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Weighted Function Spaces with Applications to Boundary Value Problems

WEIGHTED FUNCTION SPACES WITH APPLICATIONS TO BOUNDARY VALUE PROBLEMS

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. dr. ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
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door

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PREFACE

This thesis is based on the following five papers:

- N. Lindemulder. An Intersection Representation for a Class of Anisotropic Vector-valued Function Spaces. *ArXiv e-prints*, arXiv:1903.02980, March 2019.
- N. Lindemulder. Difference norms for vector-valued Bessel potential spaces with applications to pointwise multipliers. *Journal of Functional Analysis*, 272(4):1435–1476, 2017.
- N. Lindemulder, M. Meyries, and M.C. Veraar. Complex interpolation with dirichlet boundary conditions on the half line. *Mathematische Nachrichten*, 291(16):2435-2456, 2018.
- N. Lindemulder and M.C. Veraar. The heat equation with rough boundary conditions and holomorphic functional calculus. *ArXiv e-prints*, arXiv:1805.10213, May 2018.
- F.B. Hummel and N. Lindemulder. Elliptic and Parabolic Boundary Value Problems in Weighted Function Spaces. *In preparation*

These works form a selection of the output of the research I have carried out during my appointment as a PhD candidate of Mark Veraar (daily supervisor and promotor) and Jan van Neerven (promotor) in the Analysis Group of the Delft Institute of Applied Mathematics at the Delft University of Technology from September 2014 to January 2019. This PhD position was part of Mark Veraar's Vidi Project "Harmonic Analysis for Stochastic Partial Differential Equations" subsidized by the Dutch Organisation for Scientific Research (NWO) under project number 639.032.427.

CONTENTS

1

INTRODUCTION

This thesis is concerned with the maximal regularity problem for parabolic boundary value problems with inhomogeneous boundary conditions in the setting of weighted function spaces and related function space theoretic problems. This in particularly includes weighted L_q-L_p-maximal regularity but also weighted L_q-maximal regularity in weighted Triebel-Lizorkin spaces. The weights we consider are power weights in time and in space, and yield flexibility in the optimal regularity of the initial-boundary data and allow to avoid compatibility conditions at the boundary. Moreover, the use of scales of weighted Triebel-Lizorkin spaces also provides a quantitative smoothing effect for the solution on the interior of the domain.

Section 1.1 introduces the subject of this thesis by discussing the weighted L_q-L_p-maximal regularity problem for parabolic boundary value problem.

Section 1.2 subsequently gives a systematic outline of the main part of the thesis, which consists of five chapters (based on and corresponding to the respective five papers [161], [158], [164], [166] and [122]) with their own introductions and preliminaries.

In this chapter we only provide the most important references for the purpose of introducing the subject of the present thesis. More extensive citations can be found in the main part of the thesis.

1.1. GENERAL INTRODUCTION

During the last 25 years, maximal regularity has become an important tool in the theory of nonlinear parabolic partial differential equations. Maximal regularity means that there is an isomorphism between the data and the solution of the linear problem in suitable function spaces. Having established such sharp estimates for the linearized problem (in fact the best possible), the nonlinear problem can be treated with quite simple tools as the contraction principle and the implicit function theorem (see [198]). Let us mention [11, 52] for approaches in spaces of continuous functions, [1, 168] for approaches in Hölder spaces and [5, 8, 49, 50, 86, 196, 198] for approaches in L_p-spaces (with $p \in (1,\infty)$). Concretely, the concept of maximal regularity has found its application in a great variety of physical, chemical and biological phenomena, like reaction-diffusion processes, phase field models, chemotactic behaviour, population dynamics, phase transitions and the behaviour of two phase fluids, for instance (see e.g. [178, 198, 199, 204]).

An abstract Cauchy problem

$$u'(t) + Au(t) = f(t) \ (t \in J), \quad u(0) = 0, \tag{1.1}$$

in a Banach space E on a time interval $J = (0, T)$ with $T \in (0, \infty]$, where A is a densely defined closed linear operator on E with domain $D(A)$, is said to have the property of *maximal L_q-regularity*, $q \in (1, \infty)$, if for each function $f \in L_q(J; E)$ there exists a unique solution $u \in W^1_q(J; E) \cap L_q(J; D(A))$ of (1.1). Having maximal L_q-regularity for (1.1), the corresponding version

$$u'(t) + Au(t) = f(t) \ (t \in J), \quad u(0) = u_0, \tag{1.2}$$

with a non-zero initial value can be easily treated via an application of related trace theorems. As a consequence of the closed graph theorem[1], an equivalent formulation of maximal L_q-regularity for (1.1) is that the map

$$\frac{d}{dt} + A : {}_0W^1_q(J; E) \cap L_q(J; D(A)) \longrightarrow L_q(J; E)$$

is an isomorphism of Banach spaces, where ${}_0W^1_q(J; E)$ denotes the closed subspace of $W^1_q(J; E)$ consisting of all functions which have a vanishing time trace at $t = 0$. It was already observed in [229] that (1.1) has maximal L_q-regularity for some $q \in (1, \infty)$ if and only if it has maximal L_q-regularity for every $q \in (1, \infty)$.

As an application of its operator-valued Fourier multiplier theorem, Weis [244] characterized maximal L_q-regularity in terms of R-sectoriality in the setting of Banach spaces E which are of class UMD (see Section 6.3.2). A second approach to the maximal L_q-regularity problem is via the operator sum method, as initiated by Da Prato & Grisvard [53] and extended by Dore & Venni [78] and Kalton & Weis [134] (see Sections 5.2.3 and 6.3.2).

Many concrete linear parabolic PDE's can be formulated as an abstract Cauchy problem (1.1) (or (1.2)). For this thesis an important class of examples are the autonomous vector-valued parabolic initial-boundary value problems with boundary conditions of static type subject to homogeneous initial-boundary data, i.e. problems of the form

$$\begin{aligned} \partial_t u(x,t) + \mathscr{A}(x,D)u(x,t) &= f(x,t), && x \in \mathscr{O}, && t \in J, \\ \mathscr{B}_j(x,D)u(x,t) &= 0, && x \in \partial\mathscr{O}, && t \in J, \quad j = 1,\dots,n, \\ u(x,0) &= 0, && x \in \mathscr{O}, \end{aligned} \tag{1.3}$$

where $J = (0, T)$ for some $T \in (0, \infty)$, $\mathscr{O}$ is a domain in $\mathbb{R}^d$ with a compact smooth boundary $\partial\mathscr{O}$, $\mathscr{A}(x, D)$ is partial differential operator of order $2n$ having $\mathscr{B}(X)$-valued smooth variable coefficients, and the $\mathscr{B}_j(x, D)$ are partial differential boundary operators of order $n_j < 2n$ having $\mathscr{B}(X)$-valued smooth variable coefficients, where X a fixed Banach space. One could for instance take $X = \mathbb{C}^N$, describing a system of N initial-boundary value problems.

[1] In concrete situations one can often obtain explicit estimates.

For these problems an abstract formulation of the form (1.1) is possible in the L_p-setting, $p \in (1,\infty)$: just take A to be the L_p-realization of the corresponding differential boundary value problem, i.e., consider the Banach space $E = L_p(\mathscr{O};X)$ and the operator A on E given by

$$\begin{aligned} D(A) &= \{v \in W^{2n}_p(\mathscr{O};X) : \mathscr{B}_j v = 0 \,(\text{on}\, \partial\mathscr{O}), j = 1,\dots,n\}, \\ Av &= \mathscr{A} v. \end{aligned}$$

Then the associated abstract Cauchy problem (1.1) has maximal L_q-regularity if and only if for each $f \in L_q(J;L_p(\mathscr{O};X))$ there exists a unique solution $u \in W^1_q(J;L_p(\mathscr{O};X)) \cap L_q(J;W^{2n}_p(\mathscr{O};X))$ of (1.3), in which case we say that (1.3) enjoys the property of *maximal L_q-L_p-regularity.*

Denk, Hieber & Prüss [59] proved maximal L_q-L_p-regularity for a large class of problems of the form (1.3), with as structural assumptions an ellipticity condition and a condition of Lopatinskii-Shapiro type, in the setting of UMD spaces; in fact, also non-autonomous versions were treated in which the top order coefficients of the operators are assumed to be bounded and uniformly continuous (allowing for perturbation arguments). Earlier works in this direction include [58, 80–82, 120, 121, 202], all concerning scalar-valued 2nd order problems having special boundary conditions (mainly Dirichlet).

The linear parabolic initial-boundary value problems (1.3) include linearizations of reaction-diffusion systems and of phase field models with Dirichlet, Neumann and Robin conditions. However, if one wants to use linearization techniques to treat such problems with non-linear boundary conditions, then one needs to study versions (1.3) with boundary inhomogeneities. It is in fact crucial to have a sharp theory for the fully inhomogeneous version of the linear problem (1.3): the problem

$$\begin{aligned} \partial_t u(x,t) + \mathscr{A}(x,D)u(x,t) &= f(x,t), & x &\in \mathscr{O}, & t &\in J, & & \\ \mathscr{B}_j(x,D)u(x,t) &= g_j(x,t), & x &\in \partial\mathscr{O}, & t &\in J, & j &= 1,\dots,n, \\ u(x,0) &= u_0(x), & x &\in \mathscr{O}, & & & & \end{aligned} \tag{1.4}$$

is said to enjoy the property of *maximal L_q-L_p-regularity* if there exists a (necessarily unique) space of initial-boundary data $\mathscr{D}_{i.b.} \subset L_q(J;L_p(\partial\mathscr{O};X))^n \times L_p(\mathscr{O};X)$ such that for every $f \in L_q(J;L_p(\mathscr{O};X))$ it holds that (1.4) has a unique solution $u \in W^1_q(J;L_p(\mathscr{O};X)) \cap L_q(J;W^{2n}_p(\mathscr{O};X))$ if and only if $(g = (g_1,\dots,g_n),u_0) \in \mathscr{D}_{i.b.}$. In this situation there exists a Banach norm on $\mathscr{D}_{i.b.}$, unique up to equivalence, with

$$\mathscr{D}_{i.b.} \hookrightarrow L_q(J;L_p(\partial\mathscr{O};X))^n \oplus L_p(\mathscr{O};X)$$

which makes the associated solution operator a topological linear isomorphism between the data space $L_q(J;L_p(\mathscr{O};X)) \oplus \mathscr{D}_{i.b.}$ and the solution space $W^1_q(J;L_p(\mathscr{O};X)) \cap L_q(J;W^{2n}_p(\mathscr{O};X))$. The *maximal L_q-L_p-regularity problem* for (1.4) consists of establishing maximal L_q-L_p-regularity for (1.4) and explicitly determining the space $\mathscr{D}_{i.b.}$ together with a Banach norm as above.

The maximal L_q-L_p-regularity problem for (1.4) was solved by Denk, Hieber & Prüss [61], who used operator sum methods in combination with tools from vector-valued harmonic analysis; as in [59], also non-autonomous versions were considered in which the top order coefficient of the operators are assumed to be bounded and uniformly continuous. Earlier works on this problem are [152] ($q = p$) and [243] ($p \leq q$) for scalar-valued 2nd order problems with Dirichlet and Neumann boundary conditions. Later, the results of [61] for the case that $q = p$ have been extended by Meyries & Schnaubelt [180] to the setting of temporal power weights $v_\mu(t) = t^\mu$, $\mu \in [0, q-1)$ (also see [176]). After that, the results of [61, 180] were simultaneously extended by myself in [159] (also see [156]) for the full range $q, p \in (1,\infty)$ to the setting of the temporal and spatial power weights

$$v_\mu(t) = t^\mu \quad \text{and} \quad w_\gamma^{\partial\mathscr{O}}(x) = \operatorname{dist}(x, \partial\mathscr{O})^\gamma \tag{1.5}$$

with $\mu \in (-1, q-1)$ and $\gamma \in (-1, p-1)$. Works in which maximal L_q-L_p-regularity of other problems with inhomogeneous boundary conditions are studied, include [54, 65, 66, 86, 180] (the case $q = p$) and [186, 227] (the case $q \neq p$). Some of the results from [159] have been applied in [72, 73] to the study of maximal L_q-L_p-regularity for parabolic boundary value problems on the half-space in which the elliptic operators have leading coefficients from the VMO class in both the time and the space variables.

Preceding the weighted maximal regularity approach in [180], Prüss & Simonett [197] had already initiated a weighted maximal L_q-regularity approach for abstract Cauchy problems (1.1)/(1.2). Here it is proposed to work in the weighted Lebesgue-Bochner spaces

$$L_{q,\mu}(J;E) = L_q(J, v_\mu; E) = \left\{ u \in L_0(J;E) : \int_J ||u(t)||_E^q v_\mu(t)\, dt < \infty \right\},$$

equipped with the natural norm, for the power weights $v_\mu(t) = t^\mu$, $\mu \in [0, q-1)$.[2] Having maximal $L_{q,\mu}$-regularity for (1.1),[3] the problem (1.2) can be solved for initial values u_0 belonging to the real interpolation space $(E, D(A))_{1-\frac{1}{q}(1+\mu),q}$. The space of initial values $(E, D(A))_{1-\frac{1}{q}(1+\mu),q}$ gets closer to the space E when μ gets closer to $q-1$, giving a reduction in the required initial regularity. Here the intuition is that the weight v_μ gives more compensation for rough behaviour near the initial time as the weight parameter μ increases. Besides this extra flexibility of treating rougher initial data, the weights also give an inherent smoothing effect of the solutions.

The temporal power weights v_μ give corresponding benefits in [180] for (1.4). Furthermore, these weights allow to avoid compatibility conditions at the boundary. In [176, 177], this weighted maximal regularity approach was used to establish convergence to equilibria and the existence of global attractors in high norms.

The spatial power weights $w_\gamma^{\partial\mathscr{O}}$ in [159] additionally yield flexibility in the boundary data. In order to make this explicit, let us for reasons of exposition state [159, Theo-

[2] The authors actually use a different parametrization of the weights.

[3] Maximal $L_{q,\mu}$-regularity is defined analogously to maximal L_q-regularity with the natural modifications.

rem 3.4], the main result of [159], for the easy case of the scalar-valued heat equation

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{on} \quad J \times \mathscr{O}, \\ u_{|\partial\mathscr{O}} &= g \quad \text{on} \quad J \times \partial\mathscr{O}, \\ u(0) &= u_0 \quad \text{on} \quad \mathscr{O}. \end{cases} \tag{1.6}$$

Theorem 1.1.1. *([159, Example 3.6]) Let $J=(0,T)$ with $T \in (0,\infty)$ and let $\mathscr{O}$ be a C^∞-domain in $\mathbb{R}^d$ with compact boundary $\partial\mathscr{O}$. Let $q,p \in (1,\infty)$, $\mu \in (-1,q-1)$ and $\gamma \in (-1,p-1)$ be such that $2-\frac{2}{q}(1+\mu) \neq \frac{1}{p}(1+\gamma)$. Let v_μ and $w_\gamma^{\partial\mathscr{O}}$ be as in* (1.5). *Then the problem* (1.6) *has the property of $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity with space of initial-boundary data*

$$\mathscr{D}_{i.b.} = \left\{ \begin{pmatrix} g \\ u_0 \end{pmatrix} \in \begin{array}{c} F_{q,p}^{1-\frac{1}{2p}(1+\gamma)}(J,v_\mu;L_p(\partial\mathscr{O})) \cap L_q(J,v_\mu;F_{p,p}^{2-\frac{1}{p}(1+\gamma)}(\partial\mathscr{O})) \\ \times \\ B_{p,q}^{2-\frac{2}{q}(1+\mu)}(\mathscr{O},w_\gamma^{\partial\mathscr{O}}) \end{array} \right.$$
$$\left. : \quad \mathrm{tr}_{t=0}\, g = \mathrm{tr}_{\partial\mathscr{O}}\, u_0 \text{ when } 2-\tfrac{2}{q}(1+\mu) > \tfrac{1}{p}(1+\gamma) \right\},$$

that is, $u \mapsto (\partial_t u - \Delta u, \mathrm{tr}_{\partial\mathscr{O}} u, \mathrm{tr}_{t=0} u)$ defines an isomorphism of Banach spaces

$$W_q^1(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \cap L_q(J,v_\mu;W_p^2(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \longrightarrow L_q(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \times \mathscr{D}_{i.b.}.$$

In particular, (1.6) *has a unique solution $u \in W_q^1(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \cap L_q(J,v_\mu;W_p^2(\mathscr{O},w_\gamma^{\partial\mathscr{O}}))$ if and only the data (f,g,u_0) satisfy:*

- $f \in L_q(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}}))$;
- $g \in F_{q,p}^{1-\frac{1}{2p}(1+\gamma)}(J,v_\mu;L_p(\partial\mathscr{O})) \cap L_q(J,v_\mu;F_{p,p}^{2-\frac{1}{p}(1+\gamma)}(\partial\mathscr{O}))$ (boundary regularity)*;*
- $u_0 \in B_{p,q}^{2-\frac{2}{q}(1+\mu)}(\mathscr{O},w_\gamma^{\partial\mathscr{O}})$ (initial regularity)*;*
- $\mathrm{tr}_{t=0}\, g = \mathrm{tr}_{\partial\mathscr{O}}\, u_0$ *when* $2-\frac{2}{q}(1+\mu) > \frac{1}{p}(1+\gamma)$ (compatibility condition).

