

PLENARY LECTURE

ZAMM · Z. angew. Math. Mech. 70 (1990) 5, T 337–T 347

Akademie-Verlag Berlin

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Multigrid Methods in Computational Fluid Dynamics

The basic nonlinear multigrid algorithm is presented and its role as a fast solution method in computational fluid dynamics is outlined. Applications to the transonic potential equation, the compressible Euler and Navier-Stokes equations and to the incompressible Navier-Stokes equations are discussed.

1. Important issues in computational fluid dynamics

A principal aim of computational fluid dynamics (CFD) is the computation of flows in complicated three-dimensional geometries. Thanks to advances in computer technology and numerical algorithms, this aim is now coming into reach. For example, in 1986 the Euler equations were solved numerically for the flow around a complete civilian four-engined aircraft [J7], probably for the first time. The principal obstacles to be overcome are computing time requirements and the difficulty of generating grids in complex geometries. Multigrid (MG) methods can be a big help in overcoming these obstacles.

Grid generation can be assisted by MG by using overlays of local grids in difficult subregions. By comparing solutions on overlapping local grids of different mesh-size, local errors can be assessed and local adaptive grid refinements can be implemented. Some publications in this area are [B2, B4, G2, H10, H20, H21, M4, M8, S1, S12].

In this paper we will restrict ourselves to the aspect of computing time. Computing time may be reduced by using faster computers with novel architectures (vector and parallel processors). CFD is one of the major pacing items for the development of faster computers. Table 1 (from [G1]) gives estimates of the required number of floating point operations for certain CFD codes (by JAMESON c.s.) to compute steady compressible inviscid flows. Typical computations that one would like to carry out with complete models (Euler or Navier-Stokes) in three dimensions and that would be of great technological value, involve a computing task of the order of a teraflop (10^{12} floating point operations) and a memory requirement of the order of a Gword (10^9 words). Peak rates of commercially available supercomputers have gone from 10^2 Mflops (1 Mflops = 10^6 floating point operations/sec) in 1980 to over 1.5 Gflops (1 Gflops = 10^9 flops) in 1989; the practically achieved rates are generally half of this or less. Teraflops general purpose machines are expected by the mid-nineties. Size and access times of computer memories also need and are expected to improve.

Table 1.1 Computing work for compressible inviscid flow computation

Model	Flop/cell/cycle	No. of cells	No of cycles	Total Gigaflop
Potential, 3D	500	10^4	100–200	5–10
Euler, 2D	400	$5 \cdot 10^3$	500–1000	1–2
Euler, 3D	950	10^5	200–500	20–50

Efficiency improvements in algorithms are equally important in meeting computing time requirements. Here MG is the prime source of improvement at the moment. Let us define a work unit (WU) as the number of arithmetic operations involved in the definition of the discrete operator in one cell or grid point of the computational grid, times N : the total number of cells or grid points. Then a reasonable estimate of the minimum computing work required, which cannot be undercut, is a few WU. MG makes it possible to indeed attain this lower bound. Taking as a very rough guess 1 WU = 500 N for a typical fluid mechanics problem and assuming the work required to be 10 WU, we obtain the estimated lower bounds quoted in table 1.2. Comparison of tables 1.1 and 1.2 gives an indication that much is still to be gained from algorithmic improvements.

Table 1.2 Estimate of lower bound for computing work

Model	No of cells	10 Wu Gigaflop
Potential, 3D	10^4	0.050
Euler, 2D	$5 \cdot 10^3$	0.025
Euler, 3D	10^5	0.500

The best way to proceed is of course to combine computer and algorithm improvements. It is interesting to note that in the SUPRENUM projekt [T3] the computer architecture being designed for large scale scientific computation takes MG principles into account. Furthermore, a significant research effort is under way to implement MG on various types of parallel architecture; see for instance the contributions on this topic in [M5, M7].

In the following an effort has been made to give an introduction to the literature, but because of the rapid growth and the many aspects of the subject the author found it impossible to be complete.

