Generalized Pólya urn model with fitness and non-linear reinforcement

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Summary

There exist many different types of competitions, urn models can be used to describe the behavior of these competitions. This report will provide an answer to the question "How does a generalized Pólya urn model, with fitness and non-linear reinforcement behave when time goes to infinity?".

The classical Pólya urn model is a model with balls of different colours contained in an urn. At each time step a ball will be drawn and put back in the urn, along with an additional ball of the same colour. Each colour represents a component in a competition and the amount of balls of a colour is their share in the competition. This is a discrete-time Markov chain. In this report a generalized version of the model with a fitness parameter and a non-linear reinforcement are considered.

Each component has two main properties, fitness and current share in the competition. The fitness denotes how suitable that component is in the competition and does not change over time. This means a component with a larger fitness has a higher chance of winning. Secondly, the share could be denoted by an amount of balls or as a proportion of the total amount. When a component has a larger share it is more likely to win the competition. The probability distribution of the colour of the next ball in the urn is proportional to the fitness of the components, and depends in a non-linear way on their current shares. Together the fitness and non-linear reinforcement will form a feedback function for each component, that determine the transition probabilities in the Pólya urn model.

To analyse the model, the discrete-time Markov chains will be translated to continuous-time Markov chains using a standard method in the research of urn models, called exponential embedding.

When time goes to infinity the model will almost always reach a stable state, depending on the value of the non-linear reinforcement. The stable state can either be an equilibrium where all components have a non-zero proportion of the total amount, or a monopoly can be reached. When a monopoly is reached one component essentially has all but a finite amount of the total amount of balls in the urn, which means it has proportion 1, whereas the rest has proportion 0. Whether the model reaches a stable equilibrium or a monopoly depends on the strength of the non-linear reinforcement. However, the proportion of each component in a stable state depends on the fitness each component has. In the equilibrium case the stationary point is determined by the fitness and reinforcement strength and is independent of the current share in the competition. The higher the fitness of a component, the more likely it is this will component reach monopoly, or have a larger proportion in the stable equilibrium. Apart from the fitness, the initial condition of the components is also of influence on the stable state in case of monopoly. Whenever a component has a higher initial condition, it is more likely to win the competition, when time goes to infinity.

Summary for layman

Consider a competition consisting of market share, such as the market share of web searchers. Currently Google has over 90% of the market share in this field and has left all of its competitors behind. In this report the distribution of the market share of such a competition will be described with a model called the Pólya urn model. Each competitor has a fitness which denotes how suitable the competitor is, this means the higher the fitness the better. In addition, there exists a reinforcement in the dynamics, which is the same for all competitors. This reinforcement models the advantage of having a larger market share. After a large amount of time, call it an infinite amount, the model will almost always reach a stable state. Meaning that the proportion of market share each competitor has, will no longer change. In this report it can be seen for what strength of reinforcement the model reaches either a monopoly or another stable state in which all competitors have a non-zero proportion of the total amount. The proportion each competitor has in the end depends on the fitness and their initial condition. This is only one example of a competition for which the Pólya urn model can be used, but the model can be adapted for many other types of competitions.

Contents

1 Introduction

This report will provide an answer to the question "How does the Pólya urn model, with fitness and non-linear reinforcement behave when time goes to infinity?". The probable outcomes of competitions between multiple components and the interplay between fitness and non-linear reinforcement can be described using the results in this report.

To illustrate what a Pólya urn model can describe the competition between web searches can be looked at. Whenever you need an answer to a question you will probably use Google to find it. Google is the biggest web searcher and currently has a market share of over 90%. It has left all of it's competitors like Yahoo behind, even though for example Yahoo started out with a significant market share advantage. Google managed to win that competition, presumably due to a better product.

This is only one example, but there are many types of competitions and hence models. The outcome of such a competition depends on multiple variables and this report contains results for a generalized Pólya urn model with fitness and non-linear reinforcement that give a clear understanding as to when monopoly or a different stable state is reached in the model. As well as those results, flow fields of a specific model are given to understand the asymptotic dynamics within the model.

The report is structured as follows. First the generalized Pólya urn model that is used will be defined and explained. This is a discrete-time Markov chain and using exponential embedding it will be translated to a continuous-time Markov chain. Next, the concept of strong monopoly will be formally defined and it will be proven for which functions the process reaches strong monopoly. In Section 3 the attractive domains will be described for a specific kind of function using a lemma on expected explosion times, which is also proven in that section. In Section 4 the flow fields for the model are determined and some general properties for these flow fields are given. As well as those results, some examples will be given to illustrate the flow fields in different cases. Lastly, in Section 5 a result to substantiate the function of the flow fields will be given and a Lyapunov function to substantiate results from Section 4.

1.1 Pólya urn model

An urn model is a probabilistic model consisting of a number of particles of different types, often denoted by balls of different colours contained in an urn. At each time step, a set of these balls is drawn from the urn after which the contents of the urn might be altered.

The first Pólya urn model appeared in Eggenberger and Pólya in 1923 [5]. Since then many variations of urn models have been studied. Contrary to Benaim [1] who studied a model with multiple urns interacting with each other, this report focuses on models with a single urn with balls of different colours, the generalized Pólya urn model, variations of which are discussed in Pemantle [7].

Throughout this report colour/type/agent/component will be used interchangeably. This model can represent biological or economical applications of which some examples are given by Pemantle [7]. From the set of balls of different colours, which will also be called components, one ball will be drawn at random, at each time step. The colour of this ball will be noted, after which the ball is returned to the urn along with an additional ball of the drawn colour. The probability of drawing a ball of a certain colour will not only depend on the proportion of that colour in the urn but also on a fitness each colour has and a non-linear reinforcement.

In this report each of the colours or components, has a positive fitness and feedback function that does not depend on other colours. Therefore those with a high fitness are often more likely to win and contrary to Costa and Jordan [3], the types do not interact among each other. A process of "Balls-in-Bins" with similar feedback was also studied by Oliveira [6], however Oliveira did not consider a fitness for each component.

The model will now be formally defined. One urn will be considered, consisting of balls of k colours. The number of balls of each colour at a time $n =$ 0,1,2,... will be denoted by the vector $X(n) = (X_1(n), X_2(n), \dots, X_k(n)) \in \mathbb{N}_0^k$. Since the colour that is drawn does not depend on previous draws, but only on the current configuration of balls in the urn, the process $(X(n) : n \in \mathbb{N}_0)$ is a discrete-time Markov chain on state space $\mathbb{N}_0^k$. Each colour $i = 1, \dots, k$ has a feedback function $F_i(x_i) > 0$, depending on the number of balls x_i of type i. The transition probabilities are then given by

$$p(x, i) = \mathbb{P}[X(n+1) = x + e^{(i)} | X(n) = x] = \frac{F_i(x_i)}{\sum_{j=1}^k F_j(x_j)}, \tag{1}$$

where $e^{(i)}$ is a unit vector in direction i. Note that in every time step we add exactly one new ball, and equation (1) gives the probability that it is of type i and $X(n+1) = X(n) + e^{(i)}$. Whenever a particular feedback function is considered in this report instead of a one, it will be,

$$F_i(x_i) = \alpha_i x_i^{\beta}. \tag{2}$$

Here $\alpha_i > 0$ is a fitness parameter corresponding to each $i = 1, \ldots, k$, the reinforcement strength is a positive real number $\beta > 0$ and thus $F_i : \mathbb{N}_0 \to [0, \infty)$. Whenever the reinforcement is taken $\beta = 1$, the model corresponds to the classical linear Pólya urn. The initial number of balls in denoted by $n_0 = \sum_{i=1}^{k} X_i(0)$.

