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RESEARCH ARTICLE

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Ratio product model: A rank-preserving normalization-agnostic multi-criteria decision-making method

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Abstract

This paper presents a new multi-criteria decision-making (MCDM) method, namely the ratio product model (RPM). We first overview two popular aggregating models: the weighted sum model (WSM) and the weighted product model (WPM). Then, we argue that the two models suffer from some fundamental issues mainly due to ignoring the ratio nature of the alternatives' scores with respect to the criteria and the importance weights of the criteria. Building on the notion of compositional data analysis, the developed RPM regards performance scores and criteria weights as compositions, which solves the issues around the WSM and WPM. Using several examples, we show that the WSM and WPM could lead to erroneous conclusions, whereas the RPM could lead to fully reliable conclusions. Since many MCDM methods rely on some aggregation approaches, the proposed method is a significant contribution to the field and puts forward the correct way to analyze decision problems while respecting the nature and constraints of the input data.

KEYWORDS

compositional data, rank reversal, weighted product model, weighted sum model

1 | INTRODUCTION

Multi-criteria decision-making (MCDM) is a scientific field that studies the decision-making problems entailing a set of alternatives and multiple, typically conflicting, criteria. The goal is to rank, sort or select the best alternatives based on a chosen set of criteria. For doing so, the relative importance of criteria needs to be quantified by eliciting the preferences of single or multiple decision-makers (DMs). There are several methods for the preference elicitation and computing the weights of criteria, among which are analytic hierarchy process (AHP) (Saaty, 1977), analytic network process (ANP) (Saaty, 2001), best–worst method (BWM) (Rezaei, 2015), Tradeoff (Keeney et al., 1976), simple multi-attribute rating technique (SMART) (Edwards, 1977) and Swing (Mustajoki et al., 2005).

For processing the alternatives, the performance of alternatives for each criterion needs to be identified, resulting in the so-called *performance matrix* that includes the performance of all the alternatives over all the criteria. There are two ways to build the performance matrix. The first way consists of a data collection step, gathering information regarding the alternatives. When such a data collection is infeasible, the performance matrix can be created by using an MCDM method, such as AHP or BWM, and conducting the pairwise comparison between the alternatives for each criterion. The performance matrix is then created by solving the MCDM problem for each criterion. In this case, each alternative is assigned a weight concerning a criterion, also referred to as local priority.

Given the weights of the criteria and the performance matrix, there are multiple MCDM methods that process the alternatives. A class of such methods is outranking methods, where the goal is to rank the alternatives given a set of criteria and their relative importance. The well-known outranking methods are TOPSIS (Technique for order preference by similarity to ideal solution) (Tzeng & Huang, 2011), VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR; Opricovic, 1998), ELimination and Choice Expressing REality (ELECTRE; Figueira et al., 2005), and Preference Ranking Organization METHod for Enrichment of Evaluations (PROMETHEE; Brans & Mareschal, 2005). Another class evaluates the alternatives based on the given criteria and their weights. The two widely used methods are the weighted sum model (WSM) and the weighted product model (WPM). The WSM and WPM aggregate the performance of criteria and their weights and summarize the alternatives' overall performance in a score, also referred to as global priority. Perhaps needless to say that the alternatives can be ranked according to global priorities of the alternatives. The main focus of this study is on the latter class, where the alternatives' values, considering the weights of the criteria, are aggregated into an overall score.

Many studies in the literature of MCDM consider the pros and cons of the WSM and WPM. Specifically, the WSM is criticized because the ranking obtained based on the aggregated score can be reversed if, for example, an alternative or an indiscriminating criterion is added or deleted (Barzilai & Golany, 1994; Leskinen & Kangas, 2005; Triantaphyllou, 2001). This phenomenon is called rank reversal. The rank reversal has been long studied in MCDM, arguably started by Belton and Gear (1983) (Belton et al., 1985), in which the rank reversal of the WSM in the standard AHP was studied. Ever since many studies have scrutinized this phenomenon in more detail (see the study by Aires & Ferreira, 2018) for a review. Along the same line, the WSM needs to normalize the columns of the performance matrix, that is the scores of different alternatives for a criterion, and the way to make the normalization can also affect the ranking. Thus, the selection of the normalization techniques can potentially change the ranking of the alternatives even when the weights of the criteria and the values in the performance matrix remain the same. It is proved that there exists at least an MCDM problem for which the rank reversal indeed happens if a particular normalization technique is used, where the normalization is assumed to be in the form of dividing the values of a performance matrix by a number (Barzilai & Golany, 1994).

As a remedy to the rank reversal, the WPM is typically put forward as it has been shown to preserve the alternatives ordering under different circumstances (Triantaphyllou, 2001). A recent study indicates that the WPM is particularly preferred to the WSM in AHP for evaluating the alternatives, primarily due to its robustness against normalization (Krejčí & Stoklasa, 2018).

