Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

The critical setting of non-autonomous unbounded operators

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfillment of the requirements

for the degree

MASTER OF SCIENCE
in
APPLIED MATHEMATICS

by

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Delft, the Netherlands
June 2024

MSc THESIS APPLIED MATHEMATICS

"The critical setting of non-autonomous unbounded operators"

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June, 2024 Delft, the Netherlands

Preface

This thesis is written as part of the MSc AM Thesis project EEMCS, the graduation project for the Master Applied Mathematics on the TU Delft. The goal is:

- to design a research project and then execute it, where the chosen research project is to find existence and uniqueness criteria for the critical setting of non-autonomous unbounded operators;
- to write a research report, which you are reading now;
- to present and defend the research project, which occurs on the 28th of June, 2024.

First and foremost, I would like to thank Mark Veraar and Sebastian Bechtel. Mark has personally inspired me to research PDEs and has co-written the book (Hytönen, van Neerven, Veraar, & Weis, 2024) on which most of the theory in this thesis is dependent. My goal has always been to assist in writing the next chapter, in which the settings are extended as described below. This thesis is entirely based on his view of PDEs and which area can be expanded with local well-posedness results. Ideas for the proofs of Proposition 2.1.3, Theorem 2.1.4, and Lemma A.1.3 are also directly attributed to his insights. Sebastian, on the other hand, has given instrumental input in the finalising stage of the project, and has given proposals directly resulting in Lemma 1.2.6, Lemma A.3.1, Eq. (2.10), Lemma 2.3.6, Lemma 2.3.9, Lemma 2.3.12, the calculations in Section 2.4.3 and Proposition A.2.1, as well as giving input for many more of the results. I am grateful to Yves van Gennip for agreeing to read my thesis and accompanying Mark and Sebastian on my thesis defence in order to decide on my graduation.

My thanks go out to all my dedicated readers, Isabel van Geldern, Matto Leeuwis, Leonard in 't Veen, and Janessa Vleghert, who have tirelessly helped me improve and correct my thesis when it was nearing completion and gave crucial tips on presentation and style.

The following thesis is incredibly reliant on the following work: Hytönen, van Neerven, Veraar, and Weis (2024), who gave a base setting in which maximal L^p-regularity and the critical setting are defined; Di Giorgio, Lunardi, and Schnaubelt (2005), who extended the setting to non-constant domains; Yagi (2010), who developed essential ideas and estimates in the more complicated parts of non-linear problem-solving.

Summary

We tackle the well-posedness of certain dynamical systems that result in non-autonomous quasi-linear problems in a critical setting, where the coefficients defining the flux and the Neumann boundary conditions depend on the solution itself. We want to show the existence and uniqueness of these solutions on a very short timescale.

The local well-posedness of quasi-linear problems in a critical setting by the maximal L^p-regularity theory from Chapter 18 of Hytönen, van Neerven, Veraar, and Weis (2024) are investigated, where we use the non-autonomous setting with non-constant domains from Di Giorgio, Lunardi, and Schnaubelt (2005). Dominant examples in the literature of such problems are problems with multidimensional, non-constant Neumann boundary conditions influencing the domain of the operator. In this thesis, we look for ways to ensure the short-timescale existence and uniqueness of solutions to these problems and research the possibility of applying them to the model problem with Neumann boundary conditions. By applying non-autonomous linear theory from Chapter 3 Part II of Yagi (2010), we find a result that allows us to determine the existence and uniqueness of short timescale mild solutions. When applied, however, we see that, unlike the work of Yagi (2010), we can only guarantee the local well-posedness of the model Neumann problem by using averaging functions because of our more strict critical setting. In the future, results can be based on different regularity types, or the given result can be applied on spaces with negative smoothness.

Contents

Introduction

When working on dynamical systems, systems of Partial Differential Equations (PDEs) of various forms are found. When examining the heat in a closed room with a radiator for instance, we find we can model the heat $u = u(t, x)$ for time t and position in space x using a generalised equation $\partial_t u(t, x) - c\Delta u(t, x) = f(t, x)$, where ∂_t is a derivative for the time variable, Δ is the Laplacian operator differentiating twice for the space variable, $c > 0$ is some constant and $f(t, x)$ is the source function representing the heater in the room. If we want one unique solution, we have to include initial data, say $u(0, x) = u_0(x)$ as a distribution of the initial heat over the room, and a boundary condition $Bu(t, x) = g(t, x)$ for values x on the edges of the room, which tells us what happens at the edges of the room based on u and g.

Fundamental to progressing the research on these systems is determining whether or not the PDE systems we find can be solved and whether the determined solution is unique, as expected from examining our physical examples. While viewing each problem separately becomes laborious, working in a general setting will allow us to explain many problems simultaneously. One such generalisation is a description of problems which live on a defined Banach space X (which can be viewed as a type of space containing functions of the space variable x) and time interval $I \subseteq [0, \infty)$, featuring a time derivative ∂_t, spatial derivative A, which is an unbounded linear operator with domain $D(A)$, and source function f. The described problems then need to have functions $u(t)$ with values taken in X_0 and in the domain $D(A)$ so that the following holds:

$$\begin{cases} \partial_t u(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{cases}$$

Such a problem is an autonomous linear problem, where autonomous means that the only time-dependence is in the solution u and the source function f, and linear refers to the linearity of the equation with reference to the solution u. One crucial quality for A to have when looking for solutions to autonomous linear problems is maximal regularity, which entails solutions living in the same space as the source function or right-hand side function. For instance, If A has maximal L^p-regularity (Hytönen, van Neerven, Veraar, & Weis, 2024), we know that f being an L^p function[1] is enough to know the problem above has exactly one solution u, which has $\partial_t u$, Au as L^p functions such that

$$\|\partial_t u\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leq C\|\partial_t u + Au\|_{L^p(I;X)} = C\|f\|_{L^p(I;X)}$$

for some constant $C > 0$. As stated by Hytönen, van Neerven, Veraar, and Weis (2024), such an estimate is often crucial to fixed point arguments that appear when proving the existence and uniqueness of solutions to non-linear problems or, in our case, quasi-linear problems. Such problems are described starting from an initial data point u_0, where the linear part A depends on u non-linearly, and where the right-hand side function F has a non-linear dependency on u as well:

$$\begin{cases} \partial_t u(t) + A(u(t))u(t) &= F(u(t)), \quad t \in I, \\ u(0) &= u_0. \end{cases}$$

One of the key questions to answer is what happens to the domain $D(A(u(t)))$. Clearly, if we treat a different operator for each u or each t, we have to contemplate whether or not we also need to consider a different domain for each operator. For problems with constant domains, meaning every operator acts on the same domain $D(A(u(t))) = X_1$, problems like this have unique solutions by results like LeCrone, Prüss, and Wilke (2014). For linear problems with non-autonomous operator $(A(t), D(A(t)))$, where the domains

[1] Being integrable to the power p, less smooth than continuous functions.

$D(A(t))$ are non-constant for time t, we can use the maximal L^p-regularity result from Di Giorgio, Lunardi, and Schnaubelt (2005). This result requires a set of conditions known in the literature as the Acquistapace Terreni conditions to find a unique solution exists. In essence, these are conditions set on the operator A such that we can view solutions of these equations as an evolution of the initial condition u_0 over a part determined by F. However, when we want to find existence and uniqueness for non-linear problems with $(A(u), D(A(u)))$ and changing domains $D(A(u))$, we need stronger conditions than Acquistapace Terreni conditions, like the Hölder regularity conditions as in Yagi (2010), which features a setting in the Hölder continuous functions[2] instead of in the L^p-functions, and uses maximal L^∞-regularity. These conditions are essentially set in such a way that they imply the previous Acquistapace Terreni conditions and improve the evolution quality of solutions.

In recent research springing from Prüss and Wilke (2017), which is refined further in Prüss, Simonett, and Wilke (2018) and Hytönen, van Neerven, Veraar, and Weis (2024), an improvement is found of the non-linear L^p-regularity setting by including a critical part in the right-hand side function F. What this means is that we will split F into a trace part F_{Tr}, which acts as the continuous part of F and allows for initial data u_0 to be treated, and a critical part F_c, which acts on an L^r space with $r > p$ and allows for stronger non-linearities. However, all of this research is done on non-linear equations with constant domains $D(A(u))$, and results that show the existence and uniqueness of solutions to quasi-linear problems in the critical setting on non-constant domains $D(A(u))$ are currently missing. This means we can not easily treat example problems with $X_0 = L^q(\Omega)$ and domains $D(A(u)) \subseteq W^{2,q}(\Omega)$ a Sobolev space[3] for some $q \in (1, \infty)$ of the following form:

$$\begin{cases} \partial_t u(t,x) + A(u(t,x))u(t,x) &= F(u(t,x)), \quad & t \in I,\ x \in \Omega, \\ B(u(t,x))u(t,x) &= 0, & t \in I,\ x \in \partial\Omega, \\ u(0,x) &= u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

Here $B(u)$ is a certain Neumann boundary condition depending on u, $\Omega \subseteq \mathbb{R}^d$ is some open domain, and $F(u)$ is a non-linear function in the critical setting described as by Prüss, Simonett, and Wilke (2018) and Chapter 18 of Hytönen, van Neerven, Veraar, and Weis (2024). The linear part $A(u)$ and boundary condition $B(u)$ are defined as follows:

$$[A(v(t,\cdot))u(t,\cdot)](x) = \sum_{i,j=1}^{d} \partial_i \left[a_{ij}(v(t,\cdot))\partial_j u(t,\cdot)\right](x),$$

$$[B(v(t,\cdot))u(t,\cdot)](x) = \sum_{i,j=1}^{d} n_i(x) a_{ij}(v(t,x))\partial_j[u(t,\cdot)](x).$$

Here, $a_{ij} \in C^2(L^q(\mathbb{R}^d); \mathbb{R})$ are the matrix coefficients and $\mathbf{n}(x)$ is the outward vector on the boundary $\partial\Omega$. What domain $D(A(u))$ is chosen as a subspace of $W^{2,q}(\Omega)$ is, of course, entirely dependent on which boundary condition $B(u)$ currently applies to. Since the boundary condition is dependent on u, the domain of A will thus be dependent on u. Problems like this model problem Eq. (1) show up when modelling honeybee colonies (Yagi, 2010, Section 5.7) or chemotaxis of a cell, (Yagi, 2010, Section 5.8) to name a few examples from the literature.

In this thesis, we formulate conditions that ensure the short timescale existence and uniqueness of solutions to (non-)autonomous quasi-linear problems on non-constant domains $D(A(u))$ in the critical setting. In order to do this, we attempt to extend the critical setting, as is described in Chapter 18 of Hytönen, van Neerven, Veraar, and Weis (2024) for constant domains, to the setting with non-constant domains. We will first consider the non-autonomous linear setting on non-constant domains of Di Giorgio, Lunardi, and Schnaubelt (2005), then use the linear theory of Yagi (2010) to get a result that works on the same conditions as the Hölder regularity conditions, but working with the L^p-regularity of the critical setting. Our research questions are as follows:

- Can a set of conditions be derived on which we are able to show short timescale existence and uniqueness of solutions to quasi-linear problems on non-constant domains in a critical setting?

[2]Functions more smooth than continuous functions but less smooth than continuously differentiable functions.
[3]Functions with derivatives that are integrable to the power q.

- Can our result ensure a unique short timescale solution to the model problem Eq. (1)?

It should be noted that chapter 5 of Yagi (2010) already features short timescale existence and uniqueness of solutions to quasi-linear problems on non-constant domains and has a result which can be applied to a quasi-linear problem on non-constant domains in section 5.6. Therefore, it is crucial to show how our inclusion of the critical setting affects our results.

In Chapter 1, we develop the theory of maximal L^p-regularity as found in Chapter 17 of Hytönen, van Neerven, Veraar, and Weis (2024), and treat the case of non-autonomous linear problems with non-constant domains as found by Di Giorgio, Lunardi, and Schnaubelt (2005).

In Chapter 2, weighted regularity is introduced using the power weights from Section 17.2.e of Hytönen, van Neerven, Veraar, and Weis (2024), and non-autonomous semi-linear and quasi-linear cases on non-constant domains are treated, where the critical setting is introduced for the quasi-linear problem on non-constant domains. The non-linear problems on non-constant domains use the Hölder regularity assumptions as made in Chapter 3 part II and in Chapter 5 of Yagi (2010) and require results described in Appendix A.

The research questions are answered, and the results are discussed in Chapter 3.

Chapter 1

Definitions and properties

In this chapter, we set up the tools that will allow us to conduct new research. In Section 1.1, we will introduce the concept of maximal L^p-regularity on the simplest form of problems, namely the problems which are:

1. autonomous, meaning there is no dependence on time except in the solution itself and the right-hand side function;

2. linear, meaning there is no dependence on the solution itself besides the linear part Au.

Because of the linearity, we can work on the zero initial condition. The theory for this will be built on the work of chapter 17 of Hytönen et al. (2024), with occasional references to Chapter 13 of van Neerven (2022) and the Harmonic Analysis lecture notes. Next, in Section 1.2, we will extend the concept to non-autonomous problems. In the case that the domains do not depend on time, an existence-uniqueness result may be shown by perturbation as in Theorem 1.2.1 from Section 17.2.g of Hytönen et al. (2024). On the time-dependent domains, we are going work from the setting of Di Giorgio, Lunardi, and Schnaubelt (2005) and Chapter 3 part II of Yagi (2010), with references to Schnaubelt (2004). In these settings, the concept of the initial condition is also introduced using the work of Di Giorgio, Lunardi, and Schnaubelt (2005), in order to have a working initial condition for the non-linear problems of Chapter 2.

1.1 Maximal L^p-regularity on autonomous linear problems

We will start from the inhomogeneous heat equation on $\mathbb{R} \times \mathbb{R}^d$, since it is the simplest example of the problems we treat:

$$\partial_t u + \lambda u - \Delta u = f.$$

Here, $f(t, x)$ is taken in $L^p(\mathbb{R} \times \mathbb{R}^d)$ for $p \in (1, \infty)$, $\lambda > 0$, and the solutions u are in some space such that this equation makes sense as an equality in $L^p(\mathbb{R} \times \mathbb{R}^d)$. For this equation, you can show using multiplier theory[1] that there is maximal L^p-regularity for this equation, meaning that if $f \in L^p(\mathbb{R} \times \mathbb{R}^d)$, then $\partial_t u$, λu and $-\Delta u$ are all also $L^p(\mathbb{R} \times \mathbb{R}^d)$ functions, where an estimate of the L^p norm of $\partial_t u$, λu and $-\Delta u$ is given based on the L^p norm of f:

$$\|\partial_t u\|_p + \sum_{|\alpha| \leq 2} \lambda^{1-\frac{1}{2}|\alpha|} \|\partial^\alpha u\|_p \leq C_{d,p} \|f\|_p.$$

It is essentially the most regularity one can expect given $f \in L^p(\mathbb{R} \times \mathbb{R}^d)$, and it allows for convenient estimates of the solutions to be made based only on the known source function. In fact, using this estimate, it is shown in the Harmonic Analysis lecture notes that the solution exists and is unique for all $p \in (1, \infty)$. We want to be able to apply this concept to much more abstract problems in order to derive important existence uniqueness results of solutions to these problems. We will start by looking at a more general autonomous linear problem.

[1] See the Harmonic Analysis lecture notes for a complete version of this proof.

1.1.1 The Abstract Cauchy Problem

The abstract Cauchy problem we will begin researching is defined as follows:

$$\begin{cases} \partial_t u(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{cases} \tag{1.1}$$

We consider u and f as functions in the function space $L^p(I; X_0)$, where $I \subseteq (0, \infty)$ is some (un)bounded interval in time, and X_0 is any Banach space. A good example Banach space would be $X_0 = L^q(\mathbb{R}^d)$, so that $\tilde{u}(t, x) = (u(t))(x)$ and $\tilde{f}(t, x) = (f(t))(x)$ are also functions in space. In the previously mentioned heat equation, we considered $A = (\lambda - \Delta)$ with domain $D(A) = W^{2,q}(\mathbb{R}^d)$ and Banach space $X_0 = L^q(\mathbb{R}^d)$, but we will now be able to model many different differential equations by filling in any unbounded operator $(A, D(A))$ and Banach space X_0. Then we can see A as a linear operator in $\mathscr{L}(D(A), X_0)$, where we assume a continuous embedding exists $D(A) \hookrightarrow X_0$

We define proper solutions to the problem Eq. (1.1) based on the definitions of Hytönen et al. (2024):

Definition 1.1.1. 1. u is called a *strongly measurable function* if there exists a sequence of simple functions u_n which converges $u_n \uparrow u$ pointwise.

2. $u : I \to X_0$ is called a *strong solution* to the problem Eq. (1.1) for a $f \in L^1_{loc}(\bar{I}; X_0)$ if:
   (a) u is strongly measurable,
   (b) u takes values in $D(A)$ a.e.,
   (c) $Au \in L^1_{loc}(\bar{I}; X_0)$,
   (d) and u solves the integrated version of Eq. (1.1), meaning for almost all $t \in I$, we have

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s.$$

3. u is called an L^p*-Solution* to the problem Eq. (1.1) for a $f \in L^p(I; X_0)$ if u is a strong solution that has $Au \in L^p(I; X_0)$ as well.

With these definitions in mind, we are ready to define what we will mean by maximal L^p-regularity:

Definition 1.1.2 (Maximal L^p regularity)**.** The unbounded operator A has *maximal L^p regularity on I* if there exists a constant $C \geq 0$ s.t. for all $f \in L^p(I; X_0)$, the problem Eq. (1.1) admits a unique L^p-solution u_f on I which has

$$\|Au_f\|_{L^p(I;X_0)} \leq C\|f\|_{L^p(I;X_0)}.$$

The least admissible constant for which the above equation holds is denoted as $M^{reg}_{p,A}(I)$. We will denote A having maximal L^p-regularity on I as $A \in \mathcal{MR}_p(I)$ for convenience.

Below will follow a few subsections which are helpful as background material.

1.1.2 C_0-semigroups and sectoriality

Often, these unbounded operators A will have C_0-semigroups $S(t)$ such that $-A$ generates this C_0-semigroup.[2] These C_0-semigroups are often used as the solutions to the problem Eq. (1.1), and thus, the existence of solutions is very important for research into the problem. We will need some information on C_0-semigroups, and specifically analytic C_0-semigroups. We will use the work of van Neerven (2022), and consider the open sector $\Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \omega\}$ for some angle $\omega \in (0, \pi)$.

Definition 1.1.3 (C_0-semigroups)**.** 1. A is said to *generate* (or to be a *generator* of) a C_0-semigroup $S(t)$ if A is the closed operator defined by

$$D(A) = \left\{ x \in X_0 : \lim_{t \downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X_0 \right\},$$

$$Ax = \lim_{t \downarrow 0} \frac{1}{t}(S(t)x - x).$$

[2]For the full definition of a C_0-semigroup, see chapter 13 of van Neerven (2022). Essentially these are the solutions operators mapping initial data to solutions.

2. A C_0-semigroup $S(t)$ is called *analytic on* Σ_ω if for all $x \in X_0$ the function $t \mapsto S(t)x$ extends holomorphically to Σ_ω, and satisfies
$$\lim_{z \in \Sigma_\omega, z \to 0} S(z)x = x.$$

The following result by van Neerven (2022) shows why these Analytical C_0-semigroups are of interest:

Theorem 1.1.4 (Bounded analytic semigroups, complex characterization)**.** *For a densely defined closed operator A in X_0 the following assertions are equivalent:*

1. *$-A$ generates a bounded analytic C_0-semigroup $S(\cdot)$ on Σ_η for some $\eta \in (0, \frac{1}{2}\pi)$;*
2. *there exists $\zeta \in (\frac{1}{2}\pi, \pi)$ such that $\Sigma_\zeta \subseteq \varrho(-A)$ and*
$$\sup_{\lambda \in \Sigma_\zeta} \|\lambda R(\lambda, -A)\| < \infty.$$

Denoting the suprema of all admissible η and ζ by $\omega_{holo}(-A)$ and $\omega_{res}(-A)$ respectively, we have
$$\omega_{res}(-A) = \frac{1}{2}\pi + \omega_{holo}(-A).$$

Under the equivalent conditions above, we have the inverse Laplace transform representation
$$S(t)x = e^{-tA}x := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, -A)x \; d\lambda, \quad t > 0, x \in X_0, \tag{1.2}$$
where $\Gamma = \Gamma_{\zeta', B}$ is the upwards oriented boundary of $\Sigma_{\zeta'} \setminus B$, for any $\zeta' \in (\frac{1}{2}\pi, \zeta)$ and any closed ball B centered at the origin.

Proof. Theorem 13.30 from van Neerven (2022). □

The property that $-A$ has $\sup_{\lambda \in \Sigma_\zeta} \|\lambda R(\lambda, -A)\| < \infty$ is often referred to as $-A$ being a *sectorial* operator. We set $\omega(A)$ as the angle descried above in $(0, \frac{1}{2}\pi)$. Theorem 1.1.4 tells us that sectorial operators always generate bounded analytic C_0-semigroups. We will often also denote e^{-tA} as the bounded analytic semigroup generated by $-A$, as we will see this is a convenient notation in upcoming properties.

Proposition 1.1.5. *Let A be a sectorial operator with angle $\omega(A) \in (0, \frac{1}{2})$. For $\phi \in [0, 1)$, $\psi \in [-1, -\phi)$ and $s, t \in [0, T]$, we have*
$$\begin{aligned} \|A^\phi [e^{-tA} - 1] A^\psi\|_{\mathscr{L}(X_0)} &\leq \left\| \int_0^t A^{\phi+\psi+1} e^{-\tau A} d\tau \right\|_{\mathscr{L}(X_0)} \\ &\leq C \int_0^t \tau^{-\phi-\psi-1} = Ct^{-\phi-\psi}. \end{aligned} \tag{1.3}$$

Proof. This is Equation (2.129) on page 102 from Yagi (2010), which we can apply since A is sectorial. □

With the definition of sectoriality in mind, we will also have a short look at *R-sectoriality*,[3] which we will require for one of the characterizations of $\mathcal{MR}_p$ operators as seen in Theorem 1.1.20. In the special case of L^p spaces, it means that for any $N \in \mathbb{N}$, any $\lambda_n \in \Sigma_{\omega_{\text{res}}}$ and any $x_n \in L^p$. we can have the bound
$$\left\| \left(\sum_{n=1}^N |\lambda_n R(\lambda_n, A) x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq M \left\| \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.$$
Typically, R-sectoriality is defined using Rademacher sequences. The implication that sectorial operators are R-sectorial is straightforward to prove. However, the other way around usually is not true, as seen in examples in Chapter 10 of Hytönen et al. (2017).

[3]See Section 10.3.a of Hytönen et al. (2017).

1.1.3 Mild Solutions

In upcoming problems on non-constant domains, it tends to be unpractical to look for strong solutions. However, we can use the C_0-semigroups, and later evolution families, to help us get from a less strong version of a solution, which will correspond to the desired strong solution under the right circumstances.

Definition 1.1.6. Suppose $-A$ generates a locally bounded strongly measurable semigroup S on X_0. For $f \in L^1_{loc}(\bar{I}; X_0)$, the continuous function $u \in L^1_{loc}(\bar{I}; X_0)$ defined by

$$u(t) := S * f(t), \quad t \in \bar{I},$$

is called the *mild solution* of the problem Eq. (1.1).

The following results from Hytönen et al. (2024) show why this is useful:

Proposition 1.1.7. *Let $-A$ generate a locally bounded strongly measurable semigroup S on a Banach space X_0. Let $f \in L^p(I; X_0)$ with $p \in [1, \infty]$. Then for any function $u \in C(\bar{I}; X_0)$, the following assertions are equivalent:*

1. *u is a strong solution on I of Eq.* (1.1)*;*
2. *u is the mild solution on I of Eq.* (1.1)*, u is differentiable a.e., and $\frac{\mathrm{d}}{\mathrm{d}t}u \in L^1_{loc}(\bar{I}; X_0)$;*
3. *u is the mild solution on I of Eq.* (1.1)*, u takes values in $D(A)$ a.e., and $Au \in L^1_{loc}(\bar{I}; X_0)$.*

In particular, a strong solution, if it exists, is unique and equals the mild solution.

Proof. Proposition 17.1.3 from Hytönen et al. (2024). Note it is done for any initial condition. □

Now, we get a proposition that allows us to go from mild to strong solutions. Recall C^μ stands for the space of Hölder continuous functions, with a norm

$$\|f\|_{C^\mu([0,T];X_0)} = \sup_{0 \le s < t \le T} \frac{\|f(t) - f(s)\|_{X_0}}{(t-s)^\mu} + \|f\|_{C([0,T];X_0)}$$

Proposition 1.1.8. *Let A be sectorial of angle $\omega(A) \in (0, \frac{1}{2})$ and let S be the analytic C_0-semigroup generated by $-A$. Then for all $f \in C^\mu([0,T]; X_0)$ with $\mu > 0$, the mild solution $u = S * f$ to the problem Eq.* (1.1) *satisfies*

$$u \in C([0,T]; X_0) \cap L^\infty((0,T); D(A)).$$

In particular, u is a strong solution.

Proof. Proposition 17.1.4 from Hytönen et al. (2024). □

1.1.4 Solution spaces

In this part, feasible solution spaces are defined, which allow for solutions to Eq. (1.1).

Definition 1.1.9 (Solution spaces)**.** The *solution space of the time derivative* for problem Eq. (1.1) is defined as

$${}_0\dot{W}^{1,p}(I; X_0) := \left\{ v \in W^{1,p}_{\mathrm{loc}}(\bar{I}; X_0) : \ \frac{\mathrm{d}}{\mathrm{d}t} v \in L^p(I; X_0), \ v(0) = 0 \right\}.$$

The *solution space of the equation* is then defined as

$${}_0\dot{W}^{1,p}_A(I; X_0) := \left\{ v \in \ {}_0\dot{W}^{1,p}(\bar{I}; X_0) : \ v(\cdot) \in D(A) \text{ a.e. on I}, \ Av \in L^p(I; X_0) \right\}.$$

Most of the time, we will, however work on finite time intervals, on which we can use the following *maximal regularity space*:

$$\mathcal{P}^p(I) := W^{1,p}(I; X_0) \cap L^p(I; D(A)),$$

which is equipped with the norm

$$\|x\|_{\mathcal{P}^p(I)} = \|x\|_{W^{1,p}(I;X_0)} + \|Ax\|_{L^p(I;X_0)}.$$

The spaces defined here will serve as the range for the closed solution operator $\mathcal{M} : f \mapsto u_f$. We will show a few properties of the spaces here, fully derived in Hytönen et al. (2024).

Proposition 1.1.10. *If u is an L^p-solution of problem Eq.* (1.1) *on I, then*

1. ${}_0\dot{W}^{1,p}(I; X_0) \supseteq {}_0\dot{W}^{1,p}_A(I; X_0)$ *are both normed spaces, with*

$$\|v\|_{{}_0\dot{W}^{1,p}(I;X_0)} := \left\| \frac{\mathrm{d}}{\mathrm{d}t} v \right\|_{L^p(I;X_0)},$$
$$\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)} := \max\left\{ \left\| \frac{\mathrm{d}}{\mathrm{d}t} v \right\|_{L^p(I;X_0)}, \ \|Av\|_{L^p(I;X_0)} \right\}.$$

2. *For all bounded subintervals $(0, T) \subseteq I$, we have*

$${}_0\dot{W}^{1,p}_A(I; X_0) \hookrightarrow \mathcal{P}^p((0, T)),$$

with

$$\|v\|_{\mathcal{P}^p((0,T))} \leq (T + 1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)}.$$

In particular, the L^p-solution u belongs to this maximal regularity space $\mathcal{P}^p((0, T))$.

Additionally, if $A : D(A) \to X_0$ is a closed operator, then ${}_0\dot{W}^{1,p}_A(I; X_0)$ is a Banach space.

Proof. Proposition 17.2.2 in Hytönen et al. (2024). □

The space $\mathcal{P}^p((0, T)) := L^p((0, T); D(A)) \cap W^{1,p}((0, T); X_0)$ is a form of maximal regularity space because any L^p-solution u will also have L^p regularity for both Au and $\partial_t u$ by definition. If $A \in \mathcal{MR}_p((0, T))$, then the unique solution u_f has $\|u_f\|_{\mathcal{P}^p((0,T))} \leq C\|f\|_{L^p((0,T);X_0)}$ for any $f \in L^p((0, T); X_0)$.

1.1.5 Properties of $\mathcal{MR}_p$ operators

Proposition 1.1.11 ($\mathcal{MR}_p$ implies closed)**.** *If $A \in \mathcal{MR}_p(I)$ for any $p \in (1, \infty)$ and $I \subseteq [0, \infty)$, then A is a closed operator.*

Proof. Proposition 17.2.5 in Hytönen et al. (2024). □

This closedness result is then applied to show that the solution operator $\mathcal{M}$ is an isomorphism.

Corollary 1.1.12 (Solution operator)**.** *For $f \in L^p(I; x)$, define the solution operator*

$$\mathcal{M} : L^p(I; X_0) \to {}_0\dot{W}^{1,p}_A(I; X_0)$$

by $\mathcal{M}f = u_f$, where $u_f \in \mathcal{P}^p(I)$ is the solution to problem Eq. (1.1) *on I for given function f. If A is an operator with maximal L^p-regularity on I, then $\mathcal{M}$ is an isomorphism from $L^p(I; X_0)$ to ${}_0\dot{W}^{1,p}_A(I; X_0)$, with*

$$\frac{1}{2}\|f\|_{L^p(I;X_0)} \leq \|\mathcal{M}f\|_{{}_0\dot{W}^{1,p}_A(I;X_0)} \leq (M^{reg}_{p,A}(I) + 1)\frac{1}{2}\|f\|_{L^p(I;X_0)}.$$

Proof. Corollary 17.2.6 in Hytönen et al. (2024). □

This corollary means that maximal L^p-regularity gives enough information to determine existence and uniqueness of solutions, which clearly means maximal regularity is a useful concept for analysing the existence and uniqueness of solutions to PDEs. The next result uses the given bounds to allow us to move from the more difficult regularity space of ${}_0\dot{W}^{1,p}_A(I; X_0)$ to the simpler regularity space $\mathcal{P}^p(I)$.

Proposition 1.1.13. *Let $A \in \mathcal{MR}_p(I)$, $f \in L^p(I; X_0)$ and let $u_f \in {}_0\dot{W}^{1,p}_A(I; X_0)$ be the unique solution to problem Eq.* (1.1).

1. *For all bounded sub-intervals* $(0,T) \subseteq I$*, we have* $u_f \in \mathcal{P}^p((0,T))$*, and*

$$\|u_f\|_{\mathcal{P}^p((0,T))} \leq (T+1)(M^{reg}_{p,A}(I)+1)\|f\|_{L^p(I;X_0)}.$$

   Additionally, if $I = (0,T)$ *a bounded interval, then*

$${}_0\dot{W}^{1,p}_A(I;X_0) \eqsim \mathcal{P}^p((0,T)),$$

   and for all v *in this space, we have*

$$\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)} \leq \|v\|_{\mathcal{P}^p((0,T))} \leq (T+1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)};$$

2. *If* $0 \in \varrho(A)$*, we have* $u_f \in L^p(I;D(A)) \cap {}_0\dot{W}^{1,p}(I;X_0)$*, and*

$$\|u_f\|_{L^p(I;D(A))\cap\, {}_0\dot{W}^{1,p}(I;X_0)} \leq (\|A^{-1}\|+1)(M^{reg}_{p,A}(I)+1)\|f\|_{L^p(I;X_0)}.$$

   We have

$${}_0\dot{W}^{1,p}_A(I;X_0) \eqsim L^p((0,T);D(A)) \cap {}_0\dot{W}^{1,p}((0,T);X_0),$$

   and for all v *in this space, we have*

$$\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)} \leq \|v\|_{L^p((0,T);D(A))\cap\, {}_0\dot{W}^{1,p}((0,T);X_0)} \leq (\|A^{-1}\|+1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X_0)}.$$

Proof. Proposition 17.2.8 and Corollary 17.2.9 of Hytönen et al. (2024). □

The first part of this proposition uses finite intervals to deduce bounds as seen before. The second part uses invertibility of the operator[4] to deduce a similar bound. Note that Proposition 1.1.13 allows us to work with $\mathcal{P}^p((0,T))$ on finite intervals as our closed space for the $\mathcal{MR}_p((0,T))$ operators.

We will also take a result from Hytönen et al. (2024) which allows us to find solutions of the problem Eq. (1.1) with $Af \in L^p(\mathbb{R}_+;X_0)$ as the right hand side function.

Lemma 1.1.14. *Suppose that* A *has maximal* L^p*-regularity on* $\mathbb{R}_+ := [0,\infty)$*, then for every* $f \in L^p(\mathbb{R}_+;D(A))$ *one has* $f \in L^p(\mathbb{R}_+;D(A))$ *and*

$$A\mathcal{M}f = \mathcal{M}Af$$

as functions in $L^p(\mathbb{R}_+;X_0)$*. In particular, the solution to the problem Eq.* (1.1) *with* $Af \in L^p(\mathbb{R}_+;X_0)$ *is* Au_f*, where* $u_f = \mathcal{M}f$ *is the solution to the problem Eq.* (1.1) *with* $f \in L^p(\mathbb{R}_+;D(A))$*.*

Proof. Lemma 17.2.12 of Hytönen et al. (2024) □

Dore theorem on sectoriality

The following result will show that all $\mathcal{MR}_p(I)$ operators are also sectorial.

Theorem 1.1.15 (Dore)**.** *Let* A *be an unbounded operator on a Banach space* X_0*, and let* $p \in [1,\infty]$ *be fixed. Then:*

1. *if* A *has maximal* L^p*-regularity on a bounded interval* $(0,T)$*, then* $-A$ *generates an analytic semigroup on* X_0*, and* $\lambda + A$ *is sectorial of angle* $\omega(\lambda + A) < \frac{1}{2}\pi$ *for* $\lambda \in \mathbb{R}$ *large enough. Moreover, for* $\operatorname{Re}\lambda$ *large enough,*

$$\|AR(\lambda,A)\| \leq 2M^{reg}_{p,A}(\mathbb{R}_+).$$

2. *if* A *has maximal* L^p*-regularity on* $\mathbb{R}_+$*, then* $-A$ *generates a bounded analytic* C_0*-semigroup on* X_0*, and* A *is sectorial of angle* $\omega(A) < \frac{1}{2}\pi$*. Moreover, for* $\operatorname{Re}\lambda > 0$*,*

$$\|AR(\lambda,A)\| \leq M^{reg}_{p,A}(\mathbb{R}_+).$$

[4] See van Neerven (2022) for more information on how to invert unbounded operators.

