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Harmonic analysis, Representation theory / *Analyse harmonique, Théorie des représentations*

Completeness of coherent state subsystems for nilpotent Lie groups

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Abstract. Let G be a nilpotent Lie group and let π be a coherent state representation of G . The interplay between the cyclicity of the restriction $\pi|_{\Gamma}$ to a lattice $\Gamma \leq G$ and the completeness of subsystems of coherent states based on a homogeneous G -space is considered. In particular, it is shown that necessary density conditions for Perelomov's completeness problem can be obtained via density conditions for the cyclicity of $\pi|_{\Gamma}$.

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1. Introduction

Let G be a connected unimodular Lie group and let (π, \mathcal{H}_{π}) be an irreducible unitary representation of G . For a unit vector $\eta \in \mathcal{H}_{\pi}$, consider its orbit under the action π on \mathcal{H}_{π} ,

$$\pi(G)\eta = \{\pi(g)\eta : g \in G\}. \quad (1)$$

As π is irreducible, $\pi(G)\eta$ is complete in \mathcal{H}_{π} . Two elements $\pi(g_1)\eta$ and $\pi(g_2)\eta$ differ from one another up to a phase factor, i.e. determine the same state or ray, only if $\pi(g_2^{-1}g_1)\eta \in \mathbb{C}\eta$.

Let $H \leq G$ be a closed subgroup that stabilises the state defined by $\eta \in \mathcal{H}_{\pi}$, i.e.

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H, \quad (2)$$

where $\chi : H \rightarrow \mathbb{T}$ is a unitary character of H . Denote by $X = G/H$ the associated homogeneous G -space and let $\sigma : X \rightarrow G$ be a cross-section for the canonical projection $p : G \rightarrow X$. Then the system of coherent vectors

$$\{\eta_x\}_{x \in X} = \{\pi(\sigma(x))\eta\}_{x \in X}, \quad (3)$$

determine a π -system of coherent states based on X , in the sense of [24, 29].

It will be assumed that $X = G/H$ is unimodular, i.e. X admits a G -invariant positive Radon measure μ_X , and that η is *admissible*, that is,

$$\int_X |\langle \eta, \eta_x \rangle|^2 d\mu_X(x) < \infty. \quad (4)$$

Then there exists an admissibility constant $d_{\pi, \eta} > 0$ such that

$$\int_X |\langle f, \eta_x \rangle|^2 d\mu_X(x) = d_{\pi, \eta}^{-1} \|f\|_{\mathcal{H}_\pi}^2, \quad \text{for all } f \in \mathcal{H}_\pi. \quad (5)$$

The identity (5) implies, in particular, that the system (3) is overcomplete, i.e. the system $\{\eta_x\}_{x \in X}$ contains proper subsystems which are complete in \mathcal{H}_π .

For an irreducible representation (π, \mathcal{H}_π) of G that is square-integrable modulo the center $Z = Z(G)$ (resp. the kernel $K = \ker(\pi)$), any vector $\eta \in \mathcal{H}_\pi$ satisfies (2) and (4) for $H = Z$ (resp. $H = K$). Another common choice [12, 22, 26, 29] for the index space $X = G/H$ is a symplectic G -space or a homogeneous Kähler manifold that arises as a phase space in geometric quantization [34]. Subgroups $H \leq G$ defining such a phase space do not need to satisfy (2) for all $\eta \in \mathcal{H}_\pi$ and might not be contained in the isotropy group of a chosen η .

In [24, 26], a particular focus is on coherent states for which the stabilising subgroup $H \leq G$ is assumed to be maximal with the property (2), that is, $H = G_{[\eta]}$, where

$$G_{[\eta]} := \{g \in G : \pi(g)\eta = e^{i\phi(g)}\eta\} \quad (6)$$

is the stabiliser of η for the G -action in the projective Hilbert space $P(\mathcal{H}_\pi)$. The associated coherent states are so-called *Perelomov-type coherent states*; see Section 4.

Perelomov's completeness problem [24, 26] concerns the completeness of subsystems arising from discrete subgroups $\Gamma \leq G$ for which the volume of $\Gamma \backslash X$ is finite. More explicitly, subsystems parametrised by an orbit $\Gamma' := \Gamma \cdot o$ of the base point $o := eH \in X$,

$$\{\eta_{\gamma'}\}_{\gamma' \in \Gamma'} = \{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma'}. \quad (7)$$

Criteria for the completeness of subsystems (7) involving the volume of the coset space $\Gamma \backslash X$ and the admissibility constant $d_{\pi, \eta} > 0$ were posed as a problem in [24, p. 226] and [26, p. 44]. Note that if $H = G_{[\eta]}$, then $X = G/G_{[\eta]}$ depends on η , and so does the volume of $\Gamma \backslash G/G_{[\eta]}$.