Despite their differences, what is common between WSM and WPM is that they ignore the nature of the weights and the performance matrix. In particular, the outcome of the MCDM methods like AHP is a weight vector whose sum is one. For such vectors, the magnitude of each part is of no importance but rather the ratio between its different parts. These sort of variables with the constant-sum constraints is called compositional data in statistics (Aitchison, 1982; Pawlowsky-Glahn & Buccianti, 2011), whose analyses should be done differently in comparison to other multivariate variables with no constraints. In particular, it has been well-studied that the proper methods for analyzing compositional data must be based on the ratios between different parts rather than working with the parts themselves (Aitchison, 1981; Aitchison, 1982). This observation is in line with criteria weights and performance matrices computed by MCDM methods since what is gauged and conveyed by the outcome of many MCDM methods (i.e., a non-negative vector with unit-sum constraint) is the ratios highlighting the relative importance between different criteria or alternatives. That being said, the WSM seems to improperly use the weights and the performance matrix since it takes the values in a weight vector and applies standard arithmetic operations (i.e., addition and multiplication) as if the weights lie in the real space, while they lie indeed on a simplex. This improper use of operations can lead to incorrect analyses with disastrous side effects, including rank reversal.

The WPM is a step in the right direction since it considers the ratios in the performance matrix. This is why the WPM prevents rank reversal and is agnostic to normalization. However, a significant problem of the WPM is that, while the ratios between the performance of alternatives are considered, it does not do the same for the criteria weights. This is because the criteria weights are treated as real values and used as the power of ratios computed from the performance matrices. Since the ratios between the criteria weights are not considered, the ranking of alternatives accordingly could be erroneous. In addition, if the scale of criteria weights differs, while the ratios among them remain unchanged, the outcome of the WPM alters, even though the rankings of alternatives remain the same. Besides, if two alternatives have the same value for multiple criteria, the WPM ignores those criteria, and they do not influence the difference between the final aggregated scores. However, it is expected that the alternatives with similar values for criteria have a closer aggregated score and thus lower differences.

In this paper, we first discuss the WSM's and WPM's drawbacks and issues. To the best of our knowledge, this is the first article showing the problems in the WPM, while the issues of the WSM have been discussed in the literature. Then, we propose the ratio product model (RPM), a new method for analyzing alternatives, given a set of criteria, their weights, and a performance matrix. This method respects the compositional nature of both the weights and the performance matrix and is agnostic to the normalization of the performance matrix and the scale of the criteria weights. It is also proved that the RPM prevents the rank reversal phenomenon when a new alternative or criterion is added or removed.

The rest of the paper is structured as follows. Section 2 is dedicated to the prerequisite of the article; we first review the WSM and WPM models and the normalization techniques being used and then provide a rudimentary introduction to compositional data. Next, Section 3 is dedicated to the pitfalls and issues of the WSM and WPM, primarily because of the ignorance of the compositional nature of the data. Next, Section 4 presents the RPM and several examples, and the theoretical insights of the proposed method is presented in Section 5. Finally, Section 6 concludes the article and puts forward lines for future research.

2 | PRELIMINARIES

In this section, we first study two ways for computing global priorities of alternatives, that is the WSM and WPM, based on multiple criteria and their importance, and their local priorities or a given performance matrix. We then review the basic notions and concepts of compositional data.

2.1 | Weighted sum and weighted product models

There are several normalization techniques for computing the local priorities in the literature, two of which are typically used, especially for the AHP method. The first one is the distributive normalization, where the performance of alternatives for a criterion is divided by the sum of all the values for the corresponding criterion so that the unit-sum constraint is assumed. The second technique, on the other hand, is called ideal normalization, in which the performance of alternatives for a criterion is divided by the maximum performance values of alternatives for that criterion. Thus, for a criterion c_j, it means that:

$$\begin{aligned} &\sum_{i=1}^{m} P_{ij} = 1 \quad \text{(distributive)}, \\ &\max_i P_{ij} = 1 \quad \text{(ideal)}, \end{aligned} \tag{1}$$

where $P \in R^{m \times n}$ is the normalized performance matrix, containing the performance of m alternatives for n criteria, and P_{ij} is the normalized performance of alternative i for criterion j.

In the literature, there are two widely-used aggregation methodologies for computing the global priorities, shown by for alternative i. The first one is the WSM, which is defined as:

$$g_i^{WSM} = \sum_{j=1}^{n} w_j P_{ij}, \quad \forall i = 1,\ldots,m, \tag{2}$$

where g_i^{WSM} is the global priority of alternative i by the WSM, w_j is the weight of criterion j, and P_{ij} is the local priority of alternative i for criterion j.

Since the WSM method does not take into account the ratios in form of $P_{ij}/P_{i'j}$ for the local priorities of alternatives, Lootsma (1993) proposed the WPM as follows:

$$g_i^{WPM} = \prod_{j=1}^{n} P_{ij}^{w_j}, \quad \forall i = 1,\ldots,m, \tag{3}$$

where g_j^{WPM} is the global priority computed by the WPM method. Using Equation (3), comparing the value of two alternatives i and i' can be done by dividing their global priorities:

$$\frac{g_i^{WPM}}{g_{i'}^{WPM}} = \prod_{j=1}^{n} \left(\frac{P_{ij}}{P_{i'j}}\right)^{w_j}, \tag{4}$$

where the ratios of the local priorities are taken into account. We now show that under a specific normalization technique (i.e. normalization by logarithm), the WSM and WPM are equivalent.

Lemma 2.1. *By taking logarithm from the local priorities in matrix P*, the outcome of the WSM becomes equivalent to the WPM.