The main contribution of the above result is the treatment of the boundary inhomogeneity g. So let us focus on this and for simplicity assume $\mu = 0$. Note that, setting $\delta = \delta_{p,\gamma} = 1 - \frac{1+\gamma}{2p}$ and using the trivial identity $B_{p,p}^s = F_{p,p}^s$, the boundary datum g has to be in the intersection space

$$F_{q,p}^{\delta}(J;L_p(\partial\mathscr{O})) \cap L_q(J;B_{p,p}^{2\delta}(\partial\mathscr{O})), \tag{1.7}$$

which in the case $q=p$ coincides with

$$B_{p,p}^{\delta}(J;L_p(\partial\mathscr{O})) \cap L_p(J;B_{p,p}^{2\delta}(\partial\mathscr{O})) = W_p^{\delta}(J;L_p(\partial\mathscr{O})) \cap L_p(J;W_p^{2\delta}(\partial\mathscr{O})); \tag{1.8}$$

here $F_{q,p}^s$ denotes a Triebel-Lizorkin space and $W_p^s = B_{p,p}^s$ a non-integer order Sobolev-Slobodeckii space or Besov space.

The space (1.8) for the special case $q = p$ and $\mu = 0$ already nicely shows the effect of the spatial weight $w_{\gamma}^{\partial\mathscr{O}}$ on the sharp regularity of the boundary inhomogeneity g. In particular, we see that $\delta = \delta_{p,\gamma} = 1 - \frac{1+\gamma}{2p} \in (\frac{1}{2}, 1) \setminus \{1 - \frac{1}{p}\}$ can be taken arbitrarily close to $\frac{1}{2}$ by choosing γ arbitrarily close to $p-1$.

However, it is desirable to have maximal L_q-L_p-regularity for the full range $q, p \in (1,\infty)$, as this enables one to treat more nonlinearities. For instance, one often requires large q and p due to better Sobolev embeddings, and $q \neq p$ due to criticality and/or scaling invariance (see e.g. [97, 141, 199, 203, 204]). The latter has in particularly turned out to be crucial in applications to the Navier-Stokes equations, convection-diffusion equations, the Nernst-Planck-Poisson equations in electro-chemistry, chemotaxis equations and the MHD equations (see [199, 204]).

For (1.4) the case $q \neq p$ is much more involved than the case $q = p$ on the function space theoretic part of the problem due to the inhomogeneous boundary conditions. This is not only already reflected in the space of initial-boundary data for the special case (1.6) through the appearance of the intersection space describing the sharp boundary regularity ((1.7) versus (1.8)), but also in the proof(s) due to a lack of Fubini in the form of $L_q[L_p] = L_p[L_q]$ when $q \neq p$.

Let us say something about the difficulties in the proof for (1.4) in the case $q \neq p$. In [61] the proof of the solution to the L_q-L_p-maximal regularity problem for (1.4) is treated separately for the cases $q \neq p$ and $q = p$ with completely different proofs ([61, Theorem 2.3] versus [61, Theorem 2.2]). Whereas the proof for the case $q = p$ (see [176, 180] for more details) is reasonably natural and uses a Fourier transform in time in combination with representation formulae for the corresponding elliptic problems, the proof for the case $q \neq p$ relies on very complicated and clever ad hoc arguments (already knowing how the space of initial-boundary data should look like thanks to Weidemaier [243]). In [159] there is no separation into the cases $q \neq p$ and $q = p$: there is one proof that also uses a Fourier transform in time in combination with representation formulae for the corresponding elliptic problems (slightly different from the ones in [61, 180], see [159, Remark 6.4]), but additionally uses the theory of anisotropic mixed-norm function spaces as considered in [131] (partly developed in [159] as well). Here we have to remark that some of the underlying anisotropic function space theory used in [159] simplifies a lot in the case $q = p$ thanks to the availability of Fubini in the form of $L_q[L_p] = L_p[L_q]$.

Whereas the maximal regularity space

$$W_q^1(J; L_p(\mathscr{O})) \cap L_q(J; W_p^2(\mathscr{O})) \tag{1.9}$$

and the space of boundary data

$$F_{q,p}^{\delta}(J; L_p(\partial\mathscr{O})) \cap L_q(J; F_{p,p}^{2\delta}(\partial\mathscr{O})), \qquad \delta = \delta_p = 1 - \frac{1}{2p}, \tag{1.10}$$

are only viewed as intersection spaces in [61, 180], in [159] they are also viewed as anisotropic mixed-norm function spaces, described in a distribution space theoretic or Fourier analytic way, as considered in [131].

For (1.9) it is very easy to give such a description. Indeed, the maximal regularity space in (1.9) can naturally be identified with the anisotropic mixed-norm Sobolev space

$$W^{(2,1)}_{(p,q)}(\mathscr{O} \times J) = \left\{ u \in \mathscr{D}'(\mathscr{O} \times J) : \partial_t, \partial_x^\alpha u \in L_{(p,q)}(\mathscr{O} \times J), |\alpha| \leq 2 \right\}, \tag{1.11}$$

where the mixed-norm Lebesgue space

$$L_{(p,q)}(\mathscr{O} \times J) = \left\{ f \in L_0(\mathscr{O} \times J) : \left(\int_J \left(\int_{\mathscr{O}} |f(x,t)|^p \, dx \right)^{p/q} dt \right)^{1/q} < \infty \right\}$$

can be naturally identified with the Lebesgue Bochner space $L_q(J; L_p(\mathscr{O}))$. However, for (1.10) a description as a suitable mixed-norm anisotropic function space is highly non-trivial and will be treated in Chapter 2.

The main result of Chapter 2 actually is an intersection representation for a class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg & Netrusov [119], which includes weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. In Theorem 1.1.2 below we state a special case of weighted anisotropic mixed-norm Triebel-Lizorkin spaces.

Let us first introduce the setting of Theorem 1.1.2. To this end, recall that, as a consequence of [61, Theorem 2.3] for the special case (1.6) (see Theorem 1.1.1 for the extension to the weighted setting), the intersection space (1.10) is the spatial trace space of the maximal regularity space (1.9). On the other hand, this spatial trace space could be determined by viewing (1.9) as the anisotropic mixed-norm Sobolev space (1.11) and reducing the situation to the full Euclidean space $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ by standard localization arguments. This leads us to determining the spatial trace space of $W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R})$. The latter has actually been done by Johnsen & Sickel [131] using anisotropic Littlewood-Paley decompositions.

Anisotropic Littlewood-Paley decompositions for $W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R})$ can be formulated by means of anisotropic mixed-norm Triebel-Lizorkin spaces: for instance,

$$W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R}) = F^{1,(\frac{1}{2},1)}_{(p,q),2}(\mathbb{R}^d \times \mathbb{R}) \tag{1.12}$$

with an equivalence of norms. Instead of smoothness $s = 1$ and anisotropy $\boldsymbol{a} = (\frac{1}{2}, 1)$ on the right-hand side, we could take the scaled version $s = \lambda$ and $\boldsymbol{a} = \lambda(\frac{1}{2}, 1)$ for any $\lambda > 0$. However, smoothness 1 with respect to the anisotropy $(\frac{1}{2}, 1)$ seems to be a natural way to think of $W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R})$ as it nicely fits with the viewpoint of it being of order 1 with respect to the parabolic operator $\partial_t - \Delta_x$.

The anisotropic mixed-norm Triebel-Lizorkin space $F^{s,(\frac{1}{2},1)}_{(p,q),r}(\mathbb{R}^d \times \mathbb{R})$ for $s \in \mathbb{R}$, $r \in [1,\infty]$ is defined analogously to the classical isotropic Triebel-Lizorkin space $F^s_{p,r}(\mathbb{R}^d)$ (see Section 6.3.4), but with an underlying Littlewood-Paley decomposition of $\mathbb{R}^d \times \mathbb{R}$ that is adapted to the $(\frac{1}{2}, 1)$-anisotropic (or 2nd order parabolic) scalings

$$\delta^{(\frac{1}{2},1)}_\lambda(\xi, \tau) = (\lambda^{1/2}\xi, \lambda\tau), \qquad \lambda \in (0,\infty). \tag{1.13}$$

Intuitively the dilation structure (1.13) causes a decay behaviour on the Fourier side at different rates in the two components of $\mathbb{R}^d \times \mathbb{R}$ in such a way that smoothness $s \in (0,\infty)$ with respect to the anisotropy $(\frac{1}{2},1)$ corresponds to smoothness $2s$ in the spatial direction and smoothness s in the time direction. One way to look at the intersection representation (1.18) is as a way to make this intuition precise.

Regarding spatial traces, by [131, Theorem 2.2], the trace operator $\mathrm{Tr} : u \mapsto u_{[\{0\}\times\mathbb{R}^{d-1}]\times\mathbb{R}}$ defines a retraction

$$\mathrm{Tr} : F^{s,(\frac{1}{2},1)}_{(p,q),r}(\mathbb{R}^d \times \mathbb{R}) \longrightarrow F^{s-\frac{1}{2p},(\frac{1}{2},1)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}) \tag{1.14}$$

for every $s \in (\frac{1}{2p},\infty)$ and $r \in [1,\infty]$. Combining this with the Littlewood-Paley decomposition (1.12), a corollary to this result is that

$$\mathrm{Tr} : W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R}) \longrightarrow F^{\delta,(\frac{1}{2},1)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}), \qquad \delta = \delta_p = 1 - \frac{1}{2p}, \tag{1.15}$$

is a retraction as well. The intersection space (1.10) being the spatial trace space of the maximal regularity space (1.9), this suggests that

$$F^{\delta,(\frac{1}{2},1)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}) = F^{\delta}_{q,p}(\mathbb{R}; L_p(\mathbb{R}^{d-1})) \cap L_q(\mathbb{R}; F^{2\delta}_{p,p}(\mathbb{R}^{d-1})). \tag{1.16}$$

The intersection representation (1.16) with a general anisotropy (a,b) instead of $(\frac{1}{2},1)$ was proved by Denk & Kaip [63, Proposition 3.23]: for every $q,p \in (1,\infty)$, $a,b \in (0,\infty)$ and $s \in (0,\infty)$,

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}) = F^{s/b}_{q,p}(\mathbb{R}; L_p(\mathbb{R}^{d-1})) \cap L_q(\mathbb{R}; F^{s/a}_{p,p}(\mathbb{R}^{d-1})). \tag{1.17}$$

This identity was obtained by comparing the trace result [131, Theorem 2.2] by Johnsen & Sickel with a trace result by Berkolaiko [24, 25].

In (1.17) it is crucial that the microscopic parameter p coincides with the inner component of the integrability parameter (p,q) in $F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R})$. Besides that the proof given in [63, Proposition 3.23] heavily relies on that, it is also very important for the statement itself. One way to look at this is through Fubini in the form of $L_{(p,q)}[\ell_p] = L_q[\ell_p](L_p)$: inspecting (1.17) and recalling the definition of (anisotropic mixed-norm) Triebel-Lizorkin spaces (see Section 6.3.4), we realize that the order of $L_p(\mathbb{R}^{d-1})$ and $\ell_p(\mathbb{N})$ is interchanged in the first space on the right-hand side. Theorem 1.1.2 in particularly says that this is actually not necessary for the result itself, at the cost of working with a more complicated function space: It is formulated in the setting of weighted mixed-norm anisotropic Banach space-valued function spaces (see Section 6.3.4).

Theorem 1.1.2. *Let X be a Banach space, $a,b \in (0,\infty)$, $s \in (0,\infty)$, $p,q \in (1,\infty)$, $r \in [1,\infty]$, $w \in A_p(\mathbb{R}^n)$ and $v \in A_q(\mathbb{R}^m)$. Then*

$$F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m,(w,v);X) = \mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m, v; L_p(\mathbb{R}^n,w);X) \cap L_q(\mathbb{R}^m, v; F^{s/a}_{p,r}(\mathbb{R}^n,w;X)), \tag{1.18}$$

where, for $E = L_p(\mathbb{R}^n,w)$,

$$\mathbb{F}^{\sigma}_{q,r}(\mathbb{R}^m;E;X) = \left\{ f \in \mathscr{S}'(\mathbb{R}^m;E(X)) : (2^{k\sigma}S_k f)_k \in L_q(\mathbb{R}^n;E[\ell_r(\mathbb{N})](X)) \right\}$$

with $(S_k)_{k\in\mathbb{N}}$ *a Littlewood-Paley decomposition of* $\mathbb{R}^m$.

The above theorem was established in my master thesis [156, Proposition 5.2.38] under the restriction $r > 1$. As already mentioned earlier, in Chapter 2 we will actually treat a much more general and more systematic intersection representation, see Section 2.5. In particular, Theorem 1.1.2 extends to the setting of general A_∞-weights, in which the statement becomes more technical.

In the case $p = r$, Fubini yields $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m, v; L_p(\mathbb{R}^n, w); X) = F^{s/b}_{q,p}(\mathbb{R}^m, v; L_p(\mathbb{R}^n, w; X))$ and $F^{s/a}_{p,r}(\mathbb{R}^n, w; X) = B^{s/a}_{p,p}(\mathbb{R}^n, w; X)$, and we obtain an extension of the intersection representation (1.17) to decompositions $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ in the weighted Banach space-valued setting:

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}^m, (w, v); X) = F^{s/b}_{q,p}(\mathbb{R}^m, v; L_p(\mathbb{R}^n, w; X)) \cap L_q(\mathbb{R}^m, v; B^{s/a}_{p,p}(\mathbb{R}^n, w; X)). \quad (1.19)$$

In the form of (1.19), Theorem 1.1.2 is one of the main ingredients in the proof of [159, Theorem 3.4] (a version of Theorem 1.1.1 for the general case (1.4)). Another main ingredient is [159, Theorem 4.6], an extension of (1.15) to the weighted Banach space-valued setting.

Crucial to the proof of Theorem 1.1.2 are difference norm characterizations for the spaces $F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m, (w, v); X)$, $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m, v; L_p(\mathbb{R}^n, w); X)$ and $L_q(\mathbb{R}^m, v; F^{s/a}_{p,r}(\mathbb{R}^n, w; X))$. This is especially quite involved for $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m, v; L_p(\mathbb{R}^n, w); X)$. Let us for illustrational purposes state such a difference norm characterization for $\mathbb{F}^{s}_{p,q}(\mathbb{R}^d, w; E; X)$ (see Theorem 2.4.7).