The classical example of coherent states arises from the Heisenberg group $G = \mathbb{H}^1$ and the Schrödinger representation $(\pi, L^2(\mathbb{R}))$ of \mathbb{H}^1 . For any $\eta \in L^2(\mathbb{R}) \setminus \{0\}$, the stabiliser $G_{[\eta]}$ defined in (6) coincides with the centre $Z(\mathbb{H}^1)$ of \mathbb{H}^1 , and $X = G/G_{[\eta]} \cong \mathbb{R}^2$. Therefore, the coherent state system (3) is parametrised by the classical phase space \mathbb{R}^2 and the subsystem (7) associated to $\Gamma \subset \mathbb{H}^1$ is parametrised by a lattice $\Gamma' \subset \mathbb{R}^2$. If the square-integrable representation (mod Z) π is treated as a projective representation ρ of $G/G_{[\eta]} \cong \mathbb{R}^2$, then the coherent vectors (3) and the subsystem (7) arise as orbits of \mathbb{R}^2 and Γ' , respectively. In particular, a subsystem $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma'}$ is complete in $L^2(\mathbb{R})$ if, and only if, η is a cyclic vector for $\rho|_{\Gamma'}$, i.e. the linear span of $\rho(\Gamma')\eta$ is dense in $L^2(\mathbb{R})$. This shows that Perelomov's completeness problem for the Heisenberg group is equivalent to determining whether a vector is cyclic for the restriction $\rho|_{\Gamma'}$. If η is the Gaussian, the cyclicity of η has been completely characterised in [2, 23] (see also [21]) in terms of the co-volume or density of the lattice. The necessity of these density conditions have been shown to hold for arbitrary vectors and in arbitrary dimensions [28], but a density condition alone is not sufficient for describing the cyclicity of the Gaussian in higher-dimensions [7, 27]. The criteria [2, 23, 28] coincide with the density conditions characterising the cyclicity of the restricted projective representations as obtained in, e.g. [3, 30].

In other settings than the Heisenberg group, the stabilisers $G_{[\eta]}$ defined in (6) do not need to be normal subgroups and could depend crucially on the vector $\eta \in \mathcal{H}_\pi \setminus \{0\}$. For example, this occurs for the holomorphic discrete series π of $G = \text{PSL}(2, \mathbb{R})$, where $G_{[\eta]} = \text{PSO}(2)$ for a class of rotation-invariant vectors η . Hence, the coherent vectors (3) do not arise as orbits of a (projective)

representation of $G/G_{[\eta]}$ and the subsystems (7) are not parametrised by an associated discrete subgroup. Perelomov's problem for the highest weight vector has been studied for this setting in [9, 10, 25], and the criteria for the cyclicity of $\pi|_{\Gamma}$ are quite different from the completeness of coherent state subsystems; see [31, Section 9.1] for an overview.

Of particular interest are representations and vectors that support a system of coherent states based on an index manifold $X = G/H$ with additional properties, such as a symplectic [16, 17] or complex structure [13, 18]. For nilpotent Lie groups, another common choice (cf. [26, Section 10]) is the manifold X to be the corresponding coadjoint orbit \mathcal{O}_{π} of the representation π , which forms the classical phase space, like in the special case of the Heisenberg group.

The purpose of this note is to combine characterisations of coherent state representations [13, 16, 18] and criteria for the cyclicity of restricted representations [3, 31] to obtain necessary density conditions for (variants of) Perelomov's completeness problem on nilpotent Lie groups.

The first result on the completeness of subsystems concerns π -systems of coherent states based on the coadjoint orbit \mathcal{O}_{π} . (cf. Section 2 for the precise definitions.)

Theorem 1. *Let G be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose (π, \mathcal{H}_{π}) is an irreducible representation of G that admits an admissible vector $\eta \in \mathcal{H}_{\pi} \setminus \{0\}$ defining a π -system of coherent states based on a homogeneous G -space $X = G/H \cong \mathcal{O}_{\pi}$, with admissibility constant $d_{\pi, \eta} > 0$. Then*

- (i) $H = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_{\pi}}\};$
- (ii) *If $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete in \mathcal{H}_{π} , then $\text{covol}(p(\Gamma))d_{\pi, \eta} \leq 1$.*

(The value $\text{covol}(p(\Gamma))d_{\pi, \eta}$ is independent of the normalisation of G -invariant measure on X .)