Proof. By using Equation (2) and taking logarithm of the local priorities, we have:

$$\begin{aligned} g_i^{\text{WSM}} &= \sum_{j=1}^{n} w_j \log(P_{ij}) \\ &= \sum_{j=1}^{n} \log\left(P_{ij}^{w_j}\right) \\ &= \log\left(\prod_{j=1}^{n} P_{ij}^{w_j}\right), \end{aligned} \tag{5}$$

which would result in the same ratio between alternatives since logarithm is a monotone function. In addition, the difference between two alternatives is computed as:

$$\begin{aligned} &g_i^{\text{WSM}} - g_{i'}^{\text{WSM}} = \log\left(\prod_{j=1}^{n} P_{ij}^{w_j}\right) - \log\left(\prod_{j=1}^{n} P_{i'j}^{w_j}\right) \\ &= \log\left(\prod_{j=1}^{n} \frac{P_{ij}^{w_j}}{P_{i'j}^{w_j}}\right), \end{aligned} \tag{6}$$

which again results in the same ratio as the WPM. □

2.2 | Compositional data

Compositional data refer to the multivariate variables whose sum is constrained with a fixed number. The difficulty of analyzing such data was pointed out as early as 1897 by Pearson in his famous article on the spurious correlation (Pearson, 1897), which refers to the correlation of ratios with similar nominators/denominators. However, it was not until the 1970s where proper statistical analyses for such data were developed (Aitchison, 1982). We first provide a formal definition of composition.

Definition 2.2. (Aitchison, 1982) *A composition w of n parts is a vector with positive components $w_1,\ldots,w_n$ whose sum is a fixed number like K.*

The fixed number in a composition can be any positive number, like one or 100. Since the ratios between the parts in a composition are the only relevant information, multiplying a composition to a real positive number would not change it. Thus, if $\alpha > 0$ is a real number, then the compositions $[w_1,\ldots,w_n]$ and $[\alpha w_1,\ldots,\alpha w_n]$ are indiscernible and convey identical information. This means that a composition is a class of equivalent compositional data (Aitchison, 1982).It becomes evident that the weights and the local priorities in MCDM are compositional data, meaning that their analysis should be based on the ratios between the parts. For example, a weight vector w should be represented by all the pairwise ratios between its components. This means that w can be rewritten as a matrix showing the ratios between its components. The following example clarifies this point.

Example 2.3. *Let* $w=[0.4,0.3,0.2,0.1]$ *be the weights of criteria* $C=\{c_1,c_2,c_3,c_4\}$. Then, the weights *represent the following ratios between the criteria*:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{pmatrix} 1 & 4/3 & 2 & 4 \\ 3/4 & 1 & 3/2 & 3 \\ 1/2 & 2/3 & 1 & 2 \\ 1/4 & 1/3 & 1/2 & 1 \end{pmatrix} \end{array}$$

Example 2.3 shows that a matrix can represent all the ratios between the components of a composition. This matrix is similar to the pairwise comparison matrix (PCM), which means that any weight vector can be represented by a PCM. We first investigate an essential property of this matrix.

Definition 2.4. (Saaty, 1977) *Matrix M is a fully consistent PCM if and only if it satisfies the following property*:

$$m_{ij}=m_{ih}\times m_{hj}, \quad i,j,h=1,\ldots,n. \tag{7}$$

According to this definition, it is evident that the resulting PCM of a weight vector is always fully consistent.

3 | CRITIQUES ON THE WSM AND WPM

In this section, we discuss the pitfalls of the WSM and WPM in computing the global priorities. We specifically show that rank reversal in MCDM happens due to neglecting the compositional nature of priorities.

The rank reversal in MCDM is generally defined as a change in the ranking of alternatives under a particular circumstance, for example adding or removing an alternative or indiscriminating criteria. The reason for introducing the ideal normalization is to handle rank reversal in MCDM problems. However, from a compositional perspective, changing the normalization technique should not affect any analysis since the ratio between the priorities of the alternatives for each criterion is the same. In fact, the DM has expressed the preferences in the form of ratios, and these ratios should be taken into account for a meaningful aggregation of local priorities. In addition, it is proved that any normalization in the form of dividing the performance matrix by a number (the same as the distributive and ideal normalization) would lead to rank reversal in one way or another (Barzilai & Golany, 1994). Therefore, instead of looking into the normalization techniques, we need to devise a proper method for aggregation that takes into account the compositional nature of the priorities and the performance matrix. We now discuss the pitfalls of the WSM and WPM using several examples.

Example 3.1. *Assume that we have two criteria* c_1 *and* c_2 *with the same importance, that is* $w_1=w_2=0.5$, *and three alternatives* A_1, A_2 *and* A_3. Table 1 *shows the value of criteria for the alternatives in the first two columns with the distributive normalization. The columns of* g^{WSM} *and* g^{WPM} *are the global priorities computed by the WSM and WPM, respectively, and the remaining two columns are the ratios of global priorities computed by the WSM and WPM. A similar analysis is conducted by using the ideal normalization as shown in Table 2*.

It is readily seen that the ratio of global priorities is not identical for the WSM when different normalization techniques are applied. Simultaneously, the ratio of the global priorities for the WPM is the same for both normalization techniques, simply because the WPM considers the ratios between the local priorities that remain the same under different normalization. Besides, this means that the WSM is sensitive to the normalization being used. In addition to the difference in the ratios of global priorities, the rankings of alternatives change as well: A_1 is preferred over A_2 for the distributive normalization, while A_2 is preferred over A_1 for the ideal normalization.