Proof. Theorem 17.2.15 of Hytönen et al. (2024). ☐

This result can for instance be applied to relate mild solutions to L^p-solutions, as is done in Hytönen et al. (2024).

Theorem 1.1.16. *Let $-A$ generate an analytic semigroup on a Banach space X_0 and let $p \in [1, \infty]$. Let F be any dense subspace of $L^p(I; X_0)$. Then A has maximal L^p-regularity on I iff the mapping $f \to Vf$ defined for functions $f \in F$ by*

$$Vf(t) := A[S * f(t)], \quad t \in I$$

is well defined, maps F into $L^p(I; X_0)$, and there is a constant $C \geq 0$ s.t.

$$\|Vf\|_{L^p(I;X_0)} \leq C\|f\|_{L^p(I;X_0)}, \quad f \in F.$$

In this situation, V uniquely extends to a bounded operator on $L^p(I; X_0)$ s.t. $Vf = Au_f$, where u_f is the mild AND L^p-solution to the problem Eq. (1.1) *associated with f. In this situation the least admissible C from the above inequality also coincides with $M^{reg}_{p,A}(I)$.*

Proof. Theorem 17.2.19 of Hytönen et al. (2024). ☐

Permanence properties

Here we will list off the permanence properties shown in Hytönen et al. (2024).

Theorem 1.1.17 (Permanence properties)**.** *Let A be a linear operator on a Banach space X_0 and let $p \in [1, \infty]$. If A has maximal L^p-regularity on I, the following assertions hold:*

1. ***Translation**: $\lambda + A$ has maximal L^p-regularity on I in both of two situations:*
   (a) $I = \mathbb{R}_+$ and $\operatorname{Re} \lambda > 0$;
   (b) $I = (0, T)$ and $\lambda \in \mathbb{C}$.
2. ***Change of interval**: A has maximal L^p-regularity on every bounded interval $(0, T')$.*
3. ***Scalar multiples**: λA has maximal L^p-regularity on I for all $\lambda > 0$.*
4. ***Extrapolation of exponent**: A has maximal L^q-regularity on I for all $q \in (1, \infty)$.*
5. ***Duality**: If A is densely defined, then A^* has maximal $L^{p'}$-regularity on I, where $1 = \frac{1}{p} + \frac{1}{p'}$ like in van Neerven (2022).*

Proof. Theorem 17.2.26 of Hytönen et al. (2024). ☐

Remark. Because of Item 4 of Theorem 1.1.17, we will use L^p-regularity as a broad term instead of restricting it to a specific p.

1.1.6 Characterizations of Maximum Regularity

The first characterization we are interested in is Proposition 17.2.10 from Hytönen et al. (2024) which deals with extension from a dense subspace of $L^p(I; X_0)$.

Proposition 1.1.18 (Extension from dense function subspace)**.** *Let A be a closed operator, and let F be a dense subspace of $L^p(I; X_0)$. Suppose that for all $f \in F$ there exists a unique solution u_f to problem Eq.* (1.1)*, and that this solution satisfies the maximal regularity bound*

$$\|Au_f\|_{L^p(I;X_0)} \leq C\|f\|_{L^p(I;X_0)},$$

with a constant C independent of $f \in F$. Then A has maximal L^p-regularity on I with $M^{reg}_{p,A}(I) \leq C$.

Proof. Proposition 17.2.10 from Hytönen et al. (2024). ☐

We can also get maximal regularity from closed operators by showing both existence and uniqueness of the L^p solution u_f, as we see in Proposition 17.2.11 from Hytönen et al. (2024).

Proposition 1.1.19. *If A is closed and for all $f \in L^p(I; X_0)$, there exists a unique solution u_f to the problem Eq.* (1.1)*, then for all $f \in L^p(I; X_0)$ the Maximum regularity estimate,*

$$\|Au_f\|_{L^p(I;X_0)} \leq C\|f\|_{L^p(I;X_0)},$$

holds for a constant C independent of f. In particular, A has maximal L^p regularity on I.

Proof. Proposition 17.2.11 from Hytönen et al. (2024). □

Lastly and most importantly, we can categorize maximal L^p-regularity by R-sectoriality of the operator A, as seen in Theorem 17.3.1 of Hytönen et al. (2024).

Theorem 1.1.20 (Maximal L^p-regularity and R-sectoriality)**.** *Suppose A is a linear operator on a Banach space X_0 and $p \in [1, \infty]$.*

1. *If $A \in \mathcal{MR}_p([0, \infty))$, then A is R-sectorial with angle $\omega_R(A) < \frac{1}{2}\pi$.*
2. *If X_0 is UMD*[5]*, $p \in (1, \infty)$ and A is R-sectorial with angle $\omega_R(A) < \frac{1}{2}\pi$, then $A \in \mathcal{MR}_p([0, \infty))$.*
3. *If X_0 is UMD, $p \in (1, \infty)$, and $-A$ generates an analytic semigroup e^{-tA} s.t. the sets $\{e^{-tA} : t \geq 0\}$ and $\{tAe^{-tA} : t \geq 0\}$ are R-bounded,*[6] *then $A \in \mathcal{MR}_p([0, \infty))$.*

Proof. Theorem 17.3.1 of Hytönen et al. (2024). □

1.2 Non-autonomous linear problems

We move to a new set of equations for us to solve: equations where $A(t)$ is a non-autonomous operator and has a domain $D(A(t))$, possibly depending on time as well. We define the operators as $A : [0, T] \to \mathscr{L}(D(A(\cdot)), X_0)$, where $D(A(t)) \hookrightarrow X_0$ are the domains of each of the operators in time.[7] The equation is as follows:

$$\begin{cases} \partial_t u(t) + A(t)u(t) &= f(t), \quad t \in (0, T), \\ u(0) &= 0. \end{cases} \tag{1.4}$$

Here, we will not consider infinite time intervals I, and generally, we will have only small T for which we can easily prove maximal L^p-regularity. We will first consider time-independent domains $D(A(t)) = X_1 \hookrightarrow X_0$, over which we can prove maximal regularity through perturbation. Later we will consider time-dependent domains, with the additional restraint of the Acquistapace-Terreni conditions from Assumption 1.2.2.

Theorem 1.2.1 (Maximal L^p-regularity for time-dependent A)**.** *Let X_0 and X_1 be Banach spaces with continuous embedding $X_1 \hookrightarrow X_0$, and let $A \in C([0, T]; \mathscr{L}(X_1, X_0))$ be a mapping with the following two properties:*

- *there exists a constant $L > 0$ s.t. for all $t \in [0, T]$,*

$$L^{-1}\|x\|_{X_1} \leq \|A(t)x\|_{X_0} + \|x\|_{X_0} \leq L\|x\|_{X_1}, \quad x \in X_1,$$

  meaning we can consider the operators as $A(t)$ working on X_0 due to equivalent graph norms.

- *for all $t \in [0, T]$ the unbounded operator $A(t)$ on X_0 with domain $D(A(t)) = X_1$ has maximal L^p-regularity on $(0, T)$ with*

$$M := \sup_{t \in [0,T]} M^{reg}_{p,A(t)}(0, T) < \infty.$$

[5] See Chapter 4 of Hytönen et al. (2016) for information on UMD spaces.
[6] See Chapter 8 of Hytönen et al. (2017) for information on R-boundedness.
[7] Note that this tells us very little about the domains, so we only treat this as an extremely general definition of possible non-autonomous operators.

Then there exists a $0 \leq \tau \leq T$ *and a unique strong solution*

$$u \in L^p((0,\tau);X_1) \cap W^{1,p}((0,\tau);X_0)$$

to the problem Eq. (1.4) *s.t.*

$$\|u\|_{L^p((0,\tau);X_1)\cap W^{1,p}((0,\tau);X_0)} \leq C\|f\|_{L^p((0,\tau);X_0)}$$

for a certain constant $C \geq 0$. *In particular,* $A(t)$ *has maximal* L^p*-regularity.*

Proof. This is Theorem 17.2.51 in Hytönen et al. (2024), and since the proof is interesting for our end goal, we will re-state it here.

We will prove this by perturbation. First, we will rewrite the problem Eq. (1.4) such that we have a semi-linear problem with a time-independent operator:

$$\begin{cases} \partial_t u(t) + A(0)u(t) &= (A(0)-A(t))u(t) + f(t), \quad t \in (0,T), \\ u(0) &= 0. \end{cases} \tag{1.5}$$

Now we define a perturbation operator

$$\mathcal{L} : L^p((0,T);X_1) \cap W^{1,p}((0,T);X_0) \to L^p((0,T);X_1) \cap W^{1,p}((0,T);X_0),$$

which sets $\mathcal{L}(v) = u$ where u and v satisfy

$$\begin{cases} \partial_t u(t) + A(0)u(t) &= (A(0)-A(t))v(t) + f(t), \quad t \in (0,T), \\ u(0) &= 0. \end{cases} \tag{1.6}$$

Then, finding a solution u to the nonlinear problem Eq. (1.5) is the same as solving the fixed point problem $\mathcal{L}(v) = v$. For $v_1, v_2 \in L^p((0,T);X_1) \cap W^{1,p}((0,T);X_0)$, consider $u = \mathcal{L}(v_1) - \mathcal{L}(v_2)$. Since $u_1 := \mathcal{L}(v_1)$ and $u_2 := \mathcal{L}(v_2)$ both satisfy problem Eq. (1.6), we can see that $u := u_1 - u_2$ and $v := v_1 - v_2$ must satisfy

$$\begin{aligned} \partial_t u(t) + A(0)u(t) &= \partial_t u_1(t) - \partial_t u_2(t) + A(0)u_1(t) - A(0)u_2(t) \\ &= (A(0)-A(t))v_1(t) + f(t) - ((A(0)-A(t))v_2(t) + f(t)) \\ &= (A(0)-A(t))(v_1(t) - v_2(t)) \\ &= (A(0)-A(t))v(t). \end{aligned}$$

By the maximal L^p-regularity of $A(0)$, we can conclude that

$$\begin{aligned} \|u\|_{L^p((0,T);X_1)\cap W^{1,p}((0,T);X_0)} &\overset{\mathcal{MR}_p}{\leq} M\|(A(0)-A(\cdot))v(\cdot)\|_{L^p((0,T);X_0)} \\ &= M\left(\int_0^T \|(A(0)-A(t))v(t)\|_{X_0}^p dt\right)^{\frac{1}{p}} \\ &\leq \sup_{t\in[0,T]} M\|A(0)-A(t)\|_{\mathscr{L}(X_1,X_0)} \left(\int_0^T \|v(t)\|_{X_1}^p dt\right)^{\frac{1}{p}} \\ &= \sup_{t\in[0,T]} M\|A(0)-A(t)\|_{\mathscr{L}(X_1,X_0)} \|v\|_{L^p((0,T);X_1)} \\ &\leq \sup_{t\in[0,T]} M\|A(0)-A(t)\|_{\mathscr{L}(X_1,X_0)} \|v\|_{L^p((0,T);X_1)\cap W^{1,p}((0,T);X_0)}. \end{aligned}$$

Setting $0 \leq \tau \leq T$ s.t. $M\sup_{t\in[0,T]} \|A(0)-A(t)\|_{\mathscr{L}(X_1,X_0)}\| \leq C_\tau < 1$ then gives the estimate

$$\|u_1 - u_2\|_{L^p((0,\tau);X_1)\cap W^{1,p}((0,\tau);X_0)} \leq C_\tau \|v_1 - v_2\|_{L^p((0,\tau);X_1)\cap W^{1,p}((0,\tau);X_0)}. \tag{1.7}$$

We can now see that we can apply the Banach fixed point theorem to $\mathcal{L}$, since:

- $\mathcal{L}$ maps an element $v \in L^p((0,\tau);X_1) \cap W^{1,p}((0,\tau);X_0)$ to an element $u \in L^p((0,\tau);X_1) \cap W^{1,p}((0,\tau);X_0)$, meaning $\mathcal{L}$ is *self-mapping.*

- $\mathcal{L}$ has the estimate Eq. (1.7) for $C_\tau < 1$, meaning $\mathcal{L}$ is a *contraction mapping.*

The Banach Fixed Point theorem allows us to conclude there indeed exists exactly one $v \in L^p((0,\tau);X_1) \cap W^{1,p}((0,\tau);X_0)$ s.t. $\mathcal{L}(v) = v$. Since we have therefore found an L^p-solution $u_f := v$ to the problem Eq. (1.4) for $(0,\tau)$, we use Proposition 1.1.13 to get the desired bound on $\|u_f\|$. □

This result is helpful to get from maximal regularity from the individual operators to maximal regularity of the time-dependent operator. However, a proof by perturbation of the above form will not work on time-dependent domains. Most of the equalities written down above need to be seen in an entirely new light, and picking a v_1, v_2 like above becomes especially challenging. To solve the problem in this setting, we will have to work towards evolution equations using mild solutions.

1.2.1 Time-dependent domains

Suppose $D(A(t))$ are separate domains, and not equal to some set X_1, and consider the problem

$$\begin{cases} \partial_t u(t) + A(t)u(t) &= f(t), \quad t \in (0,T), \\ u(0) &= 0. \end{cases} \tag{1.8}$$

for $\partial_t u, A(\cdot)u(\cdot), f \in L^p((0,T);X_0)$.

Assumption 1.2.2. *We assume:*

(AS i) $\exists \zeta \in (\frac{1}{2}\pi, \pi), K > 0$ *such that* $\varrho(-A(t)) \supset \Sigma_\zeta$ *for all* $t \in [0,T]$, *and*

$$\|R(\lambda, -A(t))\|_{\mathscr{L}(X_0)} \le \frac{K}{1+|\lambda|}, \; t \in [0,T], \lambda \in \Sigma_\zeta.$$

(AS ii) $\forall t > s \in (0,T), \lambda \in \Sigma_\zeta, \exists \alpha_i, \beta_i, i \in \{1,\dots,k\}, 0 \le \beta_i < \alpha_i \le 2$ *s.t.* $\delta := \min_i\{\alpha_i - \beta_i\} \in (0,1)$ *and*

$$\|A(t)R(\lambda,-A(t))[A^{-1}(t) - A^{-1}(s)]\|_{\mathscr{L}(X_0)} \le K \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1}.$$

(AS iii) X_0 *is UMD. and* $A(t)$ *are uniformly R-sectorial with angle* $\omega_R(A(t)) < \frac{1}{2}\pi$, *meaning in particular that* $A(t)$ *are sectorial operators and* $\sup_{t\in[0,T]}\{M(t)\} < \infty$ *for* $M(t)$ *the sectoriality bound.*

As in Di Giorgio, Lunardi, and Schnaubelt (2005), we then get a new form of maximal regularity space that's functionally similar to $\mathcal{P}^p((0,T))$.

Definition 1.2.3 (Solution space in time-dependent domains)**.** Let Assumption 1.2.2 hold for $A(t)$, we define the *Maximal regularity space* $\mathcal{E}^p((0,T))$ as

$$\mathcal{E}^p((0,T)) := \{v \in W^{1,p}((0,T);X_0) : A(t)v(t) \in D(A(t)) \text{ a.e.}, A(\cdot)v(\cdot) \in L^p((0,T);X_0)\},$$

with a norm

$$\|v\|_{\mathcal{E}^p((0,T))} := \|v\|_{W^{1,p}((0,T);X_0)} + \|A(\cdot)v(\cdot)\|_{L^p((0,T);X_0)}.$$

From this it is natural to denote maximal L^p-regularity of $A(\cdot)$ as meaning there exists a unique solution $u_f \in \mathcal{E}^p((0,T))$ to Eq. (2.1) with $\|u_f\|_{\mathcal{E}^p((0,T))} \le C\|f\|_{L^p((0,T);X_0)}$ for all $f \in L^p((0,T);X_0)$.

Note (AS iii) of Assumption 1.2.2 implies $A(t) \in \mathcal{MR}^p((0,T))$ by Theorem 1.1.20.

1.2.2 Evolution families

In this section we derive a time-dependent version of semigroups $e^{-(t-s)A}$ based on the works of Pazy (1983) and Schnaubelt (2004). The goal is to use them to derive mild solutions, and using the mild solution to proof results which are otherwise difficult to see. We will see that the solution to the problem equation Eq. (2.1) is given by $u(t) = \int_0^t G(t,s)f(s)\,\mathrm{d}s$, where $G(t,s)$ is a certain evolution family.

Definition 1.2.4 (Evolution family)**.** An *evolution family* $G(\cdot,\cdot)$ is a family of bounded linear operators $G(t,s)$, $t \geq s$ that satisfy

$$\begin{aligned} G(s,s) &= I, \\ G(t,s) &= G(t,r)G(r,s), \quad t \geq r \geq s. \end{aligned}$$

The evolution family is called *strongly continuous* if the mapping $(t,s) \mapsto G(t,s)$ is strongly continuous on the triangle $\{(t,s) \in \mathbb{R}^2 : t \geq s\}$.

The following result is a simplified version of results from theorem 2.2 in Schnaubelt (2004). These properties will be used throughout the thesis.

Theorem 1.2.5. *Let Assumption 1.2.2 hold for $A(t)$, then there exists a strongly continuous evolution family $G(\cdot,\cdot)$ on X_0 with time interval $(0,T)$ such that for all $0 \leq s < t \leq T$,*

1. $G(t,s)X_0 \subseteq D(A(t))$,
2. $\|A(t)G(t,s)\| \leq \frac{C}{t-s}$ *for some* $C \geq 0$,
3. $\partial_t G(t,s) = -A(t)G(t,s)$,
4. $\partial_s^+ G(t,s)x = G(t,s)A(s)x$ *for* $x \in D(A(s))$.

Proof. Theorem 2.2 from Schnaubelt (2004). □

We also discuss the Yosida approximations. Let $A_n(t)$ be the Yosida approximations $A_n(t) = A(t)J_n(t)$, where we defined $J_n(t) := nR(n, A(t))$. Then there is an important result to mention for $G_n(t,s)$ the evolution family of $A_n(t)$:

$$\partial_s G_n(t,s)x = G(t,s)A(s)x \text{ for } x \in D(A(s)). \tag{1.9}$$

Essentially, the derivative from this side is no longer one-sided if you take the Yosida approximation. Lemma 2.3 from Schnaubelt (2004) also gives a convergence of A_n to A, and Proposition 2.5 from Schnaubelt (2004) gives a convergence of G_n to G.

1.2.3 Improved linear theory under stronger assumptions

The condition (AS ii) from Assumption 1.2.2 can be simplified somewhat using the interpolation spaces from Appendix A.1, as is seen in the following lemma.

Lemma 1.2.6. *Suppose the following holds for $A(t)$:*

- $\exists \gamma_0 \in (0,1)$ *s.t.* $\forall r \in [1,\infty], \beta \leq \gamma_0$, $(X_0, D(A(t)))_{\beta,r} = (X_0, D(A(0)))_{\beta,r} =: X_{\beta,r}$, $[X_0, D(A(t))]_\beta = [X_0, D(A(0))]_\beta =: X_\beta$ *for all* $t \in [0,T]$, *and for all* $t,s \in [0,T]$, $x \in X_\beta$ *and* $x \in X_{\beta,r}$ *we have*

$$\begin{aligned} \|x\|_{(X_0,D(A(t)))_{\beta,r}} &\leq C_1 \|x\|_{(X_0,D(A(s)))_{\beta,r}}, \\ \|x\|_{[X_0,D(A(t))]_\beta} &\leq C_2 \|x\|_{[X_0,D(A(s))]_\beta}, \end{aligned}$$

  where $C_1, C_2 > 0$ are independent of t, s.

- *For $t \in [0,T]$ and $\beta \leq \gamma_0$, assume $D(A^\beta(t)) = X_\beta$ with the norm*

$$\|x\|_{D(A^\beta(t))} = \|A^\beta(t)x\|_{X_0}, \quad x \in X_\beta.$$

- *$\exists \gamma \in (0, \gamma_0], \mu \in (0,1]$ s.t. $\gamma + \mu > 1$ and*

$$\|A^\gamma(t)\left[A^{-1}(t) - A^{-1}(s)\right]\|_{\mathscr{L}(X_0)} \leq N(t-s)^\mu, \quad 0 \leq s < t \leq T.^8$$

Then (AS ii) holds for $A(t)$.

Proof. To show this, we use interpolation with the operator $A(t)$, which gives

$$\|(A(t))^{1-\gamma}R(\lambda, -A(t))\|_{\mathscr{L}(X_0)} \leq C|\lambda|^{-\gamma}$$

for $\lambda \in \Sigma_\zeta$ with $\zeta \in (\frac{1}{2}\pi, \pi)$. We also use the commutativity $R(\lambda, -A(t))(A(t))^\gamma = R(\lambda, -A(t))(A(t))^\gamma$ on the domains $D(A^\gamma(t)) = X_\gamma$ with proposition 15.1.12 from Hytönen et al. (2024) to get

$$\begin{aligned}
&\|A(t)R(\lambda, -A(t))[A^{-1}(t) - A^{-1}(s)]\|_{\mathscr{L}(X_0)} \\
&\leq \|(A(t))^{1-\gamma}R(\lambda, -A(t))(A(t))^\gamma[A^{-1}(t) - A^{-1}(s)]\|_{\mathscr{L}(X_0)} \\
&\leq C|\lambda|^{-\gamma}\|(A(t))^\gamma[A^{-1}(t) - A^{-1}(s)]\|_{\mathscr{L}(X_0)} \\
&\leq C|\lambda|^{-\gamma}N(t-s)^\mu = \tilde{C}(t-s)^\mu|\lambda|^{-\gamma}
\end{aligned}$$

Setting $k = 1$, $\alpha_1 = \mu$, $\beta_1 = 1 - \gamma$, we see that we have to require $\mu + \gamma - 1 \in (0,1)$. This gives

$$1 < \gamma + \mu < 2.$$

Since $\mu, \gamma < 1$, this holds by our assumption in the lemma. □

Because of that, we can also create a stronger assumption, which we need for some linear theory that is useful when dealing with the non-linearity. Because of Lemma 1.2.6, Assumption 1.2.7 implies Assumption 1.2.2.

Assumption 1.2.7. *(AS' i) $\exists \zeta \in (\frac{1}{2}\pi, \pi), K > 0$ such that $\varrho(-A(t)) \supset \Sigma_\zeta$ for all $t \in [0,T]$, and*

$$\|R(\lambda, -A(t))\|_{\mathscr{L}(X_0)} \leq \frac{K}{1 + |\lambda|}, \; t \in [0,T], \lambda \in \Sigma_\zeta.$$

(AS' ii) X_0 is UMD and $A(t)$ are uniformly R-sectorial with angle $\omega_R(A(t)) < \frac{1}{2}\pi$.

(AS' iii) $\exists \gamma_0 \in (0,1)$ s.t. $\forall r \in [1,\infty], \beta \leq \gamma_0$, $(X_0, D(A(t)))_{\beta,r} = (X_0, D(A(0)))_{\beta,r} =: X_{\beta,r}$, $[X_0, D(A(t))]_\beta = [X_0, D(A(0))]_\beta =: X_\beta$ for all $t \in [0,T]$, and for all $t, s \in [0,T]$, $x \in X_\beta$ and $x \in X_{\beta,r}$ we have

$$\begin{aligned}
\|x\|_{(X_0,D(A(t)))_{\beta,r}} &\leq C_1\|x\|_{(X_0,D(A(s)))_{\beta,r}}, \\
\|x\|_{[X_0,D(A(t))]_\beta} &\leq C_2\|x\|_{[X_0,D(A(s))]_\beta},
\end{aligned}$$

where $C_1, C_2 > 0$ are independent of t, s.

(AS' iv) For $t \in [0,T]$ and $\beta \leq \gamma_0$, assume $D(A^\beta(t)) = X_\beta$ with the norm

$$\|x\|_{D(A^\beta(t))} = \|A^\beta(t)x\|_{X_0}, \quad x \in X_\beta.$$

(AS' v) $\exists \gamma \in (0, \gamma_0], \mu \in (0,1]$ s.t. $\gamma + \mu > 1$ and

$$\|A^\gamma(t)\left[A^{-1}(t) - A^{-1}(s)\right]\|_{\mathscr{L}(X_0)} \leq N(t-s)^\mu, \quad 0 \leq s < t \leq T,$$

where N is independent of t, s.

If we compare these to the assumptions made in Section 3.4.1 of Yagi (2010), we see that our Assumption 1.2.7 implies the structural assumptions of Section 3.4.1 of Yagi (2010). Therefore, we have the following estimates from Section 3.8.1:

[8] In the theory from Section 2.3 we can only consider $\mu \in (0, \sigma]$ because of the $\mu + \theta = \sigma$ constraint.

Lemma 1.2.8. *Suppose Assumption 1.2.7 holds for $A(t)$, and let $G(t,s)$ be the strongly continuous evolution operator of $A(t)$. Then:*

1. *For $\phi \in [0, \mu+\gamma)$, $\psi \in [-\phi, \min\{0, 1-\phi\}] \cup [0,\mu)$, and $0 \le s < t \le T$,*
$$\|A^{\phi}(t)G(t,s)A^{\psi}(s)\|_{\mathscr{L}(X_0)} \le C_1(t-s)^{-\phi-\psi}. \tag{1.10}$$

2. *For $\phi \in [0,\gamma]$, $\psi \in [-1, 0]$ and $0 \le s < t \le T$,*
$$\|A^{\phi}(t)(G(t,s)) - e^{-(t-s)A(s)})A^{\psi}(s)\|_{\mathscr{L}(X_0)} \le C_2(t-s)^{\gamma+\mu-1-\phi-\psi}. \tag{1.11}$$

3. *For $\phi \in [0,\gamma]$ and $\psi \in [-1],0]$, $\tau > 0$ and $0 \le s < t \le T$,*
$$\|A^{\phi}(t)\left(e^{-\tau A(t)} - e^{-\tau A(s)}\right)A^{\psi}(s)\|_{\mathscr{L}(X_0)} \le C_3\tau^{\gamma-\phi-\psi-1}(t-s)^{\mu}. \tag{1.12}$$

Proof.

1. If $\psi \in [0,\mu)$, combine Equations (3.81) and (3.82) on page 154 of Yagi (2010), and the $G(t,s) = G(t,r)G(r,s)$ property of evolution families from Theorem 1.2.5, to obtain
$$\begin{aligned}
\|A^{\phi}(t)G(t,s)A^{\psi}(s)\|_{\mathscr{L}(X_0)} &= \left\|A_u^{\phi}(t)G\left(t, \frac{t+s}{2}\right)G\left(\frac{t+s}{2}, s\right)A^{\psi}(s)\right\|_{\mathscr{L}(X_0)} \\
&\overset{((3.82),\text{Yagi }(2010))}{\le} C_1\left(t - \frac{t+s}{2}\right)^{-\phi}\left\|G\left(\frac{t+s}{2}, s\right)A^{\psi}(s)\right\|_{\mathscr{L}(X_0)} \\
&\overset{((3.81),\text{Yagi }(2010))}{\le} C_2\left(t - \frac{t+s}{2}\right)^{-\phi}\left(\frac{t+s}{2} - s\right)^{-\psi} \\
&= C_2\left(\frac{1}{2}\right)^{-\phi-\psi}(t-s)^{-\phi-\psi}.
\end{aligned}$$
If $\psi \in [-\phi, \min\{0, 1-\phi\}]$, the result follows from Equation (3.83) of Yagi (2010) instead. Note that for $\phi \le 1$, which will be true for all the cases in the proof of Lemma 2.3.12, we can just use $[-\phi, 0]$.

2. Equation (3.87) on page 154 from Yagi (2010).

3. Equation (3.91) on page 156 from Yagi (2010).

□

1.2.4 Initial condition

We want to consider the Initial Value Problem for a smart choice of u_0 the initial condition:
$$\begin{cases} \partial_t u(t) + A(t)u(t) &= f(t), \quad t \in (0,T) \\ u(0) &= u_0. \end{cases} \tag{1.13}$$

We will assume the Assumption 1.2.2 holds. Then the following result for the un-weighted case gives us that we should choose $u_0 \in (X_0, D(A(0)))_{1-\frac{1}{p},p}$, a real-interpolation space between X_0 and the domain of $A(0)$ so that $A^{1-\frac{1}{p}}(0)u_0$ is well defined. For more information on interpolation spaces, which will appear a lot in this thesis, see Appendix A.1. In Chapter 2, we will consider the weighted equation with weight $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$, we will see we can take $u_0 \in (X_0, D(A(0)))_{\sigma,p}$, where $\sigma = 1 - \alpha - \frac{1}{p}$.

Below stated is the theorem 2.2 from Di Giorgio, Lunardi, and Schnaubelt (2005) which gives us a definition of maximal L^p-regularity for the operator family $A(\cdot)$

Theorem 1.2.9. *Assume Assumption 1.2.2 holds for $A(t)$. Let $p \in (1,\infty)$, $T > 0$, $f \in L^p((0,T);X_0)$ and $u_0 \in (X_0, D(A(0)))_{1-\frac{1}{p},\frac{1}{p}}$. Then the problem Eq. (1.13) has a unique mild solution $u \in \mathcal{E}^p((0,T))$ given by*
$$u(t) = G(t,0)u_0 + \int_0^t G(t,s)f(s)\, ds, \qquad t \in (0,T).$$
There exists a bound of the form
$$\|u\|_{\mathcal{E}^p((0,T))} \le C(\|u_0\|_{X_{1-\frac{1}{p},p}} + \|f\|_{L^p((0,T);X_0)}).$$

Proof. This is theorem 2.2 from Di Giorgio, Lunardi, and Schnaubelt (2005). □

From now having an unique **mild** solution $u \in \mathcal{E}^p((0,T))$ with $\|u\|_{\mathcal{E}^p((0,T))} \leq C(\|u_0\|_{X_{1-\frac{1}{p},p}} + \|f\|_{L^p((0,T);X_0)})$ will be our new definition of $A(\cdot)$ having maximal L^p-regularity, with an estimate now not only depending on f but also on our initial condition u_0. Note that we, as a consequence, do not consider strong solutions in this non-constant domains definition of L^p-regularity.

Chapter 2

Weighted non-autonomous problems on non-constant domains

This chapter extends the non-autonomous setting of Di Giorgio, Lunardi, and Schnaubelt (2005) with non-constant domains in various ways.

- In Section 2.1, we extend the non-autonomous linear setting by using power weights from Section 17.2.e of Hytönen et al. (2024). The concluding result Corollary 2.1.6 allows us to find existence and uniqueness for solutions starting from an initial condition u_0 from the bigger interpolation space $X_{\sigma,p}$.

- In Section 2.2, the weighted non-autonomous linear theory is applied to a setting with a semi-linearity in the right-hand side function. Theory from section 18.1 of Hytönen et al. (2024) is used to define a simple non-linearity which acts on the same Trace space $X_{\sigma,p}$ as the initial condition. This result is set in the non-autonomous linear theory from Chapter 3, part II of Yagi (2010) with constant domains when sufficiently close to the boundary condition. This setting has important results as described in Lemma 1.2.8 for estimating the self-mapping and contraction mapping qualities that enable the application of the Banach fixed point theorem when using a mapping Φ that reduces the non-linear problem to a non-autonomous linear problem from Section 2.1. The concluding result Theorem 2.2.3 then gives short timescale existence and uniqueness of solutions to the semi-linear problem.

- in Section 2.3, the weighted non-autonomous linear setting is further extended to quasi-linear problems with a critical part in the non-linear right-hand side function, as seen in section 18.2 of Hytönen et al. (2024). By using a Hölder-continuous setting as in chapter 5 of Yagi (2010) which allows for the reduction to linear problems as in Section 2.1 with the results of Lemma 1.2.8, we can use operators $A(u(t), t)$, which besides being non-autonomous are non-linearly dependent on the solution and gain constant domains when sufficiently close to the boundary condition. Again, the Banach fixed point theorem is used with a mapping to the setting of Section 2.1 in order to prove short timescale existence and uniqueness of solutions in Theorem 2.3.7, which is the main result of this thesis. Some proofs reference Appendix A for results explaining specific steps.

In the examples of Section 2.4, we then look at examples and see if our main result Theorem 2.3.7 can be applied. The model Neumann problem Eq. (1) of our research question is investigated in Section 2.4.1.

2.1 Weighted L^p-regularity

When working with non-linear problems, it is helpful to consider weighted spaces $L^p_{\omega_\alpha}$. The reasons given for that are as follows: (Hytönen, van Neerven, Veraar, & Weis, 2024)

1. it allows initial data u_0 belonging to the space $X_{\sigma,p} = (X_0, X_1)_{1-\alpha-\frac{1}{p},p}$, where $\alpha > 0$ is a parameter associated with the weight;

2. global existence of solutions can be proven under milder blow-up criteria;

3. it allows the inclusion of the endpoint $p = \infty$, which will only be considered for problems on constant domains.

Definition 2.1.1 ($L^p_{\omega_\alpha}$ and $W^{1,p}_{\omega_\alpha}$)**.** For f strongly measurable, define a *power weight* ω_α as $\omega_\alpha f(t) := t^\alpha f(t)$. Then the *weighted L^p-space* $L^p_{\omega_\alpha}(I; X_0)$ is defined as

$$L^p_{\omega_\alpha}(I; X_0) := \{f \text{ strongly measurable} : \|\omega_\alpha f\|_{L^p(I;X_0)} < \infty\}.$$

This space has the norm $\|f\|_{L^p_{\omega_\alpha}(I;X_0)} := \|\omega_\alpha f\|_{L^p(I;X_0)}$. The *weighted Sobolev space* $W^{1,p}_{\omega_\alpha}(I; X_0)$ is defined as

$$W^{1,p}_{\omega_\alpha}(I; X_0) := \{f \in L^p_{\omega_\alpha}(I; X_0) : \|\partial_t \omega_\alpha f\|_{L^p(I;X_0)} < \infty\},$$

with the norm $\|f\|_{W^{1,p}_{\omega_\alpha}(I;X_0)} := \|\omega_\alpha f\|_{L^p(I;X_0)} + \|\partial_t \omega_\alpha f\|_{L^p(I;X_0)}$.

We can use the following result from Hytönen et al. (2024) for autonomous operators. A non-autonomous version is derived in Theorem 2.1.4.

Proposition 2.1.2 (Extrapolation with power weights)**.** *Let A be a linear operator on a Banach space X_0, let $I = (0, T)$ or $I = [0, \infty)$, let $\frac{1}{p} + \frac{1}{p'} = 1$, and consider the weight $\omega_\alpha(t) = t^\alpha$ with $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$. Then the following assertions hold:*

1. *If $p \in (1, \infty]$, then A has maximal L^p-regularity on I if and only if A has maximal $L^p_{\omega_\alpha}$-regularity on I.*

2. *If $p = 1$, then A has maximal $L^1_{\omega_\alpha}$-regularity on I if A has maximal L^1-regularity on I.*

Proof. Proposition 17.2.36 in Hytönen et al. (2024). □

Remark. Here, we see we can include $p = \infty$. However, in Di Giorgio, Lunardi, and Schnaubelt (2005), $p = \infty$ is not included. To have useful theory without restating results like in Chapter 18 of Hytönen et al. (2024), we will avoid $p = \infty$ and $p = 1$ in this thesis. However, the $p = \infty$ endpoint is likely feasible to be included because of Hytönen et al. (2024) and Yagi (2010) treating it.