Theorem 1 considers π -systems of coherent states parametrised by the canonical phase space \mathcal{O}_{π} (cf. [26, Section 10]), and provides a necessary condition for the completeness of associated subsystems. The representations satisfying the hypothesis of Theorem 1 are called *coherent state representations* in [16], and are characterised as those being an irreducible representation whose associated coadjoint orbit is a linear variety. The considered representations are therefore essentially square-integrable, like in the special case of the Heisenberg group.

The second result concerns π -systems of coherent states associated to vectors yielding a symplectic projective orbit (cf. Section 4 for the precise definitions.)

Theorem 2. *Let G be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose (π, \mathcal{H}_{π}) is an irreducible representation of G that admits an admissible vector $\eta \in \mathcal{H}_{\pi} \setminus \{0\}$ yielding a symplectic orbit and defines a π -system of coherent states based on $X = G/G_{[\eta]}$, with admissibility constant $d_{\pi, \eta} > 0$. Then*

- (i) $G_{[\eta]} = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_{\pi}}\};$
- (ii) *If $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete in \mathcal{H}_{π} , then $\text{covol}(p(\Gamma))d_{\pi, \eta} \leq 1$.*

In contrast to Theorem 1, the index manifold $X = G/G_{[\eta]}$ in Theorem 2 is selected via the maximal subgroup (6) stabilising the state determined by $\eta \in \mathcal{H}_{\pi} \setminus \{0\}$. The vectors $\eta \in \mathcal{H}_{\pi}$ yielding a symplectic orbit play a distinguished role in geometric quantization [12, 22]. Theorem 2 applies, in particular, to smooth vectors of a square-integrable representation (see Proposition 10) and to so-called *highest weight vectors* (see Remark 12).

The proofs of Theorem 1 and Theorem 2 are relatively simple and short, but they hinge on a combination of several non-trivial statements on coherent state representations [13, 16, 18] and density conditions for restricted discrete series [3, 31]. More explicitly, exploiting results of [13, 16, 18], it will be shown that the completeness of coherent state subsystems is equivalent to the admissible vector being a cyclic vector for a restricted *projective* representation; the necessary density conditions then being a direct consequence of [31].

Notation

For a complex vector space \mathcal{H} , the notation $P(\mathcal{H})$ will be used for its projective space, i.e. the space of all one-dimensional subspaces. The subspace or ray generated by $\eta \in \mathcal{H} \setminus \{0\}$ will be denoted by $[\eta] := \mathbb{C}\eta$. Henceforth, unless stated otherwise, G is a connected, simply connected nilpotent Lie group with exponential map $\exp : \mathfrak{g} \rightarrow G$. Haar measure on G is denoted by μ_G . If $\Lambda \leq G$ is a discrete subgroup, then the co-volume is defined as $\text{covol}(\Lambda) := \mu_{G/\Lambda}(G/\Lambda)$, where $\mu_{G/\Lambda}$ denotes G -invariant Radon measure on G/Λ .

2. Coherent state representations of nilpotent Lie groups

This section provides preliminaries on irreducible representations of nilpotent Lie groups and associated coherent states. References for these topics are the books [6] and [1, 26].

2.1. Coadjoint orbits

Let \mathfrak{g}^* denote the dual vector space of \mathfrak{g} . The coadjoint representation $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$ is defined by $\text{Ad}^*(g)\ell = \ell \circ \text{Ad}(g)^{-1}$ for $g \in G$ and $\ell \in \mathfrak{g}^*$. The stabiliser of $\ell \in \mathfrak{g}^*$ is the connected closed subgroup $G(\ell) = \{g \in G : \text{Ad}^*(g)\ell = \ell\}$, its Lie algebra is the annihilator subalgebra $\mathfrak{g}(\ell) = \{X \in \mathfrak{g} : \ell([Y, X]) = 0, \forall Y \in \mathfrak{g}\}$.

For $\ell \in \mathfrak{g}^*$, its *coadjoint orbit* is denoted by $\mathcal{O}_\ell := \text{Ad}^*(G)\ell$ and endowed with the relative topology from \mathfrak{g}^* . The orbit \mathcal{O}_ℓ is homeomorphic to $G/G(\ell)$; in notation: $\mathcal{O}_\ell \cong G/G(\ell)$.