Example 3.2. *Assume that we have three criteria* (i.e. c_1, c_2 *and* c_3) *with the same importance,* $w_1=w_2=w_3=0.333$, *and three alternatives* A_1, A_2 *and* A_3. *Table 3 tabulates the analysis of this problem by using the WSM and WPM. According to the WSM, the ranking of alternatives is as follows*: $A_2 \succ A_1 \succ A_3$. *The same ranking is obtained based on the WPM. Table 3 also shows the ratio between different alternatives for both the WSM and WPM.*

We now want to investigate how the rankings of alternatives change if we add another alternative. For doing so, we add alternative A_4 with identical performance as A_2. Table 4 shows the evaluation of this problem by using the WSM and WPM. According to the WSM, $A_1 \succ A_2 = A_4 \succ A_3$. Thus, the ranking of A_1 and A_2 is reversed when A_4 is added (i.e. rank reversal happens). However, according to the WPM, the ranking and the ratios between different alternatives remain unchanged.

Similar examples to Examples 3.1 and 3.2 have been the base for favoring the WPM over WSM, the primary reason of which is WPM's agnosticism to normalization and avoiding rank reversal. However, the WPM also has several problems. Before looking into the issues of the WPM, we first provide an example.

Example 3.3. *Consider Example 3.1 with the weights of criteria being* $w_1=0.6$ *and* $w_2=0.4$ *and only two alternatives* A_1 *and* A_2. Table 5 *shows the global priorities with*

TABLE 1 Aggregation of local priorities for Example 3.1 with the distributive normalization.

A	c_1	c_2	g^{WSM}	WSM ranking	Ratio of g^{WSM}	g^{WPM}	WPM ranking	Ratio of g^{WPM}
A_1	8/13	1/7	69/91	1	$g_1^{WSM}/g_2^{WSM}=69/67$	$(8/91)^{0.5}$	2	$g_1^{WPM}/g_2^{WPM}=(2/3)^{0.5}$
A_2	4/13	3/7	67/91	2	$g_1^{WSM}/g_3^{WSM}=69/46$	$(12/91)^{0.5}$	1	$g_1^{WPM}/g_3^{WPM}=(8/3)^{0.5}$
A_3	1/13	3/7	46/91	3	$g_2^{WSM}/g_3^{WSM}=67/46$	$(3/91)^{0.5}$	3	$g_2^{WPM}/g_3^{WPM}=4^{0.5}$

Note: The WSM and WPM are computed with $w_1=w_2=0.5$.

TABLE 2 Aggregation of local priorities for Example 3.1 with the ideal normalization.

A	c_1	c_2	g^{WSM}	WSM ranking	Ratio of g^{WSM}	g^{WPM}	WPM ranking	Ratio of g^{WPM}
A_1	1	1/3	2/3	2	$g_1^{WSM}/g_2^{WSM}=1/2$	$(1/3)^{0.5}$	2	$g_1^{WPM}/g_2^{WPM}=(2/3)^{0.5}$
A_2	1/2	1	3/4	1	$g_1^{WSM}/g_3^{WSM}=32/27$	$(1/2)^{0.5}$	1	$g_1^{WPM}/g_3^{WPM}=(8/3)^{0.5}$
A_3	1/8	1	9/16	3	$g_2^{WSM}/g_3^{WSM}=4/3$	$(1/8)^{0.5}$	3	$g_2^{WPM}/g_3^{WPM}=4^{0.5}$

Note: The WSM and WPM are computed with $w_1=w_2=0.5$.

TABLE 3 Aggregation of local priorities for Example 3.2 with the distributive normalization and three alternatives.

A	c_1	c_2	c_3	g^{WSM}	WSM ranking	Ratio of g^{WSM}	g^{WPM}	WPM ranking	Ratio of g^{WPM}
A_1	1/11	9/11	8/18	0.451	2	$g_1^{WSM}/g_2^{WSM}=0.96$	0.321	2	$g_1^{WPM}/g_2^{WPM}=0.96$
A_2	9/11	1/11	9/18	0.470	1	$g_1^{WSM}/g_3^{WSM}=5.70$	0.333	1	$g_1^{WPM}/g_2^{WPM}=4.16$
A_3	1/11	1/11	1/18	0.080	3	$g_2^{WSM}/g_3^{WSM}=5.93$	0.077	3	$g_1^{WPM}/g_2^{WPM}=4.32$

Note: The criteria weights are $w_1=w_2=w_3=0.333$.

TABLE 4 Aggregation of local priorities for Example 3.2 with the distributive normalization and four alternatives.

A	c_1	c_2	c_3	g^{WSM}	WSM ranking	Ratio of g^{WSM}	g^{WPM}	WPM ranking	Ratio of g^{WPM}
A_1	1/20	9/12	8/27	0.365	1	$g_1^{WSM}/g_2^{WSM}=1.26$	0.223	3	$g_1^{WPM}/g_2^{WPM}=0.95$
A_2	9/20	1/12	9/27	0.289	2	$g_1^{WSM}/g_3^{WSM}=6.43$	0.232	1	$g_1^{WPM}/g_3^{WPM}=4.16$
A_3	1/20	1/12	1/27	0.057	4	$g_2^{WSM}/g_3^{WSM}=5.08$	0.053	4	$g_2^{WPM}/g_3^{WPM}=4.32$
A_4	9/20	1/12	9/27	0.289	2	$g_2^{WSM}/g_4^{WSM}=1$	0.232	1	$g_2^{WPM}/g_4^{WPM}=1$

Note: The first three alternatives are the same as those in Table 3, and the fourth alternative is a repetition of alternative A_2. The criteria weights are $w_1=w_2=w_3=0.333$.

