The Interpretation and Application of 300 Years of Optimal Control in Economics

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Abstract

Neoclassical economics dictates the decision-making process of economic agents as the mathematical problem of maximizing utility over a prescribed planning horizon. The mathematical similarities with optimal control theory lead to a new interpretation of economic agents as optimal controllers. Pontryagin's maximum principle generates the necessary conditions, but the economic consequences become clear when its historical development is followed. It is found that the Euler-Lagrange equations result in a no-arbitrage condition in economics, and Hamilton's canonical equations describe the change in asset allocation and the asset price over time. The Hamiltonian itself is equivalent to the economic surplus of the agent, and the maximum principle requires that it is maximized along the optimal trajectory with respect to the control actions. This gives a different, myopic perspective to the economic agent, being an agent that maximizes economic surplus instantaneously instead of utility over an entire planning period.

Contents

Preface

The thesis that lies before you focuses on the interpretation and the application of optimal control theory in economics. It is the result of a year of hard work, that once started with the desire to combine my background in *Econometrics and Management Science* and *Systems & Control* into a thesis for the partial fulfillment of the requirements for the Master of Science Systems & Control at the Delft University of Technology.

This happened to be an enormous challenge and I am grateful for the fruitful discussions with my supervisor dr. ir. Max Mendel that eventually let me to formulate the topic of this thesis. With it, I hope to provide an important step towards the mechanical analogy for economic systems that is being constructed by dr. ir. Max Mendel. The hope and goal is that this mechanical analogy (with the possibility to apply the know-how from mechanical engineering and systems and control) eventually will lead to a better understanding of the dynamic behavior of individuals and the economy as a whole.

Of course, I also want to acknowledge and show my gratitude for the support, motivation and supervision that dr. ir. Max Mendel provided over the course of my thesis. This gratitude extends to the weekly meetings with all the other graduate students that worked on related topics. The discussions that we had, provided me with very valuable insights. Also, special thanks to Rebel for providing an inspiring work environment and the numerous distractions that kept me motivated to finish this thesis. A special word of thanks to Jochem Kop, my supervisor from Rebel, who suffered the figments of my imagination while discussing the use and interpretation of optimal control in economics. I enjoyed the discussions that we had, leading to valuable economic insights, and hope that my references to mechanical and control engineering where not too daunting. To all my colleagues, friends, and family, thank you for the support, motivation and the very necessary distractions that you provided. I soon hope to forget the mockery questions that you all asked about finishing my thesis.

Closing, I want to thank you as reader for at least making it to the end of my preface. For you I hope that this thesis will have a lasting effect on the way you will make your choices in the future, weighting the advantages and disadvantages of each option and imagining how each option will effect your path of life. For me personally, the writing of this thesis has been a true investment in my future which I am sure will pay off throughout my career.

I hope you will enjoy your reading.

Delft, University of Technology — Ruud Smit
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1 Introduction

Optimal control theory searches for a control strategy that renders the behavior of a controlled system optimal according to a certain criterion. The systems under evaluation can be found in all kind of fields, ranging from the smallest particles to the control of large chemical plants. The description of households, firms, governments, and entire economies in the field of neoclassical economics is remarkably similar to the theory of optimal control. Firms manage their business to maximize their profit or market share, while households allocate their assets and manage their consumption to maximize utility, a measure for psychological satisfaction. The mathematical similarities allow me to interpret economic agents as optimal controllers and use optimal control techniques to describe their economic behavior over time.

In this thesis I will specifically focus on the optimal control description of economic agents that maximize utility over a planning horizon. By coincidence or not, both the field of optimal control and the concept of utility originates in the very gifted Bernoulli family. It was Johann Bernoulli who in 1696 (more then 300 years ago) was the first to describe and solve the brachystochrone problem[1], a problem that is now known as an optimal control problem (see Sussmann and Willems (1997)). With this contribution Johann Bernoulli is seen today as the father of the calculus of variations and optimal control. His son, Daniel Bernoulli, made an fundamental contribution to preference theory and economics with the introduction of the concept of utility. While analyzing the St. Petersburg Paradox, Daniel Bernouilli wrote [2] "*The determination of the value of an item must not be based on the price, but rather on the utility it yields. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.*". The concept of utility plays today a key role in the behavioral characterization of the economic agents, and is closely related to the (shadow) price of an asset.

Although the origins of optimal control theory and utility maximization started within the same family, the development of both fields occurred relatively separate. The purpose of this thesis is therefore threefold. First, I describe the economic agent as an optimal controller, and view the maximization of utility as an optimal control problem. Here I take the perspective of a control engineer, and specifically interpret the economic variables in terms of their control theory equivalents. Second, I construct a thorough interpretation of Pontryagin's maximum principle in economics. Kamien and Schwartz (1981) and Seierstad and Sydsaeter (1986) apply the maximum principle in economics, and provide some economic intuition, but use it mostly as a mathematical technique. I show that the maximum principle has an intuitive interpretation for the economic agent such that it is much more than a mere mathematical technique, but rather a new, myopic description of the economic agent. Third, I will focus on the historical development of the maximum principle and relate it to the problem of utility maximization. Sussmann and Willems (1997) narrate the development of the maximum principle and show, with the brachystochrone problem as example, how it naturally evolved from a Lagrangian and Hamiltonian system description. I will take a similar approach, but use the optimal control problem of the economic agent to show how Pontryagin's maximum principle is related to the Euler-Lagrange equations and Hamilton's canonical equations. This approach allows me to construct the economic interpretation of maximum principle, starting with the fundamentals and gradually increasing the complexity.

Chapter 2 starts with the neoclassical description of the economic agent as an optimal controller. To fix ideas I specifically view the economic agent as a household that maximizes utility. The state of the economic agent is his asset allocation, and like a physical system this state changes dynamically over time, both autonomously and by the control of the economic agent. In agreement with Kamien and Schwartz (1981), utility is derived from both holding assets and from the control actions. This is more general than the utility functional of Merton (1975) and Samuelson (1948) were the utility is only a functional of the control actions, specifically interpreted as consumption. The utility functional itself consist of two parts, the accrual of the running utility over the planning

[1]The brachystochrone problem searches for the arc between point A and a lower point B down which a bead rolls in the least amount of time, starting at rest and accelerated by gravity without any friction.

[2]The original work is written in Latin, the translated reference is Bernoulli (1954).

period, and the bequest or salvage utility at the end of the planning horizon. The economic agent behaves as an optimal controller, planning his asset allocation and control actions over the prescribed planning horizon to maximize utility, while taking the asset dynamics and other restrictions into account.

Chapter 3 goes back to the fundamentals of optimal control by simplifying the asset dynamics to describe the economic agent as a Lagrangian system. In this setting, the economic agent behaves as a player in the economy who drifts on the economic flow, equivalently to a mechanical system that moves through space under the influence of a force field. The classical texts Landau and Lifshitz (1972) and Arnold (1978) describe the dynamics of those mechanical systems with Lagrangian mechanics. In line with the research of my supervisor Dr. Ir. Max Mendel, the isomorphic structure between mechanical and economic systems, allows the use of Lagrangian mechanics to derive the dynamics of the economic agent. It is found that the generalized coordinates correspond to the agent's assets, and the generalized momenta to the shadow prices. The Euler-Lagrange equations are interpreted as a no-arbitrage condition in economics.

Chapter 4 builds upon the Lagrangian characterization of the economic agent and introduces the Hamiltonian. However, rather than using the original definition in Hamilton (1834) and Hamilton (1835), I adopt the control Hamiltonian[3] of Sussmann and Willems (1997). As Sussmann and Willems (1997) show, the application of the control Hamiltonian will later turn out to be vital to make the step towards the maximum principle. Kamien and Schwartz (1981) refer to the Hamiltonian as a device for remembering or generating the Euler-Lagrange equations. I oppose this view, and interpret it as the economic surplus. That is, the net benefit for the agent, defined as the difference between the (instantaneous) gained utility and the cost of consumption. With the Hamiltonian representation of the economic agent, Hamilton's canonical equations describe the change in the shadow price and asset allocation over time. In addition to Hamilton's equations, the control Hamiltonian is maximized along the optimal trajectory with respect to the control input. This has an important economic implication. Initially, the economic agent maximizes utility over a planning horizon, but the Hamiltonian characterization shows that the agent can equivalently be represented myopically. This myopic perspective demonstrates an agent that maximizes economic surplus instantaneously given his current asset allocation and the shadow prices. The changes in the asset positions and the shadow prices depend on the conducted control actions of the agent, and are expressed by Hamilton's canonical equations of the controlled Hamiltonian.

The Hamiltonian characterization of the economic agent translated to the general optimal control problem of Chapter 2. Pontryagin (1962) develops the maximum principle, being mathematical necessary conditions for the optimal control problem, and Sussmann and Willems (1997) show that the maximum principle is a natural extension of Hamilton's canonical equations. Building on the work of Sussmann and Willems (1997), I extend the economic interpretation from the Hamiltonian system towards the maximum principle. I show that the maximum principle is a myopic description of the economic agent that takes the asset dynamics into consideration. The maximum principle consist of 4 necessary conditions that the optimal control and asset trajectory should satisfy. Each of these conditions can be interpreted from the agent's perspective. The maximum condition dictates that the economic agent maximizes economic surplus along the optimal trajectory with respect to his control actions. This results in a myopic interpretation of the agent. In addition, the first-order condition with respect to the control input, implies that the agent performs an marginal cost-benefit analysis instantaneously. The Hamiltonian system condition requires that the asset and shadow price evolve as a Hamiltonian system over time. The state equation simply returns the asset dynamics, the co-state equation describe the evolution of the asset prices over time. The latter one is an no-arbitrage condition for the economic agent. The non-triviality condition requires that the optimal control and asset trajectory at least depend on either the asset dynamics or the utility functional. The transversality condition is the fourth and final condition and generates the boundary condition for the asset prices. Chapter 6 concludes this thesis and Chapter 7 finishes with a short discussion on the current method and possibilities for further research.

[3] In the remainder of this thesis I will always use the control Hamiltonian, and refer to it as the Hamiltonian.

2 The Economic Agent as an Optimal Controller

Economists are always searching for a better and more complete description of the behavior of households, firms, governments and central banks in the economy. The behavior of these economic agents is often interpreted as the decisions that they make over time. These decisions will affect the agent's asset positions and future income, as well as non-financial things such as leisure time. Additionally, the decisions of one economic agent will likely affect others such that there is an interaction between the different players in the economy.

A fundamental question is how these economic players determine their decisions over time. The standard choice in the field of *neoclassical economics* is to assume that economic agents behave rationally, acting in a self-interested way to maximize their utility or personal satisfaction given all available information. These stylized, rational agents are often called *homo economicus*, but are certainly not equivalent to money-hungry monomaniacs. Instead, the homo economicus will take the satisfaction of factors like leisure time, and procreation into account to make optimal decisions over time, see Persky (1995) for an in-depth discussion.

The description of the homo economicus is closely related to the theory of optimal control. In optimal control, we try to stabilize a system or track a reference by selecting the control that maximizes a certain objective functional or minimizes a certain cost functional. Analogous the homo economicus controls his assets over the planning horizon by maximizing his utility over time and making decisions accordingly. To fix ideas while constructing the analogy between optimal control theory and economics, I specifically focus on the description of the household as an optimal controller.

2.1 The Economic Agent

Before we formalize the notion of utility maximization, we need to take a closer look at the assets of the household. The assets or possessions of the household are very diverse, ranging from physical goods (like real estate and an inventory of consumption goods) to financial products (like a bank account, bonds, stocks and insurances) and immaterial goods like human capital. The agent is at each point in time fully characterized by his assets portfolio, meaning that the asset portfolio is the state of the economic agent. We will consider n-different assets, such that $x(t) \in \mathbb{R}^n$. The units of $x(t)$ depend on the asset under consideration, but in general we see $x(t)$ as a quantity and not as the value of the assets. [4]

Similar to the state of a mechanical system, the assets change dynamically over time. Theses change occur through autonomous dynamics and by control decisions of the agent. Assets diminish autonomously over time by aging, wear and write-offs, while assets accumulate naturally through interest, pay-offs and production. The control decisions include the possibility to investment or divest in assets, to consume part of the assets, or generate an income by performing labor. The control decisions are denoted as $u(t)$ and in general we have m-different control decisions such that $u(t) \in \mathbb{R}^m$. The evolution of the assets allocation over time is then described by the first-order dynamics (see also Weitzman (2009), Samuelson (1948) and Ramsey (1928)).

$$\dot{x}(t) = f(x(t), u(t), t) \tag{2.1}$$

where $f(x(t), u(t), t)$ is some, possibly nonlinear, function of the assets $x(t)$, the control decisions $u(t)$ and time t. Recall that the assets $x(t)$ are measures as an quantity, and not in terms of its value or price. Equation (2.1) describes the change in the number of assets over time, and is certainly not equivalent to the depreciation or appreciation of the asset. The first-order, non-linear state-space description of the assets in (2.1) is very common in control engineering and mechanics. However, it implies that the economic agent cannot change his asset allocation $x(t)$ instantaneously. Instead, the homo economicus changes the number of assets gradually over time, by adjusting the rate at which the assets change.

[4] The price or shadow price is endogenous in the problem. It is conceptually hard to measure the number of assets in terms of its dollar price while the value is endogenous in the allocation problem.

In the current set up it is important to realize that every component of the state $x(t)$ is equal to one of the assets of the household, meaning that the state $x(t)$ does not incorporate any derivatives of the assets. This is different from the description of, for example, a mass-spring-damper system. The second-order dynamics of this mechanical system can equivalently be written in a system of first-order differential equations, but then the state includes both the position and the velocity of the cart. In this thesis I will specifically assume that the state only includes the asset "positions" and not the "velocity" of assets. This assumption will later be necessary to interpret the generalized momenta or the co-state as the asset (shadow) prices.