We investigate the weighted non-autonomous linear problem:

$$\begin{cases} \partial_t u(t) + A(t)u(t) &= f(t), \quad t \in (0, T), \\ u(0) &= 0. \end{cases} \tag{2.1}$$

In this section, A having maximal L^p-regularity means there exists an unique solution mild u with $\|u\|_{\mathcal{E}((0,T))} \leq C\|f\|_{L^p((0,T);X_0)}$ for all $f \in L^p((0, T); X_0)$, as from Theorem 1.2.9. We now want to find maximal $L^p_{\omega_\alpha}$-regularity for $A(t)$, which will entail the same definition as above except with $L^p_{\omega_\alpha}$ spaces where there were L^p spaces before.

We introduce the *maximum regularity space*

$$\mathcal{E}^p_\alpha((0, T)) := \{v \in W^{1,p}_{\omega_\alpha}([0, T]; X_0) : v(t) \in D(A(t)) \text{ a.e.}, A(\cdot)v(\cdot) \in L^p_{\omega_\alpha}([0, T]; X_0)\} \tag{2.2}$$

with the norm

$$\|v\|_{\mathcal{E}^p_\alpha((0,T))} = \|v\|_{W^{1,p}_{\omega_\alpha}((0,T);X_0)} + \|A(\cdot)v(\cdot)\|_{L^p_{\omega_\alpha}((0,T);X_0)}.$$

Note this is the weighted version of Definition 1.2.3, and so A having maximal $L^p_{\omega_\alpha}$-regularity means there exists a unique solution u with $\|u\|_{\mathcal{E}^p_\alpha((0,T))} \leq C\|f\|_{L^p_{\omega_\alpha}((0,T);X_0)}$ for all $f \in L^p_{\omega_\alpha}((0, T); X_0)$. This property will be shown in Corollary 2.1.6.

Firstly, we will assume Assumption 1.2.2 of Section 1.2.1, and derive a non-autonomous version of Theorem 1.1.16.

Proposition 2.1.3. *Let Assumption 1.2.2 hold for $A(t)$ and assume that $\mathcal{F}$ is a dense subspace of $L^p((0, T); X_0)$ under the same norm. Define*

$$V : \mathcal{F} \to L^p((0, T); X_0),$$

$$Vf(t) = A(t)\int_0^t G(t,s)f(s)\ ds.$$

Then $A(\cdot)$ has maximal L^p-regularity on $(0,T)$ if and only if V is a well-defined operator and there exists a $C \geq 0$ such that $\|Vf\|_{L^p((0,T);X_0)} \leq C\|f\|_{L^p((0,T);X_0)}$.

If this holds, then V extends to a bounded operator on $L^p((0,T);X_0)$, and we have

$$Vf(\cdot) = A(\cdot)u_f(\cdot),$$

where u_f is the mild solution to the problem Eq. (2.1) coming from Theorem 1.2.9.

Proof. "Only if": If $A(\cdot)$ has maximal regularity, meaning there exists a unique mild solution $u_f \in W^{1,p}((0,T);X_0)$ to problem Eq. (2.1) with $u_f \in D(A(t))$ a.e., $A(\cdot)u_f(\cdot) \in L^p((0,T;X_0)$ and

$$\|\partial_t u_f\|_{L^p((0,T);X_0)} + \|A(\cdot)u_f(\cdot)\|_{L^p((0,T);X_0)} \leq M^{reg}_{p,A}\|f\|_{L^p((0,t);X_0)}.$$

By the mild formulation, $u_f(t) = \int_0^t G(t,s)f(s)\ \mathrm{d}s$ is equal to this unique strong solution u_f. Since $Vf(t) = A(t)u_f(t)$ for this form, we get the bound

$$\|Vf\|_{L^p((0,T);X_0)} \leq M^{reg}_{p,A}\|f\|_{L^p((0,t);X_0)}.$$

"If": let $w(t) = \int_0^t G(t,s)f(s)\ ds$ for $f \in \mathcal{F}$, then by assuming V is bounded from $\mathcal{F}$ to $L^p((0,T);X_0)$ we have that $w \in D(A(t))$ a.e. with $A(\cdot)w(\cdot) = Vf(\cdot) \in L^p((0,T);X_0)$. This allows us to use Theorem 1.2.9, that states that there is a unique mild solution u_f to Eq. (2.1) equal to w, and

$$\|A(\cdot)u_f(\cdot)\|_{L^p((0,T);X_0)} = \|Vf\|_{L^p((0,T);X_0)} \leq C\|f\|_{L^p((0,T);X_0)}.$$

□

We can use this V to show the equivalence of L^p-regularity and weighted L^p regularity in the non-autonomous case.

Theorem 2.1.4. *Let Assumption 1.2.2 hold for $A(t)$, and let I be a bounded interval. We have that $A(\cdot)$ having Maximum L^p-regularity on I is equivalent to $A(\cdot)$ having Maximum $L^p_{\omega_\alpha}$-regularity on I for $p \in (1,\infty)$ and $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$.*

Proof. By Proposition 2.1.3, we need to show that V being bounded on $L^p((0,T);X_0)$ is equivalent to V being bounded on $L^p_{\omega_\alpha}((0,T);X_0)$. We will argue that V being bounded on $L^p((0,T);X_0)$ is equivalent to $V_\alpha f(t) := A(t)\int_0^t t^\alpha G(t,s)s^{-\alpha}f(s)\ \mathrm{d}s$ being bounded on $L^p((0,T);X_0)$. We then show that this again is equivalent to V being bounded on $L^p_{\omega_\alpha}((0,T);X_0)$.

First equivalence: we will show the difference $V_\alpha - V$ is bounded on $L^p((0,T);X_0)$, then show this implies V bounded $\Leftrightarrow$ V_α bounded. Take $f \in L^p((0,T);X_0)$, then

$$\begin{aligned} V_\alpha f(t) - Vf(t) &= A(t)\int_0^t t^\alpha G(t,s)s^{-\alpha}f(s)\ \mathrm{d}s - A(t)\int_0^t G(t,s)f(s)\ \mathrm{d}s \\ &= A(t)\int_0^T 1_{t>s}G(t,s)\left(\left(\frac{t}{s}\right)^\alpha - 1\right)f(s)\ \mathrm{d}s \\ &= \int_0^T k(t,s)f(s)\ \mathrm{d}s. \end{aligned}$$

Here we defined $k(t,s) := 1_{t>s}A(t)G(t,s)\left(\left(\frac{t}{s}\right)^\alpha - 1\right)$, and by lemma 17.2.35 from Hytönen et al. (2024), we only have to show that $K(t,s) := 1_{t>s}(t-s)A(t)G(t,s)$ is an $L^\infty(\mathbb{R}^2;\mathscr{L}(X_0))$ function in order to conclude that $V_\alpha - V$ is bounded on $L^p((0,T);X_0)$. To show this is indeed the case, consider the inequality $\|A(t)G(t,s)\|_{\mathscr{L}(X_0)} \leq \frac{C}{t-s}$ from Theorem 1.2.5. This allows us to conclude that

$$\|K\|_{L^\infty(\mathbb{R}^2;\mathscr{L}(X_0))} = \sup_{t>s}\|(t-s)A(t)G(t,s)\|_{\mathscr{L}(X_0)}$$

$$= \sup_{t>s}(t-s)\|A(t)G(t,s)\|_{\mathscr{L}(X_0)} \leq C.$$

We now show that $V_\alpha - V$ bounded gives the equivalence we want. Take $f \in L^p((0,T);X_0)$ with W.L.O.G. $\|f\|_{L^p((0,T);X_0)} \leq 1$, then

$$\begin{aligned}
\|Vf\|_{L^p((0,T);X_0)} &= \|Vf - V_\alpha f + V_\alpha f\|_{L^p((0,T);X_0)} \\
&\leq \|Vf - V_\alpha f\|_{L^p((0,T);X_0)} + \|V_\alpha f\|_{L^p((0,T);X_0)} \\
&\leq C_1\|f\|_{L^p((0,T);X_0)} + C_2\|f\|_{L^p((0,T);X_0)},
\end{aligned}$$

if V_α is assumed to be bounded. This shows that V is bounded. The other side of the equivalence is replacing V with V_α and vice versa.

Second equivalence: take $g \in L^p_{\omega_\alpha}((0,T);X_0)$ and define $f(t) = t^\alpha g(t)$, then

$$\begin{aligned}
\|Vg\|^p_{L^p_{\omega_\alpha}((0,T);X_0)} &= \int_0^T \left\| t^\alpha A \int_0^T 1_{t>s} G(t,s)g(s)\ \mathrm{d}s \right\|^p_{X_0} \mathrm{d}t \\
&= \int_0^T \left\| A \int_0^T 1_{t>s} t^\alpha G(t,s) s^{-\alpha} s^\alpha g(s)\ \mathrm{d}s \right\|^p_{X_0} \mathrm{d}t. \\
&= \int_0^T \left\| A \int_0^T 1_{t>s} t^\alpha G(t,s) s^{-\alpha} f(s)\ \mathrm{d}s \right\|^p_{X_0} \mathrm{d}t = \|V_\alpha f\|^p_{L^p((0,T);X_0)}.
\end{aligned}$$

If V_α is bounded in $L^p((0,T);X_0)$, this equality shows that

$$\|Vg\|_{L^p_{\omega_\alpha}((0,T);X_0)} = \|V_\alpha f\|_{L^p((0,T);X_0)} \leq \|V_\alpha\|_{\mathscr{L}(L^p((0,T);X_0))}\|f\|_{L^p((0,T);X_0)},$$

since $f \in L^p_{\omega_\alpha}((0,T);X_0)$ by the definition of $L^p_{\omega_\alpha}((0,T);X_0)$. The other way around, for any $f \in L^p((0,T);X_0)$ we take $g(t) = t^{-\alpha}f(t)$ as our $L^p_{\omega_\alpha}((0,T);X_0)$ function, and then

$$\|V_\alpha f\|_{L^p((0,T);X_0)} = \|Vg\|_{L^p_{\omega_\alpha}((0,T);X_0)} \leq \|V\|_{\mathscr{L}(L^p_{\omega_\alpha}((0,T);X_0))}\|g\|_{L^p_{\omega_\alpha}((0,T);X_0)}.$$

Therefore, the boundedness of one is equivalent to the boundedness of the other. □

As a consequence, we can now set up a weighted version of lemma 2.1 and theorem 2.2 from Di Giorgio, Lunardi, and Schnaubelt (2005). We first need an argument for the initial condition u_0.

Lemma 2.1.5. *Let Assumption 1.2.2 hold for $A(t)$, let $p \in (1,\infty)$, $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$ and $T > 0$. For all $x \in X_0$, the function $t \mapsto G(t,0)x$ belongs to $W^{1,p}_{\omega_\alpha}((0,T);X_0)$ if $x \in (X_0, D(A(0)))_{\sigma,p}$ for $\sigma = 1-\alpha-\frac{1}{p}$. If true, we get an estimate of the following form:*

$$\|G(\cdot,0)x\|_{W^{1,p}_{\omega_\alpha}((0,T);X_0)} \leq C\|x\|_{(X_0,D(A(0)))_{\sigma,p}}.$$

Proof. As described in Di Giorgio, Lunardi, and Schnaubelt (2005), we get for $t \in (0,T)$

$$-A(t)G(t,0)x \overset{(1.2.5)}{=} \frac{\mathrm{d}}{\mathrm{d}t}G(t,0)x = A(0)e^{-tA(0)}x + Z(t,0)x,$$

where Z is some operator with estimates

$$\begin{aligned}
\|Z(r,s)\|_{\mathscr{L}(X_0)} &\leq C(r-s)^{\delta-1}, \\
\|Z(r,s)\|_{\mathscr{L}((X_0,D(A(s)))_{\nu,p})} &\leq C(r-s)^{\delta+\nu-1},
\end{aligned}$$

with $\nu \in [0,1)$ and $\delta = \min\{\alpha_i - \beta_i\} > 0$ from Assumption 1.2.2. By these estimates, $Z(\cdot,0)x \in L^p_{\omega_\alpha}((0,T);X_0)$ whenever $x \in (X_0, D(A(0)))_{\sigma,p}$, since

$$\int_0^T \|t^\alpha Z(t,0)x\|^p_{X_0} dt \leq \int_0^T t^{\alpha p}\|Z(t,0)\|^p_{\mathscr{L}((X_0,D(A(0)))_{\sigma,p})}\|x\|^p_{(X_0,D(A(0)))_{\sigma,p}} dt$$

$$\begin{aligned}
&\leq C^p \|x\|^p_{(X_0,D(A(0)))_{\sigma,p}} \int_0^T t^{\alpha p} t^{(\delta+\sigma-1)p} dt \\
&= C^p \|x\|^p_{(X_0,D(A(0)))_{\sigma,p}} \int_0^T t^{(\delta-\frac{1}{p}-\alpha+\alpha)p} dt \\
&= C^p \|x\|^p_{(X_0,D(A(0)))_{\sigma,p}} \int_0^T t^{\delta p-1} dt < \infty.
\end{aligned}$$

We know $A(0)e^{-tA(0)}x \in L^p_{\omega_\alpha}((0,T);X_0)$ because it is the $L^p_{\omega_\alpha}$-solution to the weighted autonomous IVP

$$\begin{cases} \partial_t u(t) + A(0)u(t) &= 0, \quad t \in (0,T), \\ u(0) &= x, \end{cases}$$

so we can use Corollary 17.2.37 of Hytönen et al. (2024). Therefore we conclude $G(t,0)x \in W^{1,p}_{\omega_\alpha}((0,T);X_0)$. $\square$

This gives us the result we wanted to show as our weighted L^p-regularity to use, with the initial condition u_0 included.

Corollary 2.1.6 (Existence and uniqueness for weighted spaces)**.** *Let Assumption 1.2.2 hold for $A(t)$, let $p \in (1,\infty)$, $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$, $T > 0$, $f \in L^p_{\omega_\alpha}((0,T);X_0)$ and $u_0 \in (X_0, D(A(0)))_{\sigma,p}$. Then the problem*

$$\begin{cases} \partial_t u(t) + A(t)u(t) &= f(t), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases} \tag{2.3}$$

has a unique MILD solution $u \in \mathcal{E}^p_\alpha((0,T))$, given by

$$u(t) = G(t,0)u_0 + \int_0^t G(t,\tau)f(\tau)\ d\tau.$$

We have the following estimate:

$$\|u\|_{\mathcal{E}^p_\alpha((0,T))} \leq C(\|u_0\|_{X_{\sigma,p}} + \|f\|_{L^p_{\omega_\alpha}((0,T);X_0)})$$

Proof. The result follows from the same reasoning as in Theorem 2.2 of Di Giorgio, Lunardi, and Schnaubelt (2005), where we use the unweighted Theorem 1.2.9 to show maximal L^p-regularity holds for $u_0 = 0$, then using Theorem 2.1.4 we show $L^p_{\omega_\alpha}$-regularity meaning a unique mild solution $\tilde{u}$ exists to Eq. (2.1), so $\tilde{u}(t) \int_0^t G(t,s)f(s)\ ds$.

Then, we perturb $\tilde{u}$ by $G(t,0)u_0$ to solve Eq. (2.3) with $u(t) = G(t,0)u_0 + \tilde{u}(t)$. Now $G(t,0)u_0 \in \mathcal{E}^p_\alpha((0,T))$, since Lemma 2.1.5 gives $G(t,0)u_0 \in W^{1,p}_{\omega_\alpha}((0,T);X_0)$, and

$$-A(\cdot)G(\cdot,0)u_0 \overset{(1.2.5)}{=} \partial_t G(\cdot,0)u_0 \in L^p_{\omega_\alpha}((0,T);X_0)$$

for the same reason. Therefore, $u \in \mathcal{E}^p_\alpha((0,T))$ is a unique mild solution to Eq. (2.3), with the given maximal $L^p_{\omega_\alpha}$-regularity bound as above. $\square$

2.2 Semi-linear problems with $F = F_{Tr}$

We will now work with the following semi-linear equations, meaning there will be a non-linearity on the right-hand side function.

$$\begin{cases} \partial_t u(t) + A(t)u(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases} \tag{2.4}$$

Define the following space for a fixed $p \in (1,\infty)$, $\alpha \in [0, \frac{1}{p'})$ and $\sigma = 1 - \alpha - \frac{1}{p}$:

$$\begin{aligned} E_0 &:= L^p_{\omega_\alpha}((0,T);X_0), \\ Y_{Tr}(0,T) &:= C([0,T];X_{\sigma,p}). \end{aligned}$$

In this proof, we will assume Assumption 1.2.7, which allows the use of the properties from Lemma 1.2.8. We will apply the problem with the following assumptions on the initial condition u_0 and the function F, where $B_R(0;X)$ is defined as the open ball on space X with radius R around 0.

Assumption 2.2.1. *The following conditions must hold for all $R > 0$:*

(1) $\sigma < \gamma_0$, and $u_0 \in B_R(0; X_{\sigma,p})$, meaning $\exists L_{u_0} > 0$ s.t.

$$\|A^\sigma(0)u_0\|_{X_0} \leq L_{u_0} \tag{2.5}$$

(2) $F : X_{\sigma,p} \to X_0$ is locally Lipschitz continuous on $X_{\sigma,p}$, meaning $\exists L_{Tr} \geq 0$ s.t. $\forall u, v \in B_R(0; X_{\sigma,p})$,

$$\|F(u) - F(v)\|_{X_0} \leq L_{Tr}\|u - v\|_{X_{\sigma,p}}. \tag{2.6}$$

We can now define what it means to be a solution.

Definition 2.2.2. $u \in \mathcal{E}^p_\alpha((0,T))$ is called a *mild solution to the problem Eq.* (2.4) *on* $(0,T)$ if $u(0) = u_0$, $F(u(\cdot)) \in L^p_{\omega_\alpha}((0,T); X_0)$, and for all $t \in [0,T]$,

$$u(t) = G(t,0)u_0 + \int_0^t G(t,s)F(u(s))\, ds.$$

We find that $u \in Y_{Tr}(0,T)$ becomes the appropriate condition such that $F(u(\cdot)) \in L^p_{\omega_\alpha}((0,T); X_0)$ because like in Lemma 18.2.8 of Hytönen et al. (2024) for $v_1, v_2 \in Y_{Tr}(0,T)$,

$$\|F(v_1(t)) - F(v_2(t))\|_{X_0} \overset{(2.6)}{\leq} L_{Tr}\|v_1(t) - v_2(t)\|_{X_{\sigma,p}} < \infty, \tag{2.7}$$

which indicates $F(v(\cdot)) \in C([0,T]; X_0) \hookrightarrow L^p_{\omega_\alpha}((0,T); X_0) =: E_0$ for $v \in Y_{Tr}(0,T)$.

Theorem 2.2.3 (Local-wellposedness for non-autonomous semi-linear PDE's on time-dependent domains)**.** *Set $\sigma = 1 - \alpha - \frac{1}{p}$ for $p \in (1,\infty)$ and $\alpha \in [0, \frac{1}{p'})$. Choose $R > 0$ on which Assumption 1.2.7 and Assumption 2.2.1 hold. Let the following conditions hold:*

$$\begin{aligned} \sigma < \gamma &\leq \gamma_0, \\ 1 &< \gamma + \mu. \end{aligned}$$

Then for a very small $\tilde{T} > 0$ there exists a unique mild solution u to the problem Eq. (2.4) *in the space $\mathcal{E}^p_\alpha((0,\tilde{T})) \cap Y_{Tr}(0,\tilde{T})$.*

Proof. For this proof, we will need to do the following.

- Set up Φ and a small closed domain $B^T(u_0)$, which will be the core components of the Banach Fixed Point theorem. This is done below.
- Show Φ maps $B^T(u_0)$ to itself.
- Show Φ is a uniformly contracting map.
- Apply the Banach Fixed Point Theorem to find a local solution in $B^T(u_0)$, and ensure the uniqueness of the solution in the full space $\mathcal{E}^p_\alpha((0,\tilde{T})) \cap Y_{Tr}(0,\tilde{T})$ for $\tilde{T} \leq T$.

We will begin working on defining the domain. We start by defining an object, which will act as a measure of how close we are to the initial condition u_0.

Definition 2.2.4. We will define the *reference solution* z_{u_0} as the $L^p_{\omega_\alpha}$-solution to the following autonomous linear problem:

$$\left\{ \begin{array}{rcl} \partial_t u(t) + A(0)u(t) &=& 0, \quad t \in (0,T), \\ u(0) &=& u_0. \end{array} \right. \tag{2.8}$$

This means by Proposition 1.1.7,

$$z_{u_0}(t) = e^{-tA(0)}u_0 \tag{2.9}$$

as an analytic semi-group acting on u_0. $D(A(0))$ is a time-independent domain, so we can see that $z_{u_0} \in W^{1,p}_{\omega_\alpha}((0,T); X_0) \cap L^p_{\omega_\alpha}((0,T); D(A(0))) =: \mathcal{P}^p_{\alpha,A(0)}((0,T))$ as it is a solution to the problem Eq. (2.8). We can then use Proposition A.1.4 to conclude $z_{u_0} \in Y_{Tr}(0,T)$.

We can then define the Fixed point space, a closed bounded ball $B^T(u_0)$ in the Banach space $Y_{Tr}(0,T)$:

$$B^T(u_0) := \{u \in Y_{Tr}(0,T) : u(0) = u_0, \|u - z_{u_0}\|_{Y_{Tr}(0,T)} \leq 1\}.$$

Define $\Phi : B^T(u_0) \to \mathcal{E}^p_\alpha((0,T))$ as

$$\Phi(v) = u,$$

where u is the mild solution of the non-autonomous linear problem

$$\begin{cases} \partial u(t) + A(t)u(t) &= F(v(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases}$$

Note the definition of existence and uniqueness of the mapping Φ relies on Corollary 2.1.6, which means Assumption 1.2.2 is needed, which we obtain from Lemma 1.2.6 combined with Assumption 1.2.7. Secondly note that $F(v(t))$ is an admissable right-hand side function by Eq. (2.7) showing $F(v(\cdot)) \in L^p_{\omega_\alpha}((0,T);X_0)$.

The following estimate for $v \in Y_{Tr}(0,T)$ will allow us to do self-mapping and contraction estimates for Φ:

$$\left\| t \mapsto \int_0^t G(t,\tau)F(v(\tau))d\tau \right\|_{Y_{Tr}(0,T)} \leq C\|F(v(\cdot))\|_{E_0} \tag{2.10}$$

We can show this estimate the following way

$$\begin{aligned}
&\left\| \int_0^t G_v(t,\tau)F(v(\tau))d\tau \right\|_{X_{\sigma,p}} \\
\lesssim &\int_0^t \left\| A^\sigma(t)\left(G(t,\tau) - e^{-(t-\tau)A(\tau)}\right)F(v(\tau)) \right\|_{X_0} d\tau \\
&+ \int_0^t \left\| A^\sigma(\tau)\left(e^{-(t-\tau)A(\tau)}f(\tau) - e^{-(t-\tau)A(0)}\right)F(v(\tau)) \right\|_{X_0} d\tau \\
&+ \left\| \int_0^t e^{-(t-\tau)A(0)}F(v(\tau))d\tau \right\|_{X_{\sigma,p}} \\
=: &L_1 + L_2 + L_3.
\end{aligned}$$

For L_1, use Eq. (1.11) to obtain

$$\begin{aligned}
L_1 &\leq C\int_0^t (t-\tau)^{\gamma+\mu-1-\sigma}\|F(v(\tau))\|_{X_0}d\tau \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma+\mu-1-\sigma}\|_{L^r(0,t)} \cdot \left\|\tau \mapsto \tau^{-\alpha}\|\tau^\alpha F(v(\tau))\|_{X_0}\right\|_{L^{r'}(0,t)} \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma+\mu-1-\sigma}\|_{L^r(0,t)} \cdot \|\tau \mapsto \tau^{-\alpha}\|_{L^h(0,t)} \cdot \|F(v(\cdot))\|_{L^p_{\omega_\alpha}((0,t);X_0)} \\
&\leq Ct^{\gamma+\mu-1-\sigma+\frac{1}{r}}(t-s)^{-\alpha+\frac{1}{h}}\|F(v(\cdot))\|_{E_0} \\
&\leq Ct^{\gamma+\mu-1}\|F(v(\cdot))\|_{E_0}.
\end{aligned}$$

Here, $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} + \frac{1}{p} + \frac{1}{h}$. These must satisfy $\gamma + \mu - 1 - \sigma > -\frac{1}{r}$ and $-\alpha > -\frac{1}{h}$, so

$$\gamma + \mu - 1 - \sigma > \frac{1}{p} + \frac{1}{h} - 1 > \frac{1}{p} + \alpha - 1 = -\sigma.$$

By $\gamma + \mu > 1$, this is satisfied.

For L_2, use Eq. (1.12) and the same r, h as above to obtain

$$\begin{aligned}
L_2 &\leq C\int_0^t (t-\tau)^{\gamma-\sigma-1}(\tau-s)^\mu\|F(v(\tau))\|_{X_0}d\tau \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma-1-\sigma}(z-s)^\mu\|_{L^r(0,t)} \cdot \left\|\tau \mapsto \tau^{-\alpha}\|\tau^\alpha f(\tau)\|_{X_0}\right\|_{L^{r'}(0,t)} \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma-1-\phi}(z-s)^\mu\|_{L^r(0,t)} \cdot \|\tau \mapsto \tau^{-\alpha}\|_{L^h(0,t)} \cdot \|F(v(\cdot))\|_{L^p_{\omega_\alpha}((0,t);X_0)}
\end{aligned}$$

$$\begin{aligned}&\leq Ct^{\gamma+\mu-1-\sigma+\frac{1}{r}}(t-s)^{-\alpha+\frac{1}{h}}\|F(v(\cdot))\|_{E_0}\\&\leq Ct^{\gamma+\mu-1}\|F(v(\cdot))\|_{E_0}.\end{aligned}$$

For L_3, we use the maximal $L^p_{\omega_\alpha}$-regularity of $A(0)$, which comes from Theorem 1.1.20 combined with Proposition 2.1.2, to conclude that the mild solution w to the autonomous linear problem

$$\left\{\begin{array}{rcl}\partial_t w(t)+A(0)w(t) &=& F(v(t)),\quad t\in(0,T)\\ w(0)&=&0\end{array}\right.$$

has $L_3 = \left\|\int_0^t e^{-(t-\tau)A(0)}f(\tau)d\tau\right\|_{X_{\sigma,p}} = \|w(t)\|_{X_{\sigma,p}}$. Since $w\in\mathcal{P}^p_{\alpha,A(0)}((0,T))\hookrightarrow Y_{Tr}(0,T)$ with time-independent constants by Eq. (A.2),

$$\|w(t)\|_{X_{\sigma,p}}\leq\|w\|_{Y_{Tr}(0,T)}\overset{(A.2)}{\leq}\tilde{C}\|w\|_{\mathcal{P}^p_{\alpha,A(0)}((0,T))}\overset{A(0)\in\mathcal{MR}_p}{\leq}C\|f\|_{E_0},$$

so Eq. (2.10) holds when L_1, L_2 and L_3 are combined.

We can use Eq. (2.10) with the following estimate for $F(v(\cdot))$ for $v\in B^T(u_0)$:

$$\|F(v(\cdot))\|_{E_0}\leq T^{1-\sigma}(\|F(u_0)\|_{X_0}+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}(0,T)}+L_{Tr}). \tag{2.11}$$

This follows from

$$\begin{aligned}\|F(v(t))\|_{X_0}&\leq\|F(v(t))-F(z_{u_0}(t))\|_{X_0}+\|F(z_{u_0}(t))+F(u_0)\|_{X_0}+\|F(u_0)\|_{X_0}\\&\overset{(2.6)}{\leq}L_{Tr}(\|v(t)-z_{u_0}(t)\|_{X_{\sigma,p}}+\|z_{u_0}(t)-u_0\|_{X_{\sigma,p}})+\|F(u_0)\|_{X_0}\\&\leq L_{Tr}(1+\|z_{u_0}-u_0\|_{Y_{Tr}(0,T)})+\|F(u_0)\|_{X_0}.\end{aligned}$$

The result Eq. (2.11) follows from taking the $L^p_{\omega_\alpha}((0,T);X_0)$ norm:

$$\begin{aligned}\|F(v(\cdot))\|_{E_0}&\leq(\|F(u_0)\|_{X_0}+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}(0,T)}+L_{Tr})\left(\int_0^T t^\alpha dt\right)^{\frac{1}{p}}\\&=(\|F(u_0)\|_{X_0}+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}(0,T)}+L_{Tr})T^{\alpha+\frac{1}{p}}\\&=(\|F(u_0)\|_{X_0}+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}(0,T)}+L_{Tr})T^{1-\sigma}.\end{aligned}$$

We can show the self-mapping property of Φ, meaning Φ maps $B^T(u_0)$ into itself, as follows: take $u=\Phi(v)$ and use the mild solution formula from Corollary 2.1.6, so

$$u(t)=G(t,0)u_0+\int_0^t G(t,s)F(v(s))\,ds,$$

to get

$$u(t)-z_{u_0}(t)=\left(G(t,0)-e^{-tA(0)}\right)u_0+\int_0^t G(t,s)F(v(s))\,ds.$$

Applying the $Y_{Tr}(0,T)$ norm then gives

$$\|u-z_{u_0}\|_{Y_{Tr}(0,T)}\leq S_1+S_2.$$

S_1 can be estimated by

$$\begin{aligned}S_1&=\left\|(G(t,0)-e^{-tA(0)})A^{-\sigma}(0)A^\sigma(0)u_0\right\|_{C([0,T];X_{\sigma,p})}\\&\leq\sup_{t\in[0,T]}D\left\|A^\sigma(t)(G(t,0)-e^{-tA(0)})A^{-\sigma}(0)A^\sigma(0)u_0\right\|_{X_0}\\&\overset{(1.11)}{\leq}\sup_{t\in[0,T]}Ct^{\mu+\gamma-1}\left\|A^\sigma(0)u_0\right\|_{X_0}\end{aligned}$$

$$=CT^{\gamma+\mu-1}L_{u_0}.$$

By $\gamma + \mu > 1$, this holds. For S_2 we can use Eq. (2.10) and Eq. (2.11):

$$S_2 = \left\| \int_0^t G(t,s)F(v(s))\ ds \right\|_{Y_{Tr}(0,T)} \leq C\|F(v(\cdot))\|_{E_0} \leq CT^{1-\sigma}(\|F(u_0)\|_{X_0} + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}(0,T)} + L_{Tr}).$$

All these terms in S_1 and S_2 go to 0 as T decreases:

- $T^{\gamma+\mu-1}$, $T^{1-\sigma} \downarrow 0$ as $T \downarrow 0$ since these powers are positive
- $\|F(u_0)\|_{X_0} < \infty$ since $u_0 \in X_{\sigma,p}$,
- $\|z_{u_0} - u_0\|_{Y_{Tr}} \downarrow 0$ as $T \downarrow 0$ by strong continuity from Proposition 2.2.8 of Lunardi (1995).

Therefore, we can choose a T_{self} s.t. $\|u - z_{u_0}\|_{Y_{Tr}(0,T_{self})} \leq 1$.

To show Φ is a contraction mapping, meaning $\|\Phi(v_1) - \Phi(v_2)\|_{Y_{Tr}(0,T)} < \|v_1 - v_2\|_{Y_{Tr}(0,T)}$, take v_1 and v_2 in $B^T(u_0)$, and write $u_1 := \Phi(v_1)$ and $u_2 := \Phi(v_2)$. With the mild formulation,

$$u_1(t) - u_2(t) = \int_0^t G(t,s)\left(F(v_1(s)) - F(v_2(s))\right)\ ds.$$

To estimate such a difference in F, note

$$\|F(v_1(\cdot)) - F(v_2(\cdot))\|_{E_0} \leq L_{Tr}T^{1-\sigma}\|v_1 - v_2\|_{Y_{Tr}(0,T)}. \tag{2.12}$$

This follows from Eq. (2.6), since

$$\|F(v_1(t)) - F(v_2(t))\|_{X_0} \leq L_{Tr}\|v_1(t) - v_2(t)\|_{X_{\sigma,p}} \leq \|v_1 - v_2\|_{Y_{Tr}(0,T)},$$

and applying $L^p_{\omega_\alpha}((0,T);X_0)$ therefore gives the above estimate. Then we can apply Eq. (2.10) together with Eq. (2.12) to conclude

$$\begin{aligned}
\|u_1 - u_2\|_{Y_{Tr}(0,T)} &= \left\| t \mapsto \int_0^t G(t,s)\left(F(v_1(s)) - F(v_2(s))\right)\ ds \right\|_{Y_{Tr}(0,T)} \\
&\leq C\|F(v_1(\cdot)) - F(v_2(\cdot))\|_{E_0} \\
&\leq CL_{Tr}T^{1-\sigma}\|v_1 - v_2\|_{Y_{Tr}(0,T)}
\end{aligned}$$

By decreasing T down to $T_{Lipschitz} < (CL_{Tr})^{\sigma-1}$, we have $\|u_1 - u_2\|_{Y_{Tr}(0,T)} < \|v_1 - v_2\|_{Y_{Tr}(0,T)}$.