2. Historical development of multigrid methods

Consider the following general linear elliptic partial differential equation, in Cartesian tensor notation, with the summation convention:

$$-(a_{\alpha\beta} \varphi_{,\alpha})_{,\beta} + (b_{\alpha} \varphi)_{,\alpha} + c\varphi = f \quad \text{in } \Omega. \quad (2.1)$$

Let $\Omega \subset \mathbb{R}^d$ be rectangular and let (2.1) be discretized in the standard way with central differences. In what probably was the first "true" MG publication, in 1964 FEDORENKO [F1] formulated a MG algorithm for the special case of the Poisson equation with $d = 2$ and showed that the computing work required to reach a given precision is $O(N)$, with N the number of unknowns. This work was extended to the general case (2.1) ($d = 2$, Ω rectangular, smooth coefficients) by BACHVALOV [B1] in 1966. Apparently, the MG method was not put in practice at the time. This was done by BRANDT [B8] in 1972, with mixed results, but in 1977 much more convincing results were published in a paper [B9] which drew wide attention and marked the beginning of a rapid development. The MG method was discovered independently by HACKBUSCH [H1] in 1976. It is interesting to note that in 1974 FREDERICKSON [F5] published a clear account of MG with impressive results, but this report was not accepted for journal publication. This report sparked the interest of the present author in MG, resulting in rapid mesh-size independent convergence for the Navier-Stokes equations [W1, W3, W5].

The prospect of being able to solve discretizations of the general equation (2.1) and more complicated problems like Navier-Stokes is of course very tempting. Hence the rapid growth of the MG literature during the last decade presented in table 2.1. This table was compiled using the extensive MG bibliography in [M6].

Table 2.1 Yearly number of MG publications

year	64	66	71	72	73	75	76	77	78	79	80	81	82	83	84	85
number	1	1	1	1	1	1	3	11	10	22	31	70	78	96	94	149

Around 1980 there was still much debate about the true merits of MG. Only after sufficient initiation satisfactory results could be obtained. This led a number of researchers to develop simpler and stronger rate of convergence proofs. See [H3] for a survey of theoretical developments. Although MG proofs are complicated, their structure has now become more or less standardized and transparent. Proofs for nonlinear problems are beginning to appear ([R1]). Other authors have tried to spread confidence in MG by providing efficient and reliable computer codes, as much as possible of "black-box" type, for discretizations of (2.1), for uninitiated users. MG00 [F3, F4, S11] is available in ELLPACK [R2]; codes MGD1 and MGD5 [H14, H15, H16, W2] are available in the NAG library; BOXMG [D1, D2, D3] is especially suited for problems with discontinuous coefficients.

Introductions to MG can be found in [B11, B14, H3, H4, M6]. The reader is assumed to be familiar with the basic principles. A survey of recent developments and of the wideranging applicability of multigrid, or, more generally, multilevel principles is given in [B12].

3. The fundamental multigrid algorithm

Let there be given a sequence of computational grids G^1, G^2, \dots, G^l with G^k finer (smaller mesh size) than G^{k-1} . Let $U^k \rightarrow \mathbb{R}$ be the space of grid functions on G^k , let $P^k: U^{k-1} \rightarrow U^k$ be a prolongation operator and let $R^k: U^{k+1} \rightarrow U^k$ be a restriction operator. Suppose we have a nonlinear (system of) partial differential equations (s), discretized on G^1, G^2, \dots, G^l . On G^k the nonlinear algebraic problem to be solved is denoted as

$$A^k(u^k) = f^k. \quad (3.1)$$

The aim is to solve (3.1) for $k = l$. A simple iterative method called *smoother* is available on every grid. Its application to (3.1) is denoted by $S(k, u, f)$; S involves only a given small number of iterations. The basic MG idea is to reduce short wavelength error components with S and to approximate long wavelength error components on coarser grids.

For ease of exposition we first present the nonlinear two-grid algorithm:

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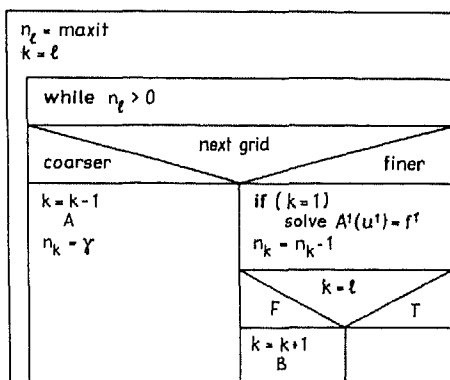
Nonlinear two grid algorithm
S1(l, u, f)
Choose sl-1 ∈ ℝ,  ũl-1 ∈ Ul-1
f̃l-1 = Al-1(ũl-1)
fl-1 = f̃l-1 + sl-1Rl-1(fl - Al(ul))
Solve Al-1(ul-1) = fl-1
ul = ul + Pl(ul-1 - ũl-1)/sl-1
S2(l, u, f)
    