The fundamental question in economics is how economic agents control their assets. In this paper I follow the neoclassical explanation that dictates the decision-making process of economic agents as the mathematical problem of maximizing utility over a prescribed planning horizon. Utility is in this setting interpreted as a measure of psychological satisfaction, and the agent pursuits happiness by maximizing it. Specifically, we follow Kamien and Schwartz (1981) and model the household as an economic agent who gains utility from both holding the assets $x(t)$ and making the control decisions $u(t)$. The agent now takes those decision $u(t)$ that maximize his forecasted, accumulated utility over a fixed planning horizon.

The total utility over the planning period consist out of two parts; the accumulation of the running utility $v(x(t), u(t), t)$ over the planning horizon, and a bequest utility $\Phi(x(T), T)$ at the end of the planning horizon. The running utility $v(x(t), u(t), t)$ measures the satisfaction from holding assets $x(t)$ and decisions $u(t)$ at a specific time t. The bequest utility $\Phi(x(T), T)$ measures the utility from leaving a bequest $x(T)$ at the end of the planning horizon. The total utility for an arbitrary control trajectory $\{u(t) \mid t \in [0, T]\}$ with corresponding asset trajectory $\{x(t) \mid t \in [0, T]\}$ is equal to

$$\mathcal{V}\left[x(\cdot), u(\cdot)\right] = \Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t)dt \tag{2.2}$$

Here, we use the shorthand notation $x(\cdot)$ and $u(\cdot)$ to indicate the asset trajectory and control trajectory over the planning horizon $[0, T]$. The formulation of the total utility in (2.2) is more general than the papers Ramsey (1928), Samuelson (1948) and Merton (1975). In these papers the running utility $v(x(t), u(t), t)$ is only a function of the consumption $u(t)$ and time t. I follow Kamien and Schwartz (1981), and argue that the economic agent also gains a satisfactory feeling (utility) from holding the assets $x(t)$. Owning the assets $x(t)$ may provide the household with a certain status within the community or give the confirmation of being successful. This means that the running utility is in general a function of the asset position $x(t)$, the consumption $u(t)$ and time t. In section Section 2.4 the usual assumptions for the utility function are provided.

The objective of the economic agent is to maximize his utility (2.2). However, the decision trajectory $u(\cdot)$ and the asset trajectory $x(\cdot)$ are not independent quantities, but are coupled through the asset dynamics (2.1). These assets dynamics constrain the optimization (2.3). Starting with some initial asset allocation $x(0) = x_0$, the asset trajectory is (assumed to be) uniquely described by the control $u(\cdot)$ and the state space description (2.1). The agent thus searches for the control trajectory $u(\cdot)$ that maximizes the utility function. This can be mathematically summarized as

$$\begin{aligned} \mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] &= \max_{u(\cdot)} \left[\Phi\left(x(t), T\right) + \int_0^T v(x(t), u(t), t)dt\right] \\ \text{s.t.} \quad \dot{x}(t) = f\big(x(t), u(t), t\big), \quad x(0) &= x_0, \qquad u(\cdot) \in \mathcal{U}, \quad x(\cdot) \in \mathcal{X} \end{aligned} \tag{2.3}$$

Denoting the optimal decision trajectory as $u^*(\cdot)$ and the corresponding optimal asset trajectory as $x^*(\cdot)$, we naturally have that the total utility corresponding to the optimal trajectories exceeds the utility for all other possible asset trajectories.

$$V^* = \mathcal{V}\left[x^*(\cdot), u^*(\cdot)\right] \geq \mathcal{V}\left[x(\cdot), u(\cdot)\right] \quad \forall \quad u(\cdot) \in \mathcal{U} \quad \text{and} \quad x(\cdot) \in \mathcal{X}$$

Similar to mechanical system where the input signal can saturate, it is possible that the control decisions of agent are constraint to some admissible set. To illustrate this, let us consider the "investment" in human capital and denoted it for the moment as u_{HC}. Now imagine that the economic agent invest in his "human capital" to increase his skill set and market value, which eventually leads to a higher wage. This investment in human capital will be positive such that $u_{HC} > 0$. It is however difficult to imagine that an economic agent can actively divest in his "human capital". This means that at least for human capital not all control decisions $u(t)$ are admissible. We will denoted the set of all admissible control signals as $\mathcal{U}$ and require that control decisions are constraint to this set

$$u(t) \in \mathcal{U} \quad \forall \quad t \in [0, T] \tag{2.4}$$

And use the shorthand notation $u(\cdot) \in \mathcal{U}$ to indicate that the full control trajectory should lie in the allowable control set.

Similar, the asset allocation may be confined by legislation or physical restrictions to some admissible set. Short selling of a financial asset may not be allowed, or limited to some extend, and having a negative number of physical assets (real estate, consumption goods) does not seem very plausible. This means that the state trajectories are also limited to the set of allowable states $\mathcal{X}$. This poses the requirement that all admissible state trajectory satisfy

$$x(t) \in \mathcal{X} \quad \forall \quad t \in [0, T] \tag{2.5}$$

and again I use the shorthand notation $x(\cdot) \in \mathcal{X}$ to indicate that the full state trajectory should lie in the allowable state set.

2.2 The Economic Agent as an Optimal Controller

The essence of the neoclassical description of the economic agent is the utility maximization problem of (2.3) with the assets dynamics in (2.1) as constraints, and with the initial asset allocation, the admissible controls in (2.4) and feasible states in (2.5) as additional constraints. This is mathematically summarized as

$$\begin{aligned} \mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] &= \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t) dt\right] \\ \text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), t), \quad x(0) &= x_0, \quad x(\cdot) \in \mathcal{X}, \quad u(\cdot) \in \mathcal{U} \end{aligned} \tag{2.6}$$

This mathematical description is very well known to a control engineer, and is equivalent to the optimal control problem of a physical system. The economic agent therefore behaves as an optimal controller. The asset allocation $x(t)$ is the state of the controlled system and the investment and consumption decisions are the control variables $u(t)$ of the agent.

Solving the optimization problem is in general difficult as we are not looking for a single value for the control input $u^*(t)$, but instead we are looking for the optimal trajectory of the state $x^*(\cdot)$ and the control $u^*(\cdot)$ over the planning horizon $t \in [0, T]$. In this sense, the problem is infinite-dimensional as the space of feasible paths is an infinite-dimensional function space.

2.3 The Objective Function

Section 2.1 and 2.2 introduce the economic agent as an optimal controller with the goal to maximize the utility functional. This functional incorporates both a running utility $v(x(t), u(t), t)$ and a bequest utility $\Phi\left(x(T), T\right)$.

$$\mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] = \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t) dt\right] \tag{2.7}$$

The optimal control problem with this objective functional is known as the Bolza Problem. The Lagrange and Mayer problem are defined similarly to the Bolza problem, but differ in the formulations of the utility functional. The Lagrange problem only includes the running utility and not the bequest utility. It is specified as

$$\mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] = \max_{u(\cdot)} \left[\int_0^T v(x(t), u(t), t)dt\right] \tag{2.8}$$

The Mayer problem on the other hand only includes the bequest utility and not the running utility. Its objective is defined as

$$\mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] = \max_{u(\cdot)} \Phi\left(x(T), T\right) \tag{2.9}$$

The problem of Bolza seems at first glance to be more general than the Lagrange and Mayer formulation. The Lagrange problem lacks the bequest utility $\Phi(\cdot) = 0$ and the problem of Mayer is missing the accrued instantaneous utility, $v(x(t), u(t), t) = 0$. However, we can actually show that the three problem formulations are equivalent, meaning that we can rewrite the problem of Bolza under some mathematical regularity conditions in terms of the problem of Lagrange or Mayer and vice versa (see Bertsekas et al. (1995) and Appendix B). This of course also means that the Lagrange and Mayer problem are equivalent.

2.4 The Utility Function

The utility functional (2.3) consist of the accrued running utility and the bequest utility. These functions describe the behavior of the household in the economy. In principle, these functions could be anything, not necessarily restricted to any specific form. There are however compelling arguments to assume that both the running utility $v(x(t), u(t), t)$ and the bequest utility $\Phi(x(T), T)$ are convex. In Becker (2011) this is discussed for simple choice model. However, we should not blindly assume that the utility function is convex as this certainly will depend on the assets and control possibilities under evaluation.

More-is-Better. The "more-is-better" property describes an economic agent that always values additional assets $x(t)$ and additional control $u(t)$. Mathematically, this means that the marginal running utility is positive in both the assets $x(t)$ and the control $u(t)$ [5]

$$\frac{\partial v}{\partial x}\Big(x(t), u(t), t\Big) \geq 0, \quad \frac{\partial v}{\partial u}\Big(x(t), u(t), t\Big) \geq 0 \tag{2.10}$$

And for the bequest utility

$$\frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \geq 0 \tag{2.11}$$

The "more-is-better" property should not be used blindly. If the control $u(t)$ is interpreted as consumption then more consumption can make the agent better off. However, not all control actions will give the economic agent a positive experience. If a component of $u(t)$ is equal to the fraction of labor and leisure time, then more labor will not benefit the agent directly (of course, labor can indirectly provide a benefit by generating income). However, if we express the problems in terms of the fraction leisure versus labor time then the "more-is-better" property is not unlikely.

Diminishing Marginal Utility. The utility of holding more assets $x(t)$ and making control decisions $u(t)$ increases with their amount. The marginal utility from the assets and decision will however decrease

$$\frac{\partial^2 v}{\partial x^2}\Big(x(t), u(t), t\Big) \leq 0, \quad \frac{\partial^2 v}{\partial u^2}\Big(x(t), u(t), t\Big) \leq 0 \tag{2.12}$$

and for the bequest utility

$$\frac{\partial^2 \Phi}{\partial x^2}\Big(x(T), T\Big) \leq 0 \tag{2.13}$$

[5] For ease of notation I do not show the time dependency of the assets $x(t)$ and the control decisions $u(t)$ in the partial derivatives.

Time Preference Similar to the time-value of money, agents also have a clear time preference. This means that utility derived from assets and consumption in the nearly future is valued higher. Mathematically, the running utility decreases over time, and the bequest utility decreases with an increasing planning horizon.

$$\frac{\partial v}{\partial t}\Big(x(t), u(t), t\Big) \leq 0, \quad \frac{\partial \Phi}{\partial T}\Big(x(T), T\Big) \leq 0 \tag{2.14}$$

3 The Economic Agent as a Lagrangian System

Building on the work of Sussmann and Willems (1997), I do not focus immediately on the solution method for the general optimal control problem. Instead, I follow the historical development of the maximum principle and start in this chapter with the Lagrangian characterization of the economic agent. This will lead to the Euler-Lagrange equations which have the interpretation of a no-arbitrage condition in economics.

The Lagrangian characterization of the economic agent is generated by the simplification of the asset dynamics in (2.1) which become

$$\begin{aligned} \dot{x}(t) &= u(t) \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

These (uncontrolled) assets dynamics imply that the amount of assets and the asset allocation does not mutate autonomously over time. The assets can only change by a control action of the agent, which is best understood as the negative of consumption[6]. I will refer to these assets as "simple consumption assets". Assuming that both the control and state are continuous time functions that are not restricted by some admissible control or state set, we can write the mathematical problem of utility maximization as

$$\begin{gathered} \mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] = \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t) dt\right] \\ \text{s.t.} \quad \dot{x}(t) = u(t), \quad x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad u(\cdot) \in \mathbb{R}^m \end{gathered} \tag{3.2}$$

This is the description of an economic agent who is endowed with an initial amount of simple consumption assets x_0, and who consumes (part of) his assets to maximize utility over the planning horizon. The total utility in (3.2) is equivalently written as a functional of the asset trajectory determined by $(x(t), \dot{x}(t), t)$ instead of $(x(t), u(t), t)$ by substitution of the asset dynamics (3.1)[7].

$$\begin{gathered} \mathcal{V}^* = \max_{x(\cdot)} \mathcal{V}\left[x(\cdot)\right] = \max_{x(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), \dot{x}(t), t) dt\right] \\ \text{s.t.} \quad x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad \dot{x}(\cdot) \in \mathbb{R}^n \end{gathered} \tag{3.3}$$

The distinctive feature of this Lagrangian characterization is that the economic agent maximizes utility over all possible asset trajectories $x(\cdot)$. The optimal asset trajectory is completed described by the utility functional, and its solution can be found by techniques from the calculus of variations.

This is different from the initial optimal control problem, where the maximization of utility takes place over all possible asset trajectories $x(\cdot)$ that satisfy the asset dynamics for some choice of the control trajectory $u(\cdot)$. The maximization is hence performed over the asset-control pair $(x(\cdot), u(\cdot))$. In the optimal control problem, both the asset dynamics and the utility functional give the problem of utility maximization an interesting structure.

The Lagrangian characterization of the economic agent is analogous to the description of a mechanical system that moves through space. The analogy will be developed in this chapter, but is summarized in Table 3.1. The solution of the optimal asset trajectory can be described both by Euler-Lagrange equations and by Hamilton's canonical equations. Arnold (1978) refers to the application of these two solution methods in mechanics as Lagrangian mechanics and Hamiltonian mechanics. We will adopt this terminology and refer to it, in the economic application under consideration, as the Lagrangian economic agent and the Hamiltonian economic agent. The Lagrangian method is discussed in the current chapter, the Hamiltonian approach in the next chapter.

[6] The assets of the agents will decreases through consumption, such that $u(t) \leq 0$ corresponds to consumption of the assets.

[7] In classical mechanics it is customary to denote the generalized coordinate and its derivative as $q(t)$ and $\dot{q}(t)$. To keep the notation consistent throughout this document, I stick to $x(t)$ and $\dot{x}(t)$ in this chapter.