Therefore, we can now apply the Banach Fixed Point Theorem to conclude that Φ has a unique fixed point u in the fixed point space $B^T(u_0) \subseteq Y_{Tr}(0,T)$, where T is taken as the minimum of T_{self} and $T_{Lipschitz}$. Since $u = \Phi(u) \in \mathcal{E}^p_\alpha((0,T))$ by Corollary 2.1.6, this gives us a mild solution u to problem Eq. (2.4), and a candidate for the unique solution in $\mathcal{E}^p_\alpha((0,T)) \cap Y_{Tr}(0,T)$ we want. We will now proof u is indeed unique in the space $\mathcal{E}^p_\alpha((0,\tilde{T})) \cap Y_{Tr}(0\tilde{T})$ for an even smaller $\tilde{T} \leq T$. We can similarly to the uniqueness result in the proof of Theorem 18.2.6 in Hytönen et al. (2024) set

$$\tilde{T} := \inf\left\{ t \in [0,T] : \|u - z_{u_0}\|_{Y_{Tr}(0,t)} \geq \frac{1}{2} \right\},$$

where we use the convention $\inf \emptyset = T$. Take $\tilde{u} \in \mathcal{E}^p_\alpha((0,T)) \cap Y_{Tr}(0,T)$ as some mild solution to the problem Eq. (2.4) on $(0,\tilde{T})$. Let

$$\tau_{\tilde{u}} := \inf\left\{ t \in [0,\tilde{T}] : \|\tilde{u} - z_{u_0}\|_{Y_{Tr}(0,t)} \geq 1 \right\},$$

where now the convention $\inf \emptyset = \tilde{T}$ is used. Then we can view $\tilde{u}$ on the interval $[0,\tau_{\tilde{u}}]$ to see

$$\tilde{u}|_{[0,\tau_{\tilde{u}}]} \in B_1^{\tau_{\tilde{v}}}(u_0),$$

which by the proof of the existence and uniqueness with $\tau_{\tilde{u}} \leq T$ allows us to conclude

$$\tilde{u}|_{[0,\tau_{\tilde{u}}]} = u|_{[0,\tau_{\tilde{u}}]}.$$

Then note the following holds:

$$\|\tilde{u} - z_{u_0}\|_{Y_{Tr}(0,\tau_{\tilde{u}})} = \|u - z_{u_0}\|_{Y_{Tr}(0,\tau_{\tilde{u}})} \leq \|u - z_{u_0}\|_{Y_{Tr}(0,\tilde{T})} < 1.$$

Therefore, we can conclude $\tilde{T} = \tau_{\tilde{u}}$, and so u is unique in the full space $\mathcal{E}^p_\alpha((0,\tilde{T})) \cap Y_{Tr}(0,\tilde{T})$. $\square$

2.3 Critical theory on quasi-linear problems

We are now going to work with the following quasi-linear equations, where $A(u,t)$ will be a u-dependent operator in addition to being non-autonomous:

$$\begin{cases} \partial_t u(t) + A(u(t),t)u(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases} \tag{2.13}$$

Define $A_u(\cdot) := A(u(\cdot),\cdot)$ as an unbounded operator family with time-dependent domains. Then $\mathcal{E}^p_{\alpha,u}((0,T))$ is defined as the maximal regularity space of the time-dependent operator family A_u, so

$$\mathcal{E}^p_{\alpha,u}((0,T)) := \{v \in W^{1,p}_{\omega_\alpha}((0,T);X_0) : v(t) \in D(A(u(t),t) \text{ a.e.}, A(u(\cdot),\cdot)v(\cdot) \in L^p_{\omega_\alpha}((0,T);X_0)\}. \tag{2.14}$$

G_u is defined as the evolution family of the operator family A_u, which exists if and only if we can show that we satisfy Assumption 1.2.2. We also define the function in time $F_u(\cdot)$ as $F(u(\cdot))$ for notational convenience.

Let $p \in (1,\infty)$ and $\alpha \in [0,\frac{1}{p'})$. For some $m \in \mathbb{N}$ as an index, we will define for all $j \in \{1,\dots,m\}$ the constants $\rho_j > 0$ determining the critical parts of F. We will work on the interpolation spaces X_θ, $X_{\sigma,p}$ and X_{β_j} for some Banach space X_0 and $X_1^u := D(A(u))$. On these spaces, we will define the following function spaces, where $\alpha_j := \frac{\alpha}{\rho_j+1}$, $p_j = (\rho_j+1)p$, and μ,θ s.t. $\mu+\theta=\sigma$:

$$\begin{aligned} E_0 &:= L^p_{\omega_\alpha}((0,T);X_0), \\ Y_{\{0\}} &:= C^\mu_{\{0\}}([0,T];X_\theta), \\ Y_{Tr} &:= C([0,T];X_{\sigma,p}), \\ Y_j &:= L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j}), \\ Y_\mu &:= C^\mu([0,T];X_\theta), \end{aligned}$$

Here, $C^\mu_{\{0\}}([0,T];X_\theta)$ is the space of functions v where

$$\|v\|_{C^\mu_{\{0\}}([0,T];X_\theta)} := \sup_{t\in[0,T]} t^{-\mu}\|v(t)-v(0)\|_{X_\theta} < \infty.$$

This definition is very useful since in all cases throughout Section 2.3, this norm will be applied to an element v with $v(0)=0$, so we do not have to work with the Hölder norms and can instead use it as a sort of negative weight on the continuous functions.

Define $Y(0,T) := Y_{\{0\}} \cap Y_{Tr} \cap \bigcap_{j=1}^m Y_j$, with the norm

$$\|u\|_{Y(0,T)} = \max\{\|u\|_{Y_{\{0\}}}, \|u\|_{Y_{Tr}}, \|u\|_{Y_1}, \dots, \|u\|_{Y_m}\}.$$

We will work with this space $Y(0,T)$ as the space on which all viable solutions live, but we need to clarify which solutions live up to our expectations.

We will assume that solutions u of the problem Eq. (2.13) will have the quality that

$$u \in \mathcal{E}^p_{\alpha,u}((0,T)).$$

It takes some thought to show this is incredibly hard to show: the domain of $A(u(\cdot),\cdot)$ depends on u, so it is not that easy to see which u will have $u \in D(A(u(\cdot),\cdot))$. We will define a set $\mathcal{Q}^p_\alpha((0,T))$ as the set of all u that have the following special quality:

$$\mathcal{Q}^p_\alpha((0,T)) = \{u \in W^{1,p}_{\omega_\alpha}((0,T);X_0) : u \in \mathcal{E}^p_{\alpha,u}((0,T))\}. \tag{2.15}$$

This means we can reduce to a specific semi-linear problem if needed if we can indeed show that indeed the time-dependent operators A_u satisfy the proper conditions. However, in some cases, we need unions of maximal regularity spaces $\mathcal{E}^p_{\alpha,u}((0,T))$, which the following definition covers.

Definition 2.3.1. For some open bounded set $G \subseteq Y(0,T) \cap Y_\mu$, initial condition $x_0 \in X_{\sigma,p}$, right-hand side function $f \in L^p_{\omega_\alpha}((0,T);X_0)$ and time $T>0$, define the solution space $SOL(G,x_0,f,T)$ as the union over $v \in G$ of the mild solutions u over $(0,T)$ to the linear problem

$$\begin{cases} \partial_t u(t) + A_v(t)u(t) &= f(t), \quad t \in (0,T), \\ u(0) &= x_0, \end{cases} \tag{2.16}$$

The following assumptions hold for the operator and the spaces we use. Note that it is parallel to Assumption 1.2.7, and we will show

Assumption 2.3.2. *For this proof, we use the following assumptions for some $R > 0$ and $\theta \in [0, \sigma]$ with $\sigma = 1 - \alpha - \frac{1}{p}$:*

1. *$\exists \zeta \in (\frac{1}{2}\pi, \pi), K > 0$ such that $\varrho(-A(t,x)) \supset \Sigma_\zeta$ for $x \in B_R(0; X_\theta)$ and $t \in [0,T]$, and*
$$\|R(\lambda, A(t,x))\|_{\mathscr{L}(X_0)} \leq \frac{K}{1+|\lambda|}, \; x \in B_R(0;X_\theta), t \in [0,T], \lambda \in \Sigma_\zeta.$$
2. *X_0 is UMD and $A(x,t)$ are uniformly R-sectorial with angle $\omega_R(A(x,t)) < \frac{1}{2}\pi$ for all $x \in B_R(0; X_\theta)$ and $t \in [0,T]$.*
3. *$\exists \gamma_0 \in (0,1)$ s.t. $\forall r \in [1,\infty], \beta \leq \gamma_0$, $(X_0, D(A(u,t)))_{\beta,r} = (X_0, D(A(0,0)))_{\beta,r} =: X_{\beta,r}$, $[X_0, D(A(u,t))]_\beta = [X_0, D(A(0,0))]_\beta =: X_\beta$ for all $u \in B_R(0; X_\theta)$ and $t \in [0,T]$, and for all $u, v \in B_R(0;X_\theta)$, $t, s \in [0,T]$, $x \in X_\beta$ and $x \in X_{\beta,r}$ we have*
$$\|x\|_{(X_0,D(A(u,t)))_{\beta,r}} \leq C_1 \|x\|_{(X_0,D(A(v,s)))_{\beta,r}},$$
$$\|x\|_{[X_0,D(A(u,t))]_\beta} \leq C_2 \|x\|_{[X_0,D(A(v,s))]_\beta},$$
where $C_1, C_2 > 0$ are independent of u, v, t, s.
4. *For $x \in B_R(0; X_\theta)$, $t \in [0,T]$ and $\beta \leq \gamma_0$, assume $D(A^\beta(x,t)) = X_\beta$ with the norm*
$$\|x\|_{D(A^\beta(x,t))} = \|A^\beta(x,t)x\|_{X_0}, \quad x \in X_\beta.$$
5. *$\exists \gamma \in (0, \gamma_0]$ s.t. for all $x, y \in B_R(0; X_\theta)$ and $t, s \in [0,T]$ we have*
$$\left\|A^\gamma(x,t)\left[A^{-1}(x,t) - A^{-1}(y,s)\right]\right\|_{\mathscr{L}(X_0)} \leq N_1 \|x-y\|_{X_\theta} + N_2|t-s|^\mu, \tag{2.17}$$
where N_1, N_2 are independent of x, y, t, s, and where $\mu = \sigma - \theta$.
6. *We additionally assume $\theta \in [0,\gamma)$, $\sigma \in (\theta, \gamma)$, and*
$$\gamma + \sigma > 1 + \theta \tag{2.18}$$

Using these assumptions, we should show $A_u(t)$ is a family of operators that satisfies the proper conditions, namely the Acquistapace Terreni conditions we defined earlier. In order for this to be true, we need to require our solutions u also satisfy $u \in C^\mu([0,T]; X_\theta)$, so that Eq. (2.17) combined with Assumption 2.3.2 will give the result. Therefore, from now on, we will only consider solutions in $\mathcal{AT}^p_\alpha(0,T) := B_R(0; C^\mu([0,T]; X_\theta)) \cap \mathcal{Q}^p_\alpha((0,T))$.

We will apply the problem with the following assumptions on the initial condition u_0 and the function F.

Assumption 2.3.3. *The following conditions must hold for all $R > 0$:*

(1) $\sigma < \gamma_0$, and $u_0 \in B_R(0; X_{\sigma,p})$, meaning $\exists L_{u_0} > 0$ s.t.
$$\|A^\sigma(u_0, 0)u_0\|_{X_0} \leq L_{u_0} \tag{2.19}$$

(2) The mapping $F : X_{\gamma_0} \to X_0$ admits a decomposition $F = F_{Tr} + F_c$, where:

(a) $F_{Tr} : X_{\sigma,p} \to X_0$ is locally Lipschitz continuous on $X_{\sigma,p}$, meaning $\exists L_{Tr} \geq 0$ s.t. $\forall u, v \in B_R(0; X_{\sigma,p})$,
$$\|F_{Tr}(u) - F_{Tr}(v)\|_{X_0} \leq L_{Tr}\|u - v\|_{X_{\sigma,p}}. \tag{2.20}$$

(b) For $j \in \{1, \ldots, m\}$, let $\beta_j \in [\sigma, \gamma_0)$ s.t.

$$\beta_j \leq \frac{\rho_j \sigma + 1}{\rho_j + 1} =: \beta_j^*,$$

and $F_c : X_{\gamma_0} \to X_0$ has some $L_c \geq 0$ s.t. $\forall u, v \in B_R(0; X_{\gamma_0})$,

$$\|F_c(u) - F_c(v)\|_{X_0} \leq L_c \sum_{j=1}^{m} \left(1 + \|u\|_{X_{\beta_j}}^{\rho_j} + \|v\|_{X_{\beta_j}}^{\rho_j}\right) \|u - v\|_{X_{\beta_j}}, \tag{2.21}$$

We define our solutions the following way:

Definition 2.3.4. $u \in \mathcal{AT}_\alpha^p(0,T)$ is called a *mild solution to the problem Eq.* (2.13) *on* $(0,T)$ if $u(0) = u_0$, $F_u(\cdot) \in L^p_{\omega_\alpha}((0,T); X_0)$, and for all $t \in [0,T]$,

$$u(t) = G_u(t,0)u_0 + \int_0^t G_u(t,s)F_u(s)\ ds$$

We will find that $u \in Y(0,T)$ becomes the appropriate condition such that $F_u(\cdot) \in L^p_{\omega_\alpha}((0,T); X_0)$ in the following lemma.

Lemma 2.3.5. *For $v_1, v_2 \in Y(0,T)$, $F_{Tr}(v(\cdot)) \in C([0,T]; X_0) \hookrightarrow L^p_{\omega_\alpha}((0,T); X_0)$ and $F_c(v(\cdot)) \in L^p_{\omega_\alpha}((0,T); X_0)$, with the following estimates:*

$$\|F_{Tr}(v_1(\cdot)) - F_{Tr}(v_2(\cdot))\|_{C([0,T];X_0)} \leq L_{Tr}\|v_1 - v_2\|_{Y_{Tr}},$$
$$\|F_c(v_1(\cdot)) - F_c(v_2(\cdot))\|_{L^p_{\omega_\alpha}((0,T);X_0)} \leq L_c \sum_{j=1}^{m} \left[T^{\delta_j} + \|v_1\|_{Y_j}^{\rho_j} + \|v_2\|_{Y_j}^{\rho_j}\right] \|v_1 - v_2\|_{Y_j}.$$

Here, $\delta_j = \frac{\rho_j}{\rho_j+1}(\alpha + \frac{1}{p})$. In particular, $F_v \in L^p_{\omega_\alpha}((0,T); X_0)$ for all $v \in Y(0,T)$.

Proof. Take $v_1, v_2 \in Y(0,T)$, meaning $v_1, v_2 \in Y_{Tr}$ and for all $j \in \{1, \ldots, m\}$, $v_1, v_2 \in Y_j$. Note that like in the proof of Eq. (2.7),

$$\|F_{Tr}(v_1(t)) - F_{Tr}(v_2(t))\|_{X_0} \overset{(2.20)}{\leq} L_{Tr}\|v_1(t) - v_2(t)\|_{X_{\sigma,p}} < \infty,$$

which indicates $F_{Tr}(v(\cdot)) \in C([0,T]; X_0)$ for $v \in Y(0,T)$. Setting $\alpha_j = \frac{\alpha}{\rho_j+1}$, $\overline{\alpha_j} = \frac{\rho_j\alpha}{\rho_j+1}$, $p_j = p(\rho_j+1)$ and $\overline{p_j} = \frac{p(\rho_j+1)}{\rho_j}$, we see that $\frac{1}{p} = \frac{1}{p_j} + \frac{1}{\overline{p_j}}$ and $\alpha = \alpha_j + \overline{\alpha_j}$. Noting $\overline{\alpha_j} \cdot \overline{p_j} = \alpha p = \alpha_j p_j$, we therefore get:

$$\begin{aligned}
&\|F_c(v_1) - F_c(v_2)\|_{L^p_{\omega_\alpha}((0,T);X_0)} \\
&\overset{(2.21)}{\leq} L_c \sum_{j=1}^{m} \left\| (1 + \|v_1\|_{X_{\beta_j}}^{\rho_j} + \|v_2\|_{X_{\beta_j}}^{\rho_j})\|)\|v_1 - v_2\|_{X_{\beta_j}} \right\|_{L^p_{\omega_\alpha}(0,T)} \\
&\overset{\text{Hölder}}{\leq} L_c \sum_{j=1}^{m} \left\| 1 + \|v_1\|_{X_{\beta_j}}^{\rho_j} + \|v_2\|_{X_{\beta_j}}^{\rho_j} \right\|_{L^{\overline{p_j}}_{\omega_{\overline{\alpha_j}}}(0,T)} \|v_1 - v_2\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \\
&= L_c \sum_{j=1}^{m} \left(\int_0^T t^{\overline{\alpha_j p_j}} + t^{\alpha_j p_j}\|v_1(t)\|_{X_{\beta_j}}^{p_j} + t^{\alpha_j p_j}\|v_2(t)\|_{X_{\beta_j}}^{p_j}\, dt \right)^{\frac{\rho_j}{p_j}} \|v_1 - v_2\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \\
&\leq L_c \sum_{j=1}^{m} \left[T^{\rho_j\alpha_j + \frac{\rho_j}{p_j}} + \|v_1\|^{\rho_j}_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} + \|v_2\|^{\rho_j}_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \right] \|v_1 - v_2\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \\
&= L_c \sum_{j=1}^{m} \left[T^{\delta_j} + \|v_1\|_{Y_j}^{\rho_j} + \|v_2\|_{Y_j}^{\rho_j} \right] \|v_1 - v_2\|_{Y_j} < \infty.
\end{aligned}$$

This indicates $F_c(v(\cdot)) \in L^p_{\omega_\alpha}((0,T); X_0)$ for $v \in Y(0,T)$. ∎

So, for our definition, we will see that $u \in Y(0,T)$ is a reasonable outcome for being a feasible solution. Now that we have an idea of where solutions will live, we should show that we can reduce to our previous linear theory from Section 1.2.1, and that the Acquistapace Terreni conditions from Assumption 1.2.7 indeed hold.

Lemma 2.3.6. *Assume Assumption 2.3.2 holds with $R > 0$. Let $v \in B_R(0; Y_\mu)$, then Assumption 1.2.7 holds for $A_v(\cdot)$.*

Proof. Write $N = N_1 R + N_2$, then for $0 \le s < t \le T$

$$\begin{aligned} \|A^\gamma(v(t),t)[A^{-1}(v(t),t) - A^{-1}(v(s),s)]\|_{\mathscr{L}(X_0)} &\underset{v\in Y_\mu}{\overset{(2.17)}{\le}} N_1\|v(t)-v(s)\|_{X_\theta} + N_2|t-s|^\mu \\ &\le (N_1[v]_{Y_\mu} + N_2)(t-s)^\mu \\ &\le N(t-s)^\mu. \end{aligned}$$

This means we have all the conditions from Assumption 1.2.7 met. Therefore, the result follows for the quasi-linear solution u and the linear solution w. This means we can use the evolution family G_u with the properties from Theorem 1.2.5 and Lemma 1.2.8 to define our mild solutions. $\square$

Remark. By Lemma 2.3.6 and Corollary 2.1.6, we also have maximal $L^p_{\omega_\alpha}$-regularity of $A_v(\cdot)$ on finite-time intervals $(0,\tilde{T})$ for any $\tilde{T} > 0$, meaning that for any $R > 0$ and mild solution $u \in SOL(B_R(0;Y_\mu), x_0, f, T)$ to the problem Eq. (2.16) that the following bounds must hold:

$$\|u\|_{\mathcal{E}^p_{\alpha,v}((0,\tilde{T}))} \le M^{reg}_{t,p,v}\left(\|x_0\|_{X_{\sigma,p}} + \|f\|_{L^p_{\omega_\alpha}((0,\tilde{T}),X_0)}\right). \tag{2.22}$$

where $M^{reg} := \sup_{v\in B_R(0;Y_\mu)} \sup_{t\in[0,\tilde{T}]} M^{reg}_{t,p,v} < \infty$, $x_0 \in X_{\sigma,p}$, and $f \in L^p_{\omega_\alpha}((0,\tilde{T}); X_0)$.

We are now ready to state the main theorem of this section, which is also the main result of the thesis.

Theorem 2.3.7 (Local-wellposedness for non-autonomous quasi-linear PDE's on non-constant domains)**.** *Set $\sigma = 1 - \alpha - \frac{1}{p}$ for some $p \in (1,\infty)$ and $\alpha \in [0, \frac{1}{p'})$. Choose $R > 0$ on which Assumption 2.3.2 and Assumption 2.3.3 hold. Let the following conditions hold for all $j \in \{1,\dots,m\}$:*

$$\begin{gathered} 0 \le \theta < \sigma < \beta_j < \gamma \le \gamma_0, \\ 1 + \theta < \gamma + \sigma, \\ \beta_j \le \beta_j^*. \end{gathered}$$

Then, for a very small $\tilde{T} > 0$, there exists a unique mild solution u to the problem Eq. (2.13) *in the space $\mathcal{AT}^p_\alpha(0,\tilde{T}) \cap Y(0,\tilde{T})$.*

Proof. For this proof, we will need to do the following.

- Set up Φ and a small closed domain $B^T_r(u_0)$, which will be the core components of the Banach Fixed Point theorem. This is done below.
- Show Φ maps $B^T_r(u_0)$ to itself. This is shown in Section 2.3.1.
- Show Φ is a uniformly contracting map. This is shown in Section 2.3.2.
- Apply the Banach Fixed Point Theorem to find a local solution in $B^T_r(u_0)$ and ensure the uniqueness of the solution in the full space $\mathcal{AT}^p_\alpha(0,\tilde{T}) \cap Y(0,\tilde{T})$ for $\tilde{T} \le T$. This is done in Section 2.3.3.

We define the domain using the reference solution

$$z_{u_0}(t) := e^{-tA(u_0,0)}u_0, \tag{2.23}$$

similar to Definition 2.2.4, and we note that we can use Proposition A.1.4 to conclude $z_{u_0} \in Y(0,T)$. We can then define the fixed point space and the operator.

Definition 2.3.8. For $r < 1$, define the closed bounded ball $B_r^T(u_0)$ in the Banach space $Y(0,T)$ as:

$$B_r^T(u_0) := \{u \in Y(0,T) : u(0) = u_0, \|u - z_{u_0}\|_{Y(0,T)} \le 1, \sup_{0\le s<t\le T} \frac{\|u(t)-u(s)\|_{X_\theta}}{(t-s)^\mu} \le 1\}.$$

Then $B_r^T(u_0) \subseteq B_1(0;Y_\mu)$, so the setting of Lemma 2.3.6 applies. Then define $\Phi : B_r^T(u_0) \to SOL(B_r^T(u_0), u_0, F_v, T)$ as

$$\Phi(v) = u,$$

where u is the unique mild solution to Eq. (2.16) with initial condition u_0 and right-hand side function F_v, so

$$u(t) = G_v(t,0)u_0 + \int_0^t G_v(t,s)F_v(s)\ ds.$$

Note the definition of existence and uniqueness of the mapping $v \mapsto u$ relies on the existence and uniqueness of the mild solution from Corollary 2.1.6, which means Assumption 1.2.2 is needed, which we obtain from Lemma 2.3.6. This lemma also gives $\Phi(u) \in \mathcal{E}^p_{\alpha,v}((0,T))$. Secondly, note that we have shown F_v is an admissable right-hand side function in Lemma 2.3.5.

It is important to show that the following estimates hold, which allows us to deal with solutions to linear problems more easily.

Lemma 2.3.9 (Estimates of solutions to linear problems)**.** *Let $f \in E_0$, $v \in B_R(0;Y(0,T)\cap Y_\mu)$ and $\tilde{v} \in B_r^T(u_0)$. For $0 \le s < t \le T$, define*

$$U_{f,v}(t) := \int_0^t G_v(t,\tau)f(\tau)d\tau.$$

Let $u = \Phi(\tilde{v})$ and $w_s(t) = \int_0^t e^{-(t-\tau)A_v(s)}f(\tau)d\tau$. Then we have the following for all $\phi \in [0,\sigma]$:

$$\|U_{f,v}(t) - U_{f,v}(s)\|_{X_\phi} \le C(t-s)^{\gamma+\mu-1-\phi+\sigma}\|f\|_{E_0} + \|w_s(t) - w_s(s)\|_{X_\phi}. \tag{2.24}$$

As a consequence, the following holds for all $j \in \{1,\dots,m\}$:

$$\|U_{f,v}(t) - U_{f,v}(s)\|_{X_\theta} \le C_1(t-s)^\mu\|f\|_{E_0}, \tag{2.25}$$
$$\|U_{f,v}(\cdot)\|_{Y_{\{0\}}} \le C_2\|f\|_{E_0}, \tag{2.26}$$
$$\|U_{f,v}(\cdot)\|_{Y_{Tr}} \le C_3\|f\|_{E_0}, \tag{2.27}$$
$$\|U_{f,v}(\cdot)\|_{Y_j} \le C_4\|f\|_{E_0}, \tag{2.28}$$
$$\|A^\sigma_{\tilde{v}}(t)u(t)\|_{X_0} \le C_5(L_{u_0} + \|F_{\tilde{v}}\|_{E_0}). \tag{2.29}$$

Proof. To show Eq. (2.25), note that similar to the proof of Eq. (2.10), we can write

$$\begin{aligned}
&\left\|\int_s^t G_v(t,\tau)f(\tau)d\tau\right\|_{X_\phi} \\
\le &\int_s^t \left\|A_v^\phi(t)\left(G_v(t,\tau) - e^{-(t-\tau)A_v(\tau)}\right)f(\tau)\right\|_{X_0} d\tau \\
&+ \int_s^t \left\|A_v^\phi(\tau)\left(e^{-(t-\tau)A_v(\tau)}f(\tau) - e^{-(t-\tau)A_v(s)}\right)f(\tau)\right\|_{X_0} d\tau \\
&+ \left\|\int_s^t e^{-(t-\tau)A_v(s)}f(\tau)d\tau\right\|_{X_\phi} \\
=:&L_1 + L_2 + L_3.
\end{aligned}$$

For L_1, use Eq. (1.11) to obtain

$$L_1 \le C\int_s^t (t-\tau)^{\gamma+\mu-1-\phi}\|f(\tau)\|_{X_0}d\tau$$

$$
\begin{aligned}
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma+\mu-1-\phi}\|_{L^r(s,t)} \cdot \left\|\tau \mapsto \tau^{-\alpha}\|\tau^\alpha f(\tau)\|_{X_0}\right\|_{L^{r'}(s,t)} \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma+\mu-1-\phi}\|_{L^r(s,t)} \cdot \|\tau \mapsto \tau^{-\alpha}\|_{L^h(s,t)} \cdot \|f\|_{L^p_{\omega_\alpha}((s,t);X_0)} \\
&\leq C(t-s)^{\gamma+\mu-1-\phi+\frac{1}{r}}(t-s)^{-\alpha+\frac{1}{h}}\|f\|_{E_0} \\
&\leq C(t-s)^{\gamma+\mu-1-\phi+\sigma}\|f\|_{E_0}.
\end{aligned}
$$

Here, $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} + \frac{1}{p} + \frac{1}{h}$. These must satisfy $\gamma + \mu - 1 - \phi > -\frac{1}{r}$ and $-\alpha > -\frac{1}{h}$, so

$$
\gamma + \mu - 1 - \phi > \frac{1}{p} + \frac{1}{h} - 1 > \frac{1}{p} + \alpha - 1 = -\sigma.
$$

By $\phi \leq \sigma$ and $\gamma + \mu > 1$, which follows from Eq. (2.18) with $\mu = \sigma - \theta$, this is satisfied.

For L_2, use Eq. (1.12) and the same r, h as above to obtain

$$
\begin{aligned}
L_2 &\leq C\int_s^t (t-\tau)^{\gamma-\phi-1}(\tau-s)^\mu \|f(\tau)\|_{X_0} d\tau \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma-1-\phi}(z-s)^\mu\|_{L^r(s,t)} \cdot \left\|\tau \mapsto \tau^{-\alpha}\|\tau^\alpha f(\tau)\|_{X_0}\right\|_{L^{r'}(s,t)} \\
&\overset{\text{Hölder}}{\leq} C\|z \mapsto (t-z)^{\gamma-1-\phi}(z-s)^\mu\|_{L^r(s,t)} \cdot \|\tau \mapsto \tau^{-\alpha}\|_{L^h(s,t)} \cdot \|f\|_{L^p_{\omega_\alpha}((s,t);X_0)} \\
&\leq C(t-s)^{\gamma+\mu-1-\phi+\frac{1}{r}}(t-s)^{-\alpha+\frac{1}{h}}\|f\|_{E_0} \\
&\leq C(t-s)^{\gamma+\mu-1-\phi+\sigma}\|f\|_{E_0}.
\end{aligned}
$$

For L_3, we use the maximal $L^p_{\omega_\alpha}$-regularity of $A_v(s)$ (which is due to Theorem 1.1.20 and Proposition 2.1.2) to conclude that the mild solution w_s (and therefore strong solution by Proposition 1.1.7) to the autonomous linear problem

$$
\begin{cases} \partial_t w(t) + A_v(s)w(t) &= f(t), \quad t \in (0,T) \\ w(0) &= 0 \end{cases}
$$

has $L_3 = \left\|\int_s^t e^{-(t-\tau)A_v(s)} f(\tau) d\tau\right\|_{X_\phi} = \|w_s(t) - w_s(s)\|_{X_\phi}$.

This combined gives Eq. (2.24), which we will now apply to show Eqs. (2.25) to (2.27) and (2.29). Since

$$
w_s \in \mathcal{P}^p_{\alpha,A_v(s)}((0,T)) := W^{1,p}_{\omega_\alpha}((0,T);X_0) \cap L^p_{\omega_\alpha}((0,T);D(A_v(s))),
$$

we can use Eq. (A.1) to get $w_s \in C^\mu([0,T];X_\theta)$ and Eq. (A.2) to conclude $w_s \in C([0,T];X_{\sigma,p})$ with time-independent constants and thus a maximal $L^p_{\omega_\alpha}$-regularity estimate. This means:

$$
\begin{aligned}
\|U_{f,v}(t) - U_{f,v}(s)\|_{X_\theta} &\leq C_1(t-s)^{\mu+\gamma+\mu-1}\|f\|_{E_0} + (t-s)^\mu\|w_s\|_{C^\mu([0,T];X_\theta)} \\
&\overset{(A.1)}{\leq} C_1(t-s)^{\mu+\gamma+\mu-1}\|f\|_{E_0} + C_2(t-s)^\mu\|w_s\|_{\mathcal{P}^p_{\alpha,A_v(s)}((0,T))} \\
&\overset{A_v(s)\in\mathcal{MR}_p}{\leq} \tilde{C}(t-s)^\mu\|f\|_{E_0}, \\
\|U_{f,v}(\cdot)\|_{Y_{\{0\}}} &\leq \sup_{t\in[0,T]} t^{-\mu}\tilde{C}t^\mu\|f\|_{E_0} = \tilde{C}\|f\|_{E_0}, \\
\|U_{f,v}(\cdot)\|_{Y_{Tr}} &\leq C_3 T^{\gamma+\mu-1}\|f\|_{E_0} + \|w_0\|_{Y_{Tr}} \\
&\overset{(A.2)}{\leq} C_3 T^{\gamma+\mu-1}\|f\|_{E_0} + C_4\|w_0\|_{\mathcal{P}^p_{\alpha,A_v(0)}((0,T))} \\
&\overset{A_v(0)\in\mathcal{MR}_p}{\leq} \tilde{C}\|f\|_{E_0}.
\end{aligned}
$$

For Eq. (2.29), note

$$
\|G_{\tilde{v}}(t,0)u_0 + U_{F_{\tilde{v}},\tilde{v}}(t)\|_{X_\sigma} \leq \|A^\sigma_{\tilde{v}} G_{\tilde{v}}(t,0)A^{-\sigma}_{\tilde{v}}(0)A^\sigma(u_0)u_0\|_{X_0} + \|U_{F_{\tilde{v}},\tilde{v}}(t)\|_{Y_{Tr}}
$$

$$\leq \tilde{C}(\|A^\sigma(u_0)u_0\|_{X_0} + \|F_{\tilde{v}}\|_{E_0})$$
$$\leq \tilde{C}(L_{u_0} + \|F_{\tilde{v}}\|_{E_0}).$$

Lastly, we should show Eq. (2.28) seperately, since $\beta_j \geq \sigma$. Taking $j \in \{1, \dots, m\}$,

$$\begin{aligned}
&\left\| t \mapsto \int_0^t G_v(t,\tau) f(\tau) d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \\
\leq &\left\| t \mapsto \int_0^t \left\| A_v^{\beta_j}(t)(G_v(t,\tau) - e^{(t-\tau)A_v(\tau)}) f(\tau) \right\|_{X_0} d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)} \\
&+ \left\| t \mapsto \int_0^t \left\| A_v^{\beta_j}(\tau)(e^{(t-\tau)A_v(\tau)} - e^{(t-\tau)A_v(0)}) f(\tau) \right\|_{X_0} d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)} \\
&+ \left\| t \mapsto \int_0^t e^{-(t-\tau)A(u_0)} f(\tau) d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})} \\
\leq &\, L_{1,j} + L_{2,j} + L_{3,j}.
\end{aligned}$$

For $L_{1,j}$, use Eq. (1.11), Proposition A.2.1 and Hölder's inequality to get

$$\begin{aligned}
L_{1,j} &\leq C \left\| t \mapsto \int_0^t (t-\tau)^{\gamma+\mu-1-\beta_j} \|f(\tau)\|_{X_0} d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)} \\
&\leq \tilde{C} \left\| z \mapsto z^{\alpha_j} \|f(z)\|_{X_0} \right\|_{L^r(0,T)} \leq \|z^{\alpha_j - \alpha}\|_{L^h(0,T)} \cdot \|f\|_{L^p_{\omega_\alpha}((0,T);X_0)} \\
&\leq \tilde{C} T^{\alpha_j - \alpha + \frac{1}{h}} \|f\|_{E_0} = \tilde{C} T^{\beta_j^* + \gamma + \mu - 1 - \beta_j} \|f\|_{E_0}.
\end{aligned}$$

Here, $\frac{1}{p_j} = \frac{1}{r} - (\gamma + \mu - \beta_j) = \frac{1}{p} + \frac{1}{h} - (\gamma + \mu - \beta_j)$, therefore the integrability condition becomes

$$\begin{aligned}
\alpha_j - \alpha &> -\frac{1}{h} = \frac{1}{p} - \frac{1}{p_j} - (\gamma + \mu - \beta_j), \\
\alpha_j + \frac{1}{p_j} - \alpha - \frac{1}{p} + \gamma + \mu &> \beta_j \\
\frac{1-\sigma}{\rho_j + 1} + \sigma + \gamma + \mu &> 1 + \beta_j, \\
\beta_j^* + \gamma + \mu &> 1 + \beta_j.
\end{aligned}$$

This is satisfied by $\beta_j \leq \beta_j^*$ and $\gamma + \mu > 1$ from Eq. (2.18) and $\mu = \sigma - \theta$.

For $L_{2,j}$, using Eq. (1.12) instead,

$$\begin{aligned}
L_{2,j} &\leq C \left\| t \mapsto \int_0^t (t-\tau)^{\gamma-1-\beta_j} \tau^\mu \|f(\tau)\|_{X_0} d\tau \right\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)} \\
&\leq \tilde{C} \left\| z \mapsto z^{\mu+\alpha_j} \|f(z)\|_{X_0} \right\|_{L^r(0,T)} \leq \|z^{\mu+\alpha_j-\alpha}\|_{L^h(0,T)} \|f\|_{E_0} \\
&\leq \tilde{C} T^{\beta_j^* + \gamma + \mu - 1 - \beta_j} \|f\|_{E_0}.
\end{aligned}$$

Now, we have $\frac{1}{p_j} = \frac{1}{r} - (\gamma - \beta_j) = \frac{1}{p} + \frac{1}{h} - (\gamma - \beta_j)$ The integrability condition is

$$\mu + \alpha_j - \alpha > -\frac{1}{h} = \frac{1}{p} - \frac{1}{p_j} - (\gamma - \beta_j),$$

which is the same as above.

