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HIGH ORDER LAX-WENDROFF-TYPE SCHEMES FOR LINEAR WAVE PROPAGATION

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Abstract. *The second order accurate Lax-Wendroff scheme is based on the first three terms of a Taylor expansion in time in which the time derivatives are replaced by space derivatives using the governing evolution equations. The space derivatives are then approximated by central difference formulas. In this paper we extent this idea and truncate the Taylor expansion at an arbitrary even order. We use then the so called Cauchy-Kovalevskaya procedure to replace all the time derivatives by space derivatives.*

The linear case is the main focus of this paper, because the proposed high order schemes are good candidates for the approximation of linear wave motion over long distances and times with the important applications in aeroacoustics and electromagnetics. We formulate the scheme for a general system of linear equations with arbitrary order and in two or three space dimensions. The numerical results are compared with a standard scheme for aeroacoustical applications with respect to their quality and the computational effort.

1 INTRODUCTION

Lax and Wendroff proposed in their classical paper[4] a numerical scheme for hyperbolic conservation laws being second order accurate in space and time. The time accuracy is established by the Taylor expansion in time

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \Delta t\, u_t(x_i, t_n) + \frac{\Delta t^2}{2!} u_{tt}(x_i, t_n) + \cdots . \tag{1}$$

Here, t_n and t_{n+1} denote the different time levels, $\Delta t = t_{n+1} - t_n$ denotes the time step, and x_i is a fixed grid point in the space interval. This Taylor expansion is truncated after the first three terms and the time derivatives are replaced by space derivatives using the governing evolution equation. The first order time derivative is directly obtained from the governing equation. The second order time derivative is obtained by differentiating the equation with respect to time once again. In the resulting equation all the time derivatives are eliminated by using the expression for u_t. After this procedure the Lax-Wendroff scheme is then obtained by approximating the space derivatives with central finite differences. The Lax-Wendroff scheme is second order accurate in space and time.

In the general case, u may be a vector valued function of x, y, z and t and the replacement of the time derivatives by space derivatives become a cumbersome and time consuming procedure. Hence, a general extension of the Lax-Wendroff scheme to higher order of accuracy and systems in multiple space dimensions seems not to be attractive. In this paper we will show that some of these difficulties can be solved based on recent results obtained in the framework of finite volume schemes. Here, Toro and his co-workers, see for example[8] and,[9] introduced the class of ADER-FV schemes, the abbreviation for Arbitrary high order accuracy using DERivatives Finite Volume schemes. Toro and his co-workers construct a numerical flux with high order accuracy in space and time by solving the so-called generalized Riemann problem. Here, the high order accuracy in time is also based on a Taylor expansion in time in which all the time derivatives are replaced by space derivatives. This is called the Cauchy-Kovalevskaya (CK) procedure, because this idea has already been introduced by Cauchy and later extended by Sonja Kovalevskaya for the proof of a general existence theorem of analytic solutions for analytic initial data, for further information see.[1]

For general linear systems in three space dimensions the Cauchy-Kovalevskaya procedure is formulated in[2] as a fast enrolled algorithm. In the following we combine this procedure with a general interpolation algorithm to generate differences for the approximation of the space derivatives on a Cartesian grid. By this we obtain an extension of the Lax-Wendroff scheme to arbitrary even order of accuracy in space and time for linear hyperbolic systems with constant coefficients in one, two and three space dimensions. We present a stability analysis in two and three dimensions. The extension to the general linear case and to boundary-fitted structured grids is addressed.

We believe that the main field of applications for the generalized Lax-Wendroff schemes will be linear wave propagation. The numerical simulation of wave propagation over long distances become quite interesting in recent years, because the numerical methods as well as the computer power becomes so strong that the propagation of noise in aeroacoustics, seismic waves in geophysics or electromagnetic waves can be resolved in the time domain. With the proposed Lax-Wendroff-type schemes we derive an easy to implement finite difference scheme of arbitrary high order. We will compare our numerical results with a scheme widely used for the solution of linear hyperbolic problems on structured grids, the DRP (Dispersion Relation Preservation) scheme of Tam, see.[7] In its original form it is based on a sixth order difference stencil in space, which is modified to approximate the dispersion relation in a better way, while reducing the formal order of accuracy to four. This stencil in space is combined with the Runge-Kutta time stepping of fourth order accuracy in time.

The format of the paper is the following. In section 2, the basic equations and the interpolation procedure for the space discretization is introduced. The Taylor series in time and the Cauchy-Kovalevskaya procedure is discussed in section 3. The stability of the schemes for linear advection and wave propagation is examined in the section 4 and their efficiency in section 5. The paper is finished by conclusions in section 6.

2 GOVERNING EQUATIONS AND THE SPATIAL DISCRETIZATION

All definitions and equations are given in the following for the two-dimensional case and can be extended to the three dimensions in a straightforward manner. In this section we restrict ourselves to Cartesian grids.