Symbol	Lagrangian & Hamiltonian Mechanics	Economic Interpretation	Units
$x(t)$	Generalized Coordinates	Asset Positions	[asset]
$u(t)$	Control Variable	Consumption	$\left[\frac{\text{asset}}{\text{year}}\right]$
$p(t)$	Generalized Momenta	Asset Prices	$\left[\frac{\text{utils}}{\text{asset}}\right]$
$F(t)$	Potential Force	Utility Rental	$\left[\frac{\text{utils}}{\text{asset year}}\right]$
$v\Big(x(t), \dot{x}(t), t\Big)$	Lagrangian	Instantaneous Utility	$\left[\frac{\text{utils}}{\text{year}}\right]$
$\Phi\Big(x(T), T\Big)$	-	Bequest Utility	[utils]
$\mathcal{V}^*$	Negative of Action	Maximized Total Utility	[utils]
$H\Big(x(t), p(t), u(t), t\Big)$	Control Hamiltonian	Economic Surplus	$\left[\frac{\text{utils}}{\text{year}}\right]$

Table 3.1: The analogy between Lagrangian and Hamiltonian Mechanics and the utility maximization problem of an economic agent with simple consumption goods.

3.1 Economic Interpretation of the Euler-Lagrange Equation

The formulation of the economic agent in (3.3) has many similarity with Lagrangian mechanics. Landau and Lifshitz (1972) and Arnold (1978) explain that the dynamics of a mechanical system can be derived from the action functional, and this is called Hamilton's principle. The action functional is defined similarly as the utility over a planning horizon[8]. The true state trajectory of such a mechanical system is described by the stationary points that minimize the action. These stationary points are described by the Euler-Lagrange equations.

The isomorphic structure between the Lagrangian economic agent and Lagrangian mechanics enables the use of Lagrangian mechanics to derive the agent's behavior. The action of a mechanical system and the utility of an agent are equivalent concepts here, but with the difference that action is minimized and utility is maximized. The agent's asset trajectory is, similar to the mechanical system, described by its stationary points. However, in contrast with mechanics, I look for the stationary points that maximize the utility functional[9]. The stationary points are obtained with the Euler-Lagrange equations which are

$$\frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big)\right) = 0 \tag{3.4}$$

In mechanics, this forms of the Euler-Lagrange equations is only applicable when all forces can be derived from some potential function. In general handling dissipative forces, and therefore also transactions costs in economics, will be hard to include in the Euler-Lagrange equations. See Appendix A for a short digression on the inclusion of dissipative forces in the Euler-Lagrange equations. The derivation of (3.4) is a standard application of the calculus of variation and can be found in the classical textbooks Arnold (1978) and Landau and Lifshitz (1972). They assume that the generalized coordinates have a boundary condition at the end of the planning horizon. For the economic agent, the final asset allocation is however part of the utility maximization problem. Luckily, the Euler-Lagrange equations still hold true without the boundary condition, but then require the additional

[8]The utility functional incorporates a bequest utility, while the action does not incorporate a final cost. This is not problematic as the Bolza utility functional can be transformed into the Lagrangian objective functional, see Section 2.3.

[9]A simple step would be to convert the maximization of utility into the minimization of the negative of utility also called disutility.

transversality condition.

$$p(T) = -\frac{\partial v}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) = \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \tag{3.5}$$

To be certain that the obtain trajectory actually maximizes the utility over the planning, we should also check the second-order conditions. This is checked with the Legendre conditions (see Gelfand and Fomin (1963)), which is

$$\frac{\partial^2 v}{\partial \dot{x}^2}\Big(x(t), \dot{x}(t), t\Big) \leq 0 \tag{3.6}$$

As in mechanics, we can give an economic interpretation to the partial derivatives in the Euler-Lagrange equations and the transversality condition. In mechanics, we define the generalized momenta that correspond to the generalized coordinates $x(t)$, as

$$p_m(t) = \frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big) \tag{3.7}$$

This is the definition of the shadow price as in the work Kamien and Schwartz (1981). However, we should remember that the action was minimized and the utility maximized. This means that we should be very careful with the sign convention here. It is better to define the shadow price as

$$p(t) = -\frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big) \tag{3.8}$$

This definition corresponds to the incremental increase of the utility functional due to an incremental increase in the asset allocation, see also Section 3.3. This is called a shadow price in economics. Remember also that the "more-is-better" property dictates that the marginal increase in utility is positive or at least nonnegative for a marginal increase in consumption. However, the control $u(t)$ in the asset dynamics (3.1) equals the negative of consumption, meaning that $u(t) \geq 0$ corresponds to the agent buying assets, and not consuming them. To ensure a positive shadow price, as one would expect it to be, the shadow price should be defined as in (3.8). The units of this shadow price are $\left[\frac{\text{utils}}{\text{asset}}\right]$. The transversality condition in (3.5) is then interpreted as a boundary condition for the asset price at the planning horizon. At that point, the price of the asset is determined by the bequest utility of the economic agent. As the "more-is-better" property also holds for the bequest utility, the price at the end of the planning horizon will be positive. The definition of the shadow price as in (3.8) will in the next chapter turn out to be essential to give a proper interpretation of the Hamiltonian.

The second term of the Euler-Lagrange equation is in mechanics interpreted as the applied force on the mechanical system. It has the definition

$$F(t) = \frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big) \tag{3.9}$$

In economics, I interpret this force as the utility rental that the agent receives while holding the assets $x(t)$. It is the incremental benefit or psychological satisfactions that the economic agent receives from an incremental amount of additional assets. The Euler-Lagrange equation (3.4) is with the definition for the shadow price and the utility rental written as

$$\dot{p}(t) = -F(t) \tag{3.10}$$

and I interpret this as the *no arbitrage condition* for the economic agent. It states that any change in the shadow price $p(t)$ of an asset $x(t)$ is due to the utility rental of that asset. The depreciation of the asset must be equal to the instantaneous payoff of holding the asset $x(t)$. The asset price decreases as the asset already pays out some of its value to the agent. To make this more clear, let us assume that this equality does not hold such that

$$\dot{p}(t) < -F(t) \tag{3.11}$$

Then the instantaneous benefit of holding asset $x(t)$ is greater than the price change of the asset. An economic agent could take advantage of this situation by buying an additional amount of the asset δx at time t for (shadow) price $p(t)$, holding it for a small period Δt, and selling it at time $t + \Delta t$. This will cost the agent in terms of the shadow price

$$p(t)\delta x - p(t + \Delta t)\delta x \approx \dot{p}(t)\Delta t \delta x$$

But he will receive the additional utility

$$\delta x \int_{t}^{t+\Delta t} F(t)dt \approx F(t)\Delta t \delta x$$

Due to (3.11), the benefit of holding an additional amount of $x(t)$ exceeds the shadow costs of buying these assets. The economic agent should always take advantage of this opportunity. The agent will buy the additional amount of asset $x(t)$ until the incremental benefit equals the shadow price of the asset $x(t)$. Therefore, the no arbitrage condition must hold.

The Euler-Lagrange equations (3.4) result in a second-order differential equation that describes the controlled asset position of the agent. This means that the controlled behavior of the Lagrangian economic agent can be described in a state-space form with 2 state variables. The state variables will be the asset price $p(t)$ and the asset allocation $x(t)$, as we will show with Hamilton's canonical equations. This is similar to a mechanical system that is at a time instant described by both its position $x(t)$ and its momentum $p(t)$ or its velocity $\dot{x}(t)$.

3.2 Economic Interpretation of the Derivation of the Euler-Langrange Equation

The proof of the Euler-Lagrange equation can be found in many textbooks such as Landau and Lifshitz (1972) and Arnold (1978). Here, we will repeat this proof using the calculus of variations and give an economic interpretation of it. Recall that it is the objective of the agent to maximize his total utility over the planning horizon $t \in [0, T]$, and in the setting of the Lagrangian economic agent, it equals

$$\begin{aligned} \mathcal{V}^* = \max_{x(\cdot)} \mathcal{V}\left[x(\cdot)\right] &= \max_{x(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), \dot{x}(t), t)dt\right] \\ \text{s.t.} \quad x(0) &= x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad \dot{x}(\cdot) \in \mathbb{R}^n \end{aligned} \tag{3.12}$$

The agent starts with an initial amount of assets x_0 and is free to choose his asset path $x(\cdot)$ over the planning horizon, without a boundary condition for $x(T)$. This is called a free-endpoint, fixed-time problem in the calculus of variations. The optimal asset trajectory $x^*(\cdot)$ is the trajectory that maximizes the total utility of the economic agent such that

$$\mathcal{V}\left[x^*(\cdot)\right] \geq \mathcal{V}\left[x(\cdot)\right] \quad \forall \quad x(\cdot) \in \mathbb{R}^n \tag{3.13}$$

for all other possible trajectories $x(\cdot)$. If the agent starts with the optimal trajectory $x^*(\cdot)$, and considers to vary this optimal path by an arbitrary change $\eta(t)$ over the entire planning horizon (such that $\eta(t) \mid t \in [0, T]$, abbreviated as $\eta(\cdot)$), then the agent will find the new asset trajectory $x^*(\cdot)+\eta(\cdot)$ less interesting as it will decrease his total utility over the planning horizon. To formalize this optimization mathematically, we consider the varied asset trajectory $x^*(\cdot) + \epsilon\eta(\cdot)$, where ϵ is some constant weight and $\eta(t)$ is an arbitrary and smooth function that is defined over the full planning horizon and atleast nonzero for some t over the planning horizon. The agent can of course not change his initial asset allocation, requiring that the initial variation equals $\eta(0) = 0$. Now for ease of notation, let us denote in the sequel of this section the varied trajectory as $x(\cdot) + \epsilon\eta(\cdot)$. The accrued utility over the planning horizon is for this varied path equal to

$$\mathcal{V}\left[x(\cdot) + \epsilon\eta(\cdot)\right] = \Phi\Big(x(T) + \epsilon\eta(T), T\Big) + \int_0^T v\Big(x(t) + \epsilon\eta(t), \dot{x}(t) + \epsilon\dot{\eta}(t), t\Big)dt \tag{3.14}$$

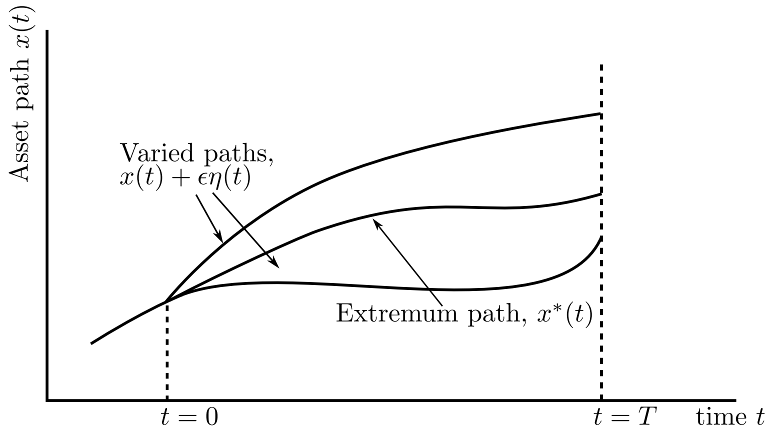

Figure 3.1: Schematic of the optimal and varied asset trajectories

The utility functional should by definition be maximized by the optimal asset trajectory $x^*(\cdot)$. Any variation of the asset allocation $\epsilon\eta(\cdot)$ of the optimal trajectory should result in a decreased well-being of the agent. This means that the change in the agent's well-being should be at extremal with respect to ϵ at the point $\epsilon = 0$. The first-order condition is equal to

$$\left.\frac{\mathrm{d}\mathcal{V}\left(x(\cdot) + \epsilon\eta(\cdot)\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0 \tag{3.15}$$

Manipulating this expression, we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{V}\left(x(\cdot) + \epsilon\eta(\cdot)\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left(\Phi\Big(x(T) + \epsilon\eta(T), T\Big) + \int_0^T v(x(t) + \epsilon\eta(t), \dot{x}(t) + \epsilon\dot{\eta}(t), t)dt\right) \\ &= \left[\frac{\partial\Phi(x,t)}{\partial x}\eta(T)\right]_{t=T} + \int_0^T \left(\frac{\partial v\left(x,\dot{x},t\right)}{\partial x}\eta(t) + \frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}}\dot{\eta}(t)\right)dt\end{aligned}$$

For the second term in the integral, we use the chain rule to write

$$\int_0^T \left(\frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}}\dot{\eta}(t)\right)dt = \left[\frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}}\eta\right]_0^T - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}}\right)\eta(t)\,\mathrm{d}t \tag{3.16}$$

Using the chain rule result, and the boundary condition that the agent is unable to vary his initial asset position $\eta(0) = 0$, the expression for the first variation of the utility functional equals

$$\eta(T)\left[\frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}} + \frac{\partial\Phi(x,t)}{\partial x}\right]_{t=T} + \int_0^T \left(\frac{\partial v\left(x,\dot{x},t\right)}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v\left(x,\dot{x},t\right)}{\partial\dot{x}}\right)\right)\eta(t)dt = 0$$

This equality only holds true for an arbitrary function $\eta(t)$ when the integral is equal to zero, resulting in the Euler-Lagrange equation

$$\frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial\dot{x}}\Big(x(t), \dot{x}(t), t\Big)\right) = 0$$

And the first term on the right side is equal to zero

$$\eta(T)\left[\frac{\partial v}{\partial\dot{x}}\Big(x(T), \dot{x}(T), T\Big) + \frac{\partial\Phi}{\partial x}\Big(x(T), T\Big)\right] = 0$$

This must hold for any variation at the end of the planning T, such that in general $\eta(T) \neq 0$. Recognizing the definition of the shadow price, this implies

$$p(T) = -\frac{\partial v}{\partial\dot{x}}\Big(x(T), \dot{x}(T), T\Big) = \frac{\partial\Phi}{\partial x}\Big(x(T), T\Big) \tag{3.17}$$

This tranversality condition simply states that without an bequest function the terminal asset stock $x(T)$ must be valueless for the economic agent. That is, the economic agent does not plan to leave anything valuable after the planning horizon. With an bequest utility, the price at the end of the planning horizon is determined by the marginal (bequest) utility.