For $L_{3,j}$, note that by the maximal $L^p_{\omega_\alpha}$-regularity of $A(u_0, 0)$ and Proposition 1.1.7, $w_0(t) = \int_0^t e^{-(t-\tau)A(u_0,0)} f(\tau) d\tau$ is the $L^p_{\omega_\alpha}$-solution to the autonomous problem

$$\begin{cases} \partial_t w(t) + A(u_0, 0) w(t) &= f(t), \quad t \in (0,T), \\ w(0) &= 0. \end{cases}$$

As before, $w_0 \in \mathcal{P}^p_{\alpha,A(u_0,0)}((0,T)) \hookrightarrow Y_j$ with time-independent constants by Eq. (A.3), so

$$L_{3,j} = \|w_0\|_{Y_j} \leq C\|w_0\|_{\mathcal{P}^p_{\alpha,A(u_0,0)}((0,T))} \leq \tilde{C}\|f\|_{E_0}$$

by maximal $L^p_{\omega_\alpha}$-regularity from Proposition 2.1.2 and Theorem 1.1.20. Combined, this gives Eq. (2.28). $\square$

2.3.1 Self-mapping estimates

Take $v \in B^T_r(u_0)$, we intend to show $\Phi(v)$ is also an element of $B^T_r(u_0)$. Therefore, we need to show $\|u - z_{u_0}\|_{Y_j} \leq r$, $\|u - z_{u_0}\|_{Y_{Tr}} \leq r$, $\|u - z_{u_0}\|_{Y_{\{0\}}} \leq r$, and $\|u(t) - u(s)\|_{X_\theta} \leq (t-s)^\mu$ for all $0 \leq s < t \leq T$. This last estimate is done in the section labelled Hölder estimate. Note that by our mild solution formulas from Corollary 2.1.6 and Eq. (2.23):

$$u(t) - z_{u_0}(t) = \left(G_v(t,0) - e^{-tA_v(0)}\right) u_0 + \int_0^t G_v(t,s)F_v(s)\, ds$$

We get $A(u_0,0) = A_v(0)$ by the fact that $v(0) = u_0$ as initial condition from $v \in B^T_r(u_0)$. Applying the $Y(0,T)$ norm then gives for all $j \in \{1,\dots,m\}$,

$$\|u - z_{u_0}\|_{Y_{\{0\}}} \leq S_{1,\{0\}} + S_{2,\{0\}},$$

$$\|u - z_{u_0}\|_{Y_{Tr}} \leq S_{1,Tr} + S_{2,Tr}.$$

$$\|u - z_{u_0}\|_{Y_j} \leq S_{1,j} + S_{2,j},$$

We need a version of the smallness Lemma 18.2.10 of Hytönen et al. (2024), which supports our set-up of $B^T_r(u_0)$. This allows us to make E_0 estimates of our right-hand side function.

Lemma 2.3.10 (Smallness). *For $v \in B^T_r(u_0)$, the following estimates hold:*

$$\begin{aligned}\|F_{Tr}(v)\|_{E_0} &\leq T^{1-\sigma}(\|F_{Tr}(u_0)\|_{X_0} + L_{Tr}r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}),\\ \|F_c(v)\|_{E_0} &\leq \|F_c(z_{u_0})\|_{E_0} + L_c r\sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|^{\rho_j}_{Y_j} + r^{\rho_j}\right].\end{aligned}$$

Here $\delta_j = \rho_j(\alpha_j + \frac{1}{p_j})$.

Proof. For the trace part, note that since $\|v - z_{u_0}\|_{Y(0,T)} \leq r$, we can use the same method as in the proof of Eq. (2.11) to get

$$\begin{aligned}\|v(t) - u_0\|_{X_{\sigma,p}} &\leq \|v(t) - z_{u_0}(t)\|_{X_{\sigma,p}} + \|z_{u_0}(t) - u_0\|_{X_{\sigma,p}}\\ &\leq r + \|z_{u_0} - u_0\|_{Y_{Tr}}\end{aligned}$$

Therefore,

$$\begin{aligned}\|F_{Tr}(v(t))\|_{X_0} &\leq \|F_{Tr}(u_0)\|_{X_0} + \|F_{Tr}(v(t)) - F_{Tr}(u_0)\|_{X_0}\\ &\overset{(2.20)}{\leq} \|F_{Tr}(u_0)\|_{X_0} + L_{Tr}\|v(t) - u_0\|_{X_{\sigma,p}}\\ &\leq \|F_{Tr}(u_0)\|_{X_0} + L_{Tr}r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}\end{aligned}$$

Taking the $L^p_{\omega_\alpha}((0,T);X_0)$ norm gives

$$\begin{aligned}\|F_{Tr}(v)\|_{L^p_{\omega_\alpha}((0,T);X_0)} &\leq (\|F_{Tr}(u_0)\|_{X_0} + L_{Tr}r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}})\left(\int_0^T t^{\alpha p}dt\right)^{\frac{1}{p}}\\ &= T^{\alpha+\frac{1}{p}}(\|F_{Tr}(u_0)\|_{X_0} + L_{Tr}r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}})\\ &= T^{1-\sigma}(\|F_{Tr}(u_0)\|_{X_0} + L_{Tr}r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}).\end{aligned}$$

Here we can note again that $\|z_{u_0} - u_0\|_{Y_{Tr}} \downarrow 0$ as $T \downarrow 0$ by strong continuity from Proposition 2.2.8 of Lunardi (1995). For the critical part, we examine $\|F_c(z_{u_0})\|_{E_0}$ and $\|F_c(v) - F_c(z_{u_0})\|_{E_0}$ separately, then use the estimates like done for the trace part.

$$\begin{aligned}\|F_c(v) - F_c(z_{u_0})\|_{E_0} &\overset{(2.3.5)}{\leq} L_c \sum_{j=1}^{m} \left[T^{\delta_j} + \|v\|_{Y_j}^{\rho_j} + \|z_{u_0}\|_{Y_j}^{\rho_j}\right] \|v - z_{u_0}\|_{Y_j} \\ &\leq L_c \sum_{j=1}^{m} \left[T^{\delta_j} + C\|z_{u_0}\|_{Y_j}^{\rho_j} + \|v - z_{u_0}\|_{Y_j}^{\rho_j} + \|z_{u_0}\|_{Y_j}^{\rho_j}\right] \|v - z_{u_0}\|_{Y_j} \\ &\leq L_c r \sum_{j=1}^{m} \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + r^{\rho_j}\right]\end{aligned}$$

Therefore,

$$\begin{aligned}\|F_c(v)\|_{E_0} &\leq \|F_c(z_{u_0})\|_{E_0} + \|F_c(v) - F_c(z_{u_0})\|_{E_0} \\ &\leq \|F_c(z_{u_0})\|_{E_0} + L_c r \sum_{j=1}^{m} \left[T^{\rho_j(\alpha_j + \frac{1}{p_j})} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + r^{\rho_j}\right].\end{aligned}$$

Note this term relies of $\|F_c(z_{u_0})\|_{E_0}$. Since $z_{u_0} \in Y_j$, we can use the argument of Lemma 2.3.5 to determine $F_c(z_{u_0}(\cdot)) \in L^p_{\omega_\alpha}((0,T);X_0) =: E_0$, and therefore $\|F_c(z_{u_0})\|_{E_0} \downarrow 0$. □

Estimate of $S_{1,\{0\}}$:

$$\begin{aligned}S_{1,\{0\}} &= \left\|(G_v(t,0) - e^{-tA_v(0)})A_v^{-\sigma}(0)A_v^{\sigma}(0)u_0\right\|_{C^{\mu}_{\{0\}}([0,T];X_\theta)} \\ &\leq \sup_{t\in[0,T]} t^{-\mu} \left\|A_v^{\theta}(t)(G_v(t,0) - e^{-tA_v(0)})A_v^{-\sigma}(0)A_v^{\sigma}(0)u_0 - 0\right\|_{X_0} \\ &\overset{(1.11)}{\leq} \sup_{t\in[0,T]} Ct^{-\mu+\mu+\gamma-\theta+\sigma-1} \|A^{\sigma}(u_0,0)u_0\|_{X_0} \\ &\leq \sup_{t\in[0,T]} Ct^{\gamma-\theta+\sigma-1} \|A^{\sigma}(u_0,0)u_0\|_{X_0} \\ &= CT^{\gamma-\theta+\sigma-1}L_{u_0}.\end{aligned}$$

$\gamma - \theta + \sigma - 1 > 0$ holds by Eq. (2.18).

Estimate of $S_{1,Tr}$:

$$\begin{aligned}S_{1,Tr} &= \left\|(G_v(t,0) - e^{-tA_v(0)})A_v^{-\sigma}(0)A_v^{\sigma}(0)u_0\right\|_{C([0,T];X_{\sigma,p})} \\ &\leq \sup_{t\in[0,T]} D \left\|A_v^{\sigma}(t)(G_v(t,0) - e^{-tA_v(0)})A_v^{-\sigma}(0)A_v^{\sigma}(0)u_0\right\|_{X_0} \\ &\overset{(1.11)}{\leq} \sup_{t\in[0,T]} Ct^{\mu+\gamma-1} \|A^{\sigma}(u_0,0)u_0\|_{X_0} \\ &= CT^{\gamma+\mu-1}L_{u_0}.\end{aligned}$$

$\gamma + \mu - 1 > 0$ holds by Eq. (2.18) and $\mu = \sigma - \theta$.

Estimate of $S_{1,j}$:

$$S_{1,j} = \left\|(G_v(t,0) - e^{-tA_v(0)})A_v^{-\sigma}(0)A_v^{\sigma}(0)u_0\right\|^{p_j}_{L^{p_j}_{\omega_{\alpha_j}}((0,T;X_{\beta_j})}$$

$$
\begin{aligned}
&\leq \int_0^T \left\| t^{\alpha_j} A_v^{\beta_j}(t)(G_v(t,0) - e^{-tA_v(0)}) A_v^{-\sigma}(0) A_v^{\sigma}(0) u_0 \right\|_{X_0}^{p_j} dt \\
&\overset{(1.11)}{\leq} C \int_0^T t^{\alpha p} t^{p_j(\mu+\gamma-1+\sigma-\beta_j)} \left\| A^{\sigma}(u_0, 0) u_0 \right\|_{X_0}^{p_j} dt \\
&= C L_{u_0}^{p_j} \int_0^T t^{p_j(\alpha_j+\mu+\gamma-1+\sigma-\beta_j)} dt.
\end{aligned}
$$

From here, we see that the integrability condition is

$$
\begin{aligned}
\gamma + \sigma + \mu - 1 &> \frac{1}{\rho_j + 1}\left(\alpha + \frac{1}{p}\right), \\
\gamma + \sigma - \beta_j + \mu - 1 &> \frac{1}{\rho_j + 1}(\sigma - 1), \\
\gamma + \mu + \frac{\rho_j \sigma + \sigma - \sigma + 1}{\rho_j + 1} &> 1 + \beta_j, \\
\gamma + \mu + \beta_j^* &> 1 + \beta_j,
\end{aligned}
\tag{2.30}
$$

which is true by Eq. (2.18), $\mu = \sigma - \theta$, and $\beta_j^* \geq \beta_j$. Then we have

$$
\left\| (G_v(t,0) - e^{-tA_v(0)}) A_v^{-\sigma}(0) A_v^{\sigma}(0) u_0 \right\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T;X_{\beta_j})} \leq C L_{u_0} T^{\gamma+\mu-\beta_j-1+\beta_j^*}.
$$

Estimate of $S_{2,\{0\}}$, $S_{2,Tr}$ and $S_{2,j}$:

Note this term is equal to $U_{F_v,v}$ for $v \in B_r^T(u_0)$ and $F_v \in L^p_{\omega_\alpha}((0,T);X_0)$ by Lemma 2.3.5. Therefore using Lemma 2.3.9, we can conclude $S_{2,\{0\}} \leq C_{\{0\}} \|F_v\|_{E_0}$, $S_{2,Tr} = \|u_2\|_{Y_{Tr}} \leq C_{Tr} \|F_v\|_{E_0}$ and $S_{2,j} \leq C_j \|F_v\|_{E_0}$.

Hölder estimate

Lastly we show $\|u(t) - u(s)\|_{X_\theta} \leq C_{T,r}(t-s)^{\mu}$ for all $0 \leq s < t \leq T$ for some small $T > 0$ and some $C_{T,r} > 0$. We will see that we can take $C_{T,r} \leq 1$ by shrinking T and r. We use the mild solution formula from Corollary 2.1.6 of $u = \Phi(v)$ to determine a form similar to that on page 206 of Yagi (2010):

$$
\begin{aligned}
u(t) - u(s) = {} & (G_v(t,s) - e^{-(t-s)A_v(s)}) A_v^{-\sigma}(s) A_v^{\sigma}(s) u(s) \\
& + (e^{-(t-s)A_v(s)} - 1) A_v^{-\sigma}(s) A_v^{\sigma}(s) u(s) \\
& + \int_s^t G_v(t,\tau) F_v(\tau) d\tau.
\end{aligned}
$$

Applying the X_θ norm gives

$$
\|u(t) - u(s)\|_{X_\theta} \leq H_1 + H_2 + H_3.
$$

Using Eq. (1.11), and keeping in mind $\mu = \sigma - \theta$ and $u_0 \in X_{\sigma,p} \hookrightarrow D(A^{\sigma}(u_0))$, we get

$$
\begin{aligned}
H_1 &\leq \|A_v^{\theta}(t)(G_v(t,s) - e^{-(t-s)A_v(s)}) A_v^{-\sigma}(s) A_v^{\sigma}(s) u(s)\|_{X_0} \\
&\leq C_1 (t-s)^{\sigma-\theta+\mu+\gamma-1} \|A_v^{\sigma}(s) u(s)\|_{X_0} \\
&\overset{(2.29)}{<} (t-s)^{\mu} \cdot \tilde{C}_1 \left(L_{u_0} + \|F_v\|_{E_0}\right).
\end{aligned}
$$

Using Eq. (1.3), which is a property of bounded analytic C_0-semigroups generated by operator $A_v(s)$, which by maximal $L^p_{\omega_\alpha}$-regularity from Proposition 2.1.2 and Theorem 1.1.20 has sectoriality by applying Theorem 1.1.15, we get

$$
\begin{aligned}
H_2 &\leq \|A_v^{\theta}(s)(e^{-(t-s)A_v(s)} - 1) A_v^{-\sigma}(s) A_v^{\sigma}(s) u(s)\|_{X_0} \\
&\leq \left\| \int_0^{t-s} (A_v(s))^{\theta+1-\sigma} e^{-\tau A_v(s)} d\tau \right\|_{\mathscr{L}(X_0)} \|A_v^{\sigma}(s) u(s)\|_{X_0}
\end{aligned}
$$

$$\begin{aligned}&\leq C_2(t-s)^{\sigma-\theta}\|A_v^\sigma(s)u(s)\|_{X_0}\\&\overset{(2.29)}{\leq}(t-s)^\mu\cdot\tilde{C}_2\left(L_{u_0}+\|F_v\|_{E_0}\right).\end{aligned}$$

For H_3, we can use Eq. (2.25) of Lemma 2.3.9 to conclude

$$\left\|\int_s^t G_v(t,\tau)F_v(\tau)d\tau\right\|_{X_\theta}\leq C_3(t-s)^\mu\|F_v\|_{E_0}.$$

Then the result will follow by shrinking T and applying smallness estimates.

Combining all estimates:

For $v\in B_r^T(u_0)$, we get using Lemma 2.3.10

$$\begin{aligned}\|u-z_{u_0}\|_{Y_{\{0\}}}&\lesssim T^{\gamma+\mu-1}L_{u_0}+\|F_v\|_{L^p_{\omega_\alpha}((0,T);X_0)}\\&\leq L_{u_0}T^{\gamma+\mu-1}+T^{1-\sigma}(\|F_{Tr}(u_0)\|_{X_0}+L_{Tr}r+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}})\\&\quad+\|F_c(z_{u_0})\|_{E_0}+L_cr\sum_{j=1}^m\left[T^{\delta_j}+2\|z_{u_0}\|_{Y_j}^{\rho_j}+r^{\rho_j}\right],\\\|u-z_{u_0}\|_{Y_{Tr}}&\lesssim L_{u_0}T^{\gamma+\mu-1}+T^{1-\sigma}(\|F_{Tr}(u_0)\|_{X_0}+L_{Tr}r+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}})\\&\quad+\|F_c(z_{u_0})\|_{E_0}+L_cr\sum_{j=1}^m\left[T^{\delta_j}+2\|z_{u_0}\|_{Y_j}^{\rho_j}+r^{\rho_j}\right],\\\|u-z_{u_0}\|_{Y_j}&\lesssim L_{u_0}T^{\gamma+\mu+\beta_j^*-1-\beta_j}+\|F_v\|_{L^p_{\omega_\alpha}((0,T);X_0)}\\&\leq L_{u_0}T^{\gamma+\mu+\beta_j^*-1-\beta_j}+T^{1-\sigma}(\|F_{Tr}(u_0)\|_{X_0}+L_{Tr}r+L_{Tr}\|z_{u_0}-u_0\|_{Y_{Tr}})\\&\quad+\|F_c(z_{u_0})\|_{E_0}+L_cr\sum_{j=1}^m\left[T^{\delta_j}+2\|z_{u_0}\|_{Y_j}^{\rho_j}+r^{\rho_j}\right].\end{aligned}$$

Here we used $\delta_j=\rho_j(\alpha_j+\frac{1}{p_j})$. All these terms go to 0 as T and r decrease:

- $T^{\gamma+\mu-1}$, $T^{\gamma+\mu+\beta_j^*-1-\beta_j}$, $T^{1-\sigma}$, $T^{\delta_j}\downarrow 0$ as $T\downarrow 0$ since these powers are positive
- $\|F_{Tr}(u_0)\|_{X_0}<\infty$ since $u_0\in X_{\sigma,p}$,
- $\|z_{u_0}-u_0\|_{Y_{Tr}}\downarrow 0$ as $T\downarrow 0$ by strong continuity from Proposition 2.2.8 of Lunardi (1995),
- $\|z_{u_0}\|_{Y_j}\downarrow 0$ as $T\downarrow 0$ since $z_{u_0}\in Y_j$,
- $\|F_c(z_{u_0})\|_{E_0}\downarrow 0$ as $T\downarrow 0$ since $F_c(z_{u_0})\in E_0$,
- $C_jr^{\rho_j+1}<\frac{r}{m+1}$ for $r<\left(\frac{C_j}{m+1}\right)^{\frac{1}{\rho_j}}$, meaning r^{ρ_j+1} shrinks faster than r^1 for all $r<1$.

Therefore, we can choose a T_{small} and r_{small} s.t. $\|u-z_{u_0}\|_{Y(0,T_{small})}\leq r_{small}$.

To summarise the Hölder estimate, we have found a bound for $\|u(t)-u(s)\|_{X_\theta}$, namely

$$\|u(t)-u(s)\|_{X_\theta}\leq(t-s)^\mu\cdot\left(C_{1,2}\left(L_{u_0}+\|F_v\|_{E_0}\right)+C_3(t-s)^\mu\|F_v\|_{E_0}\right).$$

Since $C_{1,2}<\infty$ decreases as $T>0$ decreases (Yagi, 2010, see page 207 below (5.22)), L_{u_0} is finite, and $\|F_v\|_{E_0}\downarrow 0$ as $T\downarrow 0$ and $r\downarrow 0$ by smallness estimates from Lemma 2.3.10, we can get

$$\|u(t)-u(s)\|_{X_\theta}\leq(t-s)^\mu.$$

Therefore, we can choose a $T_{self}\leq T_{small}$ and $r_{self}\leq r_{small}$ s.t. $u\in B_{r_{self}}^{T_{self}}(u_0)$.

2.3.2 Uniform contraction estimates

Take v_1 and v_2 as functions in $B_r^T(u_0)$, and write $u_1 := \Phi(v_1)$ and $u_2 := \Phi(v_2)$. Then, we write the difference as

$$\begin{aligned} u_1(t) - u_2(t) = &\left(G_{v_1}(t,0) - G_{v_2}(t,0)\right) u_0 \\ &+ \int_0^t \left(G_{v_1}(t,s) - G_{v_2}(t,s)\right) F_{v_1}(s)\ ds \\ &+ \int_0^t G_{v_2}(t,s) \left(F_{v_1}(s) - F_{v_2}(s)\right)\ ds \end{aligned}$$

Then we get the following estimates:

$$\begin{aligned} \|u_1 - u_2\|_{Y_{\{0\}}} &\leq T_{1,\{0\}} + T_{2,\{0\}} + T_{3,\{0\}}, \\ \|u_1 - u_2\|_{Y_{Tr}} &\leq T_{1,Tr} + T_{2,Tr} + T_{3,Tr}, \\ \|u_1 - u_2\|_{Y_j} &\leq T_{1,j} + T_{2,j} + T_{3,j}. \end{aligned}$$

We first need a lemma that allows us to deal with these differences in evolution families that appear in the T_1 and T_2 terms, but for this we need theory on the Yosida approximations, which are defined in Section 1.2.1. Below, the Lipschitz estimate Eq. (2.17) is extended to the Yosida approximation of $A(u(t), t)$.

Lemma 2.3.11. *Let Assumption 2.3.2 hold for $A(u,t)$ and $p \in (1,\infty)$ and $\alpha \in [0, \frac{1}{p'})$. Let $A_{u,n}(t) := A_u(t)nR(n, A_u(t))$ be the Yosida approximation of $A_u(t)$. Then $\exists \tilde{N}_1, \tilde{N}_2 > 0$ s.t. for $t, s \in (0,T)$, $u, v \in B_R(0; Y_\mu)$ and $n \in \mathbb{N}$,*

$$\|A_{u,n}^\gamma(t) \left[A_{u,n}^{-1}(t) - A_{v,n}^{-1}(s)\right]\|_{\mathcal{L}(X_0)} \leq \tilde{N}_1 \|u(t) - v(s)\|_{X_\theta} + \tilde{N}_2 |t-s|^\mu$$

Proof. By Lemma 2.3.6, A_u is a non-autonomous operator satisfying Assumption 1.2.7, meaning we can use theory from Chapter 3 Part II of Yagi (2010). From Section 4 of Chapter 3 Part II, we combine four crucial facts.

- $A_{u,n}^{-1}(t) = A_u^{-1}(t) + n^{-1}$, since

$$A_{u,n}^{-1}(t) = \left(A_u(t)n(n - A_u(t))^{-1}\right)^{-1} = n^{-1}(n - A_u(t))(A_u(t))^{-1} = A_u^{-1}(t) + n^{-1}.$$

- $A_{u,n}^{-1}(t) - A_{v,n}^{-1}(s) = A_u^{-1}(t) - A_v^{-1}(s)$ as a direct consequence.
- $A_{u,n}^\gamma(t) A_u^{-\gamma}(t) = (1 + n^{-1}A_u(t))^\gamma$. (Yagi, 2010, Proposition 3.3)
- $\|A_{u,n}^\gamma(t) A_u^{-\gamma}(t)\|_{\mathcal{L}(X_0)} \leq C$ for some $C > 0$, and for each t we have that $A_{u,n}^\gamma(t) A_u^{-\gamma}(t) \to 1$ strongly in X_0 as $n \to \infty$. (Yagi, 2010, Proposition 3.4)

Combined, we can conclude

$$\begin{aligned} \|A_{u,n}^\gamma(t) \left[A_{u,n}^{-1}(t) - A_{v,n}^{-1}(s)\right]\|_{\mathcal{L}(X_0)} &= \|A_{u,n}^\gamma(t) \left[A_u^{-1}(t) - A_v^{-1}(s)\right]\|_{\mathcal{L}(X_0)} \\ &\leq \|A_{u,n}^\gamma(t) A_u^{-\gamma}(t)\|_{\mathcal{L}(X_0)} \|A_u^\gamma(t) \left[A_u^{-1}(t) - A_v^{-1}(s)\right]\|_{\mathcal{L}(X_0)} \\ &\leq C \|A_u^\gamma(t) \left[A_u^{-1}(t) - A_v^{-1}(s)\right]\|_{\mathcal{L}(X_0)} \\ &\overset{(2.17)}{\leq} CN_1 \|u(t) - v(s)\|_{X_\theta} + CN_2 |t-s|^\mu. \end{aligned}$$

□

Then we can state the lemma we need for the T_1 and T_2 estimates.

Lemma 2.3.12. *For $v_1, v_2 \in B_r^T(u_0)$, and $\psi \in [-1, 0)$, we can write the difference of the evolution operators as*

$$(G_{v_1}(t,s) - G_{v_2}(t,s))A_{v_2}^{\psi}(s) = \int_s^t G_{v_1}(t,\tau)A_{v_1}(\tau)\left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{\psi}(s)\, d\tau, \tag{2.31}$$

which gives the following estimates for $\phi \in [0,\gamma)$:

$$\|A_{v_1}^{\phi}(t)((G_{v_1}(t,0) - G_{v_2}(t,0))u_0\|_{X_0} \le CL_{u_0}t^{\gamma+\mu+\sigma-\phi-1}\|v_1 - v_2\|_{Y_{\{0\}}}, \tag{2.32}$$

$$\left\|A_{v_1}^{\phi}(t)\int_0^t (G_{v_1}(t,s) - G_{v_2}(t,s))F_{v_1}(s)ds\right\|_{X_0} \le \int_0^t (t-\tau)^{\gamma-\phi-1}\tau^{\mu}\left\|A_{v_2}(\tau)U_{F_{v_1},v_2}(\tau)\right\|_{X_0} d\tau\|v_1 - v_2\|_{Y_{\{0\}}}. \tag{2.33}$$

Proof. Take the Yosida approximations $A_{v_1,n}(t) := A_{v_1}(t)nR(n, A_{v_1}(t))$ and $A_{v_2,n}(t) := A_{v_2}(t)nR(n, A_{v_2}(t))$, and take $G_{v_1,n}$ and $G_{v_2,n}$ as their respective evolution families. Then

$$\begin{aligned}
&\int_s^t G_{v_1,n}(t,\tau)A_{v_1,n}(\tau)\left[A_{v_1,n}^{-1}(\tau) - A_{v_2,n}^{-1}(\tau)\right]A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\, d\tau \\
&= \int_s^t \left(-G_{v_1,n}(t,\tau)A_{v_1,n}(\tau)G_{v_2,n}(\tau,s) + G_{v_1,n}(t,\tau)A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)\right)A_{v_2,n}^{\psi}(s)\, d\tau \\
&\overset{(*)}{=} \int_s^t \left(-\left(\frac{\partial}{\partial\tau}G_{v_1,n}(t,\tau)\right)G_{v_2,n}(\tau,s) - G_{v_1,n}(\tau,s)\left(\frac{\partial}{\partial\tau}G_{v_2,n}(\tau,s)\right)\right)A_{v_2,n}^{\psi}(s)\, d\tau \\
&\overset{(\%)}{=} \left[-G_{v_1,n}(t,\tau)G_{v_2,n}(\tau,s)\right]_s^t A_{v_2,n}^{\psi}(s) \\
&= (G_{v_1,n}(t,s) - G_{v_2,n}(t,s))A_{v_2,n}^{\psi}(s).
\end{aligned}$$

Note on step $(*)$ we use Theorem 1.2.5 and Eq. (1.9), and on step (%) the product rule from Lemma A.3.1, where both $\tau \mapsto G_{v_1,n}(t,\tau)$ and $\tau \mapsto G_{v_2,n}(\tau,s)$ are bounded operators with bounded derivatives. To complete the proof, take the limit $n \to \infty$ and use DCT with the following bound (where we keep in mind $\psi < 0$):

$$\begin{aligned}
&\left\|G_{v_1,n}(t,\tau)A_{v_1,n}(\tau)\left[A_{v_1,n}^{-1}(\tau) - A_{v_2,n}^{-1}(\tau)\right]A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\right\|_{X_0} \\
&= \left\|G_{v_1,n}(t,\tau)A_{v_1,n}^{1-\gamma}(\tau)A_{v_1,n}^{\gamma}(\tau)\left[A_{v_1,n}^{-1}(\tau) - A_{v_2,n}^{-1}(\tau)\right]A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\right\|_{X_0} \\
&\overset{(1.10)}{\le} C_1(t-\tau)^{\gamma-1}\left\|A_{v_1,n}^{\gamma}(\tau)\left[A_{v_1,n}^{-1}(\tau) - A_{v_2,n}^{-1}(\tau)\right]A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\right\|_{X_0} \\
&\overset{(2.3.11)}{\le} C_2(t-\tau)^{\gamma-1}\|v_1(\tau) - v_2(\tau)\|_{X_\theta}\left\|A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\right\|_{X_0} \\
&\overset{v\in B_r^T(u_0)}{\le} C_2(t-\tau)^{\gamma-1}\tau^{\mu}\left\|A_{v_2,n}(\tau)G_{v_2,n}(\tau,s)A_{v_2,n}^{\psi}(s)\right\|_{X_0}\|v_1 - v_2\|_{Y_{\{0\}}} \\
&\overset{(1.10)}{\le} C_3(t-\tau)^{\gamma-1}\tau^{\mu}(\tau-s)^{-1-\psi}\|v_1 - v_2\|_{Y_{\{0\}}}.
\end{aligned}$$

Since this function is integrable over τ on (s,t), DCT can indeed be applied. To get the result Eq. (2.31) for G_{v_1} and G_{v_2} use the convergence of the Yosida approximations found in Proposition 2.5 from Schnaubelt (2004), which shows Eq. (2.31). A note for the first use of Eq. (1.10) is that we need $1-\gamma < \mu$, which is satisfied by $\gamma + \mu > 1$ from Eq. (2.18) and $\mu = \sigma - \theta$.

Now for $\phi \in [0,\gamma)$, we use that $u_0 \in X_{\sigma,p} \hookrightarrow D(A^{\sigma}(u_0,0))$ together with Eq. (2.31) to get

$$\begin{aligned}
&\|A_{v_1}^{\phi}(t)((G_{v_1}(t,0) - G_{v_2}(t,0))u_0\|_{X_0} = \|A_{v_1}^{\phi}(t)((G_{v_1}(t,0) - G_{v_2}(t,0))A_{v_2}^{-\sigma}(0)A_{v_2}^{\sigma}(0)u_0\|_{X_0} \\
&\overset{(2.31)}{\le} \int_0^t \|A_{v_1}^{\phi}(t)G_{v_1}(t,\tau)A_{v_1}(\tau)\left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,0)A_{v_2}^{-\sigma}(0)A^{\sigma}(u_0,0)u_0\|_{X_0}d\tau \\
&\overset{(1.10)}{\le} C_1\int_0^t (t-\tau)^{\gamma-\phi-1}\|A_{v_1}^{\gamma}(\tau)\left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,0)A_{v_2}^{-\sigma}(0)A^{\sigma}(u_0,0)u_0\|_{X_0}d\tau
\end{aligned}$$

$$\begin{aligned}
&\overset{(2.17)}{\leq} C_1 \int_0^t (t-\tau)^{\gamma-\phi-1}\tau^{\mu}\|A_{v_2}(\tau)G_{v_2}(\tau,0)A_{v_2}^{-\sigma}(0)A^{\sigma}(u_0,0)u_0\|_{X_0} d\tau \|v_1-v_2\|_{Y_{\{0\}}} \\
&\overset{(1.10)}{\leq} C_2 \int_0^t (t-\tau)^{\gamma-\phi-1}\tau^{\sigma+\mu-1}\|A^{\sigma}(u_0,0)u_0\|_{X_0} d\tau \|v_1-v_2\|_{Y_{\{0\}}} \\
&= C_2 L_{u_0} \int_0^t (t-\tau)^{\gamma-\phi-1}\tau^{\sigma+\mu-1} d\tau \|v_1-v_2\|_{Y_{\{0\}}} \\
&= C_2 L_{u_0} \int_0^1 (1-z)^{\gamma-\phi-1} z^{\sigma+\mu-1} t^{\gamma+\mu+\sigma-\phi-1} dz \|v_1-v_2\|_{Y_{\{0\}}} \leq C L_{u_0} t^{\gamma+\mu+\sigma-\phi-1}\|v_1-v_2\|_{Y_{\{0\}}}
\end{aligned}$$

For $\phi \in [0,\gamma)$, $\psi \in (1-\sigma-\gamma+\phi,\gamma]$ (which has the implicit assumption that $1+\phi<\sigma+2\gamma$, which holds by $1<\sigma+\gamma-\theta\leq\sigma+\gamma$ and $\phi<\gamma$,) note that if we assume $f\in L^p_{\omega_\alpha}((0,T);X_\psi)$, we get

$$\begin{aligned}
&\left\|A_{v_1}^{\phi}(t)\int_0^t (G_{v_1}(t,s)-G_{v_2}(t,s))f(s)ds\right\|_{X_0} = \left\|A_{v_1}^{\phi}(t)\int_0^t (G_{v_1}(t,s)-G_{v_2}(t,s))A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f(s)ds\right\|_{X_0} \\
&\overset{(2.31)}{=} \left\|\int_0^t\int_s^t A_{v_1}^{\phi}(t)G_{v_1}(t,\tau)A_{v_1}(\tau)\left[A_{v_1}^{-1}(\tau)-A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f(s)\ d\tau\ ds\right\|_{X_0}. \quad (\&)
\end{aligned}$$

To show this is bounded, note that for $1=\frac{1}{r}+\frac{1}{r'}=\frac{1}{r}+\frac{1}{p}+\frac{1}{h}$,

$$\begin{aligned}
&\int_0^t\int_s^t \left\|A_{v_1}^{\phi}(t)G_{v_1}(t,\tau)A_{v_1}(\tau)\left[A_{v_1}^{-1}(\tau)-A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f(s)\right\|_{X_0} d\tau\ ds \\
&\overset{(1.10)}{\leq} C_1\int_0^t\int_s^t (t-\tau)^{\gamma-\phi-1}\left\|A_{v_1}^{\gamma}(\tau)\left[A_{v_1}^{-1}(\tau)-A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f(s)\right\|_{X_0} d\tau\ ds \\
&\overset{(2.17)}{\leq} C_1\int_0^t\int_s^t (t-\tau)^{\gamma-\phi-1}\tau^{\mu}\left\|A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f(s)\right\|_{X_0} d\tau\ ds\cdot\|v_1-v_2\|_{Y_{\{0\}}} \\
&\overset{(1.10)}{\leq} C_2\int_0^t\int_s^t (t-\tau)^{\gamma-\phi-1}\tau^{\mu}(\tau-s)^{\psi-1}\left\|A_{v_2}^{\psi}(s)f(s)\right\|_{X_0} d\tau\ ds\cdot\|v_1-v_2\|_{Y_{\{0\}}} \\
&\leq C_3 t^{\mu}\int_0^t (t-s)^{\gamma+\psi-\phi-1}\left\|A_{v_2}^{\psi}(s)f(s)\right\|_{X_0} ds\|v_1-v_2\|_{Y_{\{0\}}} \\
&\overset{\text{Hölder}}{\leq} C_3 t^{\mu}\|z\mapsto(t-z)^{\gamma+\psi-\phi-1}\|_{L^r(0,t)}\left\|\tau\mapsto\tau^{-\alpha}\|\tau^{\alpha}A_{v_2}^{\psi}(\tau)f(\tau)\|_{X_0}\right\|_{L^{r'}(0,t)}\|v_1-v_2\|_{Y_{\{0\}}} \\
&\overset{\text{Hölder}}{\leq} C_3 t^{\mu}t^{\gamma+\psi-\phi-1+\frac{1}{r}}t^{-\alpha+\frac{1}{h}}\|A_{v_2}^{\psi}f\|_{L^p_{\omega_\alpha}((0,t);X_0)}\|v_1-v_2\|_{Y_{\{0\}}} \\
&\leq C_3 t^{\mu+\gamma+\psi-\phi-\alpha-\frac{1}{p}}\|A_{v_2}^{\psi}f\|_{L^p_{\omega_\alpha}((0,T);X_0)}\|v_1-v_2\|_{Y_{\{0\}}}<\infty
\end{aligned} \tag{2.34}$$

Here we used $\gamma+\psi-\phi-1>-\frac{1}{r}$ and $-\alpha>-\frac{1}{h}$, which gives as a condition

$$-\alpha>-\frac{1}{h}=-1+\frac{1}{r}+\frac{1}{p}>-(\gamma+\psi-\phi)+\frac{1}{p},$$

so $\sigma+\gamma+\psi>1+\phi$. This is true by $\psi>1-\sigma-\gamma+\phi$. This means we can now determine (&) is bounded, we can apply Fubini for any $f\in L^p_{\omega_\alpha}((0,T);X_\psi)$.