2.2. Irreducible representations

A Lie subalgebra \mathfrak{p} of \mathfrak{g} is *subordinated* to $\ell \in \mathfrak{g}^*$ if $\ell(X) = 0$ for every $X \in [\mathfrak{p}, \mathfrak{p}]$. If \mathfrak{p} is subordinate to ℓ , then the map $\chi_\ell : \exp(\mathfrak{p}) \rightarrow \mathbb{T}$, $\chi_\ell(\exp(X)) = e^{2\pi i \ell(X)}$ defines a unitary character of $P = \exp(\mathfrak{p})$. The associated induced representation of G is denoted by $\pi_\ell = \pi(\ell, \mathfrak{p}) = \text{ind}_P^G(\chi_\ell)$.

For every π in the unitary dual \widehat{G} of G , there exists $\ell \in \mathfrak{g}^*$ and a subalgebra $\mathfrak{p} \subset \mathfrak{g}$, subordinate to ℓ , such that π is unitarily equivalent to $\pi_\ell = \pi(\ell, \mathfrak{p})$. A representation $\pi_\ell = \pi(\ell, \mathfrak{p})$, with \mathfrak{p} subordinate to $\ell \in \mathfrak{g}^*$, is irreducible if, and only if, \mathfrak{p} is a maximal subalgebra subordinated to $\ell \in \mathfrak{g}^*$ satisfying $\dim(\mathfrak{p}) = \dim(\mathfrak{g}) - \dim(\mathcal{O}_\ell)/2$, a so-called (*real*) *polarisation*.

Two irreducible induced representations $\text{ind}_{\exp(\mathfrak{p})}^G(\chi_\ell)$ and $\text{ind}_{\exp(\mathfrak{p}')}^G(\chi_{\ell'})$ are unitarily equivalent if and only if the linear functionals $\ell, \ell' \in \mathfrak{g}^*$ belong to the same coadjoint orbit. The orbit associated to the equivalence class $\pi \in \widehat{G}$ will also be denoted by \mathcal{O}_π .

2.3. Moment set

Let (π, \mathcal{H}_π) be an irreducible unitary representation of G . Denote by \mathcal{H}_π^∞ the space of smooth vectors for π , i.e. the space of $\eta \in \mathcal{H}_\pi$ for which $g \mapsto \pi(g)\eta$ is smooth.

The derived representation $d\pi : \mathfrak{g} \rightarrow L(\mathcal{H}_\pi^\infty)$ is defined by

$$d\pi(X)\eta = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(tX))\eta, \quad X \in \mathfrak{g}, \eta \in \mathcal{H}_\pi^\infty. \quad (8)$$

It can be extended complex linearly to a representation of the complexification $\mathfrak{g}_\mathbb{C}$ of \mathfrak{g} .

The *moment map* of π is the mapping $J_\pi : \mathcal{H}_\pi^\infty \rightarrow \mathfrak{g}^*$ defined by

$$J_\pi(\eta)(X) = \frac{1}{i} \frac{\langle d\pi(X)\eta, \eta \rangle}{\langle \eta, \eta \rangle}, \quad X \in \mathfrak{g}, \eta \in \mathcal{H}_\pi^\infty. \quad (9)$$

Note that the right-hand side of (9) only depends on the ray $[\eta]$ generated by $\eta \in \mathcal{H}_\pi^\infty \setminus \{0\}$.

The moment map J_π is equivariant with respect to the canonical G -actions on \mathcal{H}_π^∞ and \mathfrak{g}^* , i.e. $J_\pi(\pi(g)\eta)(X) = (\text{Ad}(g)^* J_\pi(\eta))(X)$ for $g \in G$, $X \in \mathfrak{g}$ and $\eta \in \mathcal{H}_\pi^\infty$. In particular, $J_\pi(G \cdot \eta)$ is the coadjoint orbit $\mathcal{O}_{J_\pi(\eta)}$ of $J_\pi(\eta) \in \mathfrak{g}^*$.

The *moment set* I_π of π is the closure $I_\pi := \overline{J_\pi(\mathcal{H}_\pi^\infty)}$ in \mathfrak{g}^* . Its relation to the coadjoint \mathcal{O}_π of $\pi \in \widehat{G}$ is

$$I_\pi = \overline{\text{conv}}(\mathcal{O}_\pi), \quad (10)$$

where $\overline{\text{conv}}$ denotes the closed convex hull; see [33, Theorem 4.2].

2.4. Coherent state representations

Henceforth, it is assumed that (π, \mathcal{H}_π) is non-trivial. Let $\eta \in \mathcal{H}_\pi$ be a unit vector and let $H \leq G$ be a closed subgroup such that there exists a unitary character $\chi : H \rightarrow \mathbb{T}$ satisfying

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H. \quad (11)$$

Denote $X := G/H$ and let μ_X be G -invariant Radon measure on X , which is unique up to scalar multiplication. Fix a Borel cross-section $\sigma : X \rightarrow G$ for the quotient map $p : G \rightarrow X$. The vector η is called *admissible* if

$$\int_X |\langle \eta, \pi(\sigma(x))\eta \rangle|^2 d\mu_X(x) < \infty. \quad (12)$$

A pair (η, χ) satisfying (11) and (12) is said to define a π -system of coherent states based on $X = G/H$. The condition (12) is independent of the particular choice of section σ .