Proposition 1.1.3. *Let* X *be a Banach space* E *a UMD Banach function space (e.g.* $E = L_r(S)$ *with* $r \in (1,\infty)$*),* $p \in (1,\infty)$, $q \in [1,\infty]$, $w \in A_p(\mathbb{R}^d)$ *and* $s \in (0,\infty)$. *Given* $m \in \mathbb{N}$ *with* $m > s$*, there is the equivalence of extended norms*

$$||f||_{\mathbb{F}^s_{p,q}(\mathbb{R}^d,w;E;X)} \eqsim ||f||_{L_p(\mathbb{R}^d,w;E(X))} + \left|\left|\left(\sum_{j=1}^{\infty} \left|\left|2^{js} \int_{[-1,1]^d} \Delta^m_{2^{-j}h} f \, dh \right|\right|^q_X \right)^{1/q} \right|\right|_{L_p(\mathbb{R}^d,w;E)}$$

for $f \in L_p(\mathbb{R}^d, w; E(X))$*, where*

$$\Delta_h f(x) = f(x+h) - f(x), \qquad x \in \mathbb{R}^d, h \in \mathbb{R}^d,$$

and

$$\Delta^m_h f(x) = \underbrace{\Delta_h \ldots \Delta_h}_{m \text{ times}} f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x + (m-j)h), \qquad x \in \mathbb{R}^d, h \in \mathbb{R}^d.$$

In the special case $E = \mathbb{C}$ we have $\mathbb{F}^s_{p,q}(\mathbb{R}^d, w; E; X) = F^s_{p,q}(\mathbb{R}^d, w; X)$ and the above proposition becomes an extension of the discrete version of [220, Section 2.3, Proposition 6] (considered in the proof of that result) to the weighted setting. The difference norm characterization in [220] in its own turn generalizes the classical difference norm characterizations by Strichartz [230] and Triebel [234, Theorem 2.6.3] for scalar-valued Bessel potential spaces and Triebel-Lizorkin spaces, respectively.

In the scalar-valued setting Sobolev spaces are a special case of Bessel potential spaces which are in turn a special case of Triebel-Lizorkin spaces:

$$W^k_p(\mathbb{R}^d,w) = H^k_p(\mathbb{R}^d,w) \quad \text{and} \quad H^s_p(\mathbb{R}^d,w) = F^s_{p,2}(\mathbb{R}^d,w), \tag{1.20}$$

where $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$. This breaks down in the general Banach space-valued setting: the identity $W^k_p(\mathbb{R}^d,w;X) = H^k_p(\mathbb{R}^d,w;X)$ holds provided that X is a UMD Banach space, where the UMD property may even be necessary depending on d, k (see [126]); the Littlewood-Paley decomposition $H^s_p(\mathbb{R}^d,w;X) = F^s_{p,2}(\mathbb{R}^d,w;X)$ holds true if and only if X is isomorphic to a Hilbert space.

However, for every Banach space X there still are the embedding

$$\begin{aligned} F^k_{p,1}(\mathbb{R}^d,w;X) &\hookrightarrow W^k_p(\mathbb{R}^d,w;X) \hookrightarrow F^k_{p,\infty}(\mathbb{R}^d,w;X),\\ F^s_{p,1}(\mathbb{R}^d,w;X) &\hookrightarrow H^s_p(\mathbb{R}^d,w;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d,w;X), \end{aligned} \tag{1.21}$$

that can in some instances be used through independence on the microscopic parameter q in the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d,w;X)$ to overcome the unavailability of (1.20). This idea is due to Scharf, Schmeißer & Sickel [219], who used it to determine the trace space of $W^k_p(\mathbb{R}^d;X)$ for general Banach spaces X. This idea has furthermore been powerful in works by Meyries & Veraar [182, 185, 186] in the direction of trace theory and Sobolev embedding in a weighted setting, where there are many estimates with microscopic improvement. In connection to Theorem 1.1.1, anisotropic versions of (1.21) were used in [159] to extend (1.15) to the weighted Banach space-valued setting.

Although the elementary embedding (1.21) can be quite powerful, in many instances one needs sharper information on $W^k_p(\mathbb{R}^d,w;X)$ and $H^s_p(\mathbb{R}^d,w;X)$. This is for example the case in the L_p-approach to (abstract) evolution and integral equations, both in the deterministic setting (see e.g. [5, 195, 251]) and in the stochastic setting (see e.g. [69, 191, 192]), where UMD Banach space-valued Sobolev and Bessel potential spaces play an important role (especially with $d = 1$).

In the UMD Banach function space-valued setting there still is a Littlewood-Paley decomposition like $H^s_p(\mathbb{R}^d,w) = F^s_{p,2}(\mathbb{R}^d,w)$ in terms of square functions:

$$H^s_p(\mathbb{R}^d,w;E) = \mathbb{F}^s_{p,2}(\mathbb{R}^d,w;E), \tag{1.22}$$

where E is a UMD Banach function space, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ and $s \in \mathbb{R}$. The difference norm characterization from Proposition 1.1.3 thus in particularly contains a difference norm characterization for $H^s_p(\mathbb{R}^d,w;E)$: given $m \in \mathbb{N}$ with $m > s$, there is the equivalence of extended norms

$$||f||_{H^s_p(\mathbb{R}^d,w;E)} \eqsim ||f||_{L_p(\mathbb{R}^d,w;E)} + \Big|\Big|\Big(\sum_{j=1}^{\infty}\Big|2^{js}\int_{[-1,1]^d}\Delta^m_{2^{-j}h}f\,dh\Big|^2\Big)^{1/2}\Big|\Big|_{L_p(\mathbb{R}^d,w;E)} \tag{1.23}$$

for $f \in L_p(\mathbb{R}^d, w; E)$.

The main result of Chapter 3 is a version of (1.23) in the general UMD Banach space-valued setting, see Theorem 1.1.4 below.

We denote by $\{\varepsilon_j\}_{j\in\mathbb{N}}$ a Rademacher sequence on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e. a sequence of independent symmetric $\{-1,1\}$-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem 1.1.4. *Let X be a UMD Banach space, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$, $s \in (0,\infty)$ and $m \in \mathbb{N}$, $m > s$. Suppose that*

- *$K = 1_{[-1,1]^d}$ in the unweighted case $w = 1$; or*
- *$K \in \mathscr{S}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}} K(y)\,dy \neq 0$ in the general weighted case.*

We then have the equivalence of extended norms

$$||f||_{H^s_p(\mathbb{R}^d,w;X)} \eqsim ||f||_{L_p(\mathbb{R}^d,w;X)} + \sup_{J\in\mathbb{N}} \left|\left| \sum_{j=1}^{J} \varepsilon_j 2^{js} \int_{\mathbb{R}^d} K(h)\Delta^m_{2^{-j}h} f\,dh \right|\right|_{L_p(\Omega;L_p(\mathbb{R}^d,w;X))} \tag{1.24}$$

for $f \in L_p(\mathbb{R}^d, w; X)$.

In Chapter 3 we furthermore, as an application of Theorem 1.1.4, characterize the boundedness of the indicator function $1_{\mathbb{R}^d_+}$ of the half-space $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ as a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$, $s \in (0,1)$, in terms of a continuous inclusion of the corresponding scalar-valued Bessel potential space $H^s_p(\mathbb{R}^d, w)$ into a certain weighted L_p-space:

Theorem 1.1.5. *Let $X \neq \{0\}$ be a UMD space, $s \in (0,1)$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $w_{s,p}$ be the weight on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ given by $w_{s,p}(x_1,x') := |x_1|^{-sp} w(x_1,x')$ if $|x_1| \le 1$ and $w_{s,p}(x_1,x') := w(x_1,x')$ if $|x_1| > 1$. Then $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if there is the inclusion*

$$H^s_p(\mathbb{R}^d, w) \hookrightarrow L_p(\mathbb{R}^d, w_{s,p}). \tag{1.25}$$

In the specific case of the A_p-power weights w_γ, $\gamma \in (-1, p-1)$, given by

$$w_\gamma(x_1,x') = |x_1|^\gamma, \qquad (x_1,x') \in \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}, \tag{1.26}$$

Theorem 1.1.5 gives back a result due to Meyries & Veraar [187]: given a UMD Banach space X, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$, it holds that $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w_\gamma; X)$ in the parameter range

$$\frac{1+\gamma}{p} - 1 < s < \frac{1+\gamma}{p}.$$

In Chapter 4 we provide a simplified proof of the latter (see Theorem 4.4.1), where it will be used to prove results on the complex interpolation of Sobolev spaces on the half line with Dirichlet boundary condition. In this context the half line usually stands for the time variable and X is a suitable function space for the space variable. So let us for consistency of notation within this introduction state one of the main results from Chapter 4 (see Section 4.6.2) as follows.

Theorem 1.1.6. *Let E be a UMD space, $q \in (1,\infty)$, $\mu \in (-1, q-1)$ and $v_\mu(t) = t^\mu$. Then*

$$[L_q(\mathbb{R}_+, w_\mu; E), {}_0W^1_q(\mathbb{R}_+, v_\mu; E)]_\theta = {}_0H^\theta_q(\mathbb{R}_+, v_\mu; E), \qquad \theta \in (0,1) \setminus \left\{\frac{1+\mu}{q}\right\},$$

where $W^1_q(\mathbb{R}_+, v_\mu; E) := \left\{u \in W^1_q(\mathbb{R}_+, v_\mu; E) : u(0) = 0\right\}$ and

$${}_0H^\theta_q(\mathbb{R}_+, v_\mu; E) := \begin{cases} H^\theta_q(\mathbb{R}_+, v_\mu; E), & \theta < \frac{1+\mu}{q}, \\ \left\{u \in H^\theta_q(\mathbb{R}_+, v_\mu; E) : tr_{t=0} u = 0\right\}, & \theta > \frac{1+\mu}{q}. \end{cases}$$

In the unweighted scalar-valued case $\mu = 0$ and $X = \mathbb{C}$, the result was already well-known and due to Seeley [224], where one of the advantages is that Bessel potential spaces have a simple square function characterization. The vector-valued result was already used several times in the literature without proof. The corresponding result for real interpolation is due to Grisvard [104] and more elementary to prove.

The complex interpolation result has applications in the theory of evolution equations, as it yields a characterization of the fractional power domains of the time derivative $D((d/dt)^\theta)$ and $D((-d/dt)^\theta)$ on $\mathbb{R}_+$. For instance such spaces can be used in the theory of Volterra equations (see [195, 250, 251]), in evolution equations with form methods (see [70, 89]), in stochastic evolution equations (see [192]).

As already mentioned op page 4, the $L_{q,\mu}$-maximal regularity (v_μ-weighted L_q-maximal regularity) approach to evolution equations initiated by Prüss & Simonett [197] enables one to treat rough initial values. Examples of other papers in evolution equation where such weights are used include [11, 52, 141, 159, 180, 186, 200]. The monographs [168, 198] are an excellent source for applications of weighted spaces to evolution equations.

From the viewpoint of trace theory it does not make sense to go beyond the range $(-1, q-1)$ for the temporal weight-parameter μ in the $L_{q,\mu}$-maximal regularity approach. For the treatment of rough initial values it actually already suffices to consider $\mu \in [0, q-1)$, which is reflected in $(E, D(A))_{1-\frac{1}{q}(1+\mu),q}$ being the optimal space of initial values u_0 in the $L_{q,\mu}$-maximal regularity approach to (1.2).

In the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity approach to (1.6) (see Theorem 1.1.1) the situation is different for the spatial weight parameter γ. Indeed, here it would make sense to go beyond the range $(-1, p-1)$. On the one hand, there still is a trace operator $\mathrm{tr}_{\partial\mathscr{O}}$ on $W^2_p(\mathscr{O}, w^\partial_\mathscr{O})$ for $\gamma \in (p-1, 2p-1)$, so that the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem for (1.6) still is a well-defined question for such γ. On the other hand, allowing such values of γ should enable one to treat rougher boundary data: regarding the optimal space of boundary data

$$F^\delta_{q,p}(J, v_\mu; L_p(\partial\mathscr{O})) \cap L_q(J, v_\mu; F^{2\delta}_{p,p}(\partial\mathscr{O})), \qquad \delta = \delta_{p,\gamma} = 1 - \frac{1+\gamma}{2p}, \tag{1.27}$$

note that $\delta_{p,\gamma} \in (\frac{1}{2},1)$ when $\gamma \in (-1,p-1)$ while $\delta_{p,\gamma} \in (0,1)$ can be taken arbitrarily close to 0 by choosing $\gamma \in (p-1,2p-1)$ arbitrarily close to $2p-1$.

Motivated by this, in Chapter 5 we extend Theorem 1.1.1 to $\gamma \in (p-1,2p-1)$. It turns out that (1.27) indeed still provides the correct space for the boundary data:

Theorem 1.1.7. *Let $J=(0,T)$ with $T \in (0,\infty]$ and let $\mathscr{O}$ be a bounded C^2-domain in $\mathbb{R}^d$. Let $q,p \in (1,\infty)$, $\mu \in (-1,q-1)$ and $\gamma \in (-1,2p-1)\setminus\{p-1\}$ be such that $2-\frac{2}{q}(1+\mu) \neq \frac{1}{p}(1+\gamma)$. Let v_μ and $w_\gamma^{\partial\mathscr{O}}$ be as in* (1.5)*. Then the problem* (1.6) *has the property of $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity with space of initial-boundary data*

$$\mathscr{D}_{i.b.} = \left\{ \begin{pmatrix} g \\ u_0 \end{pmatrix} \in \begin{array}{c} F_{q,p}^{1-\frac{1}{2p}(1+\gamma)}(J,v_\mu;L_p(\partial\mathscr{O})) \cap L_q(J,v_\mu;F_{p,p}^{2-\frac{1}{p}(1+\gamma)}(\partial\mathscr{O})) \\ \times \\ W_{p,q}^{2-\frac{2}{q}(1+\mu)}(\mathscr{O},w_\gamma^{\partial\mathscr{O}}) \end{array} \right.$$
$$\left. : \ \operatorname{tr}_{t=0} g = \operatorname{tr}_{\partial\mathscr{O}} u_0 \ when\, 2-\frac{2}{q}(1+\mu) > \frac{1}{p}(1+\gamma) \right\},$$

that is, $u \mapsto (\partial_t u - \Delta u, \operatorname{tr}_{\partial\mathscr{O}} u, \operatorname{tr}_{t=0} u)$ defines an isomorphism of Banach spaces

$$W_q^1(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \cap L_q(J,v_\mu;W_p^2(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \longrightarrow L_q(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \times \mathscr{D}_{i.b.}.$$

Here $W_{p,q}^s(\mathscr{O},w_\gamma^{\partial\mathscr{O}}) := (L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}}), W_p^2(\mathscr{O},w_\gamma^{\partial\mathscr{O}}))_{\frac{s}{2},q}$ for $s \in (0,2)$.

Whereas Theorem 1.1.1 has only been stated for the specific case of the heat equation (1.6) for reasons of exposition, being an example of [159, Theorem 3.4] on more general parabolic problems (1.4) as considered by Denk, Hieber & Prüss [61], in Chapter 5 we will not go beyond (1.5). The reason for this is that (1.5) is already involved enough as a first step outside the Muckenhoupt A_p-setting for $w_\gamma^{\partial\mathscr{O}}$.

Given $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$, it holds that

$$w_\gamma^{\partial\mathscr{O}} = \operatorname{dist}(\cdot,\partial\mathscr{O})^\gamma \in A_p(\mathbb{R}^d) \quad \Longleftrightarrow \quad \gamma \in (-1,p-1), \tag{1.28}$$

where $A_p(\mathbb{R}^d)$ denotes the class of Muckenhoupt A_p-weights on $\mathbb{R}^d$ (see Section 3.2.2). The main difficulty in the proof of Theorem 1.1.7 in the non-A_p setting is that standard tools from harmonic analysis are not available. For instance, the boundedness of the Hilbert transform, the boundedness of the Hardy-Littlewood maximal function operator, and the Littlewood–Paley decomposition all hold on $L_p(\mathbb{R}^d, w_\gamma^{\partial\mathscr{O}})$ if and only if $\gamma \in (-1,p-1)$ (see [103, Chapter 9] and [218]).