Remark 1. *We can give the application of the chain rule in* (3.16) *an economic interpretation when we use the definition for the asset (shadow) prices* (3.8)*.The chain rule be write as*

$$d\big(p(t)\eta(t)\big) = p(t)d\eta(t) + dp(t)\eta(t)$$

The total value change of the variation $d\big(p(t)\eta(t)\big)$ *is due to the change in the variation* $p(t)d\eta(t)$ *plus the change in the price due to the variation* $dp(t)\eta(t)$

3.3 The Shadow Price as the Generalized Momentum

In Section 3.1, the shadow price was defined as the generalized momentum[10]

$$p(t) = -\frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big) \tag{3.18}$$

We mentioned that the shadow price is equal to the incremental change of the utility functional due to an incremental increase in the asset allocation. To show that this is the case, let consider the total utility when the initial asset position increases from $x(0) = x_0$ to $x(0) = x_0 + \eta$, and denote the corresponding utility as $V^*(x_0)$ and $V^*(x_0 + \eta)$. The increased initial asset position results in a variation of the asset trajectory, denoted as $\eta(t)$. The difference between the utility functionals is approximated with a first-order Taylor expansion around the optimal trajectory $x(\cdot)$, yielding

$$V^*(x_0+\eta) - V^*(x_0) \approx \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big)\eta(T) + \int_0^T \left(\frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big)\eta(t) + \frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big)\dot{\eta}(t)\right) dt \tag{3.19}$$

The application of integration by parts (similar to the proof of the EL-equations), and the definition of the shadow price, results in

$$\eta(0)p(0) + \eta(T)\underbrace{\left[-p(T) + \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big)\right]}_{=0} + \int_0^T \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial \dot{x}}\right)\right)}_{=0}\eta(t)dt = 0$$

The two later terms vanished because of the transversality conditions and the Euler-Lagrange equations. With $\eta(0) = \eta$, this means that the increase in the utility functional is approximately

$$V^*(x_0+\eta) - V^*(x_0) \approx \eta p(0)$$

And in the limit, we have

$$\lim_{\eta \to 0} \frac{V^*(x_0+\eta) - V^*(x_0)}{\eta} = p(0)$$

An incremental increase in the initial asset position, $d\eta$, results in an incremental increase of the utility functional of the amount $p(0)d\eta$. The value of the assets at time $t = 0$ is therefore equal to $p(0)$.

A similar argument can used to show that the interpretation of a shadow price holds over the complete planning horizon. To do that, consider an agent that follows an optimal asset trajectory, but at some time τ unexpectedly receives an additional amount of assets η. This means that the variation $\eta(t) = 0$ for the interval $t \in [0, \tau\rangle$, and $\eta(\tau) = \eta$. The increase in the total utility can again be found using a Taylor approximation, resulting in

$$\begin{aligned} V\Big(x(\cdot) + \eta(\cdot)\Big) - V\Big(x(\cdot)\Big) &\approx \eta(\tau)p(\tau) + \eta(T)\underbrace{\left[-p(T) + \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big)\right]}_{=0} + \int_\tau^T \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial \dot{x}}\right)\right)}_{=0}\eta(t)dt = 0 \\ &= \eta(\tau)p(\tau) \end{aligned}$$

[10] Taking the difference between maximization of the utility and minimization of the action into account.

In the limit, we obtain

$$\lim_{\eta \to 0} \frac{V\Big(x(\cdot)+\eta(\cdot)\Big) - V\Big(x(\cdot)\Big)}{\eta} = p(\tau)$$

An incremental increase in the initial asset position at time τ, thus results in an incremental increase of the utility functional of the amount $p(\tau)d\eta$. The value of the assets at time τ is therefore equal to $p(\tau)$, and in general we can thus interpret the generalized momentum as the shadow price of the asset.

The units of the shadow price is equal to utils per asset. This is not a very convenient measure to work with as the price of an asset is generally measure in terms of a currency, e.g. dollars, euros or yens. If the dollar price of one asset is known, then we could use that as numeraire, and use it to express the value of the assets in terms of their dollar value.

3.4 Handling the Asset Dynamics in the Lagrangian

This chapter start with the asset dynamics (3.1) for the "simple consumption goods". The general asset allocation problem in Section 2.2 however incorporates the (possibly nonlinear) asset dynamics of the form

$$\dot{x}(t) = f(x(t), u(t), t) \tag{3.20}$$

The maximum principle in Chapter 5 focuses on the general asset dynamics, but incorporation of the asset dynamics is possible in the Lagrangian method. This requires the asset dynamics to be invertible, meaning that the control decisions $u(t)$ can be express as some function of $x(t)$, $\dot{x}(t)$, and t. Let this function be

$$u(t) = \hat{u}(x(t), \dot{x}(t), t) \tag{3.21}$$

Incorporation of the asset dynamics is then simply done by substitution of (3.21) in the utility functional, yielding

$$\begin{aligned} \mathcal{V}^* = \max_{x(\cdot)} \mathcal{V}\left[x(\cdot)\right] = \max_{x(\cdot)} &\left[\Phi\Big(x(T), T\Big) + \int_0^T v\left(x(t), \hat{u}\Big(x(t), \dot{x}(t), t\Big), t\right) dt\right] \\ \text{s.t.} \quad & x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad \dot{x}(\cdot) \in \mathbb{R}^n \end{aligned} \tag{3.22}$$

With the substitution of the inversed asset dynamics, the total utility is again a functional that depends on the asset trajectory $x(\cdot)$. This implies that the Euler-Lagrange equations in (3.4) must hold. Rewriting those equations, yields the Euler-Lagrange equations for the invertable asset dynamics.

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial u}\frac{\partial u}{\partial \dot{x}}\right) = 0 \tag{3.23}$$

The definition of the shadow price $p(t)$ is not changed, but accounting for the asset dynamics, it is equal to

$$p(t) = -\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} \tag{3.24}$$

And the Legendre condition can be written as

$$\frac{\partial^2 v}{\partial \dot{x}^2} = \frac{\partial}{\partial \dot{x}}\left(\frac{\partial v}{\partial u}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 v}{\partial u^2}\left(\frac{\partial u}{\partial \dot{x}}\right)^2 + \frac{\partial v}{\partial u}\frac{\partial^2 u}{\partial \dot{x}^2} \leq 0 \tag{3.25}$$

3.4.1 The Asset Dynamics of the Lagrangian Economic Agent Revised

Now let us again consider the simple consumption assets. The control in the asset dynamics (3.1) increase the asset position, and correspond with the negative of consumption. The asset dynamics of the simple consumption goods are arguably more insightful when defined as

$$\begin{aligned} \dot{x}(t) &= -u(t) \\ x(0) &= x_0 \end{aligned} \tag{3.26}$$

Where the control $u(t)$ now simply corresponds with consumption. The shadow price is now defined as

$$p(t) = \frac{\partial v}{\partial u}\Big(x(t), u(t), t\Big) \tag{3.27}$$

The Euler-Lagrange equations can be written as

$$\frac{\partial v}{\partial x}\Big(x(t), u(t), t\Big) + \frac{\partial v}{\partial u}\Big(x(t), u(t), t\Big) = 0 \tag{3.28}$$

And the Legendre condition is equal to

$$\frac{\partial^2 v}{\partial u^2} \leq 0 \tag{3.29}$$

4 The Economic Agent as a Hamiltonian System

This chapter continues with the set-up of the previous chapter. It considers the maximization of utility with simple consumption goods, and converts the Lagrangian description of the economic agent into a Hamiltonian system. It was Sir William Rowen Hamiltonian who showed in Hamilton (1834) and Hamilton (1835) that the dynamics of a mechanical system could be derived from the action integral, called Hamilton's Principle, and that the Euler-Lagrange equations can be converted into a systems of first-order differential equations. However, rather than using the original definition for the Hamiltonian, I adopt the definition of Sussmann and Willems (1997) which incorporates the control decisions $u(t)$ into the Hamiltonian, leading to *"Hamilton's equations as he should have written them"*.

To some it may seem that the Hamiltonian and Hamilton's canonical equations are a simple rewriting of the Euler-Lagrange equations. Kamien and Schwartz (1981) refer in the economic context to it as *"The device for remembering or generating these conditions (...) is the Hamiltonian"*. I oppose this idea and view the Hamiltonian not simply as a device for calculating the optimal solutions, but rather as the economic surplus of the agent. Hamilton's canonical equations, two first-order differential equations, describe the evolution of the asset positions and the shadow price over time. However, the use of the control Hamiltonian also gives us the insight that the Hamiltonian is maximized at every moment in time. This has important economic implications and results in a myopic description of the economic agent.

4.1 The Hamiltonian as the Economic Surplus

The mathematical set-up of the agent's behavior is in this chapter equivalent to the Lagrangian set-up. However, to be able to use the control Hamiltonian, the utility is expressed as functional in terms of $(x(t), u(t), t)$. The agent is then described by the optimal control problem

$$\begin{gathered} \mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] = \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t) dt\right] \\ \text{s.t.} \quad \dot{x}(t) = u(t), \quad x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad u(\cdot) \in \mathbb{R}^n \end{gathered} \tag{4.1}$$

In the previous chapter, we explained that its solution is described by the Euler-Lagrange equation in combination with the Legendre and transversality conditions. As the asset dynamics of the "simple consumption goods" ensure that $\dot{x}(t) = u(t)$, it is possible to rewrite the Euler-Lagrange as

$$\frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial u}\Big(x(t), \dot{x}(t), t\Big)\right) = 0 \tag{4.2}$$

Here we stick to the notation of Sussmann and Willems (1997), but to clarify it, with $\frac{\partial v}{\partial u}\Big(x(t), \dot{x}(t), t\Big)$ we mean the partial derivative of the running utility $v(x(t), u(t), t)$ with respect to the control action $u(t)$ and evaluated at the point $(x(t), u(t), t) = (x(t), \dot{x}(t), t)$. The Legendre and transversality condition can, in the same notation, be written as

$$\frac{\partial^2 v}{\partial u^2}\Big(x(t), \dot{x}(t), t\Big) \leq 0 \quad \text{and} \quad p(T) = \frac{\partial \Phi}{\partial x}\Big(x(T), \dot{x}(T), T\Big) \tag{4.3}$$

Now, let us reconsider the problem of utility maximization as described in (4.1). This objective states that the utility accrued over the planning period is maximized, and its solution is not a single point, but the optimal asset trajectory $x^*(\cdot)$ over the entire planning horizon. This optimal asset trajectory is described by the Euler-Lagrange equation, the Legendre condition, and the transversality condition. The Euler-Lagrange equation describes all stationary asset trajectories, and the Legendre condition guarantees that the stationary solution indeed maximizes the utility functional. The transversality condition ensures that the bequest utility is taken into account, and that the assets are correctly priced at the planning horizon.

Taking a second look at the Legendre condition, reveals that it is the usual second-order necessary condition for the instantaneous maximization of the running utility $v(x(t), u(t), t)$ with respect to the control input. However, the first-order condition, the Euler-Lagrange equation, looks not all like the first-order order condition of such an instantaneous maximization. This leads to the natural question whether there exist a new function, with an economic interpretation, that is at every instant along the optimal asset trajectory maximized with respect to the control $u(t)$. This is especially interesting from an economic perspective. The initial utility objective (4.1) describes an agent that maximizes utility over an entire planning horizon. However, if their exist a new function that is maximized at every instant with respect to control actions, and that results in the same asset and price dynamics as the Lagrangian description, then this agent can also be described myopically. This new function is found to exist, and is called the Hamiltonian in control theory, and will be defined hereafter. In economics it has the interpretation of the economic surplus. The alternative, myopic characterization describes an agent that maximizes economic surplus at every instant with respect to the allowed control actions, given the asset allocation and the asset prices.