For a π -system of coherent states, there exists an *admissibility constant* $d_{\pi, \eta} > 0$ such that, for all $f \in \mathcal{H}_\pi$,

$$\int_X |\langle f, \pi(\sigma(x))\eta \rangle|^2 d\mu_X(x) = d_{\pi, \eta}^{-1} \|f\|_{\mathcal{H}_\pi}^2. \quad (13)$$

For further properties on square-integrability modulo a subgroup, see, e.g. [17, 19].

An irreducible representation (π, \mathcal{H}_π) is called a *coherent state representation* if it admits a π -system of coherent states based on connected, simply connected homogeneous G -space X .¹

3. Completeness of coherent state subsystems

This section considers the relation between subsystems of coherent states parametrised by a simply connected G -space and lattice orbits of an associated projective representation.

3.1. Projective kernel

The *kernel* and *projective kernel* of a unitary representation (π, \mathcal{H}_π) of G are defined by

$$\ker(\pi) = \{g \in G : \pi(g) = I_{\mathcal{H}_\pi}\} \quad \text{and} \quad \text{pker}(\pi) = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_\pi}\},$$

respectively. If (π, \mathcal{H}_π) is non-trivial and irreducible, then $\text{pker}(\pi) \leq G$ is a connected, closed normal subgroup, and there exists $\chi_\pi : \text{pker}(\pi) \rightarrow \mathbb{T}$ such that $\pi(g) = \chi_\pi(g) I_{\mathcal{H}_\pi}$ for $g \in \text{pker}(\pi)$.

The following observation plays a key role in the sequel. Its proof hinges on [16, Lemma 3.5], which characterises coherent state representations π in terms of their coadjoint orbit \mathcal{O}_π .

Proposition 3. *Let $H \leq G$ be a connected subgroup. Suppose π admits a π -system of coherent states based on G/H . Then $H = \text{pker}(\pi)$. In particular, $H \leq G$ is normal.*

¹The definition of a coherent state representation used here is the same as in [16, 17, 19], but differs from the definition in [13, 14, 18], where the square-integrability assumption (12) is not part of the definition.

Proof. If π admits a pair (η, χ) satisfying (11) and (12), then π is unitarily equivalent to a subrepresentation of the induced representation $\text{ind}_H^G \chi$, see, e.g. [16, Proposition 1.2]. Since $H \leq G$ is assumed to be connected, it follows by [16, Lemma 3.5] that $H = G(\ell)$ for any $\ell \in \mathcal{O}_\pi$. By [4, Theorem 2.1], the projective kernel of an arbitrary irreducible representation π of G is given by $\text{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell)$. Therefore, $\text{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell) = H$. \square

The conclusion of Proposition 3 may fail for disconnected subgroups $H \leq G$ whenever π has a discrete kernel:

Remark 4. Let (π, \mathcal{H}_π) be an irreducible unitary representation of G .

- (a) If π is square-integrable modulo $K = \ker(\pi)$, then $\pi|_K$ satisfies (11) for the trivial character $\chi \equiv 1$ and any vector $\eta \in \mathcal{H}_\pi$ defines a π -system of coherent states based on G/K .
- (b) If π is square-integrable modulo $Z = Z(G)$, then $\pi|_Z$ satisfies (11) for the central character $\chi \in \hat{Z}$ and any vector $\eta \in \mathcal{H}_\pi$ defines a π -system of coherent states based on G/Z . Moreover, $\text{pker}(\pi) = Z(G)$ by [6, Corollary 4.5.4].

3.2. Necessary density conditions

A *uniform subgroup* $\Gamma \leq G$ is a discrete subgroup such that $\Gamma \backslash G$ is compact. For a nilpotent Lie group G , the uniformity of a discrete subgroup $\Gamma \leq G$ is equivalent to Γ being a lattice, i.e. having finite co-volume; see [6, Corollary 5.4.6].

The following result provides a criterium for cyclicity of restricted (projective) representations in terms of the lattice co-volume or density (cf. [31, Theorem 7.4]).