For a better analysis of the model, the discrete-time Markov chains (DTMC) are translated into continuous-time Markov chains (CTMC), which will be explained in the next section.

1.2 Exponential embedding

In this section the discrete-time Markov chain $(X(n) : n \in \mathbb{N}_0)$ will be defined in terms of a jump chain of a simpler continuous-time Markov chain, which is called exponential embedding. This is a standard method in the research on urn models (see [2] [4] [6] [7]). For the exponential embedding of the model, consider a birth process $(Y(t) : t \geq 0)$ on state space $\mathbb{N}_0^k$. In this process the components $Y_i(t)$ evolve independently for each $i = 1, 2, \ldots, k$ for $t \geq 0$ and can only increase by one or remain the same. First some definitions are given that will be used throughout the report.

The process starts at Y(0) with initial condition $Y_i(0) = X_i(0)$ for each i. The *birth rate* of component i is given by $F_i(y_i)$, which is the rate at which a jump is made from state y to $y + e^{(i)}$. The *waiting times* in a state $y \in \mathbb{N}_0^k$ for component i are $\tau_i(y_i) \sim \exp(F_i(y_i))$ for each i and are independent. Now for the *jump times* $J_i(z)$, out of a state $z \in \mathbb{N}_0^k$, take the sum of the τ_i, starting from the initial condition of component i, up until z. Hence this gives $J_i(z) = \sum_{y_i = y_i(0)}^{z} \tau_i(y_i)$ with a value in $[0, \infty)$, where $J_i(z) = 0$ if $z < Y_i(0)$, corresponding to the empty sum. Lastly, the *explosion time* T_i of a component i is the moment that the process $(Y_i(t) : t \geq 0)$ reaches infinity, defined as

$$T_i := \lim_{z \to \infty} J_i(z) = \sum_{y_i = y_i(0)}^{\infty} \tau_i(y_i) \in (0, \infty]. \tag{3}$$

Remark. If $T_i = \infty$, the process $Y_i(t)$ does not explode.

To see how this birth process translates into the urn model, the jump chain of the birth process will be looked at and then from the transition probabilities it can be seen that the jump chain and discrete processes have the same distribution. First the jump chain is defined.

Let $J(n) \geq 0$ be the time at which the n-th jump occurs in the CTMC $(Y(t) : t \geq 0)$. Then $J(0) = 0$ and $J(n+1) = \min\{J_i(z) > J(n) : i = 1, \ldots, k \text{ and } z \in \mathbb{N}\}$. As mentioned before, this *jump chain* $(Y(J(n)) : n \in \mathbb{N}_0)$ of the CTMC is a DTMC. The transition probabilities of the jump chain are given by the relative rates as

$$\mathbb{P}[Y(J(n+1)) = y + e^{(i)} | Y(J(n)) = y] = \frac{F_i(y_i)}{\sum_{j=1}^{k} F_j(y_i)}, \tag{4}$$

which is the same as equation (1) for the urn process. Since also $X(0) = Y(0)$, the two processes $(X(n) : n \in \mathbb{N}_0)$ and $(Y(J(n) : n \in \mathbb{N}_0)$ have the same distribution.

Hence the urn process can be defined as $(X(n) := Y(J(n)) : n \in \mathbb{N}_0)$ in terms of the jump chain of $(Y(t) : t \geq 0)$, which is called exponential embedding. With this definition of the process, events that occur in the Pólya urn model can be explained in terms of the explosion of the birth process. In the next section it will be used when the event of a monopoly is discussed.

2 Monopoly

In a Pólya urn model it can occur that after a certain time all new balls added to the urn will be of one colour. This is an important event in the model and will be discussed in this section following [4] and [6].

Consider an urn model $(X(n) : n \in \mathbb{N}_0)$ with transition probabilities as given in equation (1), with general feedback functions $F_i(z), z \in N_0$. Let $s(i) = \sum_{z=1}^{\infty} \frac{1}{F_i(z)}$ and $J = \{i : s(i) < \infty\}$. Let the urn model be such that at least for one of the feedback functions $\sum_{z=1}^{\infty} \frac{1}{F_i(z)} < \infty$, and thus $J \neq \emptyset$.

Definition 2.1 (Strong Monopoly)**.** Strong monopoly for agent j is reached whenever for

$$\sum_{i \neq j} X_i(\infty) = \lim_{n \to \infty} \sum_{i \neq j} X_i(n) =: U < \infty, \tag{5}$$

i.e. agent j has all but a finite amount U of the total amount, as $n \to \infty$.

Theorem 2.1. *Consider an urn model $(X(n) : n \in \mathbb{N}_0)$ with feedback functions $F_i(z)$ satisfying $\sum_{z=1}^{\infty} \frac{1}{F_i(z)} < \infty$ for at least one i. Then the process reaches strong monopoly as in Definition 2.1 for some agent $i \in J$ for all initial conditions $X(0) \in \mathbb{N}^k$.*

Proof. Exponential embedding is used to define the DTMC $(X(n) : n \in \mathbb{N}_0)$ in terms of the jump chain of the CTMC $(Y(t) : t \geq 0)$ with waiting times $\tau_i(z)$ and explosion times T_i as defined defined in Section 1.2. Note that for all $i \in N$,

$$\mathbb{E}\left[T_i\right] = \sum_{z=X_i(0)}^{\infty} \frac{1}{F_i(z)} \leq s(i) < \infty,$$

which implies that $\mathbb{E}[T_i] < \infty$ and therefore almost surely, $T_i < \infty$. For all $i \notin N$ almost surely $T_i = \infty$.

