Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Rooted binary level-3 phylogenetic networks are encoded by quarnets

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfillment of the requirements

for the degree

BACHELOR OF SCIENCE
in
APPLIED MATHEMATICS

by

LEONIE NIPIUS

Delft, the Netherlands
July 2020

BSc thesis APPLIED MATHEMATICS

“Rooted binary level-3 phylogenetic networks are encoded by quarnets”

LEONIE NIPIUS

Delft University of Technology

Supervisor

Dr.ir. L.J.J. van Iersel

Other thesis comittee member

Drs. E.M. van Elderen

July, 2020

Delft

Abstract

Phylogenetic networks generalize evolutionary trees and are commonly used to represent evolutionary relationships between species that undergo reticulate processes such as hybridization, recombination and lateral gene transfer. Recently, there has been great interest in knowing which networks are determined or *encoded* by their *trinets*, that are rooted networks on three species. Van Iersel and Moulton showed that recoverable rooted binary level-2 phylogenetic networks are encoded by their trinets. Based on their work for level-2 networks, we show here that *not all* recoverable rooted binary level-3 networks are weakly encoded by their trinets, but *most* networks are. Further, although not all level-3 networks are weakly encoded by their trinets, we are able to prove that *all* recoverable rooted binary level-3 networks are encoded by their *quarnets*, that are rooted networks on four species.

Contents

1 Introduction **1**

2 Preliminaries **4**
2.1 Phylogenetic network 4
2.2 Recoverable phylogenetic network 6
2.3 Encoded phylogenetic network 7
2.4 Decomposed phylogenetic network 8
2.5 Generator of phylogenetic network 9

3 Level-2 networks are encoded by trinets **12**

4 Not all level-3 networks are weakly encoded by trinets **18**

5 Most level-3 networks are weakly encoded by trinets **22**

6 All level-3 networks are encoded by quarnets **70**

7 Conclusion and discussion **74**

A Labelled level-3 generators **77**
A.1 1 crucial side, no symmetry 77
A.2 2 crucial sides, no symmetry 79
A.3 3 crucial sides, no symmetry 83
A.4 2 crucial sides, 1 set of parallel arcs, no other symmetry . . . 85
A.5 3 crucial sides, 1 set of parallel arcs, no other symmetry . . . 86
A.6 3 crucial sides, 2 sets of parallel arcs, no other symmetry . . . 87
A.7 1 crucial side, symmetry, no parallel arcs 88
A.8 2 crucial sides, symmetry, no parallel arcs 89
A.9 3 crucial sides, symmetry, no parallel arcs 89
A.10 3 crucial sides, 1 set of parallel arcs, other symmetry 90
A.11 3 crucial sides, 2 sets of parallel arcs, other symmetry 91

B Properties of level-3 generators **92**

Chapter 1

Introduction

Many biological studies use the evolutionary relationships between a given set of species. Therefore, it is important that there is a clear way to represent these relationships. Phylogenetic trees are often used for this. Formally, a *phylogenetic tree* is a rooted (graph theoretical) tree that has no indegree-1 outdegree-1 vertices, and in which the leaves are bijectively labelled by the elements of X. Note that X is a set of species. An example of a phylogenetic tree is given in Figure 1.1.

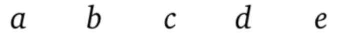


Figure 1.1: Example of a phylogenetic tree

Although phylogenetic trees are routinely used, they can not represent all evolutionary relationships between species. We need something else to represent reticulate evolutionary processes such as hybridization, recombination and lateral gene transfer. Therefore, there has been some interest in using networks instead of trees. Formally, a *(rooted phylogenetic) network* is a directed acyclic graph that has a single root, has no indegree-1 outdegree-1 vertices, and has its leaves bijectively labelled by the elements of X. Again, X is a set of species. We refer to Chapter 2 for a full definition with some remarks. This also holds for other definitions in this chapter.

An example of a rooted phylogenetic network is given in Figure 1.2. This network represents the evolutionary histories of seven different wheat species. We see that the relationships between these species can not be represented by using phylogenetic trees since there are some reticulate processes.

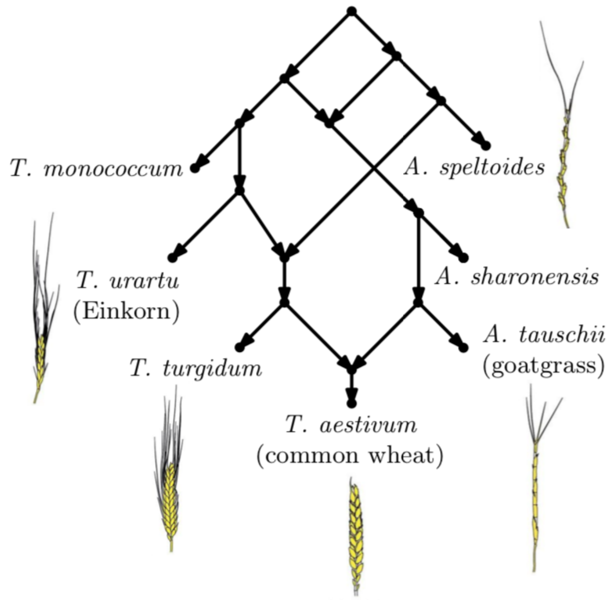


Figure 1.2: A phylogenetic network for wheat species [4]

We are often interested in the evolutionary relationships between a (large) set of species, but it is computationally not easy to research the relationships between many species at once. Therefore, it is easier to research the relationships between three or four species at once and then combine the results in such a way that we get the network that represents the relationships between the whole set of species.

A network for three different species is called a *trinet* and a network for four different species is called a *quarnet*. However, it is not easy to reconstruct the original network using trinets or quarnets. Further, we do not always get the original network [3].

Van Iersel and Moulton have done research into a specific class of networks (named binary level-2 networks) for which it is possible to reconstruct the original network by using trinets [5]. First, we explain what binary level-2 networks are. A network is *binary* if all vertices have indegree and outdegree at most two and all vertices with indegree two have outdegree one. Further, a binary network is level-2 if each biconnected component has at most two indegree-2 vertices. In general, a binary network is *level-k* if each biconnected component has at most k indegree-2 vertices.

Binary level-2 networks are encoded by trinets. In other words, knowing all possible trinets for a certain set of species is enough to reconstruct the original network. This phylogenetic network represents all relationships between the species. Then, if we have found a network that has the trinets, we know that this is the network we were searching for.

We give a complete characterization of which binary level-3 networks can be distinguished based on its trinets. There are 65 possible underlying structures of a biconnected component. We show that in 64 of the 65 cases the network can be distinguished from other level-3 networks based on its trinets. In the 65 case, this is also true except for a heavily restricted type of network. Finally, we consider quarnets (subnetworks on four leaves) and show that every binary level-3 network can be distinguished from any other network based on its quarnets. Note that we solve in this thesis an open problem that is posed in [1].

We will now give an overview of the rest of this thesis. In Chapter 2 some preliminaries will be presented. Some assumptions for the phylogenetic networks in this paper can also be found here. We will revisit the proof for level-2 networks of Van Iersel and Moulton in Chapter 3. We will explain in Chapter 4 why not all level-3 networks are weakly encoded by trinets and we will prove in Chapter 5 that most level-3 networks are weakly encoded by trinets. Then, we will prove in Chapter 6 that all level-3 networks are encoded by quarnets. Finally, there are some conclusions and a discussion in Chapter 7.

Chapter 2

Preliminaries

In this chapter we will present some preliminaries. In the first section we will discuss phylogenetic networks in general. We will give some definitions and do some assumptions that holds for the whole thesis. In the next section we will explain what a recoverable phylogenetic network is. In the third section we will explain when a phylogenetic network is encoded by trinets or quarnets. In the fourth section we will discuss two decomposition theorems for phylogenetic networks. The last section is about the generator of a phylogenetic network. In this section we will also give a useful lemma.

The whole chapter is based on [5]. Some definitions are slightly changed, but they agree with the definitions as given in [5].

2.1 Phylogenetic network

vel

First, we will give the definition of a rooted phylogenetic network. Note that in this thesis we will also use *phylogenetic network* or *network* for short. We refer then to a rooted phylogenetic network as it is defined below.

Definition 2.1. A ***rooted phylogenetic network*** on a set X is a directed acyclic graph that has a single root, has no indegree-1 outdegree-1 vertices, and has its leaves bijectively labelled by the elements of X.

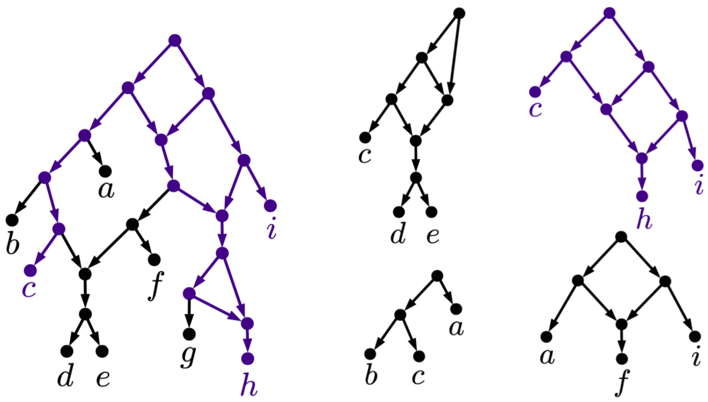
To understand this definition we need the following two definitions:

Definition 2.2. A ***root*** is a indegree-0 vertex.

Definition 2.3. A ***leaf*** is a outdegree-0 vertex.

Since the leaves are bijectively labelled by the elements of X, we can identify each leaf with its label. Often X is a set of species. So, each leaf represents then one of the species. We assume the set X to be finite throughout this thesis.

An example of a rooted phylogenetic network N is given in Figure 2.1a. The leaves of this network N are a, b, c, d, e, f, g, h and i. We will also use Figure 2.1 to explain other definitions in this chapter.



(a) Network N (b) Four trinets of N

Figure 2.1: Example of a rooted phylogenetic network N with four of its trinets. *Blue* is used to illustrate how the trinet on $\{c, h, i\}$ exhibited by N [5]

Note that a rooted phylogenetic tree is different from a rooted phylogenetic network. A *rooted phylogenetic tree* is a rooted (graph theoretical) tree that has no indegree-1 outdegree-1 vertices, and in which the leaves are bijectively labelled by the elements of X. The most important difference between a phylogenetic tree and a phylogenetic network is that a phylogenetic network can have vertices with indegree ≥ 2. We call these vertices reticulations as can be seen in the definition below.

Definition 2.4. A vertex of a directed graph with a single root is a ***reticulation*** (or: *reticulation vertex*) if it has indegree ≥ 2.

The phylogenetic networks that will be discussed in this thesis are all binary. The definition of a binary phylogenetic network can be found below.

Definition 2.5. A phylogenetic network is ***binary*** if all vertices have indegree and outdegree at most two and all vertices with indegree two have outdegree one.

Now, we know that each reticulation of a binary phylogenetic network has indegree two and outdegree one. Also, note that the network N that is given in Figure 2.1a is binary and has four reticulations.

Finally, there are different types of binary phylogenetic networks. We have, for example, level-k networks, simple networks and simple level-k networks. The definitions of these types of networks will be given later in this chapter.

2.2 Recoverable phylogenetic network

We will need some more definitions before we can define what a recoverable phylogenetic network is. First, we will give the definitions of a cut-vertex and a cut-arc:

Definition 2.6. Let D be a directed graph with a single root. A vertex v of D is a ***cut-vertex*** if its removal disconnects the underlying undirected graph of D. Similarly, an arc a of D is a ***cut-arc*** if its removal disconnects the underlying undirected graph of D.

Now, we use this definition for the definition of a biconnected graph:

Definition 2.7. A directed graph is ***biconnected*** if it has no cut-*vertices*.

Note that the definition above uses cut-vertices and not cut-arcs, as were defined in Definition 2.6. In the definition below we define what a biconnected component is, which is a special case of biconnected graphs.

Definition 2.8. A ***biconnected component*** is a maximal biconnected subgraph (i.e. a biconnected subgraph that is not contained in any other biconnected subgraph).

By this definition, each cut-arc is a biconnected component. We call such a cut-arc a *trivial biconnected component*. We will use the definition of a biconnected component in the definition of a level-k network, which is given below.

Definition 2.9. A phylogenetic network is ***level-k*** if each biconnected component has at most k reticulations.

Note that the network N that is given in Figure 2.1a is a level-3 network. Now, we give some more definitions we will need:

Definition 2.10. Let u and v be two *vertices* of a phylogenetic network N. If (u, v) is an arc of N, then u is a ***parent*** of v and v is a ***child*** of u. Furthermore, v is ***below*** u, if there is a directed path from u to v in N, or $u = v$. For two *leaves* x and y, x is ***below*** y if the parent of x is below the parent of y. For an *arc* $a = (u, v)$ and a *vertex* w, w is ***below*** a if w is below v.

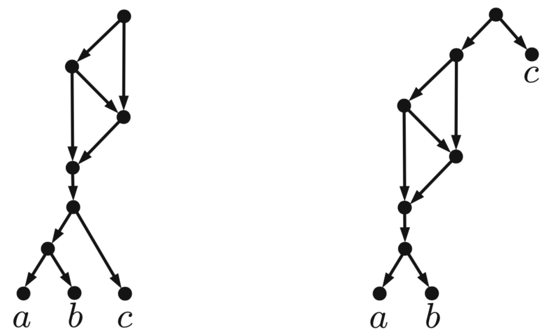
Note that this definition is different while using it for two vertices, for two leaves or for a arc and a vertex. We use this definition in the following definition of redundant and strongly redundant biconnected components:

Definition 2.11. Let B be a nontrivial biconnected component. B is ***redundant*** if it has only one outgoing arc. Furthermore, B is ***strongly redundant*** if it has only one outgoing arc (u, v) and all leaves of the network are below v.

Finally, we can define what a recoverable phylogenetic network is:

Definition 2.12. A phylogenetic network N is ***recoverable*** if it has no strongly redundant biconnected components.

Note that the network N that is given in Figure 2.1a is recoverable and has three biconnected components. Further, note that the network in Figure 2.2a is not recoverable since it has a strongly redundant biconnected component. Also, note that the network in Figure 2.2b is recoverable, because its only nontrivial biconnected component is redundant but not strongly redundant.



(a) Network is not recoverable (b) Network is recoverable

Figure 2.2: Two phylogenetic networks [5]

2.3 Encoded phylogenetic network

In this section we will explain when a phylogenetic network is encoded by its trinets or quarnets. First, we give the definition of a lowest stable ancestor:

Definition 2.13. Given a network N on X and $X' \subseteq X$, a ***lowest stable ancestor*** $LSA(X')$ is a vertex $w \notin X'$ of N for which all paths from the root to any $x \in X'$ pass through w, and such that no vertex below w has this property.

Note that the lowest stable ancestor of a certain set X' is unique. We will use the definition of a lowest stable ancestor in Definition 2.15.

A phylogenetic tree with three leaves is called a triplet. A similar definition for phylogenetic networks can be found below.

Definition 2.14. A ***trinet*** is a rooted phylogenetic network with three leaves.

Note that the phylogenetic networks that are given in Figure 2.2 are two trinets. In the following definition trinets and phylogenetic networks in general are connected:

Definition 2.15. Given a phylogenetic network N on X and $\{x, y, z\} \subseteq X$, the trinet on $\{x, y, z\}$ ***exhibited*** (or: *displayed*) by N is the trinet obtained from N by deleting all vertices that are not on any path from $LSA(\{x, y, z\})$ to x, y or z and subsequently suppressing all indegree-1 outdegree-1 vertices and parallel arcs.

Note, suppressing parallel arcs means replacing each set of parallel arcs by a single arc. Further, we will use ***Tn(N)*** to denote the set of all trinets exhibited by a phylogenetic network N.

An example of a phylogenetic network N with four of its trinets can be found in Figure 2.1. Note that *blue* is used to illustrate how the trinet on $\{c, h, i\}$ exhibited by N.

The next definition shows us when two phylogenetic networks are equal:

Definition 2.16. Given two phylogenetic networks N and N' on X, we write $\boldsymbol{N = N'}$ if there is a graph isomorphism between N and N' that preserves leaf labels, i.e. if there exists a bijective function f: $V(N) \rightarrow V(N')$ such that $f(x) = x$ for each leaf x of N and such that for every $u, v \in V(N)$ holds that (u, v) is an arc of N if and only if $(f(u), f(v))$ is an arc of N'.

Now, we can define when a phylogenetic network is encoded by its set of trinets:

Definition 2.17. A phylogenetic network N is ***encoded*** by its set of trinets $Tn(N)$ if there is no recoverable phylogenetic network $N' \neq N$ with $Tn(N) = Tn(N')$.

Note that it is important that N' in the definition above is not recoverable since we can not know if there is a strongly redundant biconnected component only using the trinets.

Finally, the definitions about *quarnets*, which are networks with four leaves, are similar to the definitions about trinets and will be given explicitly in Chapter 6.

2.4 Decomposed phylogenetic network

In this section we will discuss two decomposition theorems for phylogenetic networks. First, we give the definition of a CA-set:

Definition 2.18. Let N be a phylogenetic network on X and $A \subseteq X$. A is a ***CA-set*** (*Cut-Arc set*) of N if there exists a cut-arc (u, v) of N such that $A = \{x \in X | x \textit{ is below } v\}$.

Note, a CA-set is *not* the same as the set of cut-arcs in a network. For example, $\{g, h\}$ is a CA-set of the network N that is given in Figure

2.1a. The other CA-sets of this network are the singletons $\{a\}, \{b\}, \dots, \{i\}$. Using the definition of CA-set, which is given above, we can give the first decomposition theorem:

Theorem 2.19 (Theorem 1 in [5])**.** *Let N be a recoverable binary phylogenetic network on X, and $A \subset X$. Then, A is a CA-set of N if and only if $|A| = 1$ or, for all $z \in X \backslash A$ and $x, y \in A$ with $x \neq y$, $\{x, y\}$ is a CA-set of the trinet on $\{x, y, z\}$ exhibited by N.*

We will use this theorem in the proofs of Lemmas 3.2 and 5.2. Before we will give the other decomposition theorem, we will again give some definitions. First, we give the definition of a simple phylogenetic network:

Definition 2.20. A phylogenetic network is ***simple*** if the head of each cut-arc is a leaf.

The idea of the decomposition theorems is to restrict the problem. In the previous decomposition theorem we looked to the CA-sets of a phylogenetic network. In the next decomposition theorem we will look to the biconnected components of a phylogenetic network. First, we give the following definition:

Definition 2.21. Let N be a phylogenetic network and B a nontrivial biconnected component with b outgoing cut-arcs $a_1 = (u_1, v_1), \dots, a_b = (u_b, v_b)$. Consider the phylogenetic network N_B obtained from N by deleting all biconnected components except for $B, a_1, \dots, a_b$ and labelling $v_1, \dots, v_b$ by new labels $y_1, \dots, y_b$ that are not in X. Then, N_B is a ***restriction*** of N to B.

Note that N_B, as given in the definition above, is unique up to the choice of the new labels $y_1, \dots, y_b$. Further, note that N_B is a simple network.

Finally, we give the second decomposition for phylogenetic networks, which we will use in the proofs of Corollaries 3.3 and 5.3.

Theorem 2.22 (Theorem 2 in [5])**.** *A recoverable binary phylogenetic network N on X, with $|X| \geq 3$, is encoded by its trinets $Tn(N)$ if and only if, for each nontrivial biconnected component B of N with at least four outgoing cut-arcs, N_B is encoded by $Tn(N_B)$.*

2.5 Generator of phylogenetic network

In this section we will discuss a generator of a phylogenetic network. At the end of this section we will give a useful lemma. First, we give the definition of a simple level-k network:

Definition 2.23. A level-k phylogenetic network is a ***simple level-k*** network if it contains one nontrivial biconnected component B containing exactly k reticulations and no cut-arcs other than the ones leaving B.

Now, we give the definition of a level-k generator:

Definition 2.24. A level-k ***generator*** is a directed acyclic biconnected multigraph with exactly k reticulations with indegree 2 and outdegree at most 1, a single vertex with indegree 0 and outdegree 2, and apart from that only vertices with indegree 1 and outdegree 2.

The following definition is about the sides of a generator:

Definition 2.25. The ***sides*** of a generator are the arcs and outdegree-0 vertices of the generator.

An example of a level-3 generator with labelled sides can be found in Figure 2.3. All 65 level-3 generators are given in Appendix A.