Theorem 5 ([31]). *Let (π, \mathcal{H}_π) be an irreducible, square-integrable projective unitary representation of a unimodular group G , with formal dimension $d_\pi > 0$. Let $\Gamma \leq G$ be a lattice. If there exists $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is complete in \mathcal{H}_π , then $\text{covol}(\Gamma)d_\pi \leq 1$.*

For a genuine representation π of G that is square-integrable modulo the centre $Z(G)$, a version of Theorem 5 can also be deduced from [3, Theorem 5]; see also [3, Theorem 3] for a converse in the setting of nilpotent Lie groups. However, in order to treat a representation π that is merely square-integrable modulo $\ker(\pi)$ (equivalently, $\text{pker}(\pi)$), the projective version of Theorem 5 is particularly convenient for the purposes of the present note.

The following completeness result for coherent state subsystems can simply be obtained by combining Proposition 3 and Theorem 5.

Theorem 6. *Let $H \leq G$ be a connected subgroup. Suppose (π, \mathcal{H}_π) is an irreducible representation that admits an admissible vector $\eta \in \mathcal{H}_\pi$ defining a π -system of coherent states based on $X = G/H$, with admissibility constant $d_{\pi, \eta} > 0$. Then*

- (i) $H = \text{pker}(\pi)$;
- (ii) *If $\Gamma \leq G$ is uniform and $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete, then $\text{covol}(p(\Gamma))d_{\pi, \eta} \leq 1$.*

Proof. By Proposition 3, the admissibility of π implies that $H = \text{pker}(\pi) \leq G$ is normal. Hence, the induced mapping $\pi' : G/H \rightarrow \mathcal{U}(\mathcal{H}_\pi)$, $x \mapsto \pi(\sigma(x))$ forms an irreducible projective representation of G/H . Since the measure μ_X is Haar measure on $X = G/H$, it follows that π' is square-integrable on G/H by the admissibility condition (12). In particular, the constant $d_{\pi, \eta} > 0$ in (13) coincides with the (unique) formal dimension $d_{\pi'} > 0$ of the projective representation (π', \mathcal{H}_π) normalised according to the G -invariant measure μ_X .

Suppose $\Gamma \leq G$ is a uniform subgroup. As in the proof of Proposition 3, the admissibility of π implies that $\text{pker}(\pi) = G(\ell)$ for any $\ell \in \mathcal{O}_\pi$. A combination of [6, Proposition 5.2.6] and [6, Theorem 5.1.11] therefore yields that $\Gamma \cap H$ is a uniform subgroup of $H = \text{pker}(\pi)$. Hence, the image $p(\Gamma)$ is a uniform subgroup of G/H by [6, Lemma 5.1.4 (a)].

In combination, applying Theorem 5 to (π', \mathcal{H}_π) and $p(\Gamma) \leq G/H$ yields the result. \square

Remark 7. The constant $d_{\pi,\eta} > 0$ coincides with the formal dimension $d_{\pi'} > 0$ of the projective representation $(\pi', \mathcal{H}_{\pi'})$ of $X = G/\text{pker}(\pi)$. In particular, the product $\text{covol}(p(\Gamma))d_{\pi'}$ is independent of the choice of G -invariant measure μ_X : if $\mu'_X = c \cdot \mu_X$ for $c > 0$, then $\text{covol}'(p(\Gamma)) = c \cdot \text{covol}(p(\Gamma))$ and $d'_{\pi'} = d_{\pi'}/c$.

Theorem 1 follows directly from Proposition 3 and Theorem 6:

Proof of Theorem 1. By assumption, there exists an admissible $\eta \in \mathcal{H}_{\pi}$ and associated character $\chi : H \rightarrow \mathbb{T}$ defining a π -system of coherent states based on $G/H \cong \mathcal{O}_{\pi}$. Since \mathcal{O}_{π} is simply connected, it follows that $H \subset G$ is connected, see, e.g. [11, Proposition 1.94]. The conclusions are therefore a direct consequence of Proposition 3 and Theorem 6. \square

4. Perelomov-type coherent states

Let (π, \mathcal{H}_{π}) be an irreducible representation of G . Then π yields an action of G on the projective spaces $P(\mathcal{H}_{\pi})$ and $P(\mathcal{H}_{\pi}^{\infty})$ by $g \cdot [\eta] = [\pi(g)\eta]$.