The proof of Theorem 1.1.7 roughly speaking consists of a function space theoretic part and an operator theoretic part. In the function space theoretic part we obtain identifications of the spatial and temporal trace space of the maximal regularity space

$$W_q^1(J,v_\mu;L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})) \cap L_q(J,v_\mu;W_p^2(\mathscr{O},w_\gamma^{\partial\mathscr{O}})). \tag{1.29}$$

Having these identifications, the problem under consideration reduces to the abstract Cauchy problem (1.1) on $J = \mathbb{R}_+$ with A the realization of $-\Delta$ on $E = L_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ with domain

$$D(A) = W^2_{p,\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}) = \left\{u \in W^2_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}) : \mathrm{tr}_{\partial\mathscr{O}} u = 0\right\}.$$

In the operator theoretic part we establish $L_{q,\mu}$-maximal regularity for this Cauchy problem through the H^∞-calculus (see Section 5.2.3).

Theorem 1.1.8. *Let $\mathscr{O}$ be a bounded C^2-domain in $\mathbb{R}^d$. Let $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Let Δ_{Dir} be the realization of the Laplacian Δ on $L_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ with domain $D(\Delta_{\mathrm{Dir}}) = W^2_{p,\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$. Then Δ_{Dir} is the generator of an exponentially stable analytic C_0-semigroup and $-\Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle zero.*

The operator Δ_{Dir} and its generalizations have been studied in many papers (see [58, 59, 149]). The main contribution of Theorem 5.1.1 is the treatment of the non-A_p-case. The A_p-case $\gamma \in (-1, p-1)$ can be treated by classical methods, and it can be derived from the case of general A_p-weights which will be considered in Chapter 5 as well.

Besides $L_{q,\mu}$-maximal regularity for the Cauchy problem (1.1) on $J = \mathbb{R}_+$ with $A = -\Delta_{\mathrm{Dir}}$, the boundedness of the H^∞-calculus has many other interesting consequences for the operator Δ_{Dir} on $L_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$. Loosely speaking, the boundedness of the H^∞-calculus can be used as a black box to ensure existence of certain singular integrals. In particular, the boundedness of the H^∞-calculus implies:

- Continuous and discrete square function estimates (see [127, Theorems 10.4.4 & 10.4.23]), which are closely related to the classical Littlewood–Paley inequalities.
- Bounded imaginary powers and characterizations of fractional domains as complex interpolation spaces (see [110, Theorem 6.6.9] or [235, Theorem 1.15.3]).
- Maximal regularity for the stochastic heat equation on $L_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ (see [192, Theorem 1.1]).

Analogues of Theorems 1.1.7 and 1.1.8 for second order elliptic operators on weighted Triebel-Lizorkin spaces have been obtained by myself in [162, 163], which are independent from Theorems 1.1.7 and 1.1.8 since in the non-A_p-setting Triebel-Lizorkin spaces do not coincide with Sobolev spaces. The advantage of the scale of weighted Triebel-Lizorkin spaces is the strong harmonic analytic nature of these function spaces, leading to the availability of many powerful tools (see e.g. [38–40, 115–118, 163, 182, 185, 186, 228]). In particular, there is a Mikhlin-Hörmander Fourier multiplier theorem.

The subresult in Theorem 1.1.8 that Δ_{Dir} generates an analytic C_0-semigroup on $L_p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ with $p \in [2,\infty)$ and $\gamma \in (p-1, 2p-1)$ is used by myself & Veraar [167] to treat the heat equation with multiplicative noise of Dirichlet type at the boundary. There we use the method developed by Schnaubelt & Veraar [222] for their treatment of parabolic problems with multiplicative noise of Neumann type. A model example which fits in our framework is as follows:

$$\begin{cases} \partial_t u(t,x) &= \Delta u(t,x) & \text{on } (0,T]\times\mathscr{O}, \\ u(t,x) &= C(t,u(t,\cdot))(x)\partial_t W(t,x) & \text{on } (0,T]\times\partial\mathscr{O}, \\ u(0,x) &= u_0(x) & \text{on } \mathscr{O}, \end{cases} \tag{1.30}$$

where C is a suitable nonlinearity mapping functions on $\mathscr{O}$ to functions on $\partial\mathscr{O}$ and W is a space-time Brownian noise.

In the application to (1.30) it turns out that γ has to be taken in the non-A_p-range $(p-1,2p-1)$ in order suppresses the irregularities due to the noise near to the boundary. This goes back to Alòs and Bonaccorsi [3] and was further devoloped by Fabri & Goldys [88], who established existence and uniqueness of $L_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})$-valued solutions (with $p\geq 2$, $\gamma\in(p-1,2p-1)$ respectively $p=2$, $\gamma\in(p-1,2p-1)$) for problems with additive noise of Dirichlet type at the boundary in the one-dimensional case (in (1.30) additive noise would correspond to $C\equiv 1$). Before the results in [3, 88], Da Prato & Zabczyk [55] had already shown that an unweighted L_2-setting does not provide the right setting to obtain function-valued solutions: the solution u of the additive case of (1.30) (i.e. with $C\equiv 1$) is H^s_2-valued if and only if $s<-\frac{1}{2}$.

It would be interesting to generalize Theorems 1.1.7 and 1.1.8 to the more general setting of a higher order systems with boundary conditions of Lopatisnkii-Shapiro type (1.3)/(1.4) as considered by Denk, Hieber & Prüss [59, 61]. Regarding Theorem 1.1.8, a possible approach could proceed through an extrapolation result due to Martel [171, Theorem 7.3] in the spirit of Section 5.5.4. Having a suitable extension of Theorem 1.1.8, we obtain $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity for (1.3). As a next step, trace theory would then subsequently reduce the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem for (1.4) to solving

$$\begin{aligned} \partial_t u(x,t)+\mathscr{A}(x,D)u(x,t) &= 0, & x&\in\mathscr{O}, & t&\in J, & \\ \mathscr{B}_j(x,D)u(x,t) &= g_j(x,t), & x&\in\partial\mathscr{O}, & t&\in J, & j&=1,\dots,n, \\ u(x,0) &= 0, & x&\in\mathscr{O}, \end{aligned} \tag{1.31}$$

in an $L_{q,\mu}$-$L_{p,\gamma}$-setting.

In Chapter 6 we study the problems of $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity and $L_{q,\mu}$-$H^s_{p,\gamma}$-maximal regularity for (1.4), where $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity and $L_{q,\mu}$-$H^s_{p,\gamma}$-maximal regularity refer to $L_{q,\mu}$-maximal regularity in the Triebel-Lizorkin space $F^s_{p,r}(\mathscr{O},w_\gamma^{\partial\mathscr{O}})$ and $L_{q,\mu}$-maximal regularity in the Bessel potential space $H^s_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})$. As in Theorem 1.1.1, let us for reasons of exposition state the main result in this direction for the easy case of the scalar-valued heat equation (1.6). In view of the identity $F^s_{p,r}(\mathscr{O},w_\gamma^{\partial\mathscr{O}})=H^s_p(\mathscr{O},w_\gamma^{\partial\mathscr{O}})$ for $\gamma\in(-1,p-1)$ in the scalar-valued setting, we furthermore only formulate the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-variant.

Theorem 1.1.9. *Let $J=(0,T)$ with $T\in(0,\infty)$ and let $\mathscr{O}$ be a C^∞-domain in $\mathbb{R}^d$ with compact boundary $\partial\mathscr{O}$. Let $q,p,r\in(1,\infty)$, $\mu\in(-1,q-1)$, $\gamma\in(-1,\infty)$ and $s\in(\frac{1+\gamma}{p}-2,\frac{1+\gamma}{p})$*

be such that $s+2-\frac{2}{q}(1+\mu) \neq \frac{1}{p}(1+\gamma)$. *Let* v_μ *and* w_γ *be as in* (1.5). *Then the problem* (1.6) *has the property of* $L_{q,\mu}$-$F^s_{p,r,\gamma}$*-maximal regularity with space of initial-boundary data*

$$\mathscr{D}_{i.b.} = \left\{ \begin{pmatrix} g \\ u_0 \end{pmatrix} \in \begin{matrix} F^{\frac{s}{2}+1-\frac{1}{2p}(1+\gamma)}_{q,p}(J,v_\mu;L_p(\partial\mathscr{O})) \cap L_q(J,v_\mu;F^{s+2-\frac{1}{p}(1+\gamma)}_{p,p}(\partial\mathscr{O})) \\ \times \\ B^{s+2-\frac{2}{q}(1+\mu)}_{p,q}(\mathscr{O},w_\gamma) \end{matrix} \right.$$

$$\left. : \quad \mathrm{tr}_{t=0}\, g = \mathrm{tr}_{\partial\mathscr{O}}\, u_0 \textit{ when } s+2-\tfrac{2}{q}(1+\mu) > \tfrac{1}{p}(1+\gamma) \right\},$$

that is, $u \mapsto (\partial_t u - \Delta u, \mathrm{tr}_{\partial\mathscr{O}}u, \mathrm{tr}_{t=0}u)$ *defines an isomorphism of Banach spaces*

$$W^1_q(J,v_\mu;F^s_{p,r}(\mathscr{O},w_\gamma)) \cap L_q(J,v_\mu;F^{s+2}_{p,r}(\mathscr{O},w_\gamma)) \longrightarrow L_q(J,v_\mu;F^s_{p,r}(\mathscr{O},w_\gamma)) \times \mathscr{D}_{i.b.}.$$

The main result of Chapter 6, Theorem 6.6.2, is a version of Theorem 1.1.9 for (1.4). A version for second order elliptic operators instead of $-\Delta$ was already obtained by myself in [163].

Note that Theorem 1.1.9 contains Theorem 1.1.1 as a special case since $F^0_{p,2}(\mathscr{O},w^{\partial\mathscr{O}}_\gamma) = H^0_p(\mathscr{O},w^{\partial\mathscr{O}}_\gamma) = L_p(\mathscr{O},w^{\partial\mathscr{O}}_\gamma)$ for $\gamma \in (-1,p-1)$. In the general setting of (1.4), the $L_{q,\mu}$-$H^s_{p,\gamma}$-variant of Theorem 6.6.2 covers [159, Theorem 3.4]. Here it is worth to remark that the proof of the $L_{q,\mu}$-$L_{p,\gamma}$-case of Theorem 6.6.2 simplifies a bit on the function space theoretic side of the problem. Moreover, this in particularly yields a simplification of the previous approaches [61, 180]. than the previous ones ([61] ($\mu=0$, $\gamma=0$), [180] ($q=p$, $\mu\in[0,p-1)$, $\gamma=0$) and [159]).

Although $L_{q,\mu}$-$F^0_{p,r,\gamma}$-maximal regularity and $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity are independent notions for $\gamma \notin (-1,p-1)$, there still is a connection between the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity problem and the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem. This connection is provided by the following combination of a Sobolev embedding and an elementary embedding:

$$F^{k+\frac{\nu-\gamma}{p}}_{p,r}(\mathscr{O},w^{\partial\mathscr{O}}_\nu) \hookrightarrow F^k_{p,1}(\mathscr{O},w^{\partial\mathscr{O}}_\gamma) \hookrightarrow W^k_p(\mathscr{O},w^{\partial\mathscr{O}}_\gamma), \qquad \nu>\gamma, r\in[1,\infty].$$

Indeed, in view of the invariance

$$\delta = \delta_{p,\nu,s} = \delta_{p,\gamma}, \qquad s = \frac{\nu-\gamma}{p},$$

in connection with the optimal space of boundary data

$$F^\delta_{q,p}(J,v_\mu;L_p(\partial\mathscr{O})) \cap L_q(J,v_\mu;F^{2\delta}_{p,p}(\partial\mathscr{O}))$$

in Theorems 1.1.7 and 1.1.9, a solution operator for (1.31) with $f=0$ and $u_0=0$ in the $L_{q,\mu}$-$L_{p,\gamma}$-case could have been obtained from the $L_{q,\mu}$-$F^{\frac{\nu-\gamma}{p}}_{p,2,\nu}$-case. In the simple case of the heat equation with Dirichlet boundary condition (1.31) this actually would not simply the proof of Theorem 1.1.7. However, in the general case (1.31) this would be a good strategy, see the discussion preceding (1.31). The invariance of trace spaces under Sobolev embedding and related invariance can be a quite powerful tool and is in fact used in Chapters 5 and 6 (also see Remark 6.4.2 and the references given there).

The main technical ingredient in the proof of Theorem 6.6.2 (see the special case Theorem 1.1.9) is an analysis of anisotropic Poisson operators and their mapping properties on weighted mixed-norm anisotropic function spaces. The Poisson operators under consideration naturally occur as (or in) solution operators to the model problems

$$\begin{aligned} \partial_t u(x,t) + (1+\mathscr{A}(D))u(x,t) &= 0, & x \in \mathbb{R}^n_+, \quad & t \in \mathbb{R}, \\ \mathscr{B}_j(D)u(x',t) &= g_j(x',t), & x' \in \mathbb{R}^{n-1}, \quad & t \in \mathbb{R}, \quad j=1,\dots,n, \end{aligned} \tag{1.32}$$

where $\mathscr{A}(D)$ and $\mathscr{B}_j(D)$ are homogeneous with constant coefficients. Moreover, they are operators K of the form

$$Kg(x_1,x',t) = (2\pi)^{-n}\int_{\mathbb{R}^{n-1}\times\mathbb{R}} e^{\imath(x',t)\cdot(\xi',\tau)}\widetilde{k}(x_1,\xi',\tau)\hat{g}(\xi',\tau)\,d(\xi,\tau), \qquad g \in \mathscr{S}(\mathbb{R}^{n-1}\times\mathbb{R}), \tag{1.33}$$

for some anisotropic Poisson symbol-kernel $\widetilde{k}$.

The anisotropic Poisson operator (1.33) is an anisotropic (x',t)-independent version of the classical Poisson operator from the Boutet the Monvel calculus. The Boutet the Monvel calculus is pseudodifferential calculus that in some sense can be considered as a relatively small "algebra", containing the elliptic boundary value problems as well as their solution operators (or parametrices). The calculus was introduced by, as the name already suggests, Boutet de Monvel [32, 33], having its origin in the works of Vishik and Eskin [241], and was furhter developped in e.g. [105–107, 129, 206]; for an introduction to or an overview of the subject we refer the reader to [107, 108, 223].

A parameter-dependent version of the Boutet de Monvel calculus has been introduced and worked out by Grubb and collaborators (see [107] in the references given therein). This calculus contains the parameter-elliptic boundary value problems as well as their solution operators (or parametrices). In particular, resolvent analysis can be carried out in this calculus.

In the present paper we also consider a variant of the parameter-dependent Poisson operators from [107] in the x'-independent setting. Besides that this is one of the key ingredients in the proof of Theorem 6.6.2 (see the special case Theorem 1.1.9) through the anisotropic Poisson operators (1.33), it also forms the basis for our parameter-dependent estimates in weighted Besov, Triebel-Lizorkin and Bessel potential spaces for the elliptic boundary value problems

$$\begin{aligned} (\lambda+\mathscr{A}(x,D))u(x) &= f(x), & x \in \mathscr{O} \\ \mathscr{B}_j(x',D)u(x') &= g_j(x'), & x' \in \partial\mathscr{O}, \quad j=1,\dots,n, \end{aligned} \tag{1.34}$$

in Theorem 6.7.1. These parameter dependent estimates are an extension of [163] on second order elliptic boundary value problems subject to the Dirichlet boundary condtition, which was in turn in the spirit of [67, 109].