The random variables $\{T_i\}_{i \in [k]}$ are independent and have a continuous distribution without point masses, hence with probability 1 they are distinct. Suppose i is such that

$$T_i = \min_{1 \leq j \leq k} T_j.$$

Then for each $j \neq i$ there exists a finite number $p(j)$ such that

$$\sum_{z=X_j(0)}^{p(j)-1} \tau_j(z) < T_i < \sum_{z=X_j(0)}^{p(j)} \tau_j(z).$$

Define

$$M = \max_{j \neq i} \sum_{z=X_i(0)}^{p(j)-1} \tau_i(z).$$

This means after time M all new balls will be of type i. Thus agent i receives infinitely many balls whereas agents $j \neq i$ receive $p(j)$ balls and

$$\mathbb{P}[\text{strong monopoly for i}] = \mathbb{P}[\exists p(j) : x_j(\infty) \leq p(j) \text{ for all } j \neq i] = 1. \tag{6}$$

By Definition 2.1 agent i achieves strong monopoly almost surely. $\square$

It will be convenient to describe the state of the system by proportions $\chi = \frac{x}{\|x\|_1} \in \Delta^{k-1}$ where $\|x\| = \sum_{i=1}^{k} x_i$ is the total amount in the system. Here Δ^{k-1} is the standard simplex in $\mathbb{R}^k$ and $\sum_{i=1}^{k} \chi_i = 1$ for each χ. More on this will be explained in section 4. Note that strong monopoly for agent j implies that $\chi(n) \to \chi(\infty) = e^{(j)}$ as $n \to \infty$, which is also referred to as a weak form of monopoly.

3 Main result

In this section two results for the specific feedback function $F(z) = \alpha z^\beta$ will be formulated and proven, in the limit of diverging initial conditions.

3.1 Concentration of explosion times

The first result is on the concentration of explosion times. This lemma will be used to prove Theorem 3.2 on attractive domains.

Lemma 3.1. *Consider a birth process $(Y(t) : t \geq 0)$ as defined in Section 1.2, with rates $F(z) = \alpha z^\beta$, $\beta > 1$ and $\alpha > 0$. Let the initial condition be of the form $Y(0) = \chi(0)n_0$ for some $\chi(0) \in (0, 1)$ and the explosion time $T = \sum_{z=Y(0)}^{\infty} \tau(z)$, where $\tau(z) \sim Exp(\alpha z^\beta)$ are independent. Almost surely, $\frac{T}{\mathbb{E}[T]} \to 1$ as $n_0 \to \infty$.*

Proof. First the expected explosion time is computed, where $Y(0) = \chi(0)n_0$.

$$\begin{aligned}
\mathbb{E}[T] &= \sum_{z=Y(0)}^{\infty} \frac{1}{\alpha z^\beta} \\
&\simeq \frac{n_0^{1-\beta}}{\alpha} \frac{1}{n_0} \sum_{z=\chi(0)n_0}^{\infty} \frac{1}{(z/n_0)^\beta} \\
&\simeq n_0^{1-\beta} \frac{\chi_0^{1-\beta}}{\alpha(\beta-1)}
\end{aligned}$$

as $n_0 \to \infty$. Then compute the variance of $\frac{T}{\mathbb{E}[T]}$.

$$\begin{aligned}
Var\left[\frac{T}{\mathbb{E}[T]}\right] &= \frac{1}{n_0^{2(1-\beta)}} \frac{\alpha^2(\beta-1)^2}{\chi(0)^{2(1-\beta)}} \sum_{z=Y(0)}^{\infty} \frac{1}{\alpha^2 z^{2\beta}} \\
&= \frac{(\beta-1)^2}{(\chi(0)n_0)^{2(1-\beta)}} \frac{1}{n_0^{2\beta-1}} \frac{1}{n_0} \sum_{z=\chi(0)n_0}^{\infty} \frac{1}{(z/n_0)^{2\beta}} \\
&= \frac{(\beta-1)^2}{2\beta-1} \frac{n_0^{1-2\beta}}{n_0^{2(1-\beta)}} \frac{\chi(0)^{1-2\beta}}{\chi(0)^{2(1-\beta)}} \\
&= \frac{(\beta-1)^2}{2\beta-1} \frac{1}{n_0} \frac{1}{\chi(0)}
\end{aligned}$$

This implies only a weak law of large numbers, to get an almost sure version a fourth central moment is needed for a standard concentration argument. This is computed using the cumulant generating function where the logarithm is expanded with the Maclaurin series. This function is as follows.

$$
\begin{aligned}
K_T(\theta) &= \log(\mathbb{E}[e^{\theta T}]) \\
&= \sum_{z=Y(0)}^{\infty} \log \frac{\alpha z^{\beta}}{\alpha z^{\beta} - \theta} \\
&= \sum_{z=Y(0)}^{\infty} - \log(1 - \frac{\theta}{\lambda_z}) \\
&= \sum_{z=Y(0)}^{\infty} \frac{\theta}{\alpha z^{\beta}} + \frac{1}{2}(\frac{\theta}{\alpha z^{\beta}})^2 + \frac{1}{3}(\frac{\theta}{\alpha z^{\beta}})^3 + \frac{1}{4}(\frac{\theta}{\alpha z^{\beta}})^4.
\end{aligned}
$$

From this the fourth cumulant can be found which is the fourth derivative of the cumulant generating function at $\theta = 0$.

$$
\begin{aligned}
\frac{d^4}{d\theta^4} K_T(\theta)\Big|_{\theta=0} &= \frac{6}{\alpha^4} \sum_{z=Y(0)}^{\infty} \frac{1}{z^{4\beta}} \\
&= n_0^{1-4\beta} \frac{6}{\alpha^4} \frac{\chi(0)^{1-4\beta}}{4\beta - 1}.
\end{aligned}
$$

Then the fourth central moment can be found.

$$
\begin{aligned}
\mathbb{E}\left[(T - \mathbb{E}[T])^4\right] &= K_4 + 3K_2^2 \\
&= c_1 n_0^{1-4\beta} + c_2 n_0^{2-4\beta} \\
\mathbb{E}\left[\left(\frac{T}{\mathbb{E}[T]} - 1\right)^4\right] &= c_1 n_0^{-3} + c_2 n_0^{-2} \\
&\leq C n_0^{-2}.
\end{aligned}
$$

Now, using the Chebychev's inequality for higher moments, the following is obtained

$$
\begin{aligned}
\mathbb{P}\left[\left|\frac{T}{\mathbb{E}[T]} - 1\right| > \epsilon\right] &\leq \frac{\mathbb{E}\left[\left(\frac{T}{\mathbb{E}[T]} - 1\right)^4\right]}{\epsilon^4} \\
&\leq \frac{C}{n_0^2 \epsilon^4}.
\end{aligned}
$$

Let $\epsilon = \epsilon_{n_0} = \frac{1}{n_0^{1/8}}$ then,

$$
\mathbb{P}\left[\left|\frac{T}{\mathbb{E}[T]} - 1\right| > \epsilon_{n_0}\right] \leq C n_0^{-\frac{3}{2}}
$$

Now define the sequence of events $(A_{n_0})_{n_0 \in \mathbb{N}_0}$ as $A_{n_0} = \{|\frac{T}{\mathbb{E}[T]} - 1| > \epsilon_n - 0\}$, where $\epsilon_{n_0} = \frac{1}{n_0^{1/8}} \to 0$ as $n_0 \to \infty$, to use the Borel Cantelli lemma. As shown above $\mathbb{P}[A_{n_0}] \leq \frac{C}{n_0^2 \epsilon_{n_0}^4}$. By Borel Cantelli $\mathbb{P}[A_{n_0}$ for finitely many $n_0]$=1 hence $\mathbb{P}[\exists n'$ such that $\forall n_0 > n' : |\frac{T}{\mathbb{E}[\mathbb{T}]} - 1| \leq \frac{1}{n_0^{1/8}}] = 1$. So $\mathbb{P}[\frac{T}{\mathbb{E}[T]} \to 1] = 1$ and thus $\frac{T}{\mathbb{E}[\mathbb{T}]} \to 1$ almost surely. □

3.2 Strong monopoly and attractive domains

The next theorem characterizes the asymptotic dynamics of the urn model and the attractive domains of the flow field in Section 4 can be determined.

