

where the bracket is given by $[X, Y] = XY - YX$ for $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$

That the unitary Lie algebra is indeed a Lie algebra follows from $\mathcal{L}^\dagger(\mathcal{D})$ being an associative subalgebra of $\mathcal{L}(\mathcal{D})$. Indeed, $X^\dagger + Y^\dagger = (X+Y)^\dagger, (XY)^\dagger = Y^\dagger X^\dagger$ and $(cX)^\dagger = \bar{c}X^\dagger$ for $X, Y \in \mathcal{L}(\mathcal{D}), c \in \mathbb{C}$. This means that $\mathcal{L}^\dagger(\mathcal{D})$ is closed under addition, multiplication and scalar multiplication and is thus a subalgebra. Furthermore, as additionally $(X^\dagger)^\dagger = X$, the map $X \to X^\dagger$ is an involution.

Chapter 3

Representation Theory

In this chapter we will give a basic overview of the representation theory that is needed in the rest of this thesis. Again, the main research question is about when a symmetry is trivial in every projective unitary positive energy representation. To make sense of that question and to answer it, we need to discuss representation theory. After discussing what projective unitary positive energy representations are, we show how the second cohomology group of a Lie algebra is connected with its projective unitary positive energy representations in section 3.3. In fact, the trivial symmetries from the Wigner perspective are largely determined by the Lie algebra cohomology, and we thus cover some more Lie algebra cohomology in section 3.4 and finish the chapter with the important example of the Witt algebra.

3.1 Representations

The idea of representation theory is to describe objects, such as groups, Lie groups or Lie algebra's, as automorphisms of vector spaces. If the vector space is finite dimensional, this means that they are represented as matrices. Then one can convert problems concerning the original group or algebra into linear algebraic problems, which are typically easier to solve. We start with the definitions of a Lie group and algebra representation. For these definitions, we follow Ref. [JN24].

Definition 3.1 (Representation)**.** A *Lie group representation* of a Lie group G is a vector space $\mathbb{V}$ together with a group homomorphism π into the automorphism group of this vector space:

$$\pi : G \to \mathrm{GL}(\mathbb{V}) \tag{3.1}$$

A *Lie algebra representation* of a Lie algebra $\mathfrak{g}$ is a vector space $\mathbb{V}$ together with a Lie algebra homomorphism ρ into the endomorphism Lie algebra of this vector space:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V}) \tag{3.2}$$

As mentioned in the introduction, we want our representations to be symmetries in a quantum theory. We will adopt the viewpoint that this means that all numbers that we can measure should stay the same under a symmetry transformation. In quantum mechanics these measurable numbers are the probabilities of finding a value λ when measuring the value of an observable O. If $|\phi\rangle$ is an eigenvector of O, this probability is given by

$$\frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \tag{3.3}$$

A symmetry transformation S should thus preserve these probabilities, meaning that $|\langle S\psi|S\phi\rangle| = |\langle\psi|\phi\rangle|$ should hold for all states ψ, ϕ. By Wigner's Theorem [Wig59], all symmetries in quantum mechanics are represented either unitarily, or anti-unitarily, with the latter only occurring in disconnected groups (for a modern proof, see Ref [Wei95]). We are only interested in connected groups, as Lie algebras only give information about the identity component of a Lie group, and we will therefore only need to consider *unitary representations.*

Definition 3.2 (Unitary representation)**.** A *unitary Lie group representation* of a Lie group G is a Hilbert space $\mathcal{H}$ together with a group homomorphism π into the unitary group of the Hilbert space:

$$\pi : G \to \mathrm{U}(\mathcal{H}) \tag{3.4}$$

A *unitary Lie algebra representation* of a Lie algebra $\mathfrak{g}$ is a vector space $\mathcal{D}$ with an inner product together with a Lie algebra homomorphism ρ into the unitary algebra of $\mathcal{D}$:

$$\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{D}) \tag{3.5}$$

Remark 3.3. One might expect that a Lie algebra representation would again be a representation onto a Hilbert space. By looking at the following example however, we can see that this is far too strict of a definition: Let $\pi : \mathrm{U}(1) \to \mathrm{U}(L^2(\mathbb{S}^1))$ be given by $[\pi(e^{i\phi})\psi](\theta) \to \psi(\theta - \phi)$. This gives us a derived representation on the Lie algebra $\mathfrak{u}(1)$, as in def. 3.6:

$$[\mathrm{d}\pi(x)\psi](\theta) = \frac{d}{dt}\bigg|_{t=0} [\pi(\exp itx)\psi](\theta) = \frac{d}{dt}\bigg|_{t=0} \psi(\theta - tx) = -x\partial_\theta\psi \tag{3.6}$$

Now this would not be well defined if we would have required $\mathrm{d}\pi(x)\psi \in L^2(\mathbb{S}^1)$ again. Furthermore, as neither the differentiable or the smooth functions form a complete vector space, we would rather view them as dense subsets of $L^2(\mathbb{S}^1)$ and take this dense subset as our representations space.

Remark 3.4. This definition of a unitary Lie algebra representation only makes sense for real Lie algebra's. If we were to have a complex Lie algebra $\mathfrak{g}$, a representation π and an element in the Lie algebra $X \in \mathfrak{g}$, then $\pi(iX)^\dagger = -i\pi(X)^\dagger$, so if $\pi(X)$ is skew-Hermitian, $\pi(iX)$ is not. Instead, for complex Lie algebra's it is common to consider Lie algebra's with an involution $\dagger$ and define a unitary representation to be a *-representation, i.e. a representation such that $\pi(X^\dagger) = \pi(X)^\dagger$. This then reduces to a unitary representation in the sense of def. 3.2 on the real part of the Lie algebra.

As Lie groups have a smooth structure, we also want representations to preserve some of this structure. A first guess might be to want these representations to be bounded, i.e. continuous with respect to the norm topology. This is too much to ask however, as many interesting examples of representations are not bounded. For example, the representation π from remark 3.3 is not bounded in the norm topology. Indeed, it is discontinuous in the norm topology at $\phi = 0$:

$$\begin{aligned}\lim_{\phi\to 0} \parallel \pi(e^{i\phi}) - I \parallel &= \lim_{\phi\to 0} \sup_{\psi\in L^2(\mathbb{S}^1)} \parallel \pi(e^{i\phi})\psi - \psi \parallel_{L^2} > \lim_{\phi\to 0} \sup_{n\in\mathbb{Z}} \parallel e^{in(\theta-\phi)} - e^{in\theta} \parallel_{L^2} \\ &= \lim_{\phi\to 0} \sup_{n\in\mathbb{Z}} |e^{-in\phi} - 1| \parallel e^{in\theta} \parallel_{L^2} > 2\end{aligned} \tag{3.7}$$

where the last inequality follows as for ϕ small, we can always take n large enough such that $e^{-in\phi}$ is approximately -1. Thus π is not norm continuous, whereas it is an interesting representation that we want to include in our definition.

We therefore instead use the following definitions from Ref. [JN24], which define a representation to be continuous if all the orbit maps are continuous:

Definition 3.5 (Continuous and smooth representations).

- A unitary representation $(\pi, \mathcal{H})$ of a Lie group G is called *continuous* if for all $\psi \in \mathcal{H}$ the map $\pi_\psi : G \to \mathcal{H} : g \to \pi(g)\psi$ is continuous.
- A unitary representation $(\rho, \mathcal{D})$ of a Lie algebra $\mathfrak{g}$ is called *continuous* if for all $\psi \in \mathcal{D}$ the map $\rho_\xi : \mathfrak{g} \to \mathcal{D} : g \to \rho(g)\psi$ is continuous.
- A vector $\psi \in \mathcal{H}$ is called a *smooth vector* if the map $\pi_\psi : G \to \mathcal{H} : g \to \pi(g)\psi$ is smooth, and we denote the subspace of smooth vectors by $\mathcal{H}^\infty \subseteq \mathcal{H}$. The representation π is called *smooth* if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

Any smooth unitary representation of a Lie group induces a representation of the Lie algebra. Furthermore, for connected Lie groups, this induced, or derived representation determines the Lie group representation, and we can thus use the Lie algebra representation to solve problems regarding the Lie group.

Definition 3.6 (Derived representation). Let π be a smooth unitary representation of the Lie group G with Hilbert space $\mathcal{H}$. The *derived representation* $\mathrm{d}\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of the Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{d}\pi(\xi)\psi = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp t\xi)\psi \tag{3.8}$$

3.2 Projective representations

In quantum mechanics, states are not elements of a Hilbert space, but instead elements of the projective Hilbert space $\mathbb{P}(\mathcal{H})$, which is the quotient of the Hilbert space by the equivalence relation $\psi \sim \phi$ if there exists $\lambda \in \mathbb{C}$ such that $\psi = \lambda\phi$, for $\psi, \phi \in \mathcal{H} \setminus \{0\}$. Intuitively, this means that two quantum states are considered the same if they differ only by a complex phase and only norm one quantum states are allowed. This is based on equation 3.3, as this formula shows that the difference between two states differing only in norm or complex phase cannot be measured. As we cannot measure the difference between those states, we consider them to be equivalent. Elements of the projective Hilbert space are called rays and we will denote them as $[\xi]$ for $\xi \in \mathcal{H}$. We further define the projective unitary group by $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/\mathbb{T}\mathbf{1}$, i.e. in $\mathrm{PU}(\mathcal{H})$ two unitaries are considered the same if they differ by a complex phase.

We can now give the definition of a projective unitary representation:

Definition 3.7 (Projective unitary representation)**.** A *projective unitary representation* of a Lie group G is a projective Hilbert space $\mathbb{P}(\mathcal{H})$ together with a group homomorphism $\overline{\pi}$ into the projective unitary group of the Hilbert space:

$$\overline{\pi} : G \to \mathrm{PU}(\mathcal{H}) \tag{3.9}$$

A projective unitary representation of a Lie algebra $\mathfrak{g}$ is a projective inner product space $\mathbb{P}(\mathcal{D})$ together with a Lie algebra homomorphism $\overline{d\pi}$ into the projective unitary algebra of the inner product space

$$\overline{d\pi} : \mathfrak{g} \to \mathfrak{u}(\mathcal{D})/\mathfrak{u}(1) \tag{3.10}$$

where the projective unitary algebra $\mathfrak{u}(\mathcal{D})/\mathfrak{u}(1)$ is the quotient of $\mathfrak{u}(\mathcal{D})$ by the equivalence relation $a + i\lambda \sim a$.

We will denote projective representations with a bar.

The projective unitary group is the natural choice here, over the normal unitary group, for the following reason. Suppose we have some unitary $U \in \mathrm{U}(\mathcal{H})$ and a ray $[\psi] \in \mathbb{P}(\mathcal{H})$. Then the following two states are the same:

$$[U\psi] = [e^{i\theta} U\psi], \tag{3.11}$$

for all $\theta \in [0, 2\pi)$, as complex phases are irrelevant for quantum states. This is true for every ray $[\psi]$, and we would like two operators to be equal if they assign the same states to a state. Therefore $U \sim \mathbb{T}U$ and the projective unitary group is the correct choice.

This can also be thought of in a slightly different way; we don't need a unitary representation $\overline{\pi}$ to be an exact homomorphism into the unitary group. Instead, it only needs to be a homomorphism up to a phase:

$$\overline{\pi}(g)\overline{\pi}(h) = e^{i\boldsymbol{C}(g,h)}\overline{\pi}(gh) \tag{3.12}$$

for some function $\boldsymbol{C} : G \times G \to \mathbb{R}/2\pi\mathbb{Z}$. This function $\boldsymbol{C}$ is constrained by the need for $\overline{\pi}$ to be a group homomorphism, as group operations are associative:

$$\begin{gathered} \overline{\pi}(f)\left[\overline{\pi}(g)\pi(h)\right] = \left[\overline{\pi}(f)\overline{\pi}(g)\right]\overline{\pi}(h) \\ e^{i\boldsymbol{C}(f,gh)}e^{i\boldsymbol{C}(g,h)} = e^{i\boldsymbol{C}(f,g)}e^{i\boldsymbol{C}(fg,h)} \end{gathered} \tag{3.13}$$

Hence we need $\boldsymbol{C}$ to fulfill the following property, for all $f, g, h \in G$:

$$\boldsymbol{C}(f, gh) + \boldsymbol{C}(g, h) = \boldsymbol{C}(f, g) + \boldsymbol{C}(fg, h) \tag{3.14}$$

This property is known as the (group) 2-cocycle condition and a function $\boldsymbol{C} : G \times G \to \mathbb{R}/2\pi\mathbb{Z}$ satisfying this property is called a (group) 2-cocycle. With such a function, we can define a central extension of G:

Definition 3.8 (Central group extension)**.** A *central extension* of G by $\mathbb{T}$ is a short exact sequence:

$$1 \to \mathbb{T} \to G^\sharp \to G \to 1 \tag{3.15}$$

where the image of $\mathbb{T}$ is central in $G^\sharp$.

Intuitively, this means that $G^\sharp$ consists of pairs:

$$G^\sharp = G \times \mathbb{T} \tag{3.16}$$

such that $(1, \mathbb{T})$ is in the center of $G^\sharp$:

$$(1, \lambda)(f, \mu) = (f, \mu)(1, \lambda) \tag{3.17}$$

Any group 2-cocycle $\boldsymbol{C}$ can be used to define a central extension of G, as in this case the multiplication can be defined as:

$$(f, \lambda)(g, \mu) = (fg, \lambda + \mu + \boldsymbol{C}(f, g)) \tag{3.18}$$

This gives a well defined group multiplication because of the 2-cocycle condition.

Thus any 2-cocycle defines a group extension. The reciprocal also holds, although the 2-cocycle need not be smooth outside of a neighborhood of $(e, e) \in G \times G$, see e.g. [JN19].

We can then consider the exact (non projective) representations of $G^\sharp$. If we assume that these will be of the form:

$$\pi^\sharp((f, \lambda)) = e^{i\lambda}\overline{\pi}(f) \tag{3.19}$$

where $\overline{\pi}$ is a projective unitary representation with cocycle $\boldsymbol{C}$, as in eq. 3.14, then the multiplication:

$$\pi^\sharp((f, \lambda))\pi^\sharp((g, \mu)) = e^{i(\lambda+\mu)}\overline{\pi}(f)\overline{\pi}(g) = e^{i(\lambda+\mu+\boldsymbol{C}(f,g))}\overline{\pi}(f, g) = \pi^\sharp((f, \lambda)(g, \mu)) \tag{3.20}$$

is well defined.

Hence projective unitary representations of G give rise to exact representations of $G^\sharp$. Furthermore, any Lie group extension also gives rise to a Lie algebra extension:

Definition 3.9 (Central Lie algebra extension)**.** A central Lie algebra extension of $\mathfrak{g}$ by $\mathbb{R}$ is a short exact sequence:

$$0 \to \mathbb{R} \to \mathfrak{g}^\sharp \to \mathfrak{g} \to 0 \tag{3.21}$$

of Lie algebras such that the image of $\mathbb{R}$ is central in $\mathfrak{g}^\sharp$.

These are easier to classify and we will therefore mainly be concerned with these. Equivalently to the group case, every central Lie algebra extension gives rise to a Lie algebra *2-cocycle*, which is a continuous, bilinear, alternating map $\omega : \mathfrak{g} \to \mathbb{R}$ such that

$$0 = \omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_2, \xi_3], \xi_1) + \omega([\xi_3, \xi_1], \xi_2) \tag{3.22}$$

The projective unitary Lie algebra representations $\overline{d\pi}$ are then the representations such that

$$\overline{d\pi}([a, b]) = [\overline{d\pi(a)}, \overline{d\pi(b)}] + i\omega(a, b) \tag{3.23}$$

for a 2-cocycle ω. We will come back to these cocycles in section 3.4, but for now we conclude with the following summary: every projective unitary representation $\overline{d\pi}$ of $\mathfrak{g}$ has an associated exact unitary derived representation $d\pi$ of a central extension of $\mathfrak{g}$. This central extension has a corresponding 2-cocycle. Thus every projective unitary representation $\overline{d\pi}$ has an associated 2-cocycle ω. This 2-cocycle will be of great importance in determining the kernel of the representation $\overline{d\pi}$ as we will see in section 3.3

We first again define what it means for a projective unitary representation to be smooth. Note that the definition from before does not apply anymore, as the projective spaces are not vector spaces. We still follow Ref. [JN24].

Definition 3.10 (Continuous and smooth projective representations).

- A projective unitary representation of a Lie group G $(\overline{\pi}, \mathcal{H})$ is called *continuous* if for all $[\psi] \in \mathbb{P}(\mathcal{H})$ the map $\overline{\pi}_{[\psi]} : G \to \mathbb{P}(\mathcal{H}) : g \to \overline{\pi}(g)[\psi]$ is continuous.
- A projective unitary representation of a locally convex Lie algebra $\mathfrak{g}$ $(\overline{\pi}, \mathcal{D})$ is called *continuous* if the lift $\pi^\sharp : \mathfrak{g}^\sharp \to \mathfrak{u}(\mathcal{D})$ is continuous.
- A ray $[\psi] \in \mathbb{P}(\mathcal{H})$ is called a *smooth ray* if the map $\overline{\pi}_{[\psi]} : G \to \mathbb{P}(\mathcal{H}) : g \to \overline{\pi}(g)[\psi]$ is smooth, and we denote the subspace of smooth rays by $\mathbb{P}(\mathcal{H})^\infty \subseteq \mathbb{P}(\mathcal{H})$. The representation $\overline{\pi}$ is called *smooth* if $\mathbb{P}(\mathcal{H}^\infty)$ is dense in $\mathbb{P}(\mathcal{H})$.

3.3 Positive energy representations

In this section we will give the main proposition that we will use to determine if a symmetry is trivial in every positive energy representation. It will show that being in the kernel of a projective unitary positive energy representation is intimately connected with the corresponding cocycle being zero.

A second characteristic of quantum mechanics, next to states being elements of the projective Hilbert space, is that any quantum mechanical system is characterized by its Hamiltonian H, which is a (typically unbounded) operator with spectrum bounded from below, meaning that for some real number E_0:

$$\inf_{\psi \in \mathcal{H}} \langle \psi, H\psi \rangle \geq E_0 \tag{3.24}$$

Note that we can assume $E_0 = 0$ by redefining $H \to H - E_0$. We therefore expect a Lie algebra $\mathfrak{g}$ of a quantum symmetry group to also have such an element and with this intuition we define what it means for a representation to be of *positive energy*:

Definition 3.11 (Positive energy representations (Def 6 in Ref.[JN25]))**.** Let $\mathcal{D}$ be a complex inner product space with Hilbert space completion $\mathcal{H}$. Furthermore, let $\mathfrak{g}$ be a locally convex Lie algebra.

- A continuous unitary representation $\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{D})$ is called a positive energy representation at $\xi \in \mathfrak{g}$ if
$$\inf_{\psi \in \mathcal{D}} \langle \psi, -i\rho(\xi)\psi \rangle \geq 0 \tag{3.25}$$
- A continuous projective unitary representation $\overline{\rho} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D})$ is of positive energy at $\xi \in \mathfrak{g}$ if it has a lift $\pi : \mathfrak{g}^\sharp \to \mathfrak{u}(\mathcal{D})$ that is of positive energy at $\xi^\sharp$ covering ξ.
- A smooth unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ is of positive energy at $\xi \in \mathfrak{g}$ if the derived representation $d\pi$ of $\mathfrak{g}$ on $\mathcal{H}^\infty$ is so.

With this definition, we can prove the following proposition from Ref. [Nie23]:

Proposition 3.12 (Proposition 4.4 in Ref. [Nie23])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\overline{\rho}$ is a continuous projective unitary G-representation with derived representation $\overline{d\rho}$. Suppose the representation space of $\overline{d\rho}$ is the inner product space $\mathcal{D}$ and let the lift of $\overline{d\rho}$ be $d\rho : \mathfrak{g}^\sharp \to \mathfrak{u}(\mathcal{D})$ for some continuous central $\mathbb{R}$-extension $\mathfrak{g}^\sharp$ of $\mathfrak{g}$. Let ω represent the corresponding 2-cocycle.Let $\overline{d\rho}$ be of positive energy at $\xi \in \mathfrak{g}$. Suppose that $\eta \in \mathfrak{g}$ satisfies $[[\xi, \eta], \eta] = 0$. Then $\omega([\xi, \eta], \eta) \geq 0$ and*

$$\omega([\xi, \eta], \eta) = 0 \Leftrightarrow \overline{d\rho}([\xi, \eta]) = 0 \tag{3.26}$$

Proof. First assume that $\omega([\xi, \eta], \eta) = 0$. We define $H = -id\rho(\xi^\sharp)$. As $\overline{d\rho}$ is of positive energy at ξ, equation 3.25 holds in particular for the state $\exp(td\rho(\eta^\sharp))\psi$:

$$0 \leq \langle \exp(td\rho(\eta^\sharp))\psi, H \exp(td\rho(\eta^\sharp))\psi \rangle = \langle \psi, \exp(-td\rho(\eta^\sharp))H \exp(td\rho(\eta^\sharp))\psi \rangle \tag{3.27}$$

Here we used that $d\rho(\eta^\sharp)$ is skew-Hermitian. Now note that the right-hand side is given by $\mathrm{Ad}(\exp(-td\rho(\eta^\sharp)))H\psi$ and that $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$:

$$0 \leq \langle \psi, e^{\mathrm{ad}(-td\rho(\eta^\sharp))} H\psi \rangle \tag{3.28}$$

We can calculate the exponential term:

$$\begin{aligned} &e^{\mathrm{ad}(-td\rho(\eta^\sharp))} H = -id\rho(e^{-t\,\mathrm{ad}(\overset{\sharp}{\eta})}\xi^\sharp) = \\ &H - id\rho(-t[\eta^\sharp, \xi^\sharp] + \frac{t^2}{2}[\eta^\sharp, [\eta^\sharp, \xi^\sharp]] + \dots) \end{aligned} \tag{3.29}$$

The bracket between the centrally extended elements is given by equation 3.43:

$$[\eta^\sharp, \xi^\sharp] = ([\eta, \xi], \omega(\eta, \xi)) \tag{3.30}$$

$$[\eta^\sharp, [\eta^\sharp, \xi^\sharp]] = (0, \omega(\eta, [\eta, \xi])) \tag{3.31}$$

As the second bracket is zero up to a constant times the central element, all of the higher order iterated brackets will be zero as well. Hence the $\ldots$ in equation 3.29 are zero and the exponential is given by:

$$e^{\mathrm{ad}(-td\rho(\eta^\sharp))}H = H + itd\rho[([\eta, \xi], 0)] + itd\rho\left[(0, \omega(\eta, \xi)))\right] - \frac{t^2}{2}id\rho[(0, \omega(\eta, [\eta, \xi]))] \tag{3.32}$$

Now the representation of a central element should again be central. As the only central elements in $\mathfrak{u}(\mathcal{H})$ are constants, $id\rho\left[(0, \omega(\eta, \xi)))\right] = C\omega(\eta, \xi)$ for some constant C. Plugging the above equation into equation 3.28 we find

$$0 \leq \langle H\rangle_\psi + it\langle d\rho[([\eta, \xi], 0)]\rangle_\psi + tC\omega(\eta, \xi) + \frac{t^2}{2}C\omega(\eta, [\eta, \xi])) \tag{3.33}$$

This quadratic inequality holding for all $t \in \mathbb{R}$ is equivalent to:

$$\begin{aligned} C\omega(\eta, [\eta, \xi])) &\geq 0 \\ \langle H\rangle_\psi &\geq 0 \\ \left(i\langle d\rho[([\eta, \xi], 0)]\rangle_\psi + C\omega(\eta, \xi)\right)^2 &\leq 2\langle H\rangle_\psi C\omega(\eta, [\eta, \xi])) \end{aligned} \tag{3.34}$$

The result is immediate as $d\rho[([\eta, \xi], 0)] = \bar{d}\rho([\eta, \xi])$

Now for the conserve, note that

$$i\omega([\xi, \eta], \eta) = [d\rho([\xi, \eta]), d\rho(\eta)] - d\rho([[\xi, \eta], \eta]) = [d\rho([\xi, \eta]), d\rho(\eta)] \tag{3.35}$$

as $[[\xi, \eta], \eta] = 0$. Furthermore, by assumption $\overline{d\rho}([\xi, \eta]) = 0$, thus $d\rho([\xi, \eta])$ is scalar and $[d\rho([\xi, \eta]), d\rho(\eta)] = 0$. □

3.4 Lie algebra cohomology

In the previous section we saw that we can show that a symmetry is trivial in every projective unitary positive energy representation by a property of the corresponding cocycle. We therefore want to understand these cocycles better, which we will do by considering them from the viewpoint of cohomology. We will only cover what is necessary for this thesis and refer for a more complete reference to Ref. [Fuk86].

Definition 3.13 (Cochain)**.** Let $\mathfrak{g}$ be a locally convex Lie algebra, k a nonnegative integer and E a $\mathfrak{g}$-module. Then an E-valued *k-cochain* c on $\mathfrak{g}$ is a continuous, multilinear, alternating map:

$$c : \mathfrak{g} \times \cdots \times \mathfrak{g} \to E : (\xi_1, \ldots, \xi_k) \to c(\xi_1, \ldots, \xi_k) \tag{3.36}$$

We call the space of k-cochains on $\mathfrak{g}$ with values in E $\mathcal{C}^k_{ct}(\mathfrak{g}, E)$ and define the cochain complex $\mathcal{C}^\bullet_{ct}(\mathfrak{g}, E) = \bigoplus_{k=0}^\infty \mathcal{C}^k_{ct}(\mathfrak{g}, E)$. As the cochains are alternating, if $\mathfrak{g}$ is finite dimensional then every k-cochain with $k > \dim(\mathfrak{g})$ is zero. In addition, if we do not mention the module E, it is assumed to be $\mathbb{R}$ with trivial action. These will also be our main concern, and we will specify explicitly if we are considering a cochain with values in a module E that is not $\mathbb{R}$.

Definition 3.14 (Chevalley-Eilenberg differential [Obl17]). The Chevalley-Eilenberg differential $d_k : \mathcal{C}^k_{ct}(\mathfrak{g}, E) \to \mathcal{C}^{k+1}_{ct}(\mathfrak{g}, E)$ is given by:

$$\begin{aligned} d_k\omega(\xi_1 \ldots, \xi_{k+1}) =& \sum_{j=1}^{k+1} (-1)^{j+1} \xi_j \cdot \omega(\xi_1, \ldots, \hat{\xi_j}, \ldots, \xi_{k+1}) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi_i}, \ldots, \hat{\xi_j}, \ldots \xi_{k+1}) \end{aligned} \tag{3.37}$$

Here, the hat means that this argument is omitted. The first three differentials with trivial action will be most important to us:

$$d_0\omega_0(\xi_1) = 0 \tag{3.38}$$

$$d_1\omega_1(\xi_1, \xi_2) = -\omega([\xi_1, \xi_2]) \tag{3.39}$$

$$d_2\omega_2(\xi_1, \xi_2, \xi_3) = -\omega([\xi_1, \xi_2], \xi_3) - \omega([\xi_3, \xi_1], \xi_2) - \omega([\xi_2, \xi_3], \xi_1) \tag{3.40}$$

Clearly, $d_1 \circ d_0 = 0, d_2 \circ d_1 = 0$. In fact, it can be shown that $d_k \circ d_{k-1} = 0$ [Fuk86]. Therefore, any k cochain ω_k with the property that there exists a $(k-1)$-cochain ω_{k-1} such that $\omega_k = d_{k-1}\omega_{k-1}$, will also fulfill the property $d_k\omega_k = 0$. The converse need not be true of course. We will call k-cochains ω_k with $\omega_k = d_{k-1}\omega_{k-1}$ k-coboundaries and those with $d_k\omega_k = 0$ k-cocycles. The space of k-coboundaries is denoted by $B^k_{ct}(\mathfrak{g}, E)$ and the space of k-cocycles by $Z^k_{ct}(\mathfrak{g}, E)$. In this terminology, any k-coboundary is a k-cocycle. We define the k^{th} *cohomology space* $H^k_{ct}(\mathfrak{g})$ as the quotient of the spaces of cocycles and coboundaries:

Definition 3.15 (Lie algebra cohomology).

$$H^k_{ct}(\mathfrak{g}, E) = \mathrm{Ker}(d_k)/\mathrm{Im}(d_{k-1}) = Z^k_{ct}(\mathfrak{g}, E)/B^k_{ct}(\mathfrak{g}, E) \tag{3.41}$$

The first two cohomology groups have important and easy to understand interpretations. First, note that if the Lie algebra is perfect, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then any 1-cocycle is identically zero, as $\omega([\xi_1, \xi_2]) = 0$ for all ξ_1, ξ_2. If $\mathfrak{g}$ is not perfect, $H^1_{ct}(\mathfrak{g})$ classifies "how much" in the following sense:

$$H^1_{ct}(\mathfrak{g}) \cong \mathfrak{g}/\overline{[\mathfrak{g}, \mathfrak{g}]} \tag{3.42}$$

This follows from the fact that any element in $\mathfrak{g}/\overline{[\mathfrak{g},\mathfrak{g}]}$ uniquely defines a 1-cochain that equals 1 for that element, and is zero everywhere other than the line spanned by that element.

The second cohomology group classifies the central extensions of $\mathfrak{g}$. Any 2-cocycle ω gives rise to a central extension with the bracket:

$$[(\xi,\lambda),(\zeta,\mu)] = ([\xi,\zeta],\omega(\xi,\zeta)) \tag{3.43}$$

Now note that the 2-cocycle property, eq. 3.40 being equal to 0, is essentially a Jacobi identity, and therefore this new bracket also satisfies the Jacobi identity. The converse, that every central extension gives rise to a 2-cocycle, is also true, see Ref. [JN19]. Lastly, if two 2-cocycles differ only by a coboundary, they give identical extensions. To show this, we first have to define when two central extensions are isomorphic:

Definition 3.16 (Extension isomorphism)**.** Suppose we have two central extensions:

$$0 \to \mathbb{R} \xrightarrow{i} \mathfrak{g}^\sharp \xrightarrow{s} \mathfrak{g} \to 0 \tag{3.44}$$

$$0 \to \mathbb{R} \xrightarrow{i'} \mathfrak{g}^{\sharp'} \xrightarrow{s'} \mathfrak{g} \to 0 \tag{3.45}$$

If there exists a Lie algebra isomorphism $\phi : \mathfrak{g}^\sharp \to \mathfrak{g}^{\sharp'}$ such that $\phi \circ i = i'$, $s' \circ \phi = s$, then the extensions $\mathfrak{g}^\sharp, \mathfrak{g}^{\sharp'}$ are isomorphic.

Suppose ψ is a 1-cochain and ω is a 2-cocycle. Then the central extensions corresponding to ω and $\omega + d_1\psi$ are isomorphic as extensions, with isomorphism $f : (\xi,\lambda) \to (\xi, \lambda + \psi(\xi))$. Thus the second cohomology group indeed classifies the central extensions.

We also want to define what a homomorphism between two cochain complexes should be. This should be a map that sends k-cochains to k-cochains, coboundaries to coboundaries and cocycles to cocycles. We will show that the following definition suffices:

Definition 3.17 (Cochain homomorphism)**.** Let $\mathfrak{g},\mathfrak{h}$ be Lie algebras and E, M be $\mathfrak{g},\mathfrak{h}$-modules respectively. A map $F : \mathcal{C}^\bullet_{ct}(\mathfrak{g},E) \to \mathcal{C}^\bullet_{ct}(\mathfrak{h},M)$ is a cochain homomorphism if $F_k : \mathcal{C}^k_{ct}(\mathfrak{g},E) \to \mathcal{C}^k_{ct}(\mathfrak{h},M)$ is a well defined linear map for each k and if $d \circ F = F \circ d$.

Now F induces a map between the cohomology spaces, $F : H^\bullet_{ct}(\mathfrak{g},E) \to H^\bullet_{ct}(\mathfrak{h},M)$ as for $[\omega] \in H^\bullet_{ct}(\mathfrak{g},E)$, we have that $[dF\omega] = [Fd\omega] = [0]$ and if $\omega = d\phi$, $[F\omega] = [dF\phi] = 0$. In other words, coboundaries are mapped to coboundaries and cocycles to cocycles, meaning that $F : H^\bullet_{ct}(\mathfrak{g},E) \to H^\bullet_{ct}(\mathfrak{h},M)$ is well defined.

We will also define a notion that describes when two cochain homomorphisms are 'the same'. This is a cochain homotopy.

Definition 3.18 (Cochain homotopy)**.** Let $F, G : \mathcal{C}^\bullet_{ct}(\mathfrak{g},E) \to \mathcal{C}^\bullet_{ct}(\mathfrak{h},M)$ be two cochain homomorphisms. A map $L : \mathcal{C}^\bullet_{ct}(\mathfrak{g},E) \to \mathcal{C}^{\bullet-1}_{ct}(\mathfrak{h},M)$ is a cochain homotopy between F and G if $F - G = dL + Ld$

Cochain homomorphisms that are related by homotopy act the same on the cohomology spaces. Indeed, if F, G are homotopy related through L and $[\omega] \in H^k_{ct}(\mathfrak{g},E)$, $[(F-G)\omega] =$

$[dL\omega + Ld\omega] = [0]$. Importantly, the special case $G = 0$ shows that $dL + Ld$ itself is homotopy connected to 0.

A subset of a complex is again a complex if it is a *subcomplex.*

Definition 3.19 (Subcomplex)**.** Let $\mathcal{C}^\bullet_{ct}(\mathfrak{g}, E)$ be a cochain complex. A complex $A^\bullet$ is called a subcomplex if $A^k \subseteq \mathcal{C}^k_{ct}(\mathfrak{g}, E)$ and $dA^k \subseteq A^{k+1}$.

The image of a cochain homomorphism is a subcomplex.

Now an important class of cochain homomorphisms are given by the pullbacks of Lie algebra homomorphisms:

Definition 3.20 (Pullback)**.** Suppose $\phi : \mathfrak{g} \to \mathfrak{h}$ is a continuous Lie algebra homomorphism. Then the pullback $\phi^* : \mathcal{C}^\bullet_{ct}(\mathfrak{h}, E) \to \mathcal{C}^\bullet_{ct}(\mathfrak{g}, E)$ is defined by $\phi^* c = c \circ (\phi, \ldots, \phi)$. The action of $\mathfrak{g}$ on E is defined by $\xi \cdot x = \phi(\xi) \cdot x$ for $\xi \in \mathfrak{g}, x \in E$.

That $\phi^* c$ is again a cochain follows from ϕ being a continuous and linear map. In addition we immediately see that if $s : \mathfrak{h} \to \mathfrak{g}$ is also a Lie algebra homomorphism, then $(s\phi)^* = \phi^* s^*$. We will now show that the pullback is indeed a cochain homomorphism:

Lemma 3.21. *Suppose c is a k-cochain on $\mathfrak{h}$ with values in the $\mathfrak{h}$-module E and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a continuous Lie algebra homomorphism. Then $d_k(\phi^* c) = \phi^*(d_k c)$.*

Proof. The following computation gives the result:

$$\begin{aligned}
&\phi^*(d_k c)(\xi_1, \ldots, \xi_{k+1}) = (d_k c)(\phi(\xi_1), \ldots, \phi(\xi_{k+1})) = \\
&\sum_{j=1}^{k+1} (-1)^{j+1} \phi(\xi_j) \cdot c(\phi(\xi_0), \ldots, \phi(\hat{\xi_j}), \ldots, \phi(\xi_{k+1})) \\
&+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} c([\phi(\xi_i), \phi(\xi_j)], \phi(\xi_1), \ldots, \phi(\hat{\xi_i}), \ldots, \phi(\hat{\xi_j}), \ldots, \phi(\xi_{k+1})) = \\
&\sum_{j=1}^{k+1} (-1)^{j+1} \xi_j \cdot c(\phi(\xi_0), \ldots, \phi(\hat{\xi_j}), \ldots, \phi(\xi_{k+1})) \\
&+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} c(\phi([\xi_i, \xi_j]), \phi(\xi_1), \ldots, \phi(\hat{\xi_i}), \ldots, \phi(\hat{\xi_j}), \ldots, \phi(\xi_{k+1})) = \\
&d_k(\phi^* c)(\xi_1, \ldots, \xi_{k+1})
\end{aligned} \tag{3.46}$$

Note that we made use of ϕ preserving the Lie bracket in the second equality. □

As ϕ^* is a cochain homomorphism, ϕ^* induces a map between cohomology spaces $\phi^* : H^k_{ct}(\mathfrak{h}) \to H^k_{ct}(\mathfrak{g})$. One might now hope that a short exact sequence of Lie algebras then also induces a short exact sequence of cohomology spaces. To this end, we give the following partial result:

Proposition 3.22. *Suppose we have an exact sequence:*

$$0 \to \mathfrak{i} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0 \tag{3.47}$$

where the maps are continuous Lie algebra homomorphisms. If the sequence of Lie algebras splits, i.e. there exists a continuous Lie algebra homomorphism $s : \mathfrak{h} \to \mathfrak{g}$ *such that* $\pi \circ s = 1_{\mathfrak{g}}$*, then the following sequences are also exact:*

$$0 \to H^k_{ct}(\mathfrak{h}) \xrightarrow{\pi^*} H^k_{ct}(\mathfrak{g}) \tag{3.48}$$

$$H^k_{ct}(\mathfrak{g}) \xrightarrow{\iota^*} H^k_{ct}(\mathfrak{i}) \tag{3.49}$$

Regardless of splitting, we have $\mathrm{im}(\pi^*) \subseteq \ker(\iota^*)$*.*

Proof. Suppose π has right-inverse s. Then $\mathrm{id} = (\pi s)^* = s^*\pi^*$, so π^* has a left-inverse and is therefore injective.

Now the only thing left to prove is that $\mathrm{im}(\pi^*) \subseteq \ker(\iota^*)$. Suppose $\omega \in \mathrm{im}(\pi^*)$, then there exists $\tilde{\omega} \in H^k_{ct}(\mathfrak{h})$ such that $\omega = \pi^*\tilde{\omega}$. Now $\iota^*\omega = \iota^*\pi^*\tilde{\omega} = (\pi \circ \iota)^*\omega = 0$ as $\mathrm{im}(\iota) = \ker(\pi)$. □

If ι has a left-inverse t, ι^* is surjective. This left inverse however does not exist in general, even if the sequence of Lie algebra's splits. This is shown by the following example:

Let $\mathfrak{g}$ be a Lie algebra with basis vectors t, x, y and brackets:

$$[t, x] = x, \quad [t, y] = y, \quad [x, y] = 0 \tag{3.50}$$

Then $\mathfrak{i} = \mathrm{span}\{x, y\}$ is an abelian ideal, and the second cohomology space of $\mathfrak{i}$ is 1-dimensional with the 2-cocycle $\psi : \mathfrak{i} \times \mathfrak{i} \to \mathbb{R}$ such that $\psi(x, y) = 1$ as basis vector. Now you might expect that this can be extended to a 2-cocycle $\tilde{\psi} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. This cannot exist however, as the 2-cocycle property of $\tilde{\psi}$ gives:

$$\begin{aligned} 0 &= \tilde{\psi}([t, x], y) + \tilde{\psi}([x, y], t) + \tilde{\psi}([y, t], x) \\ 0 &= \tilde{\psi}(x, y) + \tilde{\psi}(x, y) \end{aligned} \tag{3.51}$$

This is in contradiction with $\tilde{\psi}$ being an extension of ψ.

The reverse inclusion, $\ker(\iota^*) \subseteq \mathrm{im}(\pi^*)$ also does not hold in general, which can be seen from the next counterexample. Let $\mathfrak{i} = \mathfrak{h} = \mathbb{R}$, $\mathfrak{g} = \mathbb{R}^2$, with trivial brackets. Then the sequence

$$0 \to H^2_{ct}(\mathbb{R}) \cong 0 \xrightarrow{\pi^*} H^2_{ct}(\mathbb{R}^2) \cong \mathbb{R} \xrightarrow{\iota^*} H^2_{ct}(\mathbb{R}) \cong 0 \to 0 \tag{3.52}$$

is not exact as $\ker(\iota^*) \not\subseteq \mathrm{im}(\pi^*)$.

3.4.1 Exact sequences of cohomology spaces

In the end, we want to classify the 2-cocycles, as these can be used for constructing central extensions. We will therefore restrict our attention to these for now. If we have a Lie algebra ideal, we also have a short exact sequence of Lie algebras. We can use this sequence to construct sequences of the second cohomology space.

For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the notation $H^q_{ct}(\mathfrak{g})_1$ is used for the q-cocycles in the kernel of the restriction map: $\mathrm{Res} : H^q_{ct}(\mathfrak{g}) \to H^q_{ct}(\mathfrak{h})$.

Remark 3.23. If $[\omega] \in H^q_{ct}(\mathfrak{g})_1$, then $\iota_*\omega = d\phi$ for some $(q-1)$-cochain ϕ. Assuming that there exists a continuous projection $\pi : \mathfrak{g} \to \mathfrak{h}$, we can define $\tilde{\phi} = \pi^*\phi$ and then we find that $[\omega] = [\omega - d\tilde{\phi}]$. Now $\iota_*(\omega - d\tilde{\phi}) = 0$. Thus if $\mathfrak{h}$ is complemented, $H^q_{ct}(\mathfrak{g})_1$ consists of the q-cocycles that vanish on $\mathfrak{h}$. For $q = 2$, we can always use Hahn-Banach to extend ϕ to $\mathfrak{g}$, so in this case the assumption that $\mathfrak{h}$ is complemented is unnecessary.

We now start the construction of an exact sequence that describes what happens when we inflate a 2-cocycle from the quotient $\mathfrak{g}/\mathfrak{i}$ to $\mathfrak{g}$ itself. We will assume that $\mathfrak{i}$ is complemented, but we expect that this is unnecessary if the Hahn-Banach theorem is used instead.

Lemma 3.24. *Suppose $\mathfrak{i}$ is a complemented ideal in $\mathfrak{g}$ with continuous linear section s. Then the map $\gamma : H^2_{ct}(\mathfrak{g}) \to H^1_{ct}(\mathfrak{g}/\mathfrak{i}, C^1_{ct}(\mathfrak{i}))$ induced by $\gamma(\omega)(\xi) = \omega(s(\xi), \cdot)$ is well defined.*

Proof. Let $\xi, \xi_0, \xi_1 \in \mathfrak{g}/\mathfrak{i}$ and $z, z_1, z_2 \in \mathfrak{i}$. Taking $\psi = \gamma(\omega)$, we see that $\psi(\xi)$ is clearly linear and continuous. Computing the differential:

$$\begin{aligned} &(d_1\psi(\xi_0, \xi_1))(z) = \xi_0 \cdot \psi(\xi_1)(z) - \xi_1 \cdot \psi(\xi_0)(z) - \psi([\xi_0, \xi_1])(z) = \\ &-\omega(s(\xi_1), [s(\xi_0), z]) + \omega(s(\xi_0), [s(\xi_1), z]) - \omega([s(\xi_0), s(\xi_1)]), z) = 0 \end{aligned} \tag{3.53}$$

where we find that this is zero because of the 2-cocycle identity of ω. Hence ψ is indeed a 1-cocycle.

Now suppose that ω is a 2-coboundary, so there exists $\phi \in H^1_{ct}(\mathfrak{g})$ such that $\omega = d_1\phi$ and $\psi(\xi) = -\phi([s(\xi), \cdot]) = \xi \cdot \phi = \tilde{d}_0\phi(\xi)$. The differential $\tilde{d}_0$ denotes the Chevalley-Eilenberg differential for $\mathcal{C}^0_{ct}(\mathfrak{g}/\mathfrak{i}, C^1_{ct}(\mathfrak{i}))$. Thus $\psi = \tilde{d}_0\phi$, ψ is a 1-coboundary and γ is well defined. □

Proposition 3.25. *Let $H^2_{ct}(\mathfrak{g})_1$ be the 2-cocycles vanishing on a complemented ideal $\mathfrak{i}$, $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ the projection and s the continuous linear section. Then the map $\gamma : H^2_{ct}(\mathfrak{g})_1 \to H^1_{ct}(\mathfrak{g}/\mathfrak{i}, H^1_{ct}(\mathfrak{i}))$ induced by $\gamma(\omega)(\xi) = \omega(s(\xi), \cdot)$ is well defined, independent of the section s, and* $\ker(\gamma) = \mathrm{im}(\pi^*)$

Proof. Well definedness follows from lemma 3.24 and that $\gamma(\omega)(\xi) = \psi(\xi)$ is a 1-cocycle:

$$d_1(\psi(\xi))(z_1, z_2) = -\omega(s(\xi), [z_1, z_2]) = \omega([z_1, s(\xi)], z_0) + \omega([s(\xi), z_0], z_1) = 0. \tag{3.54}$$

The last equality follows from the fact that $\mathfrak{i}$ is an ideal and ω vanishes on $\mathfrak{i}$. Independence of the section s follows the fact that for a different section $\tilde{s}$, $s(\xi) - \tilde{s}(\xi) \in \mathfrak{i}$ and ω vanishes on $\mathfrak{i} \times \mathfrak{i}$.

We will now prove $\ker(\gamma) = \mathrm{im}(\pi^*)$. Suppose $\omega \in \mathrm{im}(\pi^*)$. Then there exists $\tilde{\omega} \in H^2_{ct}(\mathfrak{g}/\mathfrak{i})$ such that $\omega = \pi^*\tilde{\omega}$. We can calculate $\gamma(\omega)(\xi)(z) = \tilde{\omega}(\pi(s(\xi)), \pi(z)) = 0$ as $\pi(z) = 0$ for all $z \in \mathfrak{i}$. For the converse, suppose $\omega \in \ker(\gamma)$. Then $\omega(s(\xi), z) = 0$ for all $\xi \in \mathfrak{g}/\mathfrak{i}, z \in \mathfrak{i}$, hence $\tilde{\omega} = s^*\omega$ is independent of the lifting operation s and therefore a well defined 2-cocycle on $\mathfrak{g}/\mathfrak{i}$. We note that $s \circ \pi = 1_{\mathfrak{g}} + \mathfrak{i}$, hence $\pi^*\tilde{\omega} = \omega(s(\pi(\cdot)), s(\pi(\cdot))) = \omega(\cdot, \cdot)$, as $\omega(\mathfrak{g}, \mathfrak{i}) = 0$. We conclude that $\ker(\gamma) = \mathrm{im}(\pi^*)$. □

In terms of exact sequences, we can now conclude

Corollary 3.26. *Suppose we have an exact sequence:*

$$0 \to \mathfrak{i} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{i} \to 0 \tag{3.55}$$

where the maps are Lie algebra homomorphisms and which splits, i.e. there exists a continuous Lie algebra homomorphism $s : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}$ such that $\pi \circ s = 1_{\mathfrak{g}}$, then the following sequence is also exact:

$$0 \to H^2_{ct}(\mathfrak{g}/\mathfrak{i}) \xrightarrow{\pi^*} H^2_{ct}(\mathfrak{g})_1 \xrightarrow{\gamma} H^1_{ct}(\mathfrak{g}/\mathfrak{i}, H^1_{ct}(\mathfrak{i})) \tag{3.56}$$

In particular, if $H^1_{ct}(\mathfrak{i}) = 0$, $H^2_{ct}(\mathfrak{g}/\mathfrak{i}) \cong H^2_{ct}(\mathfrak{g})_1$

We will need to extend the above sequence to the case where the sequence of Lie algebras does not split. This is done by considering the following cohomology space:

$$H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}} = \{[\phi] \in H^1_{ct}(\mathfrak{i}) | [t \cdot \phi] = 0 \text{ for all } t \in \mathfrak{g}/\mathfrak{i}\} \tag{3.57}$$

with the action of $\mathfrak{g}/\mathfrak{i}$ on $H^1_{ct}(\mathfrak{i})$ given by:

$$t \cdot \phi = -\phi([s(t), \cdot]) \tag{3.58}$$

for some linear continuous section s. The action is independent of the section because of the 1-cocycle property. Furthermore, as the only 1-coboundary is identically 0, it does not matter if we consider $[t \cdot \phi]$ to be 0 as a quotient class, or $t \cdot \phi$ to be 0 as a cochain. With this space defined, we extend our sequence:

Proposition 3.27. *Suppose $\mathfrak{i}$ is a complemented ideal and we have an exact sequence:*

$$0 \to \mathfrak{i} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{i} \to 0 \tag{3.59}$$

where the maps are continuous Lie algebra homomorphisms. Then the following sequence is also exact

$$H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}} \xrightarrow{\alpha} H^2_{ct}(\mathfrak{g}/\mathfrak{i}) \xrightarrow{\pi^*} H^2_{ct}(\mathfrak{g})_1 \xrightarrow{\gamma} H^1_{ct}(\mathfrak{g}/\mathfrak{i}, H^1_{ct}(\mathfrak{i})) \tag{3.60}$$

In particular, if $H^1_{ct}(\mathfrak{i}) = 0$, $H^2_{ct}(\mathfrak{g}/\mathfrak{i}) \cong H^2_{ct}(\mathfrak{g})_1$

Proof. Let $s : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}$ be a continuous linear splitting, i.e. a right inverse of π, but not necessarily a Lie algebra homomorphism, so s does not need to preserve the bracket. It then follows that $[s(x), s(y)] - s([x, y]) \in \ker \pi = \mathfrak{i}$.