1.2. Outline of the Main Part of the Thesis

Part I: Harmonic Analysis and Function Spaces

Chapter 2: An Intersection Representation for a Class of Anisotropic Vector-valued Function Spaces. In this chapter we introduce classes of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov, which includes weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. The main results are Theorem 2.5.1 and Corollary 2.5.2 on intersection representations in this setting, which contain Theorem 1.1.2 as a special case. Crucial ingredients are the estimates in terms of differences in Section 2.4, which are generalizations of Proposition 1.1.3.

Chapter 3: Difference Norms for Vector-valued Bessel Potential Spaces. In this chapter we study weighted Bessel potential spaces of tempered distributions taking values in UMD Banach spaces. The main result is Theorem 3.4.1 on a randomized difference norm characterization for such function spaces $H^s_p(\mathbb{R}^d, w; X)$. The main ingredients are R-boundedness results for Fourier multiplier operators from Section 3.3, which are of independent interest. Theorem 3.4.1 can be considered as a more general version of Theorem 1.1.4 (also see Theorem 3.1.1) thanks to Examples 3.4.4 and 3.4.5. As an application of the randomized difference norm description we characterize the pointwise multiplier property of $1_{\mathbb{R}^d_+}$ on $H^s_p(\mathbb{R}^d, w; X)$ in Theorem 4.4.1, which corresponds to Theorem 1.1.5 in this introduction.

Chapter 4: Complex interpolation with Dirichlet boundary conditions on the half line. In this chapter we prove results on the complex interpolation of weighted Sobolev spaces of distributions taking values in UMD Banach spaces spaces with Dirichlet boundary conditions. The weights that we consider are the A_p-power weights w_γ (1.26) with $\gamma \in (-1, p-1)$, where p is the integrability parameter under consideration. The main results are presented in Section 4.6.2 on the half line. These cover Theorem 1.1.6 and an application to the characterization of the fractional domain spaces of the first derivative operator on the half line. A crucial ingredient is the pointwise multiplier property of $1_{\mathbb{R}^d_+}$ on the corresponding weighted Bessel potential spaces $H^s_p(\mathbb{R}^d, w_\gamma; X)$, of which we provide a new and simpler proof as well (see Theorem 4.4.1).

Part II: Boundary Value Problems

Chapter 5: The Heat Equation subject to the Dirichlet Boundary Condition. In this chapter we consider the Laplace operator subject to Dirichlet boundary conditions on a smooth domain in a weighted L_p-setting with power weights that fall outside the classical class of Muckenhoupt A_p-weights. The first two main result are Theorem 5.6.1 and Corollary 5.6.2, corresponding to Theorem 1.1.8 in this introduction, on the boundedness of the H^∞-calculus. The third and fourth main result are Theorems 5.7.15 and 5.7.16, of which the second corresponds to Theorem 1.1.7 in this introduction, on the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem. An important role is played by Sobolev spaces with power weights outside the A_p-range, whose theory is partially developed in Sections 5.3 and 5.7.

Chapter 6: General Elliptic and Parabolic Boundary Value Problems. In this chapter we study elliptic and parabolic boundary value problems with inhomogeneous boundary conditions in weighted function spaces of Sobolev, Bessel potential, Besov and Triebel-Lizorkin type. The first main result is Theorem 6.6.2 on $L_{q,\mu}$-maximal regularity in weighted Triebel-Lizorkin spaces and Bessel potential spaces for the parabolic boundary value problems (1.4), including Theorem 1.1.9 as a special case. The second main result is Theorem 6.7.1 on parameter-dependent estimates in weighted Besov, Triebel-Lizorkin and Bessel potential spaces for the elliptic boundary value problems (1.34). The key ingredient in this chapter is an analysis of Poisson operators and their mapping properties, which is carried out in Sections 6.4 and 6.5. The anistropic Poisson operators (1.33) are a special instance of the Poisson operators that are treated here.

I

Harmonic Analysis and Function Spaces

2

An Intersection Representation for a Class of Anisotropic Vector-valued Function Spaces

This chapter is based on the paper:

> *N. Lindemulder. An Intersection Representation for a Class of Anisotropic Vector-valued Function Spaces in preparation.*

The main result of this paper is an intersection representation for a class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov, which includes weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. In the special case of the classical Triebel-Lizorkin spaces, the intersection representation gives an improvement of the well-known Fubini property. The motivation comes from the weighted L_q-L_p-maximal regularity problem for parabolic boundary value problems, where weighted anisotropic mixed-norm Triebel-Lizorkin spaces occur as spaces of boundary data.

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2.1. INTRODUCTION

The motivation for this paper comes from [159] on the weighted L_q-L_p-maximal regularity problem for parabolic boundary value problems, which provides an extension of [61] to the weighted setting.

During the last 25 years, *maximal regularity* has turned out to be an important tool in the theory of nonlinear PDEs (see e.g. [1, 5, 8, 11, 49, 50, 52, 84, 97, 141, 149, 168, 178, 180, 196, 198, 199, 203, 204]). Maximal regularity means that there is an isomorphism between the data and the solution of the problem in suitable function spaces. Having established maximal regularity for the linearized problem, many nonlinear problems can be treated with tools as the contraction principle and the implicit function theorem (see [198]). Concretely, the concept of maximal regularity has found its application in a great variety of physical, chemical and biological phenomena, like reaction-diffusion processes, phase field models, chemotactic behaviour, population dynamics, phase transitions and the behaviour of two phase fluids, for instance (see e.g. [178, 198, 199, 204]).

In order to elaborate a bit on the L_q-L_p-maximal regularity problem for parabolic boundary value problems, let us for simplicity consider the heat equation with the Dirichlet boundary condition,

$$\begin{aligned} \partial_t u(x,t) + \Delta u(x,t) &= f(x,t), & x &\in \mathscr{O}, & t &\in J, \\ u(x',t) &= g(x',t), & x' &\in \partial\mathscr{O}, & t &\in J, \\ u(x,0) &= u_0(x), & x &\in \mathscr{O}, \end{aligned} \tag{2.1}$$

where $J = (0,T)$ is a finite time interval and $\mathscr{O} \subset \mathbb{R}^d$ is a C^∞-domain with a compact boundary $\partial\mathscr{O}$. In the maximal L_q-L_p-regularity approach to (2.1) one is looking for solutions u in the *maximal regularity space*

$$W^1_q(J; L_p(\mathscr{O})) \cap L_q(J; W^2_p(\mathscr{O})). \tag{2.2}$$

The solution to the L_q-L_p-maximal regularity problem for (2.1) is classical in the case $q = p$ (see [152]). However, it is desirable to have maximal L_q-L_p-regularity for the full range $q, p \in (1,\infty)$, as this enables one to treat more nonlinearities. For instance, one often requires large q and p due to better Sobolev embeddings, and $q \neq p$ due to criticality and/or scaling invariance (see e.g. [97, 141, 199, 203, 204]). But the case $q \neq p$ is much more involved than the case $q = p$ due to a lack of Fubini in the form of $L_q[L_p] = L_p[L_q]$ when $q \neq p$.

The main difficulty in the L_q-L_p-maximal regularity approach to (2.1) is the treatment of the boundary inhomogeneity g in the case $q \neq p$. In the classical case $q = p$, g has to be in the intersection space

$$B^{\delta}_{p,p}(J; L_p(\partial\mathscr{O})) \cap L_p(J; B^{2\delta}_{p,p}(\partial\mathscr{O})) = W^{\delta}_p(J; L_p(\partial\mathscr{O})) \cap L_p(J; W^{2\delta}_p(\partial\mathscr{O}))$$

with $\delta = 1 - \frac{1}{2p}$, where $W^s_p = B^s_{p,p}$ a non-integer order Sobolev-Slobodeckii space or

Besov space. However, in the general case g has to be in the intersection space

$$F^{\delta}_{q,p}(J;L_p(\partial\mathscr{O}))\cap L_q(J;B^{2\delta}_{p,p}(\partial\mathscr{O})),\qquad \delta=1-\frac{1}{2p},\tag{2.3}$$

where $F^{s}_{q,p}$ is a Triebel-Lizorkin space. This was established in [243] in the case $p\leq q$ and extended in [61] to the full range for q,p in the more general setting of vector-valued parabolic boundary value problems with boundary conditions of Lopatinskii-Shapiro type.

The solution to the L_q-L_p-maximal regularity problem for (2.1) in particularly yields that the intersection space in (2.3) is the spatial trace space of the maximal regularity space in (2.2). However, on the one hand, this maximal regularity space (2.2) can naturally be identified with the anisotropic mixed-norm Sobolev space

$$W^{(2,1)}_{(p,q)}(\mathscr{O}\times J)=\left\{u\in\mathscr{D}'(\mathscr{O}\times J):\partial_t,\partial_x^{\alpha}u\in L_{(p,q)}(\mathscr{O}\times J),|\alpha|\leq 2\right\},$$

where the mixed-norm Lebesgue space

$$L_{(p,q)}(\mathscr{O}\times J)=\left\{f\in L_0(\mathscr{O}\times J):\left(\int_J\left(\int_{\mathscr{O}}|f(x,t)|^p\,dx\right)^{p/q}dt\right)^{1/q}<\infty\right\}$$

can be naturally identified with the Lebesgue Bochner space $L_q(J;L_p(\mathscr{O}))$. On the other hand, in [131] it was shown that the anisotropic mixed-norm Triebel-Lizorkin space $F^{s,(\frac{1}{2},1)}_{(p,q),p}(\mathbb{R}^{d-1}\times\mathbb{R})$ naturally occurs as the trace space of the anisotropic mixed-norm Sobolev space $W^{(2,1)}_{(p,q)}(\mathbb{R}^d\times\mathbb{R})$. This suggest a link between anisotropic mixed-norm Triebel-Lizorkin spaces and intersection spaces of the form (2.3).

Such a link was in fact obtained in [63, Proposition 3.23] by comparing the trace result [131, Theorem 2.2] with a trace result from [24, 25]: for every $q,p\in(1,\infty)$, $a,b\in(0,\infty)$ and $s\in(0,\infty)$,

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^{d-1}\times\mathbb{R})=F^{s/b}_{q,p}(\mathbb{R};L_p(\mathbb{R}^{d-1}))\cap L_q(\mathbb{R};B^{s/a}_{p,p}(\mathbb{R}^{d-1})).\tag{2.4}$$

It is the goal of this paper to provide a more systematic approach to the intersection representation (2.4) and obtain more general versions of it, covering the weighted Banach space-valued setting. In order to do so, we introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [119], which includes Banach space-valued weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

The main result of this paper is an intersection representation for this new class of anisotropic function spaces, from which the following theorem can be obtained as a special case:

Theorem 2.1.1. *Let $a,b\in(0,\infty)$, $p,q\in(1,\infty)$, $r\in[1,\infty]$ and $s\in(0,\infty)$. Then*

$$F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n\times\mathbb{R}^m)=\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m;L_p(\mathbb{R}^n))\cap L_q(\mathbb{R}^m;F^{s/a}_{p,r}(\mathbb{R}^n)),\tag{2.5}$$

where, for $E = L_p(\mathbb{R}^n)$*,*

$$\mathbb{F}^{\sigma}_{q,r}(\mathbb{R}^m;E) = \left\{ f \in \mathscr{S}'(\mathbb{R}^m;E) : (2^{k\sigma} S_k f)_k \in L_q(\mathbb{R}^n; E[\ell_r(\mathbb{N})]) \right\}$$

with $(S_k)_{k\in\mathbb{N}}$ *a Littlewood-Paley decomposition of* $\mathbb{R}^m$*.*

In the case $p = r$, Fubini yields $\mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m; L_p(\mathbb{R}^n)) = F^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n))$ and $F^{s/a}_{p,r}(\mathbb{R}^n) = B^{s/a}_{p,p}(\mathbb{R}^n)$, and we obtain an extension of the intersection representation (2.4) to decompositions $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$:

$$F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}^m) = F^{s/b}_{q,p}(\mathbb{R}^m; L_p(\mathbb{R}^n)) \cap L_q(\mathbb{R}^m; B^{s/a}_{p,p}(\mathbb{R}^n)).$$

In the special case that $a = b$ and $p = q$, the latter can be viewed as a special instance of Fubini property. In fact, the main result of this paper, Theorem 2.5.1/2.5.3, extends the well-known Fubini property for the classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ (see [236, Section 4] and the references given therein), see Example 2.5.4. However, as seen in Theorem 2.1.1, the availability of Fubini is unessential for intersection representations, it should just be thought of as a way to simplify the function spaces that one has to deal with in case of its availability.

Notation and convention.

$\hat{f} = \mathscr{F} f$, $\check{f} = \mathscr{F}^{-1} f$, where $\mathscr{F}$ denotes the Fourier transform, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, $\ell^s_p(\mathbb{N}) = \{(a_n)_n \in \mathbb{C}^{\mathbb{N}} : (2^{ns} a_n)_n \in \ell_p\}$, X will denote a Banach space and $(S, \mathscr{A}, \mu)$ will denote σ-finite measure space.

2.2. PRELIMINARIES

2.2.1. Anisotropy and decomposition

ANISOTROPY ON $\mathbb{R}^d$

An *anisotropy* on $\mathbb{R}^d$ is a is a symmetric real $d \times d$ matrix A with $\sigma(A) \subset \mathbb{R}_+$.[3] An anisotropy A on $\mathbb{R}^d$ gives rise to a one-parameter group of expansive dilations $(A_t)_{t\in\mathbb{R}_+}$ given by

$$A_t = t^A = \exp[A \ln(t)], \quad t \in \mathbb{R}_+,$$

where $\mathbb{R}_+$ is considered as multiplicative group.

In the special case $A = \mathrm{diag}(\boldsymbol{a})$ with $\boldsymbol{a} = (a_1, \ldots, a_d) \in (0,\infty)^d$, the associated one-parameter group of expansive dilations $(A_t)_{t\in\mathbb{R}_+}$ is given by

$$A_t = \exp[A \ln(t)] = \mathrm{diag}(t^{a_1}, \ldots, t^{a_d}), \quad t \in \mathbb{R}_+$$

Given an anisotropy A on $\mathbb{R}^d$, an A-homogeneous distance function is a Borel measurable mapping $\rho : \mathbb{R}^d \longrightarrow [0,\infty)$ satisfying

[3]For simplicity of this thesis we work with a restricted notion of anisotropy. In the paper [161] on which this chapter is based an anisotropy on $\mathbb{R}^d$ is a real $d \times d$ matrix A with $\sigma(A) \subset \mathbb{C}_+$, see [71] and the references given there.

(i) $\rho(x) = 0$ if and only if $x = 0$ (*non-degenerate*);

(ii) $\rho(A_t x) = t\rho(x)$ for all $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ ($(A_t)_{t\in\mathbb{R}_+}$-*homogeneous*);

(iii) there exists $c \in [1,\infty)$ so that $\rho(x+y) \le c(\rho(x)+\rho(y))$ for all $x, y \in \mathbb{R}^d$ (*quasi-triangle inequality*). The smallest such c is denoted c_ρ.

Any two homogeneous quasi-norms ρ_1, ρ_2 associated with an anisotropy A on $\mathbb{R}^d$ are equivalent in the sense that

$$\rho_1(x) \eqsim_{\rho_1,\rho_2} \rho_2(x), \qquad x \in \mathbb{R}^d.$$

If ρ is a quasi-norm associated associated with an anisotropy A on $\mathbb{R}^d$ and λ denotes the Lebesgue measure on $\mathbb{R}^d$, then $(\mathbb{R}^d, \rho, \lambda)$ is a space of homogeneous type.