Theorem 3.2. *Consider the urn model $(X(n) : n \in \mathbb{N}_0)$ with feedback functions $F_i(x_i) = \alpha_i x_i^\beta$ with $\beta > 1$, transition probabilities $p(x, i) = \frac{F_i(x_i)}{\sum_{j=1}^k F_j(x_j)}$ and initial conditions $X_i(0) = \lfloor n_0 \chi_i(0) \rfloor$ for some $\chi(0) \in \Delta^{k-1}$. Then, as $n_0 \to \infty$ the system exhibits strong monopoly and $\chi(\infty) = e^{(j)}$ with probability 1 if $\frac{\chi_j(0)^{1-\beta}}{\alpha_j} < \frac{\chi_i(0)^{1-\beta}}{\alpha_i}$ for all $i \neq j$. This partitions the simplex Δ^{k-1} into deterministic attractive domains for the corner points, which are convex polytopes.*

Proof. For feedback functions $F_i(x_i) = \alpha_i x_i^\beta$ with $\beta > 1$,

$$\sum_{x_i=1}^{\infty} \frac{1}{F_i(x_i)} = \sum_{x_i=1}^{\infty} \frac{1}{\alpha_i x_i^\beta} < \infty \quad \text{for all } i = 1, \ldots, k.$$

By Theorem 2.1, the process reaches strong monopoly for some $j \in \{1, \ldots, k\}$. Then as $n_0 \to \infty$ by Lemma 3.1, almost surely $\frac{T}{\mathbb{E}(t)} \to 1$ and $\frac{\chi_j(0)^{1-\beta}}{\alpha_j} < \frac{\chi_i(0)^{1-\beta}}{\alpha_i}$ implies that

$$\mathbb{E}[T_j] = \frac{(n_0 \chi_j(0))^{1-\beta}}{\alpha_j(\beta - 1)} < \frac{(n_0 \chi_i(0))^{1-\beta}}{\alpha_i(\beta - 1)} = \mathbb{E}[T_i], \tag{7}$$

for all $i \neq j$. Thus $T_j < T_i$ almost surely and the system exhibits monopoly for agent j as $n_0 \to \infty$ and $\chi(\infty) = e^{(j)}$.

The simplex Δ^{k-1} of initial conditions is a convex set. Whenever this is partitioned into two parts by a hyperplane, both parts are convex again. The hyperplanes follow from equality in equation (7), which implies the linear condition

$$\chi_j \alpha_j^{1/(\beta-1)} = \chi_i \alpha_i^{1/(\beta-1)}. \tag{8}$$

These $\frac{k(k-1)}{2}$ conditions partition, by construction, the simplex into k attractive domains for the corner points. Each of these domains is convex since these are the result of multiple splits with hyperplanes. Since the sides of these convex sets are flat, they are all polytopes. Hence the simplex is partitioned into attractive domains for the corner points, which are convex polytopes. □

For $\chi(0)$ on the boundary of an attractive domain, where equation (8) holds for some i and j, $\chi(\infty) = e^{(i)}$ and $\chi(\infty) = e^{(j)}$ are both possible outcomes and the corresponding probabilities depend on lower order scaling of the initial conditions.

4 Flow fields

Flow fields can be used to find and visualize the stationary points and domains of attraction. Recall that n_0 is the fixed initial amount of balls. In order to find the flow fields, the amounts of balls of each colour are translated to proportions as follows,

$$\chi_i(n) = \frac{X_i(n)}{n_0 + n} \in [0, 1], \tag{9}$$

where $\chi(n) = (\chi_1(n), \ldots \chi_k(n))$ and $\sum_{i=1}^{k} \chi_i(n) = 1$. Then for given $n_0 \in \mathbb{N}$ the processes $(X(n) : n \in \mathbb{N}_0)$ and $(\chi(n) : n \in \mathbb{N}_0)$ are equivalent. The process of $\chi(n)$ has state space given by the standard simplex

$$\Delta^{k-1} = \{\chi \in [0, 1]^k : \chi_1 + \ldots \chi_k = 1, \chi_i \geq 0 \text{ for } i = 1, \ldots, k\}. \tag{10}$$

Then, note that

$$\begin{aligned}
\chi(n+1) &= \frac{X(n) + e(n+1)}{n_0 + n + 1} \\
&= \frac{(n_0 + n)\chi(n) + e(n+1)}{n_0 + n + 1} \\
&= \left(1 - \frac{1}{n_0 + n + 1}\right) \chi(n) + \frac{e(n+1)}{n_0 + n + 1},
\end{aligned}$$

where $e(n + 1)$ is the random increment of the process at time $n + 1$. Hence $e(n+1) = e^{(i)}$ with probability $p(x, i)$ as in equation (1). A direct computation, see Costa and Jordan [3], leads to,

$$\chi(n+1) - \chi(n) = \frac{e(n+1) - \chi(n)}{n_0 + n + 1} = \frac{1}{n_0 + n + 1}(G(\chi(n) + \xi(n+1). \tag{11}$$

The right-hand side of equation (11) consists of two parts. The deterministic part $G(\chi(n))$ will form the vector field and a noise part, $\xi(n + 1)$ with mean zero. These two parts are given by,

$$G(\chi(n)) = \mathbb{E}[e(n+1)|\chi(n)] - \chi(n), \tag{12}$$

and

$$\xi(n+1) = e(n+1) - \mathbb{E}[e(n+1)|\chi(n)], \tag{13}$$

where indeed $\mathbb{E}[\xi(n+1)|\chi(n)] = 0$. Now let $\gamma_n = \frac{1}{n_0+n+1}$, then

$$\chi(n+1) - \chi(n) = \gamma_n(G(\chi(n)) + \xi(n+1)). \tag{14}$$

This equation (14) can be seen as a numerical approximation with step size γ_n for the ordinary differential equation (ODE) $d\chi/dt = G(\chi)$. For a small enough γ_n the ODE and asymptotic dynamics of $(\chi(n))_{n\in\mathbb{N}_0}$ are connected, this is called the dynamical system approach [2]. By Costa and Jordan [3] there exists a subset of equilibria for the flow which is induced by the vector field. Assuming that $F_i : [0.\infty) \to [0, \infty)$ are homogeneous and suppose the increment at time $n+1$ is of type i then since the distribution of $e(n+1)$ is $p(x, i)$, the vector field of equation (12) is given by,

$$G_i(\chi) = \frac{F_i(\chi_i)}{\sum_{j=1}^{k} F_j(\chi_j)} - \chi_i \tag{15}$$

for each $i = 1, \ldots, k$, where again F_i is the feedback function. The feedback function has to be homogeneous so that the rescaling factors cancel in the ratio. In particular for homogeneous feedback functions F_i, $F_i(0) = 0$.