Now let us define this new function, called the Hamiltonian, as a function of the assets $x(t) \in \mathbb{R}^n$, the control $u(t) \in \mathbb{R}^n$, the shadow price $p(t) \in \mathbb{R}^n$ and time $t \in \mathbb{R}$ and with the functional form

$$H\Big(x(t), p(t), u(t), t\Big) = v\Big(x(t), u(t), t\Big) + p(t)u(t) \tag{4.4}$$

The Hamiltonian is the sum of the direct benefit $v(x(t), u(t), t)$ plus the increase in the value of the asset portfolio when the price is assumed to be constant [11,12]. Remember that consumption in this set-up corresponds to $u(t) \leq 0$. The Hamiltonian is hence the differen1ce between the direct benefit $v(x(t), u(t), t)$ minus the cost of the consumed goods $p(t)u(t)$. This is what we will call the economic surplus of the agent, also called the consumer surplus. This definition of the Hamiltonian (4.4) is slightly different from the definition of Sussmann and Willems (1997) and it has all to do with the difference between minimizing the action and maximizing the utility. In fact, with my definition of the shadow price as

$$p(t) = -\frac{\partial v}{\partial u}\Big(x(t), \dot{x}(t), t\Big) \tag{4.5}$$

the Hamiltonian in the paper of Sussmann and Willems (1997) is the negative of the Hamiltonian of (4.4). Now let us consider the first-order conditions for maximization of the Hamiltonian with respect to its arguments $(x(t), u(t), p(t), t)$. The first derivative with respect to the assets $x(t)$ is directly interesting, and equals

$$\frac{\partial H}{\partial x}\Big(x(t), p(t), u(t), t\Big) = \frac{\partial v}{\partial x}\Big(x(t), u(t), t\Big) \tag{4.6}$$

Now recognize that term on the right sight is equal to the utility rental in (3.9), when evaluated at $(x(t), u(t), t)$ is $(x(t), \dot{x}(t), t)$. The no-arbitrage condition (the Euler-Lagrange equation in (4.2)) should still hold, meaning that the change in price can also be expressed in terms of the economic surplus as

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\Big(x(t), p(t), \dot{x}(t), t\Big) \tag{4.7}$$

The change in the economic surplus with respect to the price $p(t)$ is easy to see. An increase in the price does not directly changes the instantaneously gained utility, but it does increases the cost of consumption. This means that $\partial H/\partial p = u(t)$ and substituting the asset dynamics of the simple consumption goods, yields

$$\dot{x}(t) = \frac{\partial H}{\partial p}\Big(x(t), p(t), \dot{x}(t), t\Big) \tag{4.8}$$

The change in the economic surplus with respect to the control action $u(t)$ is the key argument to use the control Hamiltonian instead of the Hamilton's original definition. The cost of consumption increases linearly with the

[11] Recognizing the asset dynamics of the simple consumption goods, the control Hamiltonian could also be written somewhat informal as $H = v\Big(x(t), u(t), t\Big) + p(t)\dot{x}(t)$

[12] The value of the portfolio changes in two different ways due to either a change in the asset portfolio or due to a change in the asset prices.

control action, but also the derived utility changes. How much does the utility changes incrementally, well by definition (4.5) exactly with the asset price. This means that the change in economic surplus with respect to the control action is actually zero

$$\frac{\partial H}{\partial u}\Big(x(t), p(t), \dot{x}(t), t\Big) = \frac{\partial v}{\partial u}\Big(x(t), \dot{x}(t), t\Big) + p(t) = 0 \tag{4.9}$$

The economic surplus of the agent is at every instant stationary with respect to control action $u(t)$. This is the first indication that the economic surplus is maximized by the economic agent. The change of the economic surplus with respect to time, is best understood if we analyze the total time derivative of the Hamiltonian. It is equal to (omitting the arguments of the derivatives):

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial x}\dot{x}(t) + \frac{\partial H}{\partial u}\dot{u}(t) + \frac{\partial H}{\partial p}\dot{p}(t) + \frac{\partial H}{\partial t} \tag{4.10}$$

Recognizing that the first three partial derivatives of the right cancel due to (4.7), (4.8), and (4.9). Meaning that the total time derivative of the hamiltonian is equal to the partial time derivative of the hamiltonian.

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = \frac{\partial v}{\partial t}\Big(x(t), u(t), t\Big) \tag{4.11}$$

The means that the economic surplus only changes over time, if the Hamiltonian explicitly depends on time. For the problem with simple consumption goods, this means that the Hamiltonian is constant over time, if the running utility does not depend on time. For mechanical systems, this happens when all forces can be derived from a potential function. For dissipative systems, the Hamiltonian may be characterized as time-dependent as McDonald (2015) describes. For economic systems, a time-dependency often enters the running utility naturally to capture the time preference of agents.

4.2 Towards the Maximum Principle and the Myopic Agent

The definition of the Hamiltonian (4.4) ensures that the no-arbitrage condition (4.2) is compactly written into two first-order differential equations. In the Euler-Lagrange equation, the controlled evolution of the asset positions and the shadow prices are intertwined. By application of the Hamiltonian, the evolution of the asset positions and the shadow prices are entangled from each other, and are both expressed as a first-order differential equation that depends on the economic surplus.

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\Big(x(t), p(t), u(t), t\Big) \quad \text{and} \quad \dot{x}(t) = \frac{\partial H}{\partial p}\Big(x(t), p(t), u(t), t\Big) \tag{4.12}$$

In addition to this, the application of the control Hamiltonian shows that the economic surplus is at least stationary over time with respect to the control input $u(t)$

$$\frac{\partial H}{\partial u}\Big(x(t), p(t), u(t), t\Big) = 0 \tag{4.13}$$

Sussmann and Willems (1997) refer to equations (4.12) and (4.13) as "Hamilton's equations as he should have written them". They argue that the maximum principle would have been discovered by Hamilton himself, or some other 19th century mathematician, if Hamilton wrote the Hamiltonian in the form of (4.4).

Now to see that the economic surplus is indeed maximized, let us consider the Legendre condition (4.3) that states $\partial^2 v/\partial u^2 \leq 0$. Now realize that the economic surplus is the sum of the running utility $v(x(t), u(t), t)$ plus a linear term of the control $u(t)$. Then the Legendre condition is equivalently written in terms of the economic surplus as

$$\frac{\partial^2 H}{\partial u^2}\Big(x(t), p(t), u(t), t\Big) \leq 0 \tag{4.14}$$

The first-order condition in (4.13) shows that the Hamiltonian is stationary with respect to control action $u(t)$, and the Legendre condition indicated that the Hamiltonian is indeed maximizes. Section 4.6 discusses the proof that the Hamiltonian is indeed maximized along the optimal trajectory with respect to the control action, using the Weierstrass Excess Function. It shows that the economic surplus decreases for an arbitrary $u(t)$ compared to the optimal asset trajectory

$$H\Big(x^*(t), p(t), u(t), t\Big) - H\Big(x^*(t), p(t), u^*(t), t\Big) \leq 0$$

4.3 Weierstrass Excess Function to Show that Economic Surplus is Maximized.

The Weierstrass excess function was originally developed to construct conditions for a strong maximum in the calculus of variations. These solutions are continuous, but not necessarily smooth. For the agents under evaluations, it originally showed that an optimal asset trajectory $x^*(\cdot)$ is also optimal compared to another asset trajectory $x(\cdot)$ which is, at every point in time, arbitrarily close to $x^*(\cdot)$, but its derivative not necessarily. It was defined bij Weierstrass (see Sussmann and Willems (1997)) in terms of the running utility (the Lagrangian) as

$$\mathcal{E}\Big(x^*(t), \dot{x}^*(t), w, t\Big) = v\Big(x^*(t), w, t\Big) - v\Big(x^*(t), \dot{x}^*(t), t\Big) - \frac{\partial v}{\partial \dot{x}}\Big(x^*(t), \dot{x}^*(t), t\Big)\Big(w - \dot{x}^*(t)\Big)$$

But recognizing the definition of the economic surplus and the shadow price, this expression is equivalently written as the difference in economic surplus

$$\mathcal{E}\Big(x^*(t), \dot{x}^*(t), w, t\Big) = H(x^*(t), p(t), w, t) - H(x^*(t), p(t), \dot{x}^*(t), t)$$

And to proof that the economic surplus is indeed maximized, we should proof that Weierstrass excess function is not positive $\mathcal{E}\Big(x(t), \dot{x}(t), w, t\Big) \leq 0$. This proof can be set-up in different forms, but we will discuss here the proof in Leitmann (1980) and Friesz (2010), and review it from an economic perspective. The proof is constructed by considering the following variation of the optimal asset trajectory $x^*(t)$ of the form

$$x(t) = \begin{cases} x^*(t) & \text{if } t \in [0, \tau_i] \cup [\tau_e, T] \\ \zeta(t) & \text{if } t \in [\tau_i, \epsilon], \quad \text{and} \quad \epsilon \in [\tau_i, \tau_e] \\ \phi(t, \epsilon) & \text{if } t \in [\epsilon, \tau_e] \end{cases} \tag{4.15}$$

where

$$\begin{aligned} \zeta(t) &= x^*(\tau_i) + w(t - \tau_i) \qquad \text{and} \quad w \in \mathbb{R} \\ \phi(t, \epsilon) &= x^*(t) + \frac{\zeta(\epsilon) - x^*(\epsilon)}{\tau_e - \epsilon}(\tau_e - t) \end{aligned}$$

This variation of the optimal asset trajectory $x^*(\cdot)$ looks mathematically challenging, but plotted in Figure 4.1 it is actually quite simple. With the varied asset-trajectory $x(\cdot)$ the economic agent follows the optimal asset trajectory over the entire planning horizon except for the part $t \in [\tau_i, \tau_e]$. In the first part of this interval $t \in [\tau_i, \epsilon]$, the agent adjust his consumption rate to the constant rate $-w$. In the second part, the agent follows again the optimal trajectory, but adjusted for the additional or reduced consumption of the first interval. This difference is linearly spread over the interval $t \in [\epsilon, \tau_e]$ such that the amount of assets at time $t = \tau_e$ is equal in both the optimal and the varied path. Thereafter the optimal asset trajectory is again followed. The rate of change of the varied asset trajectory is equal to

$$\dot{x}(t) = \begin{cases} \dot{x}^*(t) & \text{if } \cup [0, \tau_i], \text{or } t \in [\tau_e, T] \\ w & \text{if } t \in [\tau_i, \epsilon] \\ \dot{x}^*(t) - \dfrac{\zeta(\epsilon) - x^*(\epsilon)}{\tau_e - \epsilon} & \text{if } t \in [\epsilon, \tau_e] \end{cases}$$

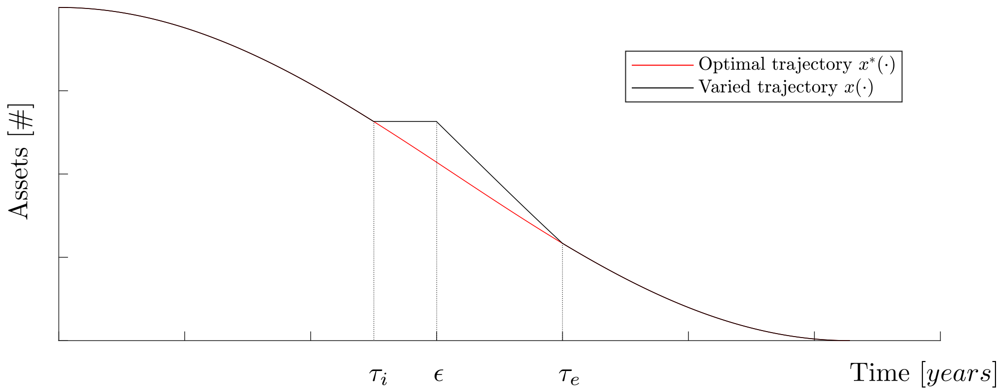

Figure 4.1: Schematic representation of the variation to proof that the economic surplus is maximized.

This formulation ensures that if the first time interval is decreased, then the varied trajectory $x(\cdot)$ approaches the optimal asset trajectory $x(\cdot)$ as $\epsilon \to \tau_i$. Now if $x^*(\cdot)$ is indeed the optimal asset trajectory then we should have that the gained utility over the optimal path is larger than over the varied path.

$$\Psi(\epsilon) = \mathcal{V}\left[x^*(\cdot)\right] - \mathcal{V}\left[x(\cdot)\right] \geq 0$$

Evaluated at $\epsilon = \tau_i$, the varied path is by its definition equal to the optimal trajectory. Because the utility will decrease when the agent leaves the optimal trajectory, this means that

$$\Psi(\tau_i) = 0 \quad \text{and} \quad \frac{\partial \Psi}{\partial \epsilon}(\tau_i) \leq 0$$

The proof is now set-up by contradiction. It show that the $\partial\Psi/\partial\epsilon$ evaluate at time τ_i is equal to Weierstrass Excess Function which proofs that the Hamiltonian is indeed maximized along the optimal trajectory. As the economic agent follows the optimal asset trajectory everywhere except for the interval $t \in [\tau_i, \tau_e]$, we obtain

$$\Psi(\epsilon) = \int_{\tau_i}^{\epsilon} \left[v\Big(\zeta(t), \dot{\zeta}(t), t\Big) - v\Big(x(t), \dot{x}(t), t\Big) \right] dt + \int_{\epsilon}^{\tau_e} \left[v\Big(\phi(t,\epsilon), \dot{\phi}(t,\epsilon), t\Big) - v\Big(x(t), \dot{x}(t), t\Big) \right] dt$$

The derivative of this expression should be evaluated using Leibniz's integral rule, resulting in

$$\frac{\partial \Psi}{\partial \epsilon}(\epsilon) = v\Big(\zeta(\epsilon), \dot{\zeta}(\epsilon), \epsilon\Big) - v\Big(\phi(\epsilon,\epsilon), \dot{\phi}(\epsilon,\epsilon), \epsilon\Big) + \int_{\epsilon}^{\tau_e} \left[\frac{\partial v}{\partial x}\frac{\partial \phi}{\partial \epsilon}(t,\epsilon) + \frac{\partial v}{\partial u}\frac{\partial \dot{\phi}}{\partial \epsilon}(t,\epsilon) \right]$$

Application of the chain rule allows us to write the later integral as

$$\int_{\epsilon}^{\tau_e} \left[\frac{\partial v}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial u}\right) \right] \frac{\partial \phi}{\partial \epsilon}(t,\epsilon) dt + \frac{\partial v}{\partial u}\Big(\phi(\tau_e,\epsilon), \dot{\phi}(\tau_e,\epsilon), \tau_e\Big)\frac{\partial \phi}{\partial \epsilon}(\tau_e,\epsilon) - \frac{\partial v}{\partial u}\Big(\phi(\epsilon,\epsilon), \dot{\phi}(\epsilon,\epsilon), \epsilon\Big)\frac{\partial \phi}{\partial \epsilon}(\epsilon,\epsilon)$$

Recognize that the $\partial\phi/\partial\epsilon(\tau_e, \epsilon) = 0$, and reducing the length of the first interval to $\epsilon \to \tau_i$. The optimal asset trajectory then approached the optimal trajectory, and the terms in the integral cancel due to the Euler-Lagrange equations. This yields the expressions

$$\frac{\partial \Psi}{\partial \epsilon}(\tau_i) = v\Big(\zeta(\tau_i), \dot{\zeta}(\tau_i), \tau_i\Big) - v\Big(\phi(\tau_i,\tau_i), \dot{\phi}(\tau_i,\tau_i), \tau_i\Big) - \frac{\partial v}{\partial u}\Big(x^*(\tau_i), \dot{x}^*(\tau_i), \tau_i\Big)\Big(\dot{\zeta}(\tau_i) - \dot{x}^*(\tau_i)\Big)$$

And with the definitions in (4.15) this simplifies to

$$\begin{aligned}
\frac{\partial \Psi}{\partial \epsilon}(\tau_i) &= v\Big(x^*(\tau_i), w, \tau_i\Big) - v\Big(x^*(\tau_i), \dot{x}^*(\tau_i), \tau_i\Big) - \frac{\partial v}{\partial u}\Big(x^*(\tau_i), \dot{x}^*(\tau_i), \tau_i\Big)\Big(w - \dot{x}^*(\tau_i)\Big) \\
&= H(x^*(\tau_i), p(\tau_i), w, \tau_i) - H(x^*(\tau_i), p(\tau_i), \dot{x}^*(\tau_i)\tau_i t) \\
&\leq 0
\end{aligned}$$

Where the latter inequality holds by definition. This proofs that the Hamiltonian is indeed maximized along the optimal asset trajectory.