TABLE 5 Aggregation of local priorities for Example 3.3 with the ideal normalization.

A	c_1	c_2	g^{WPM}	Ratio of g^{WPM}
A_1	1	1/3	$(1/3)^{0.4}$	$g_1^{WPM}/g_2^{WPM}=2^{0.6}/3^{0.4}=0.977$
A_2	1/2	1	$(1/2)^{0.6}$	

Note: The WPM are computed with $w_1=0.6$ and $w_2=0.4$.

their corresponding ratios computed by the WPM and the distributive normalization. According to this table, the global priority of the second alternative is higher than that of the first alternative, making it the most preferred one. But, based on the local priorities and the weights of criteria, A_1 is twice more preferable than A_2 with respect to c_1, but A_2 is three times more preferable to A_1 regarding c_2. If the weights of c_1 and c_2 were the same (as it was in Example 3.1), then A_2 should be favored over A_1. But because the weight of c_2 (for which A_2 is three times more preferable than A_1) is two-third of the weight of c_1 (for which A_1 is twice more preferable than A_2), we expect that A_1 and A_2 have identical global priorities. The WPM method arrives at different global priorities as tabulated in Table 5*, basically because it uses the weights of the criteria as the exponent of the local priorities. To compute a ratio, we can write:*

TABLE 6 Aggregation of local priorities for Example 3.4 with the ideal normalization.

A	c_1	c_2	g^{WPM}	Ratio of g^{WPM}
A_1	1	0.3535	$(0.3535)^{0.4}$	$g_1^{WPM}/g_2^{WPM}=2^{0.6}/(2.8282)^{0.4}\approx 1$
A_2	1/2	1	$(1/2)^{0.6}$	

Note: The WPM is computed with $w_1=0.6$ and $w_2=0.4$.

TABLE 7 Aggregation of local priorities for Example 3.4 with the ideal normalization.

A	c_1	c_2	g^{WPM}	Ratio of g^{WPM}
A_1	1	0.34	$(0.34)^{0.4}$	$g_1^{WPM}/g_2^{WPM}=2^{0.6}/2.9412^{0.4}=0.9845$
A_2	1/2	1	$(1/2)^{0.6}$	

Note: The WPM are computed with $w_1=0.6$ and $w_2=0.4$.

$$prf(A_1 \succ A_2)=\sqrt[2]{\frac{2w_1}{3w_2}}=\sqrt[2]{\frac{2\times 1.5\times w_2}{3w_2}}=1, \quad (8)$$

where equation $w_1=1.5w_2$ (i.e., $0.6=1.5\times 0.4$) is replaced in the nominator, the square root is because we have two criteria (similar to the geometric mean), and the final value of one means that A_1 is as preferred as A_2 in this problem.

Example 3.4. *We now look into two similar cases to further highlight the drawbacks of the aggregation by the WPM. Consider two alternatives with two criteria in* Table 6 *with the weights of criteria being* 0.6 *and* 0.4. *Compared to Example 3.3, the performance of A_1 over c_2 is increased. For this example, the WPM gives the same global priorities for both alternatives, thereby making equally desirable. However, the ratio, similar to Equation (8) is computed as:*

$$prf(A_1 > A_2)=\sqrt[2]{\frac{2\times 1.5}{2.8282}}=\sqrt[2]{\frac{3}{2.8282}}=1.03.$$

This equation indicates that A_1 should be preferred over A_2, while the WPM deems them as equivalent.

Consider also the performance matrix in Table 7, where the performance of A_1 over c_2 is a bit decreased with respect to that in Table 6. Computing global priorities for this performance matrix, the WPM still favors A_2 over A_1. Computing the ratios would give:

$$prf(A_1 > A_2)=\sqrt[2]{\frac{2\times 1.5}{2.9412}}=\sqrt[2]{\frac{3}{2.9412}}\approx 1.01,$$

which means that it favors A_1 over A_2. Such a preference is entirely in contrast of that computed by the WPM.

These examples show that the ranking provided by the WPM could also be fallacious: The could compute the global priorities in a way that the ranks of alternatives are entirely in contrast to what is expected. In addition, the magnitude of ratios provided by the WPM between pairs of alternatives has not been regarded. The ratio values could be misleading and does not reflect the true preference of one alternative over another. A major reason for this issue is that the criteria weights are used as the exponent of the ratios of the local priorities, which means that the ratios between the weights (as compositions) are not considered. A by-product of such an interpretation is the sensitivity to the scale of the weights, so if the unit-sum constraint is replaced by, for example, percentage, the ratio outcome of the WPM is different, despite the fact that the ratios between the weights of criteria remain the same. For instance, in Example 3.1, the ratio between A_1 to A_2 would be $2^{60}/3^{40}=0.095$ for the percentage weight, which is significantly different from $2^{0.6}/3^{0.4}=0.9767$ for the unit-sum weights. At the same time, the ratio in Equation (8) remains the same, regardless of the type of constraints on the criteria weights. On top of that, the equal criteria values of two alternatives have zero impact on the difference between the alternatives because the ratio of those criteria for the two alternatives is one, and the WPM treats them as if they do not exist. In reality, on the other hand, one expects that if two alternatives have the same values for one criterion, then it affects the difference of their aggregated score in a way that the two alternatives have closer global priorities.