Let $[\phi] \in H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$. We then define a 2-cochain on $\mathfrak{g}/\mathfrak{i}$:

$$\omega(x, y) = \phi\big([s(x), s(y)] - s([x, y])\big) \tag{3.61}$$

This is clearly linear, alternating and continuous, as s, ϕ are linear and continuous. ω is in fact a 2-cocycle:

$$\begin{aligned}
&\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)= \\
&\phi\big([s([x,y]),s(z)]-s([[x,y],z])\big)+ \\
&\phi\big([s([y,z]),s(x)]-s([[y,z],x])\big)+ \\
&\phi\big([s([z,x]),s(y)]-s([[z,x],y])\big)= \\
&\phi\big([s([x,y]),s(z)]+[s([y,z]),s(x)]+[s([z,x]),s(y)]\big)
\end{aligned} \tag{3.62}$$

where the last equality follows by the Jacobi identity. We continue the calculation

$$\begin{aligned}
&\phi\big([s([x,y]),s(z)]+[s([y,z]),s(x)]+[s([z,x]),s(y)]\big)= \\
&\phi\big(\big[s([x,y])-[s(x),s(y)],s(z)\big]+\big[[s(x),s(y)],s(z)\big]+ \\
&\quad\big[s([y,z])-[s(y),s(z)],s(x)\big]+\big[[s(y),s(z)],s(x)\big]+ \\
&\quad\big[s([z,x])-[s(z),s(x)],s(y)\big]+\big[[s(z),s(x)],s(y)\big]\big)= \\
&\phi\big(\big[s([x,y])-[s(x),s(y)],s(z)\big]+ \\
&\quad\big[s([y,z])-[s(y),s(z)],s(x)\big]+ \\
&\quad\big[s([z,x])-[s(z),s(x)],s(y)\big]\big)
\end{aligned} \tag{3.63}$$

where the last equality again follows by the Jacobi identity. Now as

$$[s(x),s(y)]-s([x,y])\in\ker\pi=\mathfrak{i}$$

and $\phi([a,x])=0$ for all $a\in\mathfrak{i}$ as $x\cdot\phi=0$, the 2-cocycle property holds.

We can now define the map $\alpha: H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}\to H^2_{ct}(\mathfrak{g}/\mathfrak{i})$ by $\alpha([\phi])=[\omega]$ with ω as defined above. Again, as 1-coboundaries in $H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$ are 0, α maps equivalence classes to equivalence classes and is well defined. Furthermore, if we were to take a different section $\tilde{s}$, then

$$\begin{aligned}
&\phi\big([\tilde{s}(x),\tilde{s}(y)]-\tilde{s}([x,y])\big)-\phi\big([s(x),s(y)]-s([x,y])\big)= \\
&\phi\big([\tilde{s}(x)-s(x),\tilde{s}(y)]+[s(x),\tilde{s}(y)-s(y)]-\tilde{s}([x,y])+s([x,y])\big)= \\
&\phi\big(-\tilde{s}([x,y])+s([x,y])\big)=(\phi\circ(s-\tilde{s}))([x,y])
\end{aligned} \tag{3.64}$$

which is a 2-coboundary. Thus α is independent of the section s we choose.

Now let $[\omega]\in\mathrm{im}(\alpha)$.

$$\begin{aligned}
&\pi^*\omega(x,y)=\phi\big([s(\pi(x)),s(\pi(y))]-s([\pi(x),\pi(y)])\big)= \\
&\phi\big([s(\pi(x))-x,s(\pi(y))]+[s(\pi(x)),s(\pi(y))-y]-[s(\pi(x))-x,s(\pi(y))-y]+[x,y]-s(\pi([x,y]))\big)= \\
&\pi(y)\cdot\phi\big(s(\pi(x))-x\big)-\pi(x)\cdot\phi\big(s(\pi(y))-y\big)-d\phi(s(\pi(x))-x,s(\pi(y))-y)+\phi\big([x,y]-s(\pi([x,y]))\big)= \\
&\phi\big([x,y]-s(\pi([x,y]))\big)
\end{aligned} \tag{3.65}$$

Here we used that ϕ is a 1-cocycle and invariant under the action of $\mathfrak{g}/\mathfrak{i}$. Thus $\pi^*\omega(x,y)=d(\phi\circ(1-s\circ\pi))(x,y)$, hence $[\pi^*\omega]=0$ and $\mathrm{im}(\alpha)\subseteq\ker(\pi^*)$. For the converse, let

$[\omega] \in \ker(\pi^*)$. Then $\pi^*\omega(x,y) = \phi([x,y])$ for some 1-cochain $\phi \in H^1(\mathfrak{g})$ and $x, y \in \mathfrak{g}$. Let $a \in \mathfrak{i}$, then $\phi([a,x]) = \pi^*\omega(a,x) = 0$ as $\pi(a) = 0$. Thus $\psi = \phi|_{\mathfrak{i}}$ is in $H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$ and $\phi\big([s([x]),s([y])] - s([[x],[y]])\big) = \phi\big([s([x]),s([y])]\big) - \phi\big(s([[x],[y]])\big) = \omega([x],[y]) - \phi\big(s([[x],[y]])\big)$. As $\phi \circ s([[x],[y]])$ is a 2-coboundary, $\alpha([\phi]) = [\omega]$ and $[\omega] \in \mathrm{im}(\alpha)$, so $\mathrm{im}(\alpha) = \ker(\pi^*)$

□

Remark 3.28. The above exact sequence is a piece of the 7-term Serre-Hothschild sequence, see for example Ref. [DHW12].

We can also describe the relation between $H^2_{ct}(\mathfrak{g})_1$ and $H^2_{ct}(\mathfrak{g})$ using an exact sequence. Before we do this, we first define $H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$, the 2-cocyles on $\mathfrak{i}$ that are invariant under the action of $\mathfrak{g}/\mathfrak{i}$, which is given by $(x \cdot \omega)(a,b) = -\omega([x,a],b) - \omega(a,[x,b])$

$$H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}} := \{[\omega] \in H^2_{ct}(\mathfrak{i}) | [t \cdot \omega] = 0\} \tag{3.66}$$

This is well defined, as if $\omega, \omega' \in [\omega]$ and $[t \cdot \omega] = 0$, $\omega' = \omega + \psi$ for some 2-coboundary $d\psi$, then $t \cdot \psi$ is again a 2-coboundary, which can be seen from the Jacobi identity. Thus $[t \cdot \omega'] = [t \cdot \omega]$.

Proposition 3.29. *Suppose we have the short exact sequence*

$$0 \to \mathfrak{i} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{i} \to 0 \tag{3.67}$$

then we get an exact sequence of cohomology spaces:

$$0 \to H^2_{ct}(\mathfrak{g})_1 \to H^2_{ct}(\mathfrak{g}) \xrightarrow{\iota^*} H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}} \tag{3.68}$$

Proof. As $H^2_{ct}(\mathfrak{g})_1$ is the kernel of the restriction map $\mathrm{Res} : H^2_{ct}(\mathfrak{g}) \to H^2_{ct}(\mathfrak{i})$, it is a subspace of $H^2_{ct}(\mathfrak{g})$ and therefore maps injectively into it. The inclusion map $\iota : \mathfrak{i} \to \mathfrak{g}$ is a continuous Lie algebra homomorphism and hence ι^* is a cochain homomorphism.

We will now show that $\mathrm{im}(\iota^*) \subseteq H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$. Suppose $[\omega] \in \mathrm{im}(\iota^*)$, then there exists $[\tilde{\omega}] \in H^2_{ct}(\mathfrak{g})$ such that $\iota^*\tilde{\omega} + d\phi = \omega$ for some 2-coboundary $d\phi$ on $\mathfrak{g}$. Now let $[x] \in \mathfrak{g}/\mathfrak{i}, a, b \in \mathfrak{i}$, then

$$\begin{aligned}&([x]\cdot\omega)(a,b) = -\omega([x,a],b) - \omega(a,[x,b]) = \\ &-\tilde{\omega}(\iota([x,a]),b) - \tilde{\omega}(\iota([b,x]),a) - \phi([[x,a],b]) - \phi([[b,x],a]) = \tilde{\omega}([a,b],x) + \phi([[a,b],x])\end{aligned} \tag{3.69}$$

as $\mathfrak{i}$ is an ideal, meaning that the Lie brackets are again in $\mathfrak{i}$. This is indeed a 2-coboundary on $\mathfrak{i}$, so $\omega \in H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$ and thus $\iota^* : H^2_{ct}(\mathfrak{g}) \to H^2_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}}$ is well defined. By definition of $H^2_{ct}(\mathfrak{g})_1$, the exactness at $H^2_{ct}(\mathfrak{g})$ follows.

□

3.4.2 Invariance under periodic flow

In this section we will assume that the Lie algebra $\mathfrak{g}$ consists of vector fields on a manifold M and contains a complete vector field E. If the flow generated by the vector field E is periodic, we will show that we only need to consider the cocycles invariant under that flow.

We define the *Lie derivative* of a k-cochain by the Cartan magic formula:

Definition 3.30 (Lie derivative of cochain)**.** The Lie derivative of a k-cochain ω along an element $x \in \mathfrak{g}$ is given by:

$$\mathcal{L}_x \omega = \iota_x d_k \omega + d_{k-1} \iota_x \omega \tag{3.70}$$

where ι denotes the interior product.

It is thus immediate that the interior product is a cochain homotopy between the Lie derivative and 0.

We then define the k-cochains that are 0 under the Lie derivative along E:

$$\mathcal{C}^k_{ct}(\mathfrak{g})_0 := \left\{ c \in \mathcal{C}^k_{ct}(\mathfrak{g}) | \mathcal{L}_E c = 0 \right\}. \tag{3.71}$$

From the Cartan magic formula we immediately find $d\mathcal{L}_E = \mathcal{L}_E d$, meaning that $d : \mathcal{C}^{\bullet}_{ct}(\mathfrak{g})_0 \to \mathcal{C}^{\bullet+1}_{ct}(\mathfrak{g})_0$ is well defined, and $\mathcal{C}^{\bullet}_{ct}(\mathfrak{g})_0$ is thus a subcomplex.

The next lemma shows that pullbacks of flows at different times are homotopy connected. We will then use this to show that cohomology of $\mathcal{C}^{\bullet}_{ct}(\mathfrak{g})$ and $\mathcal{C}^{\bullet}_{ct}(\mathfrak{g})_0$ are isomorphic.

Lemma 3.31 (lemma 3.10 from [Mia22])**.** *Let $(\phi^t)_{t>0}$ be the flow generated by the time-dependent vector-field $(X^t)_{t>0}$. Then for all $t_1, t_0 > 0$ and $c \in C^{\bullet}_{ct}(\mathfrak{g})$, we have*

$$(\phi^{t_1})^* c - (\phi^{t_0})^* c = K_{t_1,t_0} dc + dK_{t_1,t_0} c \tag{3.72}$$

where $K_{t_1,t_0} c = \int_{t_0}^{t_1} \iota_{X_t} \phi_t^* c dt$

Proof. First assume X_t is actually time independent. Then we have the following formula (proposition 12.36 in Ref. [Lee12])

$$\left.\frac{d}{dt}\right|_{t=t_0} ((\phi^t)^* c) = (\phi^{t_0})^* (\mathcal{L}_{X_{t_0}} c) \tag{3.73}$$

Now as this formula only depends on the time t_0 and the vector field X_t at $t = t_0$, it also holds when X_t is time dependent. Using the definition of the Lie derivative and that d commutes with the pullback:

$$(\phi^{t_0})^* \mathcal{L}_{X_{t_0}} c = d(\phi^{t_0})^* \iota_{X_{t_0}} c + (\phi^{t_0})^* \iota_{X_{t_0}} dc \tag{3.74}$$

Now we integrate both sides. On the left side, using the fundamental theorem of calculus, we find:

$$\int_{t_0}^{t_1} \left.\frac{d}{dt}\right|_{t=s} ((\phi^t)^* c) ds = (\phi^{t_1})^* c - (\phi^{t_0})^* c \tag{3.75}$$

On the right side, using that ϕ_t^* and ι_{X_t} commute as ϕ_t is generated by ι_{X_t},

$$\int_{t_0}^{t_1} d\iota_{X_s}(\phi^s)^*c + \iota_{X_s}(\phi^s)^*dcds = d\int_{t_0}^{t_1} \iota_{X_s}(\phi^s)^*cdt_0 + \int_s^{t_1} \iota_{X_s}(\phi^s)^*dcds = dK_{t_1,t_0}c + K_{t_1,t_0}dc \tag{3.76}$$

□

We define $\phi^t : \mathbb{R} \times M \to M$. As E is complete, the flow is global. Further assume that ϕ^t is periodic with period T, meaning that $\phi^T = \phi^0 = \text{id}$. Using lemma 3.31 with $t_0 = T$, we conclude that $(\phi^t)^*$ is homotopy connected to the identity. Then with $Nc := \frac{1}{T}\int_0^T (\phi^t)^*cdt$, N is again homotopy connected to the identity through $\frac{1}{T}\int_0^T K_t dt$. Thus for any k-cocycle c

$$[\frac{1}{T}\int_0^T (\phi^t)^*cdt] = [c] \tag{3.77}$$

Using the periodicity, we will show that $\frac{1}{T}\int_0^T (\phi^t)^*cdt$ is invariant under the flow generated by E:

$$(\phi^s)^* \int_0^T (\phi^t)^*cdt = \int_0^T (\phi^{t+s})^*cdt = \int_s^{T+s} (\phi^t)^*cdt = \int_0^T (\phi^t)^*cdt \tag{3.78}$$

where we used the periodicity of ϕ^t in the last step. Thus $\mathcal{L}_E \int_0^T (\phi^t)^*cdt = 0$ and $\int_0^T (\phi^t)^*cdt \in \mathcal{C}^k_{ct}(\mathfrak{g})_0$. Hence up to a k-coboundary, every k-cocycle is invariant under E.

Theorem 3.32. *Suppose E is a complete vector field and the flow generated by E is periodic. Then the inclusion $\iota : C^k_{ct}(\mathfrak{g})_0 \to C^k_{ct}(\mathfrak{g})$ induces an isomorphism:*

$$H^k_{ct}(\mathfrak{g}) \cong H^k_{ct}(\mathfrak{g})_0 := Z^k_{ct}(\mathfrak{g})_0/B^k_{ct}(\mathfrak{g})_0 \tag{3.79}$$

Proof. ι is injective, as if $c \in C^k(\mathfrak{g})_0$ is a coboundary, then it is also also a coboundary in $C^k(\mathfrak{g})$. Furthermore, ι is surjective as for any $[\omega] \in H^k_{ct}(\mathfrak{g})$, $[\omega] = [\int_0^T (\phi^t)^*\omega dt]$ and $[\int_0^T (\phi^t)^*\omega dt] \in H^k_{ct}(\mathfrak{g})_0$. □

Now let $E_1, \dots, E_n$ all be complete commuting vector fields that generate periodic flows $\phi_1, \dots, \phi_n$. Define $\mathcal{C}^k(\mathfrak{g})_0 := \{c \in \mathcal{C}^k(\mathfrak{g}) | \mathcal{L}_{E_i}c = 0 \quad \forall i = 1, 2, \dots, n\}$ .. We then have a similar result:

Theorem 3.33. *The inclusion $\iota : C^k_{ct}(\mathfrak{g})_0 \to C^k_{ct}(\mathfrak{g})$ induces an isomorphism:*

$$H^k_{ct}(\mathfrak{g}) \cong H^k_{ct}(\mathfrak{g})_0 := Z^k_{ct}(\mathfrak{g})_0/B^k_{ct}(\mathfrak{g})_0 \tag{3.80}$$

Proof. First note that once again $[\frac{1}{T}\int_0^T (\phi_i^t)^*cdt] = [c]$ and $\mathcal{L}_{E_i}\frac{1}{T}\int_0^T (\phi_i^t)^*cdt = 0$. Now

$$[\frac{1}{T^n}\int_0^T \cdots \int_0^T (\phi_1^{t_1})^* \dots (\phi_n^{t_n})^*cdt_1 \dots dt_n] = [\frac{1}{T^{n-1}}\int_0^T \cdots \int_0^T (\phi_1^{t_1})^* \dots (\phi_{n-1}^{t_{n-1}})^*cdt_1 \dots dt_{n-1}] = [c] \tag{3.81}$$

where we applied that $[\frac{1}{T}\int_0^T (\phi^t)^*cdt] = [c]$ n-times. Now as flows commute when their vector fields commute, $\mathcal{L}_{E_i}\frac{1}{T^n}\int_0^T \cdots \int_0^T (\phi_1^{t_1})^* \dots (\phi_n^{t_n})^*cdt_1 \dots dt_n = 0$ by the same argument as in the $n = 1$ case. □

Remark 3.34. An important case is when the Lie group corresponding to the Lie algebra has a compact abelian finite dimensional subgroup. Such a subgroup is isomorphic to $\mathbb{S}^k$ for some k, and all the vectors in the corresponding subalgebra will commute and will generate a global periodic flow. Thus the cocycles need to be invariant under the flow generated by all the vectors in this subalgebra.

3.4.3 Cohomology of complexification

As noted in section 2.3, sometimes it is easier to work with the complexification of a Lie algebra. The next proposition shows how the real and complex 2-cocycles are connected.

Proposition 3.35. *1. Suppose ω is a real 2-cocycle on $\mathfrak{g}$. Then*

$$\omega_C(x+iy, a+ib) = \big(\omega(x,a) - \omega(y,b)\big) + i\big(\omega(y,a) + \omega(x,b)\big) \tag{3.82}$$

is a complex 2-cocycle on the complexification $\mathfrak{g}_{\mathbb{C}}$

2. *Suppose ω_C is a complex 2-cocycle on $\mathfrak{g}_{\mathbb{C}}$. Then the real and imaginary parts, restricted to $\mathfrak{g}$, are real 2-cocycles:*

$$\begin{aligned} \omega_R(x,y) &= \frac{1}{2}\big(\omega_C(x,y) + \overline{\omega_C(x,y)}\big) \\ \omega_I(x,y) &= \frac{1}{2i}\big(\omega_C(x,y) - \overline{\omega_C(x,y)}\big) \end{aligned} \tag{3.83}$$

Proof. 1. That ω_C is linear, continuous and antisymmetric is immediate from the definition. Furthermore, the 2-cocycle identity clearly holds for $\omega_C(x,a)$ with $x, a \in \mathfrak{g}$. By linearity it holds for all $x+iy, a+ib \in \mathfrak{g}_{\mathbb{C}}$.

2. First note that $\mathfrak{g}$ is a real subalgebra of $\mathfrak{g}_{\mathbb{C}}$, so that restriction to $\mathfrak{g}$ is well defined. Again, linearity, continuity and antisymmetry are clear. The 2-cocycle identity is also immediate.

□

Proposition 3.35 shows that the real 2-cocycles determine the complex 2-cocycles and vice versa. Indeed, suppose ω is a real 2-cocycle. Then ω_C as in proposition 3.35 is a complex 2-cocycle and $\omega = \omega_C|_{\mathfrak{g}\times\mathfrak{g}}$. Now if ω_C is a complex 2-cocycle, ω_R, ω_I are real 2-cocycles and $\omega_C = \omega_R + i\omega_I$.

3.5 Witt algebra

We will encounter a locally convex Lie algebra called the *Witt algebra* later in the asymptotic symmetry algebra of AdS. We describe it and its first two cohomology groups here, as the theory is already known and it provides an important example of a locally convex Lie algebra.

Definition 3.36 (Witt algebra)**.** We define $\mathfrak{w}$ to be the Lie algebra of real smooth vector fields on the circle $\mathbb{S}^1$. We equip it with the following family of seminorms, turning it into a locally convex Lie algebra. Let $f\partial_x$ be a smooth vector field on $\mathbb{S}^1$, then for $k \in \mathbb{N}$ we define the k-th seminorm as:

$$p_k(f\partial_x) = \sup\{|\partial_x^k f(x)| : x \in \mathbb{S}^1\} \tag{3.84}$$

The Lie bracket is given by:

$$[f\partial_x, g\partial_x] = (f\partial_x g - g\partial_x f)\partial_x \tag{3.85}$$

To see that the Lie bracket is continuous with this topology, note that addition and scalar multiplication are continuous in all locally convex vector spaces, and the differential is continuous as well as $p_k(\partial_x f\partial_x) = p_{k+1}(f\partial_x)$. Thus if we show that multiplication is continuous, the Lie bracket is continuous as well. This is shown in the following lemma:

Lemma 3.37. *Multiplication* $\mathfrak{w} \times \mathfrak{w} \to \mathfrak{w} : (f\partial_x, g\partial_x) \to fg\partial_x$ *is continuous.*

Proof. Using lemma 2.8, we need to prove that for every seminorm p_k, there exist continuous seminorms p, q such that $p_k(fg\partial_x) \leq Cp(f\partial_x)q(g\partial_x)$ for some constant C. As $\partial_k fg = \sum_{l=0}^k \binom{k}{l} f^{(l)} g^{(k-l)}$, using the generalized Leibniz rule, we have

$$p_k(fg\partial_x) \leq \sum_{l=0}^{k} \binom{k}{l} p_l(f\partial_x)p_{k-l}(g\partial_x) \leq k!(\sum_{l=0}^{k} p_l(f\partial_x))(\sum_{l=0}^{k} p_l(g\partial_x)) \tag{3.86}$$

and the sum of continuous seminorms is again a continuous seminorm, meaning that $(\sum_{l=0}^k p_l(f\partial_x)), (\sum_{l=0}^k p_l(g\partial_x))$ are the continuous seminorms p, q that we need. $\square$

The complexification of the Witt algebra, $\mathfrak{w}_{\mathbb{C}}$ is the Lie algebra of complex smooth vector fields on the circle. For the rest of the section, when we write $\mathfrak{w}$ we mean the complexified Lie algebra. We will first show that the complex exponentials are dense in the Witt algebra, for which we will need some notation: If f is a smooth function on the circle, define the Fourier coefficients of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{3.87}$$

Then from partial integration it is immediate that $(\hat{\partial_x f})(n) = in\hat{f}(n)$. As p_0 is a norm and the family of seminorms on $\mathfrak{w}$ is countable, $\mathfrak{w}$ is metrizable by lemma 2.9. Thus the complex exponentials are dense in $\mathfrak{w}$ if each $f\partial_x \in \mathfrak{w}$ is a limit of a sequence of complex exponentials. The limit converges if it converges in each seminorm, also by lemma 2.9.

For the p_0 seminorm, this follows from the following theorem:

Theorem 3.38 (From Ref. [Nee22])**.** *For all* $f \in C(\mathbb{S}^1)$ *we have*

$$\lim_{N\to\infty} \sup_{x\in\mathbb{S}^1} |f - \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}| = 0 \tag{3.88}$$

From the p_0 case, we can derive convergence in all other seminorms as well

Lemma 3.39. *$D = \mathrm{span}\{e^{inx}|n \in \mathbb{N}\}$ is dense in $\mathfrak{w}$.*

Proof. Let $f\partial_x \in \mathfrak{w}$. Then the sequence defined by the elements $f_N\partial_x = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{f}(k)e^{ikx} \in D$ converges to $f\partial_x$:

$$\lim_{N\to\infty} p_n((f - \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{f}(k)e^{ikx})\partial_x) = \lim_{N\to\infty}\sup_{x\in\mathbb{S}^1}|\partial_x^n f - \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{f}(k)(ik)^n e^{ikx}| =$$
$$\lim_{N\to\infty}\sup_{x\in\mathbb{S}^1}|\partial_x^n f - \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}(\widehat{\partial_x^n f})(k)e^{ikx}| = 0 \tag{3.89}$$

where the last equality follows from theorem 3.38, as $\partial_x^n f$ is again a continuous function. □

We now go on to show that the Witt algebra is perfect, for which we first need the following lemma, which states that functions on the circle have an antiderivative if their mean is zero.

Lemma 3.40. *Let $f \in C^\infty(\mathbb{S}^1)$. Then there exists a function $F \in C^\infty(\mathbb{S}^1)$ such that $f = \partial_x F$ if and only $\int_0^{2\pi} f(x)dx = 0$*

Proof. Suppose $\int_0^{2\pi} f(x)dx = 0$, then we can simply take $F(x) = \int_0^x f(t)dt$. By the fundamental theorem of calculus, $f = \partial_x F$ and F is periodic as $\int_0^{2\pi} f(x)dx = 0$. Now assume that there exists a function $F \in C^\infty(\mathbb{S}^1)$ such that $f = \partial_x F$. From here we see $\int_0^{2\pi} f(x)dx = F(2\pi) - F(0) = 0$ as F is periodic. □

Using this lemma, we show that the Witt algebra is perfect.

Proposition 3.41. *$\mathfrak{w}$ is perfect, i.e. $[\mathfrak{w}, \mathfrak{w}] = \mathfrak{w}$*

Proof. First we note that $[\cos(x)\partial_x, \sin(x)\partial_x] = \partial_x$ and thus all constant vector fields are included in $[\mathfrak{w}, \mathfrak{w}]$. Now let $f\partial_x \in \mathfrak{w}$, then by lemma 3.40, there exists a vector field $F\partial_x$ such that $\partial_x F = f - \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. Now $[\partial_x, F\partial_x] = \partial_x F\partial_x = f\partial_x - \frac{1}{2\pi}\int_0^{2\pi} f(t)dt\partial_x$, so $(f - \frac{1}{2\pi}\int_0^{2\pi} f(t)dt)\partial_x \in [\mathfrak{w}, \mathfrak{w}]$. As $[\mathfrak{w}, \mathfrak{w}]$ is a vector space including the constant vector fields, $f\partial_x \in [\mathfrak{w}, \mathfrak{w}]$. □

That $\mathfrak{w}$ is perfect means that the first cohomology group is 0. We will now go on to calculate the second cohomology group.

Proposition 3.42. *The second cohomology group of $\mathfrak{w}$ is 1 dimensional, $H^2_{ct}(\mathfrak{w}) = \mathbb{R}$ and spanned by the Gelfand-Fuks cocycle:*

$$\omega(X\partial_x, Y\partial_x) = \int_0^{2\pi} X(x)\partial_x^3 Y(x)dx \tag{3.90}$$

Proof. Firstly, we will work in the complexification $\mathfrak{w}_{\mathbb{C}}$ and then conclude the result for the real part afterwards. We also drop the subscript $\mathbb{C}$ for now. Then note that $E = i\partial_x$ is a complete vector field that generates the periodic flow $\phi^t : e^{ix} \to e^{i(x+t)}$. By theorem 3.32, we only need to consider the the 2-cocycles invariant under E:

$$H^2_{ct}(\mathfrak{w}) \cong H^2_{ct}(\mathfrak{w})_0 \tag{3.91}$$

Thus for $[\omega] \in H^2_{ct}(\mathfrak{w})$, we can assume $\mathcal{L}_E\omega = 0$. Then $L_E\omega(e^{inx^+}\partial_x, e^{imx^+}\partial_x) = -(n + m)\omega(e^{inx^+}\partial_x, e^{imx^+}\partial_x) = 0$, so $\omega(e^{inx^+}\partial_x, e^{imx^+}\partial_x) = 0$ for $n + m \neq 0$. Now we can use the 2-cocycle property:

$$\begin{aligned}0 &= \omega([e^{imx}\partial_x, e^{ix}\partial_x], e^{-i(m+1)x}\partial_x) + \omega([e^{ix}\partial_x, e^{-i(m+1)x}\partial_x], e^{imx}\partial_x) + \omega([e^{-i(m+1)x}\partial_x, e^{imx}\partial_x], e^{ix}\partial_x) \\ &= i(1-m)\omega(e^{i(m+1)x}\partial_x, e^{-i(m+1)x}\partial_x) - i(m+2)\omega(e^{-imx}\partial_x, e^{imx}\partial_x) + i(2m+1)\omega(e^{-ix}\partial_x, e^{ix}\partial_x)\end{aligned} \tag{3.92}$$

Writing $c_m = \omega(e^{imx}\partial_x, e^{-imx}\partial_x)$, we get the recurrence relation:

$$0 = (m-1)c_{m+1} - (m+2)c_m + (2m+1)c_1 \tag{3.93}$$

This is a recurrence relation with a 2 dimensional solution space, as the values of c_1, c_2 determine all the other values. The solution space is spanned by $c_m = m$ and $c_m = m^3$, as it is easily checked that these are solutions of the recurrence relation.

Now define the following 2-cochains:

$$\begin{aligned}\omega'(X\partial_x, Y\partial_x) &= \int_0^{2\pi} X(x)\partial_x Y(x)dx \\ \omega(X\partial_x, Y\partial_x) &= \int_0^{2\pi} X(x)\partial_x^3 Y(x)dx\end{aligned} \tag{3.94}$$

They are clearly linear, and that they are alternating can be seen from partial integration. Continuity follows from the following formulae and lemma 2.8

$$\begin{aligned}|\omega'(X\partial_x, Y\partial_x)| &\leq 2\pi p_0(X\partial_x)p_1(Y\partial_x) \\ |\omega(X\partial_x, Y\partial_x)| &\leq 2\pi p_0(X\partial_x)p_3(Y\partial_x)\end{aligned} \tag{3.95}$$

We can see however, that ω' is a 2-coboundary as we have the following expression for ω':

$$\omega'(X\partial_x, Y\partial_x) = \frac{1}{2}\int_0^{2\pi} [X\partial_x, Y\partial_x]dx \tag{3.96}$$

which follows from partial integration. Similarly, we can show that ω is a 2-cocycle:

$$\begin{aligned}&\omega([X\partial_x, Y\partial_x], Z\partial_x) = \int_0^{2\pi} (X\partial_X - Y\partial_X)\partial_x Z^3 dx = \\ &\int_0^{2\pi} X(\partial_x(\partial_x Y\partial_x^2 Z) - \partial_x^2 Y\partial_x^2 Z) - Y(\partial_x(\partial_x X\partial_x^2 Z) - \partial_x^2 X\partial_x^2 Z)dx\end{aligned} \tag{3.97}$$

which follows from the product rule of differentiation. Now, using partial integration, we find

$$\begin{aligned}&\int_0^{2\pi} X(\partial_x(\partial_x Y\partial_x^2 Z) - \partial_x^2 Y\partial_x^2 Z) - Y(\partial_x(\partial_x X\partial_x^2 Z) - \partial_x^2 X\partial_x^2 Z)dx = \\ &\int_0^{2\pi} \partial_x X\partial_x Y\partial_x^2 Z - X\partial_x^2 Y\partial_x^2 Z - \partial_x X\partial_x Y\partial_x^2 Z + Y\partial_x^2 X\partial_x^2 Z dx = \\ -&\int_0^{2\pi} X\partial_x^2 Y\partial_x^2 Z - Y\partial_x^2 X\partial_x^2 Z dx\end{aligned} \tag{3.98}$$

From this last expression the 2-cocycle property is immediate. Furthermore, for any 2-coboundary η, $\eta(e^{inx}\partial_x, e^{-inx}\partial_x) = \phi([(e^{inx}\partial_x, e^{-inx}\partial_x]) = -2in\phi(\partial_x)$ for some 1-cochain ϕ, so any 2-coboundary has coefficients $c_n \propto n$. It is easily checked that ω has coefficients $c_n \propto n^3$ and therefore, ω is not a 2-coboundary.

Thus ω' is the 2-coboundary corresponding to $c_m \propto m$ and ω is the 2-cocycle corresponding to $c_m \propto m^3$. Hence these are the unique (up to scaling) 2-cocycles on the subspace of complex exponentials. Now as the complex exponentials are dense in the Witt algebra by lemma 3.39, they are the unique 2-cocycles on $\mathfrak{w}$.

For the complexified Witt algebra we conclude that the second cohomology space $H^2(\mathfrak{w})$ is one dimensional and is spanned by the Gelfand-Fuks cocycle:

$$\omega(X\partial_x, Y\partial_x) = \int_0^{2\pi} X(x)\partial_x^3 Y(x)dx \tag{3.99}$$

Now for the real Witt algebra, using proposition 3.35, we see that if ω_R would be a real 2-cocycle, it would define a complex 2-cocycle such that $\omega_R = \omega_C|_{\mathfrak{w}\times\mathfrak{w}}$. Then $\omega_R = \omega|_{\mathfrak{w}\times\mathfrak{w}} + d\phi_C|_{\mathfrak{w}\times\mathfrak{w}}$, thus the real Witt algebra also has a one dimensional second cohomology space spanned by the Gelfand-Fuks cocycle.

□

To close off, we note that the unique central extension of the Witt algebra is called the Virasoro algebra. The Gelfand-Fuks cocycle is accordingly additionally called the Virasoro cocycle.

Chapter 4

Weak asymptotic symmetries of three dimensional Anti-de-Sitter space

In this chapter, we first look at what an asymptotic symmetry should be in section 4.1. The description of asymptotic symmetries is typically only done on the Lie algebra level, but we show how this Lie algebra can be derived from a group of asymptotic symmetries. These group asymptotic symmetries are diffeomorphisms that preserve the metric asymptotically, which will be defined rigorously first. After describing asymptotic symmetries in general, we give a description of the geometry and isometry group of three dimensional Anti de Sitter space, also known as AdS_3, in section 4.2. Next we derive the asymptotic boundary conditions in section 4.3. To this end we follow the derivation in the seminal paper by Brown and Henneaux [BH86]. However, their boundary conditions are not mathematically rigorous, as they give no definition of what fall-off behavior such as $\mathcal{O}(r^{-1})$ means. Thus we give an exact definition of fall-off behavior in section 4.4, which allows us to finally derive the weak asymptotic symmetry algebra from these boundary conditions in section 4.5.

4.1 Asymptotic symmetries

A symmetry of spacetime is an *isometry*, which is a diffeomorphism that preserves the metric. The metric should be preserved as this is what determines the curvature of spacetime, which means it determines the behavior of gravity. In terms of equations, if M is a Lorentzian manifold with metric g, a diffeomorphism $\phi : M \to M$ is an isometry if

$$\phi^* g = g \tag{4.1}$$

where $\phi^* g$ is the pullback of the metric, which is how the g transforms under the coordinate transformation induced by ϕ. For an introduction to pullbacks and pushforwards from a physical perspective, we refer to Ref. [Car19] and for a mathematical introduction we refer to Ref. [Lee12].

As mentioned in the introduction, a realistic model of the universe will not have many symmetries. It can nevertheless have something called asymptotic symmetries, which intuitively means that the metric is preserved in the limit of $r \to \infty$. To make this exact, we will first define a subset $S \subseteq \Gamma(S^2T^*M)$ of the symmetric $(0,2)$ tensors that we want to call equal asymptotically. As S should be a subset of metrics, at each point $p \in M$, S_p should consist of symmetric $(0,2)$ tensors with the same signature as g. The signature of a metric is only preserved under linear combinations and scaling by positive constants, meaning that S_p should be a cone. We call a fiber bundle $S \subseteq \Gamma(S^2T^*M)$ a cone bundle if S_p is a cone for all $p \in M$. The tangent space at $h \in S$, T_hS is defined as the equivalence classes of smooth curves in S starting at h, with two curves being equivalent if their time derivatives at 0 are equal. These time derivatives are defined pointwise, i.e. $(\frac{\partial}{\partial t}\gamma(t))(p) = \frac{\partial}{\partial t}(\gamma(t)(p))$.

With this in mind, we define the asymptotic symmetries as follows:

Definition 4.1 (Asymptotic symmetry)**.** Let $\phi : M \to M$ be a diffeomorphism and let $S \subseteq \Gamma(S^2T^*M)$ be a cone bundle that is a subset of $\Gamma(S^2T^*M)$, which contains the metrics that we want to call asymptotically equal. Then ϕ is an asymptotic symmetry if for all $h \in S$,

$$\phi^*(h) \in S, (\phi^{-1})^*(h) \in S \tag{4.2}$$

Remark 4.2. Equivalently ϕ is an asymptotic symmetry if $\phi^*(S) = S$ and weak asymptotic symmetries are thus automorphisms on S.

These asymptotic symmetries clearly form a group, which we call the weak asymptotic symmetry group $\mathrm{WAS}(M)$. Even though this need not be a Lie group, we can still associate a Lie algebra of weak asymptotic symmetries to this group:

Definition 4.3. The weak asymptotic symmetry algebra ($\mathfrak{was}$) is given by:

$$\left\{ \left.\frac{d}{dt}\right|_{t=0} \phi_t | \phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1 \right\} \tag{4.3}$$

where smoothness means smoothness of the orbits $\phi_t(p)$ for all $p \in M$.

This definition is not that easy to work with. We want to characterize $\mathfrak{was}$ in terms of a property of the vector field $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. A natural candidate is to consider vector fields ξ such that $\left.\frac{d}{dt}\right|_{t=0} (\phi_t)^*h \in T_hS$, so we need to understand $\left.\frac{d}{dt}\right|_{t=0} (\phi_t)^*h$. If ϕ is a global flow, we know that this is the Lie derivative. It turns out that this is the Lie derivative even if ϕ is not a global flow, as we will show next, starting with the pushforward on vector fields.

Lemma 4.4. *Let $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ with $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. Then for any vector field X on M*

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_t)_*X = -\mathcal{L}_\xi X \tag{4.4}$$

Proof. Let $f \in C^\infty(M), q \in M$. Then the pushforward at q is given by

$$((\phi_t)_*X)_q f = X_{\phi_t^{-1}(q)}(f \circ \phi_t) = X^\alpha_{\phi_t^{-1}(q)} \partial_\alpha (f \circ \phi_t)\big|_{\phi_t^{-1}(q)} = X^\alpha_{\phi_t^{-1}(q)} \partial_\alpha \phi_t^\beta |_{\phi_t^{-1}(q)} \partial_\beta f|_q \tag{4.5}$$

Now taking the derivative at $t = 0$,

$$\frac{d}{dt}\bigg|_{t=0}((\phi_t)_*X)_q f = \frac{d}{dt}\bigg|_{t=0} X^\alpha_{\phi_t^{-1}(q)} \partial_\alpha \phi_t^\beta|_{\phi_t^{-1}(q)} \partial_\beta f = (\frac{d}{dt}\bigg|_{t=0} \phi_t^{-1}(q))^\gamma \partial_\gamma X^\alpha_q \partial_\alpha f + X^\alpha_q \partial_\alpha \xi^\beta_q \partial_\beta f = \\ -\xi^\gamma_q \partial_\gamma X^\alpha_q \partial_\alpha f + X^\alpha_q \partial_\alpha \xi^\beta_q \partial_\beta = [X,\xi]_q f = -(\mathcal{L}_\xi X)_q f \tag{4.6}$$

where $(\frac{d}{dt}\big|_{t=0}\phi_t^{-1}(q)) = -\xi$ follows from the following trick. Note that $q = \phi_t(\phi_t^{-1}(q))$, and thus $\frac{d}{dt}\big|_{t=0}\phi_t(\phi_t^{-1}(q)) = 0$. Computing the left hand side, we find $\frac{d}{dt}\big|_{t=0}\phi_t(\phi_t^{-1}(q)) = \xi_q + \frac{d}{dt}\phi^{-1}|_{t=0}(q) = 0$. □

Lemma 4.5. *Let $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ with $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. Then for any $(0,2)$ tensor field h on M*

$$\frac{d}{dt}\bigg|_{t=0}(\phi_t)^* h = \mathcal{L}_\xi h \tag{4.7}$$

Proof. First, we note that for $f \in C^\infty(M)$, X a vector field on M and $p \in M$, let $\gamma : [0,1] \to M$ be a curve such that $\gamma(0) = p, \dot\gamma(0) = X_p$, then

$$\frac{d}{dt}\bigg|_{t=0}(\phi_t)_* X_p = \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0}\phi_t(\gamma(s)) = \frac{d}{ds}\bigg|_{s=0}\xi^\beta_{\gamma(s)}\partial_\beta = X^\alpha_p \partial_\alpha \xi^\beta_p \partial_\beta \tag{4.8}$$

Next, the pullback on h is given by:

$$(\phi_t^* h)_p(X_p, Y_p) = h_{\phi_t(p)}((\phi_t)_* X_p, (\phi_t)_* Y_p) = (h_{\phi_t(p)})_{\alpha\beta}((\phi_t)_* X_p)^\alpha (\phi_t)_* Y_p)^\beta \tag{4.9}$$

and using the Leibniz rule we take the derivative:

$$\frac{d}{dt}\bigg|_{t=0}(\phi_t^* h)_p(X_p, Y_p) = \frac{d}{dt}\bigg|_{t=0}(h_{\phi_t(p)})_{\alpha\beta}((\phi_t)_* X_p)^\alpha ((\phi_t)_* Y_p)^\beta = \\ \xi^\gamma_p \partial_\gamma (h_p)_{\alpha\beta} X^\alpha_p Y^\beta_p + (h_p)_{\gamma\beta}\partial_\alpha \xi^\gamma_p X^\alpha_p Y^\beta_p + (h_p)_{\alpha\gamma}\partial_\beta \xi^\gamma_p X^\alpha_p Y^\beta_p \tag{4.10}$$

Now one can either recognize immediately that this last expression is the Lie derivative, or note that the last expression only depends on ξ, and not on ϕ_t. In particular, one can take ϕ_t to be the flow generated by ξ and in that case $\frac{d}{dt}\big|_{t=0}(\phi_t^* h)_p(X_p, Y_p) = (\mathcal{L}_\xi h)_p(X_p, Y_p)$ by definition. As the answer is independent of the curve as long as the curve is generated by ξ, it holds for all ϕ_t. □

Remark 4.6. This proof works for all tensor fields, not only $(0,2)$ tensor fields.

We will say that S is integrable if $\mathcal{L}_\xi h \in T_h S$ for all $h \in S$ implies that there exists a curve of diffeomorphisms ϕ_t such that $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ with $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. Under this assumption, $\mathfrak{was}$ is simply the Lie algebra of vector fields ξ such that $\mathcal{L}_\xi h \in T_h S$ for all $h \in S$.

Proposition 4.7. *Suppose S is integrable. Then the weak asymptotic symmetry algebra is given by the vector fields ξ such that $\mathcal{L}_\xi h \in T_h S$ for all $h \in S$.*

Proof. First, let ξ be a vector field in the weak asymptotic symmetry algebra. Then there exists a curve of diffeomorphisms ϕ_t such that $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ with $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. Then clearly for all $h \in S$, $\frac{d}{dt}\big|_{t=0}(\phi_t)^*h \in T_hS$ and by lemma 4.5, $\mathcal{L}_\xi h = \frac{d}{dt}\big|_{t=0}(\phi_t)^*h \in T_hS$.

The converse is given by S being integrable. $\square$

Next, we show that happens when an asymptotic symmetry acts on an element of the tangent space of S, as this will be need later. Asymptotic symmetries preserve S be definition, and the tangent space is simply shifted.

Lemma 4.8. *Let $h \in S$ and let ϕ be an asymptotic symmetry. Then*

$$\phi^*(T_hS) \subseteq T_{\phi^*(h)}S \tag{4.11}$$

Proof. Let $\xi \in T_hS$, then there exists a smooth curve $\gamma : [-1,1] \to S$ such that $\gamma(0) = h$ and $\gamma'(0) = \xi$. Now as ϕ is an asymptotic symmetry, $\phi^*(\gamma(t))$ is again a curve in S, smooth as ϕ is smooth, and $\phi^*(\gamma(0)) = \phi^*(h)$. Furthermore, as ϕ^* is linear and smooth, $\frac{d}{dt}|_{t=0}\phi^*(\gamma(t)) = \phi^*(\xi)$. We conclude that $\phi^*(\xi) \in T_{\phi^*(h)}S$. $\square$

4.1.1 Center of symmetry group

Now we show that elements in the *center* $Z(\mathrm{WAS}(M))$, which is the normal subgroup of elements that commute with every other element, impose restrictions on $\mathfrak{was}$, namely the vector fields in $\mathfrak{was}$ should be invariant under the pushforward of these elements. In fact, we slightly generalize this, by defining $\mathrm{WAS}_0(M)$ to be the component of $\mathrm{WAS}(M)$ path-connected to the identity and defining $Z_0(\mathrm{WAS}(M))$ to be the elements in $\mathrm{WAS}(M)$ that commute with $\mathrm{WAS}_0(M)$. We only need elements to commute with the component path-connected to the identity as this component uniquely determines the Lie algebra.

Proposition 4.9. *Suppose $\psi \in Z_0(\mathrm{WAS}(M))$. Then for $\xi \in \mathfrak{was}$, $\psi_*(\xi) = \xi$.*

Proof. Let $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ with $\xi = \frac{d}{dt}\big|_{t=0}\phi_t$. Then $\phi_t = \psi\phi_t\psi^{-1}$ and thus

$$\xi = \frac{d}{dt}\Big|_{t=0}\phi_t = \frac{d}{dt}\Big|_{t=0}\psi\phi_t\psi^{-1} = \psi_*(\xi) \tag{4.12}$$

$\square$

For large groups of diffeomorphisms it is not easy to see what the center should be. The next proposition shows that a discrete normal subgroup N is in $Z_0(\mathrm{WAS}(M))$. Here discrete means that the orbits $N \cdot x$ are discrete for all $x \in M$.

Proposition 4.10. *Let N be a discrete normal subgroup of* $\mathrm{WAS}(M)$*. Then $N \subseteq Z_0(\mathrm{WAS}(M))$.*

Proof. Let $\phi_t \in C^\infty([0,1], \mathrm{WAS}(M)), \phi_t(0) = 1$ and $\psi \in N$. Then $C : \mathrm{WAS}(M) \to N : \phi \to \phi\psi\phi^{-1}$ is well defined as N is normal and the orbits $(C\phi_t)x$ are discrete for $x \in M$

as N is discrete. Now as $(C\phi_t)x$ is continuous in t with discrete image, it is constant. Thus $(C\phi_t)x = (C\phi_0)x = \psi x$. As x was arbitrary, $(C\phi_t) = \psi$ from which $\phi_t\psi = \psi\phi_t$ is immediate. Thus $\psi \in Z_0(\mathrm{WAS}(M))$. □

Lastly we note that the subgroup of pointwise asymptotic symmetries, $T = \{\psi \in \mathrm{WAS}(M) | \psi(g) = g \forall g \in S\}$, is a normal subgroup.

Corollary 4.11. *Suppose $T = \{\psi \in \mathrm{WAS}(M) | \psi(g) = g \forall g \in S\}$ is discrete. Then $\psi_*(\xi) = \xi$ for all $\xi \in \mathfrak{was}, \psi \in T$.*

4.1.2 Trivial symmetries

The weak asymptotic symmetry algebra is actually too large for most applications. We only want the 'useful' symmetries. What 'useful' means can differ by application, for example if one is interested in conserved quantities, useful could mean that the symmetry defines a nontrivial conserved quantity. A trivial symmetry is then a non-useful symmetry, i.e. a symmetry that leads to a zero conserved charge. This is actually the standard definition of a trivial symmetry.

We will take a different approach, by considering a symmetry useful if it is nonzero in every projective unitary positive energy representation of the asymptotic symmetry algebra. In that case, it corresponds to a particle with energy that is bounded from below, which we expect a physical particle to satisfy. The corresponding trivial symmetries can be determined from proposition 3.12 by first calculating the cohomology of the weak asymptotic symmetry algebra. Note that the trivial symmetries form an ideal as they are given by the intersection of kernels of Lie algebra homomorphisms.

A slight issue with this approach is that $\mathrm{WAS}(M)$ might very well not be a Lie group. We can therefore not use proposition 3.12 to determine which symmetries are trivial in every projective unitary positive energy representation of the weak asymptotic symmetries. We will therefore instead define weakly trivial symmetries, which become trivial symmetries when a Lie group structure is available. These are determined more easily and by requiring that the actual asymptotic symmetry group has a Lie group structure, we ensure that the notions of weakly trivial and trivial symmetries are equivalent in the asymptotic symmetry group. As the trivial symmetries form a closed ideal, we can also define the weakly trivial symmetry algebra to be a closed ideal.

Definition 4.12 (Trivial symmetries)**.**

- Let $\mathfrak{g}$ be a Lie algebra. An element $\eta \in \mathfrak{g}$ is a trivial symmetry if $\rho(\eta) = 0$ for all projective unitary positive energy representations ρ of $\mathfrak{g}$.
- Let ξ be the positive energy element in $\mathfrak{was}(M)$. The weakly trivial symmetry algebra $\mathfrak{t}$ is the closure of the ideal of the weak asymptotic algebra generated by the vector fields $[\xi, \eta]$ such that $[[\xi, \eta], \eta] = 0$ and $\omega([\xi, \eta], \eta) = 0$ for all 2-cocycles ω.

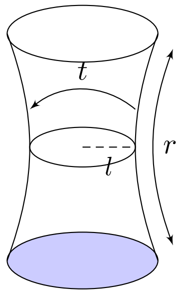

Figure 4.1: Two dimensional Anti-de Sitter space. l is the radius of the space, t is an angular coordinate and r is the radial coordinate. Identical to figure 1.1

- The asymptotic symmetry group $\mathrm{AS}(M)$ is the largest quotient group of $\mathrm{WAS}(M)$ such that $\mathrm{AS}(M)$ is a Lie group and such that the Lie algebra $\mathfrak{as}(M)$ contains no trivial symmetries.