4.4 The Myopic Economic Agent with Simple Consumption Goods

The results of Section 4.2 imply that an agent who maximizes utility over an entire planning horizon is equivalently described as an agent that maximizes his economic surplus instantaneous with respect to the control action $u(t)$, given his current asset positions $x(t)$ and shadow prices $p(t)$. This is mathematically denoted as

$$\max_{u(t)} H\Big(x(t), p(t), u(t), t\Big) = \max_{u(t)} v\Big(x(t), u(t), t\Big) + p(t)u(t) \tag{4.16}$$

The agent maximizes economic surplus only with respect to the control $u(t)$ as this is the only parameters that he can change instantaneously. The asset allocation and the shadow prices will change, but as a result of the consumption $u(t)$ of agent. The change in the asset allocation and the shadow prices is expressed in terms of the economic surplus by Hamilton's canonical equations and with the appropriate boundary conditions.

$$\dot{x}(t) = \frac{\partial H}{\partial p}\Big(x(t), p(t), u(t), t\Big) \quad \text{and} \quad x(0) = x_0 \tag{4.17}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\Big(x(t), p(t), u(t), t\Big) \quad \text{and} \quad p(T) = \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \tag{4.18}$$

From an economic point of view, this is not surprising. An increase in the shadow price does not immediately lead to an increase in the gained utility, but it does increase the cost of consumption, resulting in (4.17). Similar, an incremental change in the asset positions does not increases the cost of consumption immediately, but it does increase the utility gained instantaneously. The increase is (approximately) equal to the utility rental times the increment in the number of assets. By the no-arbitrage condition, the change in the shadow price and the utility rental are related to each other, meaning that shadow price dynamics are expressed in terms of the agent's economic surplus as in (4.18). The power of the Euler-Lagrange equation (the no-arbitrage condition), the Legendre condition and the Weierstrass Necessary Condition are all captured in this myopic perspective. Also note that it is in this formulation not necessary to define the shadow price as (4.5). Instead the shadow price is implicitly defined with the maximization of the economic surplus as in (4.16).

This myopic characterization of the economic agent should not be interpreted as an simple problem to solve. The actual asset and price trajectory are still described by the boundary value problem (4.17) and (4.18), where in addition the control $u(t)$ is determined by the surplus maximization of (4.16). Analytic solutions can only be obtained in the simplest cases, meaning that computational solvers often need to be employed.

4.5 The Control Hamiltonian versus Hamilton's Original Definition

In the previously sections, we already mentioned a couple of times that the control Hamiltonian is essential to get the insight that the economic surplus is maximized along the optimal asset trajectory. To show that this is indeed the case, let us review Hamilton's original definition in the economic setting, and follow Sussmann and Willems (1997) who gave a similar argument for the brachystochrone problem.

Hamilton considered the formulation of the action functional, where the Lagrangian is a function of $(x(t), \dot{x}(t), t)$. This corresponds with the utility maximization problem that is not explicitly a function of the control $u(t)$

$$\begin{gathered}\mathcal{V}^* = \max \mathcal{V}\left[x(\cdot)\right] = \max \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), \dot{x}(t), t)dt\right] \\ \text{s.t.} \quad x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad \dot{x}(\cdot) \in \mathbb{R}^n\end{gathered} \tag{4.19}$$

The original definition of the Hamiltonian is for the utility maximization problem written as

$$\mathcal{H}\Big(x(t), p(t), t\Big) = v\Big(x(t), \dot{x}(t), t\Big) + p(t)\dot{x}(t) \tag{4.20}$$

In this definition, we took the difference between the action minimization and utility maximization into account, as well as the sign convention for the shadow price. Notice that the original Hamiltonian can also be interpreted as the economic surplus of the agent. However, the arguments of the original Hamiltonian in (4.20) dictates that it is a function of the the assets $x(t) \in \mathbb{R}^n$, the shadow price $p(t) \in \mathbb{R}^n$ and of time $t \in \mathbb{R}$. The expression itself suggest that the Hamiltonian is also a function of the change in the asset positions $\dot{x}(t) \in \mathbb{R}^n$. Hamilton did not see the fulfilled demand $\dot{x}(t)$ as an independent variable, but as a variable that is implicitly defined by the definition of the shadow price. This means that the function for the shadow price in (4.5) should be inverted, such that the demand $\dot{x}(t)$ is a function of $x(t)$, $p(t)$ and time t. This inversion is not always possible, but lets assume that it is. In line with the asset dynamics, denote this inverted function as $\dot{x}(t) = u(x(t), p(t), t)$ and recognize that the control actions are now a function of $(x(t), p(t), t)$. The original Hamiltonian is then equal to the control Hamiltonian $H\Big(x(t), p(t), u(t), t\Big)$ with the substitution of the control $u(t) = u(x(t), p(t), t)$.

$$H\Big(x(t), p(t), u\big(x(t), p(t), t\big), t\Big) = \mathcal{H}\Big(x(t), p(t), t\Big) \tag{4.21}$$

The canonical equations that Hamilton obtained, are

$$\dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}\Big(x(t), p(t), t\Big) \quad \text{and} \quad \dot{x}(t) = \frac{\partial \mathcal{H}}{\partial p}\Big(x(t), p(t), t\Big) \tag{4.22}$$

And, if it is indeed possible to invert the function for the shadow price in (4.5), then Hamilton's original canonical equations can also be evaluated in terms of the control Hamiltonian.

$$\frac{\partial \mathcal{H}}{\partial x} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial H}{\partial x} \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial p} = \frac{\partial H}{\partial p} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial p} = \frac{\partial H}{\partial p} \tag{4.23}$$

Where $\partial H/\partial u = 0$ as the control Hamiltonian is stationary with respect to the control $u(t)$. This discussions shows that the canonical equations that result from the control Hamiltonian are equivalent to those from Hamilton's original definition. Both these Hamiltonians can in economics be interpreted as the economic surplus, but the control Hamiltonian has some clear advantages.

The first, and most obvious advantage has to with the inversion of the shadow price to obtain the fulfilled the demand $\dot{x}(t)$. The control Hamiltonian recognizes the possibility of using the asset dynamics in the Hamiltonian. The resulting canonical equations for the asset allocation and the shadow price are completely equivalent to the no-arbitrage condition of the Euler-Lagrange equation. This is not the case for the original Hamiltonian where the equivalence with the Euler-Lagrange equations only holds when the fulfilled demand $\dot{x}(t)$ can be expressed in terms of $(x(t), p(t), t)$ by inverting the function of the shadow price.

However, the most important advantage is that only the control Hamiltonian gives the insight that the economic surplus is maximized along the optimal asset trajectory with respect to the control actions $u(t)$. The original Hamiltonian does not provide this insight as it is not even a function of the control action $u(t)$ or the fulfilled demand $\dot{x}(t)$. It is therefore only the control Hamiltonian that gives the insight that the agent maximizes economic surplus along the optimal trajectory.

4.6 Hamilton's Equations and the Shadow Price from Utility Maximization

In Section 3.2 we proofed the no-arbitrage condition (the Euler-Lagrange equation) by evaluating the utility functional for a variation of the optimal trajectory. The control Hamiltonian was defined, and interpreted as the economic surplus, and we showed that the asset and price dynamics implied by the no-arbitrage condition, can also be expressed in terms of Hamilton's canonical equations (as he should have written them). However, Hamilton's canonical equations can also be derived directly from the utility functional. The utility functional is in terms of the Hamiltonian written as

$$\begin{aligned} \mathcal{V}^* = \max_{x(\cdot)} \mathcal{V}\left[x(\cdot)\right] = \max_{x(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T H\Big(x(t), p(t), \dot{x}(t), t\Big) - p(t)\dot{x}(t) dt\right] \\ \text{s.t.} \quad \dot{x}(t) = u(t), \quad x(0) = x_0, \quad x(\cdot) \in \mathbb{R}^n, \quad \dot{x}(\cdot) \in \mathbb{R}^n \end{aligned} \tag{4.24}$$

The no-arbitrage condition was derived with the variation $\epsilon\eta(\cdot)$, but let now follow the notation of Landau and Lifshitz (1972), and denote the variation as $\delta x(\cdot)$. This variation also effects the shadow price $p(t)$ and the change in the asset allocation $\dot{x}(t)$, and denoted these variations respectively as $\delta p(\cdot)$ and $\delta \dot{x}(\cdot)$. The first variation in the utility functional then equals

$$\delta \mathcal{V} = \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big)\delta x(T) + \int_0^T \left(\frac{\partial H}{\partial x}\delta x(t) + \left(\frac{\partial H}{\partial p} - \dot{x}(t)\right)\delta p(t) + \frac{\partial H}{\partial u}\delta \dot{x}(t) - p(t)\delta \dot{x}(t)\right) dt \tag{4.25}$$

The latter variation is rewritten using the chain rule and the fact that the initial position cannot change $\delta x(0) = 0$, yielding

$$\int_0^T p(t)\delta \dot{x}(t) dt = p(T)\delta x(T) - \int_0^T \dot{p}(t)\delta x(t) dt \tag{4.26}$$

Such that the first variation of the utility functional equals

$$\delta \mathcal{V} = \left(\frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) - p(T)\right)\delta x(T) + \int_0^T \left(\left(\frac{\partial H}{\partial x} + \dot{p}(t)\right)\delta x(t) + \left(\frac{\partial H}{\partial p} - \dot{x}(t)\right)\delta p(t) + \frac{\partial H}{\partial u}\delta \dot{x}(t)\right) dt \tag{4.27}$$

For a stationary asset trajectory, we require that the first variation in the utility functional is equal to zero. This means that all terms before the variations in δx, δp, and $\delta \dot{x}$ should be equal to zero, implying the Hamilton's canonical equations

$$\dot{x}(t) = \frac{\partial H}{\partial p}\Big(x(t), p(t), \dot{x}(t), t\Big), \quad \dot{p}(t) = -\frac{\partial H}{\partial p}\Big(x(t), p(t), \dot{x}(t), t\Big) \quad \text{and} \quad \frac{\partial H}{\partial u}\Big(x(t), p(t), \dot{x}(t), t\Big) = 0 \tag{4.28}$$

And the transversaility condition for the shadow price

$$p(T) = \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \tag{4.29}$$

5 The Maximum Principle as a Myopic Economic Agent

In Chapter 2, I described the utility maximization problem of the economic agent as an optimal control problem. The goal of the agent is to find the optimal control trajectory $u^*(\cdot)$ with the corresponding asset trajectory $x^*(\cdot)$. The distinctive feature of this maximization problem is that the total utility over the planning horizon depends on the asset-control pair $(x(\cdot), u(\cdot))$, and both the utility functional and the asset dynamics determine the optimal trajectory. This is different from the Lagrangian and Hamiltonian approach, where the optimal trajectory is determined by the utility functional and the agent searches over all possible assets paths. Mathematically, the problem of utility maximization is summarized as

$$\begin{aligned} \mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[x(\cdot), u(\cdot)\right] &= \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t)dt\right] \\ \text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), t), &\quad x(0) = x_0, \quad x(\cdot) \in \mathcal{X}, \quad u(\cdot) \in \mathcal{U} \end{aligned} \tag{5.1}$$

In Table 5.1 I summarize the analogous variables in utility maximization problem and control theory. The Lagrangian and Hamiltonian set-up in Chapter 3 and Chapter 4 only consider "simple consumption goods", reducing the asset dynamics to $f(x(t), u(t), t) = u(t)$, and I showed that the agent is equivalently described as a Hamiltonian system. The Hamiltonian can specifically be interpreted as the economic surplus and was defined as $H = v\left(x(t), u(t), t\right) + p(t)u(t)$ which also could be written as $H = v\left(x(t), u(t), t\right) + p(t)\dot{x}(t)$. The experience that we gained with the simple consumption goods, suggests that the general problem of utility maximization can be solved in a similar way by incorporating the asset dynamics into the Hamiltonian. This leads to a new description of the Hamiltonian as $H = v\left(x(t), u(t), t\right) + p(t)f(x(t), u(t), t)$. Pontryagin (1962) showed that the optimal control problem can indeed be solved with an Hamiltonian, but that a more general form is required. The Hamiltonian for the optimal control problem is equal to

$$H\Big(x(t), p(t), u(t), t\Big) = p_0 v\Big(x(t), u(t), t\Big) + p(t)f(x(t), u(t), t) \tag{5.2}$$

This Hamiltonian indeed incorporates the asset dynamics, but also introduces a new parameter p_0. Section 5.3 elaborates on this new parameter, but lets note here that p_0 has a constant value over the planning horizon, and that for most applications the choice $p_0 = 1$ is appropriate. For the agent under evaluation, Pontryagin's Hamiltonian can still be interpreted as the economic surplus, and it is again the sum of the (instantaneously) received utility plus the value change of the asset portfolio when the price is assumed to be constant[13].