```

S₁ or S₂ may be deleted. The parameter s_{l-1} = 1 usually, but may be changed to ensure the solvability of the problem on G^{l-1}. The function ũ^{l-1} is a guess for u^{l-1}; a possibility is ũ^{l-1} = R^{l-1}u^l. For more extensive discussions see [H3, H4]. Solving on G^{l-1} by γ MG iterations results in the following recursive formulation of the fundamental nonlinear multigrid algorithm:

```

procedure MG(k, u, f)
begin if k = 1 then solve A1(u1) = f1 else
  begin S1(k, U, f)
    Choose sk-1 ∈ ℝ,  ũk-1 ∈ Uk-1
    f̃k-1 = Ak-1(ũk-1)
    fk-1 = f̃k-1 + sk-1Rk-1(fk - Ak(uk))
    for i := 1 step 1 until γ do MG(k - 1, u, f)
    uk = uk + Pk(uk-1 - ũk-1)/sk-1
    S2(k, u, f)
  end
end MG
    
```

The statement MG(l, u, f) results in execution of one multigrid iteration. With γ = 1 or γ = 2 the so-called V-cycle or W-cycle are obtained, respectively. A non-recursive formulation using only one goto statement is given in [W4]. A structure diagram is given below.



Parts A and B are given by:

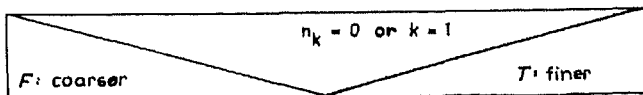
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S1(k+1, u, f)
Choose sk ∈ ℝ,  ũk ∈ Uk
f̃k = Ak(ũk)
fk = f̃k + skRk(fk+1 - Ak+1(uk+1))
    
```

```

uk = uk + Pk(uk-1 - ũk-1)/sk-1
S2(k, u, f)
    
```

With the V- or W-cycle, the choice of next grid is given by



It is easy to think of other adaptive grid switching criteria, resulting in variable instead of fixed cycles, but the structure of the algorithm remains the same.

The while statement is programmed in FORTRAN in the standard way with one goto:

```

label if (nl > 0) then
    statements
    nl = ...
    statements
    goto label
endif
    
```

Prolongation and restriction have to satisfy the following requirement. Let $m_p - 1$ be the maximum degree of polynomials interpolated exactly by P^k (e.g. for linear interpolation: $m_p = 2$). Similarly, let $m_r - 1$ be the maximum degree of polynomials interpolated exactly by $c(R^k)^T$, for some $c \in \mathbb{R}$. Then we must have [H3]

$$m_p + m_r > 2m \tag{3.2}$$

with $2m$ the order of the differential equation to be solved.

As mentioned before, the smoother must damp short wavelength error components quickly. To verify this the Fourier smoothing analysis method proposed in [B9] may be used. This is strictly valid only for constant coefficients on unbounded domains. A rigorous criterion is given by the *smoothing property* defined in [H3]:

$$\|A^k S^v\|_{\mathcal{F} \rightarrow \Phi} < \eta(v) h_k^{-\alpha}, \quad \eta(v) \rightarrow 0 \text{ as } v \rightarrow \infty \tag{3.3}$$

with $A^k: \Phi \rightarrow \mathcal{F}$, v the number of smoothing iterations and h_k a measure of the mesh-size on G^k .

The discretizations A^k on the various grids must approximate each other sufficiently well. This is measured by the *approximation property* ([H3]):

$$\|(A^k)^{-1} - P^k(A^{k-1})^{-1}R^{k-1}\|_{\Phi \rightarrow \mathcal{F}} \leq Ch_k^\alpha \tag{3.4}$$

The value of α must be the same as in the smoothing property. In applications (3.4) is usually not proven. One satisfies (3.4) by making all A^k sufficiently accurate discretizations of the differential problem.

In the remainder of this paper applications are discussed.

4. Transonic potential flow

Transonic potential flow has been treated successfully by a number of authors. See for example the KWIC index to the multigrid bibliography in [M6] and [B5, L2, V2, V3].

Generally, in the applications to be discussed in this and the following sections, boundary-fitted (composite or global) grids are used. This leads to structured computational grids, which facilitates the implementation of MG and favours computing efficiency on vector and parallel machines, but see [L3, M3] for applications of MG on unstructured grids.