We will now explain that these weakly trivial symmetries are a correct generalization of trivial symmetry. First suppose that $\mathrm{WAS}(M)$ is a Lie group. Then again by proposition 3.12 every weakly trivial symmetry is a trivial symmetry. Now if $\mathrm{WAS}(M)$ is not a Lie group, but $\mathfrak{i} \subset \mathfrak{t}$ is some ideal such that $\mathfrak{was}/\mathfrak{i}$ has a corresponding Lie group, then every weakly trivial symmetry $[\xi, \eta] \notin \mathfrak{i}$ turns into a trivial symmetry. Indeed, let $\tilde{\omega}$ be a 2-cocycle on $\mathfrak{was}/\mathfrak{i}$ and let $\pi : \mathfrak{was} \to \mathfrak{was}/\mathfrak{i}$ be the projection. Then $\pi^*\tilde{\omega}$ is a 2-cocycle on $\mathfrak{was}$ and hence $\tilde{\omega}([[\xi, \eta]], [\eta]) = \pi^*\tilde{\omega}([\xi, \eta], \eta) = 0$ and thus by proposition 3.12 η is a trivial symmetry. We conclude that whenever $\mathfrak{was}/\mathfrak{i}$ has a corresponding Lie group, the weakly trivial symmetries turn into trivial symmetries. By requiring the asymptotic symmetry group AS to be a Lie group we therefore ensure that not containing any trivial symmetries at least implies not containing any weakly trivial symmetries.

4.2 Symmetries of 3-dimensional Anti de Sitter space

As mentioned in the introduction, we are mainly interested in three dimensional Anti de Sitter space, AdS_3 in short. In this section we will give a short introduction to this space and its isometry group.

AdS_3 is defined as the maximally symmetric space with negative cosmological constant [Car19]. It is a submanifold of $\mathbb{R}^4$ with metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 - du^2 - dv^2 \tag{4.13}$$

and the submanifold being given as:

$$\mathrm{AdS}_3 = \{(x, y, u, v) \in \mathbb{R}^4 | u^2 + v^2 = l^2 + x^2 + y^2\} \tag{4.14}$$

This is a hyperboloid, which in 2 dimensions looks like figure 4.1. In three dimensions, instead of for each r there being a circle of points, there is a torus of points for each r.

The constant l is a length scale, the major radius of the torus at $r = 0$, given by the cosmological constant (note that for higher dimensional AdS spaces the proportionality factor is different):

$$l^2 = -\frac{1}{\Lambda} \tag{4.15}$$

The isometry group of AdS_3 is then the group of diffeomorphisms, which can in this case be written as 4×4 matrices, $M_{4,4}$ that preserve the metric g in equation 4.13, as in equation 4.1. This group is called $O(2,2)$:

$$O(2,2) = \{A \in M_{4,4} | A^T D A = D\} \tag{4.16}$$

where $D = \mathrm{diag}(1, 1, -1, -1)$ and we used that $A^* D = A^T D A$.

$O(2,2)$ is a matrix Lie group, the Lie algebra of which is given by

$$\mathfrak{o} = \{X \in M_{4,4} | \exp(tX) \in O(2,2) \forall t\} \tag{4.17}$$

By definition of $O(2,2)$, this means that $(e^{tX})^T D e^{tX} = D$ for all t, and in particular the t derivative at $t = 0$ should zero.

$$0 = \frac{d}{dt}|_{t=0}(e^{tX})^T D e^{tX} = X^T D + DX \tag{4.18}$$

As $D^2 = 1$, this is equivalent to:

$$\mathfrak{o}(2,2) \subseteq \{X \in M_{4,4} | DXD = -X^T\} \tag{4.19}$$

Furthermore, if $X \in \{X \in M_{4,4} | DXD = -X^T\}$, then $(e^{tX})^T D e^{tX} D = e^{tX^T} e^{-tX^T} = 1$, so $(e^{tX})^T D e^{tX} = D$ and $\exp(tX) \in O(2,2) \forall t$, thus

$$\mathfrak{o}(2,2) = \{X \in M_{4,4} | DXD = -X^T\} \tag{4.20}$$

It can easily be checked that this is a six-dimensional space, with the following matrices:

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ b & d & 0 & f \\ c & e & -f & 0 \end{pmatrix} \tag{4.21}$$

for $a, b, c, d, e, f \in \mathbb{R}$. A maximally symmetric space of dimension n should have $n + \frac{1}{2}n(n-1)$ linearly independent symmetries, n "translations" and $\frac{1}{2}n(n-1)$ "rotations" [Car19]. These "translations" and "rotations" are only global translations and rotations in a flat space. Locally however, we can always find n symmetries that act locally as translations and $\frac{1}{2}n(n-1)$ that locally act as rotations. For $n = 3$ this means that a maximally symmetric space has six linearly independent symmetries, thus AdS_3 is indeed maximally symmetric.

In terms of the coordinates u, v, x, y we can write the $\mathfrak{o}(2,2)$ symmetries as:

$$\begin{aligned} \xi_1 &= u\partial_v - v\partial_u, \quad \xi_2 = x\partial_y - y\partial_x, \quad \xi_3 = u\partial_y + y\partial_u \\ \xi_4 &= v\partial_x + x\partial_v, \quad \xi_5 = u\partial_x + x\partial_u, \quad \xi_6 = v\partial_y + y\partial_v \end{aligned} \tag{4.22}$$

The interpretation of these symmetries is not so clear. This is mainly because the coordinate system u, v, x, y does not have a clear interpretation. By instead parameterizing AdS_3 with the following coordinates r, t, ϕ:

$$\begin{aligned} u &= \sqrt{l^2 + r^2}\cos(t/l) \\ v &= \sqrt{l^2 + r^2}\sin(t/l) \\ x &= r\cos(\phi) \\ y &= r\sin(\phi) \end{aligned} \tag{4.23}$$

we get a clearer interpretation. Indeed, r is then a radial coordinate as for large r u, v, x, y are all proportional to r and t, ϕ are both angular coordinates. t is however also the time coordinate, which is typically not an angular or periodic coordinate, as otherwise time runs in a loop! Therefore the *universal cover* of AdS_3 is typically considered, which means that the hyperboloid in figure 4.1 is rolled out along the t coordinate, leading to t being an arbitrary real coordinate and no longer being periodic. For now, we will not take this approach and we keep using the original AdS_3 space.

Now we also want to write the metric, equation 4.13 and the symmetries, equation 4.22, in terms of these coordinates, so that those also get a clearer interpretation. Therefore we first need to know how the differentials and basis vectors transform under this coordinate transformation:

$$\begin{aligned} du &= \frac{r}{\sqrt{l^2 + r^2}}\cos(t/l)dr - \frac{\sqrt{l^2 + r^2}}{l}\sin(t/l)dt \\ dv &= \frac{r}{\sqrt{l^2 + r^2}}\sin(t/l)dr + \frac{\sqrt{l^2 + r^2}}{l}\cos(t/l)dt \\ dx &= \cos(\phi)dr - r\sin(\phi)d\phi \\ dy &= \sin(\phi)dr + r\cos(\phi)d\phi \end{aligned} \tag{4.24}$$

$$\begin{aligned} \partial_r &= \frac{r}{\sqrt{l^2 + r^2}}(\cos(t/l)\partial_u + \sin(t/l)\partial_v) + \cos(\phi)\partial_x + \sin(\phi)\partial_t \\ \partial_t &= \frac{\sqrt{l^2 + r^2}}{l}(\cos(t/l)\partial_v - \sin(t/l)\partial_u) \\ \partial_\phi &= r(\cos(\phi)\partial_y - \sin(\phi)\partial_x) \end{aligned} \tag{4.25}$$

Knowing this, we can now write down the metric:

$$ds^2 = -(1 + r^2/l^2)dt^2 + \frac{dr^2}{1 + r^2/l^2} + r^2 d\phi^2 \tag{4.26}$$

As mentioned in the introduction, in the limit $\Lambda \to 0$, or equivalently $l \to \infty$, we should get flat space. Indeed, in this limit the metric becomes:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 \tag{4.27}$$

which is exactly the flat metric in polar coordinates.

Now the diffeomorphism $PT : (r, t/l, \phi) \to (r, t/l + \pi, \phi + \pi)$ or $PT(u, v, x, y) = (-u, -v, -x, -y)$ is central in $O(2, 2)$ as $PT = -I$ in the u, v, x, y-coordinates, and therefore by proposition 4.9 $PT_*\xi = \xi$, where we note that $O(2, 2)$ is a weak asymptotic symmetry group with $S = \{g_{AdS}\}$. As $PT_*(\xi^\mu \partial_\mu) = \xi^\mu \circ PT\partial_\mu$ for $\mu \in \{r, t, \phi\}$, we find that for $\xi^\mu\partial_\mu$, $\xi^\mu(r, t/l + \pi, \phi + \pi) = \xi^\mu(r, t/l, \phi)$.

Lastly, in terms of the r, ϕ, t coordinates, $\mathfrak{o}(2, 2)$ is given by the basis:

$$\begin{aligned}
\xi_1 &= l\partial_t \\
\xi_2 &= \partial_\phi \\
\xi_3 &= \sqrt{l^2 + r^2}\cos(t/l)\sin(\phi)\partial_r - \frac{rl}{\sqrt{l^2 + r^2}}\sin(t/l)\sin(\phi)\partial_t + \frac{\sqrt{l^2 + r^2}}{r}\cos(t/l)\cos(\phi)\partial_\phi \\
\xi_4 &= \sqrt{l^2 + r^2}\sin(t/l)\cos(\phi)\partial_r + \frac{rl}{\sqrt{l^2 + r^2}}\cos(t/l)\cos(\phi)\partial_t - \frac{\sqrt{l^2 + r^2}}{r}\sin(t/l)\sin(\phi)\partial_\phi \\
\xi_5 &= \sqrt{l^2 + r^2}\cos(t/l)\cos(\phi)\partial_r - \frac{rl}{\sqrt{l^2 + r^2}}\sin(t/l)\cos(\phi)\partial_t - \frac{\sqrt{l^2 + r^2}}{r}\cos(t/l)\sin(\phi)\partial_\phi \\
\xi_6 &= \sqrt{l^2 + r^2}\sin(t/l)\sin(\phi)\partial_r + \frac{rl}{\sqrt{l^2 + r^2}}\cos(t/l)\sin(\phi)\partial_t + \frac{\sqrt{l^2 + r^2}}{r}\sin(t/l)\cos(\phi)\partial_\phi
\end{aligned} \tag{4.28}$$

which can be verified by plugging in the basis vectors in equation 4.25 and comparing with equation 4.22. This is quite a tedious computation and can also be done using a symbolic calculator, see appendix A.

Thus there are translational symmetries in the t, ϕ directions, which is expected as the metric in equation 4.26 does not depend on t, ϕ, and 4 additional symmetries, all of the following form:

$$\xi = \sqrt{l^2 + r^2}X(t, \phi)\partial_r + \frac{rl^2}{\sqrt{l^2 + r^2}}\partial_t X(t, \phi)\partial_t + \frac{\sqrt{l^2 + r^2}}{r}\partial_\phi X(t, \phi)\partial_\phi \tag{4.29}$$

for some function X periodic in both t and ϕ.

4.3 Boundary conditions

To define the weak asymptotic symmetry group, we first need to know what the cone bundle S of asymptotically equal metrics is, as shown in section 4.1. We will do this by defining when a metric is asymptotically Anti de Sitter, following the conventions in physics. The conditions for calling a metric asymptotically AdS are typically called boundary conditions, as they are often conditions on the metric for $r \to \infty$ and $r = \infty$ is seen as a boundary. For AdS_3, these boundary conditions were derived in Ref. [BH86], and we will partially replicate the motivation for these boundary conditions. We would like our boundary conditions to at least contain all the metrics of conic defects of AdS_3, which are the metrics corresponding to a point mass in AdS_3. Let α, A be arbitrary real numbers, then the conic defect metric corresponding to α, A is given by:

$$g_{\mu\nu}dx^\mu dx^\nu = -(\frac{r^2}{l^2} + \alpha^2)dt^2 + 2A\alpha dt d\phi + \left(\frac{r^2 - A^2}{l^2} + \alpha^2\right)^{-1} dr^2 + (r^2 - A^2)d\phi^2 \quad (4.30)$$

Furthermore, we want our definition of asymptotically AdS_3 metrics to contain at least all of the metrics obtained by applying a diffeomorphism in $O(2,2)$. Equivalently, all of the following metrics should satisfy the boundary conditions:

$$g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} \quad \text{is asymptotically AdS for } \xi \text{ in } \mathfrak{o}(2,2) \text{ and } g \text{ asymptotically AdS} \quad (4.31)$$

As $g_{\mu\nu}$ is not a function of t, ϕ, the symmetries ξ_1, ξ_2 are exact symmetries of the conic defects and $\mathcal{L}_\xi g_{\mu\nu} = 0$ in that case. The Lie derivatives for ξ_3 were calculated using a symbolic calculator and are shown in Appendix B. There, we find for $\xi = \xi_3$

$$\begin{pmatrix} \mathcal{L}_\xi g_{rr} & \mathcal{L}_\xi g_{rt} & \mathcal{L}_\xi g_{r\phi} \\ \mathcal{L}_\xi g_{tr} & \mathcal{L}_\xi g_{tt} & \mathcal{L}_\xi g_{t\phi} \\ \mathcal{L}_\xi g_{\phi r} & \mathcal{L}_\xi g_{\phi t} & \mathcal{L}_\xi g_{\phi\phi} \end{pmatrix} \sim \begin{pmatrix} \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \quad (4.32)$$

ξ_4, ξ_5, ξ_6 give the same result as ξ_3 as they are all of the same form, see equation 4.29.

The boundary conditions for an asymptotically AdS_3 space are thus:

$$\begin{pmatrix} g_{rr} & g_{r\phi} & g_{rt} \\ g_{\phi r} & g_{\phi\phi} & g_{\phi t} \\ g_{tr} & g_{t\phi} & g_{tt} \end{pmatrix} \sim \begin{pmatrix} \frac{l^2}{r^2} + \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & r^2 + \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & -\frac{r^2}{l^2} + \mathcal{O}(1) \end{pmatrix} \quad (4.33)$$

Intuitively, $f \sim \mathcal{O}(r^{-n})$ means that f behaves as r^{-n} for large r, but an exact definition will be given in the next section.

The analysis will be simpler in light-cone coordinates, defined as

$$x^\pm = \frac{t}{l} \pm \phi \quad (4.34)$$

from which we also find $dx^\pm = \frac{1}{l}dt \pm d\phi$. These are called light-cone coordinates because the unit vectors $\partial_{x^\pm}$ become orthogonal null vectors in the limit $r \to \infty$, meaning that they span the light-cone in this limit. With these coordinates, and writing $\pm$ for $x^\pm$ in the indices, the boundary conditions become:

$$\begin{pmatrix} g_{rr} & g_{r+} & g_{r-} \\ g_{+r} & g_{++} & g_{+-} \\ g_{-r} & g_{-+} & g_{--} \end{pmatrix} \sim \begin{pmatrix} \frac{l^2}{r^2} + \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) - \frac{1}{2}r^2 \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) - \frac{1}{2}r^2 & \mathcal{O}(1) \end{pmatrix} \quad (4.35)$$

We note that a periodic function in t, ϕ is not necessarily periodic in x^+, x^- separately, as for example $\sin(\phi) = \sin(\frac{x^+ - x^-}{2})$. However, $\xi \in \mathfrak{o}(2,2)$ is periodic in x^+, x^- separately due to $\xi(r, t/l + \pi, \phi + \pi) = \xi(r, t/l, \phi)$ and as the conic sections are also periodic in x^+, x^-, $\mathcal{L}_\xi g$ is as well. We thus require a general asymptotically AdS metric to be periodic in x^+, x^- as well and we take our boundary conditions to be as in equation 4.35 with $f \in \mathcal{O}(r^{-a})$ meaning that f is periodic in x^+, x^- and $\mathcal{O}(r^{-a})$ in the r coordinate, the precise meaning of which will be given in the next section.

We now define the manifold $S \subseteq \Gamma(S^2 T_* M)$ as follows. $h \in S$ if h has the same $(-, +, +)$ signature as the conic defect metrics, h is periodic in x^+, x^- and if h satisfies the boundary conditions in equation 4.35. Then for $h \in S$, $g \in T_h S$ if:

$$\begin{pmatrix} g_{rr} & g_{r+} & g_{r-} \\ g_{+r} & g_{++} & g_{+-} \\ g_{-r} & g_{-+} & g_{--} \end{pmatrix} \sim \begin{pmatrix} \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{4.36}$$

4.4 Fall-off algebra

The boundary conditions needed to specify asymptotic symmetries, are often written by specifying certain fall-off behavior of the metric, e.g. $g_{r\phi}$ is of order r^{-3}. We want a somewhat stricter behavior of what this fall-off behavior means then the standard, algorithmic, big O notation, which is commonly seen in the physics literature. The reason for this is that in the standard big O notation, we have the following[Obl17]:

$$\frac{\sin(r^{41})}{r} \text{ is of order } 1/r \tag{4.37}$$

$$\partial_r \frac{\sin(r^{41})}{r} \text{is of order } r^{40}, \tag{4.38}$$

whereas we would prefer if for any f of order r^x, $\partial_r f$ would be of order r^{x-1}. We will therefore use the following definition, where we let $\Omega = 1/r$:

Definition 4.13 (Fall-off algebra)**.** Let $a \in (0, \infty)$ be a positive real number and $n \in \mathbb{N}$. A smooth function $f : D \supseteq (a, \infty) \to \mathbb{R}$, is said to be $\mathcal{O}_a(r^{-n})$ if $\hat{f} : F \supseteq (0, 1/a) \to \mathbb{R}$: $\hat{f}(\Omega) = f(1/r)$ extends smoothly to $\hat{F} = F \cup (-\epsilon, 0]$ for some $\epsilon > 0$ and has vanishing derivatives up to order n at 0: $\partial_\Omega^k \hat{f}|_{\Omega=0} = 0$ for $0 \leq k < n$. For $n = 0$, we only need $\hat{f}$ to extend smoothly to an open neighborhood of 0 and do not put any constraints on the value or derivatives at 0. Furthermore, we say $f \sim \mathcal{O}_a(r^n)$ if $1/r^n f \sim \mathcal{O}_a(1)$. Lastly for $k \in \mathbb{Z}$, we define $\mathcal{O}_a(r^k)$ to be the set of all $\mathcal{O}_a(r^k)$ functions.

Typically we will only be interested in the asymptotic behavior of functions and the constant a will not matter. We will therefore not write it, unless strictly necessary. Do note that when we write $\mathcal{O}(r^n)$, all functions $f \in \mathcal{O}(r^n)$ share the same constant a. Let us now give some properties and examples of this definition.

Proposition 4.14. *For $n, m \in \mathbb{N}, a, b \in \mathbb{Z}$:*

1. *$\mathcal{O}(r^a)$ is a vector space with pointwise addition*

2. *If $a \leq b$, $\mathcal{O}(r^a) \subseteq \mathcal{O}(r^b)$*

3. *With pointwise multiplication, $\mathcal{O}(r^{-m})\mathcal{O}(r^{-n}) \subseteq \mathcal{O}(r^{-(m+n)})$*

4. *In particular $\mathcal{O}(r^{-m})$ with pointwise multiplication is a non-unital associative algebra.*

5. *For $n \leq m$, $r^n\mathcal{O}(r^{-m}) = \mathcal{O}(r^{n-m})$*

6. *If $n \geq 2$ and $f \in \mathcal{O}(r^{-n})$, then $\int_r^\infty f(t)dt \in \mathcal{O}(r^{-(n-1)})$*

7. *For $n \geq 1$, $\partial_r\mathcal{O}(r^{-n}) = \mathcal{O}(r^{-(n+1)})$ and $\partial_r\mathcal{O}(1) \subseteq \mathcal{O}(r^{-2})$*

8. *If $f \in \mathcal{O}(1)$, we can define $\tilde{f} = f - \hat{f}(0)$ with $\tilde{f} \in \mathcal{O}(r^{-1})$*

Proof. 1,2,8 follow immediately from the definition. 3 can be proven using the general Leibniz rule. Let $f \in \mathcal{O}(r^{-m}), g \in \mathcal{O}(r^{-n})$, then

$$\partial_\Omega^k fg = \sum_{l=0}^{k} \binom{k}{l} f^{(l)} g^{(k-l)}, \tag{4.39}$$

Now if $k < m+n, l \leq k$, either $k-l < n$ or $l < m$. In both cases $f^{(l)}g^{(k-l)}$ is zero in $\Omega = 0$. 4 simply combines 1,2,3.

For 5, we start with the forward inclusion $r^n\mathcal{O}(r^{-m}) \subseteq \mathcal{O}(r^{n-m})$ for $n = 1$, so $m \geq 1$. The inclusion for $n > 1$ then follows from repeatedly applying the $n = 1$ case. Suppose $f \in \mathcal{O}(r^{-m})$ and let U be an open neighborhood of $\Omega = 0$ on which $\hat{f}$ is defined and smooth. Then by Hadamard's lemma $\hat{f}(\Omega) = \Omega g(\Omega)$ for some smooth g defined on U. Clearly $\frac{1}{\Omega}\hat{f}(\Omega) = g(\Omega)$ is defined on U and by the general Leibniz rule:

$$\partial_\Omega^k \hat{f}(\Omega)\big|_{\Omega=0} = \partial_\Omega^k \Omega g(\Omega)\big|_{\Omega=0} = \left(\Omega g^{(k)}(\Omega) + k g^{(k-1)}(\Omega)\right)\Big|_{\Omega=0} = k g^{(k-1)}(0) \tag{4.40}$$

and hence $g^{(k)}(0) = 0$ for $0 \leq k < m-1$, which proves that $\frac{1}{\Omega}\hat{f}(\Omega) \in \mathcal{O}(r^{1-m})$. Now suppose $f \in \mathcal{O}(r^{n-m})$. Then by 3, $r^{-n}f \in \mathcal{O}(r^m)$ and hence $f \in r^n\mathcal{O}(r^m)$.

The proof of 6 is as follows: First, $\int_r^\infty f(t)dt = \int_0^\Omega \hat{f}(\tilde{\Omega})/\tilde{\Omega}^2 d\tilde{\Omega}$. By 5, there exists $\hat{g} \in \mathcal{O}(r^{-(n-2)})$ such that $\hat{g}(\Omega) = \Omega^2\hat{f}(\Omega)$, so $\int_r^\infty f(t)dt = \int_0^\Omega \hat{g}(\tilde{\Omega})d\tilde{\Omega}$. As $\hat{g}$ extends smoothly to a neighborhood of 0, its integral does as well. Furthermore, by the fundamental theorem of calculus, for $k \geq 1$:

$$\partial_\Omega^k|_{\Omega=0} \int_0^\Omega \hat{g}(\tilde{\Omega})d\tilde{\Omega} = \partial_\Omega^{k-1}|_{\Omega=0}\hat{g}(\Omega) = 0 \quad \text{for } k-1 < n-2 \tag{4.41}$$

whereas the value of the integral at $\Omega = 0$ is clearly 0. Thus the integral of g is in $\mathcal{O}(r^{-(n-1)})$ and we conclude $\int_r^\infty f(t)dt = \int_0^\Omega \hat{g}(\tilde{\Omega})d\tilde{\Omega} \in \mathcal{O}(r^{-(n-1)})$.

Next, the forward inclusion of 7 for $n \geq 1$, $\partial_r\mathcal{O}(r^{-n}) \subseteq \mathcal{O}(r^{-(n+1)})$ can be shown using

$$\partial_r f = -\Omega^2\partial_\Omega\hat{f}. \tag{4.42}$$

As $\partial_\Omega f$ is $\mathcal{O}(r^{-n+1})$ if $f \in \mathcal{O}(r^{-n})$, and $\Omega^2 \in \mathcal{O}(r^{-2})$, the result $\Omega^2\partial_\Omega f \in \mathcal{O}(r^{-(n+1)})$ follows from 3. Now assume $f \in \mathcal{O}(r^{-(n+1)})$. Then $g := -\int_r^\infty f(t)dt \in \mathcal{O}(r^{-n})$ and $f = \partial_r g$, thus $f \in \partial_r\mathcal{O}(r^{-n})$. For $n = 1$, note that $\tilde{\hat{f}}(\Omega) := \hat{f}(\Omega) - \hat{f}(0)$ is $\mathcal{O}(r^{-1})$, so $\partial_r f = \partial_r \tilde{f} \in \mathcal{O}(r^{-2})$

□

Remark 4.15. From property 5 we see that we could have alternatively defined $\mathcal{O}(r^a) = r^a\mathcal{O}(1)$ for $a \neq 1$, with $\mathcal{O}(1)$ defined in the same way as now, or using property 7 we could equivalently have defined $\mathcal{O}(r^{-(n+1)}) = \partial_r^n\mathcal{O}(r^{-1})$.

Remark 4.16. Note that $1/\sqrt{r} \notin \mathcal{O}(1)$ as $\sqrt{\Omega}$ does not extend smoothly to $\Omega = 0$. One could use an alternative definition which does include these type of fractal functions, by for example requiring a function and all its derivatives to have a smooth limit as $r \to \infty$, but this broadens the class of $\mathcal{O}(1)$ functions immensely. Furthermore, the usage of a compactification and $\Omega = \frac{1}{r}$ coordinates is standard, and we follow this convention, leading to the exclusion of fractional power functions.

As $1/r = \Omega$ clearly extends to $\Omega = 0$, with $\partial_\Omega^0\Omega|_{\Omega=0} = \Omega|_{\Omega=0} = 0$, we see that $1/r \in \mathcal{O}(r^{-1})$ as expected. By using property 6 we also find $1/r^n \in \mathcal{O}(r^{-n})$. Lastly, $\frac{\sin(r^{41})}{r} \notin \mathcal{O}(r^{-1})$ as it does not extend smoothly to $\Omega = 0$, which can be seen from the first derivative diverging as $\Omega \to 0$.

We further define $\mathcal{O}(\infty) = \bigcap_{k=0}^\infty \mathcal{O}(r^{-k})$. Clearly $\mathcal{O}(r^{-n})\mathcal{O}(\infty) \subseteq \mathcal{O}(\infty), \partial_r\mathcal{O}(\infty) \subseteq \mathcal{O}(\infty)$, and it is an algebraic ideal. The functions in $\mathcal{O}(\infty)$ are the functions that are flat at $\Omega = 0$, meaning that all derivatives at $\Omega = 0$ are zero. Examples include e^{-r} and functions that are zero for all r greater than some r'.

We will next deal with functions on $(a, \infty) \times S^1 \times S^1$ for some arbitrary real number a and we want to define when such a function is $\mathcal{O}(1)$. This is done by saying that $f : (r, x^+, x^-) \to \mathbb{R} \in \mathcal{O}(1)$ if $f(\cdot, x^+, x^-) : r \to \mathbb{R} \in \mathcal{O}(1)$ for all x^+, x^- and $\mathcal{O}(r^{-m}) = r^{-m}\mathcal{O}(1)$. Then we note that for $f : (r, x^+, x^-) \to \mathbb{R}$ smooth, if $f \in \mathcal{O}(r^{-m})$, the partial derivatives have the same fall-off behavior: $\partial_+ f, \partial_- f \in \mathcal{O}(r^{-m})$, as $r^m\partial_+ f, r^m\partial_- f \in \mathcal{O}(1)$.

4.4.1 Topology

We now give $\mathcal{O}(1)$ a topology. This topology will later be used to give the weak asymptotic symmetry algebra a topology. A standard topology for smooth functions on a manifold is the compact-open topology, which in our case is defined as follows. First note that the functions in $\mathcal{O}(1)$ are actually smooth functions on $[0, a) \times S^1 \times S^1$ in the Ω, x^+, x^- coordinates. Let $U = [0, a) \times S^1 \times S^1$ and let $(K_n)_{n\in N}$ be an increasing sequence of compact subsets of U, such that all K_i contain the boundary $\Gamma := \{0\} \times S^1 \times S^1$ and their union is U. Furthermore take $\boldsymbol{\alpha} = (k, l, m)$ to be a multi-index with $|\boldsymbol{\alpha}| = k + l + m$ and $\partial^{\boldsymbol{\alpha}} = \partial_\Omega^k\partial_{x^+}^l\partial_{x^-}^m$ to be the partial derivative corresponding to the coordinates Ω, x^+, x^+. We then define the following seminorms:

$$p_{K_j,n}(f) = \sup_{x\in K_j} \max_{|\alpha|\leq n} |\partial^{\boldsymbol{\alpha}} f(x)| \tag{4.43}$$

We define the topology to be the one induced by these seminorms:

Definition 4.17 (Topology on $\mathcal{O}(1)$)**.**
- The topology on $\mathcal{O}(1)$ is defined to be C^∞ topology, which is generated by the following family of seminorms $\mathcal{P}$:

$$p_{K_j,n}(f) = \sup_{x\in K_j} \max_{|\alpha|\leq n} |\partial^{\boldsymbol{\alpha}} f(x)| \tag{4.44}$$

  $\mathcal{P} = \{p_{K_i,n} \quad \forall i, n \geq 0\}$.

- We further define the C^k topology to be the topology generated by $\mathcal{P}_k = \{p_{K_i,n} \quad \forall i \geq 0, k \geq n \geq 0\}$.

This turns $\mathcal{O}(1)$ into a locally convex topological vector space, so addition and scalar multiplication are continuous. Furthermore, multiplication of functions is also continuous, the proof of which is the same as for lemma 3.37. Lastly, $p_{K_i,n}(\partial_\Omega^a \partial_+^b \partial_-^c f) \leq p_{K_i,n+a+b+c}(f)$, showing that derivatives are continuous as well.

We now show that these families of seminorms are equivalent to the family of seminorms given by

$$\tilde{p}_{K_j,k,l,m} = \sup_{x\in K_j} |\partial_\Omega^k \partial_+^l \partial_-^m f(x)| \tag{4.45}$$

Lemma 4.18. *The family of seminorms P is equivalent to the family of seminorms $\tilde{P} = \{\tilde{p}_{K_j,k,l,m} \quad \forall i, k, l, m \geq 0\}$*

Proof. This is immediate from $\tilde{p}_{K_j,k,l,m} \leq p_{K_j,(k+l+m)}$ and $p_{K_j,n} \leq \max_{k+l+m\leq n} p_{K_j,k,l,m}$. □

We will now go on to show that the subspaces $\mathcal{O}(\Omega^a)$ are closed in $\mathcal{O}(1)$. To this end, we note that a subset $A \subseteq X$ of a locally convex topological vector space is closed if and only if the quotient X/A is Hausdorff. We will therefore first compute the quotient:

Lemma 4.19. *The topology on $\mathcal{O}(1)/\mathcal{O}(\Omega^a)$ is given by the following family of seminorms $\mathcal{P}$:*

$$p_{k,l,m}(f) = \sup\{|\partial_\Omega^k \partial_+^l \partial_-^m f|_{\Omega=0,x^+,x^-}| x^+, x^- \in \mathbb{S}^1\} \tag{4.46}$$

$\mathcal{P} = \{p_{k,l,m} \quad \forall k < a, l, m \geq 0\}$

Proof. By lemma 2.6, the seminorms of this quotient topology are given by:

$$\bar{p}_{K_j,k,l,m}(f) = \inf_{g\in\mathcal{O}(\Omega^a)} \tilde{p}_{K_j,k,l,m}(f+g) = \inf_{g\in\mathcal{O}(\Omega^a)} \sup_{x\in K_j} |\partial_\Omega^k \partial_+^l \partial_-^m (f+g)| \tag{4.47}$$

Clearly $\bar{p}_{K_j,k,l,m}(f) \geq p_{k,l,m}$. Now take $g = \sum_{i=0}^{a-1} \frac{\Omega^i}{i!} \partial_\Omega^i f|_{\Omega=0} - f$, then $g \in \mathcal{O}(\Omega^a)$ and

$$\begin{aligned}\tilde{p}_{K_j,k,l,m}(f+g) &= \sup_{x\in K_j} |\partial_\Omega^k \partial_+^l \partial_-^m \sum_{i=0}^{a-1} \frac{\Omega^i}{i!} \partial_\Omega^i f|_{\Omega=0}| \leq \sum_{i=0}^{a-1} \sup_{x\in K_j} |\partial_\Omega^k \partial_+^l \partial_-^m \frac{\Omega^i}{i!} \partial_\Omega^i f|_{\Omega=0}| \\ &\leq \sum_{i=0}^{a-1} \sup_{\Omega\leq M} \partial_\Omega^k \frac{\Omega^i}{i!} \sup_{x\in K_j} |\partial_+^l \partial_-^m \partial_\Omega^i f|_{\Omega=0}| \leq \sum_{i=0}^{a-1} C_i p_{i,l,m}(f)\end{aligned} \tag{4.48}$$

where $M = \sup_{x \in K_j} \Omega$ and $C_i = \partial_\Omega^k \frac{\Omega^i}{i!}$ are constants independent of f. We conclude that the family of quotient seminorms is equivalent to the family of seminorms $p_{k,l,m}$.

□

This is again Hausdorff, as if $p_{k,0,0}(f) = 0$ for all $k < a$, then $f \in \mathcal{O}(\Omega^a)$ and f is thus zero in the quotient. We conclude that $\mathcal{O}(\Omega^a)$ is closed in $\mathcal{O}(1)$ and with the subspace topology $\mathcal{O}(\Omega^a)$ is also closed in $\mathcal{O}(\Omega^b)$ for all $b \leq a$.

We now demonstrate that $\mathcal{O}(\Omega^a)$ is complemented, meaning that there exists a continuous linear projection $\pi : \mathcal{O}(1) \to \mathcal{O}(\Omega^a)$. We define this projection by

$$\pi : f \to f - \sum_{k=0}^{a-1} \partial_\Omega^k f|_{\Omega=0} \frac{\Omega^k}{k!} \tag{4.49}$$

which is clearly linear. It is continuous as addition, scalar multiplication, function multiplication and differentiation are all continuous. The projection indeed maps into $\mathcal{O}(\Omega^a)$ as $\partial_\Omega^l (f - \sum_{k=0}^{a-1} \partial_\Omega^k f|_{\Omega=0} \frac{\Omega^k}{k!})|_{\Omega=0} = 0$ for all $0 \leq l < a$. The same argument shows that π is idempotent.

Proposition 4.20. *The subspaces $\mathcal{O}(\Omega^a)$ of $\mathcal{O}(1)$ are closed and complemented.*

Lastly we show that the subspace of exponentials and powers of Ω is dense in $\mathcal{O}(1)/\mathcal{O}(a)$. First we give the analogue of theorem 3.38 for this case:

Lemma 4.21. *Let $f : (\Omega, x^+, x^-) \to \mathbb{C}$ be continuous, then the following holds:*

$$\lim_{N\to\infty} \sup_{x^+,x^-\in\mathbb{S}^1} |f(\Omega, x^+, x^-) - \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{k=-n}^{n} \sum_{l=-m}^{m} \hat{\hat{f}}(\Omega, k, l) e^{ikx^+} e^{ilx^-}| = 0 \tag{4.50}$$

Proof. Let $\epsilon > 0$, then by theorem 3.38 we can find an $M_1 \in \mathbb{N}$ such that for all $N_1 > M_1$ we have

$$\sup_{x^+,x^-\in\mathbb{S}^1} |f(\Omega, x^+, x^-) - \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \hat{f}(\Omega, k, x^-) e^{ikx^+}| < \epsilon/2 \tag{4.51}$$

Next, also by theorem 3.38 we can find an $M_2 \in \mathbb{N}$ such that for all $N_2 > M_2$ we have

$$\sup_{x^+,x^-\in\mathbb{S}^1} |\frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \hat{f}(\Omega, k, x^-) e^{ikx^+} - \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{k=-n}^{n} \sum_{l=-m}^{m} \hat{\hat{f}}(\Omega, k, l) e^{ikx^+} e^{ilx^-}| < \epsilon/2 \tag{4.52}$$

Then for all $N > \max(M_1, M_2)$ we have:

$$\sup_{x^+,x^-\in\mathbb{S}^1} |f(\Omega, x^+, x^-) - \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{k=-n}^{n}\sum_{l=-m}^{m}\hat{\hat{f}}(\Omega, k, l)e^{ikx^+}e^{ilx^-}| \leq$$
$$\sup_{x^+,x^-\in\mathbb{S}^1} |f(\Omega, x^+, x^-) - \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{f}(\Omega, k, x^-)e^{ikx^+}|+$$
$$\sup_{x^+,x^-\in\mathbb{S}^1} |\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{f}(\Omega, k, x^-)e^{ikx^+} - \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{k=-n}^{n}\sum_{l=-m}^{m}\hat{\hat{f}}(\Omega, k, l)e^{ikx^+}e^{ilx^-}| < \epsilon \tag{4.53}$$

□

Proposition 4.22. *The set* $E = \operatorname{span}\left\{\Omega^b e^{ikx^+}e^{inx^-} | 0 \leq b \leq a, b \in \mathbb{N}, k, n \in \mathbb{Z}\right\}$ *is dense in* $\mathcal{O}(1)/\mathcal{O}(\Omega^a)$.

Proof. Let $f \in \mathcal{O}(1)/\mathcal{O}(\Omega^a)$, then $f = \sum_{i=0}^{a-1} \frac{f^{(i)}(0,x^+,x^-)}{i!}\Omega^i$. Then let $(f_N)_{N\geq 0}$ be the sequence defined by

$$f_N = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{k=-n}^{n}\sum_{l=-m}^{m}\sum_{i=0}^{a-1}\frac{\hat{\hat{f}}^{(i)}(0,k,l)}{i!}\Omega^i e^{ikx^+}e^{ilx^-} \tag{4.54}$$

Clearly $(f_N)_{N\geq 0} \subseteq E$ and furthermore,

$$p_{k,c,d}(f - f_N) = \sup_{x^+,x^-\in\mathbb{S}^1} |\partial_\Omega^k \partial_+^c \partial_-^d (f - f_N)|_{\Omega=0}|$$
$$= \sup_{x^+,x^-\in\mathbb{S}^1} |\partial_+^c \partial_-^d (f^{(k)}(0, x^+, x^-) - \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{j=-n}^{n}\sum_{l=-m}^{m}\hat{\hat{f}}^{(k)}(0, j, l)e^{ikx^+}e^{ilx^-})|$$
$$= \sup_{x^+,x^-\in\mathbb{S}^1} |\partial_+^c \partial_-^d f^{(k)}(0, x^+, x^-) - \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{j=-n}^{n}\sum_{l=-m}^{m}\hat{\hat{f}}^{(k)}(0, j, l)(ik)^c(il)^d e^{ikx^+}e^{ilx^-}|$$
$$= \sup_{x^+,x^-\in\mathbb{S}^1} |\partial_+^c \partial_-^d f^{(k)}(0, x^+, x^-) - \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{j=-n}^{n}\sum_{l=-m}^{m}(\partial_+^c \partial_-^d \hat{\hat{f}}^{(k)})(0, j, l)e^{ikx^+}e^{ilx^-}| \tag{4.55}$$

and thus by lemma 4.21 $\lim_{N\to\infty} p_{k,c,d}(f - f_N) = 0$ from which we conclude that $f_N \to f$. Thus E is dense in $\mathcal{O}(1)/\mathcal{O}(\Omega^a)$. □

4.5 Weak asymptotic symmetry algebra

Now that have a nice definition of $\mathcal{O}(r^{-n})$, we can specify the boundary, or fall-off conditions that we want our metric g to have. We use the boundary conditions that we determined in section 4.3. We then determine the weak asymptotic symmetry algebra by first calculating which vector fields ξ satisfy $\mathcal{L}_\xi g \in T_g S$ and then showing that S is integrable.

Definition 4.23. A metric g is called asymptotically Anti de Sitter, if its component functions have the following fall-off behavior:

$$\begin{pmatrix} g_{rr} & g_{r+} & g_{r-} \\ g_{+r} & g_{++} & g_{+-} \\ g_{-r} & g_{-+} & g_{--} \end{pmatrix} \sim \begin{pmatrix} \frac{l^2}{r^2} + \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) - \frac{1}{2}r^2 \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) - \frac{1}{2}r^2 & \mathcal{O}(1) \end{pmatrix} \tag{4.56}$$

The cone bundle S is also defined as in section 4.3. The tangent space is then given by $h \in T_g S$ if

$$\begin{pmatrix} h_{rr} & h_{r+} & h_{r-} \\ h_{+r} & h_{++} & h_{+-} \\ h_{-r} & h_{-+} & h_{--} \end{pmatrix} \sim \begin{pmatrix} \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\Omega) & \mathcal{O}(\Omega) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{4.57}$$

Note that when converting to the Ω coordinate, $dr = -\Omega^{-2}d\Omega$, so $h_{rr}dr^2 = h_{rr}(\Omega)\Omega^{-4}d\Omega^2 = h_{\Omega\Omega}d\Omega^2$.

Furthermore, we use the following: $f \sim g + \mathcal{O}(r^{-n})$ means that $f - g \in \mathcal{O}(r^{-n})$ and $f\mathcal{O}(r^{-m}) + g \sim \mathcal{O}(r^{-n})$ means that for all $h \in \mathcal{O}(r^{-m})$ there exists $\eta \in \mathcal{O}(r^{-n})$ such that $fh + g = \eta$. It follows immediately that $f\mathcal{O}(r^{-m}) \sim \mathcal{O}(r^{-n})$ with $f \in \mathcal{O}(r^{-k})$ implies $k \geq n - m$.

We will first determine the Lie algebra of vector fields ξ such that $\mathcal{L}_\xi g \in T_g S$ for all $g \in S$, and then show that S is integrable. The following is a result from Refs. [BH86] [Obl17]:

Theorem 4.24. *The Lie algebra of vector fields ξ such that $\mathcal{L}_\xi g \in T_g S$ for all $g \in S$ is given by*

$$\begin{aligned} \xi = &\left(-\frac{r}{2}\left[\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\right] + \mathcal{O}(r^{-1})\right)\partial_r \\ &+ \left(X(x^+) + \frac{l^2}{2r^2}\partial_-^2 \bar{X}(x^-) + \mathcal{O}(r^{-4})\right)\partial_+ \\ &+ \left(\bar{X}(x^-) + \frac{l^2}{2r^2}\partial_+^2 X(x^+) + \mathcal{O}(r^{-4})\right)\partial_- \end{aligned} \tag{4.58}$$

where $X, \bar{X}$ are 2π-periodic functions.

Proof. The Lie derivative is given by:

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + (\partial_\mu \xi^\lambda) g_{\lambda\nu} + (\partial_\nu \xi^\lambda) g_{\lambda\mu} \tag{4.59}$$

Clearly, if $g' \in S$, $g' = \frac{l^2}{r^2}dr^2 - r^2 dx^+ dx^- + h'$, where $h' \in T_h S$. We will show that $\mathcal{L}_\xi g \in T_g S$ holds for $g = \frac{l^2}{r^2}dr^2 - r^2 dx^+ dx^-$ and for $h' \in T_h S$, and then we are done.

First, the rr part:

$$\mathcal{L}_\xi g_{rr} = \xi^r \partial_r \frac{l^2}{r^2} + 2\partial_r \xi^r \frac{l^2}{r^2} \tag{4.60}$$

As $\mathcal{L}_\xi g_{rr}$ should be in $\mathcal{O}(r^{-4})$, for this to hold we need

$$\xi^r = r\partial_r \xi^r + \mathcal{O}(r^{-1}) \tag{4.61}$$

This differential equation can be solved, using property 6 from proposition 4.14, as follows:

$$\begin{aligned}
&-\frac{1}{r^2}\xi^r + \frac{1}{r}\partial_r \xi^r \sim \mathcal{O}(r^{-3}) \\
&\partial_r(\frac{1}{r}\xi^r) \sim \mathcal{O}(r^{-3}) \\
&\frac{1}{r}\xi^r + \mathcal{F}(x^+, x^-) \sim \mathcal{O}(r^{-2})
\end{aligned} \tag{4.62}$$

giving the solution:

$$\xi^r = r\mathcal{F}(x^+, x^-) + \mathcal{O}(r^{-1}) \tag{4.63}$$

For the $r\pm$ part, we find

$$\mathcal{L}_\xi g_{r\pm} = \partial_\pm \xi^r g_{rr} + \partial_r \xi^\mp g_{\mp\pm} \sim \mathcal{O}(r^{-3}) \tag{4.64}$$

using equation 4.63, we get

$$\frac{l^2}{r}\partial_\pm \mathcal{F}(x^+, x^-) - \frac{1}{2}r^2 \partial_r \xi^\mp \sim \mathcal{O}(r^{-3}) \tag{4.65}$$

which can again be solved to yield

$$\xi^\mp = X^\mp(x^+, x^-) - \frac{l^2}{r^2}\partial_\pm \mathcal{F}(x^+, x^-) + \mathcal{O}(r^{-4}) \tag{4.66}$$

Now let $a, b \in \{+, -\}$ and we use the notation $\neg\pm = \mp$. Then

$$\begin{aligned}
\mathcal{L}_\xi g_{ab} =& \xi^r \partial_r g_{ab} + \partial_a \xi^\lambda g_{\lambda b} + \partial_b \xi^\lambda g_{a\lambda} = \xi^r \partial_r g_{ab} + \partial_a \xi^{\neg b} g_{\neg b b} + \partial_b \xi^{\neg a} g_{a \neg a} = \\
&\begin{pmatrix} -\frac{1}{2}r^2(\partial_+\xi^- + \partial_+\xi^-) & -r\xi^r - \frac{1}{2}r^2(\partial_-\xi^- + \partial_+\xi^+) \\ -r\xi^r - \frac{1}{2}r^2(\partial_-\xi^- + \partial_+\xi^+) & -\frac{1}{2}r^2(\partial_-\xi^+ + \partial_-\xi^+) \end{pmatrix}
\end{aligned} \tag{4.67}$$

The off-diagonal part gives:

$$\mathcal{F}(x^+, x^-) + \frac{1}{2}(\partial_- X^- + \partial_+ X^+) \sim \mathcal{O}(r^{-2}) \tag{4.68}$$

Or $\mathcal{F}(x^+, x^-) = -\frac{1}{2}(\partial_- X^- + \partial_+ X^+) + \mathcal{O}(r^{-2})$. Lastly, the diagonal part shows:

$$\begin{aligned}
\partial_\pm \xi^\mp \sim& \mathcal{O}(r^{-2}) \\
\partial_\pm X^\mp(x^+, x^-) - \frac{l^2}{r^2}\partial_\pm^2 \mathcal{F}(x^+, x^-) + \mathcal{O}(r^{-4}) \sim& \mathcal{O}(r^{-2}) \\
\partial_\pm X^\mp(x^+, x^-) =& 0
\end{aligned} \tag{4.69}$$

Thus we can set $X(x^+) = X^+(x^+), \bar{X}(x^-) = X^-(x^-)$. Combining everything, we get

$$
\begin{aligned}
\xi = &\left(-\frac{r}{2}\left[\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\right] + \mathcal{O}(r^{-1})\right)\partial_r \\
&+ \left(X(x^+) + \frac{l^2}{2r^2}\partial_-^2 \bar{X}(x^-) + \mathcal{O}(r^{-4})\right)\partial_+ \\
&+ \left(\bar{X}(x^-) + \frac{l^2}{2r^2}\partial_+^2 X(x^+) + \mathcal{O}(r^{-4})\right)\partial_-
\end{aligned}
\tag{4.70}
$$

or in the coordinate $\Omega = 1/r$

$$
\begin{aligned}
\xi = &\left(\frac{\Omega}{2}\left[\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\right] + \mathcal{O}(\Omega^3)\right)\partial_\Omega \\
&+ \left(X(x^+) + \frac{l^2\Omega^2}{2}\partial_-^2 \bar{X}(x^-) + \mathcal{O}(\Omega^4)\right)\partial_+ \\
&+ \left(\bar{X}(x^-) + \frac{l^2\Omega^2}{2}\partial_+^2 X(x^+) + \mathcal{O}(\Omega^4)\right)\partial_-
\end{aligned}
\tag{4.71}
$$

Now we will show that for an arbitrary $h' \in T_hS$, $\mathcal{L}_\xi h' \in T_{h'}S$ for ξ as in equation 4.71. To this end, note that as ξ smoothly extends to $\Omega = 0$, $(\mathcal{L}_\xi h')_{\mu\nu} \in \mathcal{O}(1)$ for all $h' \in T_hS$. Thus we only need to show $(\mathcal{L}_\xi h')_{\Omega\pm} \in \mathcal{O}(\Omega)$. Now this is given by

$$
(\mathcal{L}_\xi h')_{\Omega\pm} = \xi^\lambda \partial_\lambda \mathcal{O}(\Omega) + (\partial_\Omega \xi^\lambda)h'_{\lambda\pm} + (\partial_\pm \xi^\lambda)h'_{\lambda\Omega}
\tag{4.72}
$$

The $\lambda = \Omega$ term is given by:

$$
\mathcal{O}(\Omega)\partial_\Omega \mathcal{O}(\Omega) + (\partial_\Omega \mathcal{O}(\Omega))\mathcal{O}(\Omega) + (\partial_\pm \mathcal{O}(\Omega))\mathcal{O}(1) \sim \mathcal{O}(\Omega)
\tag{4.73}
$$

and the $\lambda = +$ term

$$
\xi^+ \partial_+ \mathcal{O}(\Omega) + (\partial_\Omega \xi^+)h'_{+\pm} + (\partial_\pm \xi^+)\mathcal{O}(\Omega)
\tag{4.74}
$$

which is also clear $\mathcal{O}(\Omega)$. The same holds for $\lambda = -$. We conclude that indeed $(\mathcal{L}_\xi h')_{\Omega\pm} \sim \mathcal{O}(\Omega)$.