Overall, both the WSM and WPM have several drawbacks. The WSM is sensitive to normalization techniques being used and cannot prevent the rank reversal phenomenon. The WPM, on the other hand, is sensitive to the type of the sum constraint of the criteria weights and can result in wrong rankings of alternatives, though it can preserve the ranks and is agnostic to the normalization of the local priorities.

4 | RATIO PRODUCT MODEL

This section presents the RPM, which provides meaningfully correct rankings, is agnostic to the normalization of the performance matrix, and preserve the ranks in different situations. For the RPM, we need to translate the criteria weights as well as the performance matrix into ratio forms. Given a performance matrix P of m alternatives and n criteria with weights w, the RPM includes the following steps:

Step 1: Compute the weight ratio vector $\widehat{w}$ by dividing the criteria weights to the minimum weight value. Let $w_{min} = \min_{1 \le i \le n} w_i$, $\widehat{w}$ is defined as:

$$\widehat{w} = \left[\frac{w_1}{w_{min}}, \frac{w_2}{w_{min}}, \ldots, \frac{w_n}{w_{min}}\right]. \tag{9}$$

In such a way, the criterion with the minimum weight is considered as the unit of measure, according to which the importance of other criteria is measured.

Step 2: For each criterion, create a matrix by computing all the possible ratios between the performance values of alternatives for that criterion. The creation of such a matrix for each criterion can be computed identically to Example 2.3. As a result, we have n matrices shown by $M^k, k = 1, \ldots, n$, representing the ratios between the performance of alternatives for all the criteria. Since these matrices show the preferences of alternatives on a criterion and take no account for the importance of the criteria, we refer to them as local pairwise ratio matrix (PRM).

Step 3: Compute scaled pairwise ratio matrices $\widehat{M}$ by multiplying the weight ratios to the corresponding local RPM matrices. The weight ratio is directly multiplied to the values greater than one in the matrix, its inverse weight ratio is multiplied by the values less than one, and the entries with a value of one remain intact. Hence, the elements of matrix $\widehat{M}^k$, shown by M_{ij}^k, are computed as:

$$\widehat{M}_{ij} = \begin{cases} \widehat{w}_k M_{ij}^k & \text{if } M_{ij}^k > 1 \\ M_{ij}^k / \widehat{w}_k & \text{if } M_{ij}^k < 1 \\ M_{ij}^k & \text{if } M_{ij}^k = 1 \end{cases} \tag{10}$$

The matrices $\widehat{M}^k$ take into account the criteria weights and are comparable.

Step 4: The global preferences between alternatives are then computed by aggregating the scaled PRM $\widehat{M}^k$ by using the geometric mean. This matrix is called the global PRM and is shown by M^{agg}.

Remark 4.1. The RPM considers the ratios between the weights of criteria as well as between the performance of alternatives. So, it is agnostic to the normalization technique being used in the performance matrix, as well as the sum constrained imposed on the criteria weights. It also respects the compositional nature of criteria weights and local priorities.

Remark 4.2. If two alternatives have the same values for a criterion, then the corresponding element in the local PRM is one. While a value one does not change the computed aggregated fraction of the two alternatives both in local and scaled PRMs, it affects the ratio of the alternative in the aggregated PRM M^{agg} by increasing the root degree in the geometric mean. As a result, the aggregated ratios become closer for alternatives having criteria with identical values, as opposed to the WPM where it ignores the criteria with the same performance.

Remark 4.3. The local PRMs are fully-consistent, but the global PRMs typically have inconsistencies caused by the multiplication of the weight ratios. In fact, unless the weight ratio is one, the global PRMs have inconsistencies.

Remark 4.4. In regression analysis for the compositional data, where the predictor and predicted variables are compositional, the convex combination of compositions (i.e., *composition of compositions*) *is modeled by the logistic normal distribution* (Aitchison & Bacon-Shone, 1999), *and not a unique composition. This implies that the composition of compositions is not unique. The inconsistency in the global PRM is in line with this fact, as an inconsistent PRM implies a non-unique composition.*

Example 4.5. *In this example, we consider the MCDM problem in Example 3.1, where there are two criteria and three alternatives as tabulated in Table 1. We also assume that the weights of the two criteria are the same, similar to Example 3.1. The following steps are the result of applying the RPM to this problem:*

Step 1: Since the weights of two criteria are the same, then the weight ratio $\widehat{w}$ is computed as:

$$\widehat{w} = [1, 1]. \tag{11}$$

Step 2: We now find matrices $M^k, k = 1, 2$ for the two criteria $C = \{c_1, c_2\}$ as:

$$M^1 = \begin{matrix} & A_1 & A_2 & A_3 \\ A_1 & 1 & 2 & 8 \\ A_2 & 1/2 & 1 & 4 \\ A_3 & 1/8 & 1/4 & 1 \end{matrix}, \tag{12}$$

$$M^2 = \begin{matrix} & A_1 & A_2 & A_3 \\ A_1 & 1 & 1/3 & 1/3 \\ A_2 & 3 & 1 & 1 \\ A_3 & 3 & 1 & 1 \end{matrix} \tag{13}$$

Step 3: Since the weights of the criteria are the same and the weight ratios are one, then $\widehat{M}^1 = M^1$ and $\widehat{M}^2 = M^2$.