Let the coordinates in the computational space be ξ^i with metric tensor

$$g^{ij} = \sum_{\alpha} \partial \xi^i / \partial x^\alpha \partial \xi^j / \partial x^\alpha,$$

with x^α Cartesian coordinates. Then in ξ^i -coordinates the transonic potential equation is given by

$$\frac{\partial}{\partial \xi^i} \left[\varrho g^{-1/2} g^{ij} \frac{\partial \varphi}{\partial \xi^j} \right] = 0, \quad \varrho = \left[1 - \frac{\gamma - 1}{\gamma + 1} g^{ij} \frac{\partial \varphi}{\partial \xi^i} \frac{\partial \varphi}{\partial \xi^j} \right] \tag{4.1}$$

with ϱ density, $g = \det(g^{ij})$, φ the velocity potential and $\gamma = 7/5$ for air. In (4.1) no other assumptions are made than potential flow. To distinguish it from models in which further simplifications are made it is often called the "full potential equation". Further Mach-number dependent simplifications have played a large role in classical gas-dynamics, but these simplifications lead to inaccuracy in the transonic regime and modern CFD can handle (4.1) directly.

In two dimensions, standard central finite volume discretization leads to the 9-point stencil of fig. 4.1.

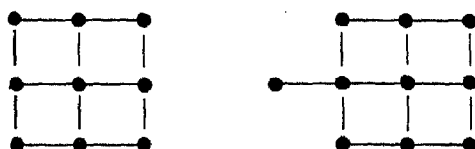


Fig. 4.1. Nine- and ten-point stencils for discretization of (4.1)

In supersonic regions thermodynamic irreversibility has to be modeled in order to obtain physically relevant solutions. A popular way to do this is to shift the point where the density ρ is evaluated in the upwind direction (retarded density concept, [H7]). Assuming that the flow is roughly in the ξ^1 -direction, this leads to the 10-point stencil of fig. 4.1 in supersonic zones.

One way to proceed is to apply Newton iteration to the nonlinear discrete system, solving the resulting linear system in each Newton iteration with a standard linear MG code. This has been done in [N2]. It is found that the linear problems are solved quite rapidly by MG. In subsonic flow Newton converges quadratically and a few iterations suffice, but in transonic flow Newton converges quite slowly. The reason is that for transonic flow the global Fréchet derivative is ill-conditioned. Therefore it is better not to use global Newton iteration, but to use the nonlinear multigrid method discussed earlier. Here only local linearization is applied, in the smoothing method. This approach has been followed in three dimensions in [V2, V3].

In three dimensions another difficulty crops up. If a single global boundary-fitted grid around a complicated geometry is used, then usually one ends up with widely varying mesh aspect ratios. The situation is modeled by the following simple linear equation:

$$\varepsilon_1 \varphi_{,11} + \varepsilon_2 \varphi_{,22} + \varepsilon_3 \varphi_{,33} = f. \quad (4.2)$$

Let this equation be discretized on a uniform mesh with equal mesh size in the three directions. Then the situation of strongly varying mesh aspect ratios is simulated by taking, for example, $\varepsilon_1 \gg \varepsilon_2, \varepsilon_3$ in part of the domain and $\varepsilon_2 \gg \varepsilon_1, \varepsilon_3$ in another part, etc.. Efficient and robust smoothers that work for all combinations of ε_i are hard to come by. Some research in this direction can be found in [K1, T1]. Alternating plane Gauss-Seidel works (solving accurately in planes, perhaps using 2D MG, choosing the planes successively in all three directions), as does alternating block LU-factorization (constructing three factorizations, permuting the coordinate directions), but these formidable smoothing methods seem a bit complicated and expensive. In [V2, V3] a middle course is taken, using a not overly complicated smoothing method of ILU/SIP type [A5, M9, M10, S10] and tailoring the mesh such that extreme aspect ratios are avoided. Large reductions (factor 10 or better) in computing time as compared to single grid methods are realized. Figure 4.2 gives an impression of the convergence behavior of the lift for the three-dimensional flow around a wing with a computational grid of $176 * 32 * 32 = 180224$ cells. Engineering accuracy is reached after 55 WU; one WU is the computational work of one ILU/SIP smoothing iteration on the finest grid.