For $F_{v_1}\in E_0$ by Lemma 2.3.5, meaning by the $L^p_{\omega_\alpha}$-maximal regularity from Eq. (2.22) that $A_{v_2}U_{F_{v_1},v_2}\in E_0$, note that $\exists(f_n)_{n\in\mathbb{N}}\subseteq L^p_{\omega_\alpha}((0,T);X_\psi)$ s.t. $f_n\to F_{v_1}$ in the E_0-norm (and therefore also pointwise a.e.) Since DCT and Fubini can be used by the upper bound found in Eq. (2.34), we conclude

$$\begin{aligned}
&\left\|A_{v_1}^{\phi}(t)\int_0^t (G_{v_1}(t,s)-G_{v_2}(t,s))F_{v_1}(s)ds\right\|_{X_0} \\
&\overset{\text{DCT}}{=} \lim_{n\to\infty}\left\|A_{v_1}^{\phi}(t)\int_0^t (G_{v_1}(t,s)-G_{v_2}(t,s))A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f_n(s)ds\right\|_{X_0} \\
&\overset{(2.31)}{=} \lim_{n\to\infty}\left\|\int_0^t\int_s^t A_{v_1}^{\phi}(t)G_{v_1}(t,\tau)A_{v_1}(\tau)\left[A_{v_1}^{-1}(\tau)-A_{v_2}^{-1}(\tau)\right]A_{v_2}(\tau)G_{v_2}(\tau,s)A_{v_2}^{-\psi}(s)A_{v_2}^{\psi}(s)f_n(s)\ d\tau\ ds\right\|_{X_0}
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{Fubini}}{=} \lim_{n\to\infty} \left\| \int_0^t \int_0^\tau A_{v_1}^{\phi}(t) G_{v_1}(t,\tau) A_{v_1}(\tau) \left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau) \right] A_{v_2}(\tau) G_{v_2}(\tau,s) f_n(s) \; ds \; d\tau \right\|_{X_0} \\
&\overset{\text{DCT}}{=} \left\| \int_0^t \int_0^\tau A_{v_1}^{\phi}(t) G_{v_1}(t,\tau) A_{v_1}(\tau) \left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau) \right] A_{v_2}(\tau) G_{v_2}(\tau,s) F_{v_1}(s) \; ds \; d\tau \right\|_{X_0} \\
&\le C \int_0^t (t-\tau)^{\gamma-\phi-1} \tau^{\mu} \left\| A_{v_2}(\tau) U_{F_{v_1},v_2}(\tau) \right\|_{X_0} d\tau \|v_1 - v_2\|_{Y\{0\}} \\
&\le \int_0^t \left\| A_{v_1}^{\phi}(t) G_{v_1}(t,\tau) A_{v_1}(\tau) \left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau) \right] A_{v_2}(\tau) \int_0^\tau G_{v_2}(\tau,s) F_{v_1}(s) \; ds \right\|_{X_0} d\tau \\
&\overset{(1.10)}{\le} C \int_0^t (t-\tau)^{\gamma-\phi-1} \left\| A_{v_1}^{\gamma}(\tau) \left[A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau) \right] A_{v_2}(\tau) \int_0^\tau G_{v_2}(\tau,s) F_{v_1}(s) \; ds \right\|_{X_0} d\tau \\
&\overset{(2.17)}{\le} C \int_0^t (t-\tau)^{\gamma-\phi-1} \tau^{\mu} \left\| A_{v_2}(\tau) \int_0^\tau G_{v_2}(\tau,s) F_{v_1}(s) \; ds \right\|_{X_0} d\tau \|v_1 - v_2\|_{Y\{0\}} \\
&= C \int_0^t (t-\tau)^{\gamma-\phi-1} \tau^{\mu} \left\| A_{v_2}(\tau) U_{F_{v_1},v_2}(\tau) \right\|_{X_0} d\tau \|v_1 - v_2\|_{Y\{0\}}.
\end{aligned}
$$

□

We will also set a lemma similar to Lemma 18.2.12 from Hytönen et al. (2024) that helps deal with Lipschitz estimates in F.

Lemma 2.3.13 (Lipschitz estimates). *For $v_1, v_2 \in B_r^T(u_0)$, $F_{v_1} - F_{v_2} \in E_0$ holds with the following estimates:*

$$
\begin{aligned}
\|F_{Tr}(v_1) - F_{Tr}(v_2)\|_{E_0} &\le L_{Tr} T^{1-\sigma} \|v_1 - v_2\|_{Y_{Tr}}, \\
\|F_c(v_1) - F_c(v_2)\|_{E_0} &\le L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + 2r^{\rho_j} \right] \|v_1 - v_2\|_{Y_j}.
\end{aligned}
$$

Here, $\delta_j = \rho_j(\alpha_j + \frac{1}{p_j})$.

Proof. Using Eq. (2.20) on F_{Tr}, we know that just as for Eq. (2.12),

$$
\|F_{Tr}(v_1(t)) - F_{Tr}(v_2(t))\|_{X_0} \le L_{Tr} \|v_1(t) - v_2(t)\|_{X_{\sigma,p}} \le L_{Tr} \|v_1 - v_2\|_{Y_{Tr}}.
$$

Applying $L^p_{\omega_\alpha}((0,T); X_0)$ therefore gives

$$
\begin{aligned}
\|F_{Tr}(v_1) - F_{Tr}(v_2)\|_{E_0} &\le L_{Tr} \|v_1 - v_2\|_{Y_{Tr}} \left(\int_0^T t^{\alpha p} ds \right)^{\frac{1}{p}} \\
&\le L_{Tr} T^{\alpha + \frac{1}{p}} \|v_1 - v_2\|_{Y_{Tr}} \\
&= L_{Tr} T^{1-\sigma} \|v_1 - v_2\|_{Y_{Tr}}.
\end{aligned}
$$

For the critical part,

$$
\begin{aligned}
\|F_c(v_1) - F_c(v_2)\|_{E_0} &\overset{(2.3.5)}{\le} L_c \sum_{j=1}^m \left[T^{\delta_j} + \|v_1\|_{Y_j}^{\rho_j} + \|v_2\|_{Y_j}^{\rho_j} \right] \|v_1 - v_2\|_{Y_j} \\
&\le L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + \|v_1 - z_{u_0}\|_{Y_j}^{\rho_j} + \|v_2 - z_{u_0}\|_{Y_j}^{\rho_j} \right] \|v_1 - v_2\|_{Y_j} \\
&\le L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + 2r^{\rho_j} \right] \|v_1 - v_2\|_{Y_j}.
\end{aligned}
$$

□

Estimate of $T_{1,\{0\}}$, $T_{1,Tr}$ and $T_{1,j}$:

$$\begin{aligned}
T_{1,\{0\}} = \| (G_{v_1}(\cdot,0) - G_{v_2}(\cdot,0))\, u_0\|_{C^{\mu}_{\{0\}}([0,T];X_\theta)} &= \sup_{t\in[0,T]} t^{-\mu}\|A^{\theta}_{v_1}(t)\,(G_{v_1}(t,0) - G_{v_2}(t,0))\,u_0 - 0\|_{X_0} \\
&\overset{(2.32)}{\leq} \sup_{t\in[0,T]} CL_{u_0} t^{-\mu} t^{\gamma+\mu+\sigma-\theta-1}\|v_1 - v_2\|_{Y_{\{0\}}} \\
&\leq CL_{u_0} T^{\gamma+\sigma-\theta-1}\|v_1 - v_2\|_{Y_{\{0\}}}, \\
T_{1,Tr} = \| (G_{v_1}(\cdot,0) - G_{v_2}(\cdot,0))\, u_0\|_{C([0,T];X_{\sigma,p})} &\leq \sup_{t\in[0,T]} D\|A^{\sigma}_{v_1}(t)\,(G_{v_1}(t,0) - G_{v_2}(t,0))\,u_0\|_{X_0} \\
&\overset{(2.32)}{\leq} \sup_{t\in[0,T]} CL_{u_0} t^{\gamma+\mu-1}\|v_1 - v_2\|_{Y_{\{0\}}} \\
&\leq CL_{u_0} T^{\gamma+\mu-1}\|v_1 - v_2\|_{Y_{\{0\}}}, \\
T_{1,j} = \| (G_{v_1}(\cdot,0) - G_{v_2}(\cdot,0))\, u_0\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T;X_{\beta_j})} &= \|t \mapsto A^{\beta_j}_{v_1}(t)\,(G_{v_1}(t,0) - G_{v_2}(t,0))\,u_0\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T;X_0)} \\
&\overset{(2.32)}{\leq} CL_{u_0}\|t \mapsto t^{\gamma+\mu+\sigma-\beta_j-1}\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)}\|v_1 - v_2\|_{Y_{\{0\}}} \\
&\leq CL_{u_0} T^{\gamma+\mu+\sigma-\beta_j-1+\alpha_j+\frac{1}{p_j}}\|v_1 - v_2\|_{Y_{\{0\}}}, \\
&\leq CL_{u_0} T^{\gamma+\mu+\beta_j^*-\beta_j-1}\|v_1 - v_2\|_{Y_{\{0\}}}.
\end{aligned}$$

The first and second equation require $\gamma + \sigma - 1 - \theta > 0$ and $\gamma + \mu - 1 > 0$, which is both true by $\mu = \sigma - \theta$ and Eq. (2.18). The last equation has an integrability condition of

$$\begin{aligned}
\gamma + \mu + \sigma - \beta_j - 1 + \alpha_j &> \frac{1}{p_j}, \\
\gamma + \mu + \frac{\alpha + \frac{1}{p}}{\rho_j + 1} + \sigma &> 1 + \beta_j, \\
\gamma + \mu + \frac{1 - \sigma}{\rho_j + 1} + \sigma &> 1 + \beta_j, \\
\gamma + \mu + \frac{1 + \rho_j\sigma}{\rho_j + 1} &> 1 + \beta_j, \\
\gamma + \mu + \beta_j^* &> 1 + \beta_j.
\end{aligned}$$

This is the same condition as Eq. (2.30) and holds for the same reason.

Estimate of $T_{2,\{0\}}$:

Set $r, h \in [1,\infty]$ s.t. $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} + \frac{1}{p} + \frac{1}{h}$

$$\begin{aligned}
T_{2,\{0\}} &= \left\| \int_0^{\cdot} (G_{v_1}(\cdot,s) - G_{v_2}(\cdot,s))\, F_{v_1}(s)\; ds \right\|_{C^{\mu}_{\{0\}}([0,T];X_\theta)} \\
&= \sup_{t\in[0,T]} t^{-\mu} \left\| A^{\theta}_{v_1}(t) \int_0^t (G_{v_1}(t,s) - G_{v_2}(t,s))\, F_{v_1}(s)\; ds - 0 \right\|_{X_0} \\
&\overset{(2.33)}{\leq} \sup_{t\in[0,T]} C_1 t^{-\mu} \int_0^t (t-\tau)^{\gamma-\theta-1}\tau^{\mu} \left\| A_{v_2}(\tau) U_{F_{v_1},v_2}(\tau)\right\|_{X_0} d\tau \|v_1 - v_2\|_{Y_{\{0\}}} \\
&\overset{\text{Hölder}}{\leq} \sup_{t\in[0,T]} C_1 t^{-\mu} \|z \mapsto (t-z)^{\gamma-\theta-1}\|_{L^r(0,t)} \left\| \tau \mapsto \tau^{\mu} \left\| A_{v_2}(\tau) U_{F_{v_1},v_2}(\tau)\right\|_{X_0} \right\|_{L^{r'}(0,t)} \|v_1 - v_2\|_{Y_{\{0\}}} \\
&\overset{\text{Hölder}}{\leq} \sup_{t\in[0,T]} C_1 t^{-\mu} \|z \mapsto (t-z)^{\gamma-\theta-1}\|_{L^r(0,t)} \left\| \tau \mapsto \tau^{\mu-\alpha}\right\|_{L^h(0,t)} \left\| A_{v_2} U_{F_{v_1},v_2}\right\|_{L^p_{\omega_\alpha}((0,T);X_0)} \|v_1 - v_2\|_{Y_{\{0\}}}
\end{aligned}$$

$$\leq \sup_{t\in[0,T]} C_1 t^{-\mu} \cdot t^{\gamma-\theta-1+\frac{1}{r}} \cdot t^{\mu-\alpha+\frac{1}{h}} \left\|U_{F_{v_1},v_2}\right\|_{\mathcal{E}^p_{\alpha,v_2}((0,T))} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{(2.22)}{\leq} C_2 T^{\gamma+\sigma-1-\theta} \|F_{v_1}\|_{E_0} \|v_1-v_2\|_{Y_{\{0\}}}.$$

The integrability conditions are $\gamma-\theta-1>-\frac{1}{r}$ and $\mu-\alpha>-\frac{1}{h}$, which gives

$$\gamma-\theta > 1-\frac{1}{r} = \frac{1}{p}+\frac{1}{h} > \frac{1}{p}+\alpha-\mu.$$

This is true by reduction to $\gamma+\sigma+\mu>1+\theta$, however the step labeled with (2.22) still requires $\gamma+\sigma>1+\theta$ so that $\sup_{t\in[0,T]} t^{\gamma+\sigma-1-\theta}\downarrow 0$ as $T\downarrow 0$.

Estimate of $T_{2,Tr}$:

Again set $r,h\in[1,\infty]$ s.t. $1=\frac{1}{r}+\frac{1}{r'}=\frac{1}{r}+\frac{1}{p}+\frac{1}{h}$

$$T_{2,Tr} = \left\|\int_0^{\cdot} \left(G_{v_1}(\cdot,s)-G_{v_2}(\cdot,s)\right) F_{v_1}(s)\; ds\right\|_{C([0,T];X_{\sigma,p})}$$
$$\leq \sup_{t\in[0,T]} D\left\|A^{\sigma}_{v_1}(t)\int_0^t \left(G_{v_1}(t,s)-G_{v_2}(t,s)\right) F_{v_1}(s)\; ds\right\|_{X_0}$$
$$\overset{(2.33)}{\leq} \sup_{t\in[0,T]} C_1 \int_0^t (t-\tau)^{\gamma-\sigma-1}\tau^{\mu}\left\|A_{v_2}(\tau)U_{F_{v_1},v_2}(\tau)\right\|_{X_0} d\tau \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{\text{Hölder}}{\leq} \sup_{t\in[0,T]} C_1 \|z\mapsto (t-z)^{\gamma-\sigma-1}\|_{L^r(0,t)} \left\|\tau\mapsto\tau^{\mu}\left\|A_{v_2}(\tau)U_{F_{v_1},v_2}(\tau)\right\|_{X_0}\right\|_{L^{r'}(0,t)} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{\text{Hölder}}{\leq} \sup_{t\in[0,T]} C_1 \|z\mapsto (t-z)^{\gamma-\sigma-1}\|_{L^r(0,t)} \left\|\tau\mapsto\tau^{\mu-\alpha}\right\|_{L^h(0,t)} \left\|A_{v_2}U_{F_{v_1},v_2}\right\|_{L^p_{\omega_\alpha}((0,T);X_0)} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\leq \sup_{t\in[0,T]} C_1 t^{\gamma-\sigma-1+\frac{1}{r}} \cdot t^{\mu-\alpha+\frac{1}{h}} \left\|U_{F_{v_1},v_2}\right\|_{\mathcal{E}^p_{\alpha,v_2}} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{(2.22)}{\leq} C_2 T^{\gamma+\mu-1} \|F_{v_1}\|_{E_0} \|v_1-v_2\|_{Y_{\{0\}}}.$$

The integrability conditions are $\gamma-\sigma-1>-\frac{1}{r}$ and $\mu-\alpha>-\frac{1}{h}$, which gives

$$\gamma-\sigma > 1-\frac{1}{r} = \frac{1}{p}+\frac{1}{h} > \frac{1}{p}+\alpha-\mu.$$

This is again true by reducing to $\gamma+\mu>1$.

Estimate of $T_{2,j}$:

Set $r,h\in[1,\infty)$ s.t. $\frac{1}{p_j}=\frac{1}{r}-(\gamma-\beta_j)=\frac{1}{p}+\frac{1}{h}-(\gamma-\beta_j)$. Then

$$T_{2,j} = \left\|\int_0^{\cdot} \left(G_{v_1}(\cdot,s)-G_{v_2}(\cdot,s)\right) F_{v_1}(s)\; ds\right\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_{\beta_j})}$$
$$= \left\|t\mapsto A^{\beta_j}(t)\int_0^t \left(G_{v_1}(t,s)-G_{v_2}(t,s)\right) F_{v_1}(s)\; ds\right\|_{L^{p_j}_{\omega_{\alpha_j}}((0,T);X_0)}$$
$$\overset{(2.33)}{\leq} C_1 \left\|t\mapsto \int_0^t (t-\tau)^{\gamma-\beta_j-1}\tau^{\mu}\left\|A_{v_2}(\tau)U_{F_{v_1},v_2}(\tau)\right\|_{X_0} d\tau\right\|_{L^{p_j}_{\omega_{\alpha_j}}(0,T)} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{(A.2.1)}{\leq} C_2 \left\|\tau\mapsto\tau^{\mu}\left\|A_{v_2}(\tau)U_{F_{v_1},v_2}(\tau)\right\|_{X_0}\right\|_{L^r_{\omega_{\alpha_j}}(0,T)} \|v_1-v_2\|_{Y_{\{0\}}}$$
$$\overset{\text{Hölder}}{\leq} C_2 \left\|\tau\mapsto\tau^{\mu+\alpha_j-\alpha}\right\|_{L^h(0,T)} \left\|A_{v_2}U_{F_{v_1},v_2}\right\|_{L^p_{\omega_\alpha}((0,T);X_0)} \|v_1-v_2\|_{Y_{\{0\}}}$$

$$\leq C_2 T^{\mu+\alpha_j-\alpha+\frac{1}{h}} \left\|U_{F_{v_1},v_2}\right\|_{\mathcal{E}^p_{\alpha,v_2}((0,T))} \|v_1 - v_2\|_{Y_{\{0\}}}$$
$$\overset{(2.22)}{\leq} C_3 T^{\mu+\alpha_j-\alpha+\gamma-\beta_j+\frac{1}{p_j}-\frac{1}{p}} \|F_{v_1}\|_{E_0} \|v_1 - v_2\|_{Y_{\{0\}}}$$
$$= C_3 T^{\gamma+\mu+\beta_j^*-1-\beta_j} \|F_{v_1}\|_{E_0} \|v_1 - v_2\|_{Y_{\{0\}}}$$

This holds for the same reasons Eq. (2.30) holds.

Estimate of $T_{3,\{0\}}$, $T_{3,Tr}$ and $T_{3,j}$:

Note this term is equal to $U_{F_{v_1}-F_{v_2},v_2}$ for $v_2 \in B_r^T(u_0)$ and $F_{v_1} - F_{v_2} \in L^p_{\omega_\alpha}((0,T);X_0)$ by Lemma 2.3.5. Therefore using Lemma 2.3.9, we can conclude $T_{3,\{0\}} \leq C_{\{0\}}\|F_{v_1} - F_{v_2}\|_{E_0}$, $T_{3,Tr} \leq C_{Tr}\|F_{v_1} - F_{v_2}\|_{E_0}$ and $T_{3,j} \leq C_j\|F_{v_1} - F_{v_2}\|_{E_0}$.

Combining all estimates:

We will use Lemma 2.3.10 and Lemma 2.3.13, and apply them on the terms of $\|u_1 - u_2\|_{Y(0,T)}$ we have estimated. For $j \in \{1, \ldots, m\}$, we have

$$\begin{aligned}
\|u_1 - u_2\|_{Y_{\{0\}}} &\lesssim L_{u_0} T^{\gamma+\mu-1}\|v_1 - v_2\|_{Y_{\{0\}}} + T^{\gamma+\mu-1}\|F_{v_1}\|_{E_0}\|v_1 - v_2\|_{Y_{\{0\}}} + \|F_{v_1} - F_{v_2}\|_{E_0} \\
&\leq T^{\gamma+\mu-1}\|v_1 - v_2\|_{Y_{\{0\}}}(L_{u_0} + T^{1-\sigma}[\|F_{Tr}(u_0)\|_{X_0} + L_{Tr} r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}] \\
&\qquad + \|F_c(z_{u_0})\|_{E_0} + L_c r \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + r^{\rho_j}\right]) \\
&\quad + L_{Tr} T^{1-\sigma}\|v_1 - v_2\|_{Y_{Tr}} + L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + 2r^{\rho_j}\right]\|v_1 - v_2\|_{Y_j}, \\
\|u_1 - u_2\|_{Y_{Tr}} &\lesssim T^{\gamma+\mu-1}\|v_1 - v_2\|_{Y_{\{0\}}}(L_{u_0} + T^{1-\sigma}[\|F_{Tr}(u_0)\|_{X_0} + L_{Tr} r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}] \\
&\qquad + \|F_c(z_{u_0})\|_{E_0} + L_c r \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + r^{\rho_j}\right]) \\
&\quad + L_{Tr} T^{1-\sigma}\|v_1 - v_2\|_{Y_{Tr}} + L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + 2r^{\rho_j}\right]\|v_1 - v_2\|_{Y_j}, \\
\|u_1 - u_2\|_{Y_j} &\lesssim L_{u_0} T^{\gamma+\mu+\beta_j^*-\beta_j-1}\|v_1 - v_2\|_{Y_{\{0\}}} + T^{\gamma+\mu+\beta_j^*-1-\beta_j}\|F_{v_1}\|_{E_0}\|v_1 - v_2\|_{Y_{\{0\}}} + \|F_{v_1} - F_{v_2}\|_{E_0} \\
&\leq T^{\gamma+\mu+\beta_j^*-1-\beta_j}\|v_1 - v_2\|_{Y_{\{0\}}}(L_{u_0} + T^{1-\sigma}[\|F_{Tr}(u_0)\|_{X_0} + L_{Tr} r + L_{Tr}\|z_{u_0} - u_0\|_{Y_{Tr}}] \\
&\qquad + \|F_c(z_{u_0})\|_{E_0} + L_c r \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + r^{\rho_j}\right]) \\
&\quad + L_{Tr} T^{1-\sigma}\|v_1 - v_2\|_{Y_{Tr}} + L_c \sum_{j=1}^m \left[T^{\delta_j} + 2\|z_{u_0}\|_{Y_j}^{\rho_j} + 2r^{\rho_j}\right]\|v_1 - v_2\|_{Y_j}.
\end{aligned}$$

So,

$$\|u_1 - u_2\|_{Y(0,T)} \leq L_{T,r,u_0}\|v_1 - v_2\|_{Y(0,T)}.$$

The contraction mapping property requires $L_{T,r,u_0} < 1$, so T and r have to be taken small enough such that this holds. We shrink the size of the constants by shrinking T to an acceptable size $T_{Lipschitz}$ as well as $r_{Lipschitz}$. This possible because:

- $T^{\gamma+\mu-1}$, $T^{1-\sigma}$, $T^{\gamma+\mu+\beta_j^*-1-\beta_j}$, $T^{\delta_j} \downarrow 0$ as $T \downarrow 0$ since these powers are positive
- $\|F_{Tr}(u_0)\|_{X_0} < \infty$ since $u_0 \in X_{\sigma,p}$,
- $\|z_{u_0} - u_0\|_{Y_{Tr}} \downarrow 0$ as $T \downarrow 0$ by strong continuity from Proposition 2.2.8 of Lunardi (1995),
- $\|z_{u_0}\|_{Y_j} \downarrow 0$ as $T \downarrow 0$ since $z_{u_0} \in Y_j$,
- $\|F_c(z_{u_0})\|_{E_0} \downarrow 0$ as $T \downarrow 0$ since $F_c(z_{u_0}) \in E_0$.

2.3.3 Existence and Uniqueness in $\mathcal{AT}^p_\alpha(0,T) \cap Y(0,T)$

We can now apply the Banach Fixed Point Theorem to conclude that Φ has a unique fixed point u in the fixed point space $B^T_r(u_0) \subseteq Y(0,T) \cap B_1(0; C^\mu([0,T]; X_\theta))$, where T is taken as the minimum of T_{self} and $T_{Lipschitz}$ and r is taken as the minimum of r_{self} and $r_{Lipschitz}$. Since $u = \Phi(u) \in \mathcal{E}^p_{\alpha,u}((0,T))$ by Lemma 2.3.6 and so $u \in \mathcal{Q}^p_\alpha((0,T))$ by Eq. (2.15), this gives us a mild solution u to problem Eq. (2.13), and a candidate for the unique solution in $\mathcal{AT}^p_\alpha(0,T) \cap Y(0,T)$ we want. We will now proof u is indeed unique in the $\mathcal{AT}^p_\alpha(0,\tilde{T}) \cap Y(0,\tilde{T})$ for a smaller $\tilde{T} \leq T$. Again, we set up a proof similar to the uniqueness result in the proof of Theorem 18.2.6 in Hytönen et al. (2024), as we have also done for the proof of Theorem 2.2.3. Set

$$\tilde{T} := \inf\left\{t \in [0,T] : \|u - z_{u_0}\|_{Y(0,t)} \geq \frac{r}{2}\right\},$$

where we use the convention $\inf \emptyset = T$.

Take $\tilde{u} \in \mathcal{AT}^p_\alpha(0,\tilde{T}) \cap Y(0,\tilde{T})$ as some mild solution to the problem Eq. (2.13) on $(0,\tilde{T})$. Let

$$\tau_{\tilde{u}} := \inf\left\{t \in [0,\tilde{T}] : \|\tilde{u} - z_{u_0}\|_{Y(0,t)} \geq r, [\tilde{u}]_{C^\mu([0,t];X_\theta)} \geq 1\right\},$$

where now the convention $\inf \emptyset = \tilde{T}$ is used. Then we can view $\tilde{u}$ on the interval $[0,\tau_{\tilde{u}}]$ to see

$$\tilde{u}|_{[0,\tau_{\tilde{u}}]} \in B^{\tau_{\tilde{v}}}_r(u_0),$$

which by the proof of the existence and uniqueness with $\tau_{\tilde{u}} \leq T$ allows us to conclude

$$\tilde{u}|_{[0,\tau_{\tilde{u}}]} = u|_{[0,\tau_{\tilde{u}}]}.$$

Then note the following holds:

$$\|\tilde{u} - z_{u_0}\|_{Y(0,\tau_{\tilde{u}})} = \|u - z_{u_0}\|_{Y(0,\tau_{\tilde{u}})} \leq \|u - z_{u_0}\|_{Y(0,\tilde{T})} < r,$$

$$\sup_{0 \leq s < t \leq \tau_{\tilde{u}}} \frac{\|\tilde{u}(t) - \tilde{u}(s)\|_{X_\theta}}{(t-s)^\mu} = \sup_{0 \leq s < t \leq \tau_{\tilde{u}}} \frac{\|u(t) - u(s)\|_{X_\theta}}{(t-s)^\mu} \leq \sup_{0 \leq s < t \leq \tilde{T}} \frac{\|u(t) - u(s)\|_{X_\theta}}{(t-s)^\mu} \leq 1.$$

Therefore, we can conclude $\tilde{T} = \tau_{\tilde{u}}$, and so u is unique in the full space $\mathcal{AT}^p_\alpha(0,\tilde{T}) \cap Y(0,\tilde{T})$. □

2.4 Examples of quasi-linear problems

In this section, we cover the model Neumann problem from Eq. (1) with a few iterations. We will see that we can handle a simplified version of the problem, but unfortunately, not the original problem as intended.

2.4.1 Model problem

Consider the problem

$$\begin{cases} \partial_t u(t,x) - A(u(t,x))u(t,x) &= F(u(t,x)), & t \in (0,T),\ x \in \Omega, \\ B(u(t,x))u(t,x) &= 0 & t \in (0,T),\ x \in \partial\Omega, \\ u(0,x) &= u_0(x), & x \in \Omega. \end{cases}$$

Here, $X_0 := L^q(\mathbb{R}^d)$ and $\Omega \subseteq \mathbb{R}^d$ is a smooth domain. We consider $F(u) = -|u|^n$ for $n \in \mathbb{N}$, and $A(u)$ and $B(u)$ defined as follows:

$$[A(v(t,\cdot))u(t,\cdot)](x) = \sum_{i,j=1}^d \partial_i \left[a_{ij}(v(t,\cdot))\partial_j u(t,\cdot)\right](x),$$

$$[B(v(t,\cdot))u(t,\cdot)](x) = \sum_{i,j=1}^d n_i(x) a_{ij}(v(t,x))\partial_j[u(t,\cdot)](x).$$

Here, $a_{ij} \in C^2(L^q(\mathbb{R}^d);\mathbb{R})$ are the matrix coefficients that define the operator A, and $\mathbf{n}(x)$ is the outward vector on the boundary $\partial\Omega$. We will consider domains $D(A(u,x,t)) \subseteq W^{2,q}_{B(u(t,x))}(\Omega)$. We will have some additional assumptions on A: it must be:

- bounded, meaning a constant $M > 0$ exists s.t.

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} |a_{ij}(u)| \leq M, \quad \forall i, j \in \{1, \dots, d\}$$

- elliptic, meaning a constant $\eta > 0$ exists s.t. for all $u, v \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a_{ij}(u) v_i v_j \geq \eta |v|^2, \quad \forall x \in \Omega,$$

Let us show that we meet the conditions set for our local well-posedness result

First assumption: structure of spaces and Acquistapace Terreni

Let us check the conditions for Assumption 2.3.2.

For the existence of γ_0, we need to look for which $X_{\gamma_0}, = [L^q(\Omega), W^{2,q}_{B(u(t,x))}(\Omega)]_{\gamma_0}$ is a space which is independent of x and t. For this we can use the result that $X_{\gamma_0} = H^{2\gamma_0,q}(\Omega)$ if $2\theta - \frac{1}{q} < 1$, where $H^{s,q}(\Omega)$ is a time- and space-independent Bessel potential space. This gives us a value of $\gamma_0 < \frac{1}{2} + \frac{1}{2q}$. If q is close to 1, we can choose γ_0 close to 1.

Secondly, we should note $L^q(\Omega)$ is a UMD space, and that $A(u)$ is uniformly R-sectioral according to Theorem 8.2 of Denk, Hieber, and Prüss (2003) (see also Example 5.1 of Di Giorgio, Lunardi, and Schnaubelt (2005)) for all $u \in B_R(0; C^\mu([0,T]; H^{2\theta,q}(\Omega)))$ for some $\mu, \theta, R > 0$ s.t. $\theta + \mu = \sigma$ and $\mu + \gamma > 1$, according to Lemma 2.3.6.