□

We denote the Lie algebra of vector fields of the form above by $\mathfrak{g}$. We further define the vector field $\xi_{(X,\bar{X})}$ to be given by:

$$
\begin{aligned}
\xi_{(X,\bar{X})} = &\left(-\frac{r}{2}\left[\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\right]\right)\partial_r \\
&+ \left(X(x^+) + \frac{l^2}{2r^2}\partial_-^2 \bar{X}(x^-)\right)\partial_+ \\
&+ \left(\bar{X}(x^-) + \frac{l^2}{2r^2}\partial_+^2 X(x^+)\right)\partial_-
\end{aligned}
\tag{4.75}
$$

Moreover, we define $\xi_n = \xi_{(e^{inx^+},0)}$ and $\bar{\xi}_n = \xi_{(0,e^{inx^-})}$ and let $\mathfrak{i}$ be the space of vector fields ξ in $\mathfrak{g}$ such that $X = \bar{X} = 0$.

4.5.1 S is integrable

We will now show that S is integrable and that $\mathfrak{g}$ is therefore the weak asymptotic symmetry algebra by proposition 4.7.

We start of by looking at the complete vector fields and show that these at least preserve the tangent space T_gS.

Lemma 4.25. *Suppose ξ is a complete vector field with $\xi^\Omega \in \mathcal{O}(\Omega)$ and $\partial_\Omega \xi^\pm \in \mathcal{O}(\Omega)$. Let ϕ_t be the flow generated by ξ, then $(\phi_t)_\Omega \in \mathcal{O}(\Omega)$ and $\frac{\partial \phi_t^\pm}{\partial \Omega} \in \mathcal{O}(\Omega)$ for all $t \in \mathbb{R}$.*

Proof. As $\xi^\Omega \in \mathcal{O}(\Omega)$, we have $\xi^\Omega = \Omega \tilde{\xi}^\Omega$ with $\tilde{\xi}^\Omega \in \mathcal{O}(1)$. Now let $p = (0, x^+, x^-)$, then $\phi_t(p)$ is the unique maximal integral curve starting at p [Lee12]:

$$\begin{aligned} &\frac{d}{dt}\phi_t^\Omega(p) = \xi^\Omega(\phi_t(p)) = \phi_t^\Omega(p)\tilde{\xi}^\Omega(\phi_t(p)) \\ &\phi_0(p) = p = (0, x^+, x^-) \end{aligned} \tag{4.76}$$

Now clearly $\tilde{\phi}_t^\Omega(p) = 0$ is a solution to the above equation and as the solution is unique, this is the solution. Thus by definition of $\mathcal{O}(\Omega)$, $\phi_t^\Omega \in \mathcal{O}(\Omega)$ for all t. Furthermore, as in fact $\partial_\Omega \xi^\pm \in \mathcal{O}(\Omega)$, $\partial_\Omega \xi^\pm = \Omega \tilde{\xi}^\pm$ with again $\tilde{\xi}^\pm \in \mathcal{O}(1)$. Now as $\phi_t^\Omega(p) = 0$

$$\begin{aligned} &\frac{d}{dt}\frac{\partial \phi_t^\pm}{\partial \Omega}\bigg|_{\Omega=0} = \frac{\partial}{\partial \Omega}\frac{d}{dt}\phi_t^\pm\bigg|_{\Omega=0} = \frac{\partial}{\partial \Omega}\xi^\pm(\phi_t)\bigg|_{\Omega=0} = \\ &\phi_t^\Omega \tilde{\xi}^\pm(\phi_t)\frac{\partial \phi_t^\Omega}{\partial \Omega}\bigg|_{\Omega=0} = 0 \end{aligned} \tag{4.77}$$

and the initial condition is clearly $\frac{\partial \phi_0^\pm(p)}{\partial \Omega} = 0$. Thus $\frac{\partial \phi_t^\pm(\Omega, x^+, x^-)}{\partial \Omega}\Big|_{\Omega=0} = 0$ for all t, and $\frac{\partial \phi_t^\pm}{\partial \Omega} \in \mathcal{O}(\Omega)$. ☐

Proposition 4.26. *Suppose ϕ_t is a curve of diffeomorphisms such that $\phi_t^\Omega \in \mathcal{O}(\Omega)$ and $\frac{\partial \phi_t^\pm}{\partial \Omega} \in \mathcal{O}(\Omega)$ for all $t \in \mathbb{R}$. Then $(\phi_t)^* T_gS \subseteq T_gS$*

Proof. The Jacobian of ϕ_t is given by:

$$J\phi_t \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\Omega) & \mathcal{O}(\Omega) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{4.78}$$

Let $h \in T_gS$, then the pullback is given by:

$$((\phi_t)^* h)_p = (J\phi_t)^T \cdot h_{\phi_t(p)} \cdot J\phi_t \tag{4.79}$$

where the superscript T denotes the transpose. Now as

$$h_{\phi_t(p)} \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\Omega) & \mathcal{O}(\Omega) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{4.80}$$

and

$$\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\Omega) & \mathcal{O}(\Omega) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}^3 \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\Omega) & \mathcal{O}(\Omega) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Omega) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{4.81}$$

we find $(\phi_t)^* h \in T_g S$.

□

The proof that S is integrable will rely on the fact that we can smoothly cutoff a vector field near $\Omega = 0$ and therefore split every vector field into one that is complete and one that is zero in a neighborhood of $\Omega = 0$. We now show that we can give a curve of diffeomorphisms that is generated by a vector field of the second category and also preserves $T_g S$. First we show that every function can be smoothly split up into non-decreasing and non-increasing parts.

Lemma 4.27. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. We can then split up f into smooth functions f^+, f^- such that $f = f^+ + f^-$ and f^+ is non-decreasing, f^- is non-increasing.*

Proof. Define $f^+(x) = h(x), f^-(x) = f(x) - h(x)$ with $h(x) = \int_0^x \tilde{h}(y)dy$. Then let $\epsilon > 0$ and define $\tilde{h}(y) = \frac{1}{2}(f'(y) + \sqrt{f'(y)^2 + \epsilon})$. Clearly $(f^+)'(x) = \tilde{h}(x) \geq 0$ and $(f^-)'(x) = f'(x) - \tilde{h}(x) = \frac{1}{2}(f'(y) - \sqrt{f'(y)^2 + \epsilon}) \leq 0$, thus f^+, f^- are non-decreasing and non-increasing respectively. □

Lemma 4.28. *Let ξ be a vector field on M such that $\mathcal{L}_\xi h \in T_h S$ and assume ξ is identically zero for $\Omega \in [0, \epsilon]$ for some $\epsilon > 0$. Then there exists a curve of diffeomorphisms ϕ_t such that $\phi_0 = \mathrm{id}$, $\dot{\phi}_0 = \xi$, and ϕ_t is an asymptotic symmetry*

Proof. Let ξ be a vector field on M such that $\mathcal{L}_\xi h \in T_h S$ and ξ is identically zero for $\Omega \in [0, \epsilon]$ for some $\epsilon > 0$. We then decompose $\xi = \xi^\Omega \partial_\Omega + \xi^\mu \partial_\mu$. For each Ω, $\xi^\mu \partial_\mu$ is a vector field on $S^1 \times S^1$ and therefore generates a global flow $\phi_t^{\|}$ on $S^1 \times S^1$. We then note that $f(\Omega, x^+, x^-) = \xi^\Omega / \Omega$ is well defined as ξ^Ω is zero near $\Omega = 0$. Now define $\psi_t^\pm(\Omega, x^+, x^-) = \Omega \exp(\pm t f^\pm(\Omega, x^+, x^-))$ for $t \in [0, 1]$, where $f^\pm$ is the splitting of f as in lemma 4.27. Clearly $\psi_0^\pm(\Omega, x^+, x^-) = \Omega$. Now

$$\partial_\Omega \psi_t^\pm(\Omega, x^+, x^-) = \exp(\pm t f^\pm(\Omega, x^+, x^-))(1 \pm t\Omega (f^\pm)'(\Omega, x^+, x^-)) > 0 \tag{4.82}$$

as $\pm t\Omega(f^\pm)'(\Omega, x^+, x^-) \geq 0$. Thus $\psi_t^\pm$ is strictly increasing and therefore a diffeomorphism for each x^+, x^-. Next, define $\psi_t = (\psi_t^-)^{-1} \circ (\psi_t^+, x^+, x^-)$. This is clearly a diffeomorphism for each x^+, x^- as $\psi_t^\pm$ is a diffeomorphism, and it is the identity at $t = 0$. We can also calculate the t derivative at $t = 0$:

$$\dot{\psi}_t = \partial_t(\psi_t^-)^{-1}|_{t=0} + \dot{\psi}_0^+ \partial_\Omega(\psi_0^-)^{-1} = \Omega(f^- + f^+) = \xi^\Omega \tag{4.83}$$

Now take $\phi_t(\Omega, x^+, x^-) = (\psi_t, \phi_t^{||})(\Omega, x^+, x^-)$. This is a diffeomorphism for each $t \in [0,1]$, the identity for $t = 0$ and $\dot{\phi}_0 = \xi$. Note that for $\Omega \in [0, \epsilon]$, $\psi_t(\Omega) = \mathrm{id}$ as ξ is zero there, meaning that for $g \in S$, $\psi_t^* g = g + h$ where h is zero for $\Omega \in [0, \epsilon)$ and $\psi_t^* g$ has the same signature as g as the signature is invariant under pullbacks, thus $\phi_t^* g \in S$. The same reasoning works for ϕ_t^{-1}. We conclude that ϕ_t is an asymptotic symmetry. □

Thus we only need to show that we can combine the two curves of diffeomorphisms from proposition 4.26 and lemma 4.28, such that we get one curve of asymptotic symmetries, which is generated by the vector field that we smoothly cut off. This is shown in the next theorem.

Theorem 4.29. *S is integrable.*

Proof. Let ξ be a vector field such that $\mathcal{L}_\xi h \in T_h S$. Then by definition of $\mathfrak{g}$, $\xi \in \mathfrak{g}$. Now using a smooth cutoff function, we can smooth split $\xi = \xi_0 + \xi_\epsilon$, where ξ_ϵ is zero for $\Omega \in [0, \epsilon]$ and ξ_0 is zero for $\Omega > \epsilon$. As ξ_0 is compactly supported and tangent to the boundary $\Omega = 0$, ξ_0 is complete and denoting its global flow by ϕ_t, $(\phi_t)^* T_g S \subseteq T_g S$ by lemma 4.25, proposition 4.26 and as the fall off conditions are satisfied for all $\xi \in \mathfrak{g}$. Using lemma 4.28 and proposition 4.26, we can also construct a curve of diffeomorphisms ψ_t that is generated by ξ_ϵ and such that ψ_t is a weak asymptotic symmetry for all $t \in [0,1]$. Define $\Phi_t = \phi_t \circ \psi_t$. Now

$$\partial_t|_{t=0} \Phi_t = \dot{\phi}_0(\psi_0) + (J\phi_0)\dot{\psi}_0 = \xi \tag{4.84}$$

and clearly $\Phi_0 = \mathrm{id}$. Thus if we show that ϕ_t is an asymptotic symmetry for $t \in [0,1]$, we are done. To that end, note that [Lee12]

$$\left.\frac{d}{dt}\right|_{t=t_0} \phi_t^* g = \phi_{t_0}^*(\mathcal{L}_\xi g) \tag{4.85}$$

and thus for $p \in M$

$$\phi_t^* g - g = \int_0^t \phi_s^*(\mathcal{L}_\xi g) ds \tag{4.86}$$

Now $\mathcal{L}_\xi g \in T_g S$ and $(\phi_s)^* T_g S \subseteq T_g S$, so the integrand is in $T_g S$ for all s. Now we can consider the components $\phi_s^*(\mathcal{L}_\xi g))_{\mu\nu}$ of the integrand, which have certain fall off behavior, say $\phi_s^*(\mathcal{L}_\xi g))_{\mu\nu} \sim \mathcal{O}(\Omega^{a_{\mu\nu}})$, where $a_{\mu\nu}$ is determined by the form of the tangent space as in equation 4.36. Therefore $\phi_s^*(\mathcal{L}_\xi g))_{\mu\nu} = \Omega^{a_{\mu\nu}} \tilde{f}_{\mu\nu}(s)$ for some $\tilde{f}_{\mu\nu}(s) \in \mathcal{O}(1)$. Hence $\int_0^t (\phi_s^*(\mathcal{L}_\xi g)) ds = \Omega^{a_{\mu\nu}} \int_0^t \tilde{f}_{\mu\nu}(s) ds \in \mathcal{O}(\Omega^{a_{\mu\nu}})$. Thus the integral is in $T_g S$, $\int_0^t \phi_s^*(\mathcal{L}_\xi g) ds \in T_g S$. Now $\phi_t^* g$ has the same signature as g as the signature is invariant under pullbacks, $g + h \in S$ for $h \in T_g S$ if $g + h$ has the correct signature, thus $\phi_t^* g \in S$. As this holds for all t and $\phi_t^{-1} = \phi_{-t}$, $\phi_{-t}^* g \in S$ as well and ϕ_t is an asymptotic symmetry. □

4.5.2 Structure, brackets and topology

As derived in theorem 4.24, the asymptotic symmetry algebra $\mathfrak{g}$ is given by the following vector fields:

$$\begin{aligned}\xi = &\left(-\frac{r}{2}\left[\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\right] + \mathcal{O}(r^{-1})\right)\partial_r \\ &+ \left(X(x^+) + \frac{l^2}{2r^2}\partial_-^2 \bar{X}(x^-) + \mathcal{O}(r^{-4})\right)\partial_+ \\ &+ \left(\bar{X}(x^-) + \frac{l^2}{2r^2}\partial_+^2 X(x^+) + \mathcal{O}(r^{-4})\right)\partial_-\end{aligned} \tag{4.87}$$

for some smooth functions on the circle $X, \bar{X}$. Furthermore we have defined $\mathfrak{i}$ to be the space of such vector fields such that $X, \bar{X}$ are zero, thus

$$\mathfrak{i} = \mathcal{O}(r^{-1})\partial_r + \mathcal{O}(r^{-4})\partial_+ + \mathcal{O}(r^{-4})\partial_- \tag{4.88}$$

We will now calculate some of the relevant brackets and thereby show that $\mathfrak{i}$ is an ideal in $\mathfrak{g}$, and that $\mathfrak{g}/\mathfrak{i}$ consists of two copies of the Witt algebra.

Proposition 4.30. *The brackets in $\mathfrak{g}$ are as follows:*

$$\begin{aligned}&[\xi_{(X,0)}, \xi_{(Y,0)}] = \xi_{([X\partial_+, Y\partial_+],0)} \\ &[\xi_{(0,\bar{X})}, \xi_{(0,\bar{Y})}] = \xi_{(0,[\bar{X}\partial_-, \bar{Y}\partial_-])} \\ &[\xi_{(X,0)}, \xi_{(0,\bar{Y})}] = \frac{l^2}{4r^4}[\partial_+^2 X\partial_-, \partial_-^2 Y\partial_+] \\ &[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}\end{aligned} \tag{4.89}$$

We conclude that $\mathfrak{i}$ is an ideal and that $\mathfrak{h} = \mathfrak{g}/\mathfrak{i}$ consists of two copies of the Witt algebra, $\mathfrak{h} = \mathfrak{w}_L \oplus \mathfrak{w}_R$.

Proof. The first bracket is calculated as follows:

$$\begin{aligned}[\xi_{(X,0)}, \xi_{(Y,0)}] = &-\frac{r}{2}\partial_+ X\left(-\frac{1}{2}\partial_+ Y\partial_r - \frac{l^2}{r^3}\partial_+^2 Y\partial_-\right) \\ &+ X\left(-\frac{r}{2}\partial_+^2 Y\partial_r + \partial_+ Y\partial_+ + \frac{l^2}{2r^2}\partial_+^3 Y\partial_-\right) \\ &+ \frac{r}{2}\partial_+ Y\left(-\frac{1}{2}\partial_+ X\partial_r - \frac{l^2}{r^3}\partial_+^2 X\partial_-\right) \\ &- Y\left(-\frac{r}{2}\partial_+^2 X\partial_r + \partial_+ X\partial_+ + \frac{l^2}{2r^2}\partial_+^3 X\partial_-\right) \\ = &-\frac{r}{2}\left(X\partial_+^2 Y - Y\partial_+^2 X\right)\partial_r \\ &+ (X\partial_+ Y - Y\partial_+ X)\partial_+ \\ &+ \frac{l^2}{2r^2}(\partial_+ X\partial_+ Y^2 + X\partial_+ Y^3 - \partial_+ Y\partial_+^2 X - Y\partial_+^3 X)\partial_- \\ =&\xi_{([X\partial_+, Y\partial_+],0)}\end{aligned} \tag{4.90}$$

The calculation of the second bracket is identical. Now the third bracket is given by:

$$\begin{aligned}
[\xi_{(X,0)}, \xi_{(0,\bar{Y})}] =& -\frac{r}{2}\partial_+ X\left(-\frac{1}{2}\partial_-\bar{Y}\partial_r - \frac{l^2}{r^3}\partial_-^2\bar{Y}\partial_+\right) \\
&+ \frac{l^2}{2r^2}\partial_+^2 X\left(-\frac{r}{2}\partial_-^2\bar{Y}\partial_r + \frac{l^2}{2r^2}\partial_-^3\bar{Y}\partial_+ + \partial_-\bar{Y}\partial_-\right) \\
&+ \frac{r}{2}\partial_-\bar{Y}\left(-\frac{1}{2}\partial_+ X\partial_r - \frac{l^2}{r^3}\partial_+^2 X\partial_-\right) \\
&- \frac{l^2}{2r^2}\partial_-^2\bar{Y}\left(-\frac{r}{2}\partial_+^2 X\partial_r + \partial_+ X\partial_+ + \frac{l^2}{2r^2}\partial_+^3 X\partial_-\right) \\
=& \frac{l^4}{4r^4}\left(\partial_+^2 X\partial_-^3\bar{Y}\partial_+ - \partial_-^2\bar{Y}\partial_+^3 X\partial_-\right) \\
=& \frac{l^4}{4r^4}[\partial_+^2 X\partial_-, \partial_-^2\bar{Y}\partial_+]
\end{aligned} \tag{4.91}$$

To show $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ we start with the bracket $[\xi_{(X,0)}, \mathfrak{i}]$. We use superscripts $\mathcal{O}(r^{-n})^x$ to denote an arbitrary function $f \in \mathcal{O}(r^{-n})$, i.e. $\mathcal{O}(r^{-n})^x$ is not a vector space, but a function.

$$\begin{aligned}
&[\xi_{(X,0)}, \mathcal{O}(r^{-1})^0\partial_r + \mathcal{O}(r^{-4})^1\partial_+ + \mathcal{O}(r^{-4})^2\partial_-] = \\
&-\frac{r}{2}\partial_+ X\left(\partial_r\mathcal{O}(r^{-1})^0\partial_r + \partial_r\mathcal{O}(r^{-4})^1\partial_+ + \partial_r\mathcal{O}(r^{-4})^2\partial_-\right) \\
&+ X\left(\partial_+\mathcal{O}(r^{-1})^0\partial_r + \partial_+\mathcal{O}(r^{-4})^1\partial_+ + \partial_+\mathcal{O}(r^{-4})^2\partial_-\right) \\
&+ \frac{l^2}{2r^2}\partial_+^2 X\left(\partial_-\mathcal{O}(r^{-1})^0\partial_r + \partial_-\mathcal{O}(r^{-4})^1\partial_+ + \partial_-\mathcal{O}(r^{-4})^2\partial_-\right) \\
&- \mathcal{O}(r^{-1})^0\left(-\frac{1}{2}\partial_+ X\partial_r - \frac{l^2}{r^3}\partial_+^2 X\partial_-\right) \\
&- \mathcal{O}(r^{-4})^1\left(-\frac{r}{2}\partial_+^2 X\partial_r + \frac{l^2}{2r^2}\partial_+^3 X\partial_- + \partial_+ X\partial_+\right) = \\
&\left(\left[-\frac{r}{2}\partial_+ X\partial_r + X\partial_+ + \frac{l^2}{2r^2}\partial_+^2 X\partial_- + \frac{1}{2}\partial_+ X\right]\mathcal{O}(r^{-1})^0 + \frac{r}{2}\partial_+^2 X\mathcal{O}(r^{-4})^1\right)\partial_r + \\
&\left(-\frac{r}{2}\partial_+ X\partial_r + X\partial_+ + \frac{l^2}{2r^2}\partial_+^2 X\partial_- - \partial_+ X\right)\mathcal{O}(r^{-4})^1\partial_+ + \\
&\left(\left[-\frac{r}{2}\partial_+ X\partial_r + X\partial_+ + \frac{l^2}{2r^2}\partial_+^2 X\partial_-\right]\mathcal{O}(r^{-4})^2 + \frac{l^2}{r^3}\partial_+^2 X\mathcal{O}(r^{-1})^0 - \frac{l^2}{2r^2}\partial_+^3 X\mathcal{O}(r^{-4})^1\right)\partial_-
\end{aligned} \tag{4.92}$$

By analyzing the orders of the last expression, we see that $[\xi_{(X,0)}, \mathfrak{i}] \subseteq \mathfrak{i}$. The calculation for $\xi_{(0,\bar{X})}$ is the same. Lastly $[\mathfrak{i}, \mathfrak{i}] \subseteq \mathfrak{i}]$ is immediate when one writes out the bracket. □

In terms of the coordinate $\Omega = 1/r$, the vector space $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathcal{O}(\Omega)\partial_\Omega \oplus \mathcal{O}(\Omega^4)\partial_+ \oplus \mathcal{O}(\Omega^4)\partial_-$. We give $\mathcal{O}(\Omega^n)$ the subspace topology inherited from $\mathcal{O}(1)$. Note that $\mathfrak{w}_L, \mathfrak{w}_R$ are also subalgebra's of $\mathcal{O}(\Omega)\partial_\Omega \oplus \mathcal{O}(1)\partial_+ \oplus \mathcal{O}(1)\partial_-$, thus $\mathfrak{g}$ itself is a subalgebra of $\mathcal{O}(\Omega)\partial_\Omega \oplus \mathcal{O}(1)\partial_+ \oplus \mathcal{O}(1)\partial_-$. Using this structure, we define the topology of $\mathfrak{g}$.

Definition 4.31 (Topology of $\mathfrak{g}$)**.** The topology of $\mathcal{O}(\Omega)\partial_\Omega \oplus \mathcal{O}(1)\partial_+ \oplus \mathcal{O}(1)\partial_-$ is taken to be the product topology of $\mathcal{O}(\Omega) \times \mathcal{O}(1) \times \mathcal{O}(1)$ We then define the topology of $\mathfrak{g}$ to be the subspace topology given by the inclusion $\mathfrak{g} \subseteq \mathcal{O}(\Omega)\partial_\Omega \oplus \mathcal{O}(1)\partial_+ \oplus \mathcal{O}(1)\partial_-$.

As direct sums of locally convex spaces are again locally convex, $\mathfrak{g}$ is a locally convex vector space. Furthermore, we saw that function multiplication and derivatives were continuous in the discussion near definition 4.17, thus the Lie bracket on $\mathfrak{g}$ is continuous, turning it into a locally convex Lie algebra.

We will now prove that $\mathfrak{g}/\mathfrak{i} \cong \mathfrak{w}_L \oplus \mathfrak{w}_R$ with the isomorphism now being a homeomorphism as well.

Lemma 4.32. *The map $\eta : \mathfrak{g}/\mathfrak{i} \to \mathfrak{w}_L \oplus \mathfrak{w}_R : \xi_{(X^+,X^-)} \to X^+\partial_+ + X^-\partial_-$ is a Lie algebra isomorphism and homeomorphism*

Proof. That η is a Lie algebra isomorphism is immediate from proposition 4.30. By lemma 2.7, the seminorms on $\mathfrak{w}_L \oplus \mathfrak{w}_R$ are given by $p_k^\pm \circ \pi^\pm$, where $\pi^\pm$ are the projections onto $\partial_\pm$ and $p_k^\pm$ are the seminorms on the Witt algebra. Now let $\xi_{(X^+,X^-)} \in \mathfrak{g}/\mathfrak{i}$. Then

$$p_k^+\pi^+(\eta(\xi_{(X,\bar{X})})) = p_k^+(X^+\partial_+) = p_{0,k,0}\pi^+(\xi_{(X^+,X^-)}) \tag{4.93}$$

and the $-$ case is analogous, meaning that η is continuous. Next, let $i \in r,+,-$ and let $f : r,+,- \to 1,0,2$ be the map given by the orders of r in $\xi_{(X^+,0)}$, so $f(r) = 1, f(+) = 0, f(-) = 2$ and let g be the same except for $\xi_{(0,X^-)}$

$$\begin{aligned} p_{k,l,m}\pi^i(\xi_{(X^+,X^-)}) \leq C_k\delta_{m,0}\delta_{f(i),k}p_{l+f(i)}\pi^+(X^+\partial_+ + X^-\partial_-) + C_k\delta_{l,0}\delta_{g(i),k}p_{m+f(i)}\pi^-(X^+\partial_+ + X^-\partial_-) = \\ C_k\delta_{m,0}\delta_{f(i),k}p_{l+f(i)}\pi^+(\eta(\xi_{(X^+,X^-)})) + C_k\delta_{l,0}\delta_{g(i),k}p_{m+g(i)}\pi^-(\eta(\xi_{(X^+,X^-)})) \end{aligned} \tag{4.94}$$

which shows that η is open. We conclude that η is a homeomorphism. $\square$

Chapter 5

Cohomology of the weak asymptotic symmetries

In this chapter we will determine the first and second cohomology groups of the weak asymptotic symmetry group. The second cohomology group will help us calculate what the trivial symmetries are, which is why we are interested in it. The first cohomology group is determined first, as this can provide information about the second cohomology group. The first cohomology group is isomorphic to the abelianization of $\mathfrak{g}$, which we thus work out. We then describe a general method that decomposes the second cohomology group as a direct limit of cohomology groups that are easier to compute. Finally we use this general method and some explicit computations to calculate the second cohomology group of $\mathfrak{g}$.

We remind the reader that $\mathfrak{g}$ is the weak asymptotic symmetry algebra as derived in theorem 4.24 and $\mathfrak{i}$ is an ideal in $\mathfrak{g}$ containing the following vector fields:

$$\mathfrak{i} = \mathcal{O}(r^{-1})\partial_r + \mathcal{O}(r^{-4})\partial_+ + \mathcal{O}(r^{-4})\partial_- \tag{5.1}$$

These are the symmetries that are trivial in the standard, Noether perspective definition of trivial symmetries, as shown by Brown and Henneaux [BH86]. Note that they need not be trivial in our new Wigner perspective of trivial symmetries, and whether they are is in fact exactly what we want to determine. The quotient has also been calculated in proposition 4.30, showing that $\mathfrak{g}/\mathfrak{i} = \mathfrak{w}_L \oplus \mathfrak{w}_R$.

5.1 Commutator ideal

We will now determine the commutator ideal and abelianization of $\mathfrak{g}$. In fact, we do this for the complexification $\mathfrak{g}_\mathbb{C}$, as it is easier to work with complex exponentials than with sines and cosines. We then take the real part, which is equal to $[\mathfrak{g}, \mathfrak{g}]$ by proposition 2.15. To calculate $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]$, we first show that the commutator contains two subspaces of $\mathfrak{g}_\mathbb{C}$, I and J. In the next part, all Lie algebras, e.g. $\mathfrak{g}, \mathfrak{w}$ are the complexified version of the real Lie algebra.

First let I be the set of the vector fields independent of x^+, x^- and of sufficiently high order in r,

$$I = \left\{ a^i\partial_i \in \mathfrak{i}_{\mathbb{C}} \middle| a^r \in \mathcal{O}(r^{-2}), a^+, a^- \in \mathcal{O}(r^{-4}) \text{ and } a^i(r, x^+, x^-) = a^i(r) \right\} \tag{5.2}$$

this is a subalgebra, which can be shown in identical fashion to the proof of proposition 4.30. Next let J be the vector fields whose component functions either have zero mean over x^+ or over x^-,

$$J = \operatorname{span}\left\{ \xi^i\partial_i \in \mathfrak{i}_{\mathbb{C}} \middle| \int_0^{2\pi} \xi^i dx^+ = 0 \text{ for } i \in \{r, x^+, x^-\} \text{ or } \int_0^{2\pi} \xi^i dx^- = 0 \text{ for } i \in \{r, x^+, x^-\} \right\} \tag{5.3}$$

J is not a Lie subalgebra however. For example $\frac{\sin(x^+)}{r}\partial_r, \frac{\sin(x^+)}{r^2}\partial_r \in J$, but $[\frac{\sin(x^+)}{r}\partial_r, \frac{\sin(x^+)}{r^2}\partial_r] = -\frac{\sin^2(x^+)}{r^4}\partial_r \notin J$. Furthermore, any vector field that is nonzero constant in both x^+ and x^- is not in J, as for $\xi^i\partial_i \in J$, $\int_0^{2\pi}\int_0^{2\pi} \xi^i(r, x^+, x^-)dx^+dx^- = 0$. In particular $1/r\partial_r \notin J$ and also clearly $1/r\partial_r \notin I$. Lastly,

$$I + J = \operatorname{span}\left\{ \left\{ a^i\partial_i \middle| a^r \in \mathcal{O}(r^{-2}), a^+, a^- \in \mathcal{O}(r^{-4}) \right\} \cup \left\{ a^r\partial_r \middle| a^r\partial_r \in J \right\} \right\} \tag{5.4}$$

We will show that these two vector spaces are in the commutator, as well as the 2 Witt algebras. In fact, we can obtain all of these as commutators with the Witt algebra's and clearly $[\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}] \subseteq [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$

Proposition 5.1. *1.* $\mathfrak{w}_L + \mathfrak{w}_R \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$

2. $J \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$

3. $I \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$

Proof. 1. As proven in proposition 3.41, $\mathfrak{w}$ is perfect, so $\mathfrak{w}_L + \mathfrak{w}_R \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$.

2. Let $\xi^i\partial_i \in J$. Let us assume that $\int_0^{2\pi} \xi^i(r, x^+, x^-)dx^+ = 0$ for $i \in \{r, x^+, x^-\}$. Then by the same proof as for lemma 3.40, we can find Ξ^i such that $\partial_+\Xi^i = \xi^i$, namely $\Xi^i = \int_0^{x^+} \xi^i(r, x, x^-)dx$. Furthermore, $\Xi^i\partial_i \in \mathfrak{i}_{\mathbb{C}}$ as we can exchange integrals and derivatives by the Leibniz integral rule. Then $[\partial_+, \Xi^i\partial_i] = \xi^i\partial_i$, so $\xi^i\partial_i \in [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$. If $\int_0^{2\pi} \xi^i(r, x^+, x^-)dx^- = 0$ we can find Ξ^i such that $\partial_-\Xi^i = \xi^i$ and $\Xi^i\partial_i \in \mathfrak{i}_{\mathbb{C}}$. Again $[\partial_-, \Xi^i\partial_i] = \xi^i\partial_i$, so $\xi^i\partial_i \in [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$. By linearity $J \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$ follows.

3. We will show this in three steps. First, we will show that $c\partial_- \in [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$ for $c \in \mathcal{O}(r^{-4})$ and c independent of x^+, x^-. To this end, we take

$$\begin{aligned} X &= e^{ix^+} \\ C &= C_{-1}(r)e^{-ix^+}\partial_- \end{aligned} \tag{5.5}$$

and calculate the following bracket:

$$\begin{aligned}
&[\xi_{(X,0)}, C] = \left[-i\frac{r}{2}e^{ix^+}\partial_r + e^{ix^+}\partial_+ - \frac{l^2}{2r^2}e^{ix^+}\partial_-, e^{-ix^+}C_{-1}\partial_-\right] = \\
&\left(-i\frac{r}{2}\partial_r C_{-1} - iC_{-1} + 0\right)\partial_- = -\frac{i}{2r}\partial_r(r^2 C_{-1})\partial_-
\end{aligned} \tag{5.6}$$

Thus we expect C_{-1} to have the following form, as the bracket should give $c\partial_-$:

$$C_{-1}(r) = \frac{2i}{r^2}\int c(r)rdr \tag{5.7}$$

By proposition 4.14, $c(r)r \in \mathcal{O}(r^{-3})$ as $c \in \mathcal{O}(r^{-4})$, and thus we can integrate $c(r)r$, also by proposition 4.14. We then find that $C_{-1} \in \mathcal{O}(r^{-4})$ so $C \in \mathfrak{i}_{\mathbb{C}}$. Thus $c\partial_- \in [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_{\mathbb{C}}]$. Now we note that x^+ and x^- are symmetric in the sense that if we exchange every x^+ and x^-, every result should stay the same. Using this intuition, we expect that

$$\begin{aligned}
\bar{X} &= e^{ix^-} \\
B &= B_{-1}(r)e^{-ix^-}\partial_+ \\
B_{-1} &= \frac{2i}{r^2}\int b(r)rdr
\end{aligned} \tag{5.8}$$

should give $[\xi_{(0,\bar{X})}, B] = b\partial_+$. Indeed,

$$\begin{aligned}
&[\xi_{(0,\bar{X})}, B] = \left[-i\frac{r}{2}e^{ix^-}\partial_r + e^{ix^-}\partial_- - \frac{l^2}{2r^2}e^{ix^-}\partial_+, e^{-ix^-}B_{-1}\partial_-\right] = \\
&\left(-i\frac{r}{2}\partial_r B_{-1} - iB_{-1} + 0\right)\partial_- = -\frac{i}{2r}\partial_r(r^2 B_{-1})\partial_- = b\partial_-
\end{aligned} \tag{5.9}$$

Lastly, for $a\partial_r$ with $a \in \mathcal{O}(r^{-2})$ and independent of x^+, x^-, we take a similar ansatz

$$\begin{aligned}
X &= e^{ix^+} \\
A &= A_{-1}(r)e^{-ix^+}\partial_r \\
A_{-1} &\in \mathcal{O}(r^{-2})
\end{aligned} \tag{5.10}$$

Again, we calculate the bracket

$$\begin{aligned}
&[\xi_{(X,0)}, A] = \left[-i\frac{r}{2}e^{ix^+}\partial_r + e^{ix^+}\partial_+ - \frac{l^2}{2r^2}e^{ix^+}\partial_-, e^{-ix^+}A_{-1}(r)\partial_r\right] = \\
&-i\left(\frac{r}{2}\partial_r A_{-1}(r) + A_{-1}(r) + 0 - \frac{A_{-1}}{2}\right)\partial_r - \frac{l^2}{r^3}A_{-1}(r)\partial_- = \\
&-\frac{i}{2}\partial_r(rA_{-1})\partial_r - A_{-1}\frac{l^2}{r^3}\partial_-
\end{aligned} \tag{5.11}$$

As $A_{-1} \in \mathcal{O}(r^{-2})$, $\frac{A_{-1}}{r^3} \in \mathcal{O}(r^{-5})$, so $A_{-1}\frac{l^2}{r^3}\partial_- \in \mathfrak{i}_{\mathbb{C}}$ and we can thus find a $C \in \mathfrak{i}_{\mathbb{C}}$

such that $[\xi_{(X,0)}, C] = A_{-1}\frac{l^2}{r^3}\partial_-$, as shown above. Using this C

$$[\xi_{(X,0)}, A + C] = -\frac{i}{2}\partial_r(rA_{-1})\partial_r \tag{5.12}$$

at which point we see that

$$A_{-1} = 2\frac{i}{r}\int a(r)dr \tag{5.13}$$

gives $[\xi_{(X,0)}, A + C] = a\partial_r$. Furthermore, by proposition 4.14, $\int a(r)dr \in \mathcal{O}(r^{-1})$, so indeed $A_{-1} \in \mathcal{O}(r^{-2})$. We conclude that $I \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{g}_\mathbb{C}]$.

□

The above proof in fact shows something stronger, namely the next proposition.

Proposition 5.2. *Let $a \geq 4$, then*

$$\mathcal{O}(r^{-a})\partial_r + \mathcal{O}(r^{-a})\partial_+ + \mathcal{O}(r^{-a})\partial_- \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathcal{O}(r^{-a})\partial_r + \mathcal{O}(r^{-a})\partial_+ + \mathcal{O}(r^{-a})\partial_-]$$

Proof. This follows from step two and three in the proof of proposition 5.1. For step two, note that $\Xi^i\partial_i$ is in $\mathcal{O}(r^{-a})\partial_r + \mathcal{O}(r^{-a})\partial_+ + \mathcal{O}(r^{-a})\partial_-$ if $\xi \in \mathcal{O}(r^{-a})\partial_r + \mathcal{O}(r^{-a})\partial_+ + \mathcal{O}(r^{-a})\partial_-$, as Ξ is simply the integral over x^+ or x^- of ξ and this does not change the order in r. For step three, it is easily checked that A_{-1}, B_{-1}, C_{-1} are of the same order in r as a, b, c, leading to the desired result. □

We now want to show that $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}] \subseteq \langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle$, where the angular brackets denote the ideal generated by the vector space $I + J + \mathfrak{w}_L + \mathfrak{w}_R \subseteq \mathfrak{g}_\mathbb{C}$. We will do this by first calculating the quotient of $\mathfrak{g}_\mathbb{C}$ and this ideal. Afterwards, we use corollary 2.17 to calculate $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle$. We will show that this is 0, meaning that $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}] \subseteq \langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle$.

Lemma 5.3. *The quotient is given by*

$$\mathfrak{g}_\mathbb{C}/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle = \operatorname{span}\{\left[\frac{1}{r}\partial_r\right]\} \tag{5.14}$$

Proof. First, let $[x] \in \mathfrak{g}_\mathbb{C}/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle$. Then $x \in \mathfrak{g}_\mathbb{C}$, so $x = \xi + (a\partial_r + b\partial_+ + c\partial_-)$ for $\xi \in \mathfrak{w}_L \oplus \mathfrak{w}_R$, $a\partial_r + b\partial_+ + c\partial_- \in \mathfrak{i}_\mathbb{C}$. Clearly, $[x] = [a\partial_r + b\partial_+ + c\partial_-]$. Furthermore, for $\bar{b} = \frac{1}{2\pi}\int_0^{2\pi} b(r, x^+, x^-)dx^+$ we have $b(r, x^+, x^-) - \bar{b} \in J$ and the same holds for a and c. Thus

$$[x] = \left[\bar{a}(r, x^-)\partial_r + \bar{b}(r, x^-)\partial_+ + \bar{c}(r, x^-)\partial_-\right] \tag{5.15}$$

We can do the same for x^-, as with $\tilde{b} = \frac{1}{2\pi}\int_0^{2\pi} \bar{b}(r, x^-)dx^-, \bar{b} - \tilde{b} \in J$,so

$$[x] = \left[\tilde{a}(r)\partial_r + \tilde{b}(r)\partial_+ + \tilde{c}(r)\partial_-\right] \tag{5.16}$$

Now $\tilde{b}(r)\partial_+ + \tilde{c}(r)\partial_- \in I$, so we find $[x] = [\tilde{a}(r)\partial_r]$. Using proposition 4.14, as $\tilde{a}(r) \in \mathcal{O}(r^{-1})$, we can write this as $\tilde{a}(r) = \frac{1}{r}\alpha(r)$ for $\alpha \in \mathcal{O}(1)$. Now $\alpha(r) = \hat{\alpha}(0) + \tilde{\alpha}(r)$ for some

$\tilde{\alpha}(r) \in \mathcal{O}(r^{-1})$ and where $\hat{\alpha}(\Omega) = \alpha(1/r)$. Thus $\tilde{a}(r) = \frac{1}{r}(\hat{\alpha}(0) + \tilde{\alpha}(r))$ and clearly $\tilde{\alpha}(r)/r \in \mathcal{O}(r^{-2})$. We then find $[\tilde{a}(r)\partial_r] = [\hat{\alpha}(0)/r\partial_r]$, conclude that $[x] \in \text{span}\{\left[\frac{1}{r}\partial_r\right]\}$ and

$$\mathfrak{g}_{\mathbb{C}}/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R\rangle = \text{span}\{\left[\frac{1}{r}\partial_r\right]\} \tag{5.17}$$

□

Note that we only used properties of the vector space quotient to show the above and $\frac{1}{r}\partial_r \notin I + J + \mathfrak{w}_L + \mathfrak{w}_R$. Hence, we have $\mathfrak{g}_{\mathbb{C}} = \text{span}\{\frac{1}{r}\partial_r\} \oplus (I + J + \mathfrak{w}_L + \mathfrak{w}_R)$ as a vector space.

We could still have that $[\frac{1}{r}\partial_r] = [0]$ however. We will now show that this is not the case.

Lemma 5.4. $\frac{1}{r}\partial_r \notin [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$

Proof. We will proof this by contradiction. Suppose $\frac{1}{r}\partial_r \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Then there exist $\xi_i, F_i \in \mathfrak{g}_{\mathbb{C}}$ such that

$$\sum_i [\xi_i, F_i] = \frac{1}{r}\partial_r \tag{5.18}$$

Now let the brackets $[\cdot]'$ denote the equivalence class brackets in the vector space quotient $\mathfrak{g}_{\mathbb{C}}/(I + J + \mathfrak{w}_L + \mathfrak{w}_R)$. Then as $\frac{1}{r}\partial_r \notin I + J + \mathfrak{w}_L + \mathfrak{w}_R$, there is at least one i such that $[[\xi_i, F_i]]' \neq 0$. As the vector space quotient is one dimensional, $[[\xi_i, F_i]]' = [\frac{c}{r}\partial_r]'$ for some constant c. Now without loss of generality we can assume that there exist $\xi, F \in \mathfrak{g}_{\mathbb{C}}$ such that

$$[[\xi, F]]' = [1/r\partial_r]' \tag{5.19}$$

If $\xi, F \in \mathfrak{w}_L + \mathfrak{w}_R$, by proposition 4.30 $[\xi, F] \in \mathfrak{w}_L + \mathfrak{w}_R + I + J$, so $[[\xi, F]]' = 0$. The same holds if $\xi, F \in \mathfrak{i}_{\mathbb{C}}$. Thus we can assume $\xi \in \mathfrak{w}_L + \mathfrak{w}_R$ and $F \in \mathfrak{i}_{\mathbb{C}}$. In that case there exist vector fields on the circle $X, \bar{X} \in C^\infty(\mathbb{S}^1)$ such that $\xi = \xi_{(X,\bar{X})}$ and we have $A \in \mathcal{O}(r^{-1}), B, C \in \mathcal{O}(r^{-4})$ such that $F = A\partial_r + B\partial_+ + C\partial_-$. Then $[\xi, F] = [\xi, F]_r\partial_r + [\xi, F]_+\partial_+ + [\xi, F]_-\partial_-$ with $[\xi, F]_+\partial_+ + [\xi, F]_-\partial_- \in I + J$ by equation 5.4, so $[[\xi, F]]' = [[\xi, F]_r\partial_r]'$

$$\begin{aligned}[\xi_{(X,\bar{X})}, F]_r = &-\frac{r}{2}(\partial_+X + \partial_-\bar{X})\partial_r A + (X + \frac{l^2}{2r^2}\partial_-^2\bar{X})\partial_+A + (\bar{X} + \frac{l^2}{2r^2}\partial_+^2 X)\partial_- A + \\ &\frac{A}{2}(\partial_+X + \partial_-\bar{X}) + \frac{r}{2}B\partial_+^2 X + \frac{r}{2}C\partial_-^2\bar{X}\end{aligned} \tag{5.20}$$

Now $\frac{l^2}{2r^2}\partial_-^2\bar{X}\partial_+A\partial_r, \frac{l^2}{2r^2}\partial_+^2 X\partial_- A\partial_r, \left(\frac{r}{2}B\partial_+^2 X + \frac{r}{2}C\partial_-^2\bar{X}\right)\partial_r \in I + J$, as all the coefficient functions are $\mathcal{O}(r^{-3})$.

$$\begin{aligned}
&\left[[\xi_{(X,\bar{X})}, F]_r \partial_r\right]' = \\
&\left[\left(-\frac{r}{2}(\partial_+ X + \partial_- \bar{X})\partial_r A + X\partial_+ A + \bar{X}\partial_- A + \frac{A}{2}(\partial_+ X + \partial_- \bar{X})\right)\partial_r\right]' = \\
&\left[\left(-\frac{r}{2}(\partial_+ X + \partial_- \bar{X})\partial_r A - \frac{A}{2}(\partial_+ X + \partial_- \bar{X}) + \partial_+(XA) + \partial_-(\bar{X}A)\right)\partial_r\right]'
\end{aligned} \tag{5.21}$$

The functions $\partial_+(XA), \partial_-(\bar{X}A)$ clearly have zero integral over x^+, x^- respectively, as $XA, \bar{X}A$ are periodic functions. Hence $\partial_+(XA)\partial_r, \partial_-(\bar{X}A)\partial_r \in J$.

$$\begin{aligned}
&\left[[\xi_{(X,\bar{X})}, F]_r \partial_r\right]' = \left[-\frac{r}{2}(\partial_+ X + \partial_- \bar{X})\partial_r A - \frac{A}{2}(\partial_+ X + \partial_- \bar{X})\right]' = \\
&\left[-\frac{1}{2}(\partial_+ X + \partial_- \bar{X})\partial_r (rA)\right]'
\end{aligned} \tag{5.22}$$

This term is also zero, as $rA \in \mathcal{O}(1)$ and $\partial_r(rA) \in \mathcal{O}(r^{-2})$ by proposition 4.14. Then $-\frac{1}{2}(\partial_+ X + \partial_- \bar{X})\partial_r(rA) \in \mathcal{O}(r^{-2})$ so $-\frac{1}{2}(\partial_+ X + \partial_- \bar{X})\partial_r(rA)\partial_r \in I + J$. We conclude that $\left[[\xi_{(X,\bar{X})}, F]\right]' = [0]'$, which is a contradiction as $[1/r\partial_r]' \neq 0$. □

Theorem 5.5. *The commutator is given by*

$$[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = \langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle \tag{5.23}$$

the quotient of $\mathfrak{g}_{\mathbb{C}}$ by this ideal is then

$$\mathfrak{g}_{\mathbb{C}}/[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = \operatorname{span}\{\frac{1}{r}\partial_r\} \tag{5.24}$$

and from lemma 2.18, we immediately find

$$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{\frac{1}{r}\partial_r\} \tag{5.25}$$

Proof. The inclusion $\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle \subseteq [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ was proven in proposition 5.1. Now let $a \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Using corollary 2.17, we have

$$[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle = [\mathfrak{g}_{\mathbb{C}}/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle, \mathfrak{g}_{\mathbb{C}}/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle] = 0 \tag{5.26}$$

as the quotient Lie algebra is 1-dimensional and therefore abelian. Furthermore the equivalence class $[a] \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]/\langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle$ and $[a] = 0$, so $a \in \langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle$. Thus $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \subseteq \langle I + J + \mathfrak{w}_L + \mathfrak{w}_R \rangle$ and equality follows.

$\mathfrak{g}_{\mathbb{C}}/[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = \operatorname{span}\{\frac{1}{r}\partial_r\}$ is then immediate from lemma 5.3 and lemma 5.4 □

Now the quotient space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is Hausdorff if and only if $[\mathfrak{g}, \mathfrak{g}]$ is closed. As $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{\frac{1}{r}\partial_r\}$, as the supremum norm separates these functions and the supremum norm is in the quotient topology, this quotient is Hausdorff. Thus $[\mathfrak{g}, \mathfrak{g}]$ is closed and $H^1_{ct}(\mathfrak{g}) = \mathbb{R}$.

5.2 Pullback to boundary

In this section we will consider cocycles ω and what happens if we smoothly pull them back to the boundary defined by $\Omega = 0$. We will see that this pullback allows us to construct a coboundary $d\phi$ that is equal to ω on a domain of vector fields that vanish "fast enough" when pulled back to the boundary, meaning that ω is cohomologous to a cocycle that is zero on this domain. This will give us a short exact sequence that will help us calculate the second cohomology group.