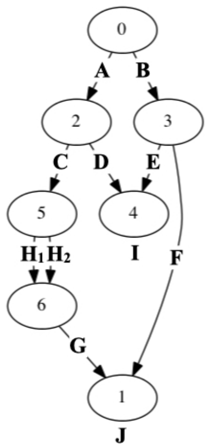


Figure 2.3: Level-3 generator with labelled sides

Note, deleting all leaves of a simple level-k network N gives a level-k generator G_N. Then, G_N is the ***underlying generator*** of N. Conversely, N can be reconstructed from G_N by 'hanging leaves' on its sides as follows:

- for each arc a of G_N, replace a by a directed path with $l \geq 0$ internal vertices $v_1, \ldots, v_l$ and, for each such interval vertex v_i, add a leaf $x_i \in X$ and an arc (v_i, x_i); and
- for each indegree-2 outdegree-0 vertex v, add a leaf $x \in X$ and an arc (v, x).

The following definition explains when a leaf of a simple level-k network is on a certain side of the underlying generator.

Definition 2.26. A leaf x ***is on side*** s if it is hung on side s in this construction of N from G_N. More precisely, for a leaf $x \in X$ of a simple level-k network N with underlying generator G_N and a side s of G_N, *x is on side s* if one of the following holds:

- s is an indegree-2 outdegree-0 vertex of G_N and (s, x) is an edge of N;
- s is an edge (u, v) of G_N and the parent of x in N lies on the directed path from u to v in N.

Now, we give another definition about the sides of a generator:

Definition 2.27. Given a level-k generator G, a set of sides of G is a ***set of crucial sides*** if it contains all indegree-2 outdegree-0 vertices together with at least one arc of each pair of parallel arcs.

Note that the generator that is given in Figure 2.3 has its sides labelled with $A, B, C, D, E, F, G, H, I, J, K, L, M$. This generator has two sets of crucial sides, namely $\{H_1, I, J\}$ and $\{H_2, I, J\}$. Further, we call a side *non-crucial* if the side is in no crucial set.

Now, using the above definition we can give the definition of a crucial trinet:

Definition 2.28. Consider any simple level-k network N on X with underlying generator G and a trinet P on $X' \subseteq X$. P is a ***crucial trinet*** of N if X' contains at least one leaf on each side in some set of crucial sides of G.

Note that for a crucial trinet the number of elements of a set of crucial sides matters. There exists level-k networks with $k \geq 4$ that has no crucial trinets since each set of crucial sides has at least four elements.

The next definition explains when a generator has symmetry:

Definition 2.29. A generator G has ***symmetry*** if it has parallel arcs or if there exists a bijective function $f : V(G) \to V(G)$ such that for all $u, v \in V$ the number of arcs from u to v is equal to the number of arcs from $f(u)$ to $f(v)$ but $f(w) \neq w$ for at least one $w \in V$. Intuitively, this means that there exists a relabelling of the sides of the generator giving an isomorphic generator.

Finally, we give the lemma, which will be very helpful in the proofs of Lemmas 3.2 and 5.2.

Lemma 2.30 (Lemma 1 in [5])**.** *Let N be a simple level-k network, G its underlying generator and $P \in Tn(N)$. Then, P is a crucial trinet of N if and only if P is a simple level-k network. Moreover, if P is a crucial trinet of N then G is its underlying generator.*

Chapter 3

Level-2 networks are encoded by trinets

In this chapter the proof that binary recoverable level-2 networks are encoded by their trinets will be revisited. The proof will be given by using the same reasoning as in [5]. Therefore, the same theorems, lemmas and observation will be used. Also, many parts of the proofs are the same. But, some parts are written in slightly different wording or are written in more detail.

First, we will give an observation, then we will give a lemma with its proof, and finally we are able to combine the results to prove that binary recoverable level-2 networks are encoded by their set of trinets.

So, first we give the following observation:

Observation 3.1 (Observation 7 in [5]). *If G is a level-2 generator, then it has a set of crucial sides of size at most two. Hence, every simple level-2 network N has at least one crucial trinet. Moreover, for every leaf x of N, there exists a crucial trinet of N containing x.*

Now, we know that for every leaf there exists a crucial trinet containing that leaf. We will see in Chapter 5 that without this useful result the proof of Lemma 3.2 will be more complicated. In the following lemma we will prove that binary, simple level-2 networks are encoded by their trinets. This result for less complex level-2 networks will be helpful to prove the main result of this chapter.

Lemma 3.2 (Lemma 3 in [5]). *Every binary, simple level-2 network on X, with $|X| \geq 3$, is encoded by its trinets.*

Proof. Let N be any binary, simple level-2 network on X, with $|X| \geq 3$. Assume that this network is not encoded by its trinets $Tn(N)$. Then, there is a recoverable network $N' \neq N$ with $Tn(N) = Tn(N')$. We will show that $N' = N$, which is a contradiction, so then the lemma follows.

We begin by showing that N' is a binary, simple, level-2 network.

- Since $Tn(N') = Tn(N)$ holds, we have by Theorem 2.19 that the set of CA-sets of N' equals the set of CA-sets of N. Note that all CA-sets of N (and also of N') are singletons, since N is a simple network. Furthermore, we claim that N' has no redundant biconnected components. If it had one, then there would be only one leaf, say x, below it. However, then all trinets containing x would have a redundant biconnected component with x directly below it. This is not possible because $Tn(N') = Tn(N)$. For each leaf x there exists a trinet in $Tn(N)$ without redundant biconnected components. So, N' has no redundant biconnected components. Since the sets of CA-sets of N' and N are the same and N' has no redundant biconnected components, we have that N' is a simple network.

- Suppose we have any simple level-k network with $k > 2$. Then, this network has exactly k reticulations. If there are at least three leaves whose parent is a reticulation, take three such leaves. Otherwise, take all leaves whose parent is a reticulation and take the remaining leaves on sides that form parallel arcs in the underlying generator of N, choosing at most one leaf per pair of parallel arcs. Then, the trinet on the chosen three leaves has at least three reticulations. Note that if a leaf is chosen on one of the parallel arcs in the underlying generator, the pair of parallel arcs will not be suppressed, and so we get a reticulation. So, a simple level-k network, with $k > 2$, has a level-k' trinet with $k' > 2$. It follows that N' is a level-2 network since $Tn(N') = Tn(N)$ contains only level-2 trinets.

- Assume that N' has a vertex with outdegree greater than 2. Let c_1, c_2 and c_3 be three of his children. Then, consider three (not necessarily different) leaves x_1, x_2 and x_3 below c_1, c_2 and c_3 respectively. Then, any trinet containing x_1, x_2 and x_3 exhibited by N' is not binary. We get a contradiction since all trinets in $Tn(N') = Tn(N)$ are binary, since N is binary. In much the same way, we can prove that each vertex in N' has indegree at most 2 and that each indegree-2 vertex has outdegree 1. Now, we can conclude that N' is binary.

So, N and N' are both binary, simple level-2 networks. Now, let G be the underlying generator of N. First, we show that G is also the underlying generator of N'. By Observation 3.1, N has at least one crucial trinet P_c. By Lemma 2.30, P_c is a simple level-2 network and its underlying generator is G. Since $Tn(N) = Tn(N')$, P_c is also a trinet of N'. Since N' and P_c are both simple level-2 networks, we have by Lemma 2.30 that P_c is a crucial trinet of N'. Then, again by Lemma 2.30, G is the underlying generator of N'.

The remainder of the proof is divided into four different cases, based on the four level-2 generators $2a$, $2b$, $2c$ and $2d$. These generators can be found in Figure 3.1 with the labels of their sides. For each generator G we show that if the generators of N' and N are the same, the networks N' and N are the same.

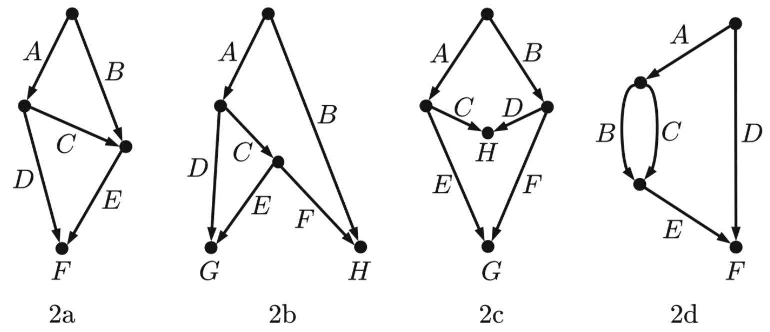


Figure 3.1: Generators $2a, 2b, 2c$ and $2d$ with labelled sides [5]

Case $G = 2a$

First, observe that there are no symmetries, i.e. no relabelling of the sides of $2a$ gives an isomorphic generator. Note that each set of crucial sides of $2a$ has only one element, namely F. Let x be the leaf on side F in N. Since x is then the leaf on side F in every crucial trinet of N, and since these crucial trinets are exhibited by N' (since $Tn(N') = Tn(N)$), and since there are no symmetries, it follows that x is also the leaf on side F in N'.

Now, consider any side $S \neq F$ of N and any leaf y on that side. Consider any crucial trinet P_c of N containing y. Then, y is on side S in P_c and, since P_c is exhibited by N' and there are no symmetries, y is on side S in N'. Hence, each leaf is on the same side in N' as it is in N.

It remains to show that the leaves on each side are in the same order in N and N'. Consider a side S of N with at least two leaves and two leaves y, z on that side such that z is below y. It follows that z is below y in the crucial trinet on $\{x, y, z\}$ and from that it follows that z is below y in N'.

We conclude that $N' = N$ since both networks have the same underlying generator, the same leaves on each side, and the same order of the leaves on each side.

Case $G = 2b$

Again, there are no symmetries. Note that in this case each set of crucial sides of $2b$ has two elements, namely G and H. Let x be the leaf on side G, y the leaf on side H and z a leaf on some other side S in N. Then, the trinet P_c on $\{x, y, z\}$ is crucial and, since there are no symmetries, it follows that leaves x, y, z are, respectively, on sides G, H, S in P_c and hence in N'.

Consequently, all leaves are on the same side in N' as in N, since S can be any side.

Now, we will show that leaves on each side are in the same order in N and N'. First, we look to the sides C, D and E, then to the sides A, B and F. Note that for sides A and C it does not matter which trinet we use since leaves x and y are both below sides A and C. So, first consider two leaves z, z' that are both on side C, D or E in N. Also, consider the (non-crucial) trinet P_1 on $\{x, z, z'\}$, which is a simple level-1 network. Now, we have that the leaves z and z' are on the same side of P_1. Moreover, if z is below z' in N, then z is below z' in P_1. Then, since $Tn(N') = Tn(N)$, we also have that z is below z' in N'. So, the order of the leaves on each of the sides C, D and E are the same in N and N'. Note that we indeed needed the leaf x to distinguish the order of the leaves, because without the leaf x a part of the generator would be left out and the leaves z and z' would become 'cherry's' without any order.

Now, consider two leaves q, q' that are both on side A, B or F in N. Also, consider the (non-crucial) trinet P_2 on $\{y, q, q'\}$, which is a simple level-1 network. As before, if q is below q' in N, then q is below q' in P_2 and hence in N', as wanted. So, the order of the leaves on each of the sides A, B and F are the same in N and N'.

Now, it follows that $N = N'$ since both networks have the same underlying generator, the same leaves on each side, and the same order of the leaves on each side.

Case $G = 2c$

In this case there is some symmetry since sides A, C and E can be interchanged with sides B, D and F, respectively, to obtain an isomorphic generator. Similarly, sides C, H and D can be interchanged with sides E, G and F, respectively, again yielding an isomorphic generator.

Note that a set of crucial sides consists again of two elements. Let x be on side G, y on side H and z on some other side S in N. Then, the crucial trinet P_c on $\{x, y, z\}$ implies that x and y are on side G and H in N', as we saw for case $2b$. Note that x and y can be interchanged because of the symmetry.

Assume without loss of generality that x is on side G and y is on side H in N'. Note that x and y are now fixed. So, again using trinet P_c, it follows that z is on side A or B in N' if it is on side A or B in N. Similarly, z is on side C or D in N' if it is on side C or D in N. Also, z is on side E or F in N' if it is on side E or F in N. So now we have fixed leaves x and y, there are less symmetries, namely only the symmetries between the left and the right of generator $2c$ (see Figure 3.1).

Now, we look to the sides of the leaves and the order of the leaves on each side. Consider two leaves z, z' that are both on A, B, C or D of N.

In view of the trinet on $\{y, z, z'\}$ where z and z' are on the same side and using $Tn(N') = Tn(N)$, z and z' are also on the same side of N' and in the same order as in N. Note here is not proved yet that the sides in N and N' are the same. Similarly, for two leaves z, z' that are both on side E or F. Also, the trinet $\{x, z, z'\}$ implies that z and z' are on the same side of N' and in the same order as in N. Thus, leaves that are on the same side in N are on the same side in N' and in the same order. Note that sides A and B of N can also be discussed using the trinet $\{x, z, z'\}$ instead of the trinet $\{y, z, z'\}$.

First, assume that there is at least one leaf on side A in N and that the leaves that are on side A in N are on side A in N'. Let a be one such leaf on side A in N.

Let c be a leaf on side C in N. Earlier we saw that then c is on side C or D in N'. Consider the trinet on $\{a, c, y\}$, which is a simple level-1 network. Then, a and c are on the same side of this trinet. Now, since $Tn(N') = Tn(N)$, c is on side C in N'.

Let z be a leaf on side $S \in \{B, D\}$ in N. Again, if z is on side B in N, then z is on side A or B in N'. Also, if z is on side D in N, then z is on side C or D in N'. Consider the trinet on $\{a, z, y\}$, which is a simple level-1 network. Then, a and z are on different sides of this trinet. Now, since $Tn(N') = Tn(N)$, z is on side S in N'.

Let z be a leaf on side E in N. Again, if z is on side E in N, then z is on side E or F in N'. Consider the trinet on $\{a, z, x\}$, which is a simple level-1 network. Then, a and z are on the same side of this trinet. Now, since $Tn(N') = Tn(N)$, z is on side E in N'.

Let z be a leaf on side F in N. Again, if z is on side F in N, then z is on side E or F in N'. Consider the trinet on $\{a, z, x\}$, which is a simple level-1 network. Then, a and z are on different sides of this trinet. Now, since $Tn(N') = Tn(N)$, z is on side F in N'.

So, all leaves are on the same side in N' as in N. It follows that $N = N'$ because all leaves are on the same side, in the same order.

Now, assume that the leaves that are on side A in N are not on side A in N'. Earlier we saw that these leaves then are on side B in N'. Then, we can argue in exactly the same way that the leaves that are on sides B, C, D, E, F in N are, respectively, on sides A, D, C, F, E in N'. Hence, again $N = N'$ by relabelling the sides appropriately (interchanging labels A and B, labels C and D, labels E and F).

Finally, if there is no leaf on side A in N, then there is a leaf on one of the sides B, C, D, E, F in N (since sides G and H have both one leaf and $|X| \geq 3$) and we can apply similar arguments based on that leaf as we did for the leaf on side A.

Now, we can conclude that $N = N'$, since (after possibly relabelling sides) both networks have the same underlying generator, the same leaves on each side, and the same order of the leaves on each side.

Case $G = 2d$

In this case the only symmetry is that sides B and C can be interchanged with C and B, respectively. Note that a set of crucial sides consists of two elements. Let x be the leaf on side F, y a leaf on side B or C and z a leaf on some side $S \in \{A, B, C, D, E\}$ in N. Note there exists at least one such leaf z since $|X| \geq 3$. Then, by the crucial trinet on $\{x, y, z\}$ and since $Tn(N') = Tn(N)$, x is on side F and y is on side B or C. So, the sides of the leaves are determined except of the symmetries. Assume y is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides B and C. Now, it follows by the crucial trinet on $\{x, y, z\}$ that z is on side S in N'. So, each leaf is on the same side in N' as in N.

Now, we only have to check the order of the leaves on same sides. Consider two leaves z, z' that are on the same side in N. Then, the trinet on $\{x, z, z'\}$ implies that the order of z and z' is the same in N' as in N, since $Tn(N') = Tn(N)$.

Also for this generator, we can conclude that $N' = N$, since (after possibly relabelling sides B and C) both networks have the same leaves on the same sides in the same order. □

Now, we have proved Lemma 3.2, we can prove the main result of this chapter, which is stated below.

Corollary 3.3 (Corollary 1 in [5]). *Every binary recoverable level-2 network N on X, with $|X| \geq 3$, is encoded by its set of trinets $Tn(N)$.*

Proof. Follows from Theorem 2.22, Lemma 3.2 and the fact that level-1 networks are encoded by their trinets [5]. □

Chapter 4

Not all level-3 networks are weakly encoded by trinets

In this chapter we will define when phylogenetic networks are weakly encoded and we will explain why not all level-3 networks are weakly encoded by trinets.

First, we define when phylogenetic networks are weakly encoded:

Definition 4.1. A class of phylogenetic networks $\mathcal{C}$ is **weakly encoded** by trinets if there are no 2 recoverable phylogenetic networks N and N', with $N \neq N'$, in class $\mathcal{C}$ such that $Tn(N) = Tn(N')$.

In Chapter 5 will be explained why we introduce this definition. For now we will just use the definition.

Consider the binary, simple level-3 networks as in Figure 4.1.

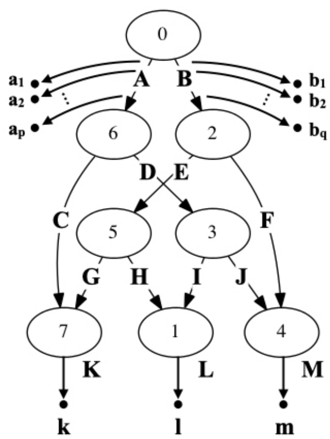


Figure 4.1: A binary, simple level-3 network with underlying generator 3.8, one leaf on each of the sides K, L and M, at least one leaf on side A or B ($p + q \geq 1$) and no leaves on sides C, D, E, F, G, H, I and J

These networks have underlying generator 3.8, at least one leaf on side A or B and no leaves on sides C, D, E, F, G, H, I and J. Note that generator 3.8 can be found in Appendix A.9.

Now, consider the network N that is given in the following figure:

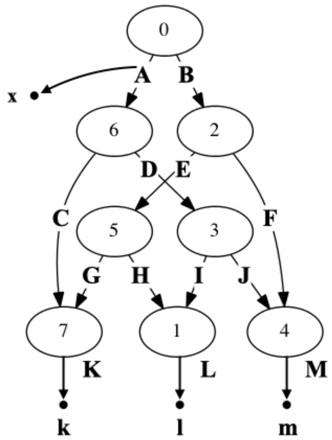


Figure 4.2: A binary, simple level-3 network N as in Figure 4.1

This network N is a network as in Figure 4.1 since it has underlying generator 3.8, one leaf on side A and no leaves on side C, D, E, F, G, H, I and J. The network N has four leaves, namely k, l, m and x. So, N has four different trinets which are given in Figures 4.3 and 4.4:

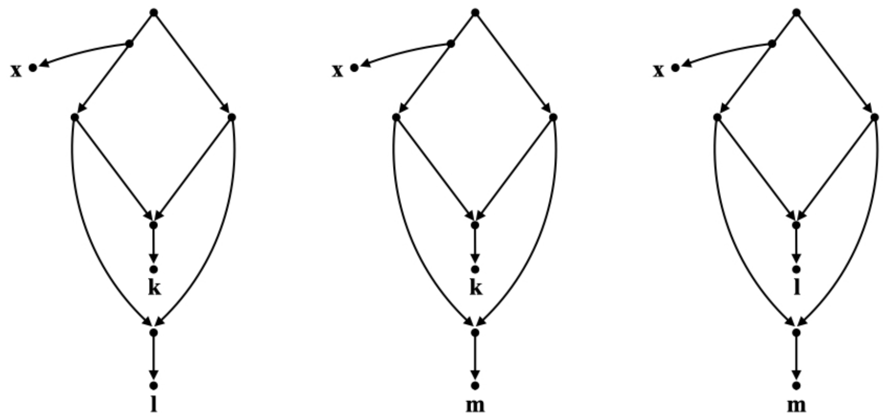


Figure 4.3: Three of the four trinets exhibited by N (and N')

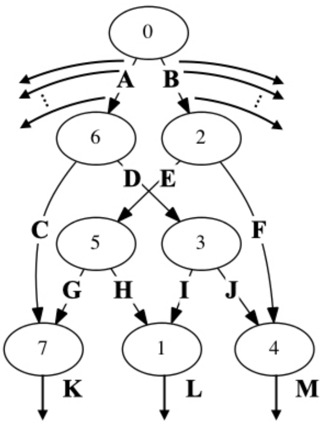


Figure 4.4: One of the four trinets exhibited by N (and N')

However, there is another network N', which is given in Figure 4.5, that has the same trinets as N. In other words, $Tn(N') = Tn(N)$ while $N \neq N'$. Intuitively, we can see that we can not distinguish leaves on sides A and B using trinets.