Lastly we will look for a possible γ and θ for the Lipschitz assumption

$$\left\| A^\gamma(u) \left[A^{-1}(u) - A^{-1}(v)\right] \right\|_{\mathscr{L}(L^q(\Omega))} \leq N \|u - v\|_{H^{2\theta,q}(\Omega)}.$$

This follows because A is an autonomous quasi-linear operator, so $N_2 = 0$ from the assumption Eq. (2.17). The space $X_\theta = H^{2\theta,q}(\Omega)$ will have to be a space that fits our needs, so we should choose it carefully based on the following calculations. Assume $u, v \in H^{2\theta,q}(\Omega)$. Note that we can follow the Integration By Parts steps from the proof of the (AT) conditions in Example 2.8 of Schnaubelt (2004). To that end, introduce the realisation A_q of the operator A with the domain

$$D(A_q(u)) = \{f \in W^{2,q}(\Omega) : B(u)f = 0 \text{ on } \partial\Omega\}.$$

For this realisation, $A_q^*(u(x)) = A_{q'}(u(x))$. For $f \in L^q(\Omega)$ and $g \in D((A_q^*(u))^\gamma)$, set $\tilde{f}(x) = (A_q^{-1}(u(x)) - A_q^{-1}(v(x)))f(x) \in W^{2,q}(\Omega)$ and $\tilde{g}(x) = (A_q^*(u(x)))^{\gamma-1} g(x) \in D(A_{q'}(u(x)))$. Then we can write, using Integration By Parts (IBP),

$$\begin{aligned}
&\langle (A_q^{-1}(u) - A_q^{-1}(v))f, (A_q^*(u))^\gamma g \rangle \\
&= \int_\Omega \tilde{f}(x) A_q(u(x)) \tilde{g}(x) \, dx \\
&= \int_\Omega \tilde{f}(x) \nabla \cdot \left[\sum_{j=1}^{d} a_{ij}(u(x)) \partial_j \tilde{g}(x) \right]_{i=1}^{d} dx \\
&\overset{\text{IBP}}{=} \int_{\partial\Omega} \tilde{f}(x) \left[\sum_{j=1}^{d} a_{ij}(u(x)) \partial_j \tilde{g}(x) \right]_{i=1}^{d} \cdot \mathbf{n} \, dS - \int_\Omega \nabla \tilde{f}(x) \cdot \left[\sum_{j=1}^{d} a_{ij}(u(x)) \partial_j \tilde{g}(x) \right]_{i=1}^{d} dx \\
&= \int_{\partial\Omega} \tilde{f}(x) \sum_{i,j=1}^{d} a_{ij}(u(x)) n_i \partial_j \tilde{g}(x) \, dS - \int_\Omega \sum_{i,j=1}^{d} a_{ij}(u(x)) \partial_i \tilde{f}(x) \partial_j \tilde{g}(x) dx \\
&= \int_{\partial\Omega} \tilde{f}(x) B(u(x)) \tilde{g}(x) \, dS - \int_\Omega \nabla \tilde{g}(x) \cdot \left[\sum_{j=1}^{d} a_{ij}(u(x)) \partial_j \tilde{f}(x) \right]_{i=1}^{d} dx
\end{aligned}$$

$$\begin{aligned}
&\overset{\tilde g\in D(A_{q'})}{=} 0 - \int_\Omega \nabla\tilde g(x)\cdot\left[\sum_{i=1}^d a_{ij}(u(x))\partial_i\tilde f(x)\right]_{j=1}^d dx\\
&\overset{\text{IBP}}{=} \int_\Omega \tilde g(x)\nabla\cdot\left[\sum_{i=1}^d a_{ij}(u(x))\partial_i\tilde f(x)\right]_{j=1}^d dx - \int_{\partial\Omega}\tilde g(x)\left[\sum_{i=1}^d a_{ij}(u(x))\partial_i\tilde f(x)\right]_{j=1}^d\cdot\mathbf{n}\; dS\\
&= \int_\Omega \tilde g(x)A_q(u(x))\tilde f(x)\;dx - \int_{\partial\Omega}\tilde g(x)B(u(x))\tilde f(x)\;dS\\
&= \int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(v(x))f(x)\;dS - \int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(u(x))f(x)\;dS\\
&\quad + \int_\Omega \tilde g(x)A_q(u(x))(A_q^{-1}(u(x)) - A_q^{-1}(v(x)))f(x)\;dx\\
&\overset{(\%)}{=} \int_\Omega \tilde g(x)A_q(u(x))(A_q^{-1}(u(x)) - A_q^{-1}(v(x)))f(x)\;dx\\
&\quad + \int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(v(x))f(x)\;dS\\
&\overset{(*)}{=} \int_\Omega \tilde g(x)(A_q(v(x)) - A_q(u(x)))A_q^{-1}(v(x))f(x)\;dx\\
&\quad + \int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(v(x))f(x)\;dS.
\end{aligned}$$

In step (%), we use the fact that A_q^{-1} maps into $D(A_q)$ to conclude that $A_q^{-1}(u)f \in D(A_q(u))$, so that

$$\int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(u(x))f(x)\;dS = 0.$$

In step (∗), we use that $A(u)(A^{-1}(u) - A^{-1}(v))h = (A(v) - A(u))A^{-1}(v)h$ for $h \in W^{2,p}(\Omega)$. This follows from applying the operators to $h = A(v)\tilde h$ and noting

$$\begin{aligned}
A(u)(A^{-1}(u) - A^{-1}(v))h &= A(u)(A^{-1}(u) - A^{-1}(v))A(v)\tilde h\\
&= [I - A(u)A^{-1}(v)]A(v)\tilde h\\
&= (A(v) - A(u))\tilde h\\
&= (A(v) - A(u))A^{-1}(v)h
\end{aligned}$$

Using $\dot f(x) := A_q^{-1}(v(x))f(x)$ and $T_{Bdr} =: \int_\Omega \tilde g(x)B(u(x))A_q^{-1}(v(x))f(x)\;dx$, we can write

$$\begin{aligned}
&\langle (A_q^{-1}(u) - A_q^{-1}(v))f, (A_q^*(u))^\gamma g\rangle\\
&= \int_\Omega \tilde g(x)(A_q(v(x)) - A_q(u(x)))\dot f(x)\;dx + T_{Bdr}\\
&= \int_\Omega \tilde g(x)\nabla\cdot\left[\sum_{i=1}^d (a_{ij}(v(x)) - a_{ij}(u(x)))\partial_i\dot f(x)\right]_{j=1}^d dx + T_{Bdr}\\
&\overset{\text{IBP}}{=} \int_{\partial\Omega}\tilde g(x)\left[\sum_{i=1}^d (a_{ij}(v(x)) - a_{ij}(u(x)))\partial_i\dot f(x)\right]_{j=1}^d\cdot\mathbf{n}\;dS + T_{Bdr}\\
&\quad - \int_\Omega \nabla\tilde g(x)\cdot\left[\sum_{i=1}^d (a_{ij}(v(x)) - a_{ij}(u(x)))\partial_i\dot f\right]_{j=1}^d dx\\
&= \int_{\partial\Omega}\tilde g(x)B(v(x))A_q^{-1}(v(x))f(x)\;dS - \int_{\partial\Omega}\tilde g(x)B(u(x))A_q^{-1}(v(x))f(x)\;dS + T_{Bdr}\\
&\quad + \sum_{i,j=1}^d \int_\Omega (a_{ij}(u(x)) - a_{ij}(v(x)))\partial_i\dot f(x)\partial_j\tilde g(x)\;dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega} \tilde{g}(x) B(v(x)) A_q^{-1}(v(x)) f(x)\; dS - T_{Bdr} + T_{Bdr} \\
&\quad + \sum_{i,j=1}^{d} \int_\Omega (a_{ij}(u(x)) - a_{ij}(v(x))) \partial_i \dot{f}(x) \partial_j \tilde{g}(x)\; dx \\
&\overset{(\%)}{=} \sum_{i,j=1}^{d} \int_\Omega (a_{ij}(u(x)) - a_{ij}(v(x))) \cdot [\partial_i A_q^{-1}(v) f](x) \cdot [\partial_j (A_q^*(u))^{\gamma-1} g](x) dx,
\end{aligned}
$$

where (%) again refers to $\int_{\partial\Omega} \tilde{g}(x) B(v(x)) A_q^{-1}(v(x)) f(x)\; dS = 0$.

From here we follow the steps of page 152 of Yagi (2010), where we also use $\gamma \in [\frac{1}{2}, \frac{1+q}{2q})$:

$$
\begin{aligned}
&|\langle (A_q^{-1}(u) - A_q^{-1}(v)) f, (A_q^*(u))^\gamma g \rangle| \\
\leq\ & \left\| (a_{ij}(u) - a_{ij}(v)) [\partial_i A_q^{-1}(v) f] \right\|_{\overset{\circ}{H}{}^{2\gamma-1,q}(\Omega)} \left\| \partial_j (A_q^*(u))^{\gamma-1} g \right\|_{H^{1-2\gamma,q'}(\Omega)} \\
\leq\ & \| (a_{ij}(u) - a_{ij}(v)) \|_{\mathscr{L}(H^{1,q}(\Omega), H^{2\gamma-1,q}(\Omega))} \left\| \partial_i A_q^{-1}(v) f \right\|_{H^{1,q}(\Omega)} \left\| \partial_j (A_q^*(u))^{\gamma-1} g \right\|_{H^{1-2\gamma,q'}(\Omega)}.
\end{aligned}
$$

We use Proposition 7.2 on page 116 from Taylor (2000) to get

$$
\begin{aligned}
&\left\| A^\gamma(u) \left[A^{-1}(u) - A^{-1}(v) \right] \right\|_{\mathscr{L}(X_0)} \\
&\leq CK_1 \|u(t) - v(t)\|_{H^{2\gamma-1,q}(\Omega)} + CK_2 (1 + \|u(t)\|_{H^{2\gamma-1,q}(\Omega)} + \|v(t)\|_{H^{2\gamma-1,q}(\Omega)}) \|u(t) - v(t)\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{2.35}
$$

We will have to use an embedding of the following form:

$$
[X_0, D(A(u))]_\theta = H^{2\theta,q}(\Omega) \hookrightarrow L^\infty(\Omega).
$$

This embedding only holds if $2\theta - \frac{d}{q} > 0$, which gives us an effective requirement on the trace space X_θ. Namely,

$$
\theta > \frac{d}{2q}. \tag{2.36}
$$

Secondly, we will need an embedding

$$
H^{2\theta,q}(\Omega) \hookrightarrow H^{2\gamma-1,q}(\Omega).
$$

This holds if $2\theta > 2\gamma - 1$, meaning

$$
\theta \geq \frac{1}{2q}. \tag{2.37}
$$

This embedding therefore holds if Eq. (2.36) holds. This means that if Eq. (2.36) holds and if $\gamma \in [\frac{1}{2}, \frac{1+q}{2q})$, we can conclude the Lipschitz estimate on A indeed holds:

$$
\left\| A^\gamma(u) \left[A^{-1}(u) - A^{-1}(v) \right] \right\|_{\mathscr{L}(X_0)} \leq C_{a,\theta,\gamma,p} \|u - v\|_{X_\theta}.
$$

All in all, we should choose our $\gamma \leq \gamma_0$ as big as possible. Therefore, we choose

$$
\gamma = \gamma_0 = \frac{1}{2} + \frac{1}{2q} - \varepsilon
$$

for a very small $\varepsilon > 0$.

We already have found a θ from Eq. (2.36), so we only have to look if it can be chosen in such a way that the conditions of Eq. (2.18) are met:

$$
\begin{aligned}
\theta < \sigma &< \gamma, \\
1 + \theta &< \gamma + \sigma.
\end{aligned}
$$

Given $\theta > \frac{d}{2q}$ from Eq. (2.36) held, $\gamma = \frac{1}{2} + \frac{1}{2q} - \varepsilon$, we get the following feasible region for θ:

$$
\frac{d}{2q} < \theta < \sigma - \frac{1}{2} + \frac{1}{2q}. \tag{2.38}
$$

In order for a feasible region of σ to exist, we must satisfy

$$1+\theta<\gamma+\sigma<2\gamma \tag{2.39}$$

By Eq. (2.38), this gives

$$\frac{d-1+q}{2q}<\sigma<\frac{1+q}{2q}. \tag{2.40}$$

Second assumption: conditions on u_0 and F

Let us check the conditions for Assumption 2.3.3.

We take $u_0 \in X_{\sigma,p} = B^{2\sigma}_{q,p}(\Omega)$, the time-independent Besov space, and we check the conditions on $F=-|u|^n$. We will set $F=F_{Tr}$ since we already have very strict conditions on $\sigma>\theta$. We see if we can get the following estimate for $u,v \in B^{2\sigma}_{q,p}(\Omega)$:

$$\||u|^n-|v|^n|\|_{L^q(\Omega)} \leq L_{Tr}\|u-v\|_{B^{2\sigma}_{q,p}(\Omega)}. \tag{2.41}$$

Note $f:\mathbb{R}\to\mathbb{R}$ defined as $f(u)=-|u|^n$ is locally Lipschitz, since for $|u|,|v|<R$, we get

$$||u|^n-|v|^n| = ||u|-|v||\cdot\sum_{j=1}^{n-1}|u|^j|v|^{n-1-j} \leq C_R|u-v|.$$

And since $f(0)=0$, we can use the Nemitskii map as in Example 18.1.3 of Hytönen et al. (2024) to show that F_{Tr} maps $u\in B^{2\sigma}_{q,p}(\Omega)$ to $F_{Tr}(u)\in L^q(\Omega)$ by setting

$$(F_{Tr}(u))(x)=f(u(x)),$$

and using the embedding

$$B^{2\sigma}_{q,p}(\Omega)\hookrightarrow C_b(\mathbb{R}^d). \tag{2.42}$$

This embedding is satisfied by $\sigma>\theta>\frac{d}{2q}$ as seen before. Then proof Eq. (2.41) the same way as in Example 18.1.3 of Hytönen et al. (2024): take $u,v\in B^{2\sigma}_{q,p}(\Omega)$, where w.l.o.g. $\|u\|_{B^{2\sigma}_{q,p}(\Omega)},\|v\|_{B^{2\sigma}_{q,p}(\Omega)}\leq N$, then there exists an embedding constant $L_{N,f,d,q}$ s.t.

$$\begin{aligned}\|F_{Tr}(u)-F_{Tr}(v)\|_{L^q(\Omega)} &= \left(\int_\Omega |f(u(x))-f(v(x))|^q dx\right)^{\frac{1}{q}}\\ &\leq L_{N,f,d,q}\left(\int_\Omega |u(x)-v(x)|^q dx\right)^{\frac{1}{q}}\\ &\leq L_{N,f,d,q}C\|u-v\|_{B^{2\sigma}_{q,p}}.\end{aligned}$$

Since we have taken $F_c=0$, the conditions

$$\begin{aligned}\beta_j&<\gamma,\\ \beta_j&\leq\beta_j^*,\end{aligned}$$

can be safely ignored by setting $\beta_j=\sigma$ and $\rho_j=0+\varepsilon$.

Results

With the above in mind, we have the following region to select for in $\sigma\in(\theta,\gamma)$, from Eq. (2.40):

$$\frac{d-1+q}{2q}<\sigma<\frac{1+q}{2q}. \tag{2.43}$$

Since $d-1>1$ for all multidimensional problems, this is the only case in which $W^{1,p}_{B(u)}$ is a time-dependent space. Therefore, we can not ensure unique short timescale solutions for this example using our Theorem 2.3.7, since Assumption 2.3.2 does not hold.

Discussion

We can see that had $\theta > \frac{d}{2q} + \frac{s}{2}$ and $\gamma < \frac{1+q}{2q} + \frac{s}{2}$ held for $s \in (0,1)$, which would have been possible if $X_0 = H^{-s,q}(\Omega)$ had been chosen and the calculations to get these θ and γ were done for this space, then Eq. (2.39) would have given

$$\frac{d-1+q}{2q} < \sigma < \frac{1+q}{2q} + \frac{s}{2}. \tag{2.44}$$

This means that increasing $s \in (0,1)$ s.t. $d-1+q < 1+q+sq$, or $d+q < 2+q(s+1)$, would have satisfied Assumption 2.3.2 and allowed application of Theorem 2.3.7

We can also think about what would have happened if $\sigma \geq \gamma$ had been allowed, similar to the theory denoted in Section 5.3.4 of Yagi (2010). Then, we would be able to find a unique solution, since

$$1 - \frac{d-1+q}{2q} = \frac{q+1-d}{2q},$$

meaning we could have used this theory for $q+1 > d$.

2.4.2 Model problem with averaging

In the previous section, we saw we could not apply Theorem 2.3.7 due to a strict condition on θ, γ and σ. In this section, we will lower the condition on θ, namely the necessity of $\theta > \frac{d}{2q}$, in order to gain more room. We consider the same problem

$$\begin{cases} \partial_t u(t,x) - A(u(t,x))u(t,x) &= F(u(t,x)), \quad t \in (0,T),\ x \in \Omega, \\ B(u(t,x))u(t,x) &= 0 \quad t \in (0,T),\ x \in \partial\Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{cases}$$

where now

$$A(v)u = \sum_{i,j=1}^{d} \partial_i \left[\tilde{a}_{ij}(v)\partial_j u\right], \qquad B(v)u = \sum_{i,j=1}^{d} n_i \tilde{a}_{ij}(v)\partial_j u.$$

Here, $\tilde{a}_{ij} : W^{2,q}(\Omega) \mapsto L^0(\Omega)$ is not defined pointwise, but instead as an operator between function spaces. A pointwise definition can be set as

$$\tilde{a}_{ij}(v)(x) = a_{ij}(v * \phi(x)),$$

for some mollifier ϕ and the same a_{ij} as described in Section 2.4.1. Before, we essentially used $\phi = \delta$ the Dirac δ-distribution, the most local convolution mollifier one can use. The idea is that we now use a mollifier $\phi \in C_c^\infty(\Omega)$.

When we re-evaluate the calculations to come to Eq. (2.35) for our current setup, we will now find the equation

$$\begin{aligned} &\left\| A^\gamma(u)\left[A^{-1}(u) - A^{-1}(v)\right]\right\|_{\mathscr{L}(X_0)} \\ &\leq CK_1 \|\phi * (u(t)-v(t))\|_{H^{2\gamma-1,q}(\Omega)} \\ &\quad + CK_2(1 + \|\phi * u(t)\|_{H^{2\gamma-1,q}(\Omega)} + \|\phi * v(t)\|_{H^{2\gamma-1,q}(\Omega)})\|\phi * (u(t)-v(t))\|_{L^\infty(\Omega)}. \end{aligned} \tag{2.45}$$

Taking $u, v \in B_R(0; C^\mu([0,T]; L^q(\Omega)))$ for $\mu = \sigma - 0$ and using

$$\begin{aligned} \|u(t) * \phi\|_{H^{2\gamma-1,q}(\Omega)} = \|u(t) * (J_{2\gamma-1}\phi)\|_{L^q(\Omega)} &\overset{\text{Young}}{\leq} \|u(t)\|_{L^q(\Omega)}\|\phi\|_{H^{2\gamma-1,1}(\Omega)} < \infty, \\ \|u(t) * \phi\|_{L^\infty(\Omega)} &\overset{\text{Young}}{\leq} \|u(t)\|_{L^q(\Omega)}\|\phi\|_{L^{q'}(\Omega)} < \infty, \end{aligned}$$

we can see that for a constant $C_{R,\phi} > 0$,

$$\|A^\gamma(u)[A^{-1}(u) - A^{-1}(v)]\|_{L^q(\Omega)} \leq C_{R,\phi}\|u - v\|_{L^q(\Omega)}.$$

This means $\theta = 0$ is a correct choice of θ, and the bound $\theta > \frac{d}{2q}$ is no longer valid. Then, $1 + \theta < \gamma + \sigma$ is satisfied by

$$\frac{1}{2} - \frac{1}{2q} < \sigma,$$

and so $\sigma \in (\theta, \gamma)$ can be chosen in the region

$$\frac{1}{2} - \frac{1}{2q} < \sigma < \frac{1}{2} + \frac{1}{2q}. \tag{2.46}$$

This means that for every $q \in [0, \infty)$, we have a viable choice of σ.

The right-hand side using $F = F_{Tr}$

Recall from Eq. (2.42) that we also needed $\sigma > \frac{d}{2q}$. Therefore, in order to have $F = F_{Tr}$, we need

$$\max\left\{\frac{1}{2} - \frac{1}{2q}, \frac{d}{2q}\right\} < \sigma < \frac{1}{2} + \frac{1}{2q}.$$

If $d+1 > q$, we need to pay attention to the restrain $\sigma > \frac{d}{2q}$ instead of $\sigma > \frac{1}{2} - \frac{1}{2q}$. If $d-1 \geq q$, no feasible region for σ exists; therefore, q must be increased. Then, we can write a constraint for q and d as

$$d - 1 < q.$$

However, we will check if we can improve this condition by setting $F = F_c$ instead in the next section since this will get rid of the $\sigma > \frac{d}{2q}$ constraint and allow for less strict requirements on the initial condition.

The right-hand side using $F = F_c$

Recall that $F(u) = f(u) = -|u|^n$. Since $f(0) = 0$ and $|f'(x)| \lesssim |x|^{n-1}$, we can use the calculations from Example 18.1.3 from Hytönen et al. (2024) (which we here apply with $s = 0$) to get

$$\|F(u) - F(v)\|_{X_0} \lesssim (\|u\|^{\rho}_{X_\beta} + \|v\|^{\rho}_{X_\beta})\|u - v\|_{X_\beta},$$

where $\rho = n-1$ and $\beta = \frac{n-1}{n} \cdot \frac{d}{2q}$. We investigate the critical σ, which is when $\beta = \beta^*$ holds. So:

$$\begin{aligned}
\frac{n-1}{n}\frac{d}{2q} &= \frac{1 + (n-1)\sigma}{n}, \\
(n-1)\frac{d}{2q} &= 1 + (n-1)\sigma, \\
\frac{d}{2q} - \frac{1}{n-1} &= \sigma.
\end{aligned}$$

So we can note that $\frac{d}{2q} - \frac{1}{n-1} < \frac{d}{2q}$, which was the previous lower bound for σ. Therefore, we definitely have an improvement on the initial condition compared to $F = F_{Tr}$. This choice of σ is valid whenever

$$\begin{aligned}
\frac{1}{2} - \frac{1}{2q} &< \frac{d}{2q} - \frac{1}{n-1}, \\
\frac{1}{2} + \frac{1}{2q} &> \frac{d}{2q} - \frac{1}{n-1}.
\end{aligned}$$

The lower bound gives $\frac{d+1}{2q} > \frac{1}{n-1} + \frac{1}{2}$. The upper bound gives $\frac{1}{n-1} + \frac{1}{2} > \frac{d-1}{2q}$. Combined, we have

$$d - 1 < \frac{q(n+1)}{n-1} < d + 1.$$

Now the constraint $q > d-1 > \frac{n-1}{n+1}(d-1)$, so we can indeed choose lower q for $F = F_c$ compared to $F = F_{Tr}$ as long as that we are careful not to choose $q \geq \frac{n-1}{n+1}(d+1)$, in which case we lose access to criticality.

2.4.3 Right-hand side function dependent on gradient

We consider the same example as in Section 2.4.2, but this time a right-hand side function of the form $F(u) = f(u, \nabla u)$ instead of $F(u) = -|u|^n$. We will assume that $\exists \rho > 0$ s.t. $\forall r \geq 1$, $\exists C_r > 0$ s.t. $\forall x_1, x_2 \in \mathbb{R}$ with $|x_1|, |x_2| \leq r$ and $\forall y_1, y_2 \in \mathbb{R}^d$, $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$|f(x_1, y_1) - f(x_2, y_2)| \leq C_r|x_1 - x_2| + C_r \left(|y_1|^\rho + |y_2|^\rho\right)|y_1 - y_2|.$$

From this, it is clear we should look for $F = F_c$ in order to get σ as low a number as possible. We can derive the following for $u, v \in H^{2\gamma,q}(\Omega)$:

$$\begin{aligned}
\|f(u, \nabla u) - f(v, \nabla v)\|_{L^q(\Omega)} &= \left(\int_\Omega |f(u(x), \nabla u(x)) - f(v(x), \nabla v(x))|^q dx\right)^{\frac{1}{q}} \\
&\leq C_1\|u - v\|_{L^q(\Omega)} + C_2\| \left(|\nabla u|^\rho + |\nabla v|^\rho\right)|\nabla u - \nabla v|\|_{L^q(\Omega)} \\
&\overset{\text{Hölder}}{\leq} C_1\|u - v\|_{L^q(\Omega)} + C_2 \left(\|\nabla u\|^\rho_{L^{q(\rho+1)}(\Omega)} + \|\nabla v\|^\rho_{L^{q(\rho+1)}(\Omega)}\right)\|\nabla u - \nabla v\|_{L^{q(\rho+1)}(\Omega)} \\
&\overset{\text{Sobolev}}{\leq} C_1\|u - v\|_{L^q(\Omega)} + C_2 \left(\|\nabla u\|^\rho_{H^{2\beta-1,q}(\Omega)} + \|\nabla v\|^\rho_{H^{2\beta-1,q}(\Omega)}\right)\|\nabla u - \nabla v\|_{H^{2\beta-1,q}(\Omega)} \\
&= C_1\|u - v\|_{L^q(\Omega)} + C_2 \left(\|u\|^\rho_{H^{2\beta,q}(\Omega)} + \|v\|^\rho_{H^{2\beta,q}(\Omega)}\right)\|u - v\|_{H^{2\beta,q}(\Omega)}.
\end{aligned}$$

The condition for Sobolev embedding $H^{k,p} \hookrightarrow H^{\ell,r}$ is $\frac{1}{p} - \frac{k}{d} = \frac{1}{r} - \frac{\ell}{d}$, as well as $k > \ell$ and $p \leq r$. $2\beta - 1 > 0$ as long as $\beta > \frac{1}{2}$ and $q < q(\rho + 1)$, so:

$$\frac{1}{q(\rho + 1)} = \frac{1}{q} - \frac{2\beta - 1}{d}.$$

To find a value of β, we write

$$\begin{aligned}
\frac{1}{q(\rho + 1)} &= \frac{1}{q} - \frac{2\beta - 1}{d}, \\
\frac{2\beta - 1}{d} &= \frac{\rho}{q(\rho + 1)}, \\
2\beta - 1 &= \frac{d}{q} \cdot \frac{\rho}{\rho + 1}, \\
\beta &= \frac{d}{2q} \cdot \frac{\rho}{\rho + 1} + \frac{1}{2}.
\end{aligned}$$

Then to reach criticality, meaning $\beta = \beta^*$, we determine the critical σ:

$$\begin{aligned}
\frac{d}{2q} \cdot \frac{\rho}{\rho + 1} + \frac{1}{2} &= \frac{1 + \rho\sigma}{1 + \rho}, \\
\frac{d\rho + q(\rho + 1)}{2q} &= 1 + \rho\sigma, \\
\frac{d\rho + q\rho}{2q} - \frac{1}{2} &= \rho\sigma, \\
\frac{d}{2q} + \frac{1}{2} - \frac{1}{2\rho} &= \sigma.
\end{aligned}$$

This gives us our condition on α and $\frac{1}{p}$. From Eq. (2.46) of Section 2.4.2, we have to check

$$\begin{aligned}
\frac{1}{2} - \frac{1}{2q} &< \frac{d}{2q} + \frac{1}{2} - \frac{1}{2\rho}, \\
\frac{1}{2} + \frac{1}{2q} &> \frac{d}{2q} + \frac{1}{2} - \frac{1}{2\rho}.
\end{aligned}$$

The lower bound becomes

$$0 < \frac{(d+1)\rho - q}{2q\rho},$$

so we need $\frac{q}{\rho} < d+1$. The upper bound becomes

$$0 < \frac{(1-d)\rho + q}{2q\rho},$$

so we need $\rho + q > d\rho$, so $\frac{q}{\rho} > d-1$. Therefore, we can apply our Theorem 2.3.7 if

$$d - 1 < \frac{q}{\rho} < d + 1.$$

Keeping in mind $d \geq 2$ and $q \geq 1$, we get the restriction $\rho > \frac{1}{d+1}$ to ensure $\frac{q}{\rho} < d+1$ is feasible. If true, then a suitable q can be chosen.

Chapter 3

Conclusion

A short recap of the thesis: Chapter 1 described the setting of maximal L^p-regularity with definitions and results from Hytönen, van Neerven, Veraar, and Weis (2024) and van Neerven (2022) and showed the extension to non-autonomous problems on non-constant domains from Di Giorgio, Lunardi, and Schnaubelt (2005) and Yagi (2010). In Chapter 2, we extended this setting to include weighted non-autonomous linear, semi-linear and quasi-linear problems in Sections 2.1 to 2.3. For each setting, we set conditions for the existence and uniqueness of solutions and showed the result to be true. We then analysed the example model problem and some other relevant examples in Section 2.4, namely the model problem Eq. (1) with right-hand side function $|u|^n$, with a power $n \geq 2$, in Section 2.4.1, an averaged version of this problem in Section 2.4.2, and the same problem but with a critical right-hand side function depending on the gradiant ∇u in Section 2.4.3. Our research questions were as follows:

- Can a set of conditions be derived on which we are able to show short timescale existence and uniqueness of solutions to quasi-linear problems on non-constant domains in a critical setting?
- Can our result ensure a unique short timescale solution to the model problem Eq. (1)?

In Section 2.3, we did indeed derive such conditions in Assumption 2.3.2 and Assumption 2.3.3. Assumption 2.3.2 entails that we need a set of Acquistapace Terreni conditions specifically tailored to the quasi-linear operator $A(u)$ by using Hölder regularity on the input u as described by Yagi (2010). Assumption 2.3.3 describes the trace part F_{Tr} and the critical part F_c of the non-linear right-hand side function $F(u)$ as explained by Hytönen, van Neerven, Veraar, and Weis (2024), as well as restricting the initial condition u_0. Together, this allowed us to state and prove Theorem 2.3.7. This result shows the short timescale existence and uniqueness of mild solutions to the prescribed problems.

However, for the model Neumann problem investigated in Section 2.4.1, we found that the model problem does not satisfy the conditions as described in Assumption 2.3.2, meaning we were not able to ensure a unique short timescale solution using our Theorem 2.3.7. This was due to the strict requirements from the prescribed Acquistapace Terreni conditions of Eq. (2.18).

We did ensure a unique short timescale solution for the quasi-linear problems of Section 2.4.2 and Section 2.4.3, which featured an averaging function in order to reduce the stress on the constants and allowed for a choice of the initial condition space $X_{\sigma,p}$, and which had right-hand side functions depending on an exponent of $|u|$ and on a gradient of u respectively.

Discussion and future research

We should discuss why ensuring a unique short timescale solution to a comparable problem was possible in Section 5.6 of Yagi (2010). One important assumption to be made is that the space of initial data $X_{\sigma,p}$ is replaced by the assumption $u_0 \in (L^q(\Omega), W^{2,p}_{B(u_0)}(\Omega))_{\sigma,p}$. We can note it becomes difficult to find initial conditions that satisfy this constraint, but not impossible: for instance, assuming $u_0 \in C_c^\infty(\Omega)$ so that it is an element of $(L^q(\Omega), W^{2,p}_{B(u)}(\Omega))_{\sigma,p}$ regardless of u. Secondly, the non-linearity on the right-hand side function is chosen as a type of Hölder continuous non-linearity, which is a much stronger assumption than the critical setting we are working in. Together, it allows for taking the space Z, which acts as their Banach contraction space and their initial condition space, much smaller than the space which ensures constant domains, meaning they have their σ larger than their γ_0. We could not do this since it loses essential information on the critical setting $F = F_{Tr} + F_c$, and we lose access to our initial data $u_0 \in X_{\sigma,p}$. A direct attempt at making the theory work directly would be to let $\zeta > \gamma$ and let the initial condition depend on $u_0 \in D(A^\zeta(u_0))$ while $F = F_c$ still depends on spaces and X_β with coefficients $\beta < \gamma$. This also means we no longer have to do Y_{Tr} estimates since F_{Tr} is absent. Since there is no need for the non-constant property of the space attached to σ, we can now increase $\zeta \geq \gamma$ and get working theory, though at a massive disadvantage of losing important information.

Another discussion point is the necessity of Hölder regularity in quasi-linear problems. Simply put, there is a Hölder regularity constraint in the Acquistapace Terreni conditions, which allows for solutions to exist when the domains are non-constant. Due to this Hölder continuous setting, working with Hölder continuous functions as done in the work of Yagi (2010) is sensible. It begs the question of whether or not there are conditions on which $H^{s,p}_{\omega_\alpha}$- or B^s_{q,p,ω_α}-regularity makes more sense than C^μ-regularity. As seen in the calculations of Lemma A.1.3, we need these spaces regardless, so using them to define our Acquistapace Terreni conditions in the quasi-linear case seems like an interesting possibility.

Lastly, we can discuss the cases $\alpha < 0$, $p = 1$ and $p = \infty$. Because of all results working for unbounded $p \in (1, \infty)$, the results can likely be extended to $p = \infty$ as is done in Chapter 18 of Hytönen, van Neerven, Veraar, and Weis (2024) and Yagi (2010). However, The other case would require negative weights α to get $\sigma > 0$, and working with negative weights gives its own problems. More research can be done to investigate this extension's utility and feasibility.

In future research, the following can be done to work on this field further.

- Show existence and uniqueness of short timescale solutions to the model problem where instead $X_0 = H^{-s,q}(\Omega)$, since likely the Acquistapace Terreni conditions from Assumption 2.3.2 will hold as long as $d + q < 2 + q(s + 1)$, as described in the conclusion of Section 2.4.1. This means our theory can be applied to another important problem closely related to our model problem.

- Research the possibility of a different set of conditions than the (AT) conditions on which we can determine solutions without the need for Hölder regularity, and with more advantages of $H^{s,p}_{\omega_\alpha}$- or B^s_{q,p,ω_α}-regularity. In this case, more needs to be changed about our proof of Theorem 2.3.7, and it will be applicable in more cases.

- Show the existence and uniqueness of global solutions. This can be done by extending the maximal solution theory of Section 18.2.d of Hytönen, van Neerven, Veraar, and Weis (2024) to our setting, but it is yet to be found how much we lose when this is applied.

- Attempt to make a short timescale existence and uniqueness proof where we abandon the critical setting to solve the Neumann problem. This should be possible by following more of the setting of chapter 5 of Yagi (2010). However, we will most likely lose information on the initial data and lose the critical setting of our non-linearity.

- Research if an extension of the results on non-linear equations to $p = \infty$, or to $p = 1$ with negative weights $\alpha \in (-\frac{1}{p}, 0]$, is helpful. If so, these extensions can be researched using the work of Hytönen, van Neerven, Veraar, and Weis (2024) and Yagi (2010).

Appendix A

Miscellaneous results

A.1 Interpolation spaces

For non-linear problems, we need to consider the initial data u_0 as well, as seen in Section 1.2.4. Often, the setting is as follows: there will be an X_1 Banach space representing the domain of the operator, and X_0 the Banach space on which everything occurs, with $X_1 \hookrightarrow X_0$ an embedding between them. We will discuss here what kind of spaces $X_{r,\theta}$ exist "between" these spaces and a few applications based on Appendix C of Hytönen et al. (2016).

The first method is complex interpolation, which is of the form

$$X_\theta := [X_0, X_1]_\theta$$

for $\theta \in (0,1)$. An example of what such a space would look like is f.i. $[L^{p_0}(\mathbb{R}^d), L^{p_1}(\mathbb{R}^d)]_\theta = L^{p_\theta}(\mathbb{R}^d)$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, or $[W^{2,q}(\mathbb{R}^d), L^q(\mathbb{R}^d)]_\theta = H^{2\theta,q}(\mathbb{R}^d)$ a Bessel potential space.

The second, more frequently used method is real interpolation, which is of the form

$$X_{\theta,p} = (X_0, X_1)_{\theta,p}$$

for $\theta \in (0,1)$ and $p \in [1,\infty]$. Some examples here are $(L^{p_0}(\mathbb{R}^d), L^{p_1}(\mathbb{R}^d))_{\theta,p_\theta} = L^{p_\theta}(\mathbb{R}^d)$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, or $(W^{2,q}(\mathbb{R}^d), L^q(\mathbb{R}^d))_{\theta,p} = B^{2\theta}_{q,p}(\mathbb{R}^d)$ a Besov space.

The following proposition will describe useful properties of interpolation spaces.