Given an anisotropy A on $\mathbb{R}^d$, we define the quasi-norm ρ_A associated with A as follows: we put $\rho_A(0) := 0$ and for $x \in \mathbb{R}^d \setminus \{0\}$ we define $\rho_A(x)$ to be the unique number $\rho_A(x) = \lambda \in (0,\infty)$ for which $A_{\lambda^{-1}} x \in S^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Then

$$\rho_A(x) := \min\{\lambda > 0 : |A_{\lambda^{-1}} x| \le 1\}, \qquad x \neq 0.$$

The quasi-norm ρ_A is C^∞ on $\mathbb{R}^d \setminus \{0\}$. We write

$$B^A(x,r) := B_{\rho_A}(x,r) = \{y \in \mathbb{R}^d : \rho_A(x-y) \le r\}, \qquad x \in \mathbb{R}^d, r \in (0,\infty).$$

Given an anisotropy A on $\mathbb{R}^d$, we write

$$\lambda^A_{\min} := \min\{\lambda : \lambda \in \sigma(A)\}, \quad \lambda^A_{\max} := \max\{\lambda : \lambda \in \sigma(A)\}.$$

Note that $0 < \lambda^A_{\min} \le \lambda^A_{\max} < \infty$. It holds that

$$\begin{aligned} t^{\lambda^A_{\min}}|x| &\le |A_t x| &\le t^{\lambda^A_{\max}}|x|, \qquad &|t| \ge 1,\\ t^{\lambda^A_{\max}}|x| &\le |A_t x| &\le t^{\lambda^A_{\min}}|x|, \qquad &|t| \le 1, \end{aligned}$$

and

$$\begin{aligned} t^{1/\lambda^A_{\max}}\rho_A(x) &\le \rho_A(tx) &\le t^{1/\lambda^A_{\min}}\rho_A(x), \qquad &|t| \ge 1,\\ t^{1/\lambda^A_{\min}}\rho_A(x) &\le \rho_A(tx) &\le t^{1/\lambda^A_{\max}}\rho_A(x), \qquad &|t| \le 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} \rho_A(x)^{\lambda^A_{\min}} &\le |x| &\le \rho_A(x)^{\lambda^A_{\max}}, \qquad &|x| \ge 1,\\ \rho_A(x)^{\lambda^A_{\max}} &\le |x| &\le \rho_A(x)^{\lambda^A_{\min}}, \qquad &|x| \le 1, \end{aligned}$$

An alternative viewpoint to anisotropy is as follows (see [34] and references given there), which is actually more general. A real $d \times d$ matrix B is an *expansive dilation* if $\min_{\lambda\in\sigma(B)} |\lambda| > 1$. A quasi-norm associated with an expansive dilation B is a Borel measurable mapping $\rho : \mathbb{R}^n \longrightarrow [0,\infty)$ satisfying

(i) $\rho(x) = 0$ if and only if $x = 0$ (*non-degenerate*);

(ii) $\rho(Bx) = |\det(B)|\rho(x)$ for all $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ (*B-homogeneous*);

(iii) there exists $c \in [1,\infty)$ so that $\rho(x+y) \le c(\rho(x)+\rho(y))$ for all $x, y \in \mathbb{R}^d$ (*quasi-triangle inequality*). The smallest such c is denoted c_ρ.

If A is an anisotropy on $\mathbb{R}^d$ and ρ is an A-homogeneous distance function, then $B = A_2 = \exp[A\ln(2)]$ is an expensive dilation and $\rho^B(x) := \rho(x)^{\mathrm{tr}(A)}$ defines a quasi-norm associated with B.

$\vec{d}$-DECOMPOSITIONS AND ANISOTROPY

Let $\vec{d} = (d_1,\dots,d_\ell) \in (\mathbb{Z}_{\ge 1})^\ell$ be such that $d = |\vec{d}|_1 = d_1 + \dots + d_\ell$. The decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_\ell}.$$

is called the *$\vec{d}$-decomposition* of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we accordingly write $x = (x_1,\dots,x_\ell)$ and $x_j = (x_{j,1},\dots,x_{j,d_j})$, where $x_j \in \mathbb{R}^{d_j}$ and $x_{j,i} \in \mathbb{R}$ ($j = 1,\dots,\ell$; $i = 1,\dots,d_j$). We also say that we view $\mathbb{R}^d$ as being *$\vec{d}$-decomposed.* Furthermore, for each $k \in \{1,\dots,\ell\}$ we define the inclusion map

$$\iota_k = \iota_{[\vec{d};k]} : \mathbb{R}^{d_k} \longrightarrow \mathbb{R}^n,\ x_k \mapsto (0,\dots,0,x_k,0,\dots,0),$$

and the projection map

$$\pi_k = \pi_{[\vec{d};k]} : \mathbb{R}^n \longrightarrow \mathbb{R}^{d_k},\ x = (x_1,\dots,x_\ell) \mapsto x_k.$$

A $\vec{d}$-anisotropy is tuple $\boldsymbol{A} = (A_1,\dots,A_\ell)$ with each A_j an anisotropy on $\mathbb{R}^{d_j}$. A $\vec{d}$-anisotropy $\boldsymbol{A}$ gives rise to a one-parameter group of expansive dilations $(\boldsymbol{A}_t)_{t\in\mathbb{R}_+}$ given by

$$\boldsymbol{A}_t x = (A_{1,t}x_1,\dots,A_{\ell,t}x_l), \qquad x \in \mathbb{R}^d, t \in \mathbb{R}_+,$$

where $A_{j,t} = \exp[A_j \ln(t)]$. Note that $\boldsymbol{A}^\oplus := \oplus_{j=1}^{\ell} A_j$ is an anisotropy on $\mathbb{R}^d$ with $\boldsymbol{A}^\oplus_t = \boldsymbol{A}_t$ for every $t \in \mathbb{R}_+$. We define the $\boldsymbol{A}^\oplus$-homogeneous distance function $\rho_{\boldsymbol{A}}$ by

$$\rho_{\boldsymbol{A}}(x) := \max\{\rho_{A_1 x},\dots,\rho_{A_\ell}(x_\ell)\}, \qquad x \in \mathbb{R}^d.$$

We write

$$B^{\boldsymbol{A}}(x,R) := B_{\rho_{\boldsymbol{A}}}(x,R), \qquad x \in \mathbb{R}^d, R \in [0,\infty),$$

and

$$B^{\boldsymbol{A}}(x,\boldsymbol{R}) := B^{A_1}(x_1,R_1) \times \dots \times B^{A_\ell}(x_\ell,R_\ell), \qquad x \in \mathbb{R}^d, \boldsymbol{R} \in [0,\infty)^\ell.$$

Note that $B^{\boldsymbol{A}}(x,R) = B^{\boldsymbol{A}}(x,\boldsymbol{R})$ when $\boldsymbol{R} = (R,\dots,R)$.

2.2.2. Vector-valued Functions and Distributions

As general reference to the theory of vector-valued distributions we mention [6] (and [5, Section III.4]).

Let G be a topological vector space. The space of G-valued tempered distributions $\mathcal{S}'(\mathbb{R}^d;G)$ is defined as $\mathcal{S}'(\mathbb{R}^d;G) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d),G)$, the space of continuous linear operators from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ to G. In this chapter we equip $\mathcal{S}'(\mathbb{R}^d;G)$ with the topology of pointwise convergence. Standard operators (derivative operators, Fourier transform, convolution, etc.) on $\mathcal{S}'(\mathbb{R}^d;G)$ can be defined as in the scalar-case.

By a combination of [6, Theorem 1.4.3] and (the proof of) [6, Lemma 1.4.6], the space of finite rank operators $\mathcal{S}'(\mathbb{R}^d)\otimes G$ is sequentially dense in $\mathcal{S}'(\mathbb{R}^d;G)$. Furthermore, as a consequence of the Banach-Steinhaus (see [214, Theorem 2.8]), if G is sequentially complete, then so is $\mathcal{S}'(\mathbb{R}^d;G)$.

Let $(T,\mathscr{B},\nu)$ be a σ-finite measure space and let G be a topological vector space. We define $L_0(T;G)$ as the space as of all ν-a.e. equivalence classes of ν-strongly measurable functions $f:T\to G$. Suppose there is a system $\mathcal{Q}$ of semi-quasi-norms generating the topology of G. We equip $L_0(T;G)$ with the topology generated by the semi-quasi-norms

$$\rho_{B,q}(f) := \int_B (q(f)\wedge 1)\,\mathrm{d}\nu, \qquad B\in\mathscr{B}, \nu(B)<\infty, q\in\mathcal{Q}.$$

This topological vector space topology on $L_0(T;G)$ is independent of $\mathcal{Q}$ and is called the topology of convergence in measure. Note that $L_0(T)\otimes G$ is sequentially dense in $L_0(T;G)$ as a consequence of the dominated convergence theorem and the definitions.

If G is an F-space, then $L_0(T;G)$ is an F-space as well. Here we could for example take $G = L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X)$ with $\boldsymbol{r}\in(0,\infty]^\ell$ and X a Banach space, where

$$L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d) = \left\{ f\in L_0(\mathbb{R}^d) : f1_B\in L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d), B\subset\mathbb{R}^d \text{ bounded Borel}\right\}$$

and

$$L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d) = L_{r_\ell}(\mathbb{R}^{d_\ell})[\dots[L_{r_1}(\mathbb{R}^{d_1})]\dots].$$

Let X be a Banach space. Then $L_0(T)\otimes\mathcal{S}'(\mathbb{R}^d)\otimes X$ is sequentially dense in both of $L_0(T;\mathcal{S}'(\mathbb{R}^d;X))$ and $\mathcal{S}'(\mathbb{R}^d;L_0(T;X))$, while the two induced topologies on $L_0(T)\otimes\mathcal{S}'(\mathbb{R}^d)\otimes X$ coincide. Therefore, we can naturally identify

$$L_0(T;\mathcal{S}'(\mathbb{R}^d;X)) \cong \mathcal{S}'(\mathbb{R}^d;L_0(T;X)).$$

2.3. DEFINITIONS AND BASIC PROPERTIES

Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed with $\mathpzc{d}\in(\mathbb{Z}_{\geq 1})^\ell$ and let $\boldsymbol{A}=(A_1,\dots,A_\ell)$ be a $\mathpzc{d}$-anisotropy. Let $\varepsilon_+,\varepsilon_-\in\mathbb{R}$ and $\boldsymbol{r}\in(0,\infty)^l$.

For $j\in\{1,\dots,\ell\}$, we define the maximal function operator $M^{A_j}_{r_j;[\mathpzc{d};j]}$ on $L_0(S\times\mathbb{R}^d)$ by

$$M^{A_j}_{r_j;[\mathpzc{d};j]}(f)(s,x) := \sup_{\delta>0} ⨍_{B^{A_j}} |f(s,x+\iota_{[\mathpzc{d};j]}y_j)|\,dy_j.$$

We define the maximal function operator $M_{\boldsymbol{r}}^{\boldsymbol{A}}$ by iteration:

$$M_{\boldsymbol{r}}^{\boldsymbol{A}}(f) := M_{r_\ell;[\mathpzc{d};\ell]}^{A_\ell}(\ldots(M_{r_1;[\mathpzc{d};1]}^{A_1}(f))\ldots).$$

The following definition is an extension of [119, Definition 1.1.1] to the anisotropic setting with some extra underlying measure space $(S,\mathscr{A},\mu)$. The extra measure space provides the right setting for intersection representations, see Section 2.5.

Definition 2.3.1. We define $\mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ as the set of all quasi-Banach function spaces E on $\mathbb{R}^d \times \mathbb{N} \times S$ with the Fatou property for which the following two properties are fulfilled:

(a) $S_+, S_- \in \mathscr{B}(E)$, the left respectively right shift on $\mathbb{N}$, with

$$\|(S_+)^k\|_{\mathscr{B}(E)} \lesssim 2^{-\varepsilon_+ k} \quad \text{and} \quad \|(S_-)^k\|_{\mathscr{B}(E)} \lesssim 2^{\varepsilon_- k}, \qquad k \in \mathbb{N}.$$

(b) $M_{\boldsymbol{r}}^{\boldsymbol{A}}$ is bounded on E:

$$\|M_{\boldsymbol{r}}^{\boldsymbol{A}}(f_n)\|_E \lesssim \|(f_n)\|_E, \qquad (f_n) \in E.$$

We similarly define $\mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r})$ without the presence of $(S,\mathscr{A},\mu)$, or equivalently, $\mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r}) = \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(\{0\},\{\varnothing,\{0\}\},\#))$.

Remark 2.3.2. Note that $\varepsilon_+ \leq \varepsilon_-$ when $E \neq \{0\}$, which can be seen by considering $(S_+)^k \circ (S_-)^k$, $k \in \mathbb{N}$.

Remark 2.3.3. Note that

$$\mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu)) \subset \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\tilde{\boldsymbol{r}},(S,\mathscr{A},\mu)), \quad \boldsymbol{r} \geq \tilde{\boldsymbol{r}}.$$

Example 2.3.4. Let us provide some examples of $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. Condition (b) in Definition 2.3.1 can be covered by means of the lattice Hardy–Littlewood maximal function operator: if F is a UMD Banach function space on S, A an ansitropy, $p \in (1,\infty)$, and $w \in A_p(\mathbb{R}^d, A)$ then (see [29, 94, 114, 211, 231])

$$Mf(x) := \sup_{\delta>0} \fint_{B^A(x,\delta)} |f(y)|\,dy$$

defines a bounded sublinear operator on $L_p(\mathbb{R}^d, w; F) = L_p(\mathbb{R}^d, w)[F]$. The latter induces a bounded sublinear operator on $L_p(\mathbb{R}^d, w)[F[\ell_\infty]]$ in the natural way. Let us furthermore remark that the mixed-norm space $F[G]$ of two UMD Banach function spaces F and G is again a UMD Banach function space (see [211, page 214]). This leads to the following examples of:

(i) Let $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $\boldsymbol{r} \in (0,\infty)^\ell$ is such that $r_j < p_1 \wedge \ldots \wedge p_j \wedge q$ for $j = 1,\ldots,\ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[\ell_q^s(\mathbb{N})] \in \mathscr{S}(s,s,\boldsymbol{A},\boldsymbol{r}).$$

(ii) Let $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $\boldsymbol{r} \in (0,\infty)^\ell$ is such that $r_j < p_1 \wedge \ldots \wedge p_j$ for $j = 1,\ldots,\ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = \ell_q^s(\mathbb{N})[L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})] \in \mathscr{S}(s, s, \boldsymbol{A}, \boldsymbol{r}).$$

(iii) Let $\boldsymbol{p} \in (0,\infty)^\ell$, $q \in (0,\infty]$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_\infty(\mathbb{R}^{d_j}, A_j)$, $s \in \mathbb{R}$ and F a quasi-Banach function space on S. If $\boldsymbol{r} \in (0,\infty)^\ell$ is such that $r_j < p_1 \wedge \ldots \wedge p_j \wedge q$ for $j = 1,\ldots,\ell$ and $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$ and $F^{\boldsymbol{r}_{\max}}$ is a UMD Banach function space,

$$F^r := \{f \in L_0(S) : |f|^{1/r} \in F\}, \qquad \|f\|_{F^r} := \||f|^{1/r}\|_F^r,$$

then

$$E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[F[\ell_q^s(\mathbb{N})]] \in \mathscr{S}(s, s, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu)).$$

For a quasi-Banach function space E on $\mathbb{R}^d \times \mathbb{N} \times S$ we define the quasi-Banach function space $E_\otimes^{\boldsymbol{A}}$ on S by

$$\|f\|_{E_\otimes^{\boldsymbol{A}}} := \|1_{B^{\boldsymbol{A}}(0,1)\times\{0\}} \otimes f\|_E, \qquad f \in L_0(S).$$

Let $\boldsymbol{p} \in (0,\infty)^\ell$ and $w : [1,\infty)^\ell \to (0,\infty)$. We define the quasi-Banach function space

$$B_{\boldsymbol{A}}^{\boldsymbol{p},w} := \Big\{f \in L_0(S) : \sup_{\boldsymbol{R}\in[1,\infty)^\ell} w(\boldsymbol{R})\|f\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,\boldsymbol{R}))} < \infty\Big\} \tag{2.6}$$

which is an extension of (a slight variant of) the space B^p considered by Beurling in [26] (see [205]).