5.2.1 Flows

Let $\mathfrak{g}$ be a subalgebra of the Lie algebra of vector fields on some manifold M with boundary ∂M and let $Y \in \mathfrak{g}$ be some vector field tangent to the boundary. We will assume that $\mathfrak{g}$ is equipped with the compact-open topology. By the Fundamental Theorem on Flows [Lee12], there exists a unique smooth maximal flow $Q : \tilde{\mathcal{D}} \to M$ whose generator is Y, where $\tilde{\mathcal{D}} \subseteq \mathbb{R} \times M$ is an open subset such that for $p \in M$, $\tilde{\mathcal{D}}^{(p)} = \{t \in \mathbb{R} | (t,p) \in \tilde{\mathcal{D}}\}$ is an open interval containing zero. Now let $k \geq 1$, $f(t) = -\frac{1}{(k-1)t^{k-1}} + \frac{1}{k-1}$ if $k > 1$ and $f(t) = \ln(t)$ if $k = 1$. Define $\mathcal{D}^{(p)} = \{t > 0 | (f(t), p) \in \tilde{\mathcal{D}}\}$, which is an open interval containing one. Next, let $\mathcal{D} = \{(t,p) | p \in M, t \in \mathcal{D}^{(p)}\}$ and $T : \mathcal{D} \to M : (t,p) \to Q^{f(t)}(p)$. Then T^t is the flow generated by $Y^t := t^{-k}Y$:

$$\dot{T}^t = \frac{d}{dt} Q^{f(t)} = f'(t) Y(Q^{f(t)}) = t^{-k} Y(T^t) = Y^t(T^t) \tag{5.27}$$

and $T^1 = Q^{f(1)} = Q^0 = 1$. The flows at different times commute, as is shown by the following calculation: $T^t T^s = Q^{f(t)} Q^{f(s)} = Q^{f(t)+f(s)} = Q^{f(s)} Q^{f(t)} = T^s T^t$. Furthermore, for $k = 1$, $(T^t)^{-1} = T^{1/t}$ as $T^t T^{1/t} = Q^{\ln(t) + \ln(1/t)} = Q^0 = 1$.

If necessary, we will call flows generated by time-independent vector fields time-independent flows, and flows generated by time-dependent vector fields time-dependent flows. Most of the time, we will just call them flows as the context will show what we mean.

For a vector field $X \in \mathfrak{g}$ and a cochain $c \in \mathcal{C}^q(\mathfrak{g})$, the pullback on X and pushforward on c under a diffeomorphism ϕ are given by:

$$(\phi^* X)_p = ((\phi^{-1})_* X)_p = (\phi^{-1}_{\phi(p)})_* X_{\phi(p)} \tag{5.28}$$

$$\phi_* c(X_1, \ldots, X_q) = c(\phi^* X_1, \ldots, \phi^* X_q) \tag{5.29}$$

Now note that ϕ^{-1} is only evaluated at $\phi(p)$, so only on the image of ϕ. Thus even if ϕ is only a local diffeomorphism, meaning that each x in its domain has an open neighborhood U such that $\phi|_U : U \to \phi(U)$ is a diffeomorphism, we can still define the pullback of vector fields and pushforward on cochains. Furthermore, every flow is a local diffeomorphism, so the pullback of a vector field by a flow is well defined.

In coordinates, $(\phi^{-1}_{\phi(p)})_*$ is given by the Jacobian $J(\phi^{-1})_{\phi(p)}$, which by the inverse function theorem is given by:

$$J(\phi^{-1})_{\phi(p)} = (J\phi_p)^{-1} \tag{5.30}$$

The pushforward on c is only well defined if all of the pullbacks are again in $\mathfrak{g}$, so we will define $\mathfrak{g}$-flows in this way:

Definition 5.6 (Half-global and $\mathfrak{g}$-flows)**.**

- A time-dependent flow S is called half-global if $(0,1] \in \mathcal{D}^{(p)}$ for all $p \in M$.
- A time-dependent flow S is called a $\mathfrak{g}$-flow if S is half-global and $(S^t)^*X \in \mathfrak{g}$ for all $X \in \mathfrak{g}$.
- A time-independent flow Q is called a $\mathfrak{g}$-flow if Q is global and $(Q^t)^*X \in \mathfrak{g}$ for all $X \in \mathfrak{g}$.

Note that for $T^t = Q^{f(t)}$ with Q a time-independent flow, T is half-global whenever Q is global. The converse does not hold, as is shown by the vector field $Y = \Omega^2\partial_\Omega$, which generates the flow $Q^t_\Omega = \frac{\Omega}{1-t\Omega}$, which is clearly not global, but $T^t = Q^{f(t)} = \frac{\Omega}{1+\Omega/t-\Omega} = \frac{t\Omega}{t+\Omega(1-t)}$ is half global. We now show that pullbacks by a weak asymptotic symmetry are again in the weak asymptotic symmetry algebra. Thus if a half-global flow is also an asymptotic symmetry for all t then it is a $\mathfrak{g}$-flow.

Lemma 5.7. *Suppose $\mathfrak{g}$ is a weak asymptotic symmetry algebra and let ψ be a diffeomorphism in the asymptotic symmetry group. Then $\psi^*(\xi) \in \mathfrak{g}$ for all $\xi \in \mathfrak{g}$.*

Proof. As $\xi \in \mathfrak{g}$, there exists a smooth curve of diffeomorphisms ϕ_t such that $\phi_0 = 1$ and $\dot{\phi}_0 = \xi$. Now $\psi^*(\xi) = \frac{d}{dt}|_{t=0}\psi^{-1}\phi_t\psi \in \mathfrak{g}$ as $\psi^{-1}\phi_t\psi$ is once again a smooth curve in the weak asymptotic symmetry group. □

We now show that the pushforward T^t_*c is again continuous. To see this, note that for any seminorm p on $\mathfrak{g}$, T^t_*p is again a continuous seminorm, as right composition and multiplication by a smooth function is continuous in the compact open topology, see Appendix C. Moreover, as c is continuous, there exist seminorms $p_1, \dots, p_q$ such that $|c(X_1, \dots, X_q)| \leq p_1(X_1)\dots p_q(X_q)$, in which case $|T^t_*c(X_1, \dots, X_q)| = |c((T^t)^*X_1, \dots, (T^t)^*X_q)| \leq T^t_*p_1(X_1)\dots T^t_*p_q(X_q)$. As these are again continuous seminorms, T^t_*c is continuous by lemma 2.8.

5.2.2 Projections

The pullback $(T^t)^*X$ might diverge in the limit $t \to 0$ for $X \in \mathfrak{g}$. We will therefore consider all cochains restricted to a smaller domain and see if these can be pulled back to the boundary. We will define projections that achieve this. Let $\mathfrak{j}$ be an ideal in $\mathfrak{g}$ such that $\bar{\mathfrak{j}}$ is complemented. Then $\mathfrak{g} = \bar{\mathfrak{j}} \oplus \mathfrak{g}/\bar{\mathfrak{j}}$ as a topological vector space, meaning that there exists a continuous projection $\pi_{\bar{\mathfrak{j}}} : \mathfrak{g} \to \bar{\mathfrak{j}}$ and a continuous section $s : \mathfrak{g}/\bar{\mathfrak{j}} \to \mathfrak{g}$. We therefore also have a continuous projection onto the lifted quotient, $\pi_{\mathfrak{q}} : \mathfrak{g} \to s(\mathfrak{g}/\bar{\mathfrak{j}})$, which is given by $\pi_{\mathfrak{q}} = 1 - \pi_{\bar{\mathfrak{j}}}$. Thus every $X \in \mathfrak{g}$ decomposes as $X = \pi_{\bar{\mathfrak{j}}}X + \pi_{\mathfrak{q}}X$. Let $I \subseteq \{1, 2, \dots, q\}$ be a set of indices, and define $X_i^I = \pi_{\bar{\mathfrak{j}}} X_i$ if $i \in I$ and $X_i^I = \pi_{\mathfrak{q}} X_i$ if $i \notin I$. Lastly we define a projection $P_p : C^q_{ct}(\mathfrak{g}) \to C^q_{ct}(\mathfrak{g})$ on the space of cochains using this notation:

$$P_p(c)(X_1, \ldots, X_q) = \sum_{\substack{I \subseteq \{1,2,\ldots,q\} \\ |I|=p}} c(X_1^I, \ldots, X_q^I) \tag{5.31}$$

The P_p operator thus projects the arguments such that p of the arguments are in $\bar{\mathfrak{j}}$. Note that $P_p(c)$ is again continuous as the projections are continuous and clearly still multilinear and alternating. We now claim that every cochain decomposes in terms of these projections.

Lemma 5.8. *Let $c \in C^q_{ct}(\mathfrak{g})$ be a continuous q-cochain. Then for $X_1, \ldots, X_q \in \mathfrak{g}$*

$$c(X_1, \ldots, X_q) = \sum_{p=0}^{q} P_p(c)(X_1, \ldots, X_q) \tag{5.32}$$

Furthermore, $P_k(P_p(c))(X_1, \ldots, X_q) = 0$ if $k \neq p$ and $P_p(P_p(c))(X_1, \ldots, X_q) = P_p(c)(X_1, \ldots, X_q)$

Proof. First, for every $I \subseteq \{1, 2, \ldots, q\}$, if I contains l, take $I' = I \setminus \{l\}$ and if not, $I' = I \cup \{l\}$. Now $c(X_1^I, \ldots, X_q^I) + c(X_1^{I'}, \ldots, X_q^{I'}) = c(X_1^I, \ldots, X_l, \ldots, X_q^I)$. Therefore

$$\sum_{I \subseteq \{l, l+1, \ldots, q\}} c(X_1, \ldots, X_l^I, \ldots, X_q^I) = \sum_{I \subseteq \{l+1, \ldots, q\}} c(X_1, \ldots, X_{l+1}^I, \ldots, X_q^I) \tag{5.33}$$

and using this formula for $l = 1$ through $l = q$ gives

$$\sum_{p=0}^{q} P_p(c)(X_1, \ldots, X_q) = \sum_{I \subseteq \{1,2,\ldots,q\}} c(X_1^I, \ldots, X_q^I) = c(X_1, \ldots, X_q) \tag{5.34}$$

Now for the second part, let $I, J \subseteq \{1, 2, \ldots, q\}$, then $(X_l^I)^J = 0$ if $l \in I \setminus J \cup J \setminus I$ thus the only way X_l can be nonzero is if $l \in I \cap J \cup (I \cup J)^c$. Therefore $c((X_1^I)^J, \ldots, (X_q^I)^J) \neq 0$ only if for every $l \in \{1, \ldots, q\}$, either l is in both I, J or in neither. In that case $I = J$. We conclude that $c((X_1^I)^J, \ldots, (X_q^I)^J) \neq 0$ only if $I = J$. Therefore

$$P_k(P_p(c))(X_1, \ldots, X_q) = \sum_{\substack{I \subseteq \{1,2,\ldots,q\} \\ |I|=p}} \sum_{\substack{J \subseteq \{1,2,\ldots,q\} \\ |J|=k}} c((X_1^J)^I, \ldots, X_q^J) = \sum_{\substack{I \subseteq \{1,2,\ldots,q\} \\ |I|=p}} \sum_{\substack{J=I \\ |J|=k}} c((X_1^J)^I, \ldots, X_q^J) \tag{5.35}$$

which is zero if $k \neq p$ and equal to $P_p(c)(X_1, \ldots, X_q)$ otherwise. $\square$

The projections P_p are thus orthonormal and from this we show that the map $P : C^q_{ct}(\mathfrak{g}) \to \bigoplus_{p=0}^{q} \operatorname{im}(P_k)$ is a linear isomorphism.

Proposition 5.9.

$$C^q_{ct}(\mathfrak{g}) \cong \bigoplus_{p=0}^{q} \operatorname{im}(P_k) \tag{5.36}$$

Proof. Let $P : C^q_{ct}(\mathfrak{g}) \to \bigoplus_{p=0}^q \mathrm{im}(P_k) : c \to \sum_{p=0}^q P_p(c)$. First, we note that if $(c_0, \ldots, c_q) \in \bigoplus_{p=0}^q \mathrm{im}(P_k)$, then each c_p is continuous as P_p maps continuous cochains to continuous cochains. Their sum is then again continuous and therefore $\bigoplus_{p=0}^q \mathrm{im}(P_k) \subseteq C^q_{ct}(\mathfrak{g})$. Next, as P_p is clearly linear, P is linear as well. Furthermore, P is idempotent, which follows from the orthogonality of P_p, and therefore surjective. For injectivity, note that if $Pc = 0$, then $P_p(P(c)) = P_p(c)$, so $P_p(c) = 0$ for all p. Then as $\sum_{p=0}^q P_p(c) = c$ by lemma 5.8, we have that $c = 0$. Thus P is injective. We conclude that P is a linear isomorphism. $\square$

Next we will show P_p how 'commutes' with the differential.

Proposition 5.10. *Let $c \in C^{q-1}_{ct}(\mathfrak{g})$ then if $p > 1$*

$$P_p dc = P_p d(P_{p+1}c + P_p c + P_{p-1}c) \tag{5.37}$$

and

$$P_1 dc = P_1 d(P_2 c + P_1 c) \tag{5.38}$$

Proof. First, note $P_p dc = P_p(d \sum_{k=0}^q P_k c) = \sum_{k=0}^q P_p d P_k c$. Now

$$\begin{aligned} P_p d P_i c(X_1, \ldots, X_q) &= \sum_{\substack{I \subseteq \{1,2,\ldots,q\} \\ |I|=p}} dP_i c(X_1^I, \ldots, X_q^I) \\ &= \sum_{\substack{I \subseteq \{1,2,\ldots,q\} \\ |I|=p}} \sum_{1 \le i < j \le q} (-1)^{i+j} P_i c([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^j, \ldots, \hat{X}_j^I, \ldots, X_q^I) \end{aligned} \tag{5.39}$$

Now $P_k c([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^I, \ldots, \hat{X}_j^I, \ldots, X_q^I)$ is only nonzero if there are at least k arguments with a nonzero projection onto $\mathfrak{j}$. If $p \le k-2$, there are at most $p + 1 < k$ with nonzero projection onto $\mathfrak{j}$ and thus $P_p d P_k c = 0$ for $p \le k-2$. Next, $P_k c([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^I, \ldots, \hat{X}_j^I, \ldots, X_q^I)$ is only nonzero if there are at least $q-1-k$ arguments with a nonzero projection onto $\mathfrak{g}/\mathfrak{j}$. If $p \ge k+2$, then at least $k+2$ of the arguments $X_1^I, \ldots, X_q^I$ are projected onto $\mathfrak{j}$ and therefore at least $k+1$ of the arguments $([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^I, \ldots, \hat{X}_j^I, \ldots, X_q^I)$ are in $\mathfrak{j}$ as well. Therefore at most $q-k-2$ of the arguments have a nonzero projection onto $\mathfrak{g}/\mathfrak{j}$. Thus $P_p d P_k c = 0$ for $p \ge k+2$. We conclude that $P_p dc = P_p d(P_{p+1}c + P_p c + P_{p-1}c)$. Now for $p = 1$, at least one of the arguments $([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^I, \ldots, \hat{X}_j^I, \ldots, X_q^I)$ is in $\mathfrak{j}$ and thus at most $q-2$ have nonzero projection onto $\mathfrak{g}/\mathfrak{j}$. Therefore $P_0 c([X_i^I, X_j^I], X_1^I, \ldots, \hat{X}_i^I, \ldots, \hat{X}_j^I, \ldots, X_q^I) = 0$ and thus $P_1 d P_0 c = 0$. $\square$

5.2.3 l-trivial ideals

We will now show that if $\mathfrak{j}$ is something called an l-trivial ideal, for each cocycle we can construct a coboundary such that they are equal on a part of the domain of the cocycle. These ideals depend on the order γ of the cochains that we are considering, which is the smallest γ such that the cochains are continuous in the the C^γ topology. Every cochain is of finite order as shown in the next lemma.

Lemma 5.11. *Let $c \in C^q_{ct}(\mathfrak{g})$. Then c is of order γ for some $\gamma \geq 0$.*

Proof. As c is continuous in the C^∞ topology, there exist seminorms $p_{K_1,n_1}, \dots, p_{K_q,n_q}$ such that $c(X_1, \dots, X_q) \leq p_{K_1,n_1}(X_1) \dots p_{K_q,n_q}(X_q)$. Then define $\gamma = \max n_i$, so that p_{K_i,n_i} is a continuous seminorm in the C^γ topology for all $1 \leq i \leq q$. Then c is continuous in the C^γ topology as well. □

We are now in position to define what an l-trivial ideal is.

Definition 5.12 (l-trivial ideal)**.** Let $l > 0, \gamma \geq 0$ be integers and suppose T^t is a $\mathfrak{g}$-flow, then we define an l-trivial ideal $\mathfrak{j} \subseteq \mathfrak{g}$ of order γ to be an ideal such that for all $c \in \mathcal{C}^q_{ct}(\mathfrak{g})$ that are continuous with respect to the C^γ-topology,

$$\lim_{t \to 0} P_p(T^t_* c)(X_1, \dots, X_q) = 0 \tag{5.40}$$

for all $p \geq l$ and

$$\lim_{t \to 0} P_p(\iota_{Y_t} T^t_* c)(X_1, \dots, X_{q-1}) \text{ is finite} \tag{5.41}$$

for all $p \geq \max(l-1, 1)$. Furthermore, we also need that for all $p \geq \max(l-1, 1)$,

$$|P_p(\iota_{Y_t} T^t_* c)(X_1, \dots, X_{q-1})| \leq g(t)\tilde{p}(Y) p_1(X_1) \dots p_{q-1}(X_{q-1}) \tag{5.42}$$

where $g(t)$ is an integrable function on $(0, 1]$ and $\tilde{p}, p_1, \dots, p_{q-1}$ are continuous seminorms. If we do not mention the order of the ideal, it is of order ∞.

Note that if $\gamma_1 \geq \gamma_2$, then any l-trivial ideal of order γ_1 is included in an l-trivial ideal of order γ_2, as the C^{γ_1} topology is finer than the C^{γ_2} topology.

We define $m = \max(l-1, 1)$ as a shorthand.

We can calculate $\mathfrak{j}$, if we assume that there exists some number ν such that $\lim_{t\to 0} t^\nu (T^t)^* X \in \mathfrak{g}$ for all $X \in \mathfrak{g}$ and the convergence is in the C^γ topology. We note that these assumptions are actually unnecessary if $l = q > 1$.

Proposition 5.13. *Let $q \geq 2, q \geq l > 0$ be some integers and take $\mathfrak{j}$ to be an ideal such that:*

$$\mathfrak{j} \subseteq \{X \in \mathfrak{g} : \lim_{t \to 0} t^{-r}(T^t)^* X \in \mathfrak{g}\} \tag{5.43}$$

where the convergence is in the C^γ topology, $m = \max(l-1, 1)$ and $r = \max\left(\frac{(q-1-m)\nu + k}{m}, \frac{(q-l)\nu+\epsilon}{l}\right)$ where $\epsilon > 0$ is arbitrary. Then if equation 5.42 holds as well, $\mathfrak{j}$ is an l-trivial ideal of order γ.

Proof. Let c be a q-cochain that is continuous with respect to the C^γ topology. Then, to prove equation 5.40, note that

$$\lim_{t \to 0} P_p(T^t_* c)(X_1, \dots, X_q) = \sum_{\substack{I \subseteq \{1,2,\dots,q\} \\ |I| = p}} T^t_* c(X^I_1, \dots, X^I_q) \tag{5.44}$$

and as $T^t_* c$ is alternating, we can always, under the possible introduction of a minus sign, reorder $(X^I_1, \ldots, X^I_q)$ such that $I = \{1, 2, \ldots, p\}$. In that case

$$\lim_{t\to 0} t^{-\epsilon} T^t_* c(X^I_1, \ldots, X^I_q) = \lim_{t\to 0} c(t^{-r'}(T^t)^* X^I_1, \ldots, t^{-r'}(T^t)^* X^I_l, t^\nu (T^t)^* X^I_{l+1}, \ldots, t^\nu (T^t)^* X^I_q) \tag{5.45}$$

with $r' = \frac{(q-l)\nu+\epsilon}{l}$ and the last limit exists as $\lim_{t\to 0} t^\nu (T^t)^* X$ and $\lim_{t\to 0} t^{-r'}(T^t)^* Y$ are again in $\mathfrak{g}$ when $Y \in \mathfrak{j}$ and as we used that $r' \leq r$, $p \geq l$. As this limit exists and $\lim_{t\to 0} t^\epsilon = 0$, we must have $\lim_{t\to 0} T^t_* c(X^I_1, \ldots, X^I_q) = 0$, and therefore $\lim_{t\to 0} P_p(T^t_* c)(X_1, \ldots, X_q) = 0$.

Then for the second limit, first note that $(T^t)^* Y^t = Y^t$. Now take $\tilde{r} = \frac{(q-1-m)\nu+k}{m}$, then

$$\lim_{t\to 0} P_p(\iota_{Y^t} T^t_* c)(Y_1, \ldots, Y_{q-1}) = \lim_{t\to 0} \sum_{\substack{I\subseteq\{1,2,\ldots,q-1\} \\ |I|=p}} t^{-k} c(Y, (T^t)^* Y^I_1, \ldots, (T^t)^* Y^I_{q-1}) \tag{5.46}$$

which exists if $\lim_{t\to 0} t^{-\tilde{r}m+\nu(q-1-m)} c(Y, (T^t)^* Y^I_1, \ldots, (T^t)^* Y^I_q)$ exists as $-k = -\tilde{r}m + \nu(q-1-m)$. Then, again without loss of generality, assume $I = \{1, \ldots, m\}$, in which case

$$\begin{aligned}&\lim_{t\to 0} t^{-\tilde{r}m+\nu(q-1-m)} c(Y, (T^t)^* Y_1, \ldots, (T^t)^* Y_q) = \\ &\lim_{t\to 0} c(Y, t^{-\tilde{r}}(T^t)^* Y_1, \ldots, t^{-\tilde{r}}(T^t)^* Y_m, t^\nu Y_{m+1}, \ldots, t^\nu (T^t)^* Y_{q-1})\end{aligned} \tag{5.47}$$

exists as $\tilde{r} \leq r$ and by continuity of c.

□

For the following proposition, note that the closure of an ideal $\mathfrak{j}$ is again an ideal, as for $x \in \bar{\mathfrak{j}}$, there exists a sequence $(x_n)_{n\geq 0} \subseteq \mathfrak{j}$ such that $x_n \to x$ and for $y \in \mathfrak{g}$, $[x, y] = \lim_{n\to\infty}[x_n, y] \in \bar{\mathfrak{j}}$.

Proposition 5.14. *Suppose T^t is a $\mathfrak{g}$-flow. Let $l > 0$ be an integer and let $\mathfrak{j}$ be an l-trivial ideal of order γ, such that its closure $\bar{\mathfrak{j}}$ is complemented. Then if c is a q-cocycle that is C^γ continuous, there exists a q-coboundary $d\phi \in \mathcal{C}^q_{ct}(\mathfrak{g})$ such that $P_p(c(X_1, \ldots, X_q)) = P_p(d\phi)(X_1, \ldots, X_q)$ whenever $p \geq l$.*

Proof. By the first condition, equation 5.40, for $p \geq l$, we have

$$\begin{aligned}P_p(c)(X_1, \ldots, X_q) &= \lim_{t\to 0}(P_p(T^1_* c) - P_p(T^t_* c))(X_1, \ldots, X_q) \\ &= \lim_{t\to 0}(P_p(K_{1,t} dc) + P_p(dK_{1,t} c))(X_1, \ldots, X_q) = \lim_{t\to 0} P_p(dK_{1,t} c)(X_1, \ldots, X_q)\end{aligned} \tag{5.48}$$

where $K_{1,t}$ is the operator from lemma 3.31 and we used that c is a cocycle. Now for

$p \geq m = \max(l-1, 1)$, the following pointwise limit exists:

$$\begin{aligned}
&\lim_{t\to 0} P_p(K_{1,t}c)(X_1,\ldots,X_{q-1}) = \lim_{t\to 0}\int_t^1 P_p(\iota_{Y^s}T_*^s c)(X_1,\ldots,X_{q-1})ds = \\
&\lim_{t\to 0}\int_t^1 P_p(\frac{1}{s^k}T_*^s c)(Y, X_1,\ldots,X_{q-1})ds =: P_p(Kc)(X_1,\ldots,X_{q-1})
\end{aligned} \tag{5.49}$$

as the limit exists in the second line because the integral exists for all $t > 0$ and $\lim_{s\to 0} P_p(\frac{1}{s^k}T_*^s c)(Y, X_1, \ldots, X_{q-1})$ is finite by the second condition, equation 5.41, meaning that the integrand is bounded in a neighborhood of 0. Moreover, $|P_p(Kc)(X_1,\ldots,X_{q-1})| \leq \int_0^1 g(t)dt\tilde{p}(Y)p_1(X_1)\ldots p_{q-1}(X_{q-1})$, where $\tilde{p}, p_1, \ldots, p_{q-1}$ are continuous seminorms, and therefore $P_p(Kc)$ is continuous.

Furthermore, if $p \geq l$, by proposition 5.10,

$$\lim_{t\to 0} P_p(dK_{1,t}c)(X_1,\ldots,X_q) = \lim_{t\to 0} P_p(d(P_{p+1}+P_p+P_{p-1}1_{p\geq 2})K_{1,t}c)(X_1,\ldots,X_q) \tag{5.50}$$

which exists as $\lim_{t\to 0} P_k K_{1,t}c$ exists whenever $k \geq \max(l-1,1)$ and as $p \geq l$. Thus $P_p(c)(X_1,\ldots,X_q) = P_p(dKc)(X_1,\ldots,X_{q-1})$. We do note that Kc is not actually a cochain on $\mathfrak{g}$, as it is not defined on all of $\mathfrak{g}$. We thus need to find a $\phi \in C_{ct}^{q-1}(\mathfrak{g})$ such that $P_p(\phi) = P_p(Kc)$ whenever $p \geq m$. $\phi = \sum_{p=m}^{q-1} P_p(Kc)$ does the job.

□

5.2.4 Direct limits

The previous proposition shows that if $\mathfrak{j}_\gamma$ is a complemented 1-trivial ideal of order γ, then the cocycles of order γ are cohomologous to a cocycle on $\mathfrak{g}/\mathfrak{j}_\gamma$. From this we can construct an exact sequence relating *direct limits* of cohomology groups of the form $H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma)$. We first define what a direct limit of groups is.

Definition 5.15 (Direct system and direct limit)**.** Let $\{A_i | i \in \mathbb{N}\}$ be a set of groups and $\pi_{i,j} : A_i \to A_j$ a homomorphism for $i \leq j$ such that $\pi_{ii} = 1$ and $\pi_{i,k} = \pi_{j,k}\pi_{i,j}$. Then $(A_i, \pi_{i,j})$ is a direct system and the direct limit $\varinjlim A_i$ is defined as $\bigsqcup_i A_i / \sim$ where $a_i \sim a_j$ for $a_i \in A_i, a_j \in A_j$ means there exists a $k \in \mathbb{N}$ such that $k \geq i, j$ and $\pi_{i,k}(a_i) = \pi_{j,k}(a_j)$

Now suppose $(\mathfrak{j}_\gamma)_{\gamma\in\mathbb{N}}$ is a decreasing sequence of complemented ideals. For $\gamma' \leq \gamma$ we define the projection $\pi_{\gamma',\gamma} : \mathfrak{g}/\mathfrak{j}_\gamma \to \mathfrak{g}/\mathfrak{j}_{\gamma'}$ which induces a cochain homomorphism $\pi_{\gamma',\gamma}^* : H_{ct}^q(\mathfrak{g}/\mathfrak{j}_{\gamma'}) \to H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma)$. It is then easily verified that $(H_{ct}^q(\mathfrak{g}/\mathfrak{j}_{\gamma'}), \pi_{\gamma',\gamma}^*)$ is a direct system. We now further define $\pi_\gamma : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}_\gamma$ and claim that $\pi_\gamma^* : H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma) \to H_{ct}^q(\mathfrak{g})$ descends to a well defined map $\varinjlim \pi_\gamma^* : \varinjlim H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma) \to H_{ct}^q(\mathfrak{g})$.

Lemma 5.16. *The map $\varinjlim \pi_\gamma^* : \varinjlim H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma) \to H_{ct}^q(\mathfrak{g})$ defined by $[[\omega]] \to \pi_\gamma^*[\omega]$ for $[\omega] \in H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma)$ is well defined.*

Proof. We need to check that for $[\omega], [\omega'] \in [[\omega]]$, with $[\omega] \in H_{ct}^q(\mathfrak{g}/\mathfrak{j}_\gamma), [\omega'] \in H_{ct}^q(\mathfrak{g}/\mathfrak{j}_{\gamma'})$, we have $\pi_\gamma^*[\omega] = \pi_{\gamma'}^*[\omega']$. To this end, note that as $[\omega] \sim [\omega']$, there exists $k \geq \gamma, \gamma'$ such that $\pi_{\gamma,k}^*[\omega] = \pi_{\gamma',k}^*[\omega']$. Furthermore, $\pi_\gamma = \pi_{\gamma,k}\pi_k$ and thus $\pi_\gamma^*[\omega] = \pi_k^*\pi_{\gamma,k}^*[\omega] = \pi_k^*\pi_{\gamma',k}^*[\omega'] = \pi_{\gamma'}^*[\omega']$. □

We now show that if $\mathfrak{j}_\gamma$ is a 1-trivial ideal of order γ for all $\gamma \in \mathbb{N}$, then $\varinjlim \pi_\gamma^*$ is surjective. We note that intuitively if $\mathfrak{j}_\gamma$ is a 1-trivial ideal of order γ, then all q-cocycles ω of order γ are zero whenever at least one of the arguments is in $\mathfrak{j}_\gamma$. The q-cocycle is thus completely determined by its values on $\mathfrak{g}/\mathfrak{j}_\gamma$, and we can hence expect that $\varinjlim \pi_\gamma^*$ is surjective.

Proposition 5.17. *Suppose $\mathfrak{j}_\gamma$ is a complemented 1-trivial ideal of order γ for all $\gamma \in \mathbb{N}$, then $\varinjlim \pi_\gamma^*$ is surjective.*

Proof. Let $[\omega'] \in H^q_{ct}(\mathfrak{g})$ then ω' has some order γ and by proposition 5.14, there exists a q-coboundary $d\phi \in C^q(\mathfrak{g})$ such that $P_p(\omega' - d\phi) = 0$ for $p \geq 1$. Defining $\omega = \omega' - d\phi$ and noting that $\omega = \sum_{p=0}^q P_p(\omega)$ by lemma 5.8, we have $\omega = P_0(\omega) = (s_\gamma \pi_\gamma)^* \omega = \pi_\gamma^* s_\gamma^* \omega$, where $s_\gamma : \mathfrak{g}/\mathfrak{j}_\gamma \to \mathfrak{g}$ is the continuous linear section. Now if we can show that $s_\gamma^* \omega$ is a q-cocycle, we are done, as then $\varinjlim \pi_\gamma^*[[s_\gamma^*\omega]] = \pi_\gamma^*[s_\gamma^*\omega] = [\omega]$. Therefore we calculate the differential:

$$\begin{aligned} &ds_\gamma^*\omega([X_1], \dots, [X_{q+1}]) = \pi_\gamma^* ds_\gamma^* \omega(X_1, \dots, X_{q+1}) = \\ &dP_0(\omega)(X_1, \dots, X_{q+1}) = d\omega(X_1, \dots, X_{q+1}) = 0 \end{aligned} \tag{5.51}$$

□

We will now specialize to $q = 2$. In this case proposition 3.27 gives the exact sequence

$$H^1_{ct}(\mathfrak{i})^{\mathfrak{g}/\mathfrak{i}} \xrightarrow{\alpha} H^2_{ct}(\mathfrak{g}/\mathfrak{i}) \xrightarrow{\pi^*} H^2_{ct}(\mathfrak{g})_1 \tag{5.52}$$

for a complemented ideal $\mathfrak{i}$ in $\mathfrak{g}$. We will now show that we can use the sequence of 1-trivial complemented ideals $(\mathfrak{j}_\gamma)_{\gamma \in \mathbb{N}}$ here and take direct limits as well, to obtain the exact sequence

$$\varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} \xrightarrow{\varinjlim \alpha_\gamma} \varinjlim H^2(\mathfrak{g}/\mathfrak{j}_\gamma) \xrightarrow{\varinjlim \pi_\gamma^*} H^2(\mathfrak{g}) \to 0 \tag{5.53}$$

We therefore first show that the direct limit $\varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}$ is well defined. The inclusion map $\iota_{\gamma',\gamma} : \mathfrak{j}_\gamma \to \mathfrak{j}_{\gamma'}$ is well defined for $\gamma' \leq \gamma$ as the sequence of ideals is decreasing and we thus obtain a map $\iota^*_{\gamma',\gamma} : H^1(\mathfrak{j}_{\gamma'})^{\mathfrak{g}/\mathfrak{j}_{\gamma'}} \to H^1(\mathfrak{j}_\gamma)$ and we need to check that for $t \in \mathfrak{g}/\mathfrak{j}_\gamma, [\omega] \in H^1(\mathfrak{j}_{\gamma'})^{\mathfrak{g}/\mathfrak{j}_{\gamma'}}$ we have $t \cdot \iota^*_{\gamma',\gamma}[\omega] = 0$. To this end, let $s_\gamma : \mathfrak{g}/\mathfrak{j}_\gamma \to \mathfrak{g}$ be a continuous linear section, and note that we can always find a $\tilde{t} \in \mathfrak{g}/\mathfrak{j}_{\gamma'}$ and a continuous linear section $s_{\gamma'}$ such that $s_\gamma(t) = s_{\gamma'}(\tilde{t})$.

$$t \cdot \iota^*_{\gamma',\gamma}[\omega] = s_\gamma(t) \cdot \iota^*_{\gamma',\gamma}[\omega] = s_{\gamma'}(\tilde{t}) \cdot \iota^*_{\gamma',\gamma}[\omega] = 0 \tag{5.54}$$

where the last equality follows as $\tilde{t} \in \mathfrak{g}/\mathfrak{j}_{\gamma'}, [\omega] \in H^1(\mathfrak{j}_{\gamma'})^{\mathfrak{g}/\mathfrak{j}_{\gamma'}}$. Thus $\iota^*_{\gamma',\gamma} : H^1(\mathfrak{j}_{\gamma'})^{\mathfrak{g}/\mathfrak{j}_{\gamma'}} \to H^1(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}$ and $(H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}, \iota^*_{\gamma',\gamma})$ is a direct system.

We now construct the map $\varinjlim \alpha_\gamma$ by setting $\varinjlim \alpha_\gamma[[\omega]] = [\alpha_\gamma[\omega]]$ for $[\omega] \in H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}$ and with α_γ being the map from proposition 3.27. We of course need to show that it is well defined.

Lemma 5.18. *The map* $\varinjlim \alpha_\gamma : \varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} \to \varinjlim H^2(\mathfrak{g}/\mathfrak{j}_\gamma) : [[\omega]] \to [\alpha_\gamma[\omega]]$ *is well defined.*

Proof. We again need to show that for $\omega, \omega' \in [\omega]$, where $\omega \in H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}, \omega' \in H^1_{ct}(\mathfrak{j}_{\gamma'})^{\mathfrak{g}/\mathfrak{j}_{\gamma'}}$, we have $[[\alpha_\gamma \omega]] = [[\alpha_{\gamma'}\omega']]$. As $\omega \sim \omega'$, there exists $k \geq \gamma, \gamma'$ such that $\iota^*_{\gamma,k}\omega = \iota^*_{\gamma',k}\omega'$, thus ω and ω' take the same values on $\mathfrak{j}_k$. Now define the section $s_k = s_\gamma \pi_{\gamma,k}$, then

$$\pi^*_{\gamma,k}\alpha_\gamma(\omega)([X_1],[X_2]) = \omega([s_\gamma\pi_{\gamma,k}[X_1], s_\gamma\pi_{\gamma,k}[X_2]] - s_\gamma\pi_{\gamma,k}([[X_1],[X_2]])) = \alpha_k(\omega)([X_1],[X_2]) \tag{5.55}$$

and as α_k is independent of the section we choose, $\pi^*_{\gamma',k}\alpha_{\gamma'}(\omega')([X_1],[X_2]) = \alpha_k(\omega')([X_1],[X_2])$ holds as well. Now $\alpha_k(\omega)$ only depends of the value of ω in $\mathfrak{j}_k$ and there ω takes the same values as ω', thus $\pi^*_{\gamma,k}\alpha_\gamma(\omega) = \alpha_k(\omega) = \alpha_k(\omega') = \pi^*_{\gamma',k}\alpha_{\gamma'}(\omega')$, meaning that $[[\alpha_\gamma\omega]] = [[\alpha_{\gamma'}\omega']]$. ☐

We are now in position to show that $\mathrm{im}(\varinjlim \alpha_\gamma) = \ker(\varinjlim \pi^*_\gamma)$. This is pretty much immediate from the definitions and the fact that $\mathrm{im}(\alpha_\gamma) = \ker(\pi^*_\gamma)$, which follows from proposition 3.27.

Proposition 5.19. $\mathrm{im}(\varinjlim \alpha_\gamma) = \ker(\varinjlim \pi^*_\gamma)$

Proof. First, let $[[\omega]] \in \mathrm{im}(\varinjlim \alpha_\gamma)$, then $[[\omega]] = [\alpha_\gamma[\phi]]$ for some $\phi \in \varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}$, so $\varinjlim \pi_\gamma[[\omega]] = [\pi_\gamma\alpha_\gamma[\phi]] = 0$ and thus $\mathrm{im}(\varinjlim \alpha_\gamma) \subseteq \ker(\varinjlim \pi^*_\gamma)$. Next, let $[[\omega]] \in \ker(\varinjlim \pi^*_\gamma)$, then $[\omega] \in \ker(\pi^*_\gamma) = \mathrm{im}(\alpha_\gamma)$ and therefore $[[\omega]] \in \mathrm{im}(\varinjlim \alpha_\gamma)$. ☐

Corollary 5.20. *Let* $(\mathfrak{j}_\gamma)_{\gamma\in\mathbb{N}}$ *is a decreasing sequence of complemented ideals such that* $\mathfrak{j}_\gamma$ *is a 1-trivial ideal of order* γ. *Then the following sequence is exact.*

$$\varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} \overset{\varinjlim \alpha_\gamma}{\to} \varinjlim H^2(\mathfrak{g}/\mathfrak{j}_\gamma) \overset{\varinjlim \pi^*_\gamma}{\to} H^2(\mathfrak{g}) \to 0 \tag{5.56}$$

Remark 5.21. This result essentially follows from the Serre-Hochschild spectral sequence and the fact that the direct limit is something called an exact functor, which means that it preserves exact sequences. The only part that does not immediately follow from these two facts is that the last term is zero, which is exactly where we used that our sequence of ideals is a sequence of 1-trivial ideals. We expect that a similar result as the above corollary can be given for higher cohomology groups if the formalism of spectral sequences is used.

If $\mathfrak{j}_\infty$ is an 1-trivial complemented ideal in $\mathfrak{g}$ of order ∞, we can simply take $\mathfrak{j}_\gamma = \mathfrak{j}_\infty$ to arrive at the following corollary:

Corollary 5.22. *Let* $\mathfrak{j}_\infty$ *be an* 1*-trivial complemented ideal in* $\mathfrak{g}$. *Then the following sequence is exact.*

$$H^1_{ct}(\mathfrak{j}_\infty)^{\mathfrak{g}/\mathfrak{j}_\infty} \overset{\alpha}{\to} H^2(\mathfrak{g}/\mathfrak{j}_\infty) \overset{\pi}{\to} H^2(\mathfrak{g}) \to 0 \tag{5.57}$$

5.3 Pullback method in AdS3

In this section we will apply the method derived in the previous section to our asymptotic symmetry algebra on AdS_3 $\mathfrak{g}$, resulting in the short exact sequence in theorem 5.33. We take $Y^t = t^{-k}\Omega^k\partial_k$ for $k \geq 2$.

A cochain $c \in C^q_{ct}(\mathfrak{g})$ is continuous in the product topology of $\mathfrak{g}$, which is a subspace topology inherited from the C^∞ topology. We will show that for each cochain there exists a $\gamma \geq 0$ such that it is in fact continuous in a product topology of subspace topologies on $\mathfrak{g}$ inherited from the C^γ topology, in similar fashion to the standard C^∞ case. We let $\mathcal{P}, \mathcal{P}_k$ be the families of seminorms of the C^∞, C^k topologies respectively, as defined in definition 4.17.

Lemma 5.23. *Let $c \in C^q_{ct}(\mathfrak{g})$ be a cochain. Then there exists a $\gamma \geq 0$ such that c is continuous in the product topology of the subspace topology on $\mathfrak{g}$ inherited from the C^γ topology.*

Proof. As c is continuous in the C^∞ topology, by lemma 2.8 there exist continuous seminorms $p_1, \dots, p_q$ on $\mathfrak{g}$ and a $C > 0$ such that $c(X_1, \dots, X_q) \leq Cp_1(X_1)\dots p_q(X_q)$ for all $X_1, \dots, X_q \in \mathfrak{g}$. The topology on $\mathfrak{g}$ is the subspace topology of $\mathcal{O}(\Omega) \times \mathcal{O}(1) \times \mathcal{O}(1)$, and therefore by lemma 2.7, the seminorms are given by $q \circ \pi_s$ for $q \in \mathcal{P}$ and π_s the projection on the s direction, with $s \in \{\Omega, +, -\}$. Now by lemma 2.4, for each p_i, there exist $q^i_1 \dots, q^i_{n_i} \in \mathcal{P}, C_1, \dots C_{n_i} > 0$ such that $p_i(X) \leq C_1 q^i_1 \circ \pi_{i,1}(X) + \dots + C_{n_i} q^i_{n_i} \circ \pi_{i,n_i}(X)$ for all $X \in \mathfrak{g}$. Now as $q^i_j \in \mathcal{P}$, there exist $K_{i,j}, k_{i,j}$ such that $q^i_j = p_{K_{i,j},k_{i,j}}$. Let k be the maximum of all those $k_{i,j}$. Then $q^i_j \in \mathcal{P}_k$ for all $q \geq i \geq 1$, $n_i \geq j \geq 1$ and therefore p_i is continuous in the C^k topology as well, by lemma 2.4. The conclusion then follows from lemma 2.8. □

We say a cochain $c \in C^q_{ct}(\mathfrak{g})$ is of order γ if c is continuous in the product topology of the subspace topology on $\mathfrak{g}$ inherited from the C^γ topology.

Lemma 5.24. *The flow T^t generated by Y^t with $T^1 = 1$ is given by:*

$$T^t(\Omega, x^+, x^-) = \left(\frac{t\Omega}{(t^{k-1} + \Omega^{k-1}(1 - t^{k-1}))^{\frac{1}{k-1}}}, x^+, x^- \right) \tag{5.58}$$

and as this is defined for all $t > 0$, T^t is half-global. Furthermore, T^t is increasing in t for $\Omega > 0$ and increasing in Ω for $t > 0$.

Proof. By definition, $\dot{T}^t(\Omega, x^+, x^-) = Y^t(T^t(\Omega, x^+, x^-))$. As Y^t is zero in the $+, -$ directions, T^t is constant in those coordinates. In the Ω coordinate, we find:

$$\dot{T}^t(\Omega)_\Omega = \frac{1}{t^k} T^t(\Omega)^k_\Omega \tag{5.59}$$

Which can easily be seen to have the solution:

$$(T^t(\Omega)_\Omega)^{-(k-1)} = \frac{1}{\Omega^{k-1}} + \frac{1}{t^{k-1}} - 1 \tag{5.60}$$

from which equation 5.58 immediately follows. The derivatives with respect to Ω and t are given by:

$$\begin{aligned}\frac{\partial (T^t)_\Omega}{\partial \Omega} &= \frac{t^k}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}} \\ \frac{\partial (T^t)_\Omega}{\partial t} &= \frac{\Omega^k}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}}\end{aligned} \tag{5.61}$$

and as the t derivative is positive for $\Omega > 0$, T^t is increasing in t for $\Omega > 0$. Similarly T^t is increasing in Ω for $t > 0$. □

For $X \in \mathfrak{g}$, with $X = A\partial_\Omega + B\partial_+ + C\partial_-$, using equations 5.28 and 5.30, the pullback on a vector field in coordinates is given by:

$$(T^t)^* X = (\frac{\partial (T^t)_\Omega}{\partial \Omega})^{-1} A \circ T^t \partial_\Omega + B \circ T^t \partial_+ + C \circ T^t \partial_- \tag{5.62}$$

We of course have to show that T^t is a $\mathfrak{g}$-flow.

Proposition 5.25. *T^t is a $\mathfrak{g}$-flow for $k \geq 3$.*

Proof. Clearly, T^t is half-global. Thus we have to show that $(T^t)^* X \in \mathfrak{g}$ for $X \in \mathfrak{g}$. As $(T^t)^*$ is linear, we can assume that either $X \in \mathfrak{w}_L, X \in \mathfrak{w}_R$ or $X \in \mathfrak{i}$. Now if $f \in \mathcal{O}(\Omega^a)$, $f(T^t(\Omega)) = T^t(\Omega)^a \tilde{f}(T^t(\Omega))$ for some $\tilde{f} \in \mathcal{O}(1)$ and

$$f(T^t(\Omega)) = T^t(\Omega)^a \tilde{f}(T^t(\Omega)) = \Omega^a (T^t(\Omega)/\Omega)^a \tilde{f}(T^t(\Omega)) \tag{5.63}$$

Now $(T^t(\Omega)/\Omega)^a \tilde{f}(T^t(\Omega) \in \mathcal{O}(1)$ and thus $f(T^t(\Omega)) \in \mathcal{O}(\Omega^a)$. From here $(T^t)^*(X) \in \mathfrak{i}$ for $X \in \mathfrak{i}$ easily follows.

Now assume $X \in \mathfrak{w}_L$. Then $X = \partial_+ f(x^+)\Omega/2\partial_\Omega + f(x^+)\partial_+ + \frac{l^2\Omega^2}{2}\partial_+^2 f(x^+)\partial_-$. We first calculate $(T^t)^*\Omega\partial_\Omega$

$$\begin{aligned}(T^t)^*\Omega\partial_\Omega =& \Omega \left(\frac{t}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{1}{k-1}}} t^{-k} (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}} \right) \partial_\Omega \\ =& \Omega \left(t^{1-k}(t^{k-1}+\Omega^{k-1}(1-t^{k-1})) \right) \partial_\Omega \\ =& \left(\Omega + \Omega^k (t^{1-k}-1) \right) \partial_\Omega\end{aligned} \tag{5.64}$$

Next, we will show that $((T^t)^*\Omega^2 - \Omega^2)\partial_- \in \mathcal{O}(\Omega^4)$. First,

$$\begin{aligned}((T^t)^*\Omega^2 - \Omega^2)\partial_- =& \Omega^2 \left(\frac{t^2}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{2}{k-1}}} - 1 \right) \partial_- \\ & \Omega^2 t^2 \left(\frac{1-(1+\Omega^{k-1}(t^{1-k}-1))^{\frac{2}{k-1}}}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{2}{k-1}}} \right) \partial_-\end{aligned} \tag{5.65}$$

Now if $1 - (1+\Omega^{k-1}(t^{1-k}-1))^{\frac{2}{k-1}} \in \mathcal{O}(\Omega^2)$ we are done. To show this, note that

$1-(1+\Omega^{k-1}(t^{1-k}-1))^{\frac{2}{k-1}}|_{\Omega=0}=0$, and that

$$\partial_\Omega|_{\Omega=0}1-(1+\Omega^{k-1}(t^{1-k}-1))^{\frac{2}{k-1}}=0 \tag{5.66}$$

for $k\geq 3$. Thus $((T^t)^*\Omega^2-\Omega^2)\partial_-\in\mathcal{O}(\Omega^4)$. Now,

$$\begin{aligned}(T^t)^*X=&\partial_+f(x^+)/2(T^t)^*\Omega\partial_\Omega+f(x^+)\partial_++\frac{l^2(T^t)^*\Omega^2}{2}\partial_+^2f(x^+)\partial_-\\=&\partial_+f(x^+)/2\Omega\partial_\Omega+f(x^+)\partial_++\frac{l^2\Omega^2}{2}\partial_+^2f(x^+)\partial_-\\&+\Omega^k(t^{1-k}-1)\partial_+f(x^+)/2\partial_\Omega+\frac{l^2g(\Omega)}{2}\partial_+^2f(x^+)\partial_-\end{aligned} \tag{5.67}$$

where $g(\Omega)\in\mathcal{O}(\Omega^4)$. As $\Omega^k(t^{1-k}-1)\partial_+f(x^+)/2\partial_\Omega+\frac{l^2g(\Omega)}{2}\partial_+^2f(x^+)\partial_-\in\mathfrak{i}$, $(T^t)^*X\in\mathfrak{g}$. If $X\in\mathfrak{w}_R$, the same holds, as $\mathfrak{w}_R$ is $\mathfrak{w}_L$ except the x^+ and x^- are interchanged. □

The goal is now to show that $T^t_*p(X)\leq g(t)\tilde{p}(X)$ for every continuous seminorm $p\in\mathcal{P}$ and where $X\in\mathfrak{h}$ where $\mathfrak{h}$ is some ideal of $\mathfrak{g}$, $\tilde{p}$ is again a continuous seminorm and $g(t)$ is an integrable function. To show this we need to understand $T^t_*p(X)=p((T^t)^*X)$ better, which involves bounding the derivatives of $(T^t)^*X$. To this end we now first show that n-th derivative of a $\mathcal{O}(\Omega^a)$ function can be expressed in terms of the a-th derivative, as long as $a>n$.