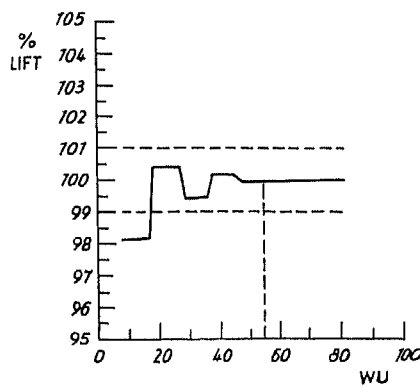


Fig. 4.2 Convergence behavior of the lift for the flow around the DFVLR-F4 wing at $M = 0.75$, $\alpha = 0.84^\circ$. WU = work unit. From [V3], with permission

5. Compressible Euler and Navier-Stokes equations

For a list of the many publications in which multigrid is applied to the Euler and Navier-Stokes equations prior to 1987, the reader may consult the KWIC index to the multigrid bibliography in [M6]. If one wishes to include the effect of vorticity generated by curved shocks, (4.1) has to be replaced by the Euler equations. If in addition vorticity generated at solid walls is to be taken into account, the Navier-Stokes equations are to be used. The numerical treatment of the compressible Euler and Navier-Stokes equations is similar, so that we will not distinguish between the two here. In two dimensions, the Navier-Stokes equations can be written as

$$\frac{\partial q}{\partial t} + \frac{\partial(f + F)}{\partial x} + \frac{\partial(g + G)}{\partial y} = 0, \quad (5.1)$$

with $q = (\rho, \rho u, \rho v, e)$ and f, g the inviscid, F, G the viscous fluxes. We have

$$f = f(q) = (\rho u, \rho u^2 + p, \rho uv, (e + p)u), \quad g = g(q) = (\rho v, \rho uv, \rho v^2 + p, (e + p)v);$$

for F and G we refer to the literature. Here ρ is the density, (u, v) is the velocity, e is the internal energy, $p = (\gamma - 1) \times (e - \frac{1}{2} \rho(u^2 + v^2))$ is the pressure.

Desirable properties of discretizations of (5.1) are: (i) second order accuracy; (ii) monotonicity; (iii) crisp resolution of discontinuities. By monotonicity we mean that the numerical scheme produces no new extrema as time progresses, so that there are no numerical "wiggles" near discontinuities. Requirements (i) and (ii) can only be satisfied simultaneously by nonlinear schemes ([H11]), i.e. schemes that are nonlinear, even if (5.1) is linear.

For the discrete approximation of (5.1), schemes of Lax-Wendroff type (see [R3]), especially the MacCormack variant [M1], have long been popular and still are widely used. They are not monotone. Ni [N1] has proposed a method of MC type to solve these discretizations efficiently; the method has been further developed in [C2, H8, J10, J11, M12, S14]. This method is specially tailored for Lax-Wendroff type discretizations and is not discussed here.

We will concentrate on finite volume discretizations of (5.1). Boundary-fitted grids are used, but it is convenient to discretize (5.1) in the physical (x, y) space. Figure 5.1 presents part of a computational grid. The unknowns q may be assigned to the vertices of the finite volumes (such as A, B, C, D), or to the centers. For the former approach, see for example [H8, J5, M3, M12]. We proceed with the cell-centered approach. Integration of (5.1) over the finite volume $\Omega_{ij} = ABCD$ gives:

$$a_{ij} \frac{d}{dt} q_{ij} + \int_{\Gamma_{ij}} H \cdot n \, ds = 0 \tag{5.2}$$

with a_{ij} the area of Ω_{ij} , q_{ij} the value of q at center of Ω_{ij} , Γ_{ij} the boundary of Ω_{ij} , $H = (f + F, g + G)$ and n the outward normal on Ω_{ij} .

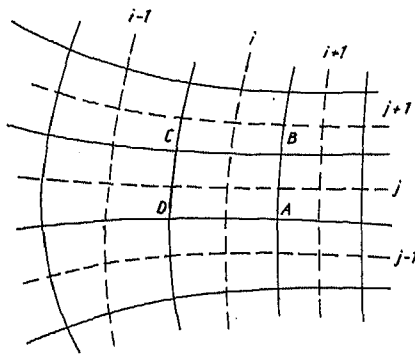


Fig. 5.1. Part of computational grid

The contour integral in (5.2) is approximated by, taking the part AB as an example,

$$\int_A^B H \cdot n \, ds \cong \{(f_{AB} + F_{AB}) n_x + (g_{AB} + G_{AB}) n_y\} |AB| \tag{5.3}$$

where f_{AB}, \dots, G_{AB} are suitable approximations of f, \dots, G on AB , on which the accuracy of the discretization depends strongly. For F and G central differences may be used, using information from neighbouring cells. A central approximation for f_{AB} is

$$f_{AB} \cong \frac{1}{2} f(q_{ij}) + \frac{1}{2} f(q_{i+1,j}) \tag{5.4}$$

resulting in second order accuracy.