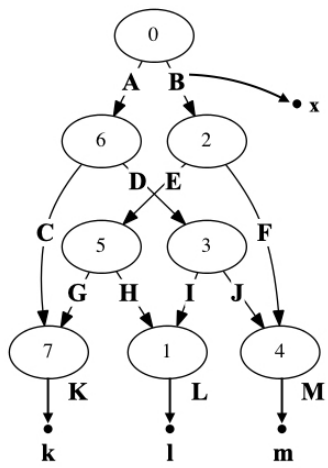


Figure 4.5: A binary, simple level-3 network N' as in Figure 4.1

Note that N and N' are both in the class of binary, simple level-3 networks with at least three leaves. Now, the class of binary, simple level-3 networks can not be weakly encoded by trinets.

We can show in the same way for each network N that is as in Figure 4.1 that there is another binary, simple level-3 network N' for which holds that $Tn(N') = Tn(N)$. So, not all simple level-3 networks are weakly encoded by trinets.

Now, consider the level-3 networks with a biconnected component as in

Figure 4.6. These networks have a biconnected component with underlying generator 3.8, at least one cut-arc on side A or B and no cut-arcs on sides C, D, E, F, G, H, I and J. Therefore, the level-3 networks with a biconnected component as in Figure 4.6 are also not weakly encoded by trinets.

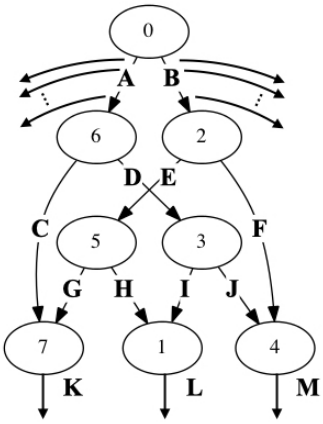


Figure 4.6: A binary, simple level-3 network with underlying generator 3.8, one cut-arc on each of the sides K, L and M, at least one cut-arc on side A or B and no cut-arcs on sides C, D, E, F, G, H, I and J

Even though not all level-3 networks are weakly encoded by trinets, most level-3 networks are weakly encoded by trinets, as we will see in the next chapter.

Chapter 5

Most level-3 networks are weakly encoded by trinets

In this chapter we will prove the corollary that the class of binary recoverable level-3 networks, except for networks with biconnected components as in Figure 4.6, is weakly encoded by trinets. This is the main result of this chapter. Before we will prove this corollary, Lemmas 5.1 and 5.2 will be discussed. The proof of Lemma 5.2 is based on the proof of Lemma 3.2 for level-2 networks, which is also given in [5]. So, some parts are the same or similar as in the proof of Lemma 3.2.

First, we give the following lemma with its proof:

Lemma 5.1. *If G is a level-3 generator, then it has a set of crucial sides of size at most 3. Hence, every simple level-3 network N has at least one crucial trinet.*

Proof. Let G be a level-3 generator. Then, G has exactly 3 reticulations with indegree 2 and outdegree at most 1. We know that a set of crucial sides contains all vertices with indegree 2 and outdegree 0 together with at least one arc of each pair of parallel arcs. Each vertex with indegree 2 and outdegree 0 is a reticulation of G. Also, each pair of parallel arcs gives a reticulation in G. Now, since G has exactly 3 reticulations, a set of crucial sides is of size at most 3. A crucial trinet contains at least one leaf on each side in some set of crucial sides. Since a set of crucial sides is of size at most 3, every simple level-3 network N has at least one crucial trinet. □

In Observation 3.1 we saw that a level-2 generator has a set of crucial sides of size at most 2. Now, we know that a level-3 generator has a set of crucial sides of size at most 3. This is an important difference between level-2 and level-3 generators. Now, there can be leaves that are not contained in any crucial trinet. This difference makes the proof for level-3 networks more complicated, as we will see in the proof of Lemma 5.2. But, we still know that every simple level-3 network has at least one crucial trinet. This will be a very useful result during the proof of Lemma 5.2.

In the beginning of the proof of Lemma 3.2 we proved that N' is a binary, simple level-2 network. For level-3 networks this is much more difficult, if it is even possible. To prevent this problem we introduced the definition of weakly encoded, which can be found in Chapter 4. If we look to weakly encoded instead of encoded itself for level-3 networks, then we do not have the problem that we have to prove that N' is a binary, simple level-3 network.

Therefore, we will prove the lemma that the class of binary, simple level-3 networks, except for networks as in Figure 4.1, is weakly encoded. This lemma is a weaker result for level-3 networks than Lemma 3.2 is for level-2 networks. Note that the exceptions to the lemma are explained in Chapter 4. Further, note that in the proof of the lemma below 65 different generators has to be discussed. This makes the proof much longer than the proof of Lemma 3.2 where just 4 different generators had to be discussed.

Lemma 5.2. *The class of binary, simple level-3 networks with at least three leaves, except for networks as in Figure 4.1 (i.e. with underlying generator* 3.8*, at least one leaf on side A or B and no leaves on sides C, D, E, F, G, H, I and J), is weakly encoded by trinets.*

Proof. Assume that the class of binary, simple level-3 networks with at least three leaves, except for networks as in Figure 4.1 (i.e. with underlying generator 3.8, at least one leaf on side A or B and no leaves on sides C, D, E, F, G, H, I and J), is not weakly encoded by trinets. Then, there are 2 recoverable, binary, simple level-3 networks N and N', which are not as in Figure 4.1, such that $Tn(N) = Tn(N')$. We will show that $N = N'$, which is a contradiction, so then the lemma follows.

Now, let G be the underlying generator of N. First, we show that G is also the underlying generator of N'. By Lemma 5.1, N has at least one crucial trinet P_c. By Lemma 2.30, P_c is a simple level-3 network and its underlying generator is G. Since $Tn(N) = Tn(N')$, P_c is also a trinet of N'. Since N' and P_c are both simple level-3 networks, we have by Lemma 2.30 that P_c is a crucial trinet of N'. Then, again by Lemma 2.30, G is the underlying generator of N'.

The remainder of the proof is divided into 65 different cases, based on the 65 level-3 generators (3.1, 3.2, ..., 3.65) that can be found in Appendix A. Note that these generators are divided in 11 groups. For each generator G we show that if the generators of N' and N are the same, the networks N' and N are the same. To do this we first prove that each generator has the same leaves on the same sides in N' as in N. Note that we will follow the cases for the generators in the way as they are ordered in Appendix B. Also, note that for generator 3.8 some binary, simple level-3 networks are excluded. Further, note that the four level-2 generators $2a$, $2b$, $2c$ and $2d$ can be found in Figure 3.1. Finally, we will prove that the leaves on each side are in the same order in N and N' in order to conclude that $N = N'$ (after possibly relabelling sides).

Group 1: 1 crucial side, no symmetry

Each of the generators 3.15, 3.19, 3.20, 3.23, 3.24, 3.25, 3.32 and 3.62 has 1 crucial side and no symmetry. Therefore, we define the set $T = \{15, 19, 20, 23, 24, 25, 32, 62\}$. Now, let $t \in R$, and consider generator $3.t$. Note that this generator with one labelled side can be found in Appendix A.1. Further, note that generator 3.15 can also be found in Figure 5.1.

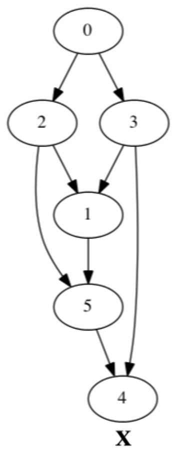


Figure 5.1: Generator 3.15 with one labelled side

Observe that there are no symmetries, i.e. no relabelling of the sides of $3.t$ gives an isomorphic generator. Note that each set of crucial sides of $3.t$ has only one element. We label this crucial side as side X. Let x be the leaf on side X in N. Since x is then the leaf on side X in every crucial trinet of N, and since these crucial trinets are exhibited by N' (since $Tn(N') = Tn(N)$), and since there are no symmetries, it follows that x is also the leaf on side X in N'.

Now, consider any side $S \neq X$ of N and any leaf y on that side. Consider any crucial trinet P_c of N containing y. Then, y is on side S in P_c and, since P_c is exhibited by N' and there are no symmetries, y is on side S in N'. Hence, each leaf is on the same side in N' as it is in N.

Group 2: 2 crucial sides, no symmetry

Each of the generators 3.4, 3.5, 3.9, 3.12, 3.13, 3.17, 3.21, 3.27, 3.29, 3.30, 3.33, 3.34, 3.35, 3.41, 3.42, 3.43, 3.44, 3.48, 3.49, 3.54, 3.55 and 3.59 has 2 crucial sides and no symmetry. Therefore, we define the set $T = \{4, 5, 9, 12, 13, 17, 21, 27, 29, 30, 33, 34, 35, 41, 42, 43, 44, 48, 49, 54, 55, 59\}$. Now, let $t \in T$, and consider generator $3.t$. Note that this generator with two labelled sides can be found in Appendix A.2. Further, note that generator 3.4 can also be found in Figure 5.2.

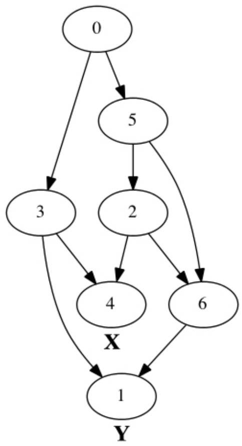


Figure 5.2: Generator 3.4 with two labelled sides

Again, observe that there are no symmetries, i.e. no relabelling of the sides of $3.t$ gives an isomorphic generator. Note that in this case a set of crucial sides has two elements. We label this crucial sides as sides X and Y. Let x be the leaf on side X, y the leaf on side Y and z a leaf on some other side S in N. Then, the trinet P_c on $\{x, y, z\}$ is crucial and, since there are no symmetries, it follows that leaves x, y, z are, respectively, on side X, Y, S in P_c and hence, since $Tn(N') = Tn(N)$, in N'. Consequently, all leaves are on the same side in N' as in N, since S can be any side.

Group 3: 3 crucial sides, no symmetry

Each of the generators 3.7, 3.10, 3.14, 3.39, 3.45, 3.46, 3.50, 3.51, 3.52, 3.57 and 3.60 has 3 crucial sides and no symmetry. First, generator 3.7 will be discussed, then the other generators (which are similar to generator 3.7).

Generator 3.7 **(group 3a)**

In Figure 5.3 generator 3.7 can be found with the labels of its sides. Note that generator 3.7 can also be found in Appendix A.3.

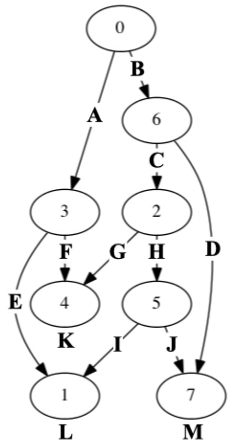


Figure 5.3: Generator 3.7 with labelled sides

Again, observe that there are no symmetries, i.e. no relabelling of the sides of 3.7 gives an isomorphic generator. Note that in this case a set of crucial sides has three elements, namely sides K, L and M. Let k be the leaf on side K, l the leaf on side L and m the leaf on side M in N. Then, the trinet P_c on $\{k, l, m\}$ is crucial and, since there are no symmetries, it follows that leaves k, l, m are, respectively, on side K, L, M in P_c, and hence, since $Tn(N') = Tn(N)$, in N'.

Now, we will prove that the other sides have also the same leaves in N and N'. For each side S of these sides we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{k, l, p_S\}, \{k, m, p_S\}, \{l, m, p_S\}$, where p_S is a leaf on side S in N. Note that for each side the used results can be found in Table 5.1.

Let p_A be a leaf on side A in N. Consider the trinet $T_{A,1}$ on $\{k, l, p_A\}$. Note that since there is no leaf on the crucial side M, a part of the underlying generator is left out, namely sides D, J and M. Now, $T_{A,1}$ is a simple level-2 network with underlying generator $2c$. Also, $T_{A,1} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{A,1}$ has the underlying generator $2c$, and since $T_{A,1} \in Tn(N)$ and $T_{A,1} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2c$ of trinet $T_{A,1}$ from the underlying generator 3.7 (after sides D, J and M being left out as mentioned before) for N and N'.

Note that the sides B and C are suppressed to one side. Also, the sides H and I are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as B/C and H/I. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2c$, after possibly relabelling sides A, E, F with, respectively, sides $B/C, H/I, G$. Remember that p_A is a leaf on side A in N. Now, p_A is a leaf on side A or B/C of generator $2c$. This means that p_A is a leaf on side A, B, C or D in N'. Note that we have to take into account side D here because of the part of generator 3.7 that is left out, and since there can be a leaf on side D, which is on side $(B/C)'$ in generator $2c$. The $'$ here denotes that a side in generator $2c$ can be labelled differently for N and N'. So, from trinet $T_{A,1}$ we get that p_A is a leaf on side A, B, C or D in N'.

Now, consider the trinet $T_{A,2}$ on $\{k, m, p_A\}$. Note that since there is no leaf on the crucial side L, a part of the underlying generator is left out, namely sides E, I and L. Now, $T_{A,2}$ is a simple level-2 network with underlying generator $2b$. Also, $T_{A,2} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{A,2}$ has the underlying generator $2b$, and since $T_{A,2} \in Tn(N)$ and $T_{A,2} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2b$ of trinet $T_{A,2}$ from the underlying generator 3.7 (after sides E, I and L being left out as mentioned before) for N and N'. Note that the sides A and F are suppressed to one side. Also, the sides H and J are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/F and H/J. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2b$. Note that the underlying generator $2b$ has no symmetries. Remember that p_A is a leaf on side A in N. Now, p_A is a leaf on side A/F of generator $2b$. This means that p_A is a leaf on side A, E or F in N'. Note that we have to take into account side E here because of the part of generator 3.7 that is left out, and since there can be a leaf on side E, which is on side A/F in generator $2b$. So, from trinet $T_{A,2}$ we get that p_A is a leaf on side A, E or F in N'. Remember that using trinet $T_{A,1}$ we got that p_A is a leaf on side A, B, C or D in N'. Since side A is the only side that follows from both trinets $T_{A,1}$ and $T_{A,2}$, we can conclude that p_A is a leaf on side A in N'.

Let $p_{B^\bullet}$ be a leaf on side $B^\bullet$ in N, with $B^\bullet \in \{B, C, D, G\}$. Consider the trinet $T_{B^\bullet,2}$ on $\{k, m, p_{B^\bullet}\}$. Note that since there is no leaf on the crucial side L, a part of the underlying generator is left out, namely sides E, I and L. Now, $T_{B^\bullet,2}$ is a simple level-2 network with underlying generator $2b$. Also, $T_{B^\bullet,2} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{B^\bullet,2}$ has the underlying generator $2b$, and since $T_{B^\bullet,2} \in Tn(N)$ and $T_{B^\bullet,2} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2b$ of trinet $T_{B^\bullet,2}$ from the underlying generator 3.7 (after sides E, I and L being left out as mentioned before) for N and N'. Note that sides A and F are suppressed to one side. Also, the sides H and J are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/F and H/J. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2b$. Note that the underlying generator $2b$ has no symmetries. Remember that $p_{B^\bullet}$ is a leaf on side $B^\bullet$ in N. Now, $p_{B^\bullet}$ is a leaf on side $B^\bullet$ of generator $2b$. This means that $p_{B^\bullet}$ is a leaf on side $B^\bullet$ in N'. Note that we do not have to take into account other sides since the sides that are left out do not encounter side $B^\bullet$ of the underlying generator $2b$. So, from trinet $T_{B^\bullet,2}$ we get that $p_{B^\bullet}$ is a leaf on side $B^\bullet$ in N'.

Let p_E be a leaf on side E in N. Consider the trinet $T_{E,1}$ on $\{k, l, p_E\}$. Note that since there is no leaf on the crucial side M, a part of the underlying generator is left out, namely sides D, J and M. Now, $T_{E,1}$ is a simple level-2 network with underlying generator $2c$. Also, $T_{E,1} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{E,1}$ has the underlying generator $2c$, and since $T_{E,1} \in Tn(N)$ and $T_{E,1} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2c$ of trinet $T_{E,1}$ from the underlying generator 3.7 (after sides D, J and M being left out as mentioned before) for N and N'. Note that the sides B and C are suppressed to one side. Also, the sides H and I are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as B/C and H/I. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2c$, after possibly relabelling sides A, E, F with, respectively, sides $B/C, H/I, G$. Remember that p_E is a leaf on side E in N. Now, p_E is a leaf on side E or H/I of generator $2c$. This means that p_E is a leaf on side E, H, I or J in N'. Note that we have to take into account side J here because of the part of generator 3.7 that is left out, and since there can be a leaf on side J, which is on side $(H/I)'$ in generator $2c$. The $'$ here denotes that a side in generator $2c$ can be labelled differently for N and N'. So, from trinet $T_{E,1}$ we get that p_E is a leaf on side E, H, I or J in N'.

Now, consider the trinet $T_{E,2}$ on $\{k, m, p_E\}$. Note that since there is no leaf on the crucial side L, a part of the underlying generator is left out, namely sides E, I and L. Note that the leaf p_E is not left out since it is on side A/F which will be defined later in the proof. Now, $T_{E,2}$ is a simple level-2 network with underlying generator $2b$. Also, $T_{E,2} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{E,2}$ has the underlying generator $2b$, and since $T_{E,2} \in Tn(N)$ and $TnE, 2inTn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2b$ of trinet $T_{E,2}$ from the underlying generator 3.7 (after sides E, I and L being left out as mentioned before) for N and N'. Note that the sides A and F are suppressed to one side. Also, the sides H and J are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/F and H/J. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2b$. Note that the underlying generator $2b$ has no symmetries. Remember that p_E is a leaf on side E in N. Now, p_E is a leaf on side A/F of generator $2b$. This means that p_E is a leaf on side A, E or F in N'. Note that we indeed have to take into account side E here because of the part of generator 3.7 that is left out, and since there is a leaf on side E, which is on side A/F in generator $2b$. So, from trinet $T_{E,2}$ we get that p_E is a leaf on side A, E or F in N'. Remember that using trinet $T_{E,1}$ we got that p_E is a leaf on side E, H, I or J in N'. Since side E is the only side that follows from both trinets $T_{E,1}$ and $T_{E,2}$, we can conclude that p_E is a leaf on side E in N'.

Let p_F be a leaf on side F in N. Consider the trinet $T_{F,1}$ on $\{k, l, p_F\}$. Note that since there is no leaf on the crucial side M, a part of the underlying generator is left out, namely sides D, J and M. Now, $T_{F,1}$ is a simple level-2 network with underlying generator $2c$. Also, $T_{F,1} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{F,1}$ has the underlying generator $2c$, and since $T_{F,1} \in Tn(N)$ and $T_{F,1} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2c$ of trinet $T_{F,1}$ from the underlying generator 3.7 (after sides D, J and M being left out as mentioned before) for N and N'. Note that the sides B and C are suppressed to one side. Also, the sides H and I are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as B/C and H/I. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2c$, after possibly relabelling sides A, E, F with, respectively, sides $B/C, H/I, G$. Remember that p_F is a leaf on side F in N. Now, p_F is a leaf on side F of generator $2c$. This means that p_F is a leaf on side F or G in N'. Note that we do not have to take into account other sides here since the sides that are left out do not encounter side F of the underlying generator $2c$. So, from trinet $T_{F,1}$ we get that p_F is a leaf on side F or G in N'.

Now, consider the trinet $T_{F,2}$ on $\{k, m, p_F\}$. Note that since there is no leaf on the crucial side L, a part of the underlying generator is left out, namely sides E, I and L. Now, $T_{F,2}$ is a simple level-2 network with underlying generator $2b$. Also, $T_{F,2} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{F,2}$ has the underlying generator $2b$, and since $T_{F,2} \in Tn(N)$ and $T_{F,2} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2b$ of trinet $T_{F,2}$ from the underlying generator 3.7 (after sides E, I and L being left out as mentioned before) for N and N'. Note that the sides A and F are suppressed to one side. Also, the sides H and J are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/F and H/J. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2b$. Note that the underlying generator $2b$ has no symmetries. Remember that p_F is a leaf on side F in N. Now, p_F is a leaf on side A/F of generator $2b$. This means that p_F is a leaf on side A, E or F in N'. Note that we have to take into account side E here because of the part of generator 3.7 that is left out, and since there can be a leaf on side E, which is on side A/F in generator $2b$. So, from trinet $T_{F,2}$ we get that p_F is a leaf on side A, E or F in N'. Remember that using trinet $T_{F,1}$ we got that p_F is a leaf on side F or G in N'. Since side F is the only side that follows from both trinets $T_{F,1}$ and $T_{F,2}$, we can conclude that p_F is a leaf on side F in N'.