Lemma 5.26. *Let $f\in\mathcal{O}(\Omega^a)$ and n a nonnegative integers such that $a>n$. Then, denoting the a-th derivative of f with respect to the first variable with $f^{(a)}$,*

$$\partial_\Omega^{(n)}f(\Omega,x^+,x^-)=\Omega^{a-n}\frac{f^{(a)}(\xi,x^+,x^-)}{(a-n)!} \tag{5.68}$$

for some unknown $\xi\in[0,\Omega]$

Proof. As $f\in\mathcal{O}(\Omega^a)$, $f^{(b)}(0,x^+,x^-)=0$ for all $0\leq b<a$, and thus by applying Taylor's theorem on $f^{(n)}$ [Lay14],

$$\partial_\Omega^{(n)}f(\Omega,x^+,x^-)=\Omega^{a-n}\frac{f^{(a)}(\xi,x^+,x^-)}{(a-n)!} \tag{5.69}$$

where $\xi\in[0,\Omega]$ is unknown. □

The next lemma calculates the derivatives of expressions related to the flow T^t.

Lemma 5.27. *Let $a,b,c\geq 0$ and h be given by*

$$h_t(\Omega)=\frac{\Omega^a}{(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{c}{k-1}}} \tag{5.70}$$

Then for $t \in (0,1]$

$$\frac{d^b}{d\Omega^b} h_t = \sum_{l=0}^{b} B_l^{a,k,c,b} (-1)^l (t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)} (1-t^{k-1})^l \Omega^{a-b+l(k-1)} \tag{5.71}$$

where $B_l^{a,k,c,b}$ is a set of nonnegative constants which are zero when $a-b+l(k-1)<0$.

Proof. We will calculate the derivative using the product and Faà di Bruno rules, see lemma C.4. To that end, we first calculate the derivatives of the denominator. Let $g(\Omega) = \Omega^{k-1}(1-t^{k-1}), f(x) = (t^{k-1}+x)^{-\frac{c}{k-1}}$, then clearly $g(\Omega) \geq 0$ for $t \in (0,1]$ and thus for $t > 0$, f is defined on the image of g. Then by the Faà di Bruno formula C.4, we have:

$$\frac{d^n}{d\Omega^n} f(g(\Omega)) = \sum \frac{n!}{m_1!1!^{m_1} m_2!2!^{m_2} \dots m_n!n!^{m_n}} \cdot f^{(m_1+\dots+m_n)}(g(\Omega)) \cdot \prod_{j=1}^{n} \left(g^{(j)}(\Omega)\right)^{m_j} \tag{5.72}$$

The derivatives of f, g are easily calculated:

$$\begin{aligned} g^{(j)}(\Omega) &= (1-t^{k-1}) \frac{(k-1)!}{(k-1-j)!} \Omega^{k-1-j} \\ f^{(l)}(x) &= (-1)^l \left(\prod_{i=0}^{l-1} \frac{c}{k-1} + i \right) (t^{k-1}+x)^{-\left(\frac{c}{k-1}+l\right)} \end{aligned} \tag{5.73}$$

for $j \leq k-1$, and for $j > k-1$ the $g^{(j)}$ is zero. Using this, with $l = \sum_i m_i$ we write

$$\begin{aligned} \frac{d^n}{d\Omega^n} f(g(\Omega)) =& \sum A_{m_1,\dots,m_n}^{k,c,n} (-1)^l (t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)} \prod_{j=1}^{n} (1-t^{k-1})^{m_j} \Omega^{m_j(k-1-j)} \\ =& \sum A_{m_1,\dots,m_n}^{k,c,n} (-1)^l (t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)} (1-t^{k-1})^l \Omega^{l(k-1)-n} \\ =& \sum_{l=0}^{n} \tilde{A}_l^{k,c,n} (-1)^l (t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)} (1-t^{k-1})^l \Omega^{l(k-1)-n} \end{aligned} \tag{5.74}$$

where $A_{m_1,\dots,m_n}^{k,c,n}, \tilde{A}_l^{k,c,n}$ are some nonnegative constants that are zero when $l(k-1)-n<0$ and we used that $\sum_i i m_i = n$. We can then calculate the derivative of h_t using the product rule:

$$
\begin{aligned}
\frac{d^b}{d\Omega^b}h_t(\Omega) = & \sum_{n=\max\{0,b-a\}}^{b} \binom{b}{n} \frac{d^n}{d\Omega^n} f(g(\Omega))\Omega^{a-b+n} \frac{a!}{(a-b+n)!} \\
= & \sum_{n=\max\{0,b-a\}}^{b} \sum_{l=0}^{n} \tilde{B}_l^{a,k,c,b,n}(-1)^l (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)}(1-t^{k-1})^l \Omega^{l(k-1)-n}\Omega^{a-b+n} \\
= & \sum_{n=\max\{0,b-a\}}^{b} \sum_{l=0}^{n} \tilde{B}_l^{a,k,c,b,n}(-1)^l (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)}(1-t^{k-1})^l \Omega^{a-b+l(k-1)} \\
= & \sum_{l=0}^{b} \left(\sum_{n=\max\{l,b-a\}}^{b} \tilde{B}_l^{a,k,c,b,n} \right) (-1)^l (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)}(1-t^{k-1})^l \Omega^{a-b+l(k-1)} \\
= & \sum_{l=0}^{b} B_l^{a,k,c,b}(-1)^l (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{c}{k-1}+l\right)}(1-t^{k-1})^l \Omega^{a-b+l(k-1)}
\end{aligned}
\tag{5.75}
$$

where $B_l^{a,k,c,b}$ is a set of nonnegative constants that are zero when $a-b+l(k-1)<0$. $\square$

Using the previous lemma we can now bound the supremum of the inverse Jacobian of T^t.

Lemma 5.28. *For $t \in (0,1]$*

$$
t^{-k} \sup_{\Omega \in K_i} |\partial_\Omega^\alpha (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}| \le \tilde{C}\alpha(Dt^{-k}+D't^{-\alpha(k-1)}) \tag{5.76}
$$

Proof. Using the Faà di Bruno formula, lemma C.4, we find

$$
\partial_\Omega^\alpha (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}} = \sum_{l=0}^{\alpha} C_l^{k,\alpha}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}(1-t^{k-1})^l \Omega^{l(k-1)-\alpha} \tag{5.77}
$$

where the constant $C_l^{k,\alpha}$ is zero whenever $l(k-1)-\alpha<0$, and

$$
t^{-k} \sup_{\Omega \in K_i} |\sum_{l=0}^{\alpha} C_l^{k,\alpha}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}(1-t^{k-1})^l \Omega^{l(k-1)-\alpha}| \le \tag{5.78}
$$

$$
t^{-k} \sup_{\Omega \in K_i} \sum_{l=0}^{\alpha} |C_l^{k,\alpha}||(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}(1-t^{k-1})^l \Omega^{l(k-1)-\alpha}| \le \tag{5.79}
$$

$$
t^{-k} \sup_{\Omega \in K_i} \Omega^{\alpha(k-1)-\alpha} \sup_{\Omega \in K_i} \sum_{l=0}^{\alpha} |C_l^{k,\alpha}||(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}| \tag{5.80}
$$

now we split the sum $\sum_{l=0}^{\alpha}|C_l^{k,\alpha}||(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}|$ into two parts, one where $\frac{k}{k-1}-l \ge 0$, and one where $\frac{k}{k-1}-l<0$. The part where $\frac{k}{k-1}-l \ge 0$ is clearly bounded for $t \in (0,1]$, say by a constant D. The part where $\frac{k}{k-1}-l<0$ attains its supremum at

$\Omega = 0$ and the summand is thus smaller than $|C_l^{k,\alpha}|t^{k-l(k-1)}$. From these considerations we conclude

$$t^{-k}\sup_{\Omega\in K_i}|\sum_{l=0}^{\alpha} C_l^{k,\alpha}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}(1-t^{k-1})^l\Omega^{l(k-1)-\alpha}| \leq \tag{5.81}$$

$$t^{-k}\sup_{\Omega\in K_i}\Omega^{\alpha(k-1)-\alpha}\sup_{\Omega\in K_i}\sum_{l=0}^{\alpha}|C_l^{k,\alpha}||(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}-l}| \leq \tag{5.82}$$

$$t^{-k}\sup_{\Omega\in K_i}\Omega^{\alpha(k-1)-\alpha}\alpha(D+D't^{k-\alpha(k-1)}) \leq \tilde{C}\alpha(Dt^{-k}+D't^{-\alpha(k-1)}) \tag{5.83}$$

□

Moreover, we can bound the supremum of derivatives of T^t itself.

Lemma 5.29.

$$\sup_{\Omega\in K}|\partial_\Omega^j T^t(\Omega)| \leq \tilde{B}_K^{k,j}t^{-(1+j(k-1))} \tag{5.84}$$

where $\tilde{B}_K^{k,j}$ is some constant.

Proof. First, we calculate the derivative using lemma 5.27, with $a=1, c=1$, so

$$\partial_\Omega^j T^t(\Omega) = \sum_{l=0}^{b} B_l^{1,k,1,j}(-1)^l(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{1}{k-1}+l\right)}(1-t^{k-1})^l\Omega^{1-j+l(k-1)} \tag{5.85}$$

where $B_l^{1,k,1,j}$ are nonnegative constants that are zero when $1-j+l(k-1)<0$. The supremum is then easily estimated by plugging in $\Omega=0$ in the $(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{-\left(\frac{1}{k-1}+l\right)}(1-t^{k-1})^l$ term:

$$\begin{aligned}\sup_{\Omega\in K}|\partial_\Omega^j T^t(\Omega)| \leq & \sum_{l=0}^{j} B_l^{1,k,1,j}t^{-(1+l(k-1))}(1-t^{k-1})^l\sup_{\Omega\in K}\Omega^{1-j+l(k-1)} \leq \\ & \sum_{l=0}^{j} B_l^{1,k,1,j}t^{-(1+j(k-1))}\sup_{\Omega\in K}\Omega^{1+j(k-2)} = \tilde{B}_K^{k,j}t^{-(1+j(k-1))}\end{aligned} \tag{5.86}$$

□

Let $\pi_\Omega, \pi_\pm$ be the projection of a vector field onto its $\partial_\Omega, \partial_\pm$ components respectively. Combining all of the above results shows that the pullback of a seminorm $p\in\mathcal{P}$ can indeed be bounded by $T_*^t p \leq g(t)\tilde{p}$, where $\tilde{p}$ is again a continuous seminorm.

Proposition 5.30. *Let $f\in\mathcal{O}(\Omega^a), g\in\mathcal{O}(\Omega^c)$. Then for $n<a, m<c$*

$$\begin{aligned}T_*^t p_{K,n}\circ\pi_\Omega(f\partial_\Omega) &\leq p_{K,a}(f)t^{a-n(k+1)-\max(k,n(k-1))}F^{n,k,K,a}(t)\\ T_*^t p_{K,m}\circ\pi_\pm(g\partial_\pm) &\leq p_{K,b}(g)t^{c-n(k+1)}G^{m,k,K,c}(t)\end{aligned} \tag{5.87}$$

and for $n \geq a, m \geq c$,

$$\begin{aligned} T^t_* p_{K,n} \circ \pi_\Omega(f\partial_\Omega) &\leq p_{K',n}(f) t^{-n(k+1)-\max(k,n)} F^{n,k,K,a}(t) \\ T^t_* p_{K,m} \circ \pi_\pm(g\partial_\pm) &\leq p_{K',m}(g) t^{-n(k+1)} G^{m,k,K,c}(t) \end{aligned} \tag{5.88}$$

where $F^{b,k,K,a}(t), G^{m,k,K,c}(t)$ are some continuous bounded functions independent of f and g.

Proof. Writing out the definitions, we find

$$\begin{aligned} T^t_* p_{K,n} \circ \pi_\Omega(f\partial_\Omega) =& p_{K,n}\left(t^{-k}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}} f \circ T^t\right) \\ =& \sup_{(\Omega,x^+,x^-)\in K} \max_{|\boldsymbol{\alpha}|\leq n} t^{-k} |\partial^{\boldsymbol{\alpha}} (t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}} f \circ T^t(\Omega,x^+,x^-)| \\ \leq& \sup_{(\Omega,x^+,x^-)\in K} \max_{|\boldsymbol{\alpha}|\leq n} \sum_{b=0}^{\alpha_\Omega} \binom{\alpha_\Omega}{b} t^{-k} |\partial_\Omega^{\alpha_\Omega - b}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}} \\ & \partial^b_\Omega(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f) \circ T^t(\Omega,x^+,x^-)| \\ \leq& \max_{|\boldsymbol{\alpha}|\leq n} \sum_{b=0}^{\alpha_\Omega} \binom{\alpha_\Omega}{b} t^{-k} \sup_{(\Omega,x^+,x^-)\in K} |\partial_\Omega^{\alpha_\Omega - b}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}| \\ & \sup_{(\Omega,x^+,x^-)\in K} |\partial^b_\Omega(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f) \circ T^t(\Omega,x^+,x^-)| \end{aligned} \tag{5.89}$$

By lemma 5.28,

$$t^{-k} \sup_{\Omega\in K} |\partial_\Omega^{\alpha_\Omega - b}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}| \leq (Dt^{-k} + D' t^{(b-\alpha_\Omega)(k-1)}) \tag{5.90}$$

for some constants D, D', and therefore

$$t^{-k} \sup_{\Omega\in K} |\partial_\Omega^{\alpha_\Omega - b}(t^{k-1}+\Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}| \leq \tilde{D} t^{-\max(k,\alpha_\Omega(k-1))} \tag{5.91}$$

for some constant $\tilde{D}$.

For the second term, $\sup_{(\Omega,x^+,x^-)\in K} |\partial^b_\Omega(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f) \circ T^t(\Omega,x^+,x^-)|$, we use the Faà di Bruno formula from lemma C.4:

$$\partial^b_\Omega(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f) \circ T^t(\Omega,x^+,x^-) = \sum E^b_{m_1,\dots,m_b} (\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(\sum_i m_i)}(T^t(\Omega,x^+,x^-)) \prod_{j=1}^b \left((T^t)^{(j)}(\Omega,x^+,x^-)\right)^{m_j} \tag{5.92}$$

and the supremum then splits up:

$$\sup_{(\Omega,x^+,x^-)\in K} |\partial_\Omega^b(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)\circ T^t(\Omega,x^+,x^-)| \leq$$

$$\sum E^b_{m_1,\ldots,m_b} \sup_{(\Omega,x^+,x^-)\in K} |(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(\sum_i m_i)}(T^t(\Omega,x^+,x^-))| \prod_{j=1}^{b}\left(\sup_{(\Omega,x^+,x^-)\in K} |(T^t)^{(j)}(\Omega,x^+,x^-)|\right)^{m_j} \tag{5.93}$$

By lemma 5.29,

$$\sup_{(\Omega,x^+,x^-)\in K} |(T^t)^{(j)}(\Omega,x^+,x^-)| \leq \tilde{B}_K^{k,j} t^{-(1+j(k-1))} \tag{5.94}$$

so

$$\prod_{j=1}^{b}\left(\sup_{(\Omega,x^+,x^-)\in K} |(T^t)^{(j)}(\Omega,x^+,x^-)|\right)^{m_j} \leq \tilde{B}_K^k \prod_{j=1}^{b} t^{-m_j(1+j(k-1))} \tag{5.95}$$

Now defining $l=\sum_j m_j$ and noting that $\sum_j jm_j = b$, we find

$$\prod_{j=1}^{b}\left(\sup_{(\Omega,x^+,x^-)\in K} |(T^t)^{(j)}(\Omega,x^+,x^-)|\right)^{m_j} \leq \tilde{B}_K^k t^{-(l+b(k-1))} \tag{5.96}$$

Next, for $a>n$, by lemma 5.26, and taking $M=\sup_{\Omega\in K}\Omega$

$$\begin{aligned}\sup_{(\Omega,x^+,x^-)\in K} |(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(\sum_i m_i)}(T^t(\Omega,x^+,x^-))| &= \sup_{(\Omega,x^+,x^-)\in K} |\frac{T^t(\Omega)^{a-l}}{(a-l)!}(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(a)}(\xi(t),x^+,x^-)| \\ &\leq \frac{t^{a-l}}{(a-l)!}\frac{M^{a-l}}{(t^{k-1}+M^{k-1}(1-t^{k-1}))^{\frac{a-l}{k-1}}} p_{K,a}(f)\end{aligned} \tag{5.97}$$

Here we used $l=\sum_i m_i$, that $T^t(\Omega)$ is increasing in Ω and that $a>n$ so also $a>|\boldsymbol{\alpha}|$. For $a\leq n$, we instead use:

$$\sup_{(\Omega,x^+,x^-)\in K} |(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(\sum_i m_i)}(T^t(\Omega,x^+,x^-))| \leq \sup_{(\Omega,x^+,x^-)\in K'} |(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)^{(\sum_i m_i)}(\Omega,x^+,x^-)| \leq p_{n,K'}(f) \tag{5.98}$$

where $K' = T^t(K)$, which is again compact as T^t is continuous. Now using equations 5.96, 5.97 in equation 5.93, we find for $a>n$:

$$\begin{aligned}&\sup_{(\Omega,x^+,x^-)\in K} |\partial_\Omega^b(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f)\circ T^t(\Omega,x^+,x^-)| \\ &\leq \sum_{l=0}^{b} E_l^b \frac{t^{a-l}}{(a-l)!}\frac{M^{a-l}}{(t^{k-1}+M^{k-1}(1-t^{k-1}))^{\frac{a-l}{k-1}}} p_{K,a}(f)\tilde{B}_K^k t^{-(l+b(k-1))} \\ &\leq \tilde{E}^{b,k,K,a} t^{a-b-(b+b(k-1))} p_{K,a}(f)\sum_{l=0}^{b}\frac{M^{a-l}}{(t^{k-1}+M^{k-1}(1-t^{k-1}))^{\frac{a-l}{k-1}}}\end{aligned} \tag{5.99}$$

Now combining this with equations 5.91, 5.89, we conclude that for $a > n$:

$$
\begin{aligned}
T^t_* p_{K,n} \circ \pi_\Omega(f\partial_\Omega) \leq & \max_{|\boldsymbol{\alpha}|\leq n} \sum_{b=0}^{\alpha_\Omega} \binom{\alpha_\Omega}{b} t^{-k} \sup_{(\Omega,x^+,x^-)\in K} |\partial_\Omega^{\alpha_\Omega - b}(t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}| \\
& \sup_{(\Omega,x^+,x^-)\in K} |\partial^b_\Omega(\partial_+^{\alpha_+}\partial_-^{\alpha_-} f) \circ T^t(\Omega, x^+, x^-)| \\
\leq & \max_{|\boldsymbol{\alpha}|\leq n} \sum_{b=0}^{\alpha_\Omega} \binom{\alpha_\Omega}{b} \tilde{D} t^{-\max(k,\alpha_\Omega(k-1))} \tilde{E}^{b,k,K,a} t^{a-b(k+1)} \\
& \sum_{l=0}^{b} \frac{M^{a-l}}{(t^{k-1} + M^{k-1}(1-t^{k-1}))^{\frac{a-l}{k-1}}} p_{K,a}(f) \\
\leq & p_{K,a}(f) t^{a-n(k+1)-\max(k,n(k-1))} F^{n,k,K,a}(t)
\end{aligned}
\tag{5.100}
$$

where $F^{n,k,K,a}(t)$ is some continuous bounded function independent of f. The case $a \leq n$ follows similarly. The proof for $T^t_* p_{K,m} \circ \pi_\pm(g\partial_\pm)$ is also almost identical, only the $t^{-k}|\partial^{\boldsymbol{\alpha}}(t^{k-1} + \Omega^{k-1}(1-t^{k-1}))^{\frac{k}{k-1}}$ term is missing.

□

Using the bounds in the above proposition, we show that $\lim_{t\to 0} t^\nu (T^t)^* X$ converges in the C^γ topology for some ν independent of $X \in \mathfrak{g}$. This is necessary for proposition 5.13.

Lemma 5.31. *Suppose $\mathfrak{g} \subseteq \mathcal{O}(\Omega^a)\partial_\Omega + \mathcal{O}(\Omega^b)\partial_+ + \mathcal{O}(\Omega^b)\partial_-$ for some integers a, b and assume $\mathfrak{g}$ is closed in the C^∞ topology. Let $\gamma \in \mathbb{N}$. Then there exists a $\nu \in \mathbb{N}$ such that $\lim_{t\to 0} t^\nu (T^t)^* X$ converges in the C^γ topology for all $X \in \mathfrak{g}$ and is again in $\mathfrak{g}$.*

Proof. Let $X \in \mathfrak{g}$, then there exist $A \in \mathcal{O}(\Omega^a), B, C \in \mathcal{O}(\Omega^b)$ such that $X = A\partial_\Omega + B\partial_+ + C\partial_-$. Now for convergence in the C^γ topology, we need to show that $\lim_{t\to 0} p_{K,n} \circ \pi_\Omega(t^\nu (T^t)^* X), \lim_{t\to 0} p_{K,m} \circ \pi_\pm(t^\nu (T^t)^* X)$ both converge for all compact sets K and $n \leq \gamma$. For ν large enough, this follows immediately from proposition 5.30 as $p_{K,n} \circ \pi_\Omega(t^\nu (T^t)^* X) = t^\nu T^t_* p_{K,n} \circ \pi_\Omega(X)$ and similarly for $x^\pm$. As $\mathfrak{g}$ is closed in the C^∞ topology and the C^γ topology is finer, $\mathfrak{g}$ is also closed in the C^γ topology, meaning that the limit is again in $\mathfrak{g}$. □

We now show that for each order γ and q, there is a 1-trivial ideal of order γ.

Proposition 5.32. *Let $\gamma \geq 0, q \geq 2$, then there exists an a such that $\mathfrak{j}_\gamma = \mathcal{O}(\Omega^a)\partial_\Omega + \mathcal{O}(\Omega^a)\partial_+ + \mathcal{O}(\Omega^a)\partial_-$ is a 1-trivial ideal of order γ.*

Proof. Let $c \in C^q_{ct}(\mathfrak{g})$. Then first we show that for a large enough,

$$
|P_p(\iota_{Y_t} T^t_* c)(X_1, \ldots, X_{q-1})| \leq g(t)\tilde{p}(Y) p_1(X_1) \ldots p_{q-1}(X_{q-1}) \tag{5.101}
$$

where $g(t)$ is a continuous and bounded function on $[0, 1]$. To this end, note that c is of order γ, so there exist seminorms $p, p'_1, p'_2, \ldots, p'_{q-1}$ which are of order γ such that

$$
|c(Y_t, X_1, \ldots, X_{q-1})| \leq p(Y_t) p'_1(X_1) \ldots p'_{q-1}(X_{q-1}) \tag{5.102}
$$

and thus

$$|\iota_{Y_t} T^t_* c(X_1, \ldots, X_{q-1})| \leq p(Y_t) T^t_* p'_1(X_1) \ldots T^t_* p'_{q-1}(X_{q-1}) \tag{5.103}$$

Now as the topology on the vector fields is a product topology, we can assume that $p'_i = p_i \circ \pi_\mu$, where $\mu \in \{\Omega, x^+, x^-\}$. Further as every continuous seminorm can be bounded by a linear combination of seminorms in the generating family, see lemma 2.4, and using proposition 5.30, we find that if $\pi_\mu X_i \in \mathcal{O}(\Omega^a)$ for $a > \gamma$, then

$$T^t_* p'_i(X_i) \leq p_i(X_i) t^{a-\gamma(k+1)-\max(k,\gamma(k-1))} F_i(t) \tag{5.104}$$

and if $\pi_\mu X_i \in \mathcal{O}(\Omega^a)$ for $a \leq \gamma$,

$$T^t_* p'_i(X_i) \leq p_i(X_i) t^{-\gamma(k+1)-\max(k,\gamma(k-1))} G_i(t) \tag{5.105}$$

Therefore if at least one of the arguments $X_1, \ldots, X_{q-1}$, $\pi_\mu X_i$ is in $\mathcal{O}(\Omega^a)$ for $a > \gamma$ and for all μ, then

$$|\iota_{Y_t} T^t_* c(X_1, \ldots, X_{q-1})| \leq t^{a-k-(q-1)(\gamma(k+1)+\max(k,\gamma(k-1)))} F(t) p(Y) p'_1(X_1) \ldots p'_{q-1}(X_{q-1}) \tag{5.106}$$

Now for $a > k + (q-1)(\gamma(k+1) + \max(k, \gamma(k+1)))$, the right hand side is clearly continuous and bounded. Then let $I_a = \mathcal{O}(\Omega^a)\partial_\Omega + \mathcal{O}(\Omega^a)\partial_+ + \mathcal{O}(\Omega^a)\partial_-$ and P_p the projection operators with respect to this ideal. These are continuous and as they project at least one of the arguments to I_a for $p \geq 1$, equation 5.102 holds.

Now we show that with $r = \max((q-2)\nu + k, (q-1)\nu + \epsilon)$, for $X \in I_{a'}$, where a' is a number that will be defined later, we have

$$\lim_{t \to 0} t^{-r} (T^t)^* X \in \mathfrak{g} \tag{5.107}$$

for all $X \in I_{a'}$ where the convergence is of order γ. Let $p_{K,n} \circ \pi_\Omega, p_{K,n} \circ \pi_\pm$ be continuous seminorms in the generating family, then by proposition 5.30,

$$\begin{aligned} t^{-r} p_{K,n} \circ \pi_\Omega((T^t)^* X) &\leq p_{K,a'}(X^\Omega) t^{a'-r-n(k+1)-\max(k,n(k-1))} F(t) \leq p_{K,a'}(X^\Omega) t^{a'-r-\gamma(k+1)-\max(k,\gamma(k-1))} F(t) \\ t^{-r} p_{K,n} \circ \pi_\pm((T^t)^* X) &\leq p_{K,a'}(X^\pm) t^{a'-r-n(k+1)} F(t) \leq p_{K,a}(X^\pm t^{a'-r-\gamma(k+1)} G(t) \end{aligned} \tag{5.108}$$

We thus see that for a' large enough, these converge to zero, and are therefore surely in $\mathfrak{g}$. Hence for a' large enough, by proposition 5.13, $j_\gamma := I_{a'}$ is a 1-trivial ideal of order γ. $\square$

Now that we know that there exists a 1-trivial ideal of order γ for each γ, we can apply corollary 5.20. Note that complementedness follows from proposition 4.20.

Theorem 5.33.

$$\varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} \overset{\varinjlim \alpha_\gamma}{\to} \varinjlim H^2(\mathfrak{g}/\mathfrak{j}_\gamma) \overset{\varinjlim \pi^*_\gamma}{\to} H^2(\mathfrak{g}) \to 0 \tag{5.109}$$

5.4 Second cohomology group

In this section we will calculate the second cohomology group, which will later help us determine the trivial symmetries.

In the previous section we saw that every 2-cocycle ω has an ideal $\mathfrak{j}_\gamma$ such that ω is equivalent to a 2-cocycle that is zero on $\mathfrak{j}_\gamma$. Every such ideal is of the form $\mathfrak{j}_\gamma = \mathcal{O}(\Omega^a)\partial_\Omega + \mathcal{O}(\Omega^a)\partial_+ + \mathcal{O}(\Omega^a)\partial_-$, and we define $\mathfrak{J}_\gamma = \mathcal{O}(\Omega^\gamma)\partial_\Omega + \mathcal{O}(\Omega^\gamma)\partial_+ + \mathcal{O}(\Omega^\gamma)\partial_-$. Then every 2-cocycle ω belongs to a class of $H^2_{ct}(\mathfrak{g}/\mathfrak{J}_\gamma)$ for some γ. We will now calculate $H^2_{ct}(\mathfrak{g}/\mathfrak{J}_\gamma)$. We first use theorem 3.33. To that end, take $E_1 = i\partial_+, E_2 = i\partial_-$ and remember that $\mathcal{C}^k(\mathfrak{g})_0 := \{c \in \mathcal{C}^k(\mathfrak{g}) | \mathcal{L}_{E_i} c = 0 \quad \forall i = 1,2\}$. Then $H^2_{ct}(\mathfrak{g}/\mathfrak{J}_\gamma) = H^2_{ct}(\mathfrak{g}/\mathfrak{J}_\gamma)_0$. We define $\mathfrak{g}^\gamma = \mathfrak{g}/\mathfrak{J}_{\gamma+1}$ and further take $I^\gamma = (\mathcal{O}(\Omega^\gamma)\partial_\Omega + \mathcal{O}(\Omega^\gamma)\partial_+ + \mathcal{O}(\Omega^\gamma)\partial_-)/\mathfrak{J}_{\gamma+1}$. This is an ideal and we will show that for any $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{J}_\gamma)_0$, we can find a $\tilde{\omega}$ that is zero on I^γ such that $[\omega] = [\tilde{\omega}]$. To that end, we first calculate the commutation relations between I^γ and $\mathfrak{g}^\gamma$. If $x \in \mathfrak{i}^\gamma := \mathfrak{i}/\mathfrak{J}_{\gamma+1}$, then $[x, I^\gamma] = 0$. For $\xi_n \in \mathfrak{w}_L$,

$$\begin{aligned}[][\xi_n, e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_-] =&[e^{inx^+}(in\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2n^2\Omega^2}{2}\partial_-), e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_-] = \\ &e^{i(k+n)x^+}e^{imx^-}\Omega^\gamma i(n\frac{\gamma}{2}+k)\partial_-\end{aligned} \tag{5.110}$$

$$\begin{aligned}[][\xi_n, e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_+] =&[e^{inx^+}(in\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2n^2\Omega^2}{2}\partial_-), e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_+] = \\ &e^{i(k+n)x^+}e^{imx^-}\Omega^\gamma i(n\frac{\gamma}{2}+k-n)\partial_+\end{aligned} \tag{5.111}$$

$$\begin{aligned}[][\xi_n, e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_\Omega] =&[e^{inx^+}(in\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2n^2\Omega^2}{2}\partial_-), e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_\Omega] = \\ &e^{i(k+n)x^+}e^{imx^-}\Omega^\gamma i(n\frac{\gamma}{2}+k-\frac{n}{2})\partial_\Omega\end{aligned} \tag{5.112}$$

We define $\gamma_\Omega = \gamma - 1, \gamma_+ = \gamma - 2, \gamma_- = \gamma$ and then for $\mu = \Omega, \pm$, we summarize the above as

$$[\xi_n, e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_\mu] = e^{i(k+n)x^+}e^{imx^-}\Omega^\gamma i(n\frac{\gamma_\mu}{2}+k)\partial_\mu \tag{5.113}$$

We further define $\bar{\gamma}_\Omega = \gamma - 1, \bar{\gamma}_+ = \gamma, \bar{\gamma}_- = \gamma - 2$ and then for $\mu = \Omega, \pm$, the commutation relations with $\bar{\xi}_n$ are given in analogy to the commutation relations in equation 5.113

$$[\bar{\xi}_n, e^{ikx^+}e^{imx^-}\Omega^\gamma\partial_\mu] = e^{ikx^+}e^{i(m+n)x^-}\Omega^\gamma i(n\frac{\bar{\gamma}_\mu}{2}+m)\partial_\mu \tag{5.114}$$

Using the 2-cocycle identity we can then restrict the behavior of the cocycles on $\mathfrak{w}_L \times I^\gamma$.

Proposition 5.34. *For $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{J}_{\gamma+1})_0$ and if $\gamma_\mu > 2$,*

$$\omega(\xi_n, e^{-inx^+}\Omega^\gamma\partial_\mu) = nc_1^\mu \tag{5.115}$$

and if $\gamma_\mu = 2$ we have

$$\omega(\xi_n, e^{-inx^+}\Omega^\gamma \partial_\mu) = nc_1^\mu + c_0^\mu \tag{5.116}$$

In both cases c_1^μ, c_0^μ are constants, possibly depending on γ and ω.

Proof. We use the cocycle identity and the commutation relation in equation 5.113,

$$\begin{aligned}
0 =& -\omega([\xi_n, \Omega^\gamma e^{i(-n-m)x^+}\partial_\mu], \xi_m) + \omega(\Omega^\gamma e^{-i(n+m)x^+}\partial_\mu, [\xi_m, \xi_n]) + \omega([\xi_m, \Omega^\gamma e^{-i(n+m)x^+}\partial_\mu], \xi_n) \\
=& -i(n\frac{\gamma_\mu}{2} - m - n)\omega(\Omega^\gamma e^{-imx^+}\partial_\mu, \xi_m) + i(n-m)\omega(\Omega^\gamma e^{-i(n+m)x^+}\partial_\mu, \xi_{m+n}) \\
&+ i(m\frac{\gamma_\mu}{2} - m - n)\omega(\Omega^\gamma e^{-inx^+}\partial_\mu, \xi_n)
\end{aligned} \tag{5.117}$$

We then define $c_n^\mu = \omega(\Omega^\gamma e^{-inx^+}\partial_\mu, \xi_n)$, after which we find:

$$0 = (n-m)c_{n+m}^\mu + (m\left(\frac{\gamma_\mu}{2} - 1\right) - n)c_n^\mu + (n\left(1 - \frac{\gamma_\mu}{2}\right) + m)c_m^\mu \tag{5.118}$$

Then taking $m = 0$ gives

$$0 = n\left(1 - \frac{\gamma_\mu}{2}\right)c_0^\mu = 0 \tag{5.119}$$

Now if $1 - \frac{\gamma_\mu}{2} \neq 0$, this implies $c_0^\mu = 0$. Otherwise, define $\tilde{c}_n^\mu = c_n^\mu - c_0^\mu$, in which case $\tilde{c}_\mu$ also satisfies equation 5.118. For now we will assume that $c_0^\mu = 0$ and add back the constant at the end. Next, setting $m = -n$ shows

$$0 = m\frac{\gamma_\mu}{2}(c_{-m}^\mu + c_m^\mu) \tag{5.120}$$

and as $\gamma_\mu \neq 0$, $c_{-m}^\mu = -c_m^\mu$. We now show that $c_{2n}^\mu = 2c_n^\mu$. To that end, take $m = -2n$ then we find

$$0 = 3nc_{-n}^\mu + (-2n(\frac{\gamma_\mu}{2} - 1) - n)c_n^\mu + (n(1 - \frac{\gamma_\mu}{2}) - 2n)c_{-2n}^\mu \tag{5.121}$$

and we use $c_{-n}^\mu = -c_n^\mu$ to arrive at

$$0 = c_n^\mu(-2n - n\gamma_\mu) + n(1 + \frac{\gamma_\mu}{2})c_{2n}^\mu \tag{5.122}$$

Thus giving the desired result. We will now show using induction that for all non-negative integers $t \in \mathbb{N}$, $c_{tm}^\mu = tc_m^\mu$, having already done the cases $t = 0, t = 1, t = 2$. Now assume that the induction hypothesis holds for t, then by taking $n = tm$ we find

$$\begin{aligned}
0 =&(tm - m)c_{tm+m}^\mu + (m\left(\frac{\gamma_\mu}{2} - 1\right) - tm)c_{tm}^\mu + (tm\left(1 - \frac{\gamma_\mu}{2}\right) + m)c_m^\mu \\
=&m(t-1)c_{m(t+1)}^\mu + (tm\left(\frac{\gamma_\mu}{2} - 1\right) - t^2m)c_m^\mu + (tm\left(1 - \frac{\gamma_\mu}{2}\right) + m)c_m^\mu \\
=&m(t-1)c_{m(t+1)}^\mu + (m - t^2m)c_m^\mu \\
0 =&c_{m(t+1)}^\mu - (t+1)c_m^\mu
\end{aligned} \tag{5.123}$$

where in the final step we divided by $m(t-1)$, and we note that $t - 1 \neq 0$ as we can take

$t \geq 2$. We conclude by induction that $c^\mu_{tm} = tc^\mu_m$ for all non-positive integers t. Now as we have shown earlier that $c^\mu_{-m} = -c^\mu_m$, it holds for all integers. We conclude that $c^\mu_m = mc^\mu_1$. Now if $\left(1 - \frac{\gamma_\mu}{2}\right) = 0$ we have to add back the constant, so then $c^\mu_m = mc^\mu_1 + c^\mu_0$. □

We now show that we can add a coboundary to any 2-cycle such that it is zero on $\mathfrak{w}_L \times I^\gamma$ as long as $\gamma_\mu > 2$.

Lemma 5.35. *For $\gamma_\mu > 2$, the following is a continuous 1-cochain*

$$\phi(\Omega^\gamma \partial_\mu) = -\frac{ic^\mu_1}{1 - \gamma_\mu/2} \tag{5.124}$$

where ϕ is zero on all other arguments. Furthermore,

$$d\phi([\xi_n, e^{-inx^+}\Omega^\gamma \partial_\mu]) = nc^\mu_1 \tag{5.125}$$

Proof. That ϕ is a cochain should be clear. For continuity, note that

$$\phi(\Omega^a f(x^+, x^-)\partial_\nu) = C \int_0^{2\pi} \int_0^{2\pi} \partial^\gamma_\Omega \left(\pi_\mu(\Omega^a f(x^+, x^-)\partial_\nu)\right)|_{\Omega=0} dx^+ dx^- \tag{5.126}$$

where C is some constant. As integration and differentiation are continuous, ϕ is continuous. The result $d\phi([\xi_n, e^{-inx^+}\Omega^\gamma \partial_\mu]) = nc^\mu_1$ follows immediately from equation 5.113. □

We now go on to show that these 2-cocycles are also zero on $\mathfrak{i} \times I^\gamma$ for $\gamma \geq 4$. We note that there are no coboundaries here as $[\mathfrak{i}, I^\gamma] = 0$.

Proposition 5.36. *For $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma+1})_0$, $\gamma \geq 4$ and $x \in \mathfrak{i}, y \in I^\gamma$,*

$$\omega(x, y) = 0 \tag{5.127}$$

Proof. We will proof this using a kind of induction. First, note that for $x \in I^{\gamma+1}, y \in I^\gamma$, clearly $\omega(x, y) = 0$ as $I^{\gamma+1} \subseteq \mathfrak{I}_{\gamma+1}$. We will then assume that $\omega(x, y) = 0$ for $x \in I^{\gamma+1-i}$ and show that for $i \leq \gamma - 3$ this implies that $\omega(x, y) = 0$ for $x \in I^{\gamma-i}$. Thus first assume that $\omega(x, y) = 0$ for $x \in I^{\gamma+1-i}$. Then for arbitrary x, letting $\pi_{\gamma+1-i}, s_{\gamma+1-i}$ be the projections and sections, we clearly have $\omega(x, y) = \omega(s_{\gamma+1-i}(\pi_{\gamma+1-i}(x)), y)$, so from now on we will consider x to be in the quotient by $I^{\gamma+1-i}$. With the shorthand $\alpha = \gamma - i$, for $y = e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma \partial_\mu, x = e^{-i(m+n)x^+}e^{-i\tilde{l}x^-}\Omega^\alpha \partial_\nu$, in the quotient x has the commutation relations as in equation 5.113. Then using the cocycle identity, and noting that $[x, y] = 0$,

$$\begin{aligned}
0 =& \omega([x, \xi_n], y) - \omega(x, [\xi_n, y]) + \omega([y, x], \xi_n) \\
=& -\omega([\xi_n, e^{-i(m+n)x^+}e^{-i\tilde{l}x^-}\Omega^\alpha \partial_\nu], e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma \partial_\mu) - \omega(e^{-i(m+n)x^+}e^{-i\tilde{l}x^-}\Omega^\alpha \partial_\nu, [\xi_n, e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma \partial_\mu]) \\
=& -i(n\frac{\alpha_\nu}{2} - m - n)\omega(e^{-imx^+}e^{-i\tilde{l}x^-}\Omega^\alpha \partial_\nu, e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma \partial_\mu) \\
&- i(n\frac{\gamma_\mu}{2} + m)\omega(e^{-i(m+n)x^+}e^{-i\tilde{l}x^-}\Omega^\alpha \partial_\nu, e^{i(m+n)x^+}e^{i\tilde{l}x^-}\Omega^\gamma \partial_\mu)
\end{aligned} \tag{5.128}$$

Now by defining $d^{\nu,\mu}_{m,\tilde{l},\alpha} = \omega(e^{-imx^+}e^{-i\tilde{l}x^-}\Omega^\alpha\partial_\nu, e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma\partial_\mu)$, we arrive at

$$0 = (n\left(\frac{\alpha_\nu}{2} - 1\right) - m)d^{\nu,\mu}_{m,\tilde{l},\alpha} + (n\frac{\gamma_\mu}{2} + m)d^{\nu,\mu}_{m+n,\tilde{l},\alpha} \tag{5.129}$$

Then setting $m = -n \neq 0$ and noting that $\alpha \geq 3$, so $\alpha_\nu \neq 0$, shows:

$$\begin{aligned} 0 = (n\left(\frac{\alpha_\nu}{2} - 1\right) + n)d^{\nu,\mu}_{-n,\tilde{l},\alpha} + (n\frac{\gamma_\mu}{2} - n)d^{\nu,\mu}_{0,\tilde{l},\alpha} \\ -\frac{\gamma_\mu - 2}{\alpha_\nu}d^{\nu,\mu}_{0,\tilde{l},\alpha} = d^{\nu,\mu}_{-n,\tilde{l},\alpha} \end{aligned} \tag{5.130}$$

Thus $d^{\nu,\mu}_{n,\tilde{l},\alpha} = d^{\nu,\mu}_{1,\tilde{l},\alpha}$ for all $n \neq 0$. Putting this back into the recurrence relation, now assuming that $m, m+n, n \neq 0$,

$$\begin{aligned} 0 =& (n\left(\frac{\alpha_\nu}{2} - 1\right) - m)d^{\nu,\mu}_{1,\tilde{l},\alpha} + (n\frac{\gamma_\mu}{2} + m)d^{\nu,\mu}_{1,\tilde{l},\alpha} \\ 0 =& \left(\frac{\gamma_\mu}{2} + \frac{\alpha_\nu}{2} - 1\right)d^{\nu,\mu}_{1,\tilde{l},\alpha} \end{aligned} \tag{5.131}$$

and as $\gamma \geq 4, \alpha \geq 3, \frac{\gamma_\mu}{2} + \frac{\alpha_\nu}{2} - 1 > 0$, we thus have $d^{\nu,\mu}_{n,\tilde{l},\alpha} = 0$ for all nonzero n. For $\gamma_\mu \neq 2$, we also have $d^{\nu,\mu}_{0,\tilde{l},\alpha} = 0$ using equation 5.131. For $\gamma_\mu = 2$, we note that as $\gamma \geq 4$, we must have $\mu = +$. Furthermore, instead of using ξ_n in the 2-cocycle identity, we could also use $\bar{\xi}_n$ and switch around every other x^+, x^- as well, which results in:

$$\begin{aligned} 0 =& (n\left(\frac{\bar{\alpha}_\nu}{2} - 1\right) + n)d^{\nu,\mu}_{\tilde{l},-n,\alpha} + (n\frac{\bar{\gamma}_\mu}{2} - n)d^{\nu,\mu}_{\tilde{l},0,\alpha} \\ &\frac{\bar{\gamma}_\mu - 2}{\bar{\alpha}_\nu}d^{\nu,\mu}_{\tilde{l},0,\alpha} = d^{\nu,\mu}_{\tilde{l},-n,\alpha} \\ 0 =& \left(\frac{\bar{\gamma}_\mu}{2} + \frac{\bar{\alpha}_\nu}{2} - 1\right)d^{\nu,\mu}_{\tilde{l},1,\alpha} \end{aligned} \tag{5.132}$$

Thus $d^{\nu,\mu}_{\tilde{l},-n,\alpha} = d^{\nu,\mu}_{\tilde{l},1,\alpha} = 0$ for $n \neq 0$, and $\frac{\bar{\gamma}_\mu - 2}{\bar{\alpha}_\nu}d^{\nu,\mu}_{\tilde{l},0,\alpha} = 0$ as well. As $\bar{\gamma}_+ = 4$, $d^{\nu,\mu}_{\tilde{l},n,\alpha} = 0$ for all $\tilde{l}, n$. Thus for all $\gamma \geq 4$, $d^{\nu,\mu}_{n,\tilde{l},\alpha} = 0$. Equivalently $\omega(e^{-imx^+}e^{-i\tilde{l}x^-}\Omega^a\partial_\nu, e^{imx^+}e^{i\tilde{l}x^-}\Omega^\gamma\partial_\mu) = 0$ for all $a \geq \gamma - i$ and as the terms $e^{-imx^+}e^{-i\tilde{l}x^-}\Omega^a\partial_\nu$ are dense in $I_{\gamma-i}$, see proposition 4.22, we conclude that $\omega(x, y) = 0$ for all $x \in I_{\gamma-i}, y \in I_\gamma$. By induction $\omega(x, y) = 0$ for all $x \in \mathfrak{j}_3, y \in I_\gamma$ and as $\mathfrak{i} \subseteq \mathfrak{j}_3$, $\omega(x, y) = 0$ for all $x \in \mathfrak{i}$. □

For $\gamma \geq 5$, we already know that 2-cocycles are cohomologous to zero on $\mathfrak{w}_L \times I^\gamma$ by proposition 5.34 and we now show that the same holds for $\mathfrak{w}_R \times I^\gamma$.

Proposition 5.37. *Let $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma+1})_0$ with $\gamma \geq 5$. Then there exists $\omega' \in [\omega]$ such that*

$$\begin{aligned} \omega'(\xi_n, e^{-inx^+}\Omega^\gamma\partial_\mu) = 0 \\ \omega'(\bar{\xi}_n, e^{-inx^-}\Omega^\gamma\partial_\mu) = 0 \end{aligned} \tag{5.133}$$

for $\mu \in \{\Omega, x^+, x^-\}$.

Proof. First, note that by proposition 5.34, there exists c_1^μ such that $\omega(\xi_n, e^{-inx^+}\Omega^\gamma\partial_\mu) = nc_1^\mu$. Then by lemma 5.35, there exists a 2-coboundary $d\phi$ such that $(\omega+d\phi)(\xi_n, e^{-inx^+}\Omega^\gamma\partial_\mu) = 0$. Define $\omega' = (\omega + d\phi)$. Then note that $[\xi_n, \bar{\xi}_m] \in \mathfrak{i}$ and by proposition 5.36, $\omega'(x, y) = 0$ if $x \in \mathfrak{i}, y \in I^\gamma$. Thus, $\omega'([\xi_n, \bar{\xi}_m], e^{-inx^+}e^{-imx^-}\Omega^\gamma\partial_\mu) = 0$ and by the 2-cocycle identity

$$0 = \omega'([\bar{\xi}_m, e^{-inx^+}e^{-imx^-}\Omega^\gamma\partial_\mu], \xi_n) + \omega'([e^{-inx^+}e^{-imx^-}\Omega^\gamma\partial_\mu, \xi_n], \bar{\xi}_m) \tag{5.134}$$

Now $[\bar{\xi}_m, e^{-inx^+}e^{-imx^-}\Omega^\gamma\partial_\mu] = e^{-inx^+}\Omega^\gamma im(\frac{\bar{\gamma}_\mu}{2} - 1)\partial_\mu$ and $\omega'(e^{-inx^+}\Omega^\gamma\partial_\mu, \xi_n) = 0$, so

$$0 = \omega'([e^{-inx^+}e^{-imx^-}\Omega^\gamma\partial_\mu, \xi_n], \bar{\xi}_m) = -in(\frac{\gamma_\mu}{2} - 1)\omega'(e^{-imx^-}\Omega^\gamma\partial_\mu, \bar{\xi}_m) \tag{5.135}$$

and thus $\omega'(e^{-imx^-}\Omega^\gamma\partial_\mu, \bar{\xi}_m) = 0$.

□

We have now shown that for $\gamma \geq 6$ a 2-cocycle in $H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma)$ is cohomologous to a 2-cocycle that is zero on I^γ, and we thus get an isomorphism $H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1})$

Theorem 5.38. *For $\gamma \geq 6$,*

$$H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1}) \tag{5.136}$$

Proof. First, note that if we define $H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma)_0 = \ker(\text{Res} : H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \to H^2(\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma)$, then $H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma)_0 \cong H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma)$ by proposition 5.36. Next, by combining this with proposition 3.27, we have the exact sequence

$$H^1_{ct}(\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma)^{\mathfrak{g}/\mathfrak{I}_{\gamma-1}} \xrightarrow{\alpha} H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1}) \xrightarrow{\pi^*} H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \xrightarrow{\gamma} H^1_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1}, H^1_{ct}(\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma)) \tag{5.137}$$

Now for $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma)$, again by proposition 5.36 and proposition 5.37, we can find $\omega \in [\omega]$ such that $\omega(\mathfrak{g}/\mathfrak{I}_\gamma, \mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma) = 0$ and therefore $\gamma([\omega]) = 0$. We thus have

$$H^1_{ct}(\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma)^{\mathfrak{g}/\mathfrak{I}_{\gamma-1}} \xrightarrow{\alpha} H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1}) \xrightarrow{\pi^*} H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \to 0 \tag{5.138}$$

Now by proposition 5.2, $\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma \subseteq [\mathfrak{g}/\mathfrak{I}_{\gamma-1}, \mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma]$ and therefore $H^1_{ct}(\mathfrak{I}_{\gamma-1}/\mathfrak{I}_\gamma)^{\mathfrak{g}/\mathfrak{I}_{\gamma-1}} = 0$, and thus the exact sequence gives $H^2_{ct}(\mathfrak{g}/\mathfrak{I}_\gamma) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{I}_{\gamma-1})$. □

Now using the above isomorphism and proposition 5.14, we find that we can quotient out $\mathfrak{I}_5$ in the second cohomology.