Let $\boldsymbol{p}, \boldsymbol{q} \in (0,\infty)^\ell$. We define $w_{\boldsymbol{A},\boldsymbol{q}} : [1,\infty)^\ell \to \mathbb{R}_+$ by

$$w_{\boldsymbol{A},\boldsymbol{q}}(\boldsymbol{R}) := \boldsymbol{R}^{-\mathrm{tr}(\boldsymbol{A})\boldsymbol{q}^{-1}} = \prod_{j=1}^{\ell} R_j^{-\mathrm{tr}(A_j)/q_j}, \qquad \boldsymbol{R} \in [1,\infty)^\ell.$$

The quasi-Banach function space $B_{\boldsymbol{A}}^{\boldsymbol{p},w_{\boldsymbol{A},\boldsymbol{q}}} \hookrightarrow L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)$ introduced in (2.6) will be convenient to formulate some of the estimates we will obtain. Note that, if $\boldsymbol{p} \in [1,\infty)$, then

$$B_{\boldsymbol{A}}^{\boldsymbol{p},w_{\boldsymbol{A},\boldsymbol{q}}}(X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X).$$

Lemma 2.3.5. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\lambda \in (-\infty, \varepsilon_+)$. For $F = (f_n)_n \in E$ and $g := \sum_{n=0}^\infty 2^{n\lambda}|f_n|$ we have*

$$\|(\delta_{0,n} g)_n\|_E \lesssim \|F\|_E. \tag{2.7}$$

Moreover, $g \in E_\otimes^{\boldsymbol{A}}[B_{\boldsymbol{A}}^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}] \hookrightarrow E_\otimes^{\boldsymbol{A}}[L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)]$ with

$$\|g\|_{E_\otimes^{\boldsymbol{A}}[B_{\boldsymbol{A}}^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}]} \lesssim \|F\|_E. \tag{2.8}$$

Remark 2.3.6. Suppose that $\varepsilon_+ > 0$ and $\lambda \in (0,\varepsilon_+)$ in Lemma 2.3.5. Let $0 < \kappa \leq \boldsymbol{r}_{\min}$ be such that $||\cdot||_E$ is a κ-norm. Then, in particular, $2^{n\lambda} f_n \in E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]$ with $||2^{n\lambda} f_n||_{E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim ||F||_E$, so that

$$\sum_{n=0}^{\infty} ||f_n||^{\kappa}_{E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]} = \sum_{n=0}^{\infty} 2^{-n\lambda\kappa} ||2^{n\lambda} f_n||^{\kappa}_{E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]} \lesssim \sum_{n=0}^{\infty} 2^{-n\lambda\kappa} ||F||_E \lesssim ||F||_E.$$

Remark 2.3.7. Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. Similarly to the proof of Lemma 2.3.5 (but simpler) it can be shown that

$$E_i \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}].$$

Proof of Lemma 2.3.5. This can be shown similarly to [119, Lemma 1.1.4]. Let us just provide the details for (2.8). As $|B^{A_j}(x_j,R_j)| \eqsim R_j^{\mathrm{tr}(A_j)/r_j}$, $j = 1,\ldots,\ell$, for any $x \in \mathbb{R}^d$ and $\boldsymbol{R} \in (0,\infty)^{\ell}$, we have

$$1_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} \otimes ||g||_{L_{\boldsymbol{r},\mathscr{d}}(B^{\boldsymbol{A}}(0,\boldsymbol{R}))} \lesssim \prod_{j=1}^{\ell} R_j^{\mathrm{tr}(A_j)/r_j} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g), \qquad \boldsymbol{R} \in [1,\infty)^{\ell}.$$

Therefore,

$$1_{B^{\boldsymbol{A}}(0,\boldsymbol{1})} \otimes w_{\boldsymbol{A},\boldsymbol{r}}(\boldsymbol{R})||g||_{L_{\boldsymbol{r},\mathscr{d}}(B^{\boldsymbol{A}}(0,\boldsymbol{R}))} \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g), \qquad \boldsymbol{R} \in [1,\infty)^{\ell},$$

so that

$$1_{B^{\boldsymbol{A}}(0,\boldsymbol{1})} \otimes ||g||_{B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}} \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g).$$

It thus follows that

$$||g||_{E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}]} = \left|\left|1_{B^{\boldsymbol{A}}(0,\boldsymbol{1})\times\{0\}} \otimes ||g||_{B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}}\right|\right|_E \lesssim ||M^{\boldsymbol{A}}_{\boldsymbol{r}}(\delta_{0,n}g)_n||_E.$$

Using the boundedness of $M^{\boldsymbol{A}}_{\boldsymbol{r}}$ on E in combination with (2.7) we obtain the desired estimate (2.8). □

Definition 2.3.8. Suppose that $\varepsilon_+,\varepsilon_- > 0$ and let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. We define $YL^{\boldsymbol{A}}(E;X)$ as the space of all $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$$

with $(f_n)_n \subset L_0(S;\mathscr{S}'(\mathbb{R}^d;X))$ satisfying the Fourier support condition

$$\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0,2^{n+1}), \qquad n \in \mathbb{N},$$

and $(f_n)_n \in E(X)$. We equip $YL^{\boldsymbol{A}}(E;X)$ with the quasinorm

$$||f||_{YL^{\boldsymbol{A}}(E;X)} := \inf ||(f_n)||_{E(X)},$$

where the infimum is taken over all representations as above.

Definition 2.3.9. Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ as the space of all $f \in L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ for which there exists $(g_n)_n \in E_+$ such that, for all $x^* \in X^*$, $\langle f, x^* \rangle$ has a representation

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} f_{x^*,n} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$$

with $(f_{x^*,n})_n \subset L_0(S; \mathscr{S}'(\mathbb{R}^d))$ satisfying the Fourier support condition

$$\operatorname{supp} \hat{f}_{x^*,n} \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}), \qquad n \in \mathbb{N},$$

and the domination $|f_{x^*,n}| \leq ||x^*|| g_n$. We equip $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ with the quasinorm

$$||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} := \inf ||(g_n)||_E,$$

where the infimum is taken over all $(g_n)_n$ as above.

Remark 2.3.10. Suppose that $\varepsilon_+, \varepsilon_- > 0$, let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and let $0 < \kappa \leq \boldsymbol{r}_{\min}$ be such that $||\cdot||_E$ is a κ-norm. Then the following statements hold:

(i) $YL^{\boldsymbol{A}}(E;X) \subset \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with induced norm.

(ii) Let $f \in YL^{\boldsymbol{A}}(E;X)$ with $(f_n)_n$ as in Definition 2.3.8 with $||(f_n)_n||_{E(X)} \leq 2||f||_{YL^{\boldsymbol{A}}(E;X)}$. Let $\tilde{\boldsymbol{r}} \in (0,\infty)^\ell$ be such that

$$E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \tilde{\boldsymbol{r}}, (S, \mathscr{A}, \mu)). \tag{2.9}$$

Then, by Remark 2.3.6, as

$$E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S; L_{\tilde{\boldsymbol{r}},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)) \hookrightarrow L_0(S; L_{\widetilde{\boldsymbol{r}\wedge\boldsymbol{r}},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)),$$

there is the convergence $f = \sum_{n=0}^{\infty} f_n$ in $E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))$ with

$$||f||_{E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))} \lesssim ||(f_n)_n||_{E(X)} \leq 2||f||_{YL^{\boldsymbol{A}}(E;X)}.$$

In particular, $YL^{\boldsymbol{A}}(E;X)$ does not depend on $\boldsymbol{r}$ and

$$YL^{\boldsymbol{A}}(E;X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{\boldsymbol{r}, w_{\boldsymbol{A},\boldsymbol{r}}}(X)).$$

(iii) Let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with $(g_n)_n \in E_+$ and $\{f_{x^*,n}\}_{(x^*,n)}$ as in Definition 2.3.9 with $||(g_n)_n||_E \leq 2||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$. Let $\tilde{\boldsymbol{r}} \in (0,\infty)^\ell$ satisfy (2.9). Then $||f||_X \leq \sum_{n=0}^{\infty} g_n$, so that $f \in E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X)) \subset L_0(S; L_{\tilde{\boldsymbol{r}},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ with

$$||f||_{E^{\boldsymbol{A}}_{\otimes}(B^{\tilde{\boldsymbol{r}}, w_{\boldsymbol{A},\tilde{\boldsymbol{r}}}}_{\boldsymbol{A}}(X))} \lesssim ||(g_n)_n||_E \leq 2||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

by Remark 2.3.6. By (ii) it furthermore holds that

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} f_{x^*,n} \quad \text{in} \quad L_0(S; L_{\tilde{\boldsymbol{r}},\boldsymbol{d},\mathrm{loc}}(\mathbb{R}^d)).$$

Therefore, $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ does not depend on $\boldsymbol{r}$ and

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow E_{\otimes}^{\boldsymbol{A}}(B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}(X)).$$

Definition 2.3.11. Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. We define $Y^{\boldsymbol{A}}(E;X)$ as the space of all $f \in L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$$

with $(f_n)_n \subset L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$ satisfying the Fourier support condition

$$\begin{aligned} &\operatorname{supp} \hat{f}_0 \subset \overline{B}^{\boldsymbol{A}}(0,2) \\ &\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0,2^{n+1}) \setminus B^{\boldsymbol{A}}(0,2^{n-1}), \qquad n \geq 1, \end{aligned}$$

and $(f_n)_n \in E(X)$. We equip $Y^{\boldsymbol{A}}(E;X)$ with the quasinorm

$$\|f\|_{Y^{\boldsymbol{A}}(E;X)} := \inf \|(f_n)\|_{E(X)},$$

where the infimum is taken over all representations as above.

Proposition 2.3.12. *Suppose that $\varepsilon_+, \varepsilon_- > 0$ and let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Then $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ are quasi-Banach spaces with*

$$YL^{\boldsymbol{A}}(E;X) \subset \widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow E_{\otimes}^{\boldsymbol{A}}(B_{\boldsymbol{A}}^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}; X)),$$

where $YL^{\boldsymbol{A}}(E;X)$ is a closed subspace of $\widetilde{YL}^{\boldsymbol{A}}(E;X)$.

Proof. By Remark 2.3.10,

$$YL^{\boldsymbol{A}}(E;X), \widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow E_{\otimes}^{\boldsymbol{A}}(B_{\boldsymbol{A}}^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}; X)). \tag{2.10}$$

That $YL^{\boldsymbol{A}}(E;X) \subset \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with $\|f\|_{YL^{\boldsymbol{A}}(E;X)} = \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ for all $f \in YL^{\boldsymbol{A}}(E;X)$ follows easily from the definitions. So it remains to be shown that $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ are complete.

Let us first treat $YL^{\boldsymbol{A}}(E;X)$. To this end, let the subspace $E(X)_{\boldsymbol{A}}$ of $E(X)$ be defined by

$$E(X)_{\boldsymbol{A}} := \left\{ (f_n)_n \in E(X) : f_n \in L_0(S; \mathscr{S}'(\mathbb{R}^d; X)), \operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, 2^{n+1}) \right\} \tag{2.11}$$

By Lemma 2.3.5,

$$\Sigma : E(X)_{\boldsymbol{A}} \longrightarrow E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d, \boldsymbol{w})](X) \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d; X)),\ (f_n)_n \mapsto \sum_{n=0}^{\infty} f_n$$

is a well-defined continuous linear mapping. As

$$YL^{\boldsymbol{A}}(E;X) \simeq E(X)_{\boldsymbol{A}} \big/ \ker(\Sigma) \quad \text{isometrically,}$$

it suffices to show that $E(X)_{\boldsymbol{A}}$ is complete.

In order to show that $E(X)_{\boldsymbol{A}}$ is complete, we prove that it is a closed subspace of the quasi-Banach space $E(X)$. Put $w(x) := \prod_{j=1}^{\ell}(1+\rho_{A_j}(x_j))^{\mathrm{tr}(A_j)/r_j}$. Then it is enough to show that, for each $k \in \mathbb{N}$,

$$E(X)_{\boldsymbol{A}} \longrightarrow L_0(S; BC(\mathbb{R}^d, w; X)),\ (f_n)_n \mapsto f_k, \tag{2.12}$$

continuously, where $BC(\mathbb{R}^d, w; X) = \{h \in C(\mathbb{R}^d; X) : wh \in L_\infty(\mathbb{R}^d; X)\}$. Indeed, $BC(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$.

In order to establish (2.12), let $(f_n)_n \in E(X)_{\boldsymbol{A}}$. By Corollary 2.A.2,

$$\sup_{z \in B^{A}(0,2^{-n})} ||f_n||_X \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_x)(x),$$

so that

$$\begin{aligned} ||f_n(x)||_X &\lesssim \inf_{z\in B^{A}(0,2^{-n})} M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X)(x+z) \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \big|\big|M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X)\big|\big|_{L_{\boldsymbol{r},\mathpzc{d}}(B^{A}(x,2^{-n}))}. \end{aligned}$$

For $\boldsymbol{R} \in [1,\infty)^{\ell}$ we can thus estimate

$$\begin{aligned} \sup_{z\in B^{A}(0,\boldsymbol{R})} ||f_n(x)||_X &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \big|\big|M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X)\big|\big|_{L_{\boldsymbol{r},\mathpzc{d}}(B^{A}(0,\boldsymbol{c}_{\boldsymbol{A}}[\boldsymbol{R}+2^{-n}\mathbf{1}]))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \big|\big|M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X)\big|\big|_{L_{\boldsymbol{r},\mathpzc{d}}(B^{A}(0,2\boldsymbol{c}_{\boldsymbol{A}}\boldsymbol{R}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \inf_{z\in B^{A}(0,\boldsymbol{R})} \big|\big|M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X)\big|\big|_{L_{\boldsymbol{r},\mathpzc{d}}(B^{A}(0,2\boldsymbol{c}_{\boldsymbol{A}}(\boldsymbol{c}_{\boldsymbol{A}}+\mathbf{1})\boldsymbol{R}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \inf_{z\in B^{A}(0,\boldsymbol{R})} M^{\boldsymbol{A}}_{\boldsymbol{r}}(M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X))(z). \end{aligned} \tag{2.13}$$

The latter implies that

$$1_{B^{A}(0,\boldsymbol{R})} \otimes ||f_n||_{L_\infty(B^{A}(0,\boldsymbol{R});X)} \lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} M^{\boldsymbol{A}}_{\boldsymbol{r}}(M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X))$$

for $\boldsymbol{R} \in [1,\infty)^{\ell}$. It thus follows that

$$\begin{aligned} ||f_n||_{E^{\boldsymbol{A}}_{\otimes}(L_\infty(B^{A}(0,\boldsymbol{R});X))} &\leq \Big|\Big| 1_{B^{A}(0,\boldsymbol{R})\times\{0\}} \otimes ||f_n||_{L_\infty(B^{A}(0,\boldsymbol{R});X)} \Big|\Big|_E \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{r}^{-1}} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} ||(\delta_{0,k} M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f_n||_X))_k||_E \end{aligned}$$

$$\lesssim 2^{n(\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}-\varepsilon_+)}\boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}}\|(h_k)_k\|_{E(X)}.$$

Let us finally prove that $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ is complete. To this end, let $0<\kappa\leq \boldsymbol{r}_{\min}$ be such that $\|\cdot\|_E$ is a κ-norm. Then $\|\cdot\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ and $\|\cdot\|_{E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})](X)}$ are κ-norms as well. It suffices to show that, if $(f^{(k)})_{k\in\mathbb{N}}\subset \widetilde{YL}^{\boldsymbol{A}}(E;X)$ satisfies $\sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}<\infty$, then $\sum_{k=0}^{\infty}f^{(k)}$ is a convergent series in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$. So fix such a $(f^{(k)})_{k\in\mathbb{N}}$. As a consequence of (2.10),

$$\sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]}\lesssim \sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}<\infty.$$

As $E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]$ is a quasi-Banach space with a κ-norm, $\sum_{k=0}^{\infty}f^{(k)}$ converges to some F in $E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]$. To finish the proof, we show that $F\in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ with convergence $F=\sum_{k=0}^{\infty}f^{(k)}$ in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$.