Let p_H be a leaf on side H in N. Consider the trinet $T_{H,1}$ on $\{k, l, p_H\}$. Note that since there is no leaf on the crucial side M, a part of the underlying generator is left out, namely sides D, J and M. Now, $T_{H,1}$ is a simple level-2 network with underlying generator $2c$. Also, $T_{H,1} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{H,1}$ has the underlying generator $2c$, and since $T_{H,1} \in Tn(N)$ and $T_{H,1} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2c$ of trinet $T_{H,1}$ from the underlying generator 3.7 (after sides D, J and M being left out as mentioned before) for N and N'. Note that the sides B and C are suppressed to one side. Also, the sides H and I are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as B/C and H/I. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2c$, after possibly relabelling sides A, E, F with, respectively, sides $B/C, H/I, G$. Remember that p_H is a leaf on side H in N. Now, p_H is a leaf on side E or H/I of generator $2c$. This means that p_H is a leaf on side E, H, I or J in N'. Note that we have to take into account side J here because of the part of generator 3.7 that is left out, and since there can be a leaf on side J, which is on side $(H/I)'$ in generator $2c$. The $'$ here denotes that a side in generator $2c$ can be labelled differently for N and N'. So, from trinet $T_{H,1}$ we get that p_H is a leaf on side E, H, I or J in N'.

Now, consider the trinet $T_{H,3}$ on $\{l, m, p_H\}$. Note that since there is no leaf on the crucial side K, a part of the underlying generator is left out, namely sides F, G and K. Now, $T_{H,3}$ is a simple level-2 network with underlying generator $2b$. Also, $T_{H,3} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{H,3}$ has the underlying generator $2b$, and since $T_{H,3} \in Tn(N)$ and $T_{H,3} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator $2b$ of trinet $T_{H,3}$ from the underlying generator 3.7 (after sides F, G and K being left out as mentioned before) for N and N'. Not that the sides A and E are suppressed to one side. Also, the sides C and H are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/E and C/H. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator $2b$. Note that the underlying generator $2b$ has no symmetries. Remember that p_H is a leaf on side H in N. Now, p_H is a leaf on side C/H of generator $2b$. This means that p_H is a leaf on side C, G or H in N'. Note that we have take into account side G here because of the part of generator 3.7 that is left out, and since there can be a leaf on side G, which is on side C/H in generator 2b. So, from trinet $T_{H,3}$ we get that p_H is a leaf on side C, G or H in N'. Remember that using trinet $T_{H,1}$ we got that p_H is a leaf on side E, H, I or J in N'. Since side H is the only side that follows from both trinets $T_{H,1}$ and $T_{H,3}$, we can conclude that p_H is a leaf on side H in N'.

Let $p_{I^\bullet}$ be a leaf on side $I^\bullet$ in N, with $I^\bullet \in \{I, J\}$. Consider the trinet $T_{I^\bullet,3}$ on $\{l, m, p_{I^\bullet}\}$. Note that since there is no leaf on the crucial side K, a part of the underlying generator is left out, namely sides F, G and K. Now, $T_{I^\bullet,3}$ is a simple level-2 network with underlying generator 2b. Also, $T_{I^\bullet,3} \in Tn(N')$ since $Tn(N') = Tn(N)$. Since N and N' have the same underlying generator 3.7, and since $T_{I^\bullet,3}$ has the underlying generator 2b, and since $T_{I^\bullet,3} \in Tn(N)$ and $T_{I^\bullet,3} \in Tn(N')$, the same indegree-1 outdegree-1 vertices were suppressed to get the underlying generator 2b of trinet $T_{I^\bullet,3}$ from the underlying generator 3.7 (after sides F, G and K being left out as mentioned before) for N and N'. Note that sides A and E are suppressed to one side. Also, the sides C and H are suppressed to one side. So, we have two 'combined' sides. We label these sides, respectively, as A/E and C/H. From the proof of Lemma 3.2 we know there are the same leaves on each side of the underlying generator 2b. Note that the underlying generator 2b has no symmetries. Remember that $p_{I^\bullet}$ is a leaf on side $I^\bullet$ in N. Now, $p_{I^\bullet}$ is a leaf on side $I^\bullet$ of generator 2b. This means that $p_{I^\bullet}$ is a leaf on side $I^\bullet$ in N'. Note that we do not have to take into account other sides since the sides that are left out do not encounter side $I^\bullet$ of the underlying generator 2b. So, from trinet $T_{I^\bullet,3}$ we get that $p_{I^\bullet}$ is a leaf on side $I^\bullet$ in N'.

Leaf on side S in N	Trinet $T_{S,1}$ (gen. 2c)	Trinet $T_{S,2}$ (gen. 2b)	Trinet $T_{S,3}$ (gen. 2b)	Result for N'
A	$A \vee B \vee C \vee D$	$A \vee E \vee F$		A
B		B		B
C		C		C
D		D		D
E	$E \vee H \vee I \vee J$	$A \vee E \vee F$		E
F	$F \vee G$	$A \vee E \vee F$		F
G		G		G
H	$E \vee H \vee I \vee J$		$C \vee G \vee H$	H
I			I	I
J			J	J

Table 5.1: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.7$)

So, generator 3.7 has the same leaves on each side for N and N'. Note that for each non-crucial side the used results can be found in Table 5.1.

The other generators (group 3b)

In Appendix A.3 generators 3.10, 3.14, 3.39, 3.45, 3.46, 3.50, 3.51, 3.52, 3.57 and 3.60 can be found with the labels of their sides. These generators will also be given later in the proof for this group of generators.

Observe that these generators have no symmetries. Also, note that for all these generators a set of crucial sides has three elements. These crucial sides are labelled with K, L and M. Therefore, these generators are similar to generator 3.7 that is discussed before.

First, we look to leaves on sides K, L and M for each of these generators. Let k be the leaf on side K, l the leaf on side L and m the leaf on side M in N. Then, the trinet P_c on $\{k, l, m\}$ is crucial and, since there are no symmetries, it follows that leaves k, l, m are, respectively, on side K, L, M in P_c, and hence, since $Tn(N') = Tn(N)$, in N'.

Now, we will prove that the other sides have also the same leaves in N and N'. Since the generators have no symmetries, and since they have three crucial sides, this can be proved in similar way as we did for generator 3.7. In Tables 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10 and 5.11 the results for the different generators can be found. We can see that for each of the generators that if there is a leaf on a side S in N, then this leaf is on side S in N'. Note that in Figures 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11, 5.12 and 5.13 the generators with the labels of their sides can be found. So, for each generator the table with results is just below the generator itself.

First, some remarks about the used trinets in the tables. Note that each of the trinets that is marked with (Δ) consists of two level-1 generators. Since these trinets are not needed to get the wanted result, we will not discuss them.

Also, note that some trinets are marked with $(*)$. Such a trinet is not biconnected for each side S. Therefore, we can not use the reasoning for the underlying generator as we did before. By Corollary 3.3 and since the trinet is a level-2 network, we only have to look to the symmetry of the trinet. Further, note that leaves on sides A and B can not be distinguished using the trinet.

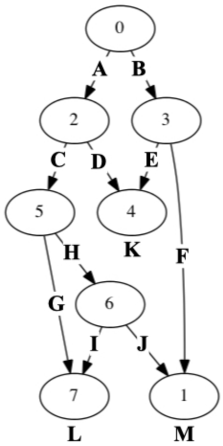


Figure 5.4: Generator 3.10 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2c$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A		$A \vee B$	$A \vee C \vee D$	A
B		$A \vee B$	$B \vee E \vee F$	B
C		$C \vee F \vee G \vee H \vee I \vee J$	$A \vee C \vee D$	C
D		$D \vee E$	$A \vee C \vee D$	D
E		$D \vee E$	$B \vee E \vee F$	E
F		$C \vee F \vee G \vee H \vee I \vee J$	$B \vee E \vee F$	F
G		$C \vee F \vee G \vee H \vee I \vee J$	G	G
H		$C \vee F \vee G \vee H \vee I \vee J$	H	H
I		$C \vee F \vee G \vee H \vee I \vee J$	I	I
J		$C \vee F \vee G \vee H \vee I \vee J$	J	J

Table 5.2: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.10$)

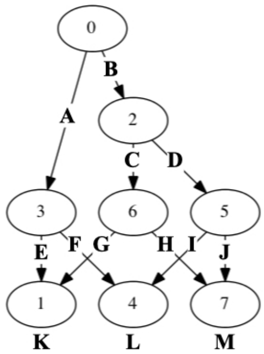


Figure 5.5: Generator 3.14 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2c$)	Trinet $T_{S,2}$ (gen. $2b$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	$A \vee B$	$A \vee E \vee F$	$A \vee E \vee F$	A
B	$A \vee B$	B	B	B
C	$C \vee E \vee G \vee H$	C	$C \vee G \vee H$	C
D	$D \vee F \vee I \vee J$	$D \vee I \vee J$	D	D
E	$C \vee E \vee G \vee H$	$A \vee E \vee F$	$A \vee E \vee F$	E
F	$D \vee F \vee I \vee J$	$A \vee E \vee F$	$A \vee E \vee F$	F
G	$C \vee E \vee G \vee H$	G	$C \vee G \vee H$	G
H	$C \vee E \vee G \vee H$	H	$C \vee G \vee H$	H
I	$D \vee F \vee I \vee J$	$D \vee I \vee J$	I	I
J	$D \vee F \vee I \vee J$	$D \vee I \vee J$	J	J

Table 5.3: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.14$)

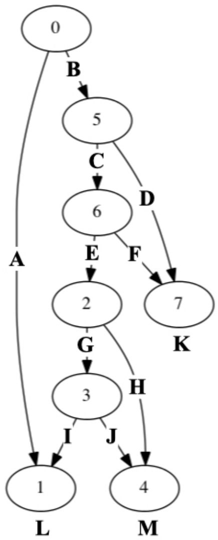


Figure 5.6: Generator 3.39 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ (Δ)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A		A	A
B	B		$B \vee C \vee D \vee E \vee F$	B
C	C		$B \vee C \vee D \vee E \vee F$	C
D	D		$B \vee C \vee D \vee E \vee F$	D
E	$E \vee G \vee H \vee I \vee J$		$B \vee C \vee D \vee E \vee F$	E
F	F		$B \vee C \vee D \vee E \vee F$	F
G	$E \vee G \vee H \vee I \vee J$		G	G
H	$E \vee G \vee H \vee I \vee J$		H	H
I	$E \vee G \vee H \vee I \vee J$		I	I
J	$E \vee G \vee H \vee I \vee J$		J	J

Table 5.4: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.39$)

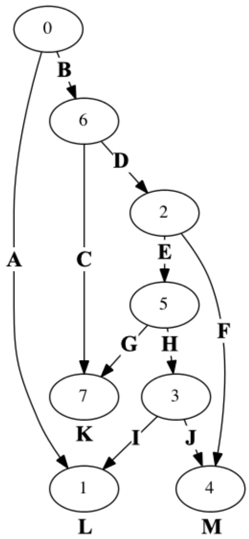


Figure 5.7: Generator 3.45 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ $(*)$	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A	$A \vee B$	A	A
B	B	$A \vee B$	$B \vee C \vee D$	B
C	C	C	$B \vee C \vee D$	C
D	$D \vee E \vee F$	D	$B \vee C \vee D$	D
E	$D \vee E \vee F$	E	$E \vee G \vee H$	E
F	$D \vee E \vee F$	F	F	F
G	G	G	$E \vee G \vee H$	G
H	$H \vee I \vee J$	$H \vee I \vee J$	$E \vee G \vee H$	H
I	$H \vee I \vee J$	$H \vee I \vee J$	I	I
J	$H \vee I \vee J$	$H \vee I \vee J$	J	J

Table 5.5: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.45$)

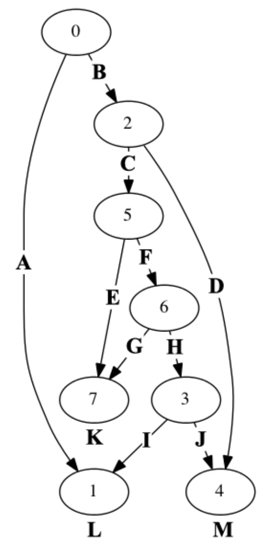


Figure 5.8: Generator 3.46 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ $(*)$	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A	$A \vee B$	A	A
B	$B \vee C \vee D$	$A \vee B$	B	B
C	$B \vee C \vee D$	C	$C \vee E \vee F \vee G \vee H$	C
D	$B \vee C \vee D$	D	D	D
E	E	E	$C \vee E \vee F \vee G \vee H$	E
F	F	F	$C \vee E \vee F \vee G \vee H$	F
G	G	G	$C \vee E \vee F \vee G \vee H$	G
H	$H \vee I \vee J$	$H \vee I \vee J$	$C \vee E \vee F \vee G \vee H$	H
I	$H \vee I \vee J$	$H \vee I \vee J$	I	I
J	$H \vee I \vee J$	$H \vee I \vee J$	J	J

Table 5.6: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.46$)

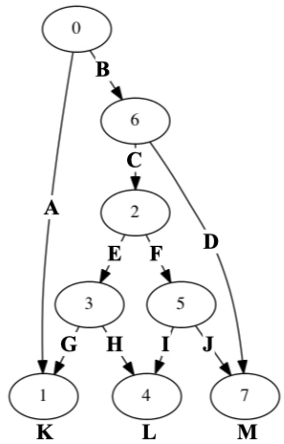


Figure 5.9: Generator 3.50 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ (gen. $2b$)	Trinet $T_{S,3}$ $(*)$	Result for N'
A	A	A	$A \vee B$	A
B	$B \vee C \vee D$	B	$A \vee B$	B
C	$B \vee C \vee D$	C	C	C
D	$B \vee C \vee D$	D	D	D
E	E	$E \vee G \vee H$	$E \vee G \vee H$	E
F	$F \vee I \vee J$	$F \vee I \vee J$	F	F
G	G	$E \vee G \vee H$	$E \vee G \vee H$	G
H	H	$E \vee G \vee H$	$E \vee G \vee H$	H
I	$F \vee I \vee J$	$F \vee I \vee J$	I	I
J	$F \vee I \vee J$	$F \vee I \vee J$	J	J

Table 5.7: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.50$)

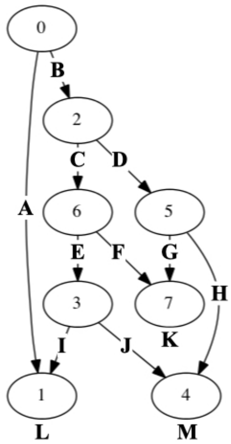


Figure 5.10: Generator 3.51 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ $(*)$	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A	$A \vee B$	A	A
B	B	$A \vee B$	B	B
C	C	$C \vee D$	$C \vee E \vee F$	C
D	$D \vee G \vee H$	$C \vee D$	$D \vee G \vee H$	D
E	$E \vee I \vee J$	$E \vee H \vee I \vee J$	$C \vee E \vee F$	E
F	F	$F \vee G$	$C \vee E \vee F$	F
G	$D \vee G \vee H$	$F \vee G$	$D \vee G \vee H$	G
H	$D \vee G \vee H$	$E \vee H \vee I \vee J$	$D \vee G \vee H$	H
I	$E \vee I \vee J$	$E \vee H \vee I \vee J$	I	I
J	$E \vee I \vee J$	$E \vee H \vee I \vee J$	J	J

Table 5.8: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.51$)

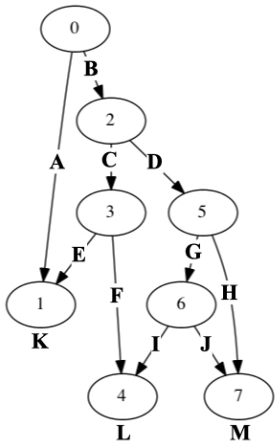


Figure 5.11: Generator 3.52 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ (Δ)	Trinet $T_{S,3}$ ($*$)	Result for N'
A	A		$A \vee B$	A
B	B		$A \vee B$	B
C	C		$C \vee E \vee F$	C
D	$D \vee G \vee H \vee I \vee J$		D	D
E	E		$C \vee E \vee F$	E
F	F		$C \vee E \vee F$	F
G	$D \vee G \vee H \vee I \vee J$		G	G
H	$D \vee G \vee H \vee I \vee J$		H	H
I	$D \vee G \vee H \vee I \vee J$		I	I
J	$D \vee G \vee H \vee I \vee J$		J	J

Table 5.9: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.52$)

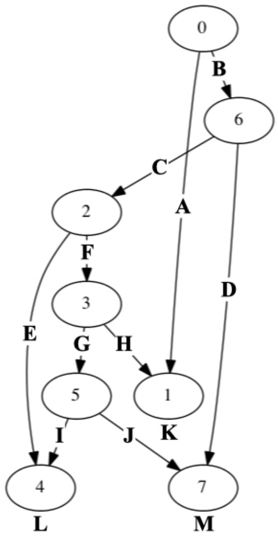


Figure 5.12: Generator 3.57 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ (gen. $2b$)	Trinet $T_{S,3}$ $(*)$	Result for N'
A	A	A	$A \vee B$	A
B	$B \vee C \vee D$	B	$A \vee B$	B
C	$B \vee C \vee D$	$C \vee E \vee F$	C	C
D	$B \vee C \vee D$	D	D	D
E	E	$C \vee E \vee F$	E	E
F	F	$C \vee E \vee F$	$F \vee G \vee H$	F
G	$G \vee I \vee J$	$G \vee I \vee J$	$F \vee G \vee H$	G
H	H	H	$F \vee G \vee H$	H
I	$G \vee I \vee J$	$G \vee I \vee J$	I	I
J	$G \vee I \vee J$	$G \vee I \vee J$	J	J

Table 5.10: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.57$)

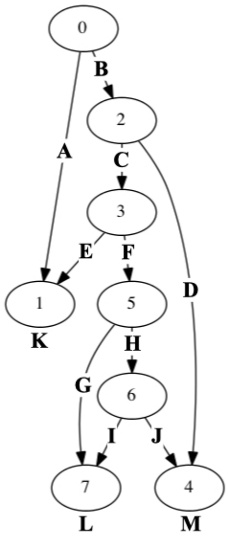


Figure 5.13: Generator 3.60 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2b$)	Trinet $T_{S,3}$ ($*$)	Result for N'
A		A	$A \vee B$	A
B		B	$A \vee B$	B
C		C	$C \vee E \vee F$	C
D		D	D	D
E		E	$C \vee E \vee F$	E
F		$F \vee G \vee H \vee I \vee J$	$C \vee E \vee F$	F
G		$F \vee G \vee H \vee I \vee J$	G	G
H		$F \vee G \vee H \vee I \vee J$	H	H
I		$F \vee G \vee H \vee I \vee J$	I	I
J		$F \vee G \vee H \vee I \vee J$	J	J

Table 5.11: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.60$)

So, each of the generators has the same leaves on each side for N and N'.

Group 4: 2 crucial sides, 1 set of parallel arcs, no other symmetry

Each of the generators 3.18, 3.22, 3.28, 3.31, 3.36 and 3.65 has 2 crucial sides, 1 set of parallel arcs and no other symmetry. Note that there is only symmetry caused by the parallel arcs. First, we define the set $T = \{18, 22, 28, 31, 36, 65\}$. Now, let $t \in T$, and consider generator $3.t$. Note that this generator with three labelled sides can be found in Appendix A.4. Further, note that generator 3.18 can also be found in Figure 5.14.

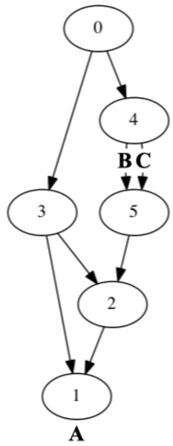


Figure 5.14: Generator 3.18 with three labelled sides

Observe that the only symmetry of generator $3.t$ is that sides B and C can be interchanged with sides C and B, respectively, to obtain an isomorphic generator. Note that sides B and C is a set of parallel arcs in generator $3.t$. Further, note that a set of crucial sides consists of two elements, namely side A and one of the sides B and C.

Let x be the leaf on side A, y a leaf on side B or C and z a leaf on some side S in N. Note there exists at least one such leaf z since N has at least 3 leaves. Then, by the crucial trinet on $\{x, y, z\}$ and since $Tn(N') = Tn(N)$, x is on side A and y is on side B or C in N'. So, the sides of the leaves are determined except of the symmetries. Assume y is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides B and C. Now, it follows by the crucial trinet on $\{x, y, z\}$ that z is on side S in N'. So, each leaf is on the same side in N' as in N (after possibly relabelling sides B and C).