Proposition A.1.1. *1. For $1 > \tilde{\theta} > \theta > 0$, we have*

$$X_1 \hookrightarrow X_{\tilde{\theta}} \hookrightarrow X_\theta \hookrightarrow X_0.$$

2. For $1 > \tilde{\theta} > \theta > 0$, $\infty \geq p \geq \tilde{p} \geq 1$ and any $q \in [1,\infty]$, we have

$$X_1 \hookrightarrow X_{\tilde{\theta},\tilde{p}} \hookrightarrow X_{\tilde{\theta},p} \hookrightarrow X_{\theta,q} \hookrightarrow X_0.$$

3. For $\theta \in (0,1)$ we have

$$(X_0, X_1)_{\theta,1} \hookrightarrow [X_0, X_1]_\theta \hookrightarrow (X_0, X_1)_{\theta,\infty}.$$

4. For $\theta \in (0,1)$ we have

$$(X_0, X_1)_{\theta,2} = [X_0, X_1]_\theta.$$

Proof. Points 1, 3 and 4 follow from Appendices C.2 and C.4 from Hytönen et al. (2016), and point 2 Proposition L.1.1 from Hytönen et al. (2024). □

We also have the following interpolation of bounded operators from Appendix C of Hytönen et al. (2016).

Theorem A.1.2. *Let T be an operator bounded from X_1 to Y_1 and bounded from X_2 to Y_2. Then both $T_\theta : X_\theta \to Y_\theta$ and $T_{\theta,p} : X_{\theta,p} \to Y_{\theta,p}$ are bounded operators for all $\theta \in (0,1)$ and $p \in [1,\infty]$, and*

$$\|T_\theta\|_{\mathscr{L}(X_\theta,Y_\theta)} \leq C_\theta \|T\|^{1-\theta}_{\mathscr{L}(X_1,Y_1)} \|T\|^{\theta}_{\mathscr{L}(X_2,Y_2)},$$
$$\|T_{\theta,p}\|_{\mathscr{L}(X_{\theta,p},Y_{\theta,p})} \leq C_{\theta,p} \|T\|^{1-\theta}_{\mathscr{L}(X_1,Y_1)} \|T\|^{\theta}_{\mathscr{L}(X_2,Y_2)},$$

for some $C_\theta, C_{\theta,p} \geq 0$.

Proof. Theorem C.2.6 and C.3.3 of Hytönen et al. (2016). □

A.1.1 Embeddings of interpolated functions spaces

For our work on the Hölder continuous functions, we need an analog of Lemma 18.2.7 from Hytönen et al. (2024) which works on all the spaces we have defined. This result is formulated here as Proposition A.1.4.

Define the real interpolation space $X_{\theta,p} = (X_0, X_1)_{\theta,p}$ and the complex interpolation space $X_\theta = [X_0, X_1]_\theta$ as above for UMD spaces $X_1 \hookrightarrow X_0$. We define the weighted Bessel potential space and weighted Besov space as the following interpolation spaces:

$$\begin{aligned} H^{1-s,p}_{\omega_\alpha}(I; X_s) &:= [L^p_{\omega_\alpha}(I; X_1), W^{1,p}_{\omega_\alpha}(I; X_0)]_s, \\ B^{1-s}_{q,p,\omega_\alpha}(I; X_s) &:= (L^p_{\omega_\alpha}(I; X_1), W^{1,p}_{\omega_\alpha}(I; X_0))_{s,q}. \end{aligned}$$

Then, we want an embedding result into the Hölder continuous functions.

Lemma A.1.3. *Let $X_1 \hookrightarrow X_0$ be UMD spaces, let $\sigma = 1 - \frac{1}{p} - \alpha$ for $p \in (1, \infty)$ and $\alpha \in [0, \frac{1}{p'})$, and let $\theta \in [0, \sigma]$. Then*

$$H^{1-\theta,p}_{\omega_\alpha}((0,T); X_\theta) \hookrightarrow C^{\sigma-\theta}((0,T); X_\theta),$$

where for $u \in B_R(0; H^{1-\theta,p}_{\omega_\alpha}((0,T); X_\theta))$ with $u(0) = 0$ and $R > 0$ we have

$$\|u\|_{C^{\sigma-\theta}([0,T];X_\theta)} \le C\|u\|_{H^{1-\theta,p}_{\omega_\alpha}((0,T);X_\theta)}$$

where C is a T-independent constant.

Proof. For the full space $\mathbb{R}$, we have the following results from Proposition 3.12 and Proposition 7.4 of Meyries and Veraar (2012):

$$H^{1-\theta,p}_{\omega_\alpha}(\mathbb{R}; X_\theta) \hookrightarrow B^{1-\theta}_{\infty,p,\omega_\alpha}(\mathbb{R}; X_\theta) \hookrightarrow C^{1-\theta-\alpha-\frac{1}{p}}(\mathbb{R}; X_\theta).$$

However, we can note that this result is formulated for the definition of Bessel potential spaces using Fourier multipliers. However, by Proposition 3.1 of Meyries and Veraar, 2014, $W^{1,p}_{\omega_\alpha}(I; X_0) = H^{1,p}_{\omega_\alpha}(I; X_0)$ and $L^p_{\omega_\alpha}(I; X_1) = H^{0,p}_{\omega_\alpha}(I; X_1)$, and by Theorem 3.18 of Lindemulder and Veraar (2020), $H^{1-s,p}_{\omega_\alpha}(I; X_s) = [H^{0,p}_{\omega_\alpha}(I; X_1), H^{1,p}_{\omega_\alpha}(I; X_0)]_s$. Therefore, it follows for our complex interpolation based definition of Bessel potential spaces too.

With the result on the full line, we can reduce to the half line $[0, \infty)$ by using the extension operator $E_\infty : C^{\sigma-\theta}([0,\infty); X_\theta) \to C^{\sigma-\theta}(\mathbb{R}; X_\theta)$ s.t. $E_\infty u(t) = u(|t|)$: take $u \in H^{1-\theta,p}_{\omega_\alpha}([0,\infty); X_\theta)$. E_∞ is bounded from $L^p_{\omega_\alpha}([0,\infty); X_1)$ to $L^p_{\omega_\alpha}(\mathbb{R}; X_1)$ and from $W^{1,p}_{\omega_\alpha}([0,\infty); X_0) = H^{1,p}_{\omega_\alpha}(I; X_0)$ to $W^{1,p}_{\omega_\alpha}(\mathbb{R}; X_0) = H^{1,p}_{\omega_\alpha}(I; X_0)$, so by Theorem A.1.2, it is bounded from $H^{1-s,p}_{\omega_\alpha}([0,\infty); X_s)$ to $H^{1-s,p}_{\omega_\alpha}(\mathbb{R}; X_s)$. Therefore:

$$\begin{aligned} \|u\|_{C^{\sigma-\theta}([0,\infty);X_\theta)} &\le \|u(|\cdot|)\|_{C^{\sigma-\theta}(\mathbb{R};X_\theta)} \\ &\le C\|u(|\cdot|)\|_{H^{1-\theta,p}_{\omega_\alpha}(\mathbb{R};X_\theta)} \\ &\le C\|E_\infty\|_{\mathscr{L}(H^{1-s,p}_{\omega_\alpha}([0,\infty);X_s), H^{1-s,p}_{\omega_\alpha}(\mathbb{R};X_s))}\|u\|_{H^{1-\theta,p}_{\omega_\alpha}([0,\infty);X_\theta)}. \end{aligned}$$

Now that the result holds on the half-line, we will use the extension operator E_T from Lemma L.4.5 of Hytönen et al. (2024) to get a result on the bounded interval $(0,T)$: take $u \in H^{1-\theta,p}_{\omega_\alpha}((0,T); X_\theta)$, where we note E_T is bounded from $H^{1-s,p}_{\omega_\alpha}((0,T); X_s)$ to $H^{1-s,p}_{\omega_\alpha}([0,\infty); X_s)$ by Theorem A.1.2:

$$\begin{aligned} \|u\|_{C^{\sigma-\theta}((0,T);X_\theta)} &\le \|E_T u\|_{C^{\sigma-\theta}([0,\infty);X_\theta)} \\ &\le C\|E_T u\|_{H^{1-\theta,p}_{\omega_\alpha}([0,\infty);X_\theta)} \\ &\le C_T\|u\|_{H^{1-\theta,p}_{\omega_\alpha}((0,T);X_\theta)}, \end{aligned}$$

where,

$$\begin{aligned} C_T &= \|E_T\|_{\mathscr{L}(H^{1-s,p}_{\omega_\alpha}((0,T);X_s), H^{1-s,p}_{\omega_\alpha}([0,\infty);X_s))} \\ &\overset{(A.1.2)}{\le} \tilde{C}\|E_T\|^s_{W^{1,p}_{\omega_\alpha}((0,T);X_0), W^{1,p}_{\omega_\alpha}([0,\infty);X_0)}\|E_T\|^{1-s}_{L^p_{\omega_\alpha}((0,T);X_1), L^p_{\omega_\alpha}([0,\infty);X_1)} \\ &\lesssim (2 + 3T^{-1}) \end{aligned}$$

from (2) and (3) of Lemma L.4.5 from Hytönen et al. (2024), but if $u(0) = 0$, we can use (4) instead of (3) to get $C_T = C_{\alpha,p}$ independent of time. ☐

Using Lemma L.4.6 and L.4.7 of Hytönen et al. (2024), we can then make a lemma that allows for embeddings out of the space $\mathcal{P}^p_\alpha$ with time-independent constants for functions with $u(0) = 0$.

Proposition A.1.4. *Let $X_1 \hookrightarrow X_0$ be UMD spaces, $\sigma = 1 - \alpha - \frac{1}{p}$, and $0 \leq \theta \leq \sigma \leq \beta \leq \frac{1+\rho\sigma}{1+\rho}$ for some $\rho > 0$. Define*

$$\mathcal{P}^p_\alpha((0,T), X_0, X_1) := W^{1,p}_{\omega_\alpha}((0,T); X_0) \cap L^p_{\omega_\alpha}((0,T); X_1).$$

Then $\mathcal{P}^p_\alpha((0,T), X_0, X_1)$ continuously embeds into $C^\mu([0,T]; X_\theta)$, $C([0,T]; X_{\sigma,p})$, and $L^{p(\rho+1)}_{\omega_{\frac{\alpha}{\rho+1}}}((0,T); X_\beta)$. For $u \in \mathcal{P}^p_\alpha((0,T), X_0, X_1)$ with $u(0) = 0$, we have embeddings with time-independent constants $C_1, C_2, C_3 > 0$ of the following form:

$$\|u\|_{C^\mu([0,T];X_\theta)} \leq C_1 \|u\|_{\mathcal{P}^p_\alpha((0,T),X_0,X_1)}, \tag{A.1}$$

$$\|u\|_{C([0,T];X_{\sigma,p})} \leq C_2 \|u\|_{\mathcal{P}^p_\alpha((0,T),X_0,X_1)}, \tag{A.2}$$

$$\|u\|_{L^{p(\rho+1)}_{\omega_{\frac{\alpha}{\rho+1}}}((0,T);X_\beta)} \leq C_3 \|u\|_{\mathcal{P}^p_\alpha((0,T),X_0,X_1)}. \tag{A.3}$$

Proof. The first embedding Eq. (A.1) follows from applying Lemma A.1.3 with the embedding time-independent embedding $\mathcal{P}^p_\alpha((0,T), X_0, X_1) \hookrightarrow H^{1-\theta,p}_{\omega_\alpha}((0,T); X_\theta)$. The other two embeddings follow from Lemma L.4.6 and L.4.7 of Hytönen et al. (2024). □

A.2 Weighted estimates of fractional integrals

Let us study integrals of the form

$$I_\phi f(t) := \int_0^t (t-\tau)^{\phi-1} f(\tau) d\tau.$$

We need results for estimates of this integral as an operator from one weighted L^p space to another weighted L^p space.

Proposition A.2.1. *Let $\phi \in (0,1)$, $q \in (1,\infty)$, $r \in [1,\infty)$ s.t. $\frac{1}{r} = \frac{1}{q} + \phi$, $\beta > -\frac{1}{q}$, and $f \in L^r_{\omega_\beta}(0,T)$. Then*

$$\|\omega_\beta I_\phi f\|_q \leq C \|\omega_\beta f\|_r$$

Proof. We can derive

$$\begin{aligned}
\|t \mapsto t^\beta I_\phi f(t)\|_q &= \left\| t \mapsto \int_0^t t^\beta (t-\tau)^{\phi-1} \tau^{-\beta} (\tau^\beta f(\tau)) d\tau \right\|_q \\
&= \left\| t \mapsto \int_0^t \tilde{K}(t,\tau)(\tau^\beta f(\tau)) d\tau \right\|_q \\
&= \|T_K(\omega_\beta f)\|_q.
\end{aligned}$$

Here, $\tilde{K}$ is the kernel of the weighted operator, and K is the kernel of the commutator. We now show the operator associated with the kernel K goes from L^r to L^q where $\phi = \frac{1}{r} - \frac{1}{q}$. We have

$$\begin{aligned}
K(t,\tau) &= ((t/\tau)^\beta - 1)(t-\tau)^{\phi-1} \\
&= \tau^{\phi-1}((t/\tau)^\beta - 1)(t/\tau - 1)^{\phi-1} \\
&= t^{\phi-\frac{1}{r}} (\tau/t)^{\phi-\frac{1}{r}} ((t/\tau)^\beta - 1)(t/\tau - 1)^{\phi-1} \tau^{\frac{1}{r}-1} \\
&= t^{-\frac{1}{q}} (t/\tau)^{\frac{1}{q}} ((t/\tau)^\beta - 1)(t/\tau - 1)^{\phi-1} \tau^{\frac{1}{r}-1}.
\end{aligned}$$

So,

$$\begin{aligned}
\|T_K(\omega_\beta f)\|_q &= \left(\int_0^T \left| \int_0^t (t/\tau)^{\frac{1}{q}} ((t/\tau)^\beta - 1)(t/\tau - 1)^{\phi-1} (\tau^{\frac{1}{r}+\beta} f(\tau)) \frac{d\tau}{\tau} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= \|c \bigstar g\|_{L^q((0,T),\frac{dt}{t})},
\end{aligned}$$

where we defined $c(z) = 1_{(0,1)}(z) \cdot z^{\frac{1}{q}}(z^\beta - 1)(z-1)^{\phi-1}$, $g(z) := z^{\frac{1}{r}+\beta} f(z)$, and ★ as the convulution product of the group $(0,T)$ using the Haar measure $\frac{dt}{t}$, similar to the set-up of the proof of Lemma 17.2.35 of Hytönen et al. (2024). Firstly we see if $c \in L^s((0,T), \frac{dt}{t})$ for $s \in [1,\infty]$ with $\frac{1}{s} = 1 - \phi$:

$$\begin{aligned}
\left(\int_0^T |c(t)|^s \frac{dt}{t}\right)^{\frac{1}{s}} &= \left(\int_0^1 (t^{\frac{1}{q}}(1-t)^{\phi-1}|t^\beta - 1|)^s t^{-1} dt\right)^{\frac{1}{s}} \\
&= \left(\int_0^1 \frac{t^{\frac{s}{q}-1}|t^\beta - 1|^s}{(1-t)^{(\phi-1)s}} dt\right)^{\frac{1}{s}} \\
&\leq 2\left(\int_0^{\frac{1}{2}} \frac{t^{\frac{s}{q}-1}\max\{1, t^{\beta s}\}}{1-t} dt\right)^{\frac{1}{s}} + \left(\int_{\frac{1}{2}}^1 \frac{t^{\frac{s}{q}-1}|t^\beta - 1|^s}{1-t} dt\right)^{\frac{1}{s}}.
\end{aligned}$$

This is true since $\frac{1}{q} + \beta > 0$ is satisfied by $\beta > -\frac{1}{q}$, and so

$$\begin{aligned}
2\left(\int_0^{\frac{1}{2}} \frac{t^{\frac{s}{q}-1}\max\{1, t^{\beta s}\}}{1-t} dt\right)^{\frac{1}{s}} &\leq \begin{cases} 2C\left(\int_0^{\frac{1}{2}} t^{s(\frac{1}{q}+\beta)} dt\right)^{\frac{1}{s}}, & \beta \in (-\frac{1}{q}, 0), \\ 2C\left(\int_0^{\frac{1}{2}} t^{s(\frac{1}{q})} dt\right)^{\frac{1}{s}}, & \beta > 0, \end{cases} \\
&= \begin{cases} 2Ct^{\frac{1}{q}+\beta} < \infty, & \beta \in (-\frac{1}{q}, 0), \\ 2Ct^{\frac{1}{q}} < \infty, & \beta > 0, \end{cases}
\end{aligned}$$

where we note $\frac{1}{q} + \beta > 0 > -\frac{1}{s} = \phi - 1$. For $f(t) := -1_{(0,1)}(t) t^{\frac{s}{q}-1}|t^\beta - 1|^s$, which is a continuous function on $[\frac{1}{2}, 1]$ and continuously differentiable function on $(\frac{1}{2}, 1)$ for the use of the mean value theorem, which will give a certain mean value $c(t) \in (t, 1)$, we have

$$\begin{aligned}
\left(\int_{\frac{1}{2}}^1 \frac{t^{\frac{s}{q}-1}|t^\beta - 1|^s}{1-t} dt\right)^{\frac{1}{s}} &= \left(\int_{\frac{1}{2}}^1 \frac{f(1) - f(t)}{1-t} dt\right)^{\frac{1}{s}} \\
&= \left(\int_{\frac{1}{2}}^1 f'(c(t)) dt\right)^{\frac{1}{s}} < \infty.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\|g\|^p_{L^r((0,T),\frac{dt}{t})} &= \int_0^T |t^{\frac{1}{r}+\beta} f(t)|^r \frac{dt}{t} \\
&= \int_0^T |t^\beta f(t)|^r \, dt = \|f\|_{L^r_\beta(0,T)}.
\end{aligned}$$

So, we can use Young's inequality with $1 + \frac{1}{q} = \frac{1}{s} + \frac{1}{r}$ w.r.t. ★ to conclude

$$\|T_K(\omega_\beta f)\|_q = \|c \bigstar g\|_{L^q((0,T),\frac{dt}{t})} \leq \|c\|_{L^s((0,T),\frac{dt}{t})} \|g\|_{L^r((0,T),\frac{dt}{t})} = \|c\|_{L^s((0,T),\frac{dt}{t})} \|f\|_{L^r_\beta(0,T)}.$$

□

A.3 Product rule

Below, a simplified version of the product rule from Lemma 5.1 of Bechtel (2023) is given, which is applied for evolution families.

Lemma A.3.1 (Evolution family product rule)**.** *Let U be a bounded differentiable family of operators with bounded derivative, and let $u \in W^{1,p}((0,T); X_0)$. Then $t \mapsto U(t)u(t) \in W^{1,p}((s,T); X_0)$, with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}(Uu)(t) = -U'(t)u(t) + U(t)u'(t).$$

Similarly, for all

Proof. Since we have $u \in W^{1,p}((0,T);X_0)$, we can write

$$u(t) - u(s) = \int_s^t u'(\tau)\ \mathrm{d}\tau, \quad 0 \leq s < t \leq T.$$

Similarly, the derivative U' can also be used to write the operator $U(t) - U(s)$ as $\int_s^t U'(\tau)\ \mathrm{d}\tau$. We expect a certain form of the derivative of $U(\cdot)u(\cdot)$, so we check whether we can rewrite this expected form to show it to indeed be equal to the derivative.

$$\begin{aligned}\int_s^t U'(\tau)u(\tau) + U(\tau)u'(\tau)\ \mathrm{d}\tau &= \int_s^t U'(\tau)\left(u(s) + \int_s^\tau u'(r)\mathrm{d}r\right) + \left(U(s) + \int_s^\tau U'(r)\mathrm{d}r\right)u'(\tau)\ \mathrm{d}\tau \\ &= \int_s^t U'(\tau)u(s)\mathrm{d}\tau + \int_s^t S(s)u'(\tau)\mathrm{d}\tau \\ &\quad + \int_s^t \int_s^\tau U'(\tau)u'(r)\mathrm{d}r\mathrm{d}\tau + \int_s^t \int_s^\tau U'(r)u'(\tau)\mathrm{d}r\mathrm{d}\tau.\end{aligned}$$

We apply Fubini to get $\int_s^t \int_s^\tau U'(\tau)u'(r)\mathrm{d}r\mathrm{d}\tau = \int_s^t \int_r^t U'(\tau)u'(r)\mathrm{d}\tau\mathrm{d}r$, and on the other integral we switch the variable names around to get $\int_s^t \int_s^\tau U'(r)u'(\tau)\mathrm{d}r\mathrm{d}\tau = \int_s^t \int_s^r U'(\tau)u'(r)\mathrm{d}\tau\mathrm{d}r$.

$$\begin{aligned}\int_s^t U'(\tau)u(\tau) + U(\tau)u'(\tau)\ \mathrm{d}\tau &= \int_s^t U'(\tau)u(s)\mathrm{d}\tau + \int_s^t S(s)u'(\tau)\mathrm{d}\tau + \int_s^t \int_s^t U'(\tau)u'(r)\mathrm{d}\tau\mathrm{d}r \\ &= \int_s^t U'(\tau)u(s)\mathrm{d}\tau + \int_s^t U(s)u'(\tau)\mathrm{d}\tau + \int_s^t U'(\tau)\mathrm{d}\tau \int_s^t u'(r)\mathrm{d}r \\ &= U(t)u(s) - U(s)u(s) + U(s)u(t) - U(s)u(s) \\ &\quad + U(t)u(t) - U(s)u(t) - U(t)u(s) + U(s)u(s) \\ &= U(t)u(t) - U(s)u(s).\end{aligned}$$

Here we used $u(t) - u(s) = \int_s^t u'(\tau)\ \mathrm{d}\tau$ and $U(t) - U(s) = \int_s^t U'(\tau)\ \mathrm{d}\tau$ to write out all the terms. This allows us to conclude that we indeed have this form of the derivative, and therefore $U(\tau)u(\tau) \in W^{1,p}((a,b);X_0)$. $\square$

Bibliography

Bechtel, S. (2023). Weighted non-autonomous $L^q(L^p)$ maximal regularity for complex systems. https://arxiv.org/abs/2208.02527

Denk, R., Hieber, M., & Prüss, J. (2003). $\mathscr{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, *166*(788), viii+114. https://doi.org/10.1090/memo/0788

Di Giorgio, D., Lunardi, A., & Schnaubelt, R. (2005). Optimal regularity and Fredholm properties of abstract parabolic operators in L^p spaces on the real line. *Proc. London Math. Soc. (3)*, *91*(3), 703–737. https://doi.org/10.1112/S0024611505015406

Hytönen, T., van Neerven, J., Veraar, M., & Weis, L. (2017). *Analysis in Banach spaces: Volume II: Probabilistic methods and operator theory*. Springer. https://doi.org/10.1007/978-3-319-69808-3

Hytönen, T., van Neerven, J., Veraar, M., & Weis, L. (2024). *Analysis in Banach spaces: Volume III: Harmonic analysis and spectral theory*. Springer. https://doi.org/10.1007/978-3-031-46598-7

Hytönen, T., van Neerven, J., Veraar, M., & Weis, L. (2016). *Analysis in Banach spaces: Volume I: Martingales and Littlewood-Paley theory*. Springer. https://doi.org/10.1007/978-3-319-48520-1

LeCrone, J., Prüss, J., & Wilke, M. (2014). On quasilinear parabolic evolution equations in weighted L_p-spaces II. *J. Evol. Equ.*, *14*(3), 509–533. https://doi.org/10.1007/s00028-014-0226-6

Lindemulder, N., & Veraar, M. (2020). The heat equation with rough boundary conditions and holomorphic functional calculus. https://arxiv.org/abs/1805.10213v3

Lunardi, A. (1995). *Analytic semigroups and optimal regularity in parabolic problems* [[2013 reprint of the 1995 original] [MR1329547]]. Birkhäuser/Springer Basel AG, Basel.

Meyries, M., & Veraar, M. (2012). Sharp embedding results for spaces of smooth functions with power weights. https://arxiv.org/abs/1112.5388

Meyries, M., & Veraar, M. (2014). Pointwise multiplication on vector-valued function spaces with power weights. https://arxiv.org/abs/1311.7404v2

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations* (Vol. 44). Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-5561-1

Prüss, J., Simonett, G., & Wilke, M. (2018). Critical spaces for quasilinear parabolic evolution equations and applications. *Journal of Differential Equations*, *264*(3), 2028–2074. https://doi.org/10.1016/j.jde.2017.10.010

Prüss, J., & Wilke, M. (2017). Addendum to the paper "on quasilinear parabolic evolution equations in weighted L^p-spaces ii". *Journal of Evolution Equations*, *17*(4), 1381–1388. https://doi.org/10.1007/s00028-017-0382-6

Schnaubelt, R. (2004). Asymptotic behaviour of parabolic nonautonomous evolution equations. In *Functional analytic methods for evolution equations* (pp. 401–472). Springer, Berlin. https://doi.org/10.1007/978-3-540-44653-8_5

Taylor, M. E. (2000). *Tools for PDE* (Vol. 81) [Pseudodifferential operators, paradifferential operators, and layer potentials]. American Mathematical Society, Providence, RI. https://doi.org/10.1090/surv/081

van Neerven, J. (2022). *Functional analysis* (Vol. 201). Cambridge University Press, Cambridge. https://doi.org/10.1017/9781009232487

Yagi, A. (2010). *Abstract parabolic evolution equations and their applications*. Springer. https://doi.org/10.1007/978-3-642-04631-5

List of symbols

Function spaces

$L^p(I;X)$	Space of functions with p-integrability for $p \in [1,\infty]$ on the interval I in the norm of the Banach space X, page 14.
$L^1_{loc}(I;X)$	Space of functions which are integrable on any finite sub-interval of I in the norm of the Banach space X, page 14.
$C(I;X)$	Space of continuous functions on I in the norm of the Banach space X, page 16.
$C^\mu(I;X)$	Space of Hölder continuous functions on I in the norm of the Banach space X with fractional power μ, page 16.
$W^{1,p}(I;X)$	Sobolev space of functions with p-integrability for $p \in [1,\infty]$ on the interval I in the norm of the Banach space X, page 16.
$W^{1,p}_{loc}(I;X)$	Sobolev space of functions with local p-integrability for $p \in [1,\infty]$ on the interval I in the norm of the Banach space X,page 16.
${}_0\dot{W}^{1,p}(I;X)$	Solution space of the time derivative on the interval I in the norm of the Banach space X,page 16.
${}_0\dot{W}^{1,p}_A(I;X)$	Solution space of the equation Eq. (1.1) on the interval I in the norm of the Banach space X,page 16.
$\mathcal{P}^p(I)$	Maximal L^p-regularity space on I, page 16.
$\mathcal{E}^p(I)$	Maximal L^p-regularity space on I for non-autonomous problems on non-constant domains, page 22.
$L^p_{\omega_\alpha}(I;X)$	Weighted L^p space with power weight ω_α on interval I in the norm of the Banach space X, page 28.
$W^{1,p}_{\omega_\alpha}(I;X)$	Weighted Sobolev space with power weight ω_α on interval I in the norm of the Banach space X, page 28.
$\mathcal{E}^p_\alpha(I)$	Maximal $L^p_{\omega_\alpha}$-regularity space on I for non-autonomous problems on non-constant domains, page 28.
E_0	$L^p_{\omega_\alpha}((0,T);X_0)$ for the time T working on X_0, page 31.
Y_{Tr}	$C([0,T];X_{\sigma,p})$ for the time T working on the real interpolation space $X_{\sigma,p}$ for $\sigma = 1-\alpha-\frac{1}{p}$, page 31.
$\mathcal{P}^p_{\alpha,A}(I)$	Maximal $L^p_{\omega_\alpha}$-regularity space on I for the operator A, page 32.
$\mathcal{E}^p_{\alpha,u}(I)$	Maximal $L^p_{\omega_\alpha}$-regularity space on I for the operator A_u with given solution u, page 36.
$C^\mu_{\{0\}}(I;X)$	Space of Hölder continuous functions with fractional power μ and difference measured on $t=0$ in the norm of the Banach space X, page 36.
$Y_{\{0\}}$	$C^\mu_{\{0\}}([0,T];X_\theta)$ for the time T working on the complex interpolation space X_θ, page 36.
Y_j	$L^{p(\rho_j+1)}_{\frac{\alpha}{\rho_j+1}}((0,T);X_{\beta_j})$ for $j \in \{1,\dots,m\}$, $\rho_j > 0$ and some complex interpolation space X_{β_j},page 36.

Y_μ	$C^\mu([0,T];X_\theta)$, page 36.
$Y(0,T)$	Union of $Y_{\{0\}}$, Y_{Tr}, and all Y_j for $j \in \{1,\dots,m\}$ on interval $(0,T)$, page 36.
$SOL(G,x_0,f,T)$	Union of solution spaces, page 36.
$W^{2,q}_{B(u(t,x))}(\Omega)$	The domain of the operator $A(u)$ with the boundary condition $B(u)$, page 54.
$H^{s,q}(\Omega)$	The Bessel potential space with fractional power s and integrability q working on Ω, page 65.
$B^s_{q,p}(\Omega)$	The Besov space with fractional power s, integrability q and interpolation p working on Ω, page 65.

Operators

∂_t	Derivative towards time, page 14.
$(A,D(A))$	Unbounded operator A with domain $D(A)$ which represents the space derivatives, page 14.
$S(t)$	C_0-semigroup depending on t, page 14.
$R(\lambda,A)$	Resolvent operator $(\lambda - A)^{-1}$ for complex number λ and operator A, page 15.
e^{-tA}	Bounded analytic C_0-semigroup generated by operator A, page 15.
$\mathcal{M}$	Solution operator of the problem Eq. (1.1), page 17.
$(A(t),D(A(t)))$	Non-autonomous linear operator $A(t)$ with domain $D(A(t))$ depending on a time t, page 20.
$G(t,s)$	Evolution family operator of $A(t)$ with dependency on times t and s, page 20.
∂_t^+	One-sided time derivative from above, page 23.
$A_n(t)$	Yosida approximation of $A(t)$, page 23.
$G_n(t,s)$	Evolution family operator of $A_n(t)$, page 23.
ω_α	Power weight with power α, page 28.
Φ	Solution operator mapping functions feasible as solutions to the non-linear problem to solutions of a given linear problem, page 33.
$(A(u,t),D(A(u,t)))$	Non-autonomous quasi-linear operators $A(u,t)$ depending on a solution u and time $t$with domains $D(A(u,t))$, page 36.
$(A_u(t),D(A_u(t)))$	Non-autonomous linear operators $A(u(t),t)$ for a given solution $u(t)$ depending on time t with domains $D(A_u(t))$, page 36.
$G_u(t,s)$	Evolution family of $A_u(t)$, page 36.
$A(u)$	The operator of the Neumann problem, page 54.
$B(u)$	The boundary condition of the Neumann problem, page 54.
$A_q(u)$	Realisation of $A(u)$, page 54.

Functions

$f(t)$	Right-hand side function depending on time t, page 14.
$F(u)$	Non-linear right-hand side function, page 31.
$z_{u_0}(t)$	Reference solution depending on time t, page 32.
$F_u(t)$	Right-hand side function with given solution u depending on time t, page 36.
$F_{Tr}(u)$	Trace part of $F(u)$, page 37.
$F_c(u)$	Critical part of $F(u)$, page 38.
$U_{f,v}(t)$	Solution to non-autonomous linear problem with 0 IC, right-hand side $f \in L^p_{\omega_\alpha}((0,T);X_0)$

	and given solution v on time t, page 40.
$w_s(t)$	Solution to autonomous linear problem with 0 IC, right-hand side $f \in L^p_{\omega_\alpha}((0,T);X_0)$ and operator $A_v(s)$ for solution v and time s, page 40.
$a_{ij}(u)$	The matrix coefficients of $A(u)$, page 54.
$\tilde{a}_{ij}(u)$	The mollified matrix coefficients of $A(u)$, page 59.

Banach spaces

X_0	Base Banach space, page 14.
$\mathscr{L}(X_0,X_1)$	Space of bounded linear operators from the Banach space X_0 to the Banach space X_1, page 14.
$\mathscr{L}(X_0)$	Space of bounded linear operators on the Banach space X_0, page 23.
$[X_0,X_1]_\theta$	Complex interpolation space between the spaces X_0 and X_1 of order θ, page 65.
$(X_0,X_1)_{\theta,r}$	Real interpolation space between the spaces X_0 and X_1 of order θ and power r, page 65.

Other

I	Time interval either equal to $(0,T)$ for some $T>0$ or equal to $[0,\infty)$, page 14.
$X_1 \hookrightarrow X_0$	Symbol representing there exists a continuous embedding from X_1 to X_0, page 14.
$M^{reg}_{p,A}(I)$	Constant of maximal L^p-regularity for the operator A on interval I, page 14.
$\mathcal{MR}_p(I)$	Set of maximally L^p-regular operators on interval I, page 14.
Σ_ω	Open sector in the complex plane of angle ω, page 14.
$\rho(A)$	Resolvent set of operator A, page 15.
$\omega(A)$	Sectoriality angle of A, page 15.
$\omega_R(A)$	R-sectoriality angle of A, page 20.
u_0	Initial condition in a given interpolation space, page 25.
σ	$1-\alpha-\frac{1}{p}$ for $p\in(1,\infty)$ and $\alpha\in(-\frac{1}{p},\frac{1}{p'})$, page 25.
$B_R(0;X)$	Open ball around 0 in the Banach space X with radius $R>0$, page 32.
$B^T(u_0)$	Open ball around z_{u_0} in the Banach space $Y_{Tr}(0,T$ with radius 1, page 33.
p_j	$p(\rho_j+1)$ for $p\in(1,\infty)$, $j\in\{1,\dots,m\}$ and $\rho_j>0$, page 36.
α_j	$\frac{\alpha}{\rho_j+1}$ for $\alpha\in[0,\frac{1}{p'})$, $j\in\{1,\dots,m\}$ and $\rho_j>0$, page 36.
$\mathcal{Q}^p_\alpha(I)$	Set of u with $u\in\mathcal{E}^p_{\alpha,u}(I)$, page 36.
$\mathcal{AT}^p_\alpha(I)$	Set of u on which $A_u(t)$ satisfies the Acquistapace Terreni conditions of Assumption 1.2.7, page 37.
β^*_j	$\frac{1+\rho_j\sigma}{1+\rho_j}$ for $j\in\{1,\dots,m\}$ and $\rho_j>0$, page 38.
$B^T_r(u_0)$	Open ball around z_{u_0} in the Banach space $Y(0,T)$ with radius r included in $B_1(0;Y_\mu)$, page 40.
Ω	A smooth domain in $\mathbb{R}^d$, page 54.
$\mathbf{n}$	Outward vector, page 54.