2.1 Linear Hyperbolic Equations

The general system of linear hyperbolic partial differential equations in two space dimensions is written as

$$\underline{U}_t + A\,\underline{U}_x + B\,\underline{U}_y = 0 \tag{2}$$

where $\underline{U} = \underline{U}(x, y, t)$ denotes the vector of the physical variables. In the following, the matrices A and B are assumed to be constant. Important members of this class of equations are the linear wave equation written as a first order system, the Maxwell equations consisting of six equations and the five equations of the globally linearized Euler equations.

2.2 Grid Definitions

Some basic definitions concerning the finite difference approach are given in the following. For the two-dimensional case an equally spaced Cartesian grids in a rectangular domain is considered. The computational domain $[a, b] \times [c, d]$ is covered with grid points

$$P_{i,j} = (x_i, y_i) \quad \text{with} \quad 1 \le i \le I,\, 1 \le j \le J\,, \tag{3}$$

where

$$a + \frac{\Delta x}{2} = x_1 < \ \dots\ < x_I = b - \frac{\Delta x}{2} \tag{4}$$

and

$$c + \frac{\Delta y}{2} = y_1 < \ \dots\ < y_J = d - \frac{\Delta y}{2}\,. \tag{5}$$

Here, $\Delta x = x_i - x_{i-1}$ and $\Delta y = y_j - y_{j-1}$ denote the constant grid sizes into the x- and y-direction, respectively.

2.3 Approximation of Space Derivatives

The first step in the construction of the generalized Lax-Wendroff scheme is a polynomial interpolation of spatial derivatives of the state vector. Here we make use of the Cartesian grid on which this can easily be done with arbitrary high order of accuracy.

In order to get a scheme with an accuracy order O in time and space, we need an interpolation polynomial of the state vector at every mesh point $P_{i,j}$ that is of O-th order in space. The basis functions used for the interpolation are defined to be monomials. In the multidimensional case, the set of all possible combinations of all 1-D monomials of order up to O is used. By this we get for the two-dimensional case

$$SM(O) = \{x^r y^s \mid \text{ for } r = 0, ..., O,\ s = 0, ..., O\}. \tag{6}$$

Hereby, $SM(O)$ is a set of two-dimensional monomials with $n_{monom} = (O+1)^2$ elements. The m-th element of $SM(O)$ is abbreviated by $SM_m(O)$. With this basis we define the polynomials as

$$F = \sum_{m=1}^{n_{monom}} C_m SM_m(O) \tag{7}$$

where the C_m are the yet unknown coefficients.

For a given scalar field U with values at all grid points we want to define an interpolating polynomial F at the grid point $P_{i,j}$. To determine the coefficients C_m we need a set of data points around $P_{i,j}$ which will be called stencil SP from now on. In order to get a stable and symmetric scheme, the stencil SP should be compact and symmetric with respect to the interpolation point $P_{i,j}$. Using the same number of points n_{points} as monomial basis functions: $n_{points} = n_{monom} = (O+1)^2$ the stencil is

$$SP(O) = \bigcup P_{i+ii,j+jj}, \text{with} \quad ii = -[\frac{O+1}{2}], \dots, [\frac{O}{2}], jj = -[\frac{O+1}{2}], \dots, [\frac{O}{2}] \tag{8}$$

where $[a]$ denotes the largest integer that is smaller than a. We note that we get symmetric stencils for any even O.

For every point of the stencil SP_l we require that the interpolation conditions

$$U(SP_l) = F(SP_l) \quad \text{with} \quad l = 1, \dots, n_{points} \tag{9}$$

are satisfied. This yields the following system of linear equation for the coefficients C_m:

$$SM_m(SP_l)C_m = U(SP_l), \quad \text{with} \quad l = 1, \dots, n_{points},\ m = 1, \dots, n_{monom} \tag{10}$$

using tensor notation with summation over equal indices. These are $n_{points} = (O+1)^2$ linear equations with the same number of unknowns C_m which allows to determine the interpolating polynomial. The differentiation of this polynomial with respect to the space variables x and y is straightforward:

$$\frac{\partial^{k+l} F}{\partial x^k y^l} = \sum_{m=1}^{n_{monom}} C_m \frac{\partial^{k+l} SM_m}{\partial x^k y^l}\,. \tag{11}$$

If a local coordinate system is chosen for interpolation, the evaluation of the derivatives in equation (11) can be done at $(x,y) = (0,0)$ and simplifies to

$$\frac{\partial^{k+l} F}{\partial x^k y^l}(0,0) = k!l!C_{m^\star} \tag{12}$$

where $m^\star$ is the index belonging to the monomial basis function with $r = k$ and $s = l$. This formulation of the interpolation will be useful in the following chapters.

The interpolation stencil described in equation (8) is symmetric to the interpolation point, if the order O is even. In this case, we get a central scheme and don't take care of the direction of wave propagation. As we will show in the following, these schemes work well in the case of linear equations with smooth solutions. If O is odd, then the stencils described in equation (8) are not symmetric to the interpolation point. In this case some upwind difference formula may be introduced. In the following we will restrict ourselves to the most efficient symmetric case.