Group 5: 3 crucial sides, 1 set of parallel arcs, no other symmetry

Each of the generators 3.11, 3.38, 3.40, 3.47, 3.53, 3.56 and 3.61 has 3 crucial sides, 1 set of parallel arcs and no other symmetry. Note that there is only symmetry caused by the parallel arcs. In Appendix A.5 the generators can be found with the labels of their sides. These generators will also be given later in the proof for this group of generators.

Observe that the only symmetry of these generators is that sides H_1 and H_2 can be interchanged with sides H_2 and H_1, respectively, to obtain an isomorphic generator. Note that sides H_1 and H_2 is a set of parallel arcs in each of the generators. Further, note that for all these generators a set of crucial sides has three elements, namely sides I, J and one of the sides H_1, H_2.

First, we look to leaves on sides H_1, H_2, I and J for each of these generators. Let h be the leaf on side H_1 or H_2, i the leaf on side I and j the leaf on side J in N. Then, the trinet P_c on $\{h, i, j\}$ is crucial and, since the only symmetry is caused by the parallel arcs H_1 and H_2, it follows that leaves i, j are, respectively, on side I, J in P_c, and hence, since $Tn(N') = Tn(N)$, in N'. Also, we get that leaf h is on side H_1 or H_2 in P_c, and hence, since $Tn(N') = Tn(N)$, in N'. Assume h is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides H_1 and H_2.

Now, we will prove that the other sides have also the same leaves in N and N'. We will do this in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{h, i, p_S\}, \{h, j, p_S\}, \{i, j, p_S\}$, where p_S is a leaf on side S in N. In Tables 5.12, 5.13, 5.14, 5.15, 5.16, 5.17 and 5.18 the results for the different generators can be found. We can see that for each of the generators that if there is a leaf on a side S in N, then this leaf is on side S in N'. Note that in Figures 5.15, 5.16, 5.17, 5.18, 5.19, 5.20 and 5.21 the generators with the labels of their sides can be found. So, for each generator the table with results is just below the generator itself. Also, note that the meaning of (Δ) and $(*)$ is explained in the proof for the generators of group 3b. Further, note that generator $2d$ now is also used, which has symmetry caused by the parallel sides.

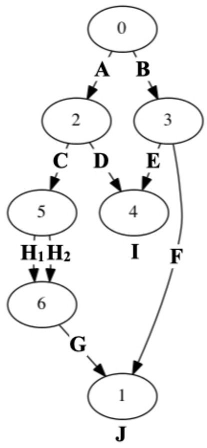


Figure 5.15: Generator 3.11 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2c$)	Result for N'
A		$A \vee C \vee D$	$A \vee B$	A
B		$B \vee E \vee F$	$A \vee B$	B
C		$A \vee C \vee D$	$C \vee F \vee G \vee H_1 \vee H_2$	C
D		$A \vee C \vee D$	$D \vee E$	D
E		$B \vee E \vee F$	$D \vee E$	E
F		$B \vee E \vee F$	$C \vee F \vee G \vee H_1 \vee H_2$	F
G		G	$C \vee F \vee G \vee H_1 \vee H_2$	G

Table 5.12: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.11$)

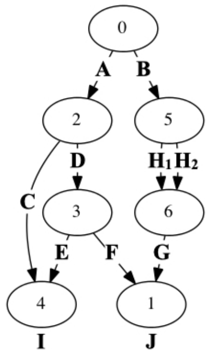


Figure 5.16: Generator 3.38 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A		$A \vee C \vee D \vee E \vee F$	A	A
B		B	$B \vee G \vee H_1 \vee H_2$	B
C		$A \vee C \vee D \vee E \vee F$	C	C
D		$A \vee C \vee D \vee E \vee F$	D	D
E		$A \vee C \vee D \vee E \vee F$	E	E
F		$A \vee C \vee D \vee E \vee F$	F	F
G		G	$B \vee G \vee H_1 \vee H_2$	G

Table 5.13: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.38$)

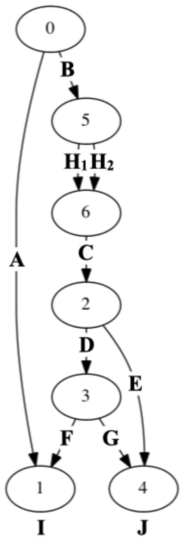


Figure 5.17: Generator 3.40 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2d$)	Trinet $T_{S,2}$ (Δ)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A		A	A
B	B		$B \vee C \vee H_1 \vee H_2$	B
C	$C \vee D \vee E \vee F \vee G$		$B \vee C \vee H_1 \vee H_2$	C
D	$C \vee D \vee E \vee F \vee G$		D	D
E	$C \vee D \vee E \vee F \vee G$		E	E
F	$C \vee D \vee E \vee F \vee G$		F	F
G	$C \vee D \vee E \vee F \vee G$		G	G

Table 5.14: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.40$)

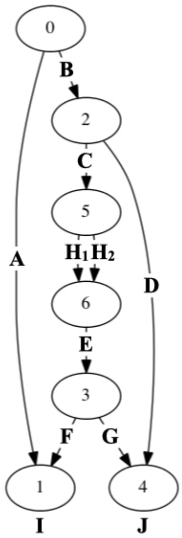


Figure 5.18: Generator 3.47 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2d$)	Trinet $T_{S,2}$ $(*)$	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A	A	$A \vee B$	A	A
B	$B \vee C \vee D$	$A \vee B$	B	B
C	$B \vee C \vee D$	C	$C \vee E \vee H_1 \vee H_2$	C
D	$B \vee C \vee D$	D	D	D
E	$E \vee F \vee G$	$E \vee F \vee G$	$C \vee E \vee H_1 \vee H_2$	E
F	$E \vee F \vee G$	$E \vee F \vee G$	F	F
G	$E \vee F \vee G$	$E \vee F \vee G$	G	G

Table 5.15: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.47$)

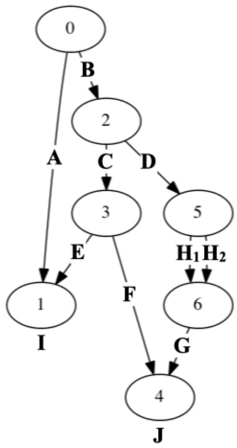


Figure 5.19: Generator 3.53 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ ($*$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A		$A \vee B$	A	A
B		$A \vee B$	B	B
C		$C \vee E \vee F$	C	C
D		D	$D \vee G \vee H_1 \vee H_2$	D
E		$C \vee E \vee F$	E	E
F		$C \vee E \vee F$	F	F
G		G	$D \vee G \vee H_1 \vee H_2$	G

Table 5.16: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.53$)

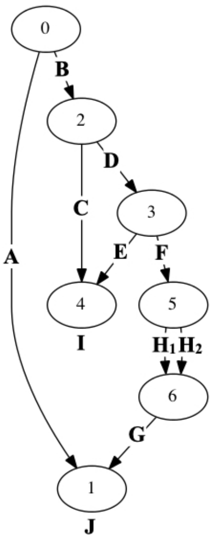


Figure 5.20: Generator 3.56 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A		A	A	A
B		$B \vee C \vee D \vee E \vee F$	B	B
C		$B \vee C \vee D \vee E \vee F$	C	C
D		$B \vee C \vee D \vee E \vee F$	D	D
E		$B \vee C \vee D \vee E \vee F$	E	E
F		$B \vee C \vee D \vee E \vee F$	$F \vee G \vee H_1 \vee H_2$	F
G		G	$F \vee G \vee H_1 \vee H_2$	G

Table 5.17: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.56$)

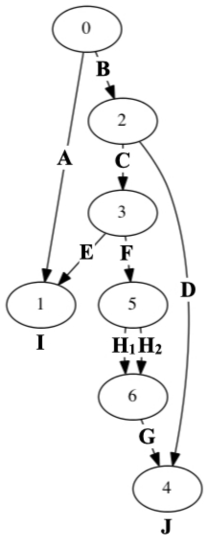


Figure 5.21: Generator 3.61 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ ($*$)	Trinet $T_{S,3}$ (gen. $2b$)	Result for N'
A		$A \vee B$	A	A
B		$A \vee B$	B	B
C		$C \vee E \vee F$	C	C
D		D	D	D
E		$C \vee E \vee F$	E	E
F		$C \vee E \vee F$	$F \vee G \vee H_1 \vee H_2$	F
G		G	$F \vee G \vee H_1 \vee H_2$	G

Table 5.18: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.61$)

So, each of the generators has the same leaves on each side for N and N' (after possibly relabelling sides H_1 and H_2).

Group 6: 3 crucial sides, 2 sets of parallel arcs, no other symmetry

Generator 3.64 has 3 crucial sides, 2 sets of parallel arcs and no other symmetry. Note that there is only symmetry caused by the parallel arcs. In Appendix A.6 the generator can be found with the labels of its sides. The generator will also be given later in the proof for this generator.

Observe that that the only symmetry of this generator is that sides E_1

and E_2 can be interchanged with sides E_2 and E_1, respectively, or that sides F_1 and F_2 can be interchanged with sides F_2 and F_1, respectively, to obtain an isomorphic generator. Note that sides E_1 and E_2 and sides F_1 and F_2 are both a set of parallel arcs. Further, note that a set of crucial sides has three elements, namely side G, one of the sides E_1, E_2 and one of the sides F_1, F_2.

First, we look to the leaves on sides E_1, E_2, F_1, F_2 and G. Let e be the leaf on side E_1 or E_2, f the leaf on side F_1 or F_2, and g the leaf on side G, in N. Then, the trinet P_c on $\{e, f, g\}$ is crucial and, since the only symmetry is caused by the parallel arcs E_1 and E_2 and the parallel arcs F_1 and F_2, it follows that leaf g is, respectively, on side G in P_c, and hence, since $Tn(N') = Tn(N)$, in N'. Also, we get that leaf e is on side E_1 or E_2 in P_c, and hence, since $Tn(N') = Tn(N)$, in N'. Further, we get that leaf f is on side F_1 or F_2 in P_c, and hence, since $Tn(N') = Tn(N)$, in N'. Assume e is on the same side in N' as in N. Also, assume f is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides E_1 and E_2 or relabelling sides F_1 and F_2.

Now, we will prove that the other sides have also the same leaves in N and N'. We will do this in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{e, f, p_S\}, \{e, g, p_S\}, \{f, g, p_S\}$, where p_S is a leaf on side S in N. In Table 5.19 the results can be found. We can see that if there is a leaf on a side S in N, then this leaf is on side S in N'. Note that in Figure 5.22 the generator with the labels of its sides can be found. So, the table with results is just below the generator itself. Also, note that the meaning of (Δ) is explained in the proof for the generators of group $3b$.

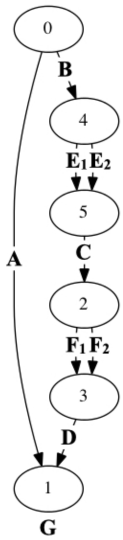


Figure 5.22: Generator 3.64 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2d$)	Result for N'
A		A	A	A
B		B	$B \vee C \vee E_1 \vee E_2$	B
C		$C \vee D \vee F_1 \vee F_2$	$B \vee C \vee E_1 \vee E_2$	C
D		$C \vee D \vee F_1 \vee F_2$	D	D

Table 5.19: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.64$)

So, the generator has the same leaves on each side for N and N' (after possibly relabelling sides E_1 and E_2 or sides F_1 and F_2).

Group 7: 1 crucial side, symmetry, no parallel arcs

Generator 3.16 has 1 crucial side, symmetry and no parallel arcs. In Appendix A.7 the generator can be found with the labels of its sides. The generator can also be found in Figure 5.23.

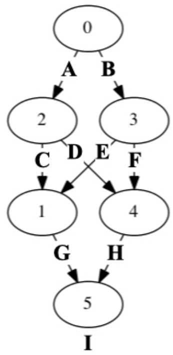


Figure 5.23: Generator 3.16 with labelled sides

Observe that there is some symmetry. Sides A, C, D can be interchanged with sides B, E, F, respectively, to obtain an isomorphic generator. Also, sides C, E, G can be interchanged with sides D, F, H, respectively, again yielding an isomorphic generator. Further, sides A, C, D, G can be interchanged with sides B, F, E, H, respectively, again yielding an isomorphic generator. Note, a set of crucial sides has one element, namely side I.

Let i be the leaf on side I, x and y two other leafs in N. Then, the trinet P_c on $\{i, x, y\}$ implies that i is on side I in N'. Now, again using trinet P_c, it follows that if there is on side A or B in N, then this leaf is on side A or B in N'. Also, if a leaf is on side C, D, E or F in N, then this leaf is on side C, D, E or F in N'. Further, if a leaf is on side G or H in N, then this leaf is on side G or H in N'. Note, we use the symmetry of the generator.

First, assume that there is at least one leaf on side C in N and that the leaves that are on side C in N are on side C in N'. Let c be one such leaf on side C in N.

Let a be a leaf on side A in N. Earlier we saw that this leaf then is on side A or B in N'. Consider the crucial trinet on $\{a, c, i\}$, which has the same underlying generator as N and N'. Then, leaves a and c are on sides that are arcs of the generator and for which holds that the end point of one of the two sides is the same as the begin point of the other side. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, a is on side A in N'.

Let b be a leaf on side B in N. Earlier we saw that this leaf then is on side A or B in N'. Consider the crucial trinet on $\{b, c, i\}$, which has the same underlying generator as N and N'. Then, leaves b and c are on sides that are arcs of the generator and for which does not hold that the end point of one of the two sides is the same as the begin point of the other side. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, b is on side B in N'.

Let d be a leaf on side D in N. Earlier we saw that this leaf then is on side C, D, E or F in N'. Consider the crucial trinet on $\{c, d, i\}$, which has the same underlying generator as N and N'. Then, leaves c and d are on sides that are arcs of the generator and that have the same begin point but different end points. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, d is on side D in N'.

Let e be a leaf on side E in N. Earlier we saw that this leaf then is on side C, D, E or F in N'. Consider the crucial trinet on $\{c, e, i\}$, which has the same underlying generator as N and N'. Then, leaves c and e are on sides that are arcs of the generator and that have the same end point but different begin points. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, e is on side E in N'.

Let f be a leaf on side F in N. Earlier we saw that this leaf then is on side C, D, E or F in N'. Consider the crucial trinet on $\{c, f, i\}$, which has the same underlying generator as N and N'. Then, leaves c and f are on sides that are arcs of the generator and that have different begin points and different end points. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, f is on side f in N'.

Let g be a leaf on side G in N. Earlier we saw that this leaf then is on side G or H in N'. Consider the crucial trinet on $\{c, g, i\}$, which has the same underlying generator as N and N'. Then, leaves c and g are on sides that are arcs of the generator and for which holds that the end point of one of the two sides is the same as the begin point of the other side. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, g is on side G in N'.

Let h be a leaf on side H in N. Earlier we saw that this leaf then is on side G or H in N'. Consider the crucial trinet on $\{c, h, i\}$, which has the same underlying generator as N and N'. Then, leaves c and h are on sides that are arcs of the generator and for which does not hold that the end point of one of the two sides is the same as the begin point of the other side. So, since c is on side C in N' and since $Tn(N') = Tn(N)$, h is on side H in N'. So, all leaves are on the same side in N' as in N.

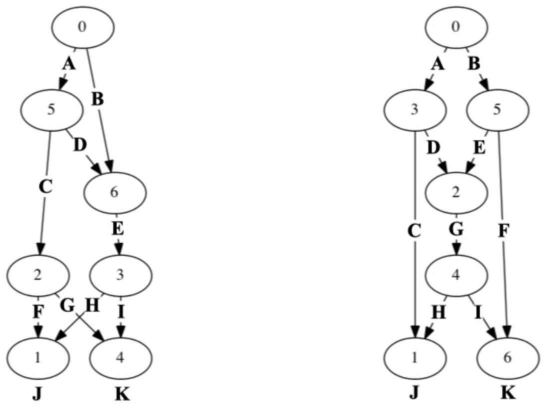
Now, assume that the leaves that are on side C in N are not on side C in N'. Earlier we saw that these leaves then are on side D, E or F in N'. First, if the leaves that are on side C in N are on side D in N', then we can argue in exactly the same way that the leaves that are on sides A, B, D, E, F, G, H in N are, respectively, on sides A, B, C, F, E, H, G in N'. Now, relabelling sides C, E, G with sides D, F, H, respectively, gives that all leaves are on the same side in N' as in N. Secondly, if the leaves that are on side C in N are on side E in N', then we can argue in exactly the same way that the leaves that are on sides A, B, D, E, F, G, H in N are, respectively, on sides B, A, F, C, D, G, H in N'. Now, relabelling sides A, C, D with sides B, E, F, respectively, gives that all leaves are on the same side in N' as in N. Thirdly, if the leaves that are on side C in N are on side F in N', then we can argue in exactly the same way that the leaves that are on sides A, B, D, E, F, G, H in N are, respectively, on sides B, A, E, D, C, H, G in N'. Now, relabelling sides A, C, D, G with sides B, F, E, H, respectively, gives that all leaves are on the same side in N' as in N. So, all leaves are on the same side in N' as in N.

Finally, if there is no leaf on side C in N, then there is a leaf on one of the sides A, B, D, E, F, G, H in N (since N has at least three leaves). First, assume that there is a leaf on one of the sides D, E or F. Then, we can apply similar arguments based on that leaf as we did for the leaf on side C in order to get that all leaves are on the same side in N' as in N.

Now, if there is no leaf on sides C, D, E and F in N, then there is a leaf on one of the sides A, B, G or H in N (since N has at least three leaves).Earlier we saw that if a leaf is on side A or B in N, this leaf is on side A or B in N'. Assume that the leaves that are on side A in N are on side A in N' and that the leaves that are on side B in N are on side B in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides A, C, D with sides B, E, F, respectively. Earlier we also saw that if a leaf is on side G or H in N, this leaf is on side G or H in N'. Assume that the leaves that are on side G in N are on side G in N' and that the leaves that are on side H in N are on side H in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides C, E, G with sides D, F, H, respectively. So, all leaves are on the same side in N' as in N (after possibly relabelling sides).

Group 8: 2 crucial sides, symmetry, no parallel arcs

Each of the generators 3.3 and 3.26 has 2 crucial sides, symmetry and no parallel arcs. In Appendix A.8 the generators can be found with the labels of their sides. These generators can also be found in Figure 5.24.



(a) Generator 3.3 (b) Generator 3.26

Figure 5.24: Generators of group 8 with labelled sides

Observe that the symmetry of generator 3.3 is that sides F, H, J can be interchanged with sides G, I, K, respectively, to obtain an isomorphic generator. Also, observe that the symmetry of generator 3.26 is that sides A, C, D, H, J can be interchanged with sides B, F, E, I, K, respectively, to obtain an isomorphic generator. Further, note that for each of the generators a set of crucial sides has two elements, namely sides J and K.

Now, we will prove that each leaf in on the same side in N' as in N for each of the generators. Let j be the leaf on side J, k the leaf on side K and s a leaf on some other side S in N. Then, the trinet P_c on $\{j, k, s\}$ implies that j and k are on sides J and K in N'. Assume that j is on side J and k is on side K in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides using the symmetry we discussed before. Now, again using trinet P_c, it follows that s is on side S in N'. So, each leaf is on the same side in N' as in N (after possibly relabelling sides).

Group 9: 3 crucial sides, symmetry

Each of the generators 3.1, 3.6, 3.8, 3.37 and 3.58 has 3 crucial sides and symmetry. First generator 3.6 will be discussed, then the other generators.

Generator 3.6 (group 9a)

In Appendix A.9 generator 3.6 can be found with the labels of its sides. The generator will also be given later in the proof for this generator.

Observe that the symmetry is that sides A, C, D, G, H, L can be interchanged with sides B, F, E, J, I, M, respectively, to obtain an isomorphic generator. Further, note that a set of crucial sides has three elements, namely sides K, L and M.

First, we look to the leaves on sides K, L and M. Let k be the leaf on side K, l the leaf on side L and m the leaf on side M in N. Then, the trinet P_c on $\{k, l, m\}$ implies that leaf k is on side K in N' and that leafs l and m are on sides L and M in N'. Assume that leaf l is on side L and leaf m is on side M in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides A, C, D, G, H, L with sides B, F, E, J, I, M, respectively.

Now, we will prove that the other sides have also the same leaves in N and N'. The first part we will do in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{k, l, p_S\}, \{k, m, p_S\}, \{l, m, p_S\}$, where p_S is a leaf on side S in N. In Table 5.20 the results can be found. We can see that if there is a leaf on a side S in N, then this leaf is on side S or S' in N', with S' the side that S can be interchanged with according the symmetry of the generator. Note that in Figure 5.25 the generator with the labels of its sides can be found. So, the table with results is just below the generator itself.