In section 5 it will be shown why only the deterministic part of equation (14) is relevant for the vector field.

4.1 General properties

In this subsection some general properties of the flow field with feedback function 2 will be given. In the following subsections these properties will be discussed in the cases of $k = 2$ and $k = 3$.

4.1.1 Stationary points

Using the vector field the stationary points of the model can be found. A point is stationary whenever the vector field equals zero in that point. For general k, $G_i(\chi) = \frac{\alpha_i \chi_i^{\beta}}{\sum_{j=1}^{k} \alpha_j \chi_j^{\beta}} - \chi_i$, so the corner points are always stationary with $G_i(e^l) = \delta_{i,l} - \delta_{i,l} = 0$ for all i, l. Existence of other stationary point, in the interior of the simplex, depends on the α_i and β.

First set $G_j(\chi) = 0$ for all j and rewrite this as $\frac{\alpha_j \chi_j^{\beta-1}}{\sum_{i=1}^{k} \alpha_i \chi_i^{\beta}} = 1$. This suggests

$$\alpha_i \chi_i^{\beta-1} = \alpha_j \chi_j^{\beta-1} \tag{16}$$

for all i and j.

Remark. For the interior stationary point of the flow field, all expected explosion times are equal.

Next, assume $\chi_j = \alpha_j^{1/(1-\beta)}$, this is correct for all $\beta \neq 1$ and solves the equation above for all pairs of i and j since the equation, $1 = 1$ will be obtained. Then the solution can be is renormalized and for the stationary point in the interior of the simplex,

$$\chi_j = \frac{\alpha_j^{1/(1-\beta)}}{\sum_{i=1}^{k} \alpha_i^{1/(1-\beta)}} \tag{17}$$

is obtained. This exists for $\beta \neq 1$ and $k \geq 2$.

In Section 5.2 a Lyapunov function is used to show, there are no limit cycles and the system actually converges to one of the stationary points.

4.1.2 Stability

The stability of the stationary points will determine whether a point is attracting or not and can be determined with the Jacobian or a local analysis. For $\beta = 1$ the stability is discussed in Section 5.2.

For $\beta > 1$ and $\beta < 1$ the flow field around the corner points will be looked at to determine the stability in these points. The case of $k = 2$ will be given which can be extended to general k. In the case of $k = 2$, instead of looking at (1,0), the point $(1 - \epsilon, \epsilon)$ for small ϵ is taken. This point is substituted into equation (18) and the following is obtained

$$G(1-\epsilon,\epsilon) = \begin{pmatrix} \epsilon - 1 + \frac{\alpha_2(1-\epsilon)^\beta}{\alpha_1\epsilon^\beta + \alpha_2(1-\epsilon)^\beta} \\ -\epsilon + \frac{\alpha_1\epsilon^\beta}{\alpha_1\epsilon^\beta + \alpha_2(1-\epsilon)^\beta} \end{pmatrix},$$

and for small $\epsilon \to 0$ to leading order,

$$G(1-\epsilon,\epsilon) \simeq \begin{pmatrix} \epsilon - \frac{\alpha_1\epsilon^\beta}{\alpha_2} \\ -\epsilon + \frac{\alpha_1\epsilon^\beta}{\alpha_2} \end{pmatrix}.$$

Then for $\beta > 1$ the first entry is positive and the second is negative. For $\beta < 1$ it is the other way around. Hence this means for $\beta > 1$ the point (1,0) is attracting and for $\beta < 1$ it is unstable. The same goes for the other corner point (0,1).

This can be generalized to k components. The point $(1-\epsilon, 0, \ldots, \epsilon, \ldots, 0)$ will be considered, close to $(1, 0, \ldots, 0)$, where ϵ can be at any entry $\chi_i, i = 2, \ldots k$. This point is then again substituted in the function for the vector field, this time $G(\chi)$. Most entries will be zero, except for $G_1(\chi)$ and $G_i(\chi)$, those will be similar to the case of k=2. Which means in general, for $\beta > 1$, $G_1(1-\epsilon, 0, \ldots, \epsilon, \ldots, 0) > 0$ and $G_i(1-\epsilon, 0, \ldots, \epsilon, \ldots, 0) < 0$ for $i \neq 1$ and thus is the corner point attractive. Whereas for $\beta < 1$, $G_1(1-\epsilon, 0, \ldots, \epsilon, \ldots, 0) < 0$ and $G_i(1-\epsilon, 0, \ldots, \epsilon, \ldots, 0) > 0$ for $i \neq 1$, hence the point is unstable.

The stability of the stationary point inside the simplex will be opposite of that of the corner points and can be determined by a standard analysis of the Jacobian of G.

4.2 Flow field of the model for k=2

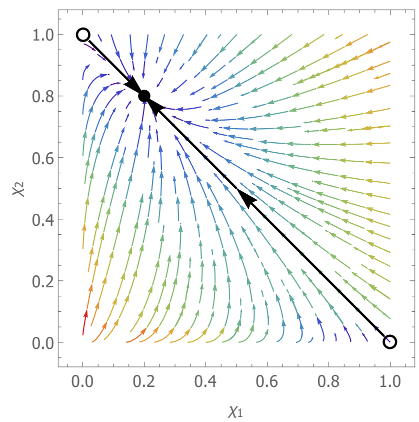
In this subsection the flow induced by a feedback function will be illustrated for a two dimensional model. Considering the case where the feedback function is defined as $F_i(\chi_i) = \alpha_i\chi_i^\beta$ and $k = 2$, hence $i = 1, 2$. This results in the two dimensional vector field

$$G(\chi_1,\chi_2) = \begin{pmatrix} \frac{\alpha_1\chi_1^\beta}{\alpha_1\chi_1^\beta + \alpha_2\chi_2^\beta} - \chi_1 \\ \frac{\alpha_2\chi_2^\beta}{\alpha_1\chi_1^\beta + \alpha_2\chi_2^\beta} - \chi_2 \end{pmatrix} \tag{18}$$