3 GENERALIZED LAX-WENDROFF-TYPE SCHEME

At a grid point $P_{i,j}$ at time t the evolution of the state vector $\underline{U}$ is expressed in the form of a Taylor series in time

$$\underline{U}_{i,j}^{n+1} = \underline{U}_{i,j}^{n} + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial^k \underline{U}_{i,j}^{n}}{\partial t^k} \tag{13}$$

where $\frac{\partial^k}{\partial t^k}$ abbreviates the k-th temporal derivative and Δt is the time step between the discrete time levels t_n and t_{n+1}, that is $t_{n+1} - t_n = \Delta t$. The truncation of the infinite sum in equation (13) up to $k = O$ leads to a numerical scheme of the order O of accuracy in time. By the interpolation step in the previous section high order approximations of the space derivatives of the state vector $\underline{U}$ can be calculated from known nodal values $\underline{U}(P_{i,j})$. All space derivatives are obtained by differentiation of the space polynomial which results from central interpolation (even order schemes).

The next step is to replace the time derivatives in (13) by these approximate space derivatives. The first order time derivative is directly given by the evolution equation (2):

$$\underline{U}_t = -A\underline{U}_x - B\underline{U}_y. \tag{14}$$

To obtain the second order time derivative, the equation (14) is differentiated with respect to x, to y, and with respect to t yielding the following three equations:

$$\underline{U}_{tx} = -A\underline{U}_{xx} - B\underline{U}_{yx}\,, \tag{15}$$

$$\underline{U}_{ty} = -A\underline{U}_{xy} - B\underline{U}_{yy}\,, \tag{16}$$

$$\underline{U}_{tt} = -A\underline{U}_{xt} - B\underline{U}_{yt}\,. \tag{17}$$

In a first step all space derivatives of the interpolating polynomial are calculated up to the polynomials order. The next step is to calculate $\underline{U}_t$ according to (14) as well as all the space-time cross derivatives $\underline{U}_{tx}$ and $\underline{U}_{ty}$ based on (15), (16), respectively. The second order time derivative is then calculated from (17).

For higher order accuracy we need the third order derivatives. By differentiating equation (14) into all space directions twice all third order derivatives involving one time differentiation can be calculated: $\underline{U}_{xxt}$, $\underline{U}_{xyt}$, and $\underline{U}_{yyt}$. These are used to get the space-time derivatives containing time derivatives of second order from the equations

$$\underline{U}_{xtt} = -A\underline{U}_{xxt} - B\underline{U}_{xyt}\,, \tag{18}$$

$$\underline{U}_{ytt} = -A\underline{U}_{xyt} - B\underline{U}_{yyt}\,. \tag{19}$$

They are obtained from the equation (**??**) differentiated with respect to x and y. The desired third order time derivative is then given by

$$\underline{U}_{ttt} = -A\underline{U}_{xtt} - B\underline{U}_{ytt} \tag{20}$$

analogously to (20).

For higher order time derivatives one continues this procedure in the same manner. The following relation can be used :

$$\begin{aligned} \frac{\partial^{n+1,i,j}\underline{U}}{\partial^{n+1}t\partial^i x\partial^j y} \quad = \quad & -A\cdot\frac{\partial^{n,i+1,j}\underline{U}}{\partial^n t\;\partial^{i+1}x\;\partial^j y} - B\cdot\frac{\partial^{n,i,j+1}\underline{U}}{\partial^n t\;\partial^i x\;\partial^{j+1}y}, \\ & n\in[0,\ldots,mO-2], \\ & i\in[0,\ldots,mO-2-n], \\ & j\in[0,\ldots,mO-2-n-i]. \end{aligned} \tag{21}$$

In order to calculate the time derivative $\partial^{n+1}/\partial t^{n+1}$ for an arbitrary order, the time derivatives of level n, $\partial^n/\partial t^n$ with one higher space derivative are needed. The derivatives for $n=0$ are known from the interpolation step itself. For example, the calculation of a 4th order time derivative looks like this:

$$\begin{array}{llllll}
\mathrm{n} & \mathrm{i{=}0} & \mathrm{i{=}1} & \mathrm{i{=}2} & \mathrm{i{=}3} & \mathrm{i{=}4} \\
0 & 0 & \mathrm{x} & \mathrm{xx} & \mathrm{xxx} & \mathrm{xxxx} \\
 & \downarrow\swarrow & \downarrow\swarrow & \downarrow\swarrow & \downarrow\swarrow & \swarrow \\
1 & 0\mathrm{t} & \mathrm{xt} & \mathrm{xxt} & \mathrm{xxxt} & \\
 & \downarrow\swarrow & \downarrow\swarrow & \downarrow\swarrow & \swarrow & \\
2 & 0\mathrm{tt} & \mathrm{xtt} & \mathrm{xxtt} & & \\
 & \downarrow\swarrow & \downarrow\swarrow & \swarrow & & \\
3 & 0\mathrm{ttt} & \mathrm{xttt} & & & \\
 & \downarrow\swarrow & \swarrow & & & \\
4 & 0\mathrm{tttt} & & & &
\end{array}$$