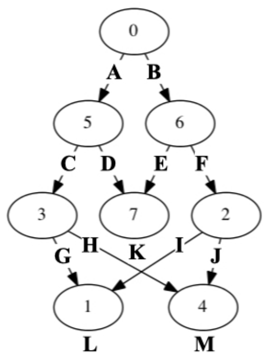


Figure 5.25: Generator 3.6 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2c$)	Trinet $T_{S,2}$ (gen. $2c$)	Trinet $T_{S,3}$ (gen. $2c$)	Result for N'
$A \vee B$	$A \vee B$	$A \vee B$	$A \vee B \vee C \vee D \vee E \vee F$	$A \vee B$
$C \vee F$	$C \vee F \vee G \vee H \vee I \vee J$	$C \vee F \vee G \vee H \vee I \vee J$	$A \vee B \vee C \vee D \vee E \vee F$	$C \vee F$
$D \vee E$	$D \vee E$	$D \vee E$	$A \vee B \vee C \vee D \vee E \vee F$	$D \vee E$
$G \vee I$	$C \vee F \vee G \vee H \vee I \vee J$	$C \vee F \vee G \vee H \vee I \vee J$	$G \vee I$	$G \vee I$
$H \vee J$	$C \vee F \vee G \vee H \vee I \vee J$	$C \vee F \vee G \vee H \vee I \vee J$	$H \vee J$	$H \vee J$

Table 5.20: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.6$)

First, assume that there is at least one leaf on side A in N and that the leaves that are on side A in N are on side A in N'. Let a be one such leaf on side A in N.

Let d be a leaf on side D in N. Earlier we saw that then d is on side D or E in N'. Consider the trinet on $\{a, d, k\}$, which is a simple level-1 network. Then, a and d are on the same side of this trinet. Now, since $Tn(N') = Tn(N)$ holds, d is on side D in N'.

Let x be a leaf on side $S \in \{B, E\}$ in N. Again, if x is on side B in N, then x is on side A or B in N'. Also, if x is on side E in N, then x is on side D or E in N'. Consider the trinet on $\{a, x, k\}$, which is a simple level-1 network. Then, a and x are on different sides of this trinet. Now, since $Tn(N') = Tn(N)$ holds, x is on side S in N'.

Let y be a leaf on side $S \in \{C, G, H\}$ in N. Again, if y is on side C in N, then y is on side C or F in N'. Also, if y is on side G in N, then y is on side G or I in N'. Further, if y is on side H in N, then y is on side H or J in N'. Consider the trinet on $\{a, y, l\}$, which is a simple level-1 network. Then, a and y are on the same side of this trinet. Now, since $Tn(N') = Tn(N)$ holds, y on side S in N'.

Let z be a leaf on side $S \in \{F, I, J\}$ in N. Again, if z is on side F in N, then z is on side C or F in N'. Also, if z is on side I in N, then z is on side G or I in N'. Further, if z is on side J in N, then z is on side H or J in N'. Consider the trinet on $\{a, z, l\}$, which is a simple level-1 network. Then, a and z are on different sides of this trinet. Now, since $Tn(N') = Tn(N)$ holds, z is on side S in N'. So, all leaves are on the same side in N' as in N.

Now, assume that the leaves that are on side A in N are not on side A in N'. Earlier we saw that these leaves then are on side B in N'. Then, we can argue in exactly the same way that the leaves that are on sides

$B, C, D, E, F, G, H, I, J$ in N are, respectively, on sides $A, F, E, D, C, I, J, G, H$ in N'. Now, relabelling sides $A, B, C, D, E, F, G, H, I, J$ with sides $B, A, F, E, D, C, I, J, G, H$, respectively, gives that all leaves are on the same side in N' as in N.

Finally, assume that there is no leaf on side A in N. If there is a leaf on one of the sides $B, C, D, E, F, G, H, I, J$ in N, then we can apply similar arguments based on that leaf as we did for the leaf on side A in order to get that all leaves are on the same side in N' as in N. If there is no leaf on one of the sides $B, C, D, E, F, G, H, I, J$ in N, then leaves k, l and m are the only leaves in N and for leaves k, l and m we already showed that they are on the same side in N' as in N.

So, each leaf is on the same side in N' as in N (after possibly relabelling sides).

The other generators (group 9b)

In Appendix A.9 generators 3.1, 3.8, 3.37 and 3.58 can be found with the labels of their sides. These generators will also be given later in the proof for this group of generators. Note that for generator 3.8 some binary, simple level-3 networks are excluded.

Observe that the symmetry of generators 3.1 and 3.58 is that sides G, I, L can be interchanged with sides H, J, M, respectively, to obtain an isomorphic generator. The symmetry of generators 3.8 and 3.37 is that sides A, C, D, G, H, K can be interchanged with sides B, F, E, J, I, M, respectively, to obtain an isomorphic generator. Further, note that for each of the generators a set of crucial sides has three elements, namely sides K, L and M.

First, we look to the leaves on sides K, L and M for each of the generators. Let k be the leaf on side K, l the leaf on side L and m the leaf on side M in N. Then, for generators 3.1 and 3.58, the trinet P_c on $\{k, l, m\}$ implies that leaf k is on side K in N' and that leafs l and m are on sides L and M in N'. Assume that leaf l is on side L and leaf m is on side M in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides using the symmetry we discussed before.

Further, for generators 3.8 and 3.37, the trinet P_c on $\{k, l, m, \}$ implies that leaf l is on side L in N' and that leafs k and m are on sides K and M in N'. Assume that leaf k is on side K and leaf m is on side M in N'. We can assume this without loss of generality, because if it is not the case, we can relabelling sides using the symmetry we discussed before.

Now, we will prove that the other sides have also the same leaves in N and N'. A large part we will do in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{k, l, p_S\}, \{k, m, p_S\}, \{l, m, p_S\}$, where p_S is a leaf on side S in N. In Tables 5.21, 5.22, 5.23 and 5.24 the results for the different generators can be found. We can see that for three of the generators that if there is a leaf on a side S in N, then this leaf is on side S in N'. For generator 3.8 we get the same result, except for the sides A and B. Note that in Figures 5.26, 5.27, 5.28 and 5.29 the generators with the labels of their sides can be found. So, for each generator the table with results is just below the generator itself. Also, note that the meaning of (Δ) and $(*)$ is explained in the proof for the generators of group 3b.

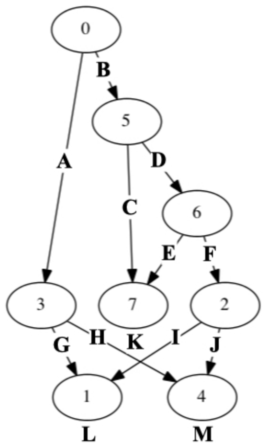


Figure 5.26: Generator 3.1 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. 2b)	Trinet $T_{S,2}$ (gen. 2b)	Trinet $T_{S,3}$ (gen. 2c)	Result for N'
A	$A \vee G \vee H$	$A \vee G \vee H$	$A \vee B \vee C \vee D \vee E \vee F$	A
B	B	B	$A \vee B \vee C \vee D \vee E \vee F$	B
C	C	C	$A \vee B \vee C \vee D \vee E \vee F$	C
D	D	D	$A \vee B \vee C \vee D \vee E \vee F$	D
E	E	E	$A \vee B \vee C \vee D \vee E \vee F$	E
F	$F \vee I \vee J$	$F \vee I \vee J$	$A \vee B \vee C \vee D \vee E \vee F$	F
G	$A \vee G \vee H$	$A \vee G \vee H$	$G \vee I$	G
H	$A \vee G \vee H$	$A \vee G \vee H$	$H \vee J$	H
I	$F \vee I \vee J$	$F \vee I \vee J$	$G \vee I$	I
J	$F \vee I \vee J$	$F \vee I \vee J$	$H \vee J$	J

Table 5.21: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.1$)

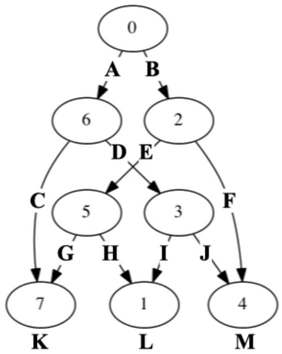


Figure 5.27: Generator 3.8 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2c$)	Trinet $T_{S,2}$ (gen. $2c$)	Trinet $T_{S,3}$ (gen. $2c$)	Result for N'
$A \vee B$	$A \vee B \vee E \vee F$	$A \vee B$	$A \vee B \vee C \vee D$	$A \vee B$
C	$C \vee G$	$C \vee E \vee G \vee H$	$A \vee B \vee C \vee D$	C
D	$D \vee H \vee I \vee J$	$D \vee F \vee I \vee J$	$A \vee B \vee C \vee D$	D
E	$A \vee B \vee E \vee F$	$C \vee E \vee G \vee H$	$E \vee G \vee H \vee I$	E
F	$A \vee B \vee E \vee F$	$D \vee F \vee I \vee J$	$F \vee J$	F
G	$C \vee G$	$C \vee E \vee G \vee H$	$E \vee G \vee H \vee I$	G
H	$D \vee H \vee I \vee J$	$C \vee E \vee G \vee H$	$E \vee G \vee H \vee I$	H
I	$D \vee H \vee I \vee J$	$D \vee F \vee I \vee J$	$E \vee G \vee H \vee I$	I
J	$D \vee H \vee I \vee J$	$D \vee F \vee I \vee J$	$F \vee J$	J

Table 5.22: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.8$)

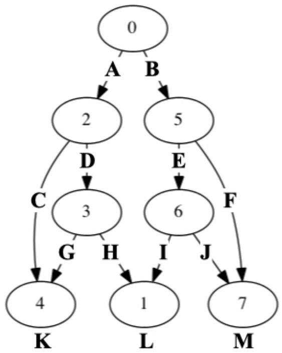


Figure 5.28: Generator 3.37 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. 2b)	Trinet $T_{S,2}$ (Δ)	Trinet $T_{S,3}$ (gen. 2b)	Result for N'
A	A		$A \vee C \vee D \vee G \vee H$	A
B	$B \vee E \vee F \vee I \vee J$		B	B
C	C		$A \vee C \vee D \vee G \vee H$	C
D	D		$A \vee C \vee D \vee G \vee H$	D
E	$B \vee E \vee F \vee I \vee J$		E	E
F	$B \vee E \vee F \vee I \vee J$		F	F
G	G		$A \vee C \vee D \vee G \vee H$	G
H	H		$A \vee C \vee D \vee G \vee H$	H
I	$B \vee E \vee F \vee I \vee J$		I	I
J	$B \vee E \vee F \vee I \vee J$		J	J

Table 5.23: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.37$)

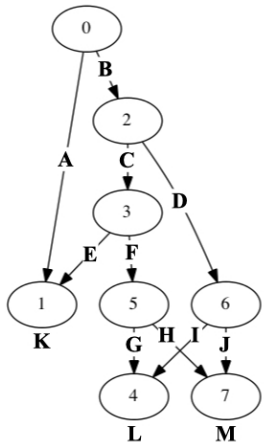


Figure 5.29: Generator 3.58 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2b$)	Trinet $T_{S,2}$ (gen. $2b$)	Trinet $T_{S,3}$ $(*)$	Result for N'
A	A	A	$A \vee B$	A
B	B	B	$A \vee B$	B
C	C	C	$C \vee D \vee E \vee F$	C
D	$D \vee I \vee J$	$D \vee I \vee J$	$C \vee D \vee E \vee F$	D
E	E	E	$C \vee D \vee E \vee F$	E
F	$F \vee G \vee H$	$F \vee G \vee H$	$C \vee D \vee E \vee F$	F
G	$F \vee G \vee H$	$F \vee G \vee H$	$G \vee I$	G
H	$F \vee G \vee H$	$F \vee G \vee H$	$H \vee J$	H
I	$D \vee I \vee J$	$D \vee I \vee J$	$G \vee I$	I
J	$D \vee I \vee J$	$D \vee I \vee J$	$H \vee J$	J

Table 5.24: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.58$)

So, for generator 3.8 leaves on sides A and B can not be distinguished in this way. Therefore, we need another trinet to distinguish leaves on these two sides. First, we look to the case that there is a leaf x on side C, D, E, F, G, H, I or J in N for sides A and B. Let a be a leaf on side A in N. Earlier we saw that then a is on side A or B in N'. Consider the trinet on $\{a, x, k\}$, which is a simple level-1 network. If x is on side C, D, I or J in N, then a and x are on the same side of this trinet. If x is on side E, F, G or H in N, then a and x are on different sides of the trinet. Now, since $Tn(N') = Tn(N)$ holds, a is on side A in N'.

Let b be a leaf on side B in N. Earlier we saw that then b is on side A or B in N'. Consider the trinet on $\{b, x, k\}$, which is a simple level-1 network. If x is on side C, D, I or J in N, then a and x are on different sides of this trinet. If x is on side E, F, G or H in N, then a and x are on the same side of the trinet. Now, since $Tn(N') = Tn(N)$ holds, b is on side B in N'.

Note, we do not have to look to the case that there are no leaves on sides $C, D, E, F,$ G, H, I and J in N since then N is a network as in Figure 4.1 (i.e. with underlying generator 3.8, at least one leaf on side A or B and no leaves on sides C, D, E, F, G, H, I and J) and therefore excluded.

So, each of the generators has the same leaves on each side for N and N' (after possibly relabelling sides).

Group 10: 3 crucial sides, 1 set of parallel arcs, other symmetry

Generator 3.2 has 3 crucial sides, 1 set of parallel arcs and other symmetry. Note that this generator has also symmetry that is not caused by the sets of parallel arcs. In Appendix A.10 the generator can be found with the labels of its sides. The generator will also be given later in the proof for this generator.

Observe that there is some symmetry. Sides H_1 and H_2 can be interchanged with sides H_2 and H_1, respectively, to obtain an isomorphic generator. Also, sides D, F, I can be interchanged with sides E, G, J, respectively, again yielding an isomorphic generator. Note that side H_1 and H_2 is a set of parallel arcs. Further, note that a set of crucial sides has three elements, namely sides I, J and one of the sides H_1, H_2.

First, we look to the leaves on sides H_1, H_2, I and J. Let h be the leaf on side H_1 or H_2, i the leaf on side I and j the leaf on side J in N. Then, the trinet P_c on $\{h, i, j\}$ implies that leaf h is on side H_1 or H_2 in N'. Also, we get that leafs i and j are on sides I and J in N'. Assume leaf h is on the same side in N' as in N. Also, assume this for leafs i and j. We can assume this without loss of generality, because if it is not the case, we can relabelling sides using the symmetry we discussed before.

Now, we will prove that the other sides have also the same leaves in N and N'. We will do this in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{h, i, p_S\}, \{h, j, p_S\}, \{i, j, p_S\}$, where p_S is a leaf on side S in N. In Table 5.25 the results can be found. We can see that if there is a leaf on a side S in N, then this leaf is on side S in N'. Note that in Figure 5.30 the generator with the labels of its sides can be found. So, the table with results is just below the generator itself.

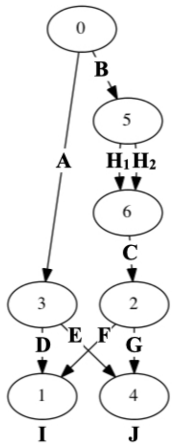


Figure 5.30: Generator 3.2 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (gen. $2d$)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2c$)	Result for N'
A	$A \vee D \vee E$	$A \vee D \vee E$	$A \vee B \vee C \vee H_1 \vee H_2$	A
B	B	B	$A \vee B \vee C \vee H_1 \vee H_2$	B
C	$C \vee F \vee G$	$C \vee F \vee G$	$A \vee B \vee C \vee H_1 \vee H_2$	C
D	$A \vee D \vee E$	$A \vee D \vee E$	$D \vee F$	D
E	$A \vee D \vee E$	$A \vee D \vee E$	$E \vee G$	E
F	$C \vee F \vee G$	$C \vee F \vee G$	$D \vee F$	F
G	$C \vee F \vee G$	$C \vee F \vee G$	$E \vee G$	G

Table 5.25: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.2$)

So, the generator has the same leaves on each side for N and N' (after possibly relabelling sides).

Group 11: 3 crucial sides, 2 sets of parallel arcs, other symmetry

Generator 3.63 has 3 crucial sides, 2 sets of parallel arcs and other symmetry. Note that this generator has also symmetry that is not caused by the sets of parallel arcs. In Appendix A.11 the generator can be found with the labels of its sides. The generator will also given later in the proof for this generator.

Observe that there is some symmetry. Sides E_1 and E_2 can be interchanged with sides E_2 and E_1, respectively, to obtain an isomorphic generator. Also, sides F_1 and F_2 can be interchanged with F_2 and F_1, respectively, again yielding an isomorphic generator. Further, sides A, C, E_1, E_2 can be interchanged with B, D, F_1, F_2, respectively, again yielding an isomorphic generator. Note that sides E_1 and E_2 and sides F_1 and F_2 are both a set of parallel arcs. Further, note that a set of crucial sides has three elements, namely side G, one of the sides E_1, E_2 and one of the sides F_1, F_2.

First, we look to the leaves on sides E_1, E_2, F_1, F_2 and G. Let e be the leaf on side E_1 or E_2, f the leaf on side F_1 or F_2, and g the leaf on side G, in N. Then, the trinet P_c on $\{e, f, g\}$ implies that leave g is on side G in N'. Also, we get that leaf e is on side E_1, E_2, F_1 or F_2 in N'. Assume leaf e is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides using the symmetry we discussed before. Now, again using trinet P_c, it follows that leaf f is on side F_1 or F_2 in N'. Assume leaf f is on the same side in N' as in N. We can assume this without loss of generality, because if it is not the case, we can relabelling sides F_1 and F_2.

Now, we will prove that the other sides have also the same leaves in N and N'. We will do this in similar way as we did for the generators of group 3. For each side S we use some of the trinets $T_{S,1}, T_{S,2}, T_{S,3}$ on, respectively, $\{e, f, p_S\}, \{e, g, p_S\}, \{f, g, p_S\}$, where p_S is a leaf on side S in N. In Table 5.26 the results can be found. We can see that if there is a leaf on a side S in N, then this leaf is on side S in N'. Note that in Figure 5.31 the generator with the labels of its sides can be found. So, the table with results is just below the generator itself. Also, note that the meaning of (Δ) is explained in the proof for the generators of group 3b.

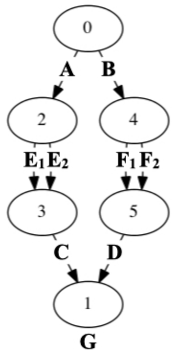


Figure 5.31: Generator 3.63 with labelled sides

Leaf on side S in N	Trinet $T_{S,1}$ (Δ)	Trinet $T_{S,2}$ (gen. $2d$)	Trinet $T_{S,3}$ (gen. $2d$)	Result for N'
A		A	$A \vee C \vee E_1 \vee E_2$	A
B		$B \vee D \vee F_1 \vee F_2$	B	B
C		C	$A \vee C \vee E_1 \vee E_2$	C
D		$B \vee D \vee F_1 \vee F_2$	D	D

Table 5.26: The resulting side(s) in N' for each leaf on a non-crucial side S in N using trinets $T_{S,1}$, $T_{S,2}$ and $T_{S,3}$ ($G = 3.63$)

So, the generator has the same leaves on each side for N and N' (after possibly relabelling sides).

The order of the leaves

It remains to show that the leaves on each side are in the same order in N and N'. The proof for this holds for all 65 different level-3 generators G.

Consider a side V of N with at least two leaves and two leaves v, v' on that side such that v' is below v. Then, there exists a side W which is a outdegree-0 vertex that is below side V. Let w be the leaf on side W in N. Now, consider the trinet P on $\{v, v', w\}$. Then, we have that the leaves v and v' are on the same side of trinet P. Moreover, since v' is below v in N, v' is below v in P. Then, since $Tn(N') = Tn(N)$, v' is below v in N'. So, the order of the leaves on each of the sides are the same in N and N'.

Note that we indeed need the leaf w to distinguish the order of the leaves, because without the leaf w a part of the generator would be left out and the leaves u and u' would become 'cherry's' without any order.

Now, we can conclude that $N = N'$, since (after possibly relabelling sides) both networks have the same underlying generator, the same leaves on each side, and the same order of the leaves on each side. $\square$

Finally, we are able to combine the results to prove the following corollary, which is the main result of this chapter.