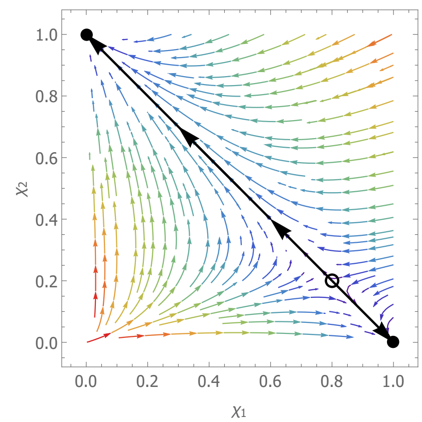
From this it can be seen immediately that both $(1,0)$ and $(0,1)$ are stationary points as well as

$$\chi = \begin{pmatrix} \frac{\alpha_1^{1/(1-\beta)}}{\sum_{i=1}^{k} \alpha_i^{1/(1-\beta)}} \\ \frac{\alpha_2^{1/(1-\beta)}}{\sum_{i=1}^{k} \alpha_i^{1/(1-\beta)}}, \end{pmatrix} \tag{19}$$

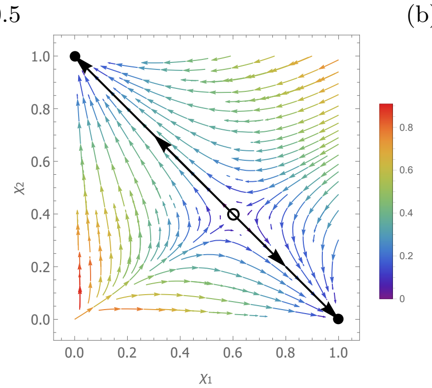
as found in equation (17). Next, the stability of these stationary points will be illustrated for $\beta > 1$ and $\beta < 1$.



(a) $\beta = 0.5$



(b) $\beta = 1.5$



(c) $\beta = 2.5$

Figure 1: *Flow fields for $\alpha_1 = 1$, $\alpha_2 = 2$ and given different β values. For $\beta < 1$ in (a) the corner points of the simplex Δ^1 are unstable and there exists a stable attracting stationary point in the interior of simplex. This stable point is closest to the corner of the component with largest value for α. For $\beta > 1$ in (b) the corner points of the simplex are attracting and the stationary point in the interior is unstable. The unstable point is closest to the corner point of the component with smallest value for α. For larger values of $\beta > 1$ in (c) the stationary point in the interior moves to the middle of the simplex since the ratio between α_1 and α_2 becomes less significant.*

In Figure 1 the vector fields are given corresponding to the values $\alpha_1 = 1$ and $\alpha_2 = 2$. The simplex Δ is given by the black line, with the stable and unstable stationary points denoted by black and white dots respectively. In Figure 1a clearly the two corner points (0,1) and (1,0) are unstable points and there is a stable attractive point at $(0.2, 0.8)$, which is computed using equation (19). This last attractive point can also be determined using the linear relation in equation (8) because in this case of k=2 the simplex is only divided once, hence the stationary point lies on the border of the two domains. Due to the choice for α_1 and α_2 the stable attractive point lies closer to (0,1) in the case of $\beta > 1$. This is because the α corresponding to the χ_2 component is larger, which means that that component has a higher fitness and thus an advantage over χ_1. Hence the attractive point lies closer to (0,1).

The value for β is changed to $\beta = 1.5 > 1$ and figure 1b is found. It can be seen from figure 1b that the corner points in Δ^1 are now stable attractive points, hence black dots. However, the third stationary point has moved to (0.8, 0.2) and is no longer attractive but is now unstable. Since still $\alpha_2 > \alpha_1$ the χ_2 component again has a higher fitness, this can be seen from the attractive domain in the figure. The unstable point has moved towards (1,0), leading to a larger attractive domain for the χ_2 component than the χ_1 component. The attractive domain of (0,1) is in this case the part of the line $\chi_1 + \chi_2 = 1$ from (0,1) to the stationary point around (0.8, 0.2).

To see what happens when a higher value for β is considered as well, Figure 1c with $\beta = 2.5$ is given. In comparison to figure 1b the corner points stay attractive, however only the position of the unstable point has changed, which is due to the higher value for β. This stationary point has moved towards the middle of the simplex, which means the attractive domain of (0,1) decreases but that of (1,0) increases. Since β is an exponent it is of large influence on the flow field. The higher β is, the less significant the difference in α_1 and α_2 becomes, which results in the unstable point moving towards the middle.

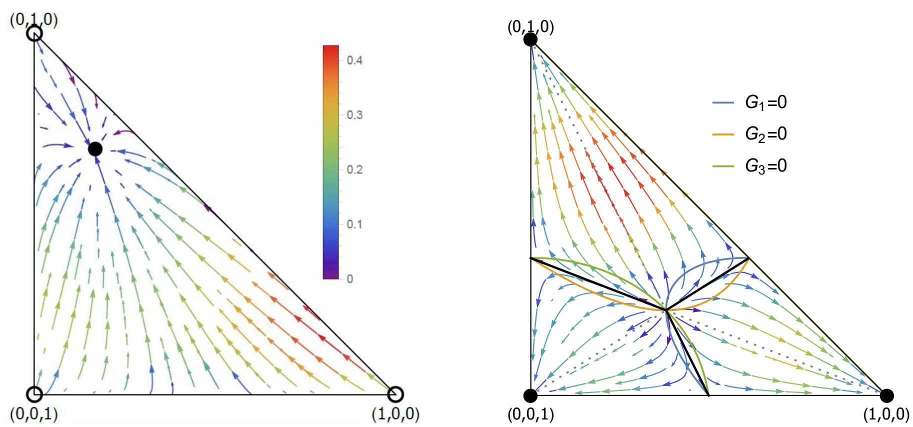
These flow fields for k=2 can easily be extended to k=3. The fields that are obtained will be explained next.

4.3 Flow field of the model for k=3

The vector field for the case of three components is very similar to G for two components and is given by the function

$$G(\chi_1, \chi_2, \chi_3) = \begin{pmatrix} \frac{\alpha_1 \chi_1^\beta}{\alpha_1 \chi_1^\beta + \alpha_2 \chi_2^\beta + \alpha_2 \chi_3^\beta} - \chi_1 \\ \frac{\alpha_2 \chi_2^\beta}{\alpha_1 \chi_1^\beta + \alpha_2 \chi_2^\beta + \alpha_2 \chi_3^\beta} - \chi_2 \\ \frac{\alpha_2 \chi_3^\beta}{\alpha_1 \chi_1^\beta + \alpha_2 \chi_2^\beta + \alpha_2 \chi_3^\beta} - \chi_3 \end{pmatrix}. \tag{20}$$

The vector field will be plotted again and in order to do so χ_3 will be rewritten as $\chi_3 = 1 - \chi_1 - \chi_2$. Furthermore $\chi_1 + \chi_2 \leq 1$ so that, in general, $\sum_i \chi_i = 1$. This will result in a triangular flow field in $\mathbb{R}^3$ which can be drawn as a triangular field in $\mathbb{R}^2$. An example of this field is given in figure 2a. Surely the corner points are stationary points again, hence in this case those are (1,0,0), (0,1,0) and (0,0,1). The stability of these points depend on β, as determined above. Two examples will now be given, showing the difference between $\beta < 1$ and $\beta > 1$.