We implemented this in this general form with the advantage that the program can be applied to any linear system by specifying the Jacobian matrices the order. For a special system this may be additionally optimized taking into account the form of the matrices. Inserting all the time derivatives into the Taylor expansion (13) completes the general Lax-Wendroff-type scheme for the linear case. In the considered case of linear equations with constant coefficients, it is easy to combine all the steps into one step, bringing the scheme into the form

$$\underline{U}_{ij}^{n+1} \;=\; \underline{U}_{ij}^{n} - \left[\sum_{ii=-[\frac{O+1}{2}]}^{[\frac{O}{2}]} \; \sum_{jj=-[\frac{O+1}{2}]}^{[\frac{O}{2}]} C^{\star}_{ii,jj} \underline{U}^{n}_{i+ii,j+jj} \right] . \tag{22}$$

On a regular Cartesian grid the coefficient matrix $C^{\star}_{ii,jj}$ is constant and the same for any grid point.

4 Stability

The simplification of the scheme (22) to the one-dimensional case and its application to the linear convection equation

$$u_t + a u_x = 0 \tag{23}$$

leads to the Lax-Wendroff-type schemes in the form

$$U_i^{n+1} \;=\; U_i^{n} - \left[\sum_{ii=-[\frac{O+1}{2}]}^{[\frac{O}{2}]} C^{\star}_{ii} U^{n}_{i+ii} \right] . \tag{24}$$

As already mentioned, only the stencils of the even order schemes are symmetric and correspond to a central difference formula, while odd orders have a non-symmetric stencil and allow to introduce some upwinding.

In this simple one-dimensional linear case the coefficients $C^{\star}_{ii}$ coincide with the coefficients of the ADER-FV approach given in[6] which reveal that the Lax-Wendroff-type schemes are in this case formally identical to the ADER-FV schemes, except the fact that the ADER-FV schemes evolve cell-averaged data, whereas the Lax-Wendroff schemes evolve nodal data. A consequence is that all stability and accuracy analysis done in[6] for the one-dimensional case hold for the generalized Lax-Wendroff schemes as well.

In the following we present results of a von Neumann stability analysis for the two- and three-dimensional cases. These stability considerations are performed for the linearized Euler equations (LEE-equations) used in computational aeroacoustics as a mathematical model for acoustic wave propagation in a fluid. They are obtained from the equations of gas dynamics, the Euler equations, by a perturbation Ansatz with the assumption that the fluctuations of the constant background flow are small. They read as

$$\rho'_t + \underline{v}_0 \cdot \nabla \rho' + \rho_0 \nabla \cdot \underline{v}' = 0 \,, \tag{25}$$

$$\underline{v}'_t + (\underline{v}_0 \cdot \nabla)\,\underline{v}' + (\underline{v}' \cdot \nabla)\,\underline{v}_0 + \frac{1}{\rho_0}\nabla p' = 0\,, \tag{26}$$

$$p'_t + \underline{v}_0 \cdot \nabla p' + \gamma p_0 \nabla \cdot \underline{v}' = 0\,, \tag{27}$$

where the primed quantities denote the acoustic fluctuations, the quantities of the constant background flow have the subscript 0. The variables ρ ,v, p denote the density, velocity, and pressure, respectively; γ is the adiabatic exponent of a perfect gas. The advantage of using this system for the stability analysis is that this model first is of practical interest as model for acoustic wave propagation (see e.g.,) and second that it contains scalar transport with the transport velocity $\underline{v}_0$ as well as wave propagation associated with the wave speeds $|\underline{v}_0| - c_0$ and $|\underline{v}_0| + c_0$.

For the von Neumann stability analysis the discrete numerical solution is represented as a superposition of harmonics of the form

$$\underline{W}(\underline{x}, t) = \hat{\underline{V}}(t)e^{I\underline{\kappa}\cdot\underline{x}}, \tag{28}$$

where $\underline{\kappa}$ is a wavenumber vector. In discretized form, we have

$$(\underline{\kappa} \cdot \underline{x})_{i,j} = i(\kappa_x \Delta x) + j(\kappa_y \Delta y) \equiv i\phi_x + j\phi_y \tag{29}$$

and the parameters ϕ_x , ϕ_y range from $-\pi$ to π in each of the space dimensions assuming periodic boundary conditions.