Necessary conditions for the optimal control trajectory and the corresponding asset trajectory are developed by Pontryagin (1962), and rigorously proven as well. These conditions are generally referred to as Pontrygan's maximum principle or simply the maximum principle. In the next section I will discuss the economic implications of the maximum principle for the agent, and not to surprisingly, it is closely related to the Hamiltonian system description of the previous chapter. With the maximum principle, the agent is again described myopically, maximizing economic surplus at every instant along the optimal asset trajectory. Hence, the goal of this chapter is not to give a full treatment of maximum principle, but rather show that it naturally evolved from the Hamiltonian system characterization and that the economic interpretation remains. The maximum principle consists of four necessary conditions that an optimal control and asset trajectory should satisfy. These conditions are in control theory called

- The maximum condition
- Hamiltonian system condition
- Non-triviality condition
- Transversality conditions

[13]The value of the portfolio changes in two different ways due to either a change in the asset portfolio or due to a change in the asset prices. The economic agent does not account for the price change in the Hamiltonian.

In the next sections I will discuss these conditions, relate it to the Lagrangian and Hamiltonian description of the previous sections and show how it is interpreted in economics. Handling state constraint in the optimal control problem is very hard, and I will not focus on it in the discussion that follows, such that $x(\cdot) \in \mathbb{R}^n$. It is furthermore noteworthy to mention that the control signal $u(t)$ can be a partial continuous function over time. The agent is under the maximum principle allowed to change his control instantaneously, as one would expect it to be.

Symbol	Control Theory	Economic Interpretation	Units
$x(t)$	State Variable	Asset Portfolio	[asset]
$u(t)$	Control Variable	Control Variable	$\left[\frac{\text{asset}}{\text{year}}\right]$
$p(t)$	Co-state	Price of Assets	$\left[\frac{\text{utils}}{\text{asset}}\right]$
$\mathcal{V}^*$	Action	Total Utility	[utils]
$\mathcal{X}$	Allowable state set	Allowable Asset Positions	-
$\mathcal{U}$	Allowable control set	Allowable Control Decisions	-
$v\big(x(t), \dot{x}(t), t\big)$	Running cost	Instantaneous Utility	$\left[\frac{\text{utils}}{\text{year}}\right]$
$\Phi\left(x(T), T\right)$	Final cost	Bequest Utility	[utils]
$H\big(x(t), u(t), p(t), t\big)$	Control Hamiltonian	Economic Surplus	[utils]

Table 5.1: The analogy between the maximum principle and the utility maximization problem

5.1 The Maximum Condition as the Myopic Maximization of Economic Surplus

The maximum condition states that the Hamiltonian is maximized along the optimal asset trajectory, while taking the allowable control set $\mathcal{U}$ into account. Mathematically, that is

$$H\Big(x^*(t), u^*(t), p(t), t\Big) = \max_{u(t)\in\mathcal{U}} H\Big(x^*(t), u(t), p(t), t\Big) \quad \text{at each } t \in [0, T] \tag{5.3}$$

where the Hamiltonian is defined as in (5.2). Here, we immediately recognize the similarity with the maximum condition for "simple consumption goods" as in (4.16). With the general asset dynamics, the agent still behaves myopically, maximization economic surplus at every time instant while taking the limitations on the control actions $u(t)$ into account. The co-state $p(t)$ can still be interpreted as the shadow price. The same argument can be applied as was used in Section 4.6. The shadow price is now however defined implicitly by (5.3) and not by equation (4.5).

If all control actions are allowed such that $u(t) \in \mathbb{R}^m$, or the optimal control signal is not located on the boundary of the control set, then the maximum condition also implies that the Hamiltonian is still stationary with respect to the control $u(t)$

$$\frac{\partial H}{\partial u}\Big(x^*(t), u(t), p(t), p_0, t\Big) = p_0 \frac{\partial v}{\partial u}\Big(x^*(t), u(t), t\Big) + p(t)\frac{\partial f}{\partial u}\Big(x^*(t), u(t), t\Big) = 0 \tag{5.4}$$

And for the economic agent this is equivalent to a marginal cost-benefit analysis. To show this, let us fix ideas and interpret the control signal $u(t)$ as consumption. The first part of the equation is than interpreted as the marginal benefit of consumption. The marginal increase would be equal to

$$p_0 \frac{\partial v}{\partial u}\Big(x^*(t), u(t), t\Big)\delta u$$

The second part is the marginal cost of consumption. It is equal to the price $p(t)$ times the change in the asset position due to the additional consumption.

$$p(t)\frac{\partial f}{\partial u}\Big(x^*(t), u(t), t\Big)\delta u$$

Equation (5.3) than states that the marginal benefit of consumption must be equal to the marginal cost of consumption. The maximum condition hence describes an agent that maximizes economic surplus, while taking the allowable control actions into account. When the control set is not restricted, than the maximizing behavior of the economic agent implies that he performs an marginal cost-marginal benefit analysis instantaneously.

5.2 Hamiltonian System for the Asset Positions and Asset Prices

The asset positions and the shadow prices evolve over the planning horizon according to a Hamiltonian system. The dynamics of both are expressed in terms of the surplus as

$$\dot{x}(t) = \frac{\partial H}{\partial p}\Big(x(t), u(t), p(t), t\Big) \quad \text{at each } t \in [0, T] \tag{5.5}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\Big(x(t), u(t), p(t), t\Big) \quad \text{at each } t \in [0, T] \tag{5.6}$$

These equations look familiar, and are equal to Hamilton's canonical equations (4.17) and (4.18) of the previous chapter. The first equation (5.5) simply returns the uncontrolled asset dynamics $\dot{x}(t) = f(x(t), u(t), t)$. The second equation (4.18) is interpreted as the no-arbitrage condition for the assets, and we can use an similar argument as in Section 3.1. To develop the no-arbitrage argument, notice that the canonical equation for the price, together with the definition of the Hamiltonian can be written as

$$\dot{p}(t) = -\frac{\partial v}{\partial x}\Big(x(t), u(t), t\Big) - p(t)\frac{\partial f}{\partial x}\Big(x(t), u(t), t\Big) \tag{5.7}$$

where for the moment the parameter p_0 is set to $p_0 = 1$. Lets assume that (5.7) does not hold such that

$$\dot{p}(t) + p(t)\frac{\partial f}{\partial x}\Big(x(t), u(t), t\Big) < -\frac{\partial v}{\partial x}\Big(x(t), u(t), t\Big) \tag{5.8}$$

Then the agent could take advantage of this situation by buying an additional amount of assets, and selling these assets a moment later. To get this insight, consider an agent that buys an incremental amount of assets $\delta x(t)$ at time t and sells these assets a moment later at time $t + \Delta t$. The assets dynamics imply that the additional assets $\delta x(t)$ increase over this short time interval approximately as

$$\delta x(t + \Delta t) \approx \delta x(t) + \frac{\partial f}{\partial x}\Big(x(t), u(t), t\Big)\delta x(t)\Delta t \tag{5.9}$$

The total costs from buying and selling these assets is equal to

$$p(t)\delta x(t) - p(t + \Delta t)\delta x(t + \Delta t) \approx p(t)\delta x(t) - p(t + \Delta t)\Big(\delta x(t) + \frac{\partial f}{\partial x}\Big(x(t), u(t), t\Big)\delta x(t)\Delta t\Big) \tag{5.10}$$

$$\approx -\dot{p}(t)\Delta t \delta x(t) - p(t + \Delta t)\frac{\partial f}{\partial x}\Big(x(t), u(t), t\Big)\delta x(t)\Delta t \tag{5.11}$$

The additionally utility gained from holding the additional utility is equal to

$$\frac{\partial v}{\partial x}\delta x \Delta t \tag{5.12}$$

And it follows that if the inequality in (5.8) holds then the incremental utility from the additional assets exceeds the cost of buying and selling them. The agent should take advantage of this situation and buy the assets until the equality (5.6) is obtained.

5.3 Non-Triviality Condition

Pontryagin's Hamiltonian uses two multipliers, the constant multiplier p_0 and the co-state $p(t)$. The non-triviality condition excludes the trivial solution for which both multipliers are equal to zero. That is

$$(p_0, p(t)) \neq (0, 0) \tag{5.13}$$

Solutions that satisfy the 4 conditions of the maximum principle are in control theory called extremals. It is possible that there exist extremals for the case where $p_0 = 0$ and for the case that $p_0 > 0$. The latter are called normal extremals, and then it is always possible to set the multiplier $p_0 = 1$ (see Schättler and Ledzewicz (2012)). However, it may occur that a solution with $p_0 = 0$ also exists and these extremals are called abnormal.

In Section 4.4 we found that the Hamiltonian for simple consumption goods only incorporates the shadow price $p(t)$, and effectively set the multiplier $p_0 = 1$. For the simple consumption goods this is always possible. An abnormal solution $p_0 = 0$ would directly imply that also the shadow price is equal to $p(t) = 0$. This violates the non-triviality condition and therefore the abnormal solution does not exist for simple consumption goods. In general, we can however not exclude the possibility of an abnormal solution. The economic interpretation of the abnormal solution is however rather odd, because such an extremal does not depend on the running utility. For normal extremals, it is always possible to set $p_0 = 1$, because the Hamiltonian is linear in its multipliers. A different choice would simply scale the shadow price $p(t)$.

5.4 Transversality Condition for the Asset Price

The boundary condition for the asset price is for the Maximum Principle very similar to the transversalility condition of Section 4.4. When $x(\cdot)$ is not restricted and the terminal time T is fixed, then the transversality condition is equal to

$$p(T) = p_0 \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \tag{5.14}$$

The shadow price at the end of the planning horizon is again determined by the bequest utility.

6 Summary and Conclusion

Economic agents face optimal control problems in their pursuit for financial gains and happiness. Their objective is to maximize profits, markets shares or utility by controlling their possessions optimally over time. The economic agents therefore behave as optimal controllers while planning their assets and control trajectories. This enables the use of control theory in economics. However, recognizing that each agent's decision-making process is an optimal control problem and solving it with optimal control theory, is not enough. Only applying optimal control theory leads to a black-box procedure that does not provide much insight into the behavior of economic agents. In this thesis I therefore contribute to the current literature by developing a thorough economic interpretation of Pontryagin's maximum principle. To construct this economic interpretation, I start with the origins of optimal control and gradually increase the complexity. To fix ideas, I focus in particular on households that maximize utility over a fixed planning horizon, and on assets that have deterministic dynamics.

The complexity of the utility maximization problem reduces when only "simple consumption goods" are considered. The economic agent is then equivalent to a Lagrangian system. An important difference is however that the action of a mechanical system is minimized, while agents maximize their utility. The difference does not discard the possibility to describe the economic agents in terms of Lagrangian mechanics, but it leads to some differences in the sign convention. In particular I find that the agent's utility over the planning horizon is similar to the negative of the action in mechanics. Also, the asset positions and shadow prices are respectively the generalized coordinates and generalized momenta of the economic agent. All obtained analogous variables are summarized in Table 3.1, and these results are in accordance with the current research of my supervisor Dr. Ir. Max Mendel. The resemblance with Lagrangian mechanics permits the application of the Euler-Lagrange equation to obtain the controlled asset dynamics. For the economic agent I interpret the Euler-Lagrange equation as a "no-arbitrage condition", and it requires that the asset prices decrease with the utility rental. The Euler-Lagrange equation does not completely specify the agent's optimal asset trajectory, because the asset positions at the horizon of the control interval are part of the utility maximization problem. The transversality condition generates the additional boundary condition, and relates the terminal prices to the bequest utility.

The transformation from Lagrangian towards Hamiltonian mechanics gives an entirely new, myopic perspective to the economic agent. The key insight is generated when the control Hamiltonian is used. By definition, it is equal to the difference in the direct utility minus the cost of consumption. The Hamiltonian is therefore a measures for the agent's instantaneous net benefit, also called economic surplus. The formulation of the Hamiltonian ensures that the no-arbitrage condition is compactly written into two first-order differential equations. From the Euler-Lagrange equation, it entangles the evolution of the asset positions and the price dynamics. In addition, the Legendre and Weierstrass necessary condition imply that the control Hamiltonian is maximized along the optimal asset trajectory with respect to the control input. I find that this gives an novel, myopic characterization of the economic agent. Instead of maximization utility over the planning horizon, the economic agent maximizes economic surplus myopically, given the current asset positions and shadow prices. The assets and shadow prices evolve over time according to a Hamiltonian system that excludes arbitrage possibilities.

The myopic interpretation of the economic agent translated to the general utility maximization problem. The control Hamiltonian incorporates the asset dynamics, and preserves its interpretation of the economic surplus. The control Hamiltonian is equal to the summation of the direct utility plus the value change of the asset portfolio (when the agent assumes that the prices are constant). The maximum principle generates four necessary conditions and I interpret each condition in terms of the agent's behavior. I find that the maximum condition dictates that the economic agent maximizes economic surplus with respect to his control actions and given the current asset positions and the shadow prices. This ensures that the myopic interpretation of the agent also holds for the maximum principle. In addition, the first-order condition with respect to the control input, implies that the agent performs a marginal cost-benefit analysis instantaneously. The Hamiltonian system condition requires that the asset and shadow price evolve as a Hamiltonian system over time. The state equation simply returns the original asset dynamics, and the co-state equation describe the evolution of the asset prices over time. The latter price equation prevents arbitrage possibilities and is therefore equivalent to a no-arbitrage condition. The non-triviality condition requires that the optimal control and asset trajectory at least depend on either the asset dynamics or the utility functional, and the transversality condition ensure that the bequest utility is correctly priced into the assets by generating a boundary condition for the shadow prices.