(a) $\beta = 0.5$

(b) $\beta = 2.5$

Figure 2: *Flow fields for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1$ and given different β values. For $\beta < 1$ in (a) the corner points of the simplex are unstable stationary points. There exists a stable stationary point as in* (17) *in the interior of the simplex, which is closest to the corner point of the component with the highest value for α. For $\beta > 1$ in (b) the corner points of the simplex are stable attractive points. The attractive domains for each of these points are marked by black lines. In the interior of the simplex is an unstable stationary point, furthest away from the component with the highest value for α.*

Contrary to the case of k=2 where the stationary points were on a line, the simplex is now a triangular region in Δ^2. In Figure 2a, $\beta = 0.5 < 1$ is considered again, now with $\alpha_1 = 1, \alpha_2 = 2$ and $\alpha_3 = 1$. The corner points are stationary and not attracting but the point (17) inside the domain is. This time computing two of the relations will give approximately (0.17, 0.67, 0.17). It is closer to the (0,1,0) α_2 is the largest and hence the most fit to win. As one can notice, the field can be interpreted in a similar way to the case of k=2, however a plane is now considered instead of a line. Then for $\beta > 1$ Figure 2b is obtained.

As could be expected for $\beta > 1$, again the corner points are attractive stationary points and there is another stationary point inside the triangle. Now if the attractive domains of the corner points are determined, instead of a line as in the case of k=2, regions are found corresponding to the different components. These regions intersect, and using the expected explosion times of each component, the boundaries of the regions can be found, as was discussed in section 3.2. These boundaries are marked by black lines and it can be seen that the sides are flat, hence indeed these domains are convex polytopes.

5 Asymptotic dynamics

In the section two results are given to substantiate the asymptotic behavior that was described before.

5.1 Noise term of the flow field

To show how the noise of the flow field in equation (14) vanishes, the time will be rescaled by the initial condition n_0, which will be send to infinity.

Theorem 5.1. *Let $(X(n) : n \in \mathbb{N}_0)$ be the urn model with transition probabilities $p(x,i)$ and initial condition $X_i(0) = \lfloor n_0 \chi_i(0) \rfloor$ for some $\chi(0) \in \Delta^{k-1}$. Then the rescaled process $\big(\chi(\lfloor tn_0 \rfloor) : t \geq 0\big)$, extended to continuous time as a step function, converges to a limit process $\big(\chi^\infty(t) : t \geq 0\big)$ as $n_0 \to \infty$. This process is deterministic and solves the following system of ordinary differential equations (ODE),*

$$\frac{d}{dt}\chi^\infty(t) = \frac{G(\chi^\infty(t))}{1+t} \text{ with } \chi^\infty(0) = \chi(0). \tag{21}$$

Proof. First the classical tightness criterion as stated by Williams in proposition C.3 [8] is checked to guarantee that the sequence of rescaled processes has limit points. This criterion consists of two parts, starting with the criterion on boundedness.

The jump size γ_{n+1} is bounded in the supremum norm, $\|\chi(\lfloor tn_0 \rfloor + 1) - \chi(\lfloor tn_0 \rfloor)\|_\infty < \frac{1}{\lfloor tn_0 \rfloor + n_0}$, hence the first criterion is satisfied. For the second criterion the number of jumps in a time interval $[t, t+\epsilon)$ is considered, this has order $n_0\epsilon$ which means the noise is bounded by ϵ. Then as $\epsilon \to 0$ the second criterion is fulfilled as well. This means it is impossible for a lot of jumps to be accumulated in a small time interval. Furthermore, the jump size tends to 0 as $n_0 \to \infty$ which means the limit paths will be continuous. Thus the sequence of processes has weak limit points on path space, which are again proccesses with continuous paths. Now it will be shown that these limit points satisfy the ODE since the noise term will vanish.

For $0 < t_1 < t_2$,

$$\chi(t_2 n_0) - \chi(t_1 n_0) = \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \gamma_{n+1} G(\chi(n)) + \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \gamma_{n+1} \xi(n+1) \tag{22}$$

First the deterministic part will be worked out.

$$\sum_{n=t_1 n_0}^{t_2 n_0 - 1} \gamma_{n+1} G(\chi(n)) = \frac{1}{n_0} \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \frac{1}{1 + n/n_0 + 1/n_0} G(\chi(\frac{n}{n_0} n_0)).$$

Now sending n_0 to infinity, then $\frac{n}{n_0} \to t$ and $\frac{1}{n_0} \to 0$ hence,

$$\int_{t_1}^{t_2} \frac{1}{1+t} G(\chi^\infty(t))\, dt,$$

where $\chi^\infty(t)$ is a limit of the rescaled processes along a subsequence.

Secondly the noise part. Let $H = \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \gamma_{n+1}\xi(n+1)$. It is already known that $\mathbb{E}[H] = 0$ and a bound on the variance is needed. It is obvious that $\xi_i(n) \in [-1, 1]$ for each $i = 1\ldots, k$, so then $\mathbb{E}[|\xi_i(n)\xi_j(m)|] \leq 1$, for all i, j and n, m. Hence the following is obtained,

$$\begin{aligned}
\mathbb{E}[H_i^2] &\leq 1 \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \frac{1}{(n + n_0 + 1)^2} \\
&= \frac{1}{n_0} \frac{1}{n_0} \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \frac{1}{(1 + n/n_0 + 1/n_0)^2}.
\end{aligned}$$

Again, as was done in the computation of the deterministic part, letting $n_0 \to \infty$ an integral is found, of which another bound will be obtained.

$$\frac{1}{n_0} \sum_{n=t_1 n_0}^{t_2 n_0 - 1} \frac{1}{(1 + n/n_0 + 1/n_0)^2} \to \int_{t_1}^{t_2} \frac{1}{(1+t)^2}\, dt \leq \int_0^\infty \frac{1}{(1+t)^2} = 1.$$

The bound that is now obtained is independent of t_1 and t_2 and is given by, $\mathbb{E}[H_i^2] \leq \frac{C}{n_0}$. When again $n_0 \to \infty$, the result is that $\mathbb{E}[H_i^2] \to 0$. To see that indeed the noise vanishes it is written as the standard deviation times a standardized random variable U, thus $H_i \simeq \frac{C}{\sqrt{n_0}} U$ which goes to 0 in probability, as n_0 goes to infinity, for all $t_1, t_2 > 0$.

Now for all $t_2 > t_1 > 0$, as $n_0 \to \infty$ along a suitable subsequence,

$$\chi(t_2 n_0) - \chi(t_1 n_0) \to \chi^\infty(t_2) - \chi^\infty(t_1) = \int_{t_1}^{t_2} \frac{1}{1+t} G(\chi^\infty(t))\, dt, \tag{23}$$

where the noise part has vanished.