Step 4: The global preferences of alternatives are computed by aggregating $\widehat{M}^1$ and $\widehat{M}^2$ using the geometric mean:

$$M^{agg} = \begin{pmatrix} 1 & 2/3 & 8/3 \\ 3/2 & 1 & 4 \\ 3/8 & 1/4 & 1 \end{pmatrix}^{1/2}, \tag{14}$$

where the power of the matrix is element-wise.

Step 5: The alternatives can be ranked based on M^{agg}, which is $A_2 \succ A_1 \succ A_3$.

Example 4.6. *Now consider Example 3.3, which is the same as the previous example but the criteria weights are different, that is* $w=[0.6,0.4]$*. Then, the steps of the RPM would be as follows.*

Step 1: We first compute the weight ratios as:

$$\widehat{w}=[1.5,1]. \tag{15}$$

Step 2: We now find matrices $M^k, k=1,2$, for the two criteria $C=\{c_1,c_2\}$ as:

$$M^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 4 \\ 1/8 & 1/4 & 1 \end{pmatrix} \end{array}, \tag{16}$$

$$M^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 1/3 & 1/3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \end{array} \tag{17}$$

Step 3: In contrast to the previous example, the scaled PRMs are different and computed as:

$$\hat{M}^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} & 3 & 12 \\ 1/3 & & 6 \\ 1/12 & 1/6 & \end{pmatrix} \end{array},$$

$$\hat{M}^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 1/3 & 1/3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \end{array} \tag{18}$$

Step 4: We now compute the global PRM as:

$$M^{agg} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 6 \\ 1/4 & 1/6 & 1 \end{pmatrix}^{1/2}. \tag{19}$$

Step 5: The ranking of alternatives is $A_1 = A_2 \succ A_3$

Example 4.7. *Now consider the MCDM problem in Example 3.2, where it entails three criteria with three alternatives. Let the criteria weights be* $[0.5,0.4,0.1]$*. The RPM steps are as follows:*

Step 1: Since the criteria weights are identical, the weight ratio $\widehat{w}$ *is computed as:*

$$\widehat{w}=[5,4,1]. \tag{20}$$

Step 2:The matrices $M^k, k=1,2,3$ *for the three criteria* $C=\{c_1,c_2,c_3\}$ *are obtained as:*

$$M^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 1/9 & 1 \\ 9 & 1 & 9 \\ 1 & 1/9 & 1 \end{pmatrix} \end{array}, \tag{21}$$

$$M^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 9 & 9 \\ 1/9 & 1 & 1 \\ 1/9 & 1 & 1 \end{pmatrix} \end{array}, \tag{22}$$

$$M^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 8/9 & 8 \\ 9/8 & 1 & 9 \\ 1/8 & 1/9 & 1 \end{pmatrix} \end{array} \tag{23}$$

Step 3: The scaled PRMs $\widehat{M}^k, k=1,2,3$ *are computed as:*

$$\hat{M}^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 1/45 & 1 \\ 45 & 1 & 45 \\ 1 & 1/45 & 1 \end{pmatrix} \end{array}, \tag{24}$$

$$\hat{M}^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 36 & 36 \\ 1/36 & 1 & 1 \\ 1/36 & 1 & 1 \end{pmatrix} \end{array}, \tag{25}$$

$$\hat{M}^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 8/9 & 8 \\ 9/8 & 1 & 9 \\ 1/8 & 1/9 & 1 \end{pmatrix} \end{array} \tag{26}$$

Step 4: The global preferences of alternatives are computed by aggregating the scaled PRMs, that is

$$M^{agg} = \begin{pmatrix} 1 & 32/45 & 36 \\ 45/32 & 1 & 405 \\ 1/36 & 1/405 & 1 \end{pmatrix}^{1/3}. \tag{27}$$

Step 5: The alternatives can be ranked based on M^{agg}*, which is* $A_2 \succ A_1 \succ A_3$*.*

5 | THEORETICAL INSIGHTS OF RATIO PRODUCT MODEL

This section investigates the features of the proposed RPM and highlights its main advantages over the WSM and WPM.

We first show that the RPM is equivalent to the WPM if the weights of criteria are the same.

Lemma 5.1. *Let* $C=\{c_1,c_2,\ldots,c_n\}$ *be n criteria with weights* $W=[w_1,w_2,\ldots,w_n]$*. If the weights of criteria are identical, then the WPM is analogous to the RPM.*

Proof. The ratio between two alternatives i and i' in the WPM is computed as:

$$\prod_{j=1}^{n}\left(\frac{P_{ij}}{P_{i'j}}\right)^{w_j}=\left(\prod_{j=1}^{n}\frac{P_{ij}}{P_{i'j}}\right)^{\frac{1}{n}}, \tag{28}$$

where the last equality is obtained since $w_1 = w_2 = \ldots = w_n = 1/n$. We now need to compare the ratio between two alternatives in the WPM with the corresponding element in M^{agg}. Since the weights of criteria are the same, the weight ratios $\widehat{w}$ will be all one in the RPM (i.e., $\widehat{M}^j = M^j, j = 1,\ldots,n$), and the elements in the global preferences PCM can be calculated as:

$$\begin{aligned} M_{ii'}^{agg} &= \left(\prod_{j=1}^{n}\widehat{M}_{ii'}^{j}\right)^{\frac{1}{n}} \\ &= \left(\prod_{j=1}^{n}M_{ii'}^{j}\right)^{\frac{1}{n}} \\ &= \left(\prod_{j=1}^{n}\frac{P_{ij}}{P_{i'j}}\right)^{\frac{1}{n}}. \end{aligned} \tag{29}$$