Theorem 5.39.

$$H^2_{ct}(\mathfrak{g}) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{I}_5) \tag{5.139}$$

Proof. Let $(\mathfrak{j}_\gamma)_{\gamma\in\mathbb{N}}$ be the sequence of complemented ideals such that $\mathfrak{j}_\gamma$ is a 1-trivial ideal of order γ constructed in proposition 5.32. Then note that $\mathfrak{j}_\gamma = \mathcal{O}(\Omega^{a_\gamma})\partial_\Omega + \mathcal{O}(\Omega^{a_\gamma})\partial_+ + \mathcal{O}(\Omega^{a_\gamma})\partial_- = J_{\alpha_\gamma}$. We will assume that $a_\gamma \geq 5$ for all $\gamma \geq 0$, if not, we can define a new sequence of 1-trivial ideals such that this is true. Now applying theorem 5.38 repeatedly, we find $H^2(\mathfrak{g}/\mathfrak{j}_\gamma) = H^2(\mathfrak{g}/\mathfrak{I}_5)$ and therefore $\varinjlim H^2(\mathfrak{g}/\mathfrak{j}_\gamma) = H^2(\mathfrak{g}/\mathfrak{I}_5)$. Thus by corollary 5.20 the following sequence is exact.

$$\varinjlim H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} \to H^2(\mathfrak{g}/\mathfrak{I}_5) \to H^2(\mathfrak{g}) \to 0 \tag{5.140}$$

Now we will show that $H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} = 0$ for all $\gamma \geq 0$, after which we are done by exactness of the sequence. Let $\phi \in H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma}$, then for $t \in \mathfrak{g}/\mathfrak{j}_\gamma, x \in \mathfrak{j}_\gamma$, $t \cdot \phi(x) = -\phi([t,x]) = 0$. Now as $a_\gamma \geq 5$, $\mathfrak{w}_L + \mathfrak{w}_R \subseteq \mathfrak{g}/\mathfrak{j}_\gamma$. Furthermore, as $\mathfrak{j}_\gamma \subseteq [\mathfrak{w}_L + \mathfrak{w}_R, \mathfrak{j}_\gamma]$ by proposition 5.2, $\phi(x) = 0$ for all $x \in \mathfrak{j}_\gamma$. Hence $H^1_{ct}(\mathfrak{j}_\gamma)^{\mathfrak{g}/\mathfrak{j}_\gamma} = 0$ □

We now only have to calculate the last few terms and we start by showing the behavior of 2-cocycle classes on $\mathfrak{w}_L \times \mathfrak{i}/\mathfrak{I}_5$.

Proposition 5.40. *Let $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_5)$. Then there exists a 2-coboundary $d\phi$ such that*

$$\begin{aligned}
(\omega + d\phi)(\xi_n, e^{-inx^+}\Omega^4\partial_+) &= nc_1^+ + c_0^+ \\
(\omega + d\phi)(\xi_n, e^{-inx^+}\Omega^4\partial_-) &= 0 \\
(\omega + d\phi)(\xi_n, e^{-inx^+}\Omega^4\partial_\Omega) &= 0 \\
(\omega + d\phi)(\xi_n, e^{-inx^+}\Omega^3\partial_\Omega) &= nc_1^\Omega + c_0^\Omega
\end{aligned} \tag{5.141}$$

Proof. The result for Ω^4 follows from proposition 5.34 and lemma 5.35. For Ω^3, the commutation relation becomes

$$\begin{aligned}
[\xi_n, e^{ikx^+}\Omega^3\partial_\Omega] =& [e^{inx^+}(in\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2n^2\Omega^2}{2}\partial_-), e^{ikx^+}\Omega^3\partial_\Omega] = \\
& e^{i(k+n)x^+}\left(i(n\frac{3}{2} + k - \frac{n}{2})\Omega^3\partial_\Omega + l^2n^2\Omega^4\partial_-\right)
\end{aligned} \tag{5.142}$$

Writing $\omega' = \omega + d\phi$, noting that $\omega'(\xi_n, e^{-inx^+}\Omega^4\partial_-) = 0$ the 2-cocycle identity gives

$$\begin{aligned}
0 =& -\omega'([\xi_n, \Omega^3 e^{i(-n-m)x^+}\partial_\Omega], \xi_m) + \omega'(\Omega^3 e^{-i(n+m)x^+}\partial_\Omega, [\xi_m, \xi_n]) + \omega'([\xi_m, \Omega^3 e^{-i(n+m)x^+}\partial_\Omega], \xi_n) \\
=& -i(n-m-n)\omega(\Omega^3 e^{-imx^+}\partial_\Omega, \xi_m) + i(n-m)\omega(\Omega^3 e^{-i(n+m)x^+}\partial_\Omega, \xi_{m+n}) \\
& + i(m-m-n)\omega(\Omega^3 e^{-inx^+}\partial_\Omega, \xi_n)
\end{aligned} \tag{5.143}$$

We then define $c_n^\Omega = \omega(\Omega^3 e^{-inx^+}\partial_\mu, \xi_n)$, after which we find:

$$0 = (n-m)c_{n+m}^\Omega - nc_n^\Omega + mc_m^\Omega \tag{5.144}$$

which is simply equation 5.118 with $\gamma_\mu = 2$, which we know has the solution $c_n^\Omega = nc_1^\Omega + c_0^\Omega$. □

We already know that 2-cocycles are zero on $\mathfrak{i} \times I^4$ by proposition 5.36, so to understand the behavior on $\mathfrak{i} \times \mathfrak{i}$ we only need to consider $\omega(\mathcal{O}(\Omega^3)\partial_\Omega, \mathcal{O}(\Omega^3)\partial_\Omega)$, which is also zero as shown next.

Proposition 5.41. *Let* $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_5)$*. Then*

$$\omega(\Omega^3 e^{inx^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^3 e^{-inx^+} e^{-i\tilde{l}x^-} \partial_\Omega) = 0 \tag{5.145}$$

Proof. First, note that by proposition 5.36, $\omega(\Omega^3 e^{inx^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^4 e^{-inx^+} e^{-i\tilde{l}x^-} \partial_-) = 0$. Then using the 2-cocycle identity, and noting that $[\Omega^3 e^{imx^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^3 e^{-i(m+n)x^+} e^{-i\tilde{l}x^-} \partial_\Omega] = 0$

$$\begin{aligned}
0 =& \omega([\xi_n, \Omega^3 e^{imx^+} e^{i\tilde{l}x^-} \partial_\Omega], \Omega^3 e^{-i(m+n)x^+} e^{-i\tilde{l}x^-} \partial_\Omega) + \omega([\Omega^3 e^{-i(m+n)x^+} e^{-i\tilde{l}x^-} \partial_\Omega, \xi_n], \Omega^3 e^{imx^+} e^{i\tilde{l}x^-} \partial_\Omega) \\
=& i(n+m)\omega(\Omega^3 e^{i(m+n)x^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^3 e^{-i(m+n)x^+} e^{-i\tilde{l}x^-} \partial_\Omega) \\
&+ i(n-m-n)\omega(\Omega^3 e^{imx^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^3 e^{-imx^+} e^{-i\tilde{l}x^-} \partial_\Omega)
\end{aligned} \tag{5.146}$$

Setting $f_{n,l} = \omega(\Omega^3 e^{inx^+} e^{i\tilde{l}x^-} \partial_\Omega, \Omega^3 e^{-inx^+} e^{-i\tilde{l}x^-} \partial_\Omega)$, we find

$$0 = (n+m) f_{n+m,\tilde{l}} - m f_{m,\tilde{l}} \tag{5.147}$$

Setting $m = 0$ shows $f_{n,\tilde{l}} = 0$ for $n \neq 0$. For $n = 0$, we note that $f_{n,l}$ should be symmetric in n, l as x^+, x^- play identical roles. Thus $f_{0,l} = f_{l,0} = 0$, for $l \neq 0$. For $f_{0,0}$ we use that the cochain is alternating which shows that $f_{0,0} = 0$. □

Lastly we show that we cannot simultaneously set a 2-cocycle to zero on $\mathfrak{w}_L \times \mathfrak{i}$ and $\mathfrak{w}_R \times \mathfrak{i}$ by adding a coboundary.

Proposition 5.42. *Let* $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{I}_5)$ *and take* $d\phi$ *as in proposition 5.40. Then*

$$\begin{aligned}
&(\omega + d\phi)(\xi_n, e^{-inx^+} \Omega^4 \partial_\mu) = 0 \\
&(\omega + d\phi)(\bar{\xi}_n, e^{-inx^-} \Omega^4 \partial_\Omega) = 0 \\
&(\omega + d\phi)(\bar{\xi}_n, e^{-inx^-} \Omega^4 \partial_-) = 0 \\
&(\omega + d\phi)(\bar{\xi}_n, e^{-inx^-} \Omega^4 \partial_+) = n\bar{c}_1^+ \\
&(\omega + d\phi)(\xi_n, e^{-inx^+} \Omega^3 \partial_\Omega) = nc_1^\Omega + c_0^\Omega \\
&(\omega + d\phi)(\bar{\xi}_n, e^{-inx^-} \Omega^3 \partial_\Omega) = -n^2 il^2 \bar{c}_1^+ + n\bar{c}_1^\Omega + \bar{c}_0^\Omega
\end{aligned} \tag{5.148}$$

Proof. Define $\omega' = \omega + d\phi$. Then note that the result $(\omega + d\phi)(\xi_n, e^{-inx^+} \Omega^4 \partial_\mu) = 0$ for $\mu = \Omega, x^-$ already follows from proposition 5.40. We will now use the 2-cocycle identity:

$$0 = \omega'([\xi_n, \bar{\xi}_m], e^{-inx^+} e^{-imx^-} \Omega^a \partial_\mu) + \omega'([\bar{\xi}_m, e^{-inx^+} e^{-imx^-} \Omega^a \partial_\mu], \xi_n) + \omega'([e^{-inx^+} e^{-imx^-} \Omega^a \partial_\mu, \xi_n], \bar{\xi}_m) \tag{5.149}$$

Now $[\xi_n, \bar{\xi}_m] \in I^4$, so by proposition 5.36, $\omega'([\xi_n, \bar{\xi}_m], e^{-inx^+} e^{-imx^-} \Omega^a \partial_\mu) = 0$. Next, we note that for $a = 4$, we can use equations 5.113, 5.114, to arrive at:

$$\begin{aligned}0 =& i(m\frac{\bar{a}_\mu}{2} - m)\omega'(e^{-inx^+}\Omega^a\partial_\mu, \xi_n) - i(n\frac{a_\mu}{2} - n)\omega'(e^{-imx^-}\Omega^a\partial_\mu, \bar{\xi}_m)\\ 0 =& m(\frac{\bar{a}_\mu}{2} - 1)\omega'(e^{-inx^+}\Omega^a\partial_\mu, \xi_n) - n(\frac{a_\mu}{2} - 1)\omega'(e^{-imx^-}\Omega^a\partial_\mu, \bar{\xi}_m)\end{aligned} \tag{5.150}$$

Now for $\mu = x^-, \Omega$, $(\frac{a_\mu}{2} - 1) \neq 0$ and $\omega'(e^{-inx^+}\Omega^a\partial_\mu, \xi_n) = 0$, so $\omega'(e^{-imx^-}\Omega^a\partial_\mu, \bar{\xi}_m) = 0$. For $\mu = x^+$, $(\frac{a_\mu}{2} - 1) = 0$ and we arrive at

$$0 = m\omega'(e^{-inx^+}\Omega^4\partial_+, \xi_n) \tag{5.151}$$

Now note that $(\omega + d\phi)(\bar{\xi}_n, e^{-inx^-}\Omega^4\partial_\mu) = n\bar{c}_1^+$ follows from proposition 5.34 as x^+, x^- are symmetric. For $a = 3$, $\mu = \Omega$, note that $(\omega + d\phi)(\xi_n, e^{-inx^+}\Omega^3\partial_\Omega) = nc_1^\Omega + c_0^\Omega$ follows from proposition 5.40. Now we use the 2-cocycle identity:

$$0 = \omega'([\bar{\xi}_n, \Omega^3 e^{-i(n+m)x^-}\partial_\Omega], \bar{\xi}_m) + \omega'([\Omega^3 e^{-i(n+m)x^-}\partial_\Omega, \bar{\xi}_m], \bar{\xi}_n) + \omega'([\bar{\xi}_m, \bar{\xi}_n], \Omega^3 e^{-i(n+m)x^-}\partial_\Omega) \tag{5.152}$$

The commutator is given by:

$$\begin{aligned}[\bar{\xi}_n, e^{ikx^-}\Omega^3\partial_\Omega] =& [e^{inx^-}(in\frac{\Omega}{2}\partial_\Omega + \partial_- - \frac{l^2n^2\Omega^2}{2}\partial_+), e^{ikx^-}\Omega^3\partial_\Omega] =\\ & e^{i(k+n)x^-}\left(i(n\frac{3}{2} + k - \frac{n}{2})\Omega^3\partial_\Omega + l^2n^2\Omega^4\partial_+\right)\end{aligned} \tag{5.153}$$

Then the 2-cocycle identity becomes

$$\begin{aligned}0 =& -im\omega'(\Omega^3 e^{-imx^-}\partial_\Omega, \bar{\xi}_m) + l^2n^2\omega'(\Omega^4 e^{-imx^-}\partial_+, \bar{\xi}_m)\\ & + in\omega'(\Omega^3 e^{-inx^-}\partial_\Omega, \bar{\xi}_n) - l^2m^2\omega'(\Omega^4 e^{-inx^-}\partial_+, \bar{\xi}_n)\\ & + i(m-n)\omega'(\Omega^3 e^{-i(n+m)x^-}\partial_\Omega, \bar{\xi}_{m+n})\end{aligned} \tag{5.154}$$

Now using $\omega'(\Omega^4 e^{-inx^-}\partial_+, \bar{\xi}_n) = -n\bar{c}_1^+$ and setting $c_m = \omega'(\Omega^3 e^{-imx^-}\partial_\Omega, \bar{\xi}_m)$, we find

$$\begin{aligned}0 =& -mc_m + il^2n^2m\bar{c}_1^+\\ & + nc_n - il^2m^2n\bar{c}_1^+\\ & + (m-n)c_{m+n}\end{aligned} \tag{5.155}$$

The homogeneous part is simply equation 5.118, so $c_n = nc_1 + c_0 + c_n^p$, where c_n^p is a particular solution. It is then easily checked that $c_n^p = n^2il^2\bar{c}_1^+$ is a particular solution. □

Now combining the above results we find that the second cohomology is at most seven dimensional.

Theorem 5.43.

$$\dim H^2_{ct}(\mathfrak{g}) \leq 7 \tag{5.156}$$

Proof. First, note that $H^2_{ct}(\mathfrak{g}) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{J}_5) \cong H^2_{ct}(\mathfrak{g}/\mathfrak{J}_5)_0$ by theorem 5.39 and theorem 3.33, so we will show that

$$\dim H^2_{ct}(\mathfrak{g}/\mathfrak{J}_5)_0 \leq 7 \tag{5.157}$$

To that end, note that there are at most 2 nontrivial cocycle classes given by the Virasoro cocycles on the two Witt algebra's, $\omega(\Omega^a e^{inx^+}e^{ilx^-}\partial_\mu, \Omega^b e^{-inx^+}e^{-ilx^-}\partial_\nu) = 0$ for $[\omega] \in H^2_{ct}(\mathfrak{g}/\mathfrak{J}_5)$, $a, b \geq 3$, by proposition 5.36 and proposition 5.41. Proposition 5.42, together with the density of the complex exponentials, proposition 4.22, then shows that there are at most 5 additional cocycle classes. Thus in total, there are at most 7 nontrivial cocycle classes. $\square$

Chapter 6

Asymptotic symmetries of three dimensional Anti-de-Sitter space

Now that we have determined the second cohomology group, we can determine the trivial symmetry algebra and the asymptotic symmetry algebra, according to definition 4.12. Remember that the trivial symmetries are the ones that are irrelevant for a classification of the fundamental particles and the asymptotic symmetry algebra is the weak asymptotic symmetry algebra without the trivial symmetries. These will be determined in section 6.2 and afterwards we will interpret the result in section 6.3. The asymptotic symmetry algebra is significantly larger than the isometry algebra, so we will explain how these additional symmetries should be interpreted in terms of the resulting fundamental particles. We are also interested in the comparison between the Noether and Wigner perspective symmetry algebras, so we actually start by discussing how the Noether symmetry algebra was derived in Ref. [BH86].

6.1 Noether perspective asymptotic symmetry algebra

In the seminal paper [BH86], the asymptotic symmetry algebra in the Noether perspective was determined. This is the quotient of the weak asymptotic symmetry algebra by the symmetries that generate trivial charges. In this section we will explain qualitatively how this result was obtained, without treading too much into the details.

First of all, we note that Brown and Henneaux work in the Arnowitt–Deser–Misner (ADM) formalism of general relativity. In this formalism D dimensional spacetime is split up into the $D-1$ dimensional slices, which should contain the spatial directions. The direction that is left is thus the time direction. In this formalism one can construct a Hamiltonian that specifies Einstein's equations. The canonical coordinates are then given by the induced metric in the spatial directions $g_{ij}(x)$, where $1 \leq i, j \leq D-1$. The canonical momentum is in turn given as normal by $\pi^{ij}(x) = \frac{\partial \mathcal{L}}{\partial g_{ij}}$, with $\mathcal{L}$ the Lagrangian density.

Moreover, Brown and Henneaux used the same boundary conditions for the metric as in this thesis. They popularized these boundary conditions and motivated them by choosing the strongest boundary conditions that contained all of the point particle metrics as well as including the metrics obtained by acting with an AdS_3 isometry on the point particle metrics.

After calculating the weak asymptotic symmetry group in the same manner as in section 4.5, they then determine the conserved charges via the Regge-Teitelboim procedure [RT74], which we will now explain.

The Hamiltonian $H[\xi] = \int_C d^2x\mathcal{H}[\xi]$ is an integral over a Hamiltonian density where C is some two dimensional slice of AdS containing the spatial directions. The variation of this Hamiltonian produces boundary terms, i.e.

$$\delta H[\xi] = \int_C d^2x A^{ij}(x)\delta g_{ij}(x) + B_{ij}(x)\delta\pi^{ij}(x) + \int_{\partial C} dx\tilde{J}[\xi] \tag{6.1}$$

for some A^{ij}, B_{ij} and $\tilde{J}$. Now these boundary terms are unwanted, for the following reason. The equations of motion are given by

$$\begin{aligned} \dot{g_{ij}}(x) &= \frac{\delta H}{\delta\pi^{ij}(x)} \\ \dot{\pi^{ij}}(x) &= -\frac{\delta H}{\delta g_{ij}(x)} \end{aligned} \tag{6.2}$$

and for the volume terms these variations are given by

$$\begin{aligned} \frac{\int_C d^2x A^{ij}(x)\delta g_{ij}(x) + B_{ij}(x)\delta\pi^{ij}(x)}{\delta\pi^{ij}(x)} &= B_{ij} \\ \frac{\int_C d^2x A^{ij}(x)\delta g_{ij}(x) + B_{ij}(x)\delta\pi^{ij}(x)}{\delta g_{ij}(x)} &= A^{ij} \end{aligned} \tag{6.3}$$

but the variations of the boundary terms are undefined. We thus add a boundary term $J[\xi]$ to the Hamiltonian that compensates these terms:

$$\tilde{H}[\xi] = H[\xi] + J[\xi] \tag{6.4}$$

such that $\delta\tilde{H}[\xi] = \int_C d^2x A^{ij}(x)\delta g_{ij}(x) + B_{ij}(x)\delta\pi^{ij}(x)$. The boundary terms $J[\xi]$ are now the conserved charges.

Actually performing the calculations then shows that the charge J is zero for all the symmetries in the ideal $\mathfrak{i}$. The asymptotic symmetry algebra in the Noether perspective is thus simply $\mathfrak{w}_L \oplus \mathfrak{w}_R$.

6.2 Wigner perspective trivial symmetry algebra

Remember that a trivial symmetry in the Wigner perspective is a symmetry ξ such that $\pi(\xi) = 0$ for every continuous projective unitary positive energy representation π of the symmetry algebra. We will take the positive energy representations to be of positive energy at ∂_t, as this is the timelike direction. As noted in section 4.1 however, $\mathfrak{g}$ might not have a corresponding Lie group, so we cannot determine what symmetries are trivial by looking at the 2-cocycles on $\mathfrak{g}$ and using proposition 3.12. Instead we determine the weakly trivial symmetry algebra $\mathfrak{t}$ generated by the vector fields $\partial_t \eta$ such that $[\partial_t \eta, \eta] = 0$ and $\omega(\partial_t \eta, \eta) = 0$. If a quotient algebra of $\mathfrak{g}$ has a corresponding Lie group, any weakly trivial symmetries contained in this quotient algebra become trivial symmetries, as shown in section 4.1. As the asymptotic symmetry algebra should be a quotient algebra of $\mathfrak{g}$ with a corresponding Lie group and should not contain any trivial symmetries, it cannot contain any weakly trivial symmetries either. We will thus determine $\mathfrak{g}/\mathfrak{t}$ and we will find that this has a corresponding Lie group, and is thus the asymptotic symmetry algebra.

We remind the reader that $\mathfrak{g}$ only contains vector fields that are 2π-periodic in both x^+, x^-. Thus for example $\Omega^3 \sin(t)\partial_\Omega = \Omega^3 \sin(\frac{x^+ + x^-}{2})\partial_\Omega$ is not in $\mathfrak{g}$.

Let us first consider the Witt algebras $\mathfrak{w}_L \oplus \mathfrak{w}_R$. These are not trivial symmetries as we can extend the projective unitary positive energy continuous Witt representations on $\mathfrak{w}$ (for the existence of these see e.g. Ref. [SN14]) to $\mathfrak{g} = \mathfrak{w}_L \ltimes (\mathfrak{w}_R \ltimes \mathfrak{i})$ using the semi-direct product structure. Thus $\mathfrak{t} \subseteq \mathfrak{i}$. Now we determine which elements generate the ideal of weakly trivial symmetries.

Proposition 6.1. *Suppose $x \in \mathfrak{i}$ such that $x = f(\Omega)g(t)p(\phi)\partial_\mu$ with $\mu \in \{\Omega, \phi\}$ and $g = \partial_t h$ for some smooth function h. Then $x \in \mathfrak{t}$*

Proof. Define $y = f(\Omega)h(t)p(\phi)\partial_\mu$, then $y \in \mathfrak{i}$ and $\partial_t y = x$. Now note that $[\partial_t y, y] = \big(fp(\partial_t h)(\partial_\mu fp)h - fph(\partial_\mu \partial_t fph)\big)\partial_\mu = 0$, and for all $[\omega] \in H^2_{ct}(\mathfrak{g})$, by proposition 5.36 and proposition 5.41, $\omega(\partial_t y, y) = 0$. Hence $x \in \mathfrak{t}$. $\square$

From now on we will work in the complexification, and note that by proposition 2.15, for any subalgebra $\mathfrak{h}$, $[\mathfrak{g}, \mathfrak{h}] = \mathrm{Re}([\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}])$. Thus if $x \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}]$, then $\mathrm{Re}(x) \in \mathfrak{t}$. Furthermore, note that $\mathfrak{t}$ is closed by definition. Thus if $D \subseteq \mathfrak{t}$, then the closure $\overline{D} \subseteq \mathfrak{t}$.

Using proposition 6.1 and the fact that $\mathfrak{t}$ is a closed ideal, we show that every term $f\partial_\mu \in \mathfrak{t}$ with $f \in \mathcal{O}(\Omega^6)$ by showing that $f\partial_\mu \in [\mathfrak{g}, \mathfrak{t}]$.

Proposition 6.2.

$$\mathcal{O}(\Omega^6)\partial_\Omega + \mathcal{O}(\Omega^6)\partial_t + \mathcal{O}(\Omega^6)\partial_\phi \subseteq \mathfrak{t} \tag{6.5}$$

Proof. We will start by showing that $\mathcal{O}(\Omega^6)\partial_\mu \subset \mathfrak{t}$ for $\mu \in \{\Omega, \phi\}$. Let $f(\Omega, t, \phi)\partial_\mu \in \mathcal{O}(\Omega^6)\partial_\mu$ and define $A(\Omega, t, \phi) = \Omega^3 \int f(\Omega, t, \phi)\Omega^{-6} d\Omega$ if $\mu = \Omega$ and $A(\Omega, t, \phi) = \int f(\Omega, t, \phi)\Omega^{-3} d\Omega$ if $\mu = \phi$. These are well defined and $A \in \mathcal{O}(\Omega^4)$ by proposition 4.14. Thus $A(\Omega, t, \phi)e^{i(x^+ + x^-)}\partial_\mu \in \mathfrak{g}$. Now by proposition 6.1, noting that $e^{i(x^+ + x^-)} = e^{i2t/l}$, $\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega \in \mathfrak{t}$ and as $\mathfrak{t}$ is an ideal, $[\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega, A(\Omega, t, \phi)e^{-i(x^+ + x^-)}\partial_\mu] \in \mathfrak{t}$. Computing this, first for $\mu = \Omega$,

$$\begin{aligned} [\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega, A(\Omega, t, \phi)e^{-i(x^+ + x^-)}\partial_\Omega] &= \left(\Omega^3 \partial_\Omega A(\Omega, t, \phi) - 3\Omega^2 A(\Omega, t, \phi)\right)\partial_\Omega \\ &= \Omega^6 \partial_\Omega (A(\Omega, t, \phi)\Omega^{-3})\partial_\Omega \\ &= f(\Omega, t, \phi)\partial_\Omega \end{aligned} \tag{6.6}$$

and for $\mu = \phi$,

$$[\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega, A(\Omega, t, \phi)e^{-i(x^+ + x^-)}\partial_\phi] = \Omega^3 \partial_\Omega A(\Omega, t, \phi)\partial_\phi = f(\Omega, t, \phi)\partial_\phi \tag{6.7}$$

Thus $f(\Omega,t,\phi)\partial_\mu \in \mathfrak{t}$.

Now let $f(\Omega,t,\phi)\partial_t \in \mathcal{O}(\Omega^6)\partial_t$ and again define $A(\Omega,t,\phi) = \int f(\Omega,t,\phi)\Omega^{-3}d\Omega$. Again $[\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega, A(\Omega,t,\phi)e^{-i(x^+ + x^-)}\partial_t] \in \mathfrak{t}$ and this is given by

$$\begin{aligned}[\Omega^3 e^{i(x^+ + x^-)}\partial_\Omega, A(\Omega,t,\phi)e^{-i(x^+ + x^-)}\partial_t] =&\Omega^3\partial_\Omega A(\Omega,t,\phi)\partial_t - 2i/l\Omega^3 A(\Omega,t\phi)\cos(t)\partial_\Omega \\ =&f(\Omega,t,\phi)\partial_t - 2i/l\Omega^3 A(\Omega,t,\phi)\partial_\Omega\end{aligned} \tag{6.8}$$

Now as $\mathcal{O}(\Omega^6)\partial_\Omega \subset \mathfrak{t}$ and $-2i/l\Omega^3 A(\Omega,t,\phi)\partial_\Omega \in \mathcal{O}(\Omega^6)\partial_\Omega$, we conclude that $f(\Omega,t,\phi)\partial_t \in \mathfrak{t}$ as well. □

For the next few proofs, we note that as $J = \mathcal{O}(\Omega^6)\partial_\Omega + \mathcal{O}(\Omega^6)\partial_t + \mathcal{O}(\Omega^6)\partial_\phi \subseteq \mathfrak{t}$, we can work in the quotient $\mathfrak{t}/J$. Then $f \in \mathfrak{t}/J$ implies $f \in \mathfrak{t}$. Furthermore, we can assume that $f = \Omega^a e^{inx^+}e^{imx^-}\partial_\mu$ as these elements are dense in $\mathfrak{t}/J$, see proposition 4.22. Lastly, note that $f = \Omega^a e^{inx^+}e^{imx^-}\partial_\mu = \Omega^a e^{it(n+m)/l}e^{i\phi(n-m)}\partial_\mu$ and thus if $n+m \neq 0$, $f \in \mathfrak{t}$ by proposition 6.1. We thus assume that $n = -m$. We conclude that if we show $\Omega^b e^{inx^+}e^{-inx^-}\partial_\mu \in \mathfrak{t}/J$ for $b \geq a$, then $\mathcal{O}(\Omega^a)\partial_\mu \subseteq \mathfrak{t}$.

We will now calculate commutators between $\mathfrak{t}$ and the Witt algebra elements, and show that $\mathfrak{i}/\text{span}\{\Omega^3\partial_\Omega\} \subseteq \mathfrak{t}$ where the quotient is a vector space quotient. As $\Omega^3\partial_\Omega \notin [\mathfrak{g},\mathfrak{g}]$, it cannot be a generator of $\mathfrak{t}$ and is also not in the closed ideal generated by the weakly trivial symmetries, thus $\Omega^3\partial_\Omega$ is not a weakly trivial symmetry.

Proposition 6.3. *Let $a \geq 4$ and suppose $\mathcal{O}(\Omega^{a+1})\partial_- \in \mathfrak{t}$. Then*

$$\mathcal{O}(\Omega^a)\partial_\Omega \in \mathfrak{t} \tag{6.9}$$

Proof. Let $f(\Omega,t,\phi)\partial_\Omega \in \mathcal{O}(\Omega^a)\partial_\Omega$. As noted above, we can assume that $f = \Omega^b e^{i2n\phi}\partial_\Omega = \Omega^b e^{in(x^+ - x^-)}\partial_\Omega$ for $b \geq a$. Now let $k \in \mathbb{Z}$, then

$$\begin{aligned}[\xi_k, \Omega^b e^{i(n-k)x^+}e^{-inx^-}\partial_\Omega] =&[e^{ikx^+}\left(ik\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2k^2\Omega^2}{2}\partial_-\right), \Omega^b e^{i(n-k)x^+}e^{-inx^-}\partial_\Omega] = \\ =&e^{inx^+}e^{-inx^-}\left((ik\frac{b}{2}\Omega^b + i(n-k)\Omega^b + i\frac{nk^2l^2}{2}\Omega^{b+2} - ik\frac{1}{2}\Omega^b)\partial_\Omega + l^2k^2\Omega^{b+1}\partial_-\right) \\ =&e^{inx^+}e^{-inx^-}\left(i(n+k\frac{b-3}{2})\Omega^b\partial_\Omega + i\frac{nk^2l^2}{2}\Omega^{b+2}\partial_\Omega + l^2k^2\Omega^{b+1}\partial_-\right)\end{aligned} \tag{6.10}$$

Now for $k \neq 0$, $\Omega^b e^{i(n-k)x^+}e^{-inx^-}\partial_\Omega \in \mathfrak{t}$ by proposition 6.1, so $e^{inx^+}e^{inx^-}\left(i(n+k\frac{b-3}{2})\Omega^b\partial_\Omega + i\frac{nk^2l^2}{2}\Omega^{b+2}\partial_\Omega + l^2k^2\Omega^{b+1}\partial_-\right) \in \mathfrak{t}$ as well. As $e^{inx^+}e^{inx^-}\left(i\frac{nk^2l^2}{2}\Omega^{b+2}\partial_\Omega + l^2k^2\Omega^{b+1}\partial_-\right) \in \mathfrak{t}$ by assumption and proposition 6.2, we conclude that $(n+k\frac{b-3}{2})e^{inx^+}e^{inx^-}\Omega^b\partial_\Omega \in \mathfrak{t}$ for $k \neq 0$. Taking $k \neq 0, n + k\frac{(b-3)}{2} \neq 0$, yields the result as $b \geq a \geq 4$. □

Corollary 6.4.

$$\mathcal{O}(\Omega^5)\partial_\Omega \in \mathfrak{t} \tag{6.11}$$

Knowing that $\mathcal{O}(\Omega^5)\partial_\Omega \in \mathfrak{t}$, we work in the vector space quotient $\mathfrak{g}/I$ with $I = \mathcal{O}(\Omega^5)\partial_\Omega + \mathcal{O}(\Omega^6)\partial_+ + \mathcal{O}(\Omega^6)\partial_-$ from now on.

Proposition 6.5.

$$\mathcal{O}(\Omega^4)\partial_\phi \subseteq \mathfrak{t} \tag{6.12}$$

Proof. Let $f(\Omega, t, \phi)\partial_\phi \in \mathcal{O}(\Omega^4)\partial_\phi$. We can again assume $f = \Omega^a e^{i2n\phi}\partial_\phi = \Omega^a e^{in(x^+ - x^-)}(\partial_+ - \partial_-)$ for $a = 4, 5$. Now for $k \neq 0$, $\Omega^a e^{i(n-k)x^+}e^{-inx^-}(\partial_+ - \partial_-) \in \mathfrak{t}$ by proposition 6.1, so commutators with this element are in $\mathfrak{t}$ as well.

$$\begin{aligned}
&[\xi_k, \Omega^a e^{i(n-k)x^+}e^{-inx^-}(\partial_+ - \partial_-)] \\
=&[e^{ikx^+}\left(ik\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2k^2\Omega^2}{2}\partial_-\right), \Omega^a e^{i(n-k)x^+}e^{-inx^-}(\partial_+ - \partial_-)] = \\
=&e^{inx^+}e^{-inx^-}\left((ik\frac{a}{2}\Omega^a + i(n-k)\Omega^a + i\frac{nk^2l^2}{2}\Omega^{a+2})(\partial_+ - \partial_-) - ik\Omega^a\left(ik\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2k^2\Omega^2}{2}\partial_-\right)\right) \\
=&e^{inx^+}e^{-inx^-}i\Omega^a\left(\left(n + k\left(\frac{a}{2} - 1\right)\right)(\partial_+ - \partial_-) - k\partial_+\right)
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
&[\bar{\xi}_m, \Omega^a e^{inx^+}e^{-i(n+m)x^-}(\partial_+ - \partial_-)] \\
=&[e^{imx^-}\left(im\frac{\Omega}{2}\partial_\Omega + \partial_- - \frac{l^2m^2\Omega^2}{2}\partial_+\right), \Omega^a e^{inx^+}e^{-i(n+m)x^-}(\partial_+ - \partial_-)] = \\
=&e^{inx^+}e^{-inx^-}\left((im\frac{a}{2}\Omega^a - i(n+m)\Omega^a - i\frac{nm^2l^2}{2}\Omega^{a+2})(\partial_+ - \partial_-) + im\Omega^a\left(im\frac{\Omega}{2}\partial_\Omega + \partial_- - \frac{l^2m^2\Omega^2}{2}\partial_+\right)\right) \\
=&e^{inx^+}e^{-inx^-}i\Omega^a\left(\left(-n + m\left(\frac{a}{2} - 1\right)\right)(\partial_+ - \partial_-) + m\partial_-\right)
\end{aligned} \tag{6.14}$$

and adding the two with $k = m$ results in:

$$\begin{aligned}
&[\xi_k, \Omega^a e^{i(n-k)x^+}e^{-inx^-}(\partial_+ - \partial_-)] + [\bar{\xi}_m, \Omega^a e^{inx^+}e^{-i(n+m)x^-}(\partial_+ - \partial_-)] \\
&= e^{inx^+}e^{-inx^-}i\Omega^a k(a-3)(\partial_+ - \partial_-)
\end{aligned} \tag{6.15}$$

As $k \neq 0, a \neq 3$, we conclude that $e^{inx^+}e^{-inx^-}\Omega^a\partial_\phi \in \mathfrak{t}$. As $\mathfrak{t}$ is closed and the exponentials are dense, we conclude that $\mathcal{O}(\Omega^4)\partial_\phi \subseteq \mathfrak{t}$. □

Proposition 6.6.

$$\mathcal{O}(\Omega^4)\partial_\pm \subseteq \mathfrak{t} \tag{6.16}$$

Proof. Let $f(\Omega, t, \phi)\partial_\pm \in \mathcal{O}(\Omega^4)\partial_\pm$. We can assume that $f(\Omega, t, \phi) = \Omega^a e^{inx^+}e^{imx^-}$ for $a = 4, 5$. By proposition 6.5 $\Omega^a e^{i(n-k)x^+}e^{imx^-}(\partial_+ - \partial_-) \in \mathfrak{t}$, so the following commutator is in $\mathfrak{t}$ as well.

$$
\begin{aligned}
&[\xi_k, \Omega^a e^{i(n-k)x^+} e^{imx^-}(\partial_+ - \partial_-)] = [e^{ikx^+}\left(ik\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2k^2\Omega^2}{2}\partial_-\right), \Omega^a e^{i(n-k)x^+} e^{imx^-}(\partial_+ - \partial_-)] = \\
=&e^{inx^+}e^{imx^-}\left((ik\frac{a}{2}\Omega^a + i(n-k)\Omega^a - i\frac{mk^2l^2}{2}\Omega^{a+2})(\partial_+ - \partial_-) - ik\Omega^a\left(ik\frac{\Omega}{2}\partial_\Omega + \partial_+ - \frac{l^2k^2\Omega^2}{2}\partial_-\right)\right) \\
=&e^{inx^+}e^{imx^-}i\Omega^a\left(\left(n + k\left(\frac{a}{2} - 1\right)\right)(\partial_+ - \partial_-) - k\partial_+\right)
\end{aligned}
\tag{6.17}
$$

Now as $e^{inx^+}e^{imx^-}i\Omega^a\left(n + k\left(\frac{a}{2} - 1\right)\right)(\partial_+ - \partial_-) \in \mathfrak{t}$ by proposition 6.5, we conclude that $e^{inx^+}e^{imx^-}\Omega^a\partial_+ \in \mathfrak{t}$ as well, as we can always take $k \neq 0$. Now $e^{inx^+}e^{imx^-}\Omega^a\partial_- \in \mathfrak{t}$ follows by switching around x^+, x^- as is shown in equation 6.14. ☐

Corollary 6.7.

$$\mathcal{O}(\Omega^4)\partial_\Omega \subseteq \mathfrak{t} \tag{6.18}$$

Proof. This is immediate from proposition 6.3 and proposition 6.6. ☐

In the next proposition and corollary, the quotient is again the vector space quotient.

Proposition 6.8.

$$(\mathcal{O}(\Omega^3)/\mathrm{span}\{\Omega^3\})\partial_\Omega \subseteq \mathfrak{t} \tag{6.19}$$

Proof. Let $f(\Omega, t, \phi)\partial_\Omega \in (\mathcal{O}(\Omega^3)/\mathrm{span}\{\Omega^3\}))\partial_\Omega$. We can once again assume $f = \Omega^3 e^{i2n\phi}\partial_\Omega = \Omega^3 e^{in(x^+ - x^-)}\partial_\Omega$. However, now as $f \neq \Omega^3$, we can assume that $n \neq 0$. Now let $k \in \mathbb{Z}$, then equation 6.10 gives

$$[\xi_k, \Omega^3 e^{i(n-k)x^+} e^{-inx^-}\partial_\Omega] = e^{inx^+}e^{-inx^-}\left(in\Omega^3\partial_\Omega + i\frac{nk^2l^2}{2}\Omega^5\partial_\Omega + l^2k^2\Omega^4\partial_-\right) \tag{6.20}$$

Moreover as $e^{inx^+}e^{-inx^-}\left(i\frac{nk^2l^2}{2}\Omega^5\partial_\Omega + l^2k^2\Omega^4\partial_-\right) \in \mathfrak{t}$ and $\Omega^b e^{i(n-k)x^+}e^{-inx^-}\partial_\Omega \in \mathfrak{t}$ for $k \neq 0$, we conclude that $e^{inx^+}e^{-inx^-}in\Omega^3\partial_\Omega \in \mathfrak{t}$. ☐

Corollary 6.9.

$$(\mathcal{O}(\Omega^3)/\mathrm{span}\{\Omega^3\})\partial_\Omega + \mathcal{O}(\Omega^4)\partial_+ + \mathcal{O}(\Omega^4)\partial_- \subseteq \mathfrak{t} \tag{6.21}$$

Proof. This is immediate from proposition 6.8 and proposition 6.6. ☐

Thus $\mathfrak{g}/\mathfrak{t} = \mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathrm{span}\{\Omega^3\partial_\Omega\}$. This has a corresponding Lie group, as $\mathfrak{w}$ is the Lie algebra of $\mathrm{Diff}(S^1)$ and $\mathrm{span}\{\Omega^3\partial_\Omega\}$ is isomorphic to the Lie algebra of $\mathbb{R}$. Moreover as $\mathfrak{w}_L \oplus \mathfrak{w}_R$ do not contain any trivial symmetries, if $\mathrm{span}\{\Omega^3\partial_\Omega\}$ is a nontrivial symmetry, then $\mathfrak{g}/\mathfrak{t}$ is thus the asymptotic symmetry algebra. Now note that any continuous projective unitary positive energy representation ρ of $\mathfrak{w}_L \oplus \mathfrak{w}_R$ can be extended to $\mathfrak{g}/\mathfrak{t}$ by letting $\rho(\Omega^3\partial_\Omega) \in \mathfrak{pu}(\mathcal{D})$ be arbitrary, for example nonzero. Thus $\Omega^3\partial_\Omega \notin \mathfrak{t}$ and we conclude that $\mathfrak{g}/\mathfrak{t}$ does not contain any trivial symmetries and has a corresponding Lie group. It is therefore the asymptotic symmetry algebra, so we denote $\mathfrak{as} = \mathfrak{g}/\mathfrak{t}$.

6.3 Physical interpretation of asymptotic symmetries

In the previous section we have shown that the asymptotic symmetry algebra in the Wigner perspective is given by $\mathfrak{as} = \mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathbb{R}$. We will now explain how this result should be interpreted physically. We will therefore also use a physicist's level of rigor in this section. We will start with the interpretation of the asymptotic symmetry algebra in the Noether perspective, where it is given by $\mathfrak{w}_L \oplus \mathfrak{w}_R$ and turn to the Wigner perspective asymptotic symmetry algebra afterwards.

6.3.1 Noether perspective asymptotic symmetry algebra

We follow the interpretation as given in Ref. [Obl17]. We first note that $\mathfrak{w}_L \oplus \mathfrak{w}_R$ contains the isometry algebra of AdS_3, $\mathfrak{o}(2,2)$, and the irreducible projective unitary positive energy representations of the isometry algebra can be interpreted as the fundamental particles in AdS_3 space, akin to Wigner's classification. However, AdS_3 is a vacuum solution, meaning that the energy-stress tensor is zero everywhere, thus no particle can truly live in AdS_3. Instead, the existence of a fundamental particle would slightly alter the spacetime and metric. Of course, if the particle moves around in spacetime, this perturbation of the gravitational field would move along with it. We thus say that the fundamental particle is *dressed* by the gravitational field, or by *soft gravitons*. The graviton is the hypothetical particle given by excitations of the gravitational field, and soft means low energy in this context. The core idea, and the terminology therefore as well, is similar to dressed particles in solid state physics such as polarons. The particle without the dressing is called the bare particle.

The bare particles are thus classified by the isometry algebra. The additional symmetries in the asymptotic symmetry algebra tell us how the bare particles can be dressed. This is sensible, as asymptotic AdS_3 spaces need not be vacuum solutions and can thus have fundamental particles in them. The asymptotic symmetries tell us which symmetries are allowed such that the spacetime stays asymptotically AdS_3, and if we assume that all asymptotically AdS_3 spaces have the same bare particles, then the additional symmetries must show how these bare particles are to be dressed.

Now the Wigner perspective asymptotic symmetry algebra has an additional factor of $\mathbb{R}$ compared to the Noether perspective one. As irreducible representations of abelian Lie algebras are one-dimensional, this means that every fundamental particle has an additional label, which is given by an arbitrary number in $\mathbb{R}$. As this factor of $\mathbb{R}$ is not included in the isometry algebra, we expect this label to tell us something about the way the particle is dressed, and not about the bare particle.

6.3.2 Wigner perspective asymptotic symmetry algebra

In Ref. [CHD95] it is shown that the asymptotic dynamics of AdS_3 are given by Liouville theory. This is a two dimensional conformal field theory (CFT). The field ψ is a real field on the torus, with action given by

$$S[\psi] = \int dt d\phi (\frac{1}{2}\partial_+\psi\partial_-\psi + 2\exp(\psi)) \tag{6.22}$$

It is a field on the torus specifically as the torus is the asymptotic boundary of AdS_3.

Inspired by these two dimensional CFT's, we will construct quantum states for the interacting boson, following the procedure for the free boson in section 6.3 of Ref. [DMS97], given by the Lagrangian

$$\mathcal{L}[\psi] = \int d\phi (\partial_t\psi)^2 - (\partial_\phi\psi)^2 + V(\psi) \tag{6.23}$$

where ψ is once again a real field on either the torus or the cylinder, so $t \in \mathbb{R}, \phi \in \mathbb{S}^1$. The field is defined on either the torus or the cylinder as the torus is the asymptotic boundary of AdS_3 and the CFT should be defined on this boundary. The cylinder is also possible for our purposes as this gives the same results. We will then find that this CFT gives us a projective unitary positive energy representation of $\mathfrak{as}$.

The states that we will construct will be the same states as for the free boson, so with $V = 0$. In the free case these states are the eigenstates of the Hamiltonian, but this is no longer true in the interacting case. Lastly, we note that Liouville theory is a special case of the interacting boson, where the potential is given by an exponential.