In order to obtain monotonicity and crisp resolution of discontinuities with this approximation of f_{AB} , artificial nonlinear dissipation terms must be added. This approach is followed by JAMESON c.s. in a widely used set of computer codes [J1, J2, J3, J4, J6, J8]. Another widespread approach is *flux splitting*. One may write, rotating the axes locally so that the x -axis is in the direction of n ,

$$(f_{AB} n_x + g_{AB} n_y) |AB| = R^{-1} f_{AB}(Rq) |AB| \tag{5.5}$$

with

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x & -n_x & 0 \\ 0 & n_y & n_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.6}$$

A splitting

$$f = f^+ + f^- \tag{5.7}$$

is introduced, such that $\partial f^+/\partial q$ and $\partial f^-/\partial q$ have non-negative and non-positive eigenvalues, respectively. Then one makes in (5.5) the following approximation:

$$f_{AB} = f_{ij}^+ + f_{i+1,j}^- \quad (5.8)$$

The splitting (5.7) is not unique, of course. For a survey of flux-splitting discretization, see [H12, V6]. The resulting scheme has first order accuracy, is monotone and has good resolution of discontinuities that are approximately aligned with the grid (at present there exist no schemes that give sharp resolution of discontinuities with general orientation, except when adaptive local grid refinement is used, as in [B4]). Second order accuracy can be obtained by assuming linear distribution of q in each volume; monotonicity is ensured by adding nonlinear "limiters" [B6, S8, S9, S13, V1, V5]. After spatial discretization, eq. (5.2) can be written as a system of ordinary differential equations:

$$dq_h/dt = N_h(q_h) \quad (5.9)$$

with q_h the vector of all unknowns and N_h a nonlinear algebraic operator. The aim is to obtain steady (time-independent) solutions of (5.9). One way to achieve this has been proposed in [J6]. Runge-Kutta time-stepping is used. Convergence to steady state is enhanced by choosing the Runge-Kutta coefficients to increase the stability domain, by choosing the maximum time-step allowed by stability (local Courant number ≈ 1) in each finite volume (since the transient behavior of q_h is not of interest) and by introducing MG: time-stepping takes place alternatingly on coarser and finer grids, driving transient waves out rapidly by the large time steps allowed on coarse grids [H8, J2, J3, J4, J5]. This MG approach is not of the standard type discussed before. It would lead too far to go into further detail here.

Another approach is to discretize (5.9) with implicit Euler:

$$(q_h^{n+1} - q_h^n)/\Delta t = N_h(q_h^{n+1}). \quad (5.10)$$

Now Δt is unconstrained by stability and one may step to " $t = \infty$ " in very few, even one, steps. The nonlinear system for q_h^{n+1} can be solved with the fundamental MG algorithm of section 3. This approach is followed in [A2, D5, D8, H17, H18, H19, J9, M13, M14, S2, S3, S9]. The main choice to make is the smoother. In grid-point (i, j) equation (5.10) gives a nonlinear algebraic relation between unknowns in neighboring grid points, which we may denote as

$$K(q_{ij}^{n+1}, q_{i+1,j}^{n+1}, q_{j+1,i}^{n+1}, \dots) = q_{ij}^n/\Delta t. \quad (5.11)$$