Since i and i' are arbitrary and the ratio between the associated alternatives is identical, the RPM and WPM are equivalent for the case where the weights of criteria are the same, and this completes the proof. □

Remark 5.2. Lemma 5.1 highlights the observation that the WPM, though is a step in the right direction, is still erroneous because it deals with the criteria weights in a wrong way. When the weights are identical for all the criteria, the WPM has no issue (and identical to the RPM). Keep in mind, though, the case where all the criteria have the same importance is quite rare in MCDM.

We now prove three important lemmas for the proposed RPM: Its agnosticism to the normalization technique and the scale of criteria weights, as well as the robustness of its ranking against adding/removing new alternative(s).

Proposition 5.3. *The RPM provides the same ratios and ranking of alternatives regardless of the normalization technique (in form of dividing/multiplying the scores) being used for the performance matrix or the scale of criteria weights.*

Proof. It is clear that by changing the scale of the criteria weights, the weight ratio vector remains the same, since the ratio between criteria weights to the minimum weight does not alter. On the other hand, any normalization technique in form of dividing/multiplying the scores in the performance matrix would not affect the local PRM. As a result, scaled PRMs for each criterion as well as the global PRM are not affected, and the final ranking remains the same. □

Proposition 5.4. *The ranks provided by the RPM are not reversed if a new alternative added/removed.*

Proof. If a new alternative is added or removed, the elements in the local PRMs are not changed, and only a row and a column are added to the matrix. This means that the ratios between the previous alternatives remain the same, and the final ratios and the corresponding ranking do not alter. □

It is also likely that some indiscriminating criteria (i.e. a criterion whose value is the same for all alternatives) get eliminated from the MCDM problems. In those case, the weights for the reduced set of criteria are normalized (to satisfy the unit-sum constraint), and the following processes, like aggregation by the WSM or WPM, are conducted accordingly. It is studied that removing indiscriminating criteria could also lead to rank reversal. We now show that ranks provided by the RPM are not affected by indiscriminating criteria, and is thus robust to adding/removing such criteria.

Proposition 5.5. *The ranks provided by the RPM are robust to adding/removing indiscriminating criteria.*

Proof. The ranking of two alternatives with n criteria is based on global PRM M^{agg}. For two alternatives A_i and A_j, A_i is ranked higher than A_2 if $M_{ij}^{agg} > 1$. Now, assume that criterion c_k is indiscriminating, the local RPM for such a criterion is a matrix of one (since all alternatives have the same performance for the criterion), as well as the scaled RPM (see Step 3 in the RPM). This means that:

$$M_{ij}^{agg} = \sqrt[n]{\alpha}, \tag{30}$$

where α denotes the products of the corresponding elements in scaled PRMs. If we remove c_k, then α remains the same, and the new global priority for the two alternatives is computed as:

$$\widehat{M}_{ij}^{agg} = \sqrt[n-1]{\alpha}. \tag{31}$$

If α is greater (less) than one (meaning that A_1 (A_2) is preferred over A_2 (A_1)), the right hand side of both Equations (30) and (31) is greater (less) than one, meaning that ranks provided by the RPM is robust to adding/removing indiscriminating criteria. This completes the proof. □.

6 | CONCLUSION AND FUTURE WORK

This paper presented RPM, a new multi-criteria decision-making (MCDM) method. The RPM development was motivated by highlighting the problems and drawbacks of two popular approaches in MCDM, namely the WPM and WSM. While the drawbacks of the WSM have already been discussed in the literature, this article also inspects the drawbacks of the WPM. In addition, we discussed the origin of the errors in the WPM and WSM, which is in principle their ignorance of nature and the constraints of the input data. Having identified the source of the problems, the RPM is developed with respect to the compositional nature of the criteria weights as well as the performance matrix.

The issues in the WSM and WPM exist in many MCDM methods. For example, the outranking methods typically treat the criteria weights as if they lie in the real space, thereby ignoring their compositional nature. This leads to issues such as rank reversal to those methods. As a result, the same approach used in this article can be applied to other MCDM methods so that they respect the nature of the input data and they are robust against rank reversal.

Aside from issues in the methods, the vision put forward in the RPM should be used in other problems inside MCDM. For instance, for MCDM problems with many criteria, the common practice is to create a tree-based hierarchy and apply an MCDM method for each of the levels. The weights of criteria at the last level are then computed by multiplying their weights to the weight of the parent criteria. Since the criteria weights are compositional, the multiplication of two weights is meaningless, making the overall weights of criteria unreliable and invalid. As a result, another venue for future research would be the use of vision propounded in the RPM for creating a proper structure for hierarchical-based MCDM problems. Further, as another venue for future, it is critical to show that the global PRM has the transitivity property so that the ranks provided based on the RPM are consistent.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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