Inserting a harmonic (28) into equation (22), one can compute a complex $4x4$ amplification matrix G with

$$\underline{V}^{n+1} = G\underline{V}^n. \tag{30}$$

The largest absolute value of the eigenvalues of G should be lower or equal to 1.0 for a stable scheme. For stability analysis, we compute the eigenvalues of G for a large set of possible wavelength modes and for different CFL numbers. A detailed description of the von Neumann stability analysis can be found, e.g., in.[3]

As the measure of the physical behavior of the solution we introduce the global Mach number $M_0 := \frac{|\underline{v}_0|}{c_0}$. If M_0 is small, then the acoustic wave propagation is dominant, while for large M_0 the transport dominates. The angle α in Figures 1 to 3 represents the angle between the x coordinate axis and the background velocity vector. The range $0, ..., \frac{\pi}{4}$ covers all possible stability limits, as angles $\alpha^\star \in \frac{\pi}{4}, ..., \frac{\pi}{2}$ correspond to angles $\alpha = \frac{\pi}{2} - \alpha^\star$ if we commute x- and y-axis. The CFL number is defined by

$$CFL = \frac{\Delta t a_{max}}{Min(\Delta x, \Delta y, \Delta z)}\,,$$

where a_{max} is the largest wave speed of the system, which is in the case of the linearized Euler equations $a_{max} = |\underline{v}_0| + c_0$.

Figure 1: Von Neumann stability for even order Lax-Wendroff-type schemes at $M_0 = \infty$.

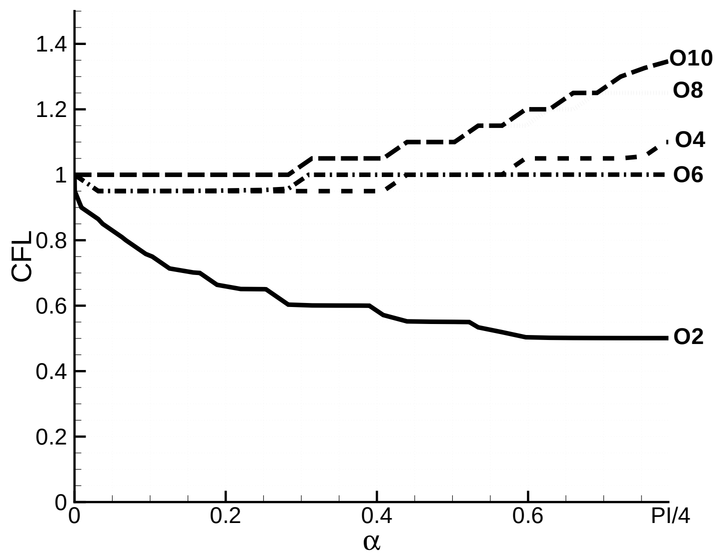
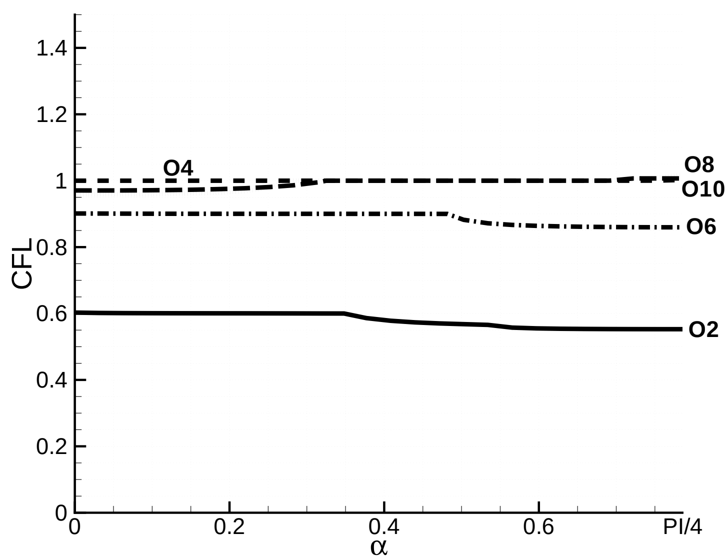
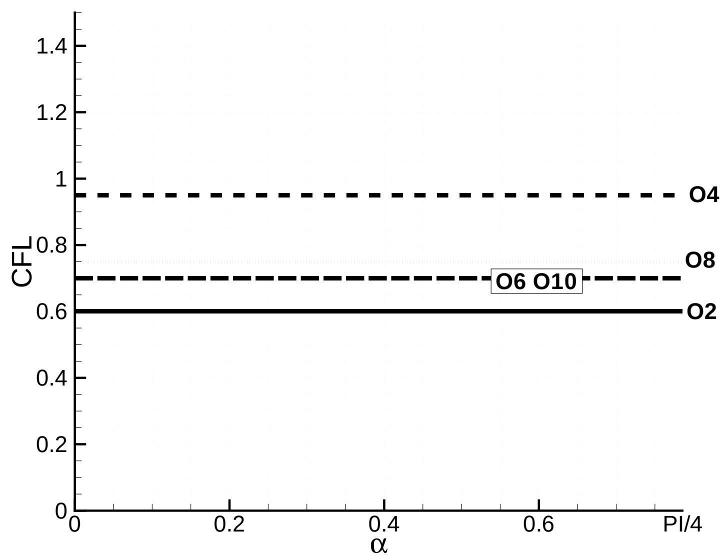


Figure 2: Von Neumann stability for even order Lax-Wendroff-type schemes at $M_0 = 0.25$.