Construction of states

We start by decomposing $\psi(\phi, t)$ into Fourier modes:

$$\psi(\phi, t) = \sum_n e^{in\phi}\psi_n(t) \tag{6.24}$$

$$\psi_n(t) = \frac{1}{2\pi}\int d\phi e^{-in\phi}\psi(\phi, t) \tag{6.25}$$

Now

$$\begin{aligned} \int d\phi(\partial_t\phi)^2 &= \int d\phi \sum_n\sum_m e^{i(n+m)\phi}\dot{\psi}_n\dot{\psi}_m = 2\pi\sum_n \dot{\psi}_n\dot{\psi}_{-n} \\ \int d\phi(\partial_\phi\psi)^2 &= \int d\phi \sum_n\sum_m e^{i(n+m)\phi}\cdot -nm\psi_n\psi_m = 2\pi\sum_n n^2\psi_n\psi_{-n} \end{aligned} \tag{6.26}$$

and the Lagrangian thus becomes

$$\mathcal{L}[\psi] = 2\pi\sum_n \dot{\psi}_n\dot{\psi}_{-n} - n^2\psi_n\psi_{-n} + V(\psi) \tag{6.27}$$

The conjugate momentum is then easily found to be

$$\pi_n = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} = 4\pi\dot{\psi}_{-n} \tag{6.28}$$

The Hamiltonian is given by

$$H = 2\pi \sum_n \dot{\psi}_n \dot{\psi}_{-n} + n^2 \psi_n \psi_{-n} - V(\psi) = \frac{1}{8\pi} \sum_n \pi_n \pi_{-n} + (4\pi n)^2 \psi_n \psi_{-n} - V(\psi) \quad (6.29)$$

Lastly we note that $\psi_n^\dagger = \psi_{-n}$ because of its definition as Fourier modes and as ψ is real, and therefore also $\pi_n^\dagger = \pi_{-n}$. Hence $\psi_n \psi_{-n} = |\psi_n|^2$ and similarly for π_n, thus for $V = 0$ we have a set of decoupled harmonic oscillators. Now we will quantize and set $[\psi_n, \pi_n] = i\delta_{n,m}$. As we want to construct the eigenstates of the free boson, we draw inspiration from the harmonic oscillator, and define the creation and annihilation operators, or ladder operators, $\tilde{a}_n, \tilde{a}_n^\dagger$

$$\tilde{a}_n = \frac{1}{\sqrt{8\pi|n|}}(4\pi|n|\psi_n + i\pi_{-n}) \quad (6.30)$$

for $n \neq 0$. For $n = 0$ the frequency of the harmonic oscillator is zero and the operator is not well defined. We now have

$$\begin{aligned} [\tilde{a}_n, \tilde{a}_m] &= \frac{1}{8\pi|n||m||}([4\pi|n|\psi_n, i\pi_{-m}] + i[\pi_{-n}, 4\pi|m|\psi_m]) = 0 \\ [\tilde{a}_n, \tilde{a}_m^\dagger] &= \frac{1}{8\pi|n||m||}([4\pi|n|\psi_n, -i\pi_m] + i[\pi_{-n}, 4\pi|m|\psi_{-m}]) = \delta_{n,m} \end{aligned} \quad (6.31)$$

as expected. We essentially have two sets of ladder operators, one for $n > 0$ and one for $n < 0$. We split these into two sectors explicitly as follows:

$$a_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_n & (n > 0) \\ i\sqrt{-n}\,\tilde{a}_{-n}^\dagger & (n < 0) \end{cases} \qquad \bar{a}_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_{-n} & (n > 0) \\ i\sqrt{-n}\,\tilde{a}_n^\dagger & (n < 0) \end{cases} \quad (6.32)$$

which have the commutation relations

$$[a_n, a_m] = n\delta_{n+m} \quad [a_n, \bar{a}_m] = 0 \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m} \quad (6.33)$$

Moreover, note that $a_{-n} = a_n^\dagger, \bar{a}_{-n} = \bar{a}_n^\dagger$. We now want to rewrite the Hamiltonian in terms of these ladder operators. To this end we compute the following expression:

$$\begin{aligned} &\sum_{n\neq 0} a_{-n}a_n + \bar{a}_{-n}\bar{a}_n = \sum_{n>0} a_{-n}a_n + a_n a_{-n} + \bar{a}_{-n}\bar{a}_n + \bar{a}_n\bar{a}_{-n} \\ =&\sum_{n>0} 2a_{-n}a_n + 2\bar{a}_{-n}\bar{a}_n + 2n = \sum_{n>0} 2n\tilde{a}_n^\dagger\tilde{a}_n + 2n\tilde{a}_{-n}^\dagger\tilde{a}_{-n} + 2n \\ =&\sum_{n>0} 2n(\frac{1}{8\pi n}(4\pi|n|\psi_{-n} - i\pi_n)(4\pi|n|\psi_n + i\pi_{-n}) + \frac{1}{8\pi n}(4\pi|n|\psi_n - i\pi_{-n})(4\pi|n|\psi_{-n} + i\pi_n) + 1) \\ =&\sum_{n>0} \frac{2n}{8\pi n}(2(4\pi n)^2\psi_{-n}\psi_n + 2\pi_n\pi_{-n} + 4\pi n i(\psi_{-n}\pi_{-n} - \pi_n\psi_n + \psi_n\pi_n - \pi_{-n}\psi_{-n}) + 8\pi_n) \\ =&\frac{1}{2\pi}\sum_{n>0} \pi_n\pi_{-n} + (4\pi n)^2\psi_{-n}\psi_n = \frac{1}{4\pi}\sum_{n\neq 0} \pi_n\pi_{-n} + (4\pi n)^2\psi_{-n}\psi_n \end{aligned} \quad (6.34)$$

Thus the Hamiltonian is given by

$$\begin{aligned}
H &= \frac{1}{8\pi}\pi_0^2 + \frac{1}{2}\sum_{n\neq 0} a_{-n}a_n + \bar{a}_{-n}\bar{a}_n - V(\psi) \\
&= \frac{1}{8\pi}\pi_0^2 + \frac{1}{2}\sum_{n\neq 0} a_n^\dagger a_n + \bar{a}_n^\dagger \bar{a}_n - V(\psi)
\end{aligned} \tag{6.35}$$

We will now show that the ladder operators are indeed raising and lowering operators for the free boson. We thus for now assume $V = 0$. Then for $m \neq 0$, using $[AB, C] = A[B, C] + [A, C]B$,

$$\begin{aligned}
[H, a_{-m}] =& \frac{1}{2}\sum_{n\neq 0}[a_{-n}a_n + \bar{a}_{-n}\bar{a}_n, a_{-m}] \\
=& \frac{1}{2}\sum_{n\neq 0} a_{-n}[a_n, a_{-m}] + [a_{-n}, a_{-m}]a_n = \frac{1}{2}(ma_{-m} + ma_{-m}) = ma_{-m}
\end{aligned} \tag{6.36}$$

Thus if $|E\rangle$ is an eigenstate with energy E, then

$$Ha_{-m}|E\rangle = a_{-m}E|E\rangle + ma_{-m}|E\rangle \tag{6.37}$$

and hence $a_{-m}|E\rangle$ is an eigenstate with energy $E + m$, unless possibly if $a_{-m}|E\rangle$ is the zero (vacuum) state. We conclude that the operators $a_m, \bar{a}_m$ are indeed ladder operators for the free boson. We now return to the interacting case, and define the Witt operators

$$\begin{aligned}
a_0 &= \frac{1}{\sqrt{8\pi}}\pi_0 \\
L_n &= \frac{1}{2}\sum_m : a_{n-m}a_m :
\end{aligned} \tag{6.38}$$

and similarly for $\bar{L}_n$. Here the colons denote normal ordering, which is defined as

$$: a_n a_k := \begin{cases} a_n a_k & \text{if} \quad k \geq n \\ a_k a_n & \text{if} \quad k < n \end{cases} \tag{6.39}$$

One can also normal order larger strings of creation/annihilation operators, in which case they are put in ascending order of index. This results in the annihilation operators always being in front on the creation operators in a normally ordered string of operators, which is desirable as that means that L_n acts as a finite sum on any state with finitely many particles in it. Furthermore, for $n \neq 0$, $a_{n-m}a_m$ commute, so we can rewrite the definition of L_n as

$$\begin{aligned} L_n &= \frac{1}{2} \sum_m a_{n-m} a_m \\ L_0 &= \sum_{m>0} a_{-m} a_m + \frac{1}{2} a_0^2 \end{aligned} \tag{6.40}$$

With this definition, clearly $[L_n, \bar{L}_n] = 0$ for all n. We also want to calculate the commutator $[L_n, L_m]$, but to handle the infinite sums in L_n we will first introduce an additional cutoff function, following Ref. [KRR13],

$$\psi(x) = 1 \quad \text{if} \quad |x| \leq 1; \psi(x) = 0 \quad \text{if} \quad |x| > 1 \tag{6.41}$$

and define

$$L_n(\epsilon) = \frac{1}{2} \sum_m : a_{n-m} a_m : \psi(\epsilon m) \tag{6.42}$$

meaning that for $\epsilon \neq 0$, $L_n(\epsilon)$ is a finite sum and $L_n(\epsilon) \to L_n$ as $\epsilon \to 0$. Now we use that $[a_k, : a_{n-m} a_m :] = [a_k, a_{n-m} a_m]$ as the difference in ordering introduces a scalar factor at most, which commutes with a_k, and we first compute $[a_k, L_n(\epsilon)]$:

$$\begin{aligned} [a_k, L_n(\epsilon)] =& \frac{1}{2} \sum_m [a_k, a_{n-m} a_m] \psi(\epsilon m) = \frac{1}{2} \sum_m [a_k, a_{n-m}] a_m \psi(\epsilon m) + \frac{1}{2} \sum_m a_{n-m} [a_k, a_m] \psi(\epsilon m) \\ =& \frac{1}{2} k a_{n+k} \psi(\epsilon(n+k)) + \frac{1}{2} k a_{n+k} \psi(-\epsilon k) \end{aligned} \tag{6.43}$$

The limit $\epsilon \to 0$ then gives $[a_k, L_n] = k a_{n+k}$.

We are now finally ready to compute $[L_n, L_m]$:

$$\begin{aligned} [L_n(\epsilon), L_m] =& \frac{1}{2} \sum_k [a_{n-k} a_k, L_m] \psi(\epsilon k) = \frac{1}{2} \sum_k a_{n-k} [a_k, L_m] \psi(\epsilon k) + \frac{1}{2} \sum_k [a_{n-k}, L_m] a_k \psi(\epsilon k) \\ =& \frac{1}{2} \sum_k a_{n-k} k a_{k+m} \psi(\epsilon k) + \frac{1}{2} \sum_k (n-k) a_{m+n-k} a_k \psi(\epsilon k) \end{aligned} \tag{6.44}$$

We now want to put the two sums in normal order. For the first sum, the current order is correct for $n - k \leq k + m$, so $k \geq (n-m)/2$, and needs to be reversed for $k < (n-m)/2$. The terms in the second sum are in the correct order for $m + n - k \leq k$ or $k \geq (m+n)/2$ and need to be reversed for $k < (m+n)/2$. Using the commutations relations we reverse the necessary terms, resulting in

$$\begin{aligned}
[L_n(\epsilon), L_m] =& \frac{1}{2}\sum_k k : a_{n-k}a_{k+m} : \psi(\epsilon k) + \frac{1}{2}\sum_k (n-k) : a_{m+n-k}a_k : \psi(\epsilon k) \\
&+ \frac{1}{2}\sum_{k<(n-m)/2} \delta_{n+m} k(n-k)\psi(\epsilon k) + \frac{1}{2}\sum_{k<(n+m)/2} \delta_{n+m}(n-k)(-k)\psi(\epsilon k) \\
=& \frac{1}{2}\sum_k (k-m) : a_{n+m-k}a_k : \psi(\epsilon(k-m)) + \frac{1}{2}\sum_k (n-k) : a_{m+n-k}a_k : \psi(\epsilon k) \\
&+ \frac{1}{2}\sum_{k<n} \delta_{n+m} k(n-k)\psi(\epsilon k) - \frac{1}{2}\sum_{k<0} \delta_{n+m} k(n-k)\psi(\epsilon k) \\
=& \frac{1}{2}\sum_k (k-m) : a_{n+m-k}a_k : \psi(\epsilon(k-m)) + \frac{1}{2}\sum_k (n-k) : a_{m+n-k}a_k : \psi(\epsilon k) \\
&+ \frac{1}{2}\delta_{n+m} \sum_{0\leq k<n} k(n-k)\psi(\epsilon k)
\end{aligned} \tag{6.45}$$

Finally we take the limit $\epsilon \to 0$, to arrive at

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{2}\delta_{n+m}\sum_{0\leq k<n} k(n-k) = (n-m)L_{n+m} + \delta_{n+m}\frac{1}{12}(n^3-n) \tag{6.46}$$

This is precisely the Gelfand-Fuks cocycle we found in proposition 3.42, thus the L_n give a projective representation of the Witt algebra. These operators L_n are then called the Virasoro operators, after the Virasoro algebra that they form, which is the unique central extension of the Witt algebra. Moreover, we note that for the free boson, $H = L_0 + \bar{L}_0$, meaning that ∂_t in $\mathfrak{as}$ gets mapped to the free boson Hamiltonian.

We will now define the quantum states, by defining the *Fock space*, which means that we define what the zero particle states are, the one particle states, two particle states et cetera. The $a_n, \bar{a}_n$ should be annihilation operators for $n > 0$ and creation operators for $n < 0$, as we saw that for the free boson $a_{-m}|E\rangle$ is an eigenstate of H with energy $E + m$ if $|E\rangle$ is an eigenstate of H. We would thus typically define the vacuum to be the state $|0\rangle$ such that for $n > 0$

$$a_n|0\rangle = \bar{a}_n|0\rangle = 0 \tag{6.47}$$

This does not define the vacuum uniquely however, as we still have the operator a_0, which commutes with a_n. Therefore, we actually have infinitely many vacua, one for each eigenvalue of a_0. We thus define the vacua as follows

$$a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \quad (n > 0) \quad \text{and} \quad a_0|\alpha\rangle = \alpha|\alpha\rangle \tag{6.48}$$

Let $(n_j)_{j\geq 1}, (m_j)_{j\geq 1} \subseteq \mathbb{N}$ such that $\sum n_j, \sum m_j$ are finite, then we define the other states as

$$|(n_j)_{j\geq 1}, (m_j)_{j\geq 1}, \alpha\rangle = C_{(n_j)_{j\geq 1},(m_j)_{j\geq 1}} a_{-1}^{n_1} a_{-2}^{n_2} \dots \bar{a}_{-1}^{m_1}\bar{a}_{-2}^{m_2} \dots |\alpha\rangle \tag{6.49}$$

$$C_{(n_j)_{j\geq 1},(m_j)_{j\geq 1}} = \frac{1}{\sqrt{\prod_j (n_j)!(m_j)! j^{m_j+n_j}}} \tag{6.50}$$

The necessity of this normalization constant will be shown later. We will write $\boldsymbol{n} = (n_j)_{j\geq 1}, \boldsymbol{m} = (m_j)_{j\geq 1}, \boldsymbol{\delta}_k = (\delta_{j,k})_{j\geq 1}$ as a shorthand. We define the inner product to be the unique inner product such that $|\boldsymbol{n}, \boldsymbol{m}, \alpha\rangle$,$|\boldsymbol{n'}, \boldsymbol{m'}, \alpha'\rangle$ are orthonormal. One might question if states with different α should indeed be orthogonal, as α is a continuous parameter, but we note that as we want a_0 to be self-adjoint, $\langle\alpha'|a_0|\alpha\rangle = \alpha\langle\alpha'|\alpha\rangle = \alpha'\langle\alpha'|\alpha\rangle$, meaning $\langle\alpha'|\alpha\rangle$ must be zero.

The operators $L_n, \bar{L}_n, a_0$ together with the Fock space thus give us a projective representation of $\mathfrak{as}$. We will now show that this is a projective unitary positive energy representation. Starting with positive energy, we note that $L_0 + \bar{L}_0$ should be the generator of time translations, as this is ∂_t in $\mathfrak{as}$. We will therefore show that L_0 is positive energy, then analogously the same holds for $\bar{L}_0$ and the sum is also of positive energy. We start by computing the following commutator, for $n, m > 0$,

$$\begin{aligned}[][a_m, a_{-m}^n] =& a_m a_{-m}^n - a_{-m}^n a_m = a_{-m} a_m a_{-m}^{n-1} - a_{-m}^n a_m + m a_{-m}^{n-1} \\ =& a_{-m}[a_m, a^{n-1} a_{-m}] + m a_m^{n-1} = nm a_m^{n-1}\end{aligned} \tag{6.51}$$

where the last step follows by applying the previous steps $n-1$ times. Using this calculation, we find

$$\begin{aligned} a_{-k} a_k |\boldsymbol{n}, \boldsymbol{m}, \alpha\rangle =& a_{-1}^{n_1} a_{-2}^{n_2} \dots a_{-k} a_k a_{-k}^{n_k} \dots \bar{a}_{-1}^{m_1} \bar{a}_{-2}^{m_2} \dots |\alpha\rangle = \\ & a_{-1}^{n_1} a_{-2}^{n_2} \dots a_{-k}^{n_k} a_{-k} a_k \dots \bar{a}_{-1}^{m_1} \bar{a}_{-2}^{m_2} \dots |\alpha\rangle + n_k k |\boldsymbol{n}, \boldsymbol{m}, \alpha\rangle = \\ & 0 + n_k k |\boldsymbol{n}, \boldsymbol{m}, \alpha\rangle \end{aligned} \tag{6.52}$$

Note that in the last step, the first term is zero as a_k annihilates the vacuum and commutes with all of the terms in front of it. We are now ready to calculate $\langle L_0\rangle$,

$$\begin{aligned}\langle \boldsymbol{n}, \boldsymbol{m}, \alpha | L_0 | \boldsymbol{n}, \boldsymbol{m}, \alpha\rangle =& \frac{1}{2}\alpha^2 + \sum_{k>0} \langle (n_j)_{j\geq 1}, (m_j)_{j\geq 1}, \alpha | a_{-k} a_k | (n_j)_{j\geq 1}, (m_j)_{j\geq 1}, \alpha\rangle \\ =& \frac{1}{2}\alpha^2 + \sum_{k>0} k n_k \geq 0\end{aligned} \tag{6.53}$$

Thus L_0 is of positive energy.

To show that this is a unitary representation, we first note that this is a complex Lie algebra, so the standard notion of the operators being skew-Hermitian does not make sense, as noted in remark 3.4. Instead, we should show that this is a *-representation. If π is the representation, thus mapping $\xi_n \in \mathfrak{ads}_3$ to L_n as an operator on the Fock space, we require $\pi(x)^\dagger = x^\dagger$. Now it is easily shown that $\xi_n^\dagger = \xi_{-n}, (1/r\partial_r)^\dagger = 1/r\partial_r$, thus we need to show that $L_n^\dagger = L_{-n}$ and $a_0^\dagger = a_0$.

We note that we have previously defined a dagger on the operators $\tilde{a}_n$, but we have not checked that this is consistent with our definition of the inner product in the Fock space. We will therefore not use this dagger, but show directly from the inner product that $a_k^\dagger = a_{-k}$ for $k > 0$. First, note that

$$\langle \boldsymbol{n'}, \boldsymbol{m'}, \alpha' | a_{-k} | \boldsymbol{n}, \boldsymbol{m}, \alpha\rangle = C_{\boldsymbol{n},\boldsymbol{m}}/C_{\boldsymbol{n}+\delta_k,\boldsymbol{m}} \langle \boldsymbol{n'}, \boldsymbol{m'}, \alpha' | \boldsymbol{n} + \boldsymbol{\delta}_k, \boldsymbol{m}, \alpha\rangle \tag{6.54}$$

If we show that the same is true for $a_n^\dagger$, then they are the same operator. To this end, note that using equation we have

$$\begin{aligned} a_k |\boldsymbol{n'}, \boldsymbol{m'}, \alpha'\rangle &= C_{\boldsymbol{n'},\boldsymbol{m'}} a_{-1}^{n'_1} a_{-2}^{n'_2} \dots a_k a_{-k}^{n'_k} \dots \bar{a}_{-1}^{m'_1} \bar{a}_{-2}^{m'_2} \dots |\alpha'\rangle \\ =& 0 + C_{\boldsymbol{n'},\boldsymbol{m'}} n'_k k a_{-1}^{n'_1} a_{-2}^{n'_2} \dots a_k a_{-k}^{n'_k - 1} \dots \bar{a}_{-1}^{m'_1} \bar{a}_{-2}^{m'_2} \dots |\alpha'\rangle = \frac{C_{\boldsymbol{n'},\boldsymbol{m'}} n'_k k}{C_{\boldsymbol{n'}-\delta_k,\boldsymbol{m'}}} |\boldsymbol{n'} - \delta_k, \boldsymbol{m'}, \alpha'\rangle \end{aligned} \tag{6.55}$$

Thus

$$\langle \boldsymbol{n'}, \boldsymbol{m'}, \alpha' | a_k^\dagger | \boldsymbol{n}, \boldsymbol{m}, \alpha\rangle = \frac{C_{\boldsymbol{n'},\boldsymbol{m'}} n'_k k}{C_{\boldsymbol{n'}-\delta_k,\boldsymbol{m'}}} \langle \boldsymbol{n'} - \delta_k, \boldsymbol{m'}, \alpha' | \boldsymbol{n}, \boldsymbol{m}, \alpha\rangle \tag{6.56}$$

We conclude that if $\frac{C_{\boldsymbol{n},\boldsymbol{m}}}{C_{\boldsymbol{n}+\delta_k,\boldsymbol{m}}} = \frac{C_{\boldsymbol{n'},\boldsymbol{m'}} n'_k k}{C_{\boldsymbol{n'}-\delta_k,\boldsymbol{m'}}}$ for $\boldsymbol{n'} = \boldsymbol{n} + \boldsymbol{\delta}_k, \boldsymbol{m'} = \boldsymbol{m}$, we are done. This is immediate from $\frac{C_{\boldsymbol{n},\boldsymbol{m}}}{C_{\boldsymbol{n}+\delta_k,\boldsymbol{m}}} = \sqrt{(n_k+1)k}$, $\frac{C_{\boldsymbol{n'},\boldsymbol{m'}} n'_k k}{C_{\boldsymbol{n'}-\delta_k,\boldsymbol{m'}}} = \sqrt{n'_k k}$.

We thus see that this form of the normalization constant $C_{\boldsymbol{n},\boldsymbol{m}}$ is necessary to obtain the correct adjoint relations for the creation and annihilation operators. The form of the normalization constant is also similar to the standard Fock space normalization constant and the difference occurs as now $[a_n, a_{-n}] = n$ instead of 1, which is standard. Now for $k < 0$, we note that $a_k^\dagger = (a_{-k}^\dagger)^\dagger = a_{-k}$ and lastly, $a_0^\dagger = a_0$ is immediate from $\langle \boldsymbol{n'}, \boldsymbol{m'}, \alpha' | a_0 | \boldsymbol{n}, \boldsymbol{m}, \alpha\rangle = \alpha \delta_{\boldsymbol{n'},\boldsymbol{n}} \delta_{\boldsymbol{m'},\boldsymbol{m}}, \delta_{\alpha',\alpha}$.

Now calculating the adjoint of L_n for $n \neq 0$ is easily done from the definition in equation 6.38. Indeed,

$$L_n^\dagger = \frac{1}{2} \sum_m a_m^\dagger a_{n-m}^\dagger = \frac{1}{2} \sum_m a_{-m} a_{m-n} = \frac{1}{2} \sum_m a_{-n-m} a_m = L_{-n} \tag{6.57}$$

For $n = 0$, it is also immediate, $L_0^\dagger = \sum_{m>0} a_m^\dagger a_{-m}^\dagger + \frac{1}{2} a_0^2 = L_0$. We conclude that this is indeed a *-representation of AdS_3 and therefore gives rise to a projective unitary positive energy representation of $\mathfrak{as}$. However, ∂_t is represented by the free boson Hamiltonian instead of the Liouville Hamiltonian, which is unexpected.

For a more rigorous approach to the Lie algebras generated by a_n, by L_n and their representations, we refer to Ref. [KRR13].

Further notes

An interesting implication of constructing the Virasoro operators like this is that $L_0 - \bar{L}_0$ is now quantized, having only integer eigenvalues. $L_0 - \bar{L}_0$ should be the generator of the angular momentum, as it is the image of ∂_ϕ, thus the angular momentum would be quantized. Noting that this is the angular momentum of a particle, we should interpret this as the spin of the particle, which should indeed be quantized.

Something very similar to this quantization of the boson occurs in string theory, where one also has Virasoro operators $L_n, \bar{L}_n$ and the center-of-mass momentum p_0 of the string, and p_0 commutes with the Virasoro operators. For more details, see for example Ref. [GSW12].

Lastly, the projective unitary positive energy representations of the Witt algebra are the highest weight representations, where we take some $h \geq 0$, define $L_0|h\rangle = h|h\rangle$ and then let L_{-n} (L_n) be the creation (annihilation) operators. The Witt modes L_n are required to have the commutation relations of the Witt algebra. We note that if we additionally define $h = \frac{1}{2}\alpha^2$, $|h\rangle = |\alpha\rangle$, $a_0|\alpha\rangle = \alpha$, then this gives us a projective unitary positive energy representation of $\mathfrak{w} \oplus \mathbb{R}$. If we also introduce $\bar{L}_n, \bar{a}_0$ in the same fashion we have a projective unitary positive energy representation of $\mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathbb{R} \oplus \mathbb{R}$. This symmetry algebra has one factor of $\mathbb{R}$ too many, so we have to set $\bar{a}_0 = a_0$. This implies $\bar{\alpha} = \alpha, \bar{h} = h$ and gives a projective unitary positive energy representation of $\mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathbb{R}$ as we were looking for. As $\bar{a}_0 = a_0$, $\bar{h} = h$, there are no winding modes in this construction, which are modes for which $\bar{a}_0 \neq a_0$.

The representations of the Noether perspective symmetry algebra thus have a natural extension to the representations of the Wigner perspective symmetry algebra, where the additional factor represents the square root of the weight of the representation. We note however that we now had to require $\bar{h} = h$, as this is not necessary for a projective unitary positive energy representation of $\mathfrak{w}_L \oplus \mathfrak{w}_R$. Without this condition, the spin $L_0 - \bar{L}_0$ is not quantized.

In both the CFT and the string theory cases, the extra factor of $\mathbb{R}$ in the Wigner perspective thus corresponds to the zero-mode or center-of-mass momentum. It also implies the quantization of spin, which is not present in arbitrary projective unitary positive energy representations of the Noether perspective symmetry algebra. This is expected in a sense, as quantization of spin is a quantum phenomenon, whereas the Noether perspective is a classical perspective as it looks at classically conserved charges. However, we do note that this is only one possible interpretation and not a definite answer. As a last bit of evidence for our interpretation, we mention that the vector field $1/r\partial_r$, which represents the factor $\mathbb{R}$, is also constant in ϕ, just as a_0 and p_0.

Chapter 7

Conclusion

Results

In the introduction, we asked the following research question:

What is the largest normal subgroup of the weak asymptotic symmetry group of AdS_3 for which every symmetry is trivial in every projective unitary positive energy representation? What is the quotient group of the weak asymptotic symmetry group by this normal subgroup and how does it differ from the standard asymptotic symmetry group?

We have answered this question by determining the trivial symmetries on the level of Lie algebras. These are given by the ideal $\mathfrak{t} \subseteq \mathfrak{i}$ such that $\mathfrak{as} = \mathfrak{g}/\mathfrak{t} = \mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \text{span}\{\Omega^3 \partial_\Omega\}$.

The asymptotic symmetry group has $\mathfrak{as}$ as its Lie algebra and the component connected to the identity is therefore given by $\text{AS}_0 = \text{Diff}(\mathbb{S}^1)_0 \oplus \text{Diff}(\mathbb{S}^1)_0 \oplus \mathbb{R}$. The standard asymptotic symmetry group, so in the Noether perspective, is given by $\text{Diff}(\mathbb{S}^1) \oplus \text{Diff}(\mathbb{S}^1)$, so the new, Wigner perspective asymptotic symmetry group is a trivial extension of the standard Noether perspective group.

We have then looked at a possible physical interpretation of this result. First of all, we saw that the extra Virasoro modes, so aside from the $\mathfrak{sl}(2)$ modes $n = -1, 0, 1$, should be interpreted as the dressing of a bare particle with gravitons. Then for the additional factor of $\text{span}\{\Omega^3 \partial_\Omega\} = \text{span}\{1/r \partial_r\}$, we looked at a conformal field theory on the boundary of AdS. This was inspired by asymptotic AdS being equivalent to Liouville theory on the boundary and by the more general AdS/CFT correspondence. We then found that the free and interacting boson on the boundary both have $\mathfrak{as}$ as symmetry algebra. In these cases the additional factor $\text{span}\{1/r\partial_r\}$ corresponded to the average momentum of the field in the representation. The same is true in string theory and we saw that in fact projective unitary positive energy representations of $\mathfrak{w}_L \oplus \mathfrak{w}_R$ have a natural extension to a projective unitary positive energy representation of $\mathfrak{w}_L \oplus \mathfrak{w}_R \oplus \mathbb{R}$. This natural extension leads to the quantization of spin.

Discussion

Of course, there could also be a mistake in our assumptions. The main assumption is that of the boundary conditions and it is sometimes said that choosing boundary conditions is more of an art than a science. Moreover, the choice of boundary conditions determines which weak asymptotic symmetry algebra is found and therefore determines the trivial symmetries as well. A suggestion of different boundary conditions that keeps the Witt algebra structure as the lowest order structure is given in remark 4.16. Currently, we use the Ω coordinate and ask of all functions in $\mathcal{O}(\Omega^a)$ to extend smoothly to $\Omega = 0$. For example $1/\sqrt{r}$ does not satisfy this, as $1/\sqrt{r} = \sqrt{\Omega}$ is not differentiable in $\Omega = 0$. Instead, one could define $\mathcal{O}(1)$ as all the functions f such that $\lim_{r\to\infty} \partial_r^k f$ converges, as then all of the fractional powers of r would be included in $\mathcal{O}(1)$.

However, despite possible errors in the boundary conditions, we think that the Wigner perspective of trivial symmetries offers a useful way of looking at trivial symmetries, for the following reasons. Especially when one is interested in the fundamental particles belonging to the asymptotic symmetry algebra, the Wigner perspective of trivial symmetries is a priori more natural. A posteriori in the case of asymptotic AdS_3 it was shown to be more useful as well, due to the additional factor of $\mathrm{span}\{1/r\partial_r\}$ having natural correspondences in 2D CFT's and string theory.

Future research

In our case of asymptotic AdS_3 the difference between the Wigner perspective and Noether perspective turned out to be "small", in the sense that the Wigner perspective asymptotic symmetry algebra is a one dimensional trivial extension of the Noether perspective algebra. We expect something similar to hold for three dimensional asymptotically flat space, as the Noether perspective asymptotic symmetry algebra in this case is connected with the AdS_3 one through the Inönü-Wigner contraction. This is the limit of the curvature going to zero, and we expect this connection to hold in the Wigner perspective as well. The four dimensional flat space case, where the Noether perspective symmetry algebra is the BMS algebra, might very well be different however. Moreover, as the fundamental particles of the BMS algebra are known to be nonphysical, as mentioned in the introduction, this new definition could be crucial in determining the correct asymptotic symmetry algebra.

Another interesting research direction would be to extend these results to the supersymmetric case. Supersymmetry is a theory where every fermion has a corresponding boson symmetry partner, and vice versa. The graviton, the putative particle associated to gravity, would thus for example have a fermion symmetry partner that is called the gravitini. In supersymmetric theories the Hamiltonian can typically be written as an anticommutator of two elements, leading to the Hamiltonian having positive energy. As our Wigner perspective analysis is based on the Hamiltonian having positive energy, these supersymmetric theories provide an interesting ground to extend our results to. One way to proceed would be to first determine the weak asymptotic symmetries in the supersymmetric theory. This could be done by extending the Witt algebras to super Witt algebras, where the bosonic elements are given by vector fields on AdS_3 and the fermionic elements are given by spinors. Then the commutators between the two sectors could be determined, leading to a minimal weak asymptotic symmetry algebra. Now one could simply look at the ideal generated by the trivial symmetries $\mathfrak{t}$ in the super Lie algebra and therefore determine which super symmetries should be trivial on these grounds alone.

Appendix A

Code

Some of the longer calculations in this thesis were checked using Python code, with the Sympy library. This library can do symbolic algebra and is therefore useful for long calculations. All of the code is in the following GitHub repository:

`https://github.com/Joost987/AsymptoticSymmetries`

Appendix B

Boundary conditions

The following were calculated using Sympy, see Appendix A. They were then simplified a little by hand.

$$
\begin{aligned}
\mathcal{L}_{\xi_3} g_{rr} =& \frac{2l^2 r\left(-A^2+\alpha^2 l^2 - l^2\right)\sin(\phi)\cos\left(\frac{t}{l}\right)}{\sqrt{l^2+r^2}\left(-A^2+\alpha^2 l^2+r^2\right)^2} \\
\mathcal{L}_{\xi_3} g_{rt} =& \frac{l\left(-A\alpha l\left(l^2+r^2\right)\left(-A^2+\alpha^2 l^2+r^2\right)\cos(\phi)\cos\left(\frac{t}{l}\right) - r^2\left(l^2+r^2\right)^2\sin(\phi)\sin\left(\frac{t}{l}\right)\right)}{r^2\left(l^2+r^2\right)^{\frac{3}{2}}\left(-A^2+\alpha^2 l^2+r^2\right)} + \\
& \frac{l\left(r^2\left(\alpha^2 l^2+r^2\right)\left(-A^2+\alpha^2 l^2+r^2\right)\sin(\phi)\sin\left(\frac{t}{l}\right)\right)}{r^2\left(l^2+r^2\right)^{\frac{3}{2}}\left(-A^2+\alpha^2 l^2+r^2\right)} \\
\mathcal{L}_{\xi_3} g_{r\phi} =& \frac{l^2\left(-A\alpha l r^2\left(-A^2+\alpha^2 l^2+r^2\right)\sin(\phi)\sin\left(\frac{t}{l}\right)\right)}{r^2\left(l^2+r^2\right)^{\frac{3}{2}}\left(-A^2+\alpha^2 l^2+r^2\right)} + \\
& \frac{l^2\left(\left(r^2(l^2+r^2)(l^2(1-\alpha^2)+A^2)+A^2(l^2+r^2)(-A^2+\alpha^2 l^2+r^2)\right)\cos(\phi)\cos\left(\frac{t}{l}\right)\right)}{r^2\left(l^2+r^2\right)^{\frac{3}{2}}\left(-A^2+\alpha^2 l^2+r^2\right)} \\
\mathcal{L}_{\xi_3} g_{tt} =& \frac{2\left(-A\alpha l\left(l^2+r^2\right)\sin\left(\frac{t}{l}\right)\cos(\phi)+r^2 l^2(\alpha^2-1)\sin(\phi)\cos\left(\frac{t}{l}\right)\right)}{l^2 r\sqrt{l^2+r^2}} \\
\mathcal{L}_{\xi_3} g_{t\phi} =& \frac{-A\alpha l r^2\sin(\phi)\cos\left(\frac{t}{l}\right) - A\alpha l\left(l^2+r^2\right)\sin(\phi)\cos\left(\frac{t}{l}\right)+r^2 l^2\left(\alpha^2-1\right)\sin\left(\frac{t}{l}\right)\cos(\phi)}{lr\sqrt{l^2+r^2}} \\
& \frac{+A^2\left(l^2+r^2\right)\sin\left(\frac{t}{l}\right)\cos(\phi)}{lr\sqrt{l^2+r^2}} \\
\\
\mathcal{L}_{\xi_3} g_{\phi\phi} =& \frac{2A\left(Al^2\sin(\phi)\cos\left(\frac{t}{l}\right)+Ar^2\sin(\phi)\cos\left(\frac{t}{l}\right)-\alpha l r^2\sin\left(\frac{t}{l}\right)\cos(\phi)\right)}{r\sqrt{l^2+r^2}}
\end{aligned}
\tag{B.1}
$$

By analyzing the orders we quickly find:

$$
\begin{pmatrix}
\mathcal{L}_\xi g_{rr} & \mathcal{L}_\xi g_{rt} & \mathcal{L}_\xi g_{r\phi} \\
\mathcal{L}_\xi g_{tr} & \mathcal{L}_\xi g_{tt} & \mathcal{L}_\xi g_{t\phi} \\
\mathcal{L}_\xi g_{\phi r} & \mathcal{L}_\xi g_{\phi t} & \mathcal{L}_\xi g_{\phi\phi}
\end{pmatrix}
\sim
\begin{pmatrix}
\mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\
\mathcal{O}(r^{-3}) & \mathcal{O}(r^{1}) & \mathcal{O}(r^{1}) \\
\mathcal{O}(r^{-3}) & \mathcal{O}(r^{1}) & \mathcal{O}(r^{1})
\end{pmatrix}
\tag{B.2}
$$

Appendix C

Compact open topology

In this appendix we discuss some properties of the compact open topology. First, let $D \subseteq \mathbb{R}^n$ be a vector space and let $(K_i)_{i\geq 0}$ be an increasing sequence of compact subsets of D with union D. We then consider the smooth functions on D, $C^\infty(D)$ with the topology generated by the following seminorms:

$$p_{K_j,n}(f) = \sup_{x\in K_j} \max_{|\boldsymbol{\alpha}|\leq n} |\partial^{\boldsymbol{\alpha}} f(x)| \tag{C.1}$$

where $f \in C^\infty(D)$ and α is a multi-index.

We will first show that multiplication of functions is jointly continuous, i.e. the map $C^\infty(D) \times C^\infty(D) \to C^\infty(D) : (f, g) \to fg$ is continuous in the product topology. To this end, note that this a multilinear map and we can therefore use lemma 2.8.

Lemma C.1. *Multiplication is jointly continuous in the compact open topology*

Proof. By lemma 2.8, all we need to show is that for all seminorms $p_{K_j,n}$, we can find continuous seminorms p, q and a constant C such that

$$p_{K_j,n}(fg) \leq Cp(f)q(g) \tag{C.2}$$

To this end, we use the general Leibniz rule:

$$\partial^{\boldsymbol{\alpha}} fg = \sum_{|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} f \partial^{\boldsymbol{\beta}} g \tag{C.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ and $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \prod_{i=1}^n \binom{\alpha_i}{\beta_i}$. Now the result follows easily from a computation:

$$\begin{aligned} p_{K_j,n}(fg) &= \sup_{x\in K_j} \max_{|\boldsymbol{\alpha}|\leq n} |\partial^{\boldsymbol{\alpha}}(fg)(x)| \leq \sup_{x\in K_j} \max_{|\boldsymbol{\alpha}|\leq n} \sum_{|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\partial^{\boldsymbol{\alpha}} f(x)||\partial^{\boldsymbol{\beta}} g(x)| \leq \\ & \sup_{x\in K_j} \max_{|\boldsymbol{\alpha}|\leq n} \sum_{|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\partial^{\boldsymbol{\alpha}} f(x)| \sup_{x\in K_j} \max_{|\boldsymbol{\alpha}|\leq n} \sum_{|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\partial^{\boldsymbol{\beta}} g(x)| \leq \\ & Cp_{K_j,n}(f)p_{K_j,n}(g) \end{aligned} \tag{C.4}$$

□

We will now also show that right composition is continuous in this topology. To this end, let $f \in C^\infty(D)$ and let $\boldsymbol{g} = (g_1, \dots, g_n)$ with $g_1, \dots g_n \in C^\infty(D)$ and $\mathrm{im}(\boldsymbol{g}) \subseteq D$. As the seminorms involve derivatives, we need to know that the n-th derivative is of a composition of functions, which is given by the Faà di Bruno formula. This formula is quite convoluted with a lot of sums, products and combinatorial constants, which we do not need. We will therefore give a simplified version of the formula and refer to Ref. [CS96] for the full formula.

Lemma C.2 (Faà di Bruno, lemma 2.3 in Ref. [CS96])**.** *Let $f \in C^\infty(D)$, let $\boldsymbol{g} = (g_1, \dots, g_n)$ with $g_1, \dots g_n \in C^\infty(D)$ and $\mathrm{im}(\boldsymbol{g}) \subseteq D$. Then for $\boldsymbol{\alpha}$ a multi-index, the $\boldsymbol{\alpha}$ derivative is given by*

$$\partial^{\boldsymbol{\alpha}} f(\boldsymbol{g}(\boldsymbol{x})) = \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} (\partial^{\boldsymbol{\lambda}} f)(\boldsymbol{g}(\boldsymbol{x})) P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(\boldsymbol{x}) \tag{C.5}$$

where $P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}$ is a polynomial in derivatives of $g_1, \dots, g_n$ and independent of f.

Now continuity follows easily:

Lemma C.3. *Let $\boldsymbol{g} = (g_1, \dots, g_n)$ with $g_1, \dots g_n \in C^\infty(D)$ and $\mathrm{im}(\boldsymbol{g}) \subseteq D$. Then right composition with $\boldsymbol{g}$ is continuous, i.e. the map $f \to f(\boldsymbol{g})$ for $f \in C^\infty(D)$ is continuous.*

Proof. We have to show that for any j, n, the corresponding seminorm on the composition is bounded by another continuous seminorm for all $f \in C^\infty(D)$:

$$p_{K_j,n}(f \circ \boldsymbol{g}) \leq C q(f) \tag{C.6}$$

where C is a constant and q a continuous seminorm. This follows from lemma C.2 and a simple computation:

$$\begin{aligned} p_{K_j,n}(f \circ \boldsymbol{g}) = \sup_{\boldsymbol{x} \in K_j} \max_{|\boldsymbol{\alpha}| \leq n} |\partial^{\boldsymbol{\alpha}} f(\boldsymbol{g}(\boldsymbol{x}))| \leq \sup_{\boldsymbol{x} \in K_j} \max_{|\boldsymbol{\alpha}| \leq n} \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} |(\partial^{\boldsymbol{\lambda}} f)(\boldsymbol{g}(\boldsymbol{x})) P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(\boldsymbol{x})| \leq \\ \sup_{\boldsymbol{x} \in K_j} \max_{|\boldsymbol{\alpha}| \leq n} \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} |(\partial^{\boldsymbol{\lambda}} f)(\boldsymbol{g}(\boldsymbol{x}))| \sup_{\boldsymbol{x} \in K_j} \max_{|\boldsymbol{\alpha}| \leq n} \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} |P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(\boldsymbol{x})| \end{aligned} \tag{C.7}$$

As $\boldsymbol{g}$ is continuous, its image under K_j is again compact. As $(K_i)_{i \geq 0}$ is an increasing sequence with union D, there exists some i such that $\boldsymbol{g}(K_j) \subseteq K_i$. Furthermore, as $P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}$ is independent of f, $\sup_{\boldsymbol{x} \in K_j} \max_{|\boldsymbol{\alpha}| \leq n} \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} |P_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(\boldsymbol{x})|$ is a constant we will call $\tilde{C}$. Then

$$p_{K_j,n}(f \circ \boldsymbol{g}) \leq \tilde{C} \sup_{\boldsymbol{x} \in K_i} \max_{|\boldsymbol{\alpha}| \leq n} \sum_{|\boldsymbol{\lambda}| \leq |\boldsymbol{\alpha}|} |(\partial^{\boldsymbol{\lambda}} f)(\boldsymbol{x})| \leq C \sup_{\boldsymbol{x} \in K_i} \max_{|\boldsymbol{\alpha}| \leq n} |(\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{x})| = C p_{K_i,n}(f) \tag{C.8}$$

□

For later calculations we also give the one-dimensional Faà di Bruno formula [Bru55]:

Lemma C.4 (Faà di Bruno, one dimensional). *Let $D \subseteq \mathbb{R}$ be some domain, $f \in C^\infty(D), g \in C^\infty(\mathbb{R})$ with $\mathrm{im}(g) \subseteq D$. Then for all nonnegative integers n, the derivative is given by*

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1!1!^{m_1} m_2!2!^{m_2} \ldots m_n!n!^{m_n}} \cdot f^{(m_1+\cdots+m_n)}(g(x)) \cdot \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{m_j} \quad \text{(C.9)}$$

where the sum is over all nonnegative integers $m_1, \ldots, m_n$ such that

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n \quad \text{(C.10)}$$

Bibliography

[AE15] Martin Ammon and Johanna Erdmenger. "The AdS/CFT correspondence". In: *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015, pages 179–218 (cited on page 5).

[AM84] A. Ashtekar and A. Magnon. "Asymptotically anti-de Sitter space-times". en. In: *Classical and Quantum Gravity* 1 (1984), page L39. ISSN: 0264-9381. `https://doi.org/10.1088/0264-9381/1/4/002` (cited on page 7).

[Ban10] E.P. van den Ban. *Lie groups*. 2010 (cited on page 16).

[BBM62] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 269.1336 (1962), pages 21–52. ISSN: 00804630 (cited on page 5).

[BH86] J. D. Brown and Marc Henneaux. "Central charges in the canonical realization of asymptotic symmetries: An example from three dimensional gravity". en. In: *Communications in Mathematical Physics* 104.2 (June 1986), pages 207–226. ISSN: 1432-0916. `https://doi.org/10.1007/BF01211590` (cited on pages 5, 7, 43, 52, 59, 68, 102).

[Bru55] C.F. Faà di Bruno. "Sullo sviluppo delle funzione". In: *Ann. di Scienze Matem. et Fisiche di Tortoloni* 6 (1855), pages 479–480 (cited on page 123).

[BTo] BTotaro. CC BY-SA 4.0 (cited on page 11).

[Car19] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019 (cited on pages 43, 48, 49).

[CHD95] Oliver Coussaert, Marc Henneaux, and Peter van Driel. "The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant". In: *Classical and Quantum Gravity* 12.12 (Dec. 1995), page 2961. `https://doi.org/10.1088/0264-9381/12/12/012` (cited on page 108).

[CS96] G. Constantine and T. Savits. "A Multivariate Faa di Bruno Formula with Applications". en. In: *Transactions of the American Mathematical Society* 348.2 (1996), pages 503–520. ISSN: 0002-9947, 1088-6850. `https://doi.org/10.1090/S0002-9947-96-01501-2` (cited on page 123).

[DHW12] Karel Dekimpe, Manfred Hartl, and Sarah Wauters. "A seven-term exact sequence for the cohomology of a group extension". In: *Journal of Algebra* 369 (2012), pages 70–95. ISSN: 0021-8693. `https://doi.org/https://doi.org/10.1016/j.jalgebra.2012.07.009` (cited on page 35).

[DMS97] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. English. Graduate texts in contemporary physics. New York: Springer, 1997. ISBN: 038794785X; 9780387947853; 9781461274759; 1461274753 (cited on page 109).

[Fuk86] D. B. Fuks. *Cohomology of Infinite-Dimensional Lie Algebras*. en. New York, NY: Springer US, 1986. ISBN: 978-1-4684-8767-1. `https://doi.org/10.1007/978-1-4684-8765-7` (cited on pages 27, 28).

[GSW12] M. Green, John H. Schwarz, and E. Witten. *Superstring theory. Volume 1, Introduction*. English. 25th anniversary edition. Cambridge: Cambridge University Press, 2012. ISBN: 9781139531207; 1139531204; 9781139248563; 1139248561; 9781139526531; 1139526537. `https://doi.org/10.1017/CBO9781139248563` (cited on page 116).

[JN19] Bas Janssens and Karl-Hermann Neeb. "Projective unitary representations of infinite-dimensional Lie groups". In: *Kyoto Journal of Mathematics* 59.2 (2019), pages 293–341. ISSN: 2156-2261, 2154-3321. `https://doi.org/10.1215/21562261-2018-0016` (cited on pages 24, 29).

[JN24] Bas Janssens and Karl-Hermann Neeb. *Positive Energy Representations of Gauge Groups I: Localization*. en. European Mathematical Society - EMS - Publishing House GmbH, Feb. 2024. ISBN: 9783985470679 9783985475674. `https://doi.org/10.4171/mems/9` (cited on pages 20, 22, 25).

[JN25] Bas Janssens and Milan Niestijl. "Generalized Positive Energy Representations of the Group of Compactly Supported Diffeomorphisms". en. In: *Communications in Mathematical Physics* 406.2 (Jan. 2025), page 45. ISSN: 1432-0916. `https://doi.org/10.1007/s00220-024-05226-w` (cited on pages 18, 26).

[Kle09] J.M. Klebanov I.R. Maldacena. *Solving quantum field theories via curved spacetimes*. en. Jan. 2009 (cited on page 5).

[KRR13] Victor G Kac, Ashok K Raina, and Natasha Rozhkovskaya. *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*. 2nd. WORLD SCIENTIFIC, 2013. `https://doi.org/10.1142/8882` (cited on pages 112, 115).

[Lay14] S.R. Lay. *Analysis with an Introduction to Proof*. Always learning. Pearson, 2014. ISBN: 978-1-292-04024-0 (cited on page 85).

[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. en. Volume 218. Graduate Texts in Mathematics. New York, NY: Springer, 2012. ISBN: 978-1-4419-9981-8. https://doi.org/10.1007/978-1-4419-9982-5 (cited on pages 9, 36, 43, 62, 64, 74).

[Mia22] Lukas Miaskiwskyi. *A local-to-global analysis of Gelfand-Fuks cohomology*. 2022 (cited on page 36).

[Mil85] J. Milnor. "Remarks on infinite dimensional Lie groups". In: *Les Houches Summer School on Theoretical Physics: Relativity, Groups and Topology*. 1985, pages 1007–1057 (cited on page 10).

[NAS] NASA (cited on page 4).

[NB10] Lawrence Narici and Edward Beckenstein. *Topological vector spaces*. English. Second edition. Boca Raton, FL: CRC Press, 2010. ISBN: 9781584888673; 1584888679 (cited on pages 13, 14).

[Nee06] Karl-Hermann Neeb. "Towards a Lie theory of locally convex groups". In: *Japanese Journal of Mathematics* 1.2 (2006), pages 291–468. ISSN: 1861-3624. https://doi.org/10.1007/s11537-006-0606-y (cited on page 10).

[Nee22] Jan van Neerven. *Functional Analysis*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022 (cited on page 39).

[Nie23] Milan Niestijl. "Generalized positive energy representations of groups of jets". en. In: *Documenta Mathematica* 28.3 (2023), pages 709–763. ISSN: 1431-0635. https://doi.org/10.4171/dm/920 (cited on page 26).

[Noe18] E. Noether. "Invariante Variationsprobleme". ger. In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1918 (1918), pages 235–257 (cited on page 3).

[Obl17] Blagoje Oblak. *BMS Particles in Three Dimensions*. Springer Theses. Cham: Springer International Publishing, 2017. ISBN: 978-3-319-61877-7. https://doi.org/10.1007/978-3-319-61878-4 (cited on pages 5, 6, 28, 53, 59, 108).

[Res68] Robert Resnick. *Introduction to Special Relativity*. en. Wiley, 1968. ISBN: 978-0-471-71724-9 (cited on page 4).

[RT74] Tullio Regge and Claudio Teitelboim. "Role of surface integrals in the Hamiltonian formulation of general relativity". In: *Annals of Physics* 88.1 (1974), pages 286–318. ISSN: 0003-4916. https://doi.org/https://doi.org/10.1016/0003-4916(74)90404-7 (cited on page 103).

[SN14] Hadi Salmasian and Karl-Hermann Neeb. "Classification of Positive Energy Representations of the Virasoro Group". In: *International Mathematics Research Notices* 2015 (Feb. 2014). https://doi.org/10.1093/imrn/rnu197 (cited on page 104).

[TN] TN. https://commons.wikimedia.org/wiki/File:Tangentialvektor.svg (cited on page 10).

[Wal09] Robert M. Wald. *General Relativity*. English. Chicago: University of Chicago Press, 2009. ISBN: 978-0-226-87033-5 (cited on page 5).

[Wei95] Steven Weinberg. *The Quantum Theory of Fields: Volume 1: Foundations*. Volume 1. Cambridge: Cambridge University Press, 1995. ISBN: 978-0-521-67053-1. https://doi.org/10.1017/CBO9781139644167 (cited on page 21).

[Wig39] E. Wigner. "On Unitary Representations of the Inhomogeneous Lorentz Group". In: *Annals of Mathematics* 40.1 (1939), pages 149–204. ISSN: 0003486X, 19398980 (cited on page 4).

[Wig59] E.P. Wigner. *Group Theory And its Application to the Quantum Mechanics of Atomic Spectra*. en. Volume 5. DOI: 10.1016/B978-0-12-750550-3.50004-5. Elsevier, 1959. ISBN: 978-0-12-750550-3. https://doi.org/10.1016/B978-0-12-750550-3.50004-5 (cited on page 21).