Corollary 5.3. *The class of binary recoverable level-3 networks with at least three leaves, except for networks with a biconnected component as in Figure 4.6 (i.e. with a biconnected component with underlying generator* 3.8*, at least one cut-arc on side A or B and no cut-arcs on sides C, D, E, F, G, H, I and J) is weakly encoded by trinets.*

Proof. Follows from Theorem 2.22, Lemma 5.2, Corollary 3.3 and the fact that level-1 networks are encoded by their trinets [5]. $\square$

Chapter 6

All level-3 networks are encoded by quarnets

In this chapter we will prove the corollary that level-3 networks are encoded by quarnets. This is the main result of this thesis. Before we will prove this corollary, some definitions for quarnets and Lemma 6.4 will be discussed. A large part of the proof of Lemma 6.4 is based on the proof of Lemma 3.2 for level-2 networks, which is also given in [5]. So, some parts are the same or similar as in the proof of Lemma 3.2.

First, we define what a quarnet is:

Definition 6.1. A ***quarnet*** is a rooted phylogenetic network with four leaves.

We also define how a quarnet can be exhibited by a phylogenetic network:

Definition 6.2. Given a phylogenetic network N on X and $\{w, x, y, z\} \subseteq X$, the quarnet on $\{w, x, y, z\}$ ***exhibited*** (or: *displayed*) by N is the quarnet obtained from N by deleting all vertices that are not on any path from $LSA(\{w, x, y, z\})$ to w, x, y or z and subsequently suppressing all indegree-1 outdegree-1 vertices and parallel arcs.

We will use ***Qn(N)*** to denote the set of all quarnets exhibited by a phylogenetic network N. Now, we can define when a phylogenetic network is encoded by its set of quarnets:

Definition 6.3. A phylogenetic network N is ***encoded*** by its set of quarnets $Qn(N)$ if there is no recoverable phylogenetic network $N' \neq N$ with $Qn(N) = Qn(N')$.

Note that there are also other definitions for quarnets, but we will use these definitions since they hold for rooted binary phylogenetic networks and are similar to the definitions for trinets.

In the previous chapter we proved that most level-3 networks are weakly encoded by trinets. In Lemma 6.4 we will prove that binary, simple level-3 networks are encoded by quarnets. Now, we have quarnets in the statement of the lemma instead of trinets as we had in Lemma 5.2. Therefore, we are able to prove a much stronger result. The simple level-3 networks are not only weakly encoded but also encoded by quarnets. Further, we do not have the exceptions for the simple level-3 networks as in Figure 4.1. Also, note that for a network N the trinets $Tn(N)$ can be obtained from the quarnets $Qn(N)$.

Lemma 6.4. *Every binary, simple level-3 network on X, with $|X| \geq 4$, is encoded by its quarnets.*

Proof. Let N be any binary, simple level-3 network on X, with $|X| \geq 4$. Assume that this network is not encoded by its quarnets $Qn(N)$. Then, there is a recoverable network $N' \neq N$ with $Qn(N) = Qn(N')$. We will show that $N' = N$, which is a contradiction, so then the lemma follows.

We begin by showing that $Tn(N) = Tn(N')$. Let x, y and z be three leaves of N. Since $|X| \geq 4$, there exists a quarnet Q in $Qn(N)$ containing leaves x, y and z. Then, the trinet on $\{x, y, z\}$ exhibited by Q is in $Tn(N)$ and $Tn(N')$ since $Qn(N) = Qn(N')$. We can conclude that $Tn(N) = Tn(N')$.

Now, we show that N' is a binary, simple, level-3 network.

- Since $Tn(N') = Tn(N)$ holds, we have by Theorem 2.19 that the set of CA-sets of N' equals the set of CA-sets of N. Note that all CA-sets of N (and also of N') are singletons, since N is a simple network. Furthermore, we claim that N' has no redundant biconnected components. If it had one, then there would be only one leaf, say x, below it. However, then all trinets containing x would have a redundant biconnected component with x directly below it. This is not possible because $Tn(N') = Tn(N)$. For each leaf x there exists a trinet in $Tn(N)$ without redundant biconnected components. So, N' has no redundant biconnected components. Since the sets of CA-sets of N' and N are the same and N' has no redundant biconnected components, we have that N' is a simple network.

- Suppose we have any simple level-k network with $k > 3$. Then, this network has exactly k reticulations. If there are at least four leaves whose parent is a reticulation, take four such leaves. Otherwise, take all leaves whose parent is a reticulation and take the remaining leaves on sides that form parallel arcs in the underlying generator of N, choosing at most one leaf per pair of parallel arcs. Then, the quarnet on the chosen three leaves has at least four reticulations. Note that if a leaf is chosen on one of the parallel arcs in the underlying generator, the pair of parallel arcs will not be suppressed, and so we get a reticulation. So, a simple level-k network, with $k > 3$, has a level-k' quarnet with $k' > 3$. It follows that N' is a level-3 network since $Qn(N') = Qn(N)$ contains only level-3 quarnets.

- Assume that N' has a vertex with outdegree greater than 2. Let c_1, c_2 and c_3 be three of his children. Then, consider three (not necessarily different) leaves x_1, x_2 and x_3 below c_1, c_2 and c_3 respectively. Then, any trinet containing x_1, x_2 and x_3 exhibited by N' is not binary. We get a contradiction since all trinets in $Tn(N') = Tn(N)$ are binary, since N is binary. In much the same way, we can prove that each vertex in N' has indegree at most 2 and that each indegree-2 vertex has outdegree 1. Now, we can conclude that N' is binary.

So, N and N' are both binary, simple level-3 networks. Now, let G be the underlying generator of N. First, we show that G is also the underlying generator of N'. By Lemma 5.1, N has at least one crucial trinet P_c. By Lemma 2.30, P_c is a simple level-3 network and its underlying generator is G. Since $Tn(N) = Tn(N')$, P_c is also a trinet of N'. Since N' and P_c are both simple level-3 networks, we have by Lemma 2.30 that P_c is a crucial trinet of N'. Then, again by Lemma 2.30, G is the underlying generator of N'. Note that the 65 different level-3 generators (3.1, 3.2, ..., 3.65) can be found in Appendix A.

First, assume that N is not a network as in Figure 4.1. Since $Tn(N) = Tn(N')$, we know from the proof of Lemma 5.2 that $N = N'$ (after possibly relabelling sides).

Now, assume that N is a network as in Figure 4.1. Then, N has underlying generator 3.8, at least one leaf on side A or B and no leaves on sides C, D, E, F, G, H, I and J. Earlier we saw that N and N' have the same underlying generator. So, N and N' have both underlying generator 3.8. Note that generator 3.8 can be found in Figure 6.1 with the labels of its sides.

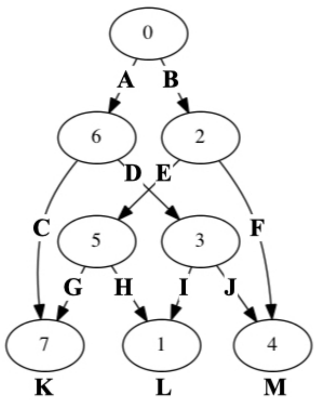


Figure 6.1: Generator 3.8 with labelled sides

A set of crucial sides has three elements, namely sides K, L and M. Let k be the leaf on side K, l the leaf on side L, m the leaf on side M and x the leaf on side A or B in N. From the proof of Lemma 5.2 we know that leaves k, l, m are, respectively, on sides K, L, M in N' (after possibly relabelling sides). Consider the quarnet on $\{k, l, m, x\}$. Then, since there are no symmetries, leaf x is on the same side in N' as it is in N. Now, generator 3.8 has the same leaves on each side for N and N'.

It remains to show that the leaves on each side are in the same order in N and N'.

Consider a side V of N with at least two leaves and two leaves v, v' on that side such that v' is below v. Then, there exists a side W which is a outdegree-0 vertex that is below side V. Let w be the leaf on side W in N. Now, consider the trinet P on $\{v, v', w\}$. Then, we have that the leaves v and v' are on the same side of trinet P. Moreover, since v' is below v in N, v' is below v in P. Then, since $Tn(N') = Tn(N)$, v' is below v in N'. So, the order of the leaves on each of the sides are the same in N and N'.

Note that we indeed need the leaf w to distinguish the order of the leaves, because without the leaf w a part of the generator would be left out and the leaves u and u' would become 'cherry's' without any order.

Now, we can conclude that $N = N'$, since (after possibly relabelling sides) both networks have the same underlying generator, the same leaves on each side, and the same order of the leaves on each side. □

Finally, we are able to combine the results to prove that level-3 networks are encoded by quarnets, which is the main result of this thesis.

Corollary 6.5. *Every binary recoverable level-3 network N on X, with $|X| \geq 4$, is encoded by its set of quarnets.*

Proof. Follows from Theorem 2.22, Lemma 6.4, Corollary 3.3 and the fact that level-1 networks are encoded by their trinets [5]. □

Chapter 7

Conclusion and discussion

In this chapter there are some conclusions and a discussion.

7.1 Conclusion

First, we give some conclusions. We have proved that *not all* recoverable rooted binary level-3 networks with at least three leaves are weakly encoded by their trinets, but *most* networks are. More precisely, the class of binary recoverable level-3 networks, except for networks with a biconnected component as in Figure 4.6 (i.e. with a biconnected component with underlying generator 3.8, at least one cut-arc on side A or B and no cut-arcs on sides C, D, E, F, G, H, I and J) is weakly encoded by trinets. Further, although not all level-3 networks are weakly encoded by their trinets, we were able to prove that *all* recoverable rooted binary level-3 networks with at least four leaves are encoded by their quarnets.

7.2 Discussion

Now, we will give a discussion. The results we proved for recoverable rooted binary level-3 networks can be extended to higher level networks in various ways. First, it would be of great interest to investigate which level-k networks ($k \geq 4$) are encoded by trinets or quarnets. Further, we can look for the largest class of level-k networks ($k \geq 4$) that is weakly encoded by trinets or quarnets. For level-k networks for which it does not hold and therefore has to be excluded, we can find a counter-example. For the other level-k networks, we can prove that they are (weakly) encoded.

Some methods that are used in the proofs for level-2 and level-3 networks can also be used for level-k networks ($k \geq 4$). This concerns level-k generators with one or two crucial side(s) and without symmetry. The proof for these generators does not depend on k. Therefore, the proofs for level-k ($k \geq 4$) generators with one or two crucial side(s) and without symmetry can be given in the same way as the proofs for level-2 and level-3 generators of this type that are given in this thesis.

In this thesis we have looked to networks that are encoded by trinets or quarnets (networks on three or four leaves). It might also be interesting to investigate which level-k networks are encoded by networks with five or more leaves. For example, we can investigate for a certain k if all level-k networks are encoded by networks on l leaves for a certain l. Note that in order to get the most strong result, we need l to be as small as possible. Further, note that if a network is encoded by networks on p leaves, then the network is also encoded by networks on q leaves if $p > q$.

Now, we have mentioned many ideas for further research about level-k networks with $k \geq 4$, but there are some large difficulties for these level-k networks. The first difficulty is that the number of level-k grows very rapidly as k increases. Therefore, it is no longer possible to write the proofs in the way we did for level-2 and level-3 networks. We have to find a shorter and more efficient way to investigate the level-k networks with $k \geq 4$. Possibly, there is another way to decompose the level-k networks in order to investigate these networks in a more efficient way. The other difficulty for level-k networks with $k \geq 4$ is that there are networks that has no crucial trinet. A solution has to be found for this.

Furthermore, this thesis about level-3 networks can give some ideas for an algorithm to reconstruct level-3 networks (that are encoded by quarnets) from their sets of quarnets.

Finally, it could be of some interest to compare the results of this thesis which holds for rooted binary level-3 networks with the current results for networks that are not rooted or not binary.

Bibliography

[1] Magnus Bordewich, Britta Dorn, Simone Linz, and Rolf Niedermeier. Algorithms and Complexity in Phylogenetics (Dagstuhl Seminar 19443). *Dagstuhl Reports*, 9(10):134–151, 2020.

[2] Philippe Gambette, Vincent Barry, and Christophe Paul. The structure of level-k phylogenetic networks. In *CPM: Combinatorial Pattern Matching*, volume 5577 of *Lecture Notes in Computer Science*, pages 289–300, Lille, France, June 2009. Received from: `http://igm.univ-mlv.fr/~gambette/ProgGenerators.php`.

[3] Katharina Huber, Leo van Iersel, Vincent Moulton, and Taoyang Wu. How much information is needed to infer reticulate evolutionary histories? *Systematic Biology*, 64(1):102–111, 2015.

[4] Leo van Iersel. Hoe zijn ze verwant? *Nieuw Archief voor Wiskunde*, 5/16(3):174–178, 2015. `http://www.nieuwarchief.nl/serie5/pdf/naw5-2015-16-3-174.pdf`.

[5] Leo van Iersel and Vincent Moulton. Trinets encode tree-child and level-2 phylogenetic networks. *Journal of Mathematical Biology*, 68(7):1707–1729, 2014.

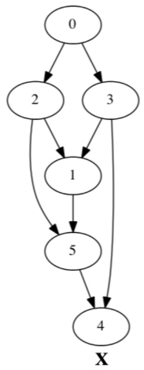
Appendix A

Labelled level-3 generators

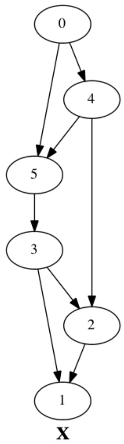
From [2] the 65 level-3 generators can be obtained. These level-3 generators are based on the work of Steven Kelk. Note that these generators are numbered $(3.1, 3.2, \ldots, 3.65)$ in the order as they are given in [2]. In the following sections the 65 level-3 generators can be found in the order as they are discussed in the proof of Lemma 5.2. This order of the generators is based on their properties, as can be found in Appendix B. Note that the labels of the sides that are explicitly used during the proof of Lemma 5.2 can be found on each of the generators. Further, note that the numbers of the vertices are not used in this thesis.

A.1 1 crucial side, no symmetry

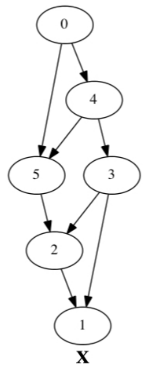
In this section the 8 level-3 generators with 1 crucial side and without symmetry can be found with some labels.



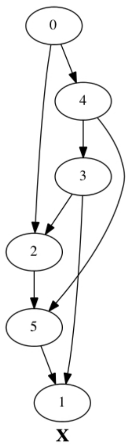
(a) Generator 3.15



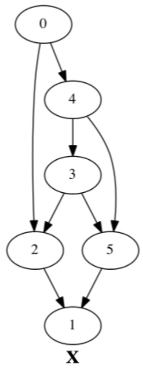
(b) Generator 3.19



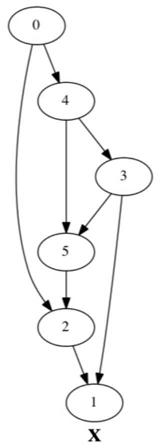
(c) Generator 3.20



(d) Generator 3.23

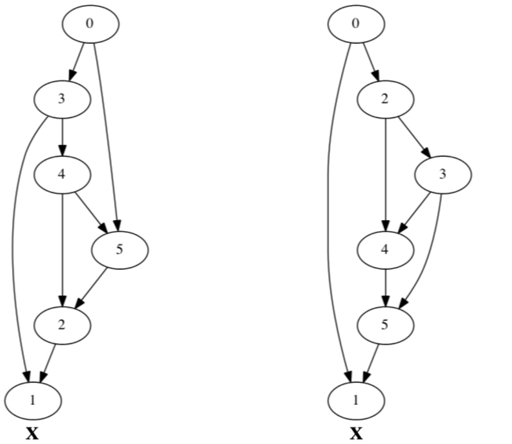


(e) Generator 3.24



(f) Generator 3.25

Figure A.1: Partly labelled level-3 generators $3.15, 3.19, 3.20, 3.23, 3.24$ and 3.25

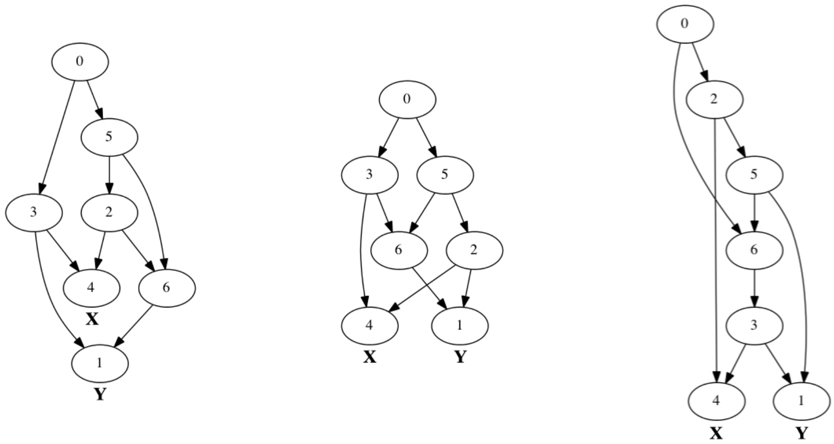


(a) Generator 3.32 (b) Generator 3.62

Figure A.2: Partly labelled level-3 generators 3.32 and 3.62

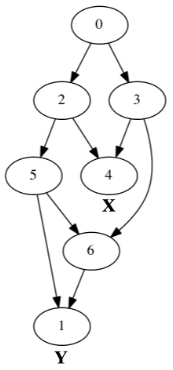
A.2 2 crucial sides, no symmetry

In this section the 22 level-3 generators with 2 crucial sides and without symmetry can be found with some labels.

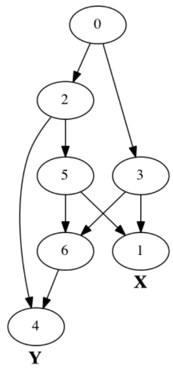


(a) Generator 3.4 (b) Generator 3.5 (c) Generator 3.9

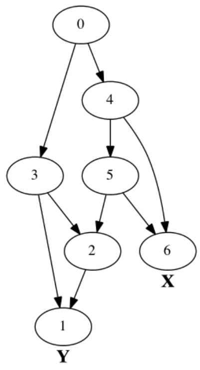
Figure A.3: Partly labelled level-3 generators $3.4, 3.5$ and 3.9



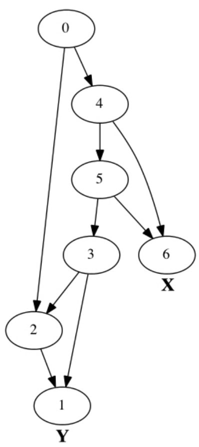
(a) Generator 3.12



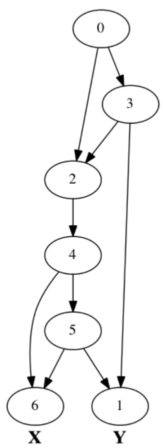
(b) Generator 3.13



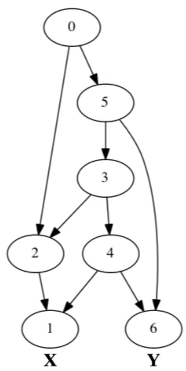
(c) Generator 3.17



(d) Generator 3.21

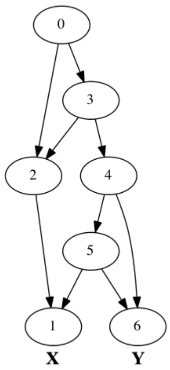


(e) Generator 3.27

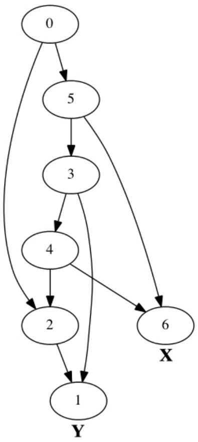


(f) Generator 3.29

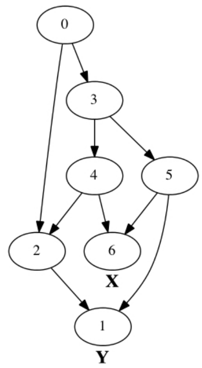
Figure A.4: Partly labelled level-3 generators $3.12, 3.13, 3.17, 3.21, 3.27$ and 3.29