In Figure 1, the stability limits of the Lax-Wendroff schemes with the orders of accuracy $2, 4, 6, 8$, and 10 for the case $M_0 = \infty$ are presented. In this case, the propagation of information is only due to convectional propagation with the given velocity vector $\underline{v}_0$. Here, the stability limits coincide with those for the scalar convection equation as given in[6] for the ADER-FV-schemes. In Figure 2, the same diagram is shown for $M_0 = 0.25$. It is obvious that the stability limits differ from those of pure convection. In Figure 3, the stability limits are presented for the pure wave motion case $M_0 = 0$. Here, no dependency

Figure 3: Von Neumann stability for even order Lax-Wendroff-type schemes at $M_0 = 0$.



of α occur, because the convectional speed is zero. These results indicate that in this case the stability limits are lower than for $M_0 = \infty$, except for the schemes of orders 2 and 4. The stability limit over the Mach number range from $M_0 = 0$ to $M_0 = 1$ is shown in Figure 4. The reduction of the stability limit of the 6th, 8th and 10th order schemes for small Mach numbers disappear with increasing Mach number.

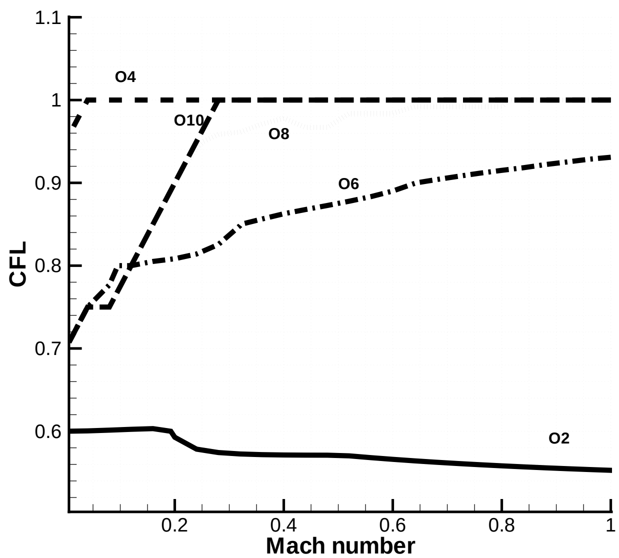
Essentially, the same behavior can be observed in three space dimensions. Here, the 2nd order scheme has a smaller stability limit than in 2-D. We note that in the case of LEE, it is sufficient to take into account the stability limit which corresponds to the angle α of the (constant) reference state, regardless the stability limits corresponding to other angles α. The graphs in Figure 4 show stability limits that are sufficient stability criterions for all α.

5 PERFORMANCE

In order to get information about the efficiency of the Lax-Wendroff-type schemes applied to LEE, we performed numerical computations with the convective transport of a density Gauss pulse in a domain with periodic boundaries. We chose the domain to be $[0, 100]^2$ in two and $[0, 100]^3$ in three space dimensions. The Gauss pulse is centered in these domains and has the initial height in density of $\rho_{max} = 1.0$ and the half width 5.0. It is transported diagonally through the computational domain by setting the velocity of the background flow to $(v_x, v_y) = (1, 1)$ in the two- and to $(v_x, v_y, v_z) = (1, 1, 1)$ in three-dimensional case. The other state variables are set to zero at $t = 0$. At time $t = 100, 200, 300, ..,$ the Gauss pulse should arrive again at the initial position and we can compute the error.

In Figure 5, the result of this simulation for $t = 1000$ (10 periods) is shown for different

Figure 4: Von Neumann stability for even order two-dimensional Lax-Wendroff-type schemes as function of the Mach number. The stability regions hold for all directions of the convection velocity.



orders of accuracy of the generalized Lax-Wendroff schemes in the two-dimensional case. It is clear to see that high order schemes lead to much more accurate results than the lower order schemes.

Figure 6 shows the L^2-norm of the error over the CPU-time for some of the Lax-Wendroff-type schemes in comparison with the DRP-scheme of[7] for the two-dimensional computations. Error reduction is achieved by mesh refinement, while the CFL-number was chosen to be always constant at 0.9. The numerical results indicate that the DRP-scheme achieves a slightly better performance than the 4-th order Lax-Wendroff-type scheme. But, if the order of the Lax-Wendroff-type schemes is increased, then the efficiency becomes much better. If high accuracy is needed, then the high order schemes always do a much better job.

Figure 7 shows the same diagram for the three-dimensional case. Here, the better efficiency is shifted more towards the DRP-scheme in comparison to two dimensions. All higher order Lax-Wendroff-type schemes become more efficient only, for errors smaller than $5.0 \cdot 10^{-5}$. The numerical results clearly indicate that the step from two to three space dimensions is more costly for Lax-Wendroff-type schemes. To explain this behavior, the growth of the computational effort per point and per time step will be analyzed in the following.