A system of *Perelomov-type coherent states* is a G -orbit in $P(\mathcal{H}_{\pi})$,

$$G \cdot [\eta] = \{[\pi(g)\eta] : g \in G\}.$$

Let $G_{[\eta]}$ be the isotropy group of $\eta \in \mathcal{H}_{\pi} \setminus \{0\}$ in the projective space $P(\mathcal{H}_{\pi})$,

$$G_{[\eta]} := \{g \in G : \pi(g)\eta \in \mathbb{C}\eta\}. \quad (14)$$

Denote by $X = G/G_{[\eta]}$ the associated homogeneous space and let $\sigma : X \rightarrow G$ be a Borel section for the quotient map $p : G \rightarrow X$. Then a Perelomov-type coherent state system is determined by the system of vectors,

$$\{\eta_x\}_{x \in X} = \{\pi(\sigma(x))\eta\}_{x \in X}.$$

See [24, Section 2] and [26, Chapter 2] for the basic properties of Perelomov-type states.

Let $\chi_{\eta} : G_{[\eta]} \rightarrow \mathbb{T}$ be the unitary character of $G_{[\eta]}$ such that $\pi(g)\eta = \chi_{\eta}(g)\eta$ for all $g \in G_{[\eta]}$. Note that $G_{[\eta]}$ is the maximal subgroup satisfying the property (11) for a chosen η .

The following sections consider Perelomov-type coherent states of vectors $\eta \in \mathcal{H}_{\pi}^{\infty} \setminus \{0\}$ with the property that $G/G_{[\eta]}$ has a symplectic or complex structure. Such systems are of particular interest for geometric quantization, see [22] and [26, Section 16].

4.1. Symplectic projective orbits

Following [12, 13], an orbit $G \cdot [\eta] = \{[\pi(g)\eta] : g \in G\}$ is called *symplectic* if $[\eta] \in P(\mathcal{H}_{\pi}^{\infty})$ and $G \cdot [\eta]$ is a symplectic submanifold of $P(\mathcal{H}_{\pi})$.

The following simple characterisation of symplectic orbits will be used below, see, e.g. [8, Theorem 26.8] or [5, Proposition 2.1] for proofs.

Lemma 8 ([8]). *Let $[\eta] \in P(\mathcal{H}_{\pi}^{\infty})$ and let $J_{\pi} : P(\mathcal{H}_{\pi}^{\infty}) \rightarrow \mathfrak{g}^*$ be the momentum map of π . The orbit $G \cdot [\eta]$ is symplectic if, and only if, the stabiliser $G_{[\eta]}$ is an open subgroup of $G(J_{\pi}(\eta))$.*

For the purposes of this note, the significance of a symplectic orbit is that its stabiliser subgroups coincides with the projective kernel, and hence does not depend on the chosen vector. This is demonstrated by the following proposition.

Proposition 9. *Suppose $\eta \in \mathcal{H}_{\pi}^{\infty} \setminus \{0\}$ is such that $G \cdot [\eta]$ is symplectic. Then $G_{[\eta]}$ is connected. In particular, if η is an admissible vector defining a π -system of coherent states based on $G/G_{[\eta]}$, then $G_{[\eta]} = \text{pker}(\pi)$.*

Proof. If $G \cdot [\eta]$ is symplectic, then $G \cdot [\eta]$ forms a Hamiltonian G -space, with momentum map $J_\pi : G \cdot [\eta] \rightarrow \mathfrak{g}^*$ given as in (9), see, e.g. [13, Section 2.5]. Set $\ell := J_\pi([\eta])$. Then, by Lemma 8, the stabiliser $G_{[\eta]}$ is an open subgroup of $G(\ell)$. Since $G(\ell)$ is connected (cf. Section 2.1), it follows that $G_{[\eta]} = G(\ell)$ is connected. The last assertion follows from Proposition 3. \square

The following provides a partial converse to Proposition 9.

Proposition 10. *Suppose (π, \mathcal{H}_π) is square-integrable modulo $\text{pker}(\pi)$. Then, for any $[\eta] \in P(\mathcal{H}_\pi^\infty)$, the orbit $G \cdot [\eta]$ is symplectic and $G_{[\eta]} = \text{pker}(\pi)$.*

Proof. Let $\eta \in \mathcal{H}_\pi^\infty \setminus \{0\}$ be fixed. The inclusion $\text{pker}(\pi) \subseteq G_{[\eta]}$ is immediate. Conversely, if $g \in G_{[\eta]}$, then

$$J_\pi([\pi(g)\eta]) = \frac{1}{i} \frac{\langle \pi(g)\eta, d\pi(X)\pi(g)\eta \rangle}{\langle \pi(g)\eta, \pi(g)\eta \rangle} = \frac{1}{i} \frac{\langle \eta, d\pi(X)\eta \rangle}{\langle \eta, \eta \rangle} = J_\pi([\eta]), \quad X \in \mathfrak{g},$$

so that by the G -equivariance of J_π it follows that $\text{Ad}^*(g)J_\pi([\eta]) = J_\pi([\eta])$. This means that $g \in G(J_\pi([\eta]))$, and it remains to show that $G(J_\pi([\eta])) \subseteq \text{pker}(\pi)$.