The collective Gauss-Seidel smoothing method consists of visiting the finite volumes in a predetermined sequence and updating q_{ij}^{n+1} , keeping q^{n+1} fixed in neighboring cells. The update may consist of a single Newton iteration. This involves solution of a linear 4×4 system for the four unknowns represented by q . The adjective "collective" refers to the fact that the four unknowns are updated together. On vector machines this should be done with a fixed algorithm that is the same for each cell, which precludes pivoting. Therefore QR-factorization or a least squares solution method are recommended. See [S3] for aspects of vectorization. Until now good MG smoothers are available only for first order discretizations that contain some form of upwinding (flux splitting) or artificial viscosity. Second order accuracy may be obtained with defect correction [H13, H19, K2, K3, K4], which works as follows. Let N_h^1, N_h^2 be a first and a second order discretization, respectively. Then one solves, instead of (5.10):

$$(q_h^* - q_h^n)/\Delta t = N_h^1(q_h^*)$$

for $i = 1$ step 1 until m do

begin solve approximately:

$$(\bar{q}_h - q_h^n)/\Delta t = N_h^1(\bar{q}_h) + N_h^2(q_h^*) - N_h^1(q_h^*) \quad (5.12)$$

$$q_h^* = \bar{q}_h$$

end

$$q_h^{n+1} = q_h^*$$

The algorithm carries out m defect corrections. Usually m can be taken small. With 1 nonlinear MG iteration (V-cycle with one symmetric collective Gauss-Seidel pre- and postsmoothing) per defect correction in [K2] second order engineering accuracy is obtained after about five defect corrections for two-dimensional supercritical airfoil flows. This amounts to about 14 work units (one work unit is the cost of one Gauss-Seidel iteration on the finest grid). The savings in computing cost due to MG are large in these applications.

We have not discussed finite element discretization. See [F2, L1].

6. Incompressible Navier-Stokes equations

The incompressible Navier-Stokes equations are characterized mathematically as a saddle-point problem, which is why the incompressible case is usually handled separately from the compressible case.

Again we refer to the KWIC index in [M6] for access to the literature prior to 1987.

The stationary Navier-Stokes in two dimensions will be discussed, restricting ourselves to Cartesian grids. A choice has to be made between staggered and non-staggered discretization. Figure 6.1 shows the well-known staggered arrangement [H9] of unknowns; in the non-staggered variant the unknowns u , v , p are located at the same point. The two approaches are compared in [P4]; this is not the place to discuss their relative merits. MG has been applied to the staggered formulation mainly, but this may change in the future.

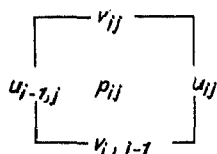


Fig. 6.1. Staggered grid

The MG methods that have been proposed (a partial list is [A4, B3, B10, B13, D6, D7, F6, F7, L4, M2, S4, S5, S6, T2, V7, V8, V9, W6, W7]) follow the basic MG algorithm presented in section 3. They differ mainly in the choice of smoothing method. As smoothing method one may consider single-grid iterative methods that have been proposed in the past, such as the SIMPLE method and its variants [P2, P3, V4]. These methods and the smoothing method proposed by BRANDT and DINAR [B13] have been brought together under the unifying viewpoint of "distributive iteration" by WITTUM [W6], who proves that these have the smoothing property.

In [A4] a number of smoothing methods are compared by numerical tests. The most attractive one seems to be one proposed by VANKA [V7, V8, V9], although for this method the smoothing property has not yet been proved. The method is of collective Gauss-Seidel type. The cells of the grid are visited in a prescribed order and the five unknowns of each cell (see fig. 6.1) are updated simultaneously, using the continuity equation, the x -momentum equation for $u_{i-1,j}$ and u_{ij} and the y -momentum equation for $v_{i,j-1}$ and v_{ij} . After local linearization and a certain simplification a 5×5 system of special form results, that can be solved with an explicit formula. In the resulting iteration method underrelaxation has to be applied with a factor that can be kept fixed (more or less). In [A4] speed-up factors compared with single grid ranging from 13 to 80 are reported. The speed-up increases as the grid is refined. Similar gains are reported by other authors, using different smoothers. First order upwind discretization of the convective terms is required, unfortunately. It is expected that defect correction may help to improve discretization accuracy, as in section 5.

MG is applied to a non-staggered formulation using flux-splitting in [D6, D7]. For an application of MG to finite element discretization of the Navier-Stokes equations, see [A1].

7. Final remarks

A short review has been given of the application of MG to compressible and incompressible flow computation. An attempt has been made to introduce the reader to the literature, citing much of it. But an exhaustive literature survey of this large field has not been given. No mention has been made of flow in porous media (reservoir engineering), where use of MG is starting (see for example [S1]). We have also neglected the topic of grid generation, where the use of MG is not yet mature, but evolving rapidly, especially for adaptive purposes. In the application areas discussed above MG has been investigated thoroughly, generating enough confidence to stimulate widespread use, permitting large gains in computing time.

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