For the Lax-Wendroff-type schemes it is easy to find the formula for the number of operations per point and time step:

$$NO_{LaxWen} = (O+1)^{ndim} nvar^2, \tag{31}$$

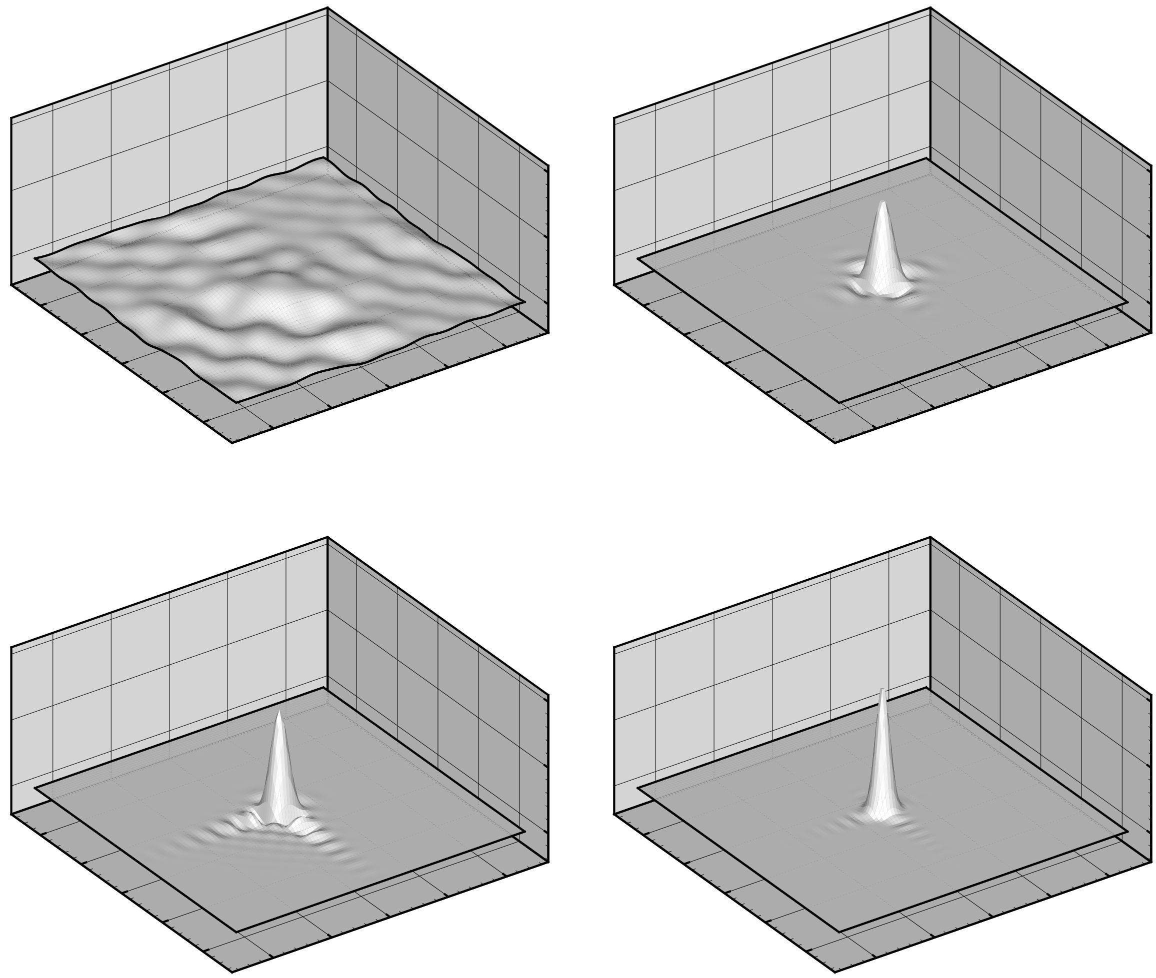


Figure 5: Advection of a Gauss pulse at $t = 1000$ computed with the generalized Lax-Wendroff schemes of orders 2,4,6, and 12

where *ndim* is the number of space dimensions and *nvar* the dimension of the state vector $\underline{U}$, operation is defined as a multiplication followed by an addition. We note that the variable *ndim* appears in the exponent, representing the large multi-dimensional interpolation stencil.

The DRP-scheme uses four Runge-Kutta stages per time step. In any of these stages, *ndim* one-dimensional space derivatives are calculated by using an interpolation over $npoints = 7$ grid points for the standard DRP-scheme. Then we find first time derivatives from the space derivatives, that is, we have *ndim* times a multiplication with a Jacobian matrix. Finally, we have to update the solution vector. Hence, a formula for the number