Since $\pi \in \widehat{G}$ is square-integrable modulo $\text{pker}(\pi)$, it is also square-integrable modulo $\ker(\pi)$, see, e.g., [4, Corollary 2.1]. It follows therefore by [6, Theorem 4.5.2] and [6, Theorem 3.2.3] that \mathcal{O}_π is a linear variety of the form $\mathcal{O}_\pi = \ell + \mathfrak{k}^\perp$ for $\ell \in \mathcal{O}_\pi$, with \mathfrak{k} being the Lie algebra of $\text{pker}(\pi)$. In addition, [6, Theorem 3.2.3] yields that $\mathfrak{g}(\ell) = \mathfrak{k}$ for $\ell \in \mathcal{O}_\pi$, so that $G(\ell) = \text{pker}(\pi)$ for $\ell \in \mathcal{O}_\pi$. By [33, Theorem 4.2] (see also Equation (10)) it follows, in particular, that

$$J_\pi([\eta]) \in J_\pi(P(\mathcal{H}_\pi^\infty)) \subseteq I_\pi = \overline{\text{conv}}(\mathcal{O}_\pi) = \mathcal{O}_\pi,$$

where $I_\pi := \overline{J_\pi(\mathcal{H}_\pi^\infty)}$ denotes the moment set of π . Therefore, $G(J_\pi([\eta])) = \text{pker}(\pi)$.

Lastly, since $G_{[\eta]} = \text{pker}(\pi) = G(J_\pi([\eta]))$ by the arguments above, the orbit $G \cdot [\eta]$ is symplectic by Lemma 8. \square

Proof of Theorem 2. If $G \cdot [\eta]$ is symplectic, then $G_{[\eta]}$ is connected by Proposition 9. Therefore, if η determines a π -system of coherent states based on $G/G_{[\eta]}$, the conclusions of Theorem 2 follow directly from Theorem 6. \square

4.2. Highest weight vectors

In [13, 18], an orbit $G \cdot [\eta] = \{[\pi(g)\eta] : g \in G\}$ is called *complex* if $[\eta] \in P(\mathcal{H}_\pi^\infty)$ and $G \cdot [\eta]$ is a complex submanifold of $P(\mathcal{H}_\pi)$.

The following lemma characterises complex orbits in terms of a (complex) stabiliser; cf. [13, Proposition 2.8] and [20, Lemma XV.2.3].

Lemma 11 ([13]). *Let $\mathfrak{s} = (\mathfrak{g})_\mathbb{C}$. For $[\eta] \in P(\mathcal{H}_\pi^\infty)$, let $\mathfrak{s}_{[\eta]} = \{X \in \mathfrak{s} : d\pi(X)\eta \in \mathbb{C} \cdot \eta\}$.*

The following assertions are equivalent:

- (i) *The orbit $G \cdot [\eta]$ is complex;*
- (ii) $\mathfrak{s}_{[\eta]} + \overline{\mathfrak{s}_{[\eta]}} = \mathfrak{s}$.

A stabiliser $\mathfrak{s}_{[\eta]}$ satisfying part (ii) of Lemma 11 is called *maximal* in [26, Section 2.4], where it is part of a principle for selecting coherent states that minimise the uncertainty principle. Such vectors and associated orbits play an important role in Berezin's quantization, see [26, Section 16]. In addition, vectors of this type are intimately related to highest weight modules and representations (cf. [18, 20]) and are also referred to as *highest weight vectors*.

Remark 12. By [13, Proposition 2.8], any complex orbit is automatically symplectic in the sense of Section 4.1. Theorem 2 applies therefore to highest weight vectors.

Remark 13. The significance of a complex orbit $G \cdot [\eta]$ is that the quotient manifold $G/G_{[\eta]}$ admits a complex structure (cf. [20, Section XV.2]). In turn, for certain (classes of) representations admitting highest weight vectors, the representation space may be realised as a space of holomorphic functions (see [26, Section 2.4] and [32]); in particular, see [15, Section 5] for complex orbits for the Heisenberg group. For nilpotent Lie groups, the existence of complex orbits appears to be restrictive, i.e. [14, Theorem 1] asserts that the only irreducible representations with a discrete kernel admitting complex orbits are those of Heisenberg groups. In contrast, symplectic orbits do exist for all groups admitting square-integrable representations by Proposition 10.

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