Thus as a result, $(\chi(n_0 t) : t \geq 0) \to (\chi^\infty(t) : t \geq 0)$ in distribution as $n_0 \to \infty$ along all subsequences, where $(\chi^\infty(t) : t \geq 0)$ is deterministic and is the unique solution of the system of ordinary differential equations (21). □

Remark. The interpolation by step function is not essential, any other linear or sufficiently regular interpolation between discrete time points can be used as well.

Remark. In (21), increasing t in the $'\frac{1}{1+t}'$ term slows down the dynamics of the flow field since the changes in fractions become smaller.

5.2 A Lyapunov function

To be sure there are no limit cycles and the system of ODE's (21) actually converges to one of the stationary points, the Lyapunov function will be analyzed. It has to be checked that the Lyapunov function is non-negative, and equals 0 if and only if χ is a stationary point with $G(\chi) = 0$. For simplicity the factor $\frac{1}{1+t}$, on the right-hand side of equation (21), can be ignored here. In analogy to results in [3] the Lyapunov function is given by

$$L(\chi_1, \chi_2, \ldots, \chi_k) = -(\chi_1 + \chi_2 + \ldots + \chi_k) + \frac{1}{\beta}\log(\alpha_1\chi_1^\beta + \alpha_2\chi_2^\beta + \ldots + \alpha_k\chi_k^\beta).$$

This is a strict Lyapunov function and

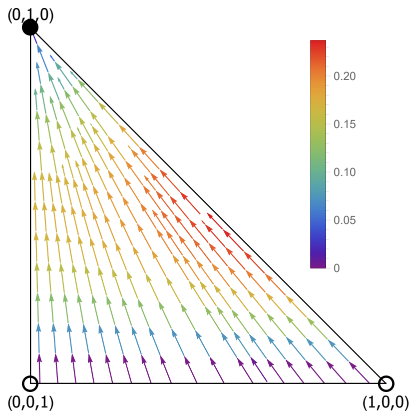
$$\frac{\partial L}{\partial \chi_i} = \frac{1}{\chi_i} G_i \text{ for all } i.$$

Let $\chi(t) = (\chi_1(t), \chi_2(t), \ldots, \chi_k(t))$ be an integral curve of G then,

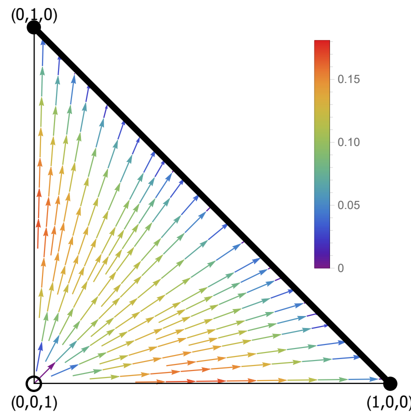
$$\begin{aligned}
\frac{d(L \circ \chi)(t)}{dt} &= \sum_{i=1}^{k} \frac{\partial L}{\partial \chi_i} \frac{d\chi_i(t)}{dt} \\
&= \sum_{i=1}^{k} \chi_i(t) \left(\frac{\partial L}{\partial \chi_i} \right)^2 \\
&= \sum_{i=1}^{k} \frac{1}{\chi_i} G_i(\chi)^2 \geq 0.
\end{aligned}$$

The equality above only holds when $G(\chi) = 0$. By Costa and Jordan [3] this means there are only single stationary points. Hence these are the corner points and a point in the interior of the simplex.

The stability of the model for $\beta < 1$ and $\beta > 2$ was discussed in Section 4.1.2. In the case of $\beta = 1$ the condition for all pairs to be equal, in equation (16), is $\alpha_i = \alpha_j$ and every point in the interior of the simplex will be stationary. In this case the flow field vanishes and the dynamics are given by only the fluctuations. However, if not all α_i are equal, all convex combinations of corner points with maximal α are attractive stationary points, which could be a single point in case of a unique maximal α. Two examples for the case of k=3 are given in Figure 3.



(a) $\alpha_1 = 1, \alpha_2 = 2$ and $\alpha_3 = 1$



(b) $\alpha_1 = 2, \alpha_2 = 2$ and $\alpha_3 = 1$

Figure 3: *Flow fields for $\beta = 1$ are given for different values of the α_i. In 3a, α_2 has the highest value and is thus the only attractive stationary point, denoted by a black dot. The other corner points are unstable. In 3b, α_1 and α_2 have the highest value, this results in attractive corner points for components* 1 *and* 2 *and a convex combination between those, denoted by a black line.*

These figures clearly show the flow fields for $\beta = 1$. When a single component has the highest fitness as in Figure 3a this will be the only attractive point and this component will certainly reach monopoly. However when multiple components have the value of the highest fitness, the corner points of these components will all be attractive. Aside from these attractive corner points, the area of the simplex between those corners will be attractive as well, this is the black line in Figure 3b. This means a monopoly can be reached or a stable equilibrium on the attractive region between and on the corner points.

6 Discussion

The main question that has been answered in this report is, "How does the generalized Pólya urn model, with fitness and non-linear reinforcement behave when time goes to infinity?". The generalized Pólya urn model can be used to describe the probable outcomes of competitions and in this report general feedback functions for the model were considered, as well as a particular feedback function.

First a proof was given for the well-known result that for general feedback functions $F_i(z)$ satisfying $\sum_{z=1}^{\infty} \frac{1}{F_i(z)} < \infty$, the process will certainly reach a monopoly but the winner will be random. It is likely that the larger $F_i(X_i(0))$ is, the higher the chances are that agent i wins. For the specific feedback function $F_i(x_i) = \alpha_i x_i^{\beta}$ with $\beta > 1$, monopoly is indeed reached. To describe the behavior of the model with this feedback function the dynamics were analyzed of the component fractions in the standard simplex, in the limit of a diverging initial number of balls in the urn.

In the case of $\beta > 1$, the corner points are stable and attractive. In this case a monopoly will always be reached. The simplex will be partitioned into deterministic convex polytopes and there exists an unstable stationary point in the interior of the simplex. This point depends on the α_i and β, and is on the intersection of the attractive domains. The relation between the fitness and initial condition is very important and will determine who wins the competition. The component with largest value for $\chi_i(0)^{\beta-1}\alpha_i$ will eventually reach monopoly. For larger values of β the value for α_i will eventually be less significant.

For $\beta < 1$ the model reaches an equilibrium. This is an attractive stable point in the interior of the simplex of the flow field. This point is determined by the fitness of the components in the model. Components with a higher fitness have a larger chance of winning. The corner points are unstable in this case, which means none of the components have a chance of reaching monopoly.

Lastly, for $\beta = 1$ every point in the interior of the simplex is stationary when all the components have equal fitness. When the components have distinct fitness there are attractive regions. If there exists a single component with largest fitness, it can reach monopoly.

For future research it could be interesting to focus more on the losing components and determine the total amount they receive, as was also considered for another model by Oliveira [6]. Another interesting question is what happens to the model when the fitness of the components change over time, the losing component could become much fitter than the winning component. Will this have a significant influence on the model when time goes to infinity?

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