Figure 6: L^2-error as function of the CPU-time for two-dimensional convection

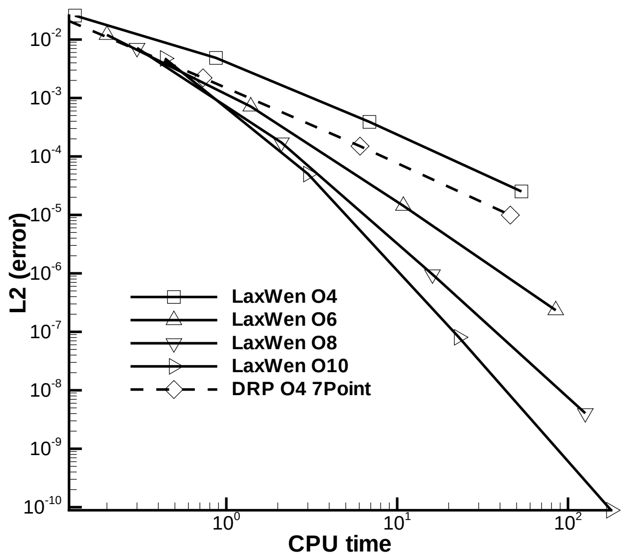
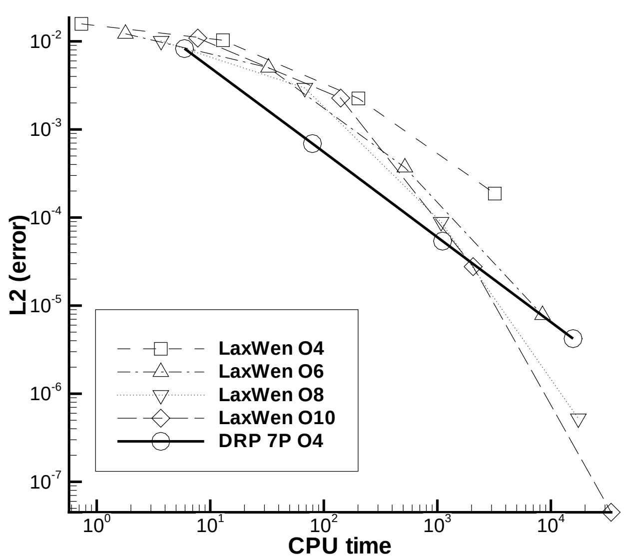


Figure 7: L^2-error as function of CPU-time for three-dimensional convection



of operations per point and time step is:

$$NO_{DRP} = 4(ndim(npoints * nvar + nvar^2) + nvar)\,. \tag{32}$$

In this formula, $ndim$ is not an exponent, but a factor only.

scheme	$2-D$	$3-D$
DRP	500	740
Lax-Wendroff $\mathcal{O}4$	400	3125
Lax-Wendroff $\mathcal{O}6$	748	8575
Lax-Wendroff $\mathcal{O}8$	1296	18225
Lax-Wendroff $\mathcal{O}10$	1936	33275

Table 1: Number of operations per point and time step for linearized Euler equations

In Table 5, the number of operations is explicitly presented for some of the schemes applied to the two-dimensional ($ndim = 2, nvar = 4$) and the three-dimensional ($ndim = 3, nvar = 5$) linearized Euler equations. For the DRP-scheme, $npoints = 7$ was chosen. The augmentation of the number of necessary operations when passing from two to three space dimensions is significantly larger for the Lax-Wendroff-type schemes than for the DRP-scheme. Here, the Lax-Wendroff-type schemes suffer especially in three space dimensions from a large interpolation stencil.

6 CONCLUSIONS

We proposed a class of the Lax-Wendroff-type schemes that have arbitrary even order accuracy in space and time. These finite difference schemes are easy to implement in such a general way that the computer code may be applied to any linear system of equations and with arbitrary even order of accuracy in space and time. The change from Maxwell equations, to seismic or acoustic equations and the change of the order of accuracy only needs the change of the input data. The basic building block of the proposed Lax-Wendroff-type schemes is the Cauchy-Kovalevskaya procedure by which the values for the time derivatives are calculated from space derivatives. For linear systems the Cauchy-Kovalevskaya procedure can be constructed in a general enrolled version that can efficiently be implemented. The increase of the order of accuracy allows to capture waves over long distances with small dispersion and dissipation errors and to get high accurate results on coarse meshes. The high order Lax-Wendroff-type schemes are able to capture acoustic waves with very small phase errors without introducing any artificial dissipation. Hence, we think that they are good candidates for computational aeroacoustics or electromagnetics in the time domain. We also see a big advantage in the Lax-Wendroff-type schemes because of their robustness. We never needed additional artificial dissipation to smooth out dispersion errors, the inherent dissipation was always large enough.

The Lax-Wendroff-type schemes suffer in the three-dimensional case from a big interpolation stencil being necessary to approximate all the cross derivatives in an appropriate way. The full cubic stencil occurs due to the fact that the finite differences are calculated by a tensor product interpolation. This interpolation may be improved in the sense that the number of points in the stencil are reduced. This strategy is proposed in.[5]

In the nonlinear case, the occurrence of strong gradients or even discontinuities may strongly complicate the numerical simulation. Although the paper of[4] was one of the pioneering work to enhance the construction of the modern shock-capturing finite volume schemes, we did not consider nonlinear equations in this paper. Stimulated by the Lax-Wendroff scheme the finite volume schemes with fluxes that take into account the nonlinear wave propagation have been discovered. High order upwind schemes can be formulated much better within the finite volume framework.

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