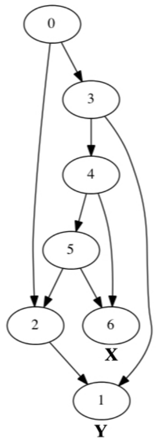
(a) Generator 3.30



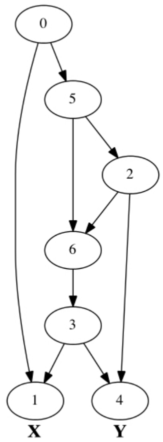
(b) Generator 3.33



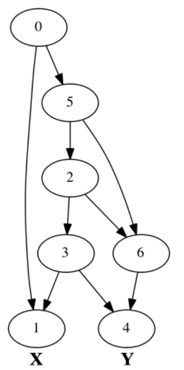
(c) Generator 3.34



(d) Generator 3.35

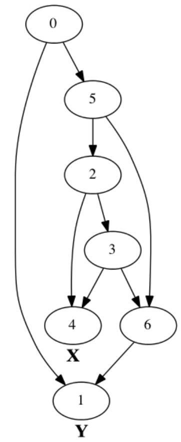


(e) Generator 3.41

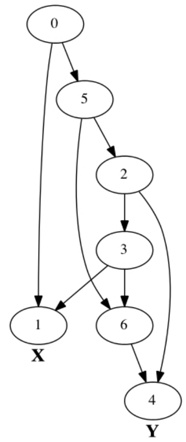


(f) Generator 3.42

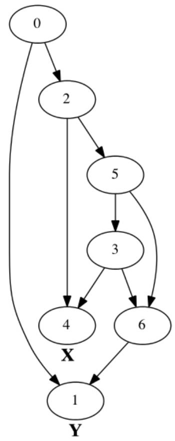
Figure A.5: Partly labelled level-3 generators $3.30, 3.33, 3.34, 3.35, 3.41$ and 3.42



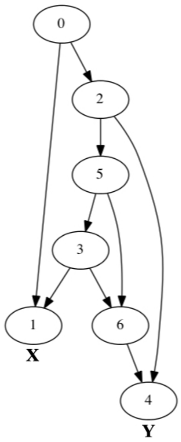
(a) Generator 3.43



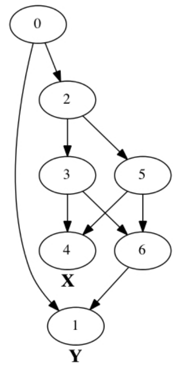
(b) Generator 3.44



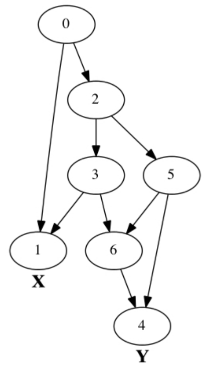
(c) Generator 3.48



(d) Generator 3.49

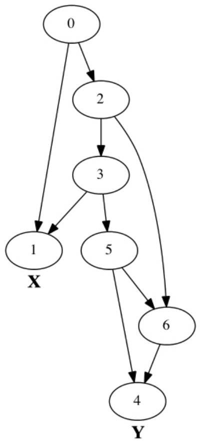


(e) Generator 3.54



(f) Generator 3.55

Figure A.6: Partly labelled level-3 generators $3.43, 3.44, 3.48, 3.49, 3.54$ and 3.55

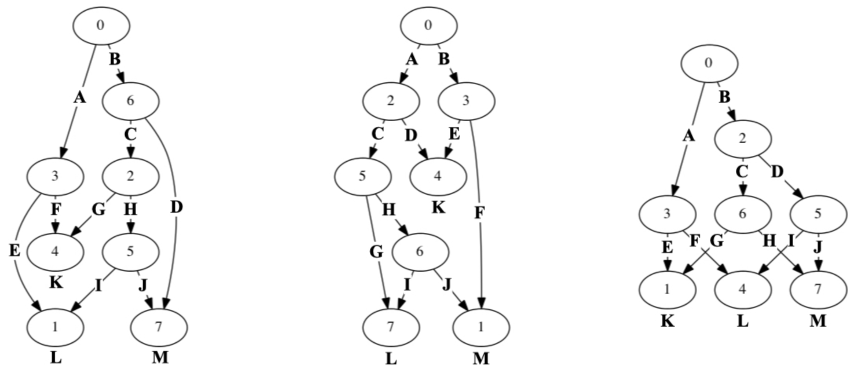


(a) Generator 3.59

Figure A.7: Partly labelled level-3 generator 3.59

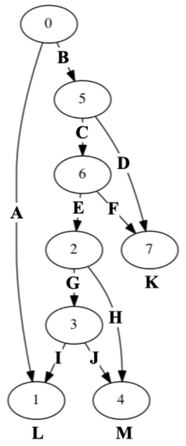
A.3 3 crucial sides, no symmetry

In this section the 11 level-3 generators with 3 crucial sides and without symmetry can be found with their labels.

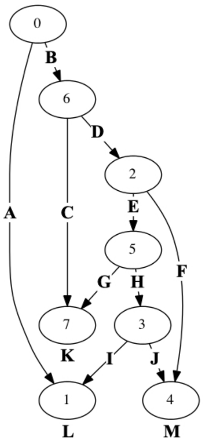


(a) Generator 3.7 (b) Generator 3.10 (c) Generator 3.14

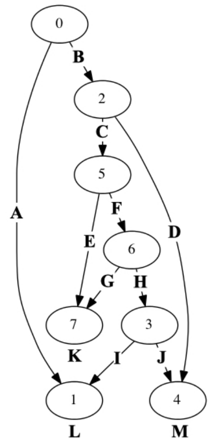
Figure A.8: Labelled level-3 generators $3.7, 3.10$ and 3.14



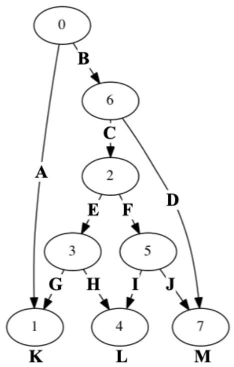
(a) Generator 3.39



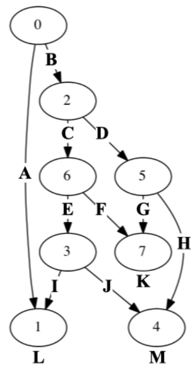
(b) Generator 3.45



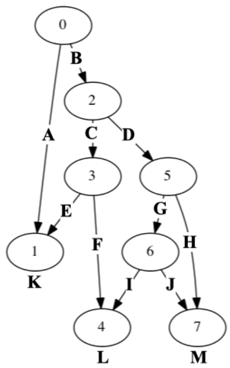
(c) Generator 3.46



(d) Generator 3.50

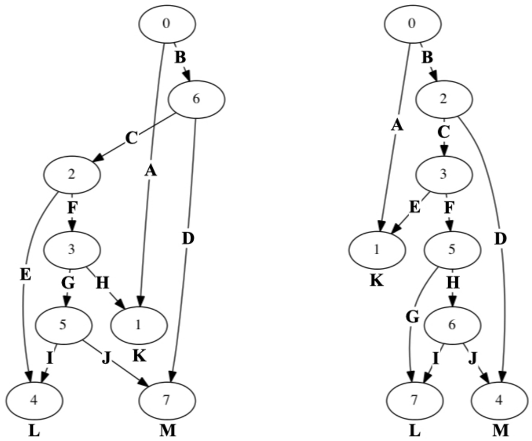


(e) Generator 3.51



(f) Generator 3.52

Figure A.9: Labelled level-3 generators $3.39, 3.45, 3.46, 3.50, 3.51$ and 3.52

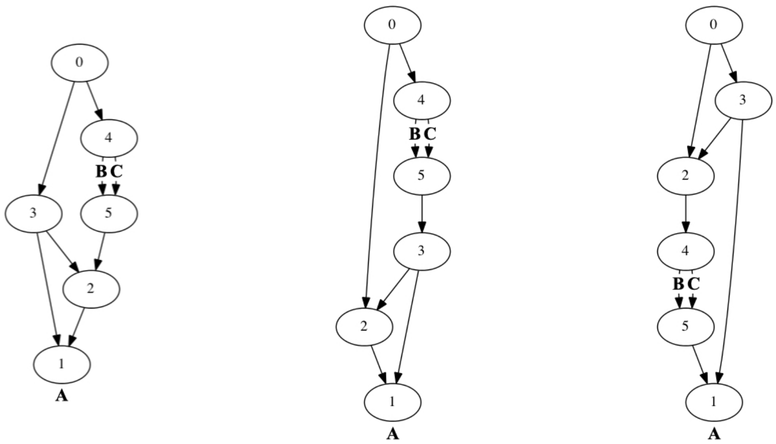


(a) Generator 3.57 (b) Generator 3.60

Figure A.10: Labelled level-3 generators 3.57 and 3.60

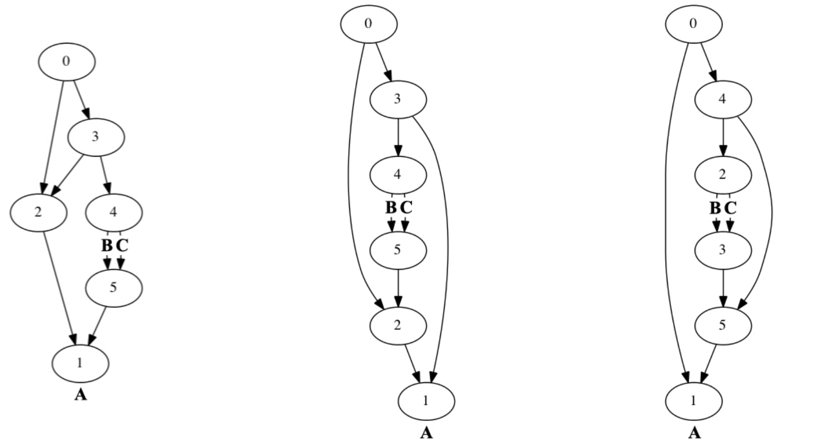
A.4 2 crucial sides, 1 set of parallel arcs, no other symmetry

In this section the 6 level-3 generators with 2 crucial sides, 1 set of parallel arcs and without other symmetry can be found with some labels. Note that there is only symmetry caused by the parallel arcs.



(a) Generator 3.18 (b) Generator 3.22 (c) Generator 3.28

Figure A.11: Partly labelled level-3 generators $3.18, 3.22$ and 3.28

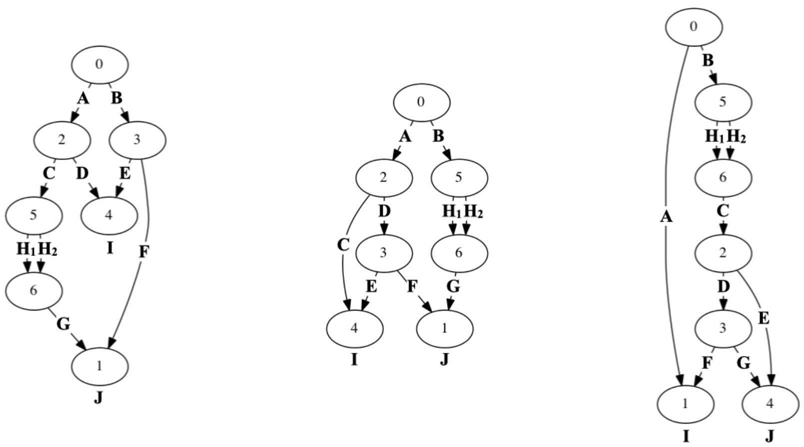


(a) Generator 3.31 (b) Generator 3.36 (c) Generator 3.65

Figure A.12: Partly labelled level-3 generators $3.31, 3.36$ and 3.65

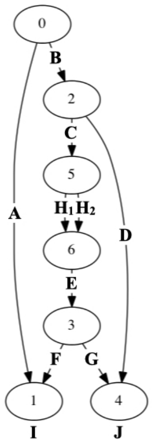
A.5 3 crucial sides, 1 set of parallel arcs, no other symmetry

In this section the 7 level-3 generators with 3 crucial sides, 1 set of parallel arcs and without other symmetry can be found with their labels. Note that there is only symmetry caused by the parallel arcs.

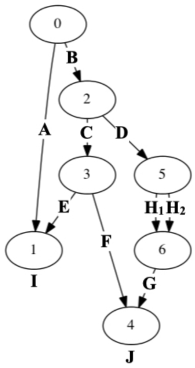


(a) Generator 3.11 (b) Generator 3.38 (c) Generator 3.40

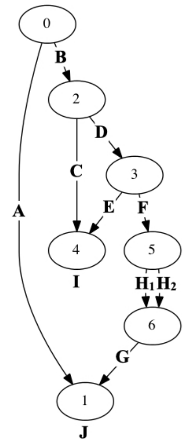
Figure A.13: Labelled level-3 generators $3.11, 3.3$ and 3.40



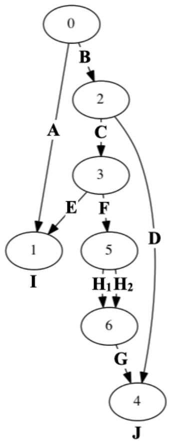
(a) Generator 3.47



(b) Generator 3.53



(c) Generator 3.56

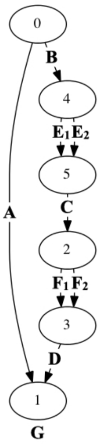


(d) Generator 3.61

Figure A.14: Labelled level-3 generators $3.47, 3.53, 3.56$ and 3.61

A.6 3 crucial sides, 2 sets of parallel arcs, no other symmetry

In this section the level-3 generator with 3 crucial sides, 2 sets of parallel arcs and without other symmetry can be found with its labels. Note that there is only symmetry caused by the parallel arcs.

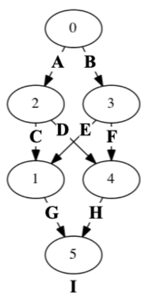


(a) Generator 3.64

Figure A.15: Labelled level-3 generator 3.64

A.7 1 crucial side, symmetry, no parallel arcs

In this section the level-3 generator with 1 crucial side, symmetry and without parallel arcs can be found with its labels.

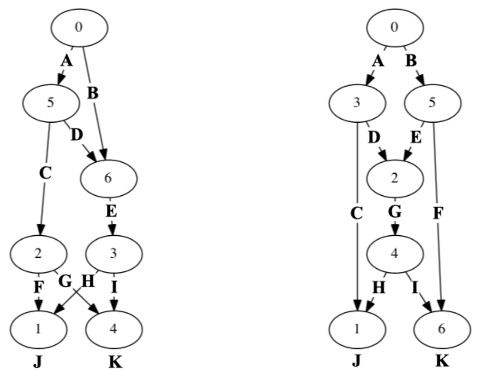


(a) Generator 3.16

Figure A.16: Labelled level-3 generator 3.16

A.8 2 crucial sides, symmetry, no parallel arcs

In this section the 2 level-3 generators with 2 crucial sides, symmetry and without parallel arcs can be found with their labels.

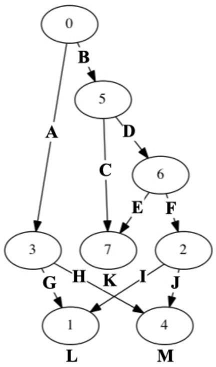


(a) Generator 3.3 (b) Generator 3.26

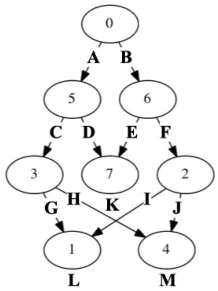
Figure A.17: Labelled level-3 generators 3.3 and 3.26

A.9 3 crucial sides, symmetry, no parallel arcs

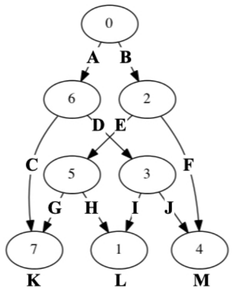
In this section the 5 level-3 generators with 3 crucial sides, symmetry and without parallel arcs can be found with their labels.



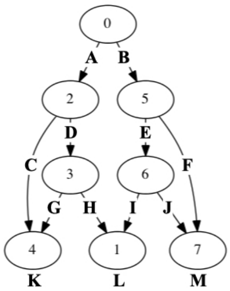
(a) Generator 3.1



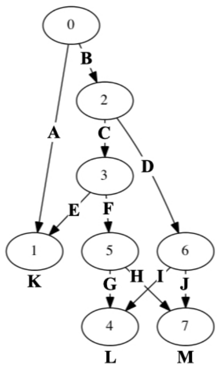
(b) Generator 3.6



(c) Generator 3.8



(d) Generator 3.37

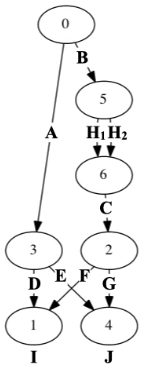


(e) Generator 3.58

Figure A.18: Labelled level-3 generators $3.1, 3.6, 3.8, 3.37$ and 3.58

A.10 3 crucial sides, 1 set of parallel arcs, other symmetry

In this section the level-3 generator with 3 crucial sides, 1 set of parallel arcs and other symmetry can be found with their labels. Note that this generator has also symmetry that is not caused by the set of parallel arcs.

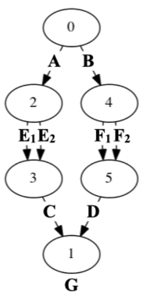


(a) Generator 3.2

Figure A.19: Labelled level-3 generator 3.2

A.11 3 crucial sides, 2 sets of parallel arcs, other symmetry

In this section the level-3 generator with 3 crucial sides, 2 sets of parallel arcs and other symmetry can be found with their labels. Note that this generator has also symmetry that is not caused by the sets of parallel arcs.



(a) Generator 3.63

Figure A.20: Labelled level-3 generator 3.63

Appendix B

Properties of level-3 generators

The level-3 generators, that can be found in Appendix A, are ordered in a way that is used in the proof of Lemma 5.2. To understand this way of ordering we will look to four different properties of a generator.

The first property, property A, is the number of pairs of parallel arcs in a generator. So, if the number for property A is 0, then the generator has no parallel arcs. Property B is the number of indegree-2 outdegree-0 vertices in a generator. Note that there is always at least one of such a vertex in a level-3 generator. Now, property C is the number of crucial sides, which can be found by adding the quantities for properties A and B. Property D and E are about symmetry. A generator has property D if the generator has any symmetry, i.e. relabelling the sides of the generator gives an isomorphic generator. If a generator has property D, it is denoted with 'yes', and otherwise it is denoted with 'no'. A generator has property E if the generator has symmetry that is not caused by parallel arcs. If a generator has property E, it is denoted with 'yes', otherwise it is denoted with 'no'.

For each of the 65 level-3 generators the properties A, B, C, D and E can be found in Tables B.1 and B.2. Note that Table B.1 contains the generators without symmetry and Table B.2 the generators with symmetry. Furthermore, note that the generators are sorted on the properties in such a way that it is useful for the proof of Lemma 5.2. In other words, in the proof of Lemma 5.2 the same order is used for the generators as in Tables B.1 and B.2.

Number	Prop. A	Prop. B	Prop. C	Prop. D	Prop. E
15	0	1	1	no	no
19	0	1	1	no	no
20	0	1	1	no	no
23	0	1	1	no	no
24	0	1	1	no	no
25	0	1	1	no	no
32	0	1	1	no	no
62	0	1	1	no	no
4	0	2	2	no	no
5	0	2	2	no	no
9	0	2	2	no	no
12	0	2	2	no	no
13	0	2	2	no	no
17	0	2	2	no	no
21	0	2	2	no	no
27	0	2	2	no	no
29	0	2	2	no	no
30	0	2	2	no	no
33	0	2	2	no	no
34	0	2	2	no	no
35	0	2	2	no	no
41	0	2	2	no	no
42	0	2	2	no	no
43	0	2	2	no	no
44	0	2	2	no	no
48	0	2	2	no	no
49	0	2	2	no	no
54	0	2	2	no	no
55	0	2	2	no	no
59	0	2	2	no	no
7	0	3	3	no	no
10	0	3	3	no	no
14	0	3	3	no	no
39	0	3	3	no	no
45	0	3	3	no	no
46	0	3	3	no	no
50	0	3	3	no	no
51	0	3	3	no	no
52	0	3	3	no	no
57	0	3	3	no	no
60	0	3	3	no	no

Table B.1: Properties of level-3 generators without symmetry

Number	Prop. A	Prop. B	Prop. C	Prop. D	Prop. E
18	1	1	2	yes	no
22	1	1	2	yes	no
28	1	1	2	yes	no
31	1	1	2	yes	no
36	1	1	2	yes	no
65	1	1	2	yes	no
11	1	2	3	yes	no
38	1	2	3	yes	no
40	1	2	3	yes	no
47	1	2	3	yes	no
53	1	2	3	yes	no
56	1	2	3	yes	no
61	1	2	3	yes	no
64	2	1	3	yes	no
16	0	1	1	yes	yes
3	0	2	2	yes	yes
26	0	2	2	yes	yes
6	0	3	3	yes	yes
1	0	3	3	yes	yes
8	0	3	3	yes	yes
37	0	3	3	yes	yes
58	0	3	3	yes	yes
2	1	2	3	yes	yes
63	2	1	3	yes	yes

Table B.2: Properties of level-3 generators with symmetry