7 Discussion and Further Research

In this thesis I model the decision-making process of economic agents in a neoclassical manner, assuming that agents act rational, have access to assets with deterministic dynamics, and maximize utility over a fixed planning horizon. Recognizing that this utility maximization problem is an optimal control problem, enables the use of control theory in economics. The major contribution of my work is to construct a thorough economic interpretation of the maximum principle. I find that the maximum principle dictates an alternative, myopic perspective where the economic agent maximizes his economic surplus instantaneously, instead of utility over a planning horizon. Further research needs to be done to complete the interpretation and analogy, but the work in this thesis already leads to ample of opportunities for the application of control theory in economics. In the following sections I address opportunities for the improvement of the current interpretation and suggest possibilities for further research and applications.

The dynamics of an entire economy Macroeconomic theory often models the dynamics of an entire economy as a single rational economic agent with an infinite planning horizon. Examples of such models are: The Ramsey model, Solow-Swam model and the Ramsey-Koopmans-Cass model. The analogy development in this thesis also applies for these models.

However, it would be interesting to model an entire economy as an collection of different agents and not as an single representative agent. This requires the modeling of different agents (household, firms, governments) and their interactions. The recent developments in distributed control theory may prove a sensible way to do this and diverge from the idea of a single representative agent. It is especially interesting to start with the modeling of a market that consist of interacting households and firms each with their own goals.

Assets The assets in this thesis are modeled as deterministic with known dynamics. The economic agent has perfect foresight and does not encounter any unexpected changes in the assets. This simplification may on average be a good representation, but it would be better to acknowledge that assets change randomly over time. Physical assets may break down earlier then expected, the value of stocks and bonds fluctuate randomly over time, and sudden technological improvements can render human capital superfluous. This means that the uncertainty should be incorporated in the asset dynamics. One of the difficulties will be that the underlying assets (remember it is a quantity) do not necessarily change, but that its price may fluctuate over time.

From a control perspective, uncertainty in the equations of motion is not uncommon. Controlled systems are often affected by disturbances or higher order dynamics that are not modeled. By introducing a stochastic component in the assets dynamics, it will be necessary to look at the expected total utility functional and interpret the risk aversion of the economic agent in control theory. In that situation the co-state corresponds to the "velocity" of the asset.

The Shadow Price Throughout this thesis, I assume that each component in the state $x(t)$ corresponds to an asset of the household. This ensures that the corresponding co-state $p(t)$ can be interpreted as a shadow price. However, it is unclear how the co-state is interpreted when the uncontrolled dynamics of one asset is of second-order.

Controlling Government The decision-making process of economic agents depends both on the asset dynamics and the utility functional. Parameters in especially the assets dynamics can be controlled by the government and the central bank. Interest rates are affected by monetary policies, and income generated by both assets and labor are taxed by governments. Changes in the tax legislation and monetary policy effects the decisions-making of economic agents, resulting in a behavioral change. It would be interesting to model the government as a controller with specific goals on the wealth accumulation, employment rates, and government revenue. The interaction with households and firms will especially be challenging.

References

V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 1978.

G. Becker. *Economic Theory.* Economics Classics. Transaction Publishers, 2011.

D. Bernoulli. Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36, 1954.

D. P. Bertsekas, D. P. Bertsekas, D. P. Bertsekas, and D. P. Bertsekas. *Dynamic programming and optimal control*, volume 1. Athena scientific Belmont, MA, 1995.

T. L. Friesz. *Dynamic optimization and differential games*, volume 135. Springer Science & Business Media, 2010.

I. Gelfand and S. Fomin. Calculus of variations. 1963. *Revised English edition translated and edited by Richard A. Silverman*, 1963.

W. R. Hamilton. Xv. on a general method in dynamics; by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central relation, or characteristic function. *Philosophical transactions of the Royal Society of London*, 124:247–308, 1834.

W. R. Hamilton. Second essay on a general method in dynamics. *Philosophical Transactions of the Royal Society of London*, 125:95–144, 1835.

M. I. Kamien and N. L. Schwartz. *Dynamic optimization: the calculus of variations and optimal control in economics and management.* Elsevier North-Holland, 1981.

L. D. Landau and E. M. Lifshitz. *Mechanics, Vol. I of Course on Theoretical Physics.* Pergamon Press, 1972.

G. Leitmann. *The calculus of variations and optimal control: an introduction*, volume 24. Springer Science & Business Media, 1980.

K. T. McDonald. A damped oscillator as a hamiltonian system. *Joseph Henry Laboratories, Princeton University*, 2015.

R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic Optimization Models in Finance*, pages 621–661. Elsevier, 1975.

J. Persky. Retrospectives: The ethology of homo economicus. *The Journal of Economic Perspectives*, 9(2): 221–231, 1995.

L. S. Pontryagin. *Mathematical theory of optimal processes.* CRC Press, 1962.

F. P. Ramsey. A mathematical theory of saving. *The economic journal*, 38(152):543–559, 1928.

P. A. Samuelson. Foundations of economic analysis. 1948.

H. Schättler and U. Ledzewicz. *Geometric optimal control: theory, methods and examples*, volume 38. Springer Science & Business Media, 2012.

A. Seierstad and K. Sydsaeter. *Optimal control theory with economic applications.* Elsevier North-Holland, Inc., 1986.

H. J. Sussmann and J. C. Willems. 300 years of optimal control: from the brachystochrone to the maximum principle. *IEEE Control Systems Magazine*, 17(3):32–44, 1997.

M. L. Weitzman. *Income, wealth, and the maximum principle.* Harvard university press, 2009.

A Friction forces in the Euler-Lagrange Equation.

The Euler-Lagrange equation for a mechanical system in a force field is equal to

$$\frac{\partial v}{\partial x}\Big(x(t), \dot{x}(t), t\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v}{\partial \dot{x}}\Big(x(t), \dot{x}(t), t\Big)\right) = 0 \tag{A.1}$$

Handling frictions forces is however hard. These forces are velocity dependent and therefore cannot be derived from a potential function. The same problems will arise when a Lagrangian economic agent is being investigated in the presence of transactions costs. Friction forces can be handled by the introduction of a Rayleigh dissipation function $R(x(t), \dot{x}(t), t)$

$$\frac{\partial \mathcal{L}\left(x, \dot{x}, t\right)}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}\left(x, \dot{x}, t\right)}{\partial \dot{x}}\right) + \frac{\partial R}{\partial \dot{x}} = 0$$

By an additional term $\tau(x, \dot{x}, t)$ called the generalized forces with

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{\partial \mathcal{L}}{\partial x} = \tau(x, \dot{x}, t)$$

These methods result for mechanical systems in the correct equations of motion, but the relation with the action principle is lost. A novel approach is to evaluate complex Lagrangians. The Euler-Lagrange equations than incorporates a half derivative of the position $x(t)$, denoted as $\hat{x}$.

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + i\frac{\partial \mathcal{L}}{\partial \hat{x}} + \frac{\partial \mathcal{L}}{\partial x} = 0$$

This is a novel approach that is currently being researched by my supervisor dr. ir. Max Mendel to model dissipative transactions costs in economics. This seems to be the most promising method to incorporate the transaction cost and retain the principle of utility maximization.

B Convert the Bolza Problem into the Lagrange Problem and Mayer Problem

B.0.1 Conversion of the Bolza formulation into the Lagrange problem

The difference between the Bolza and Lagrange objective functional in (2.7) and (2.8) is the bequest utility $\Phi(x(T), T)$ at the end of the planning horizon. In order to convert the Bolza problem in terms of the Lagrange problem, it is necessary to incorporate the bequest utility into the running utility. To do this, let us assume that the bequest utility $\Phi(x(t), t)$ is defined over the full planning horizon $t \in [0, T]$, and assume that its time derivative exist. Then we can rewrite the bequest utility as

$$\begin{aligned}\Phi\left(x(T), T\right) &= \Phi(x_0, 0) + \int_0^T \frac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t} dt \\ &= \Phi(x_0, 0) + \int_0^T \left(\frac{\partial\Phi(x,t)}{\partial t} + \frac{\partial\Phi(x,t)}{\partial x} f(x(t), u(t), t)\right) dt\end{aligned}$$

This means that Bolza functional is equivalent to

$$\begin{aligned}\mathcal{V}^* = \max_{u(\cdot)} \mathcal{V}\left[u(\cdot)\right] &= \max_{u(\cdot)} \left[\Phi\left(x(T), T\right) + \int_0^T v(x(t), u(t), t) dt\right] \\ &= \Phi(x_0, 0) + \max_{u(\cdot)} \left[\int_0^T \left(v(x(t), u(t), t) + \frac{\partial\Phi(x,t)}{\partial t} + \frac{\partial\Phi(x,t)}{\partial x} f(x(t), u(t), t)\right) dt\right]\end{aligned}$$

The first term on the right hand side, $\Phi(x_0, 0)$, only depends on the initial time $t_0 = 0$ and the initial asset allocation x_0. This terms is not affected by the control decision $u(\cdot)$, meaning that the maximization is equivalent to

$$\max_{u(\cdot)\in\mathcal{U}} \left[\int_0^T \left(v(x(t), u(t), t) + \frac{\partial\Phi(x,t)}{\partial t} + \frac{\partial\Phi(x,t)}{\partial x} f(x(t), u(t), t)\right) dt\right]$$

which is of course a Lagrange type of problem.

B.0.2 Conversion of the Bolza formulation into the Mayer problem

The difference between the Bolza and Mayer objective functional in (2.7) and (2.9) is the running utility $v(x(t), u(t), t)$ over the planning horizon. In order to convert the Bolza problem into a Mayer problem, it is necessary to incorporate the running utility in the bequest utility. To do this, let us define an auxiliary state $y(t)$ with the running utility as it equations of motion, and with the initial condition

$$\begin{aligned}\dot{y}(t) &= v(x(t), u(t), t) \\ y(0) &= 0\end{aligned}$$

The value of this auxiliary state $y(t)$ at the end of the planning horizon is equal to the accrued instantaneous utility over the planning horizon.

$$y(T) = \int_0^T v(x(t), u(t), t) dt$$

Extending the state-space $x(t)$ with this auxiliary state to $\bar{x} = \begin{bmatrix} x, y \end{bmatrix}$, the Bolza functional is equivalently written as

$$\max_{u(\cdot)\in\mathcal{U}} \left[\Phi\left(x(T), T\right) + y(T)\right] \quad \text{s.t.} \quad \begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = \begin{bmatrix} f(x,u,t) \\ v(x,u,t)\end{bmatrix}$$

Here we recognize the Mayer formulation of optimal control.

C Proof of the EL-equations by converting the Bolza Problem into Langrange's Formulation

Now, let us again consider the utility functional with a bequest utility and recall from Section 2.3 that we can rewrite the Bolza utility functional in the Lagrange formulation, yielding

$$\begin{aligned}\mathcal{V}^* = \max \mathcal{V}\left[x(\cdot), \dot{x}(\cdot)\right] = \max &\left[\int_0^T \left(v(x(t), \dot{x}(t), t) + \frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial \Phi(x,t)}{\partial x}\dot{x}\right) dt\right] \\ \text{s.t.} \quad x(0) = x_0, &\quad x(\cdot) \in \mathbb{R}^n\end{aligned} \tag{C.1}$$

This is equal to the set-up in the previous proof, but now with the bequest utility incorporates into the running utility. The Euler-Lagrange equations (3.4) derived in the previous section must also hold for this utility functional, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v(x(t), \dot{x}(t), t)}{\partial \dot{x}} + \frac{\partial \Phi(x,t)}{\partial x}\right) - \frac{\partial v(x(t), \dot{x}(t), t)}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial \Phi(x,t)}{\partial x}\dot{x}\right) = 0$$

Now, we can simply this equations by recognizing that the last expressions on the left side is equal to the total time-derivative of the bequest function.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v(x(t), \dot{x}(t), t)}{\partial \dot{x}} + \frac{\partial \Phi(x,t)}{\partial x}\right) - \frac{\partial v(x(t), \dot{x}(t), t)}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t}\right) = 0 \tag{C.2}$$

The partial derivative of x and total time derivative can be interchanged for the bequest bequest, meaning that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi(x,t)}{\partial x}\right) = \frac{\partial^2 \Phi(x,t)}{\partial t \partial x} + \frac{\partial^2 \Phi(x,t)}{\partial x^2}\dot{x}(t) = \frac{\partial}{\partial x}\left(\frac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t}\right)$$

Cancellation of these partial derivatives in (C.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial v\left(x, \dot{x}, t\right)}{\partial \dot{x}}\right) - \frac{\partial v\left(x, \dot{x}, t\right)}{\partial x} = 0 \tag{C.3}$$

Remarkably, this is equal to the previously obtained Euler-Lagrangian equations that do not depend on the bequest utility $\Phi(x(T), T)$. However, the bequest utility does influences the behavior of the economic agent through the transversality conditions. Using (3.5), the boundary condition is equal to

$$\frac{\partial}{\partial \dot{x}}\left(v(x(t), \dot{x}(t), t) + \frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial \Phi(x,t)}{\partial x}\dot{x}\right)\bigg|_{t=T} = 0$$

And recognizing the definition of the shadow price, we find that the shadow price at the end of the planning horizon is determined by the bequest utility

$$p(T) = -\frac{\partial v}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) = \frac{\partial \Phi}{\partial x}\Big(x(T), T\Big) \tag{C.4}$$