For each $k\in\mathbb{N}$ there exists $(g^{(k)}_n)_n\in E_+$ with $\|(g^{(k)}_n)_n\|_E\leq 2\|f^{(k)}\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ such that, for every $x^*\in X^*$, $\langle f^{(k)},x^*\rangle$ has the representation

$$\langle f^{(k)},x^*\rangle=\sum_{n=0}^{\infty}f^{(k)}_{x^*,n}\quad\text{in}\quad L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$$

for some $(f^{(k)}_{x^*,n})_n\in E_{\boldsymbol{A}}$ with $|f^{(k)}_{x^*,n}|\leq \|x^*\|g^{(k)}_n$. By Remark 2.3.10,

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\|f^{(k)}_{x^*,n}\|^{\kappa}_{E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]}\lesssim \sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}<\infty.$$

As $E^{\boldsymbol{A}}_{\otimes}[L_{\boldsymbol{r}}(\mathbb{R}^d,\boldsymbol{w})]\hookrightarrow L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))\hookrightarrow L_0(S\times\mathbb{R}^d)$ is a quasi-Banach space with a κ-norm, we thus find that $F=\sum_{n=0}^{\infty}F_{x^*,n}$ in $L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ with $F_{x^*,n}:=\sum_{k=0}^{\infty}f^{(k)}_{x^*,n}$ in $L_0(\mathbb{R}^d\times S)$ satisfying $|F_{x^*,n}|\leq\sum_{k=0}^{\infty}|f^{(k)}_{x^*,n}|\leq\|x^*\|\sum_{k=0}^{\infty}g^{(k)}_n$. As $E_{\boldsymbol{A}}$ is a closed subspace of the quasi-Banach function space E on $\mathbb{R}^d\times\mathbb{N}\times S$ with κ-norm, it follows from

$$\sum_{k=0}^{\infty}\|(f^{(k)}_{x^*,n})_n\|^{\kappa}_E\leq\|x^*\|^{\kappa}\sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}<\infty$$

that $(F_{x^*,n})_n=\sum_{k=0}^{\infty}f^{(k)}_{x^*,n}$ in E and thus that $(F_{x^*,n})_n\in E_{\boldsymbol{A}}$. Moreover, $G_n:=\sum_{k=0}^{\infty}g^{(k)}_n$ defines $(G_n)_n\in E_+$ with

$$\|(G_n)_n\|^{\kappa}_E\leq\sum_{k=0}^{\infty}\|(g^{k}_n)_n\|^{\kappa}_E\leq 2\sum_{k=0}^{\infty}\|f^{(k)}\|^{\kappa}_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

and $|F_{x^*,n}|\leq\|x^*\|G_n$. This shows that $F\in\widetilde{YL}^{\boldsymbol{A}}(E;X)$ with convergence $F=\sum_{k=0}^{\infty}f^{(k)}$ in $\widetilde{YL}^{\boldsymbol{A}}(E;X)$. □

The content of the following proposition is a Littlewood-Paley characterization for $Y^{\boldsymbol{A}}(E;X)$. Before we state it, we first need to introduce the set $\Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ of all $\boldsymbol{A}$-anisotropic Littlewood-Paley sequences $\varphi=(\varphi_n)_{n\in\mathbb{N}}$.

Definition 2.3.13. For $0 < \gamma < \delta < \infty$ we define $\Phi^{\boldsymbol{A}}_{\gamma,\delta}(\mathbb{R}^d)$ as the set of all sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^d)$ that can be constructed in the following way: given $\varphi_0 \in \mathscr{S}(\mathbb{R}^d)$ satisfying

$$0 \leq \hat{\varphi}_0 \leq 1,\ \hat{\varphi}_0(\xi) = 1 \text{ if } \rho_{\boldsymbol{A}}(\xi) \leq \gamma,\ \hat{\varphi}_0(\xi) = 0 \text{ if } \rho_{\boldsymbol{A}}(\xi) \geq \delta,$$

$(\varphi_n)_{n\geq 1} \subset \mathscr{S}(\mathbb{R}^d)$ is obtained through

$$\hat{\varphi}_n = \hat{\varphi}_1(\boldsymbol{A}_{2^{-n+1}}\,\cdot) = \hat{\varphi}_0(\boldsymbol{A}_{2^{-n}}\,\cdot) - \hat{\varphi}_0(\boldsymbol{A}_{2^{-n+1}}\,\cdot), \qquad n \geq 1.$$

We define $\Phi^{\boldsymbol{A}}(\mathbb{R}^d) := \bigcup_{0<\gamma<\delta<\infty} \Phi^{\boldsymbol{A}}_{\gamma,\delta}(\mathbb{R}^d)$.

Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}_{\gamma,\delta}(\mathbb{R}^d)$. Then $\sum_{n=0}^{\infty} \hat{\varphi}_n = 1$ in $\mathscr{O}_M(\mathbb{R}^d)$ with

$$\operatorname{supp}\hat{\varphi}_0 \subset \{\xi : \rho_{\boldsymbol{A}}(\xi) \leq \gamma\}, \qquad \operatorname{supp}\hat{\varphi}_n \subset \{\xi : 2^{n-1}\gamma \leq \rho_{\boldsymbol{A}}(\xi) \leq 2^n\delta\}, \quad n \geq 1,$$

To φ we associate the family of convolution operators $(S_n)_{n\in\mathbb{N}} = (S_n^{\varphi})_{n\in\mathbb{N}} \subset \mathscr{L}(\mathscr{S}'(\mathbb{R}^d;X), \check{\mathscr{E}}'(\mathbb{R}^d;X))$ given by

$$S_n f = S_n^{\varphi} f := \varphi_n * f = \mathscr{F}^{-1}[\hat{\varphi}_n \hat{f}].$$

Proposition 2.3.14. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$. Then*

$$Y^{\boldsymbol{A}}(E;X) = \{f \in L_0(S; \mathscr{S}'(\mathbb{R}^d;X)) : (S_n f)_n \in E(X)\}$$

with

$$||f||_{Y^{\boldsymbol{A}}(E;X)} \eqsim ||(S_n f)_n||_{E(X)}.$$

Before we go the proof of Proposition 2.3.14, let us first consider:

Example 2.3.15. In the following three points we let the notation be as in Example 2.3.4.(i), Example 2.3.4.(ii) and Example 2.3.4.(iii), respectively. We define:

(i) $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) := Y^{\boldsymbol{A}}(E;X)$ for $E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[\ell^s_q(\mathbb{N})]$;

(ii) $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X) := Y^{\boldsymbol{A}}(E;X)$ for $E = \ell^s_q(\mathbb{N})[L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})]$;

(iii) $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X) := Y^{\boldsymbol{A}}(E;X)$ for $E = L_{\boldsymbol{p}}(\mathbb{R}^d, \boldsymbol{w})[F[\ell^s_q(\mathbb{N})]]$.

Restricting to special cases we find, in view of Proposition 2.3.14, B- and F-spaces that have been studied in the literature:

(i)&(ii):
(a) In case $\ell = 1$, $w = 1$ and $X = \mathbb{C}$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic Besov and Triebel-Lizorkin spaces considered in e.g. [56, 71]. The latter are special cases of the anisotropic spaces from the more general [22, 34, 35] by taking 2^A as the expansive dilation in the approach there.

(b) In case $\ell = d$, $\boldsymbol{A} = \operatorname{diag}(\boldsymbol{a})$ with $\boldsymbol{a} \in (0,\infty)$, $\boldsymbol{w} = \boldsymbol{1}$ and $X = \mathbb{C}$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic mixed-norm Besov and Triebel-Lizorkin spaces considered in e.g. [130, 131].

(c) In case $\boldsymbol{A} = (a_1 I_{d_1}, \dots, a_\ell I_{d_\ell})$ with $\boldsymbol{a} \in (0,\infty)$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the anisotropic weighted mixed-norm Besov and Triebel-Lizorkin spaces considered in [156, 159] and in Chapters 5 and 6.

(d) In case $\ell = 1$ and $A = I$, $F^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ and $B^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; X)$ reduce to the weighted Besov and Triebel-Lizorkin spaces considered in e.g. [38–40, 115–118, 162, 163, 228] ($X = \mathbb{C}$) and [182, 186, 187] (X a general Banach space). In the case $w = 1$ these further reduces to the classical Besov and Triebel-Lizorkin spaces (see e.g. [219, 233, 234]).

(iii): (a) In case $\ell = 1$, $A = I$, $p \in (1,\infty)$, $q \in [1,\infty]$, $w = 1$, F is a UMD Banach function space and $X = \mathbb{C}$, $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$ reduces to a special case of the generalized Triebel-Lizorkin spaces considered in [148].

(b) In case $\ell = 1$, $A = I$, $p \in (1,\infty)$, $q = 2$, $w \in A_p(\mathbb{R}^d)$, F is a UMD Banach function space and X is a Hilbert space, $\mathbb{F}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d, \boldsymbol{w}; F; X)$ coincides with the weighted Bessel potential space $H^s_p(\mathbb{R}^d, w; F(X))$ (which follows from a combination of (3.6) and (3.13)).

The proof of Proposition 2.3.14 basically only consists of proving the estimate in the following lemma. We have extracted it as a lemma as it is interesting on its own. A consequence of the lemma for instance is that the Fourier support condition in Definition 2.3.11 could be slightly modified.

Lemma 2.3.16. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $c \in (1,\infty)$ and $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n\in\mathbb{N}}$. For all $f \in L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$ which have a representation*

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$$

with $(f_n)_n \subset L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$ satisfying the Fourier support condition

$$\begin{aligned} &\operatorname{supp} \hat{f}_0 \subset \overline{B}^{\boldsymbol{A}}(0, c) \\ &\operatorname{supp} \hat{f}_n \subset \overline{B}^{\boldsymbol{A}}(0, c2^n) \setminus B^{\boldsymbol{A}}(0, c^{-1}2^n), \qquad n \geq 1, \end{aligned}$$

there is the estimate

$$\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)}.$$

Proof. This can be established as in [156, Lemma 5.2.10] (also see [233, Section 2.3.2] and [237, Section 15.5]), using a combination of Corollary 2.A.2 and Lemma 2.A.3. □

Proof of Proposition 2.3.14. Let $f \in Y^{\boldsymbol{A}}(E; X)$. Take $(f_n)_n$ as in Definition 2.3.11 with $\|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y^{\boldsymbol{A}}(E;X)}$. Lemma 2.3.16 (with $c = 2$) then gives

$$\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)} \leq 2\|f\|_{Y^{\boldsymbol{A}}(E;X)}.$$

For the reverse direction, let $f \in L_0(S;\mathscr{S}'(\mathbb{R}^d;X))$ be such that $(S_n f)_n \in E(X)$. Pick $\psi = (\psi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\hat{\psi}_0 \subset \overline{B}^{\boldsymbol{A}}(0,2), \qquad \operatorname{supp}\hat{\psi}_n \subset \overline{B}^{\boldsymbol{A}}(0,2^{n+1}) \setminus B^{\boldsymbol{A}}(0,2^{n-1}), \quad n \geq 1,$$

and let $(T_n)_{n\in\mathbb{N}}$ denote the associated sequence of convolution operators. Then

$$\operatorname{supp}\widehat{T_0 f} \subset \overline{B}^{\boldsymbol{A}}(0,2), \qquad \operatorname{supp}\widehat{T_n f} \subset \overline{B}^{\boldsymbol{A}}(0,2^{n}) \setminus B^{\boldsymbol{A}}(0,2^{n-1}), \quad n \geq 1, \tag{2.14}$$

Picking $c \in (1,\infty)$ such that

$$\operatorname{supp}\hat{\varphi}_0 \subset \overline{B}^{\boldsymbol{A}}(0,c), \qquad \operatorname{supp}\hat{\varphi}_n \subset \overline{B}^{\boldsymbol{A}}(0,c2^{n}) \setminus B^{\boldsymbol{A}}(0,c^{-1}2^{n}), \quad n \geq 1,$$

we furthermore have

$$\operatorname{supp}\widehat{S_n f} \subset \overline{B}^{\boldsymbol{A}}(0,c), \qquad \operatorname{supp}\widehat{S_n f} \subset \overline{B}^{\boldsymbol{A}}(0,c2^{n}) \setminus B^{\boldsymbol{A}}(0,c^{-1}2^{n}), \quad n \geq 1.$$

As $f = \sum_{n=0}^{\infty} S_n f$ in $L_0(S;\mathscr{S}'(\mathbb{R}^d;X))$, Lemma 2.3.16 gives

$$||(T_n f)_n||_{E(X)} \lesssim ||(S_n f)_n||_{E(X)}.$$

Since $f = \sum_{n=0}^{\infty} S_n f$ in $L_0(S;\mathscr{S}'(\mathbb{R}^d;X))$ with (2.14), it follows that $f \in Y^{\boldsymbol{A}}(E;X)$ with

$$||f||_{Y^{\boldsymbol{A}}(E;X)} \leq ||(T_n f)_n||_{E(X)} \lesssim ||(S_n f)_n||_{E(X)}. \qquad \square$$

Theorem 2.3.17. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. Suppose that $\varepsilon_+ > \operatorname{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\mathbf{1})_+$. Then*

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S;L_{\mathbf{1}\wedge\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)) \tag{2.15}$$

and

$$\begin{aligned} Y^{\boldsymbol{A}}(E;X) &\hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X)) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;E^{\boldsymbol{A}}_{\otimes}(X)) \\ &\hookrightarrow \mathscr{S}'(\mathbb{R}^d;L_0(S;X)) = L_0(S;\mathscr{S}'(\mathbb{R}^d;X)) \end{aligned} \tag{2.16}$$

and there is the identity

$$Y^{\boldsymbol{A}}(E;X) = YL^{\boldsymbol{A}}(E;X) = \widetilde{YL}^{\boldsymbol{A}}(E;X). \tag{2.17}$$

We will use the following lemma in the proof of Theorem 2.3.17.

Lemma 2.3.18. *Let the notations and assumptions be as in Theorem 2.3.17. If $(f_n)_n \in E(X)_{\boldsymbol{A}}$ (see (2.11), then $\sum_{n\in\mathbb{N}} f_n$ is a convergent series in $L_0(S;(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X))$ with*

$$\Big|\Big|\sum_{n=0}^{\infty} f_n\Big|\Big|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}}(X))} \leq \Big|\Big|\sum_{n=0}^{\infty} ||f_n||_X\Big|\Big|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\mathbf{1}}}_{\boldsymbol{A}})} \lesssim ||(f_n)_n||_{E(X)_{\boldsymbol{A}}}.$$

Proof. It suffices to prove the second estimate. We may without loss of generality assume that $\boldsymbol{r} \in (0,1]^{\ell}$. Choose $\kappa > 0$ such that $E^{\boldsymbol{A}}_{\otimes}$ has a κ-norm. For simplicity of notation we only present the case $\ell = 2$, the general case being the same.

Let $(f_n)_n \in E(X)_{\boldsymbol{A}}$. Let $\boldsymbol{R} \in [1,\infty)^2$. As a consequence of the Paley-Wiener-Schwartz theorem,

$$\check{\mathscr{E}}_{\overline{B}^{\boldsymbol{A}}(0,2^n)}(\mathbb{R}^d;X) \hookrightarrow C^{\infty}(\mathbb{R}^{d_2};\check{\mathscr{E}}_{\overline{B}^{A_1}(0,2^n)}(\mathbb{R}^{d_1};X)) \cap C^{\infty}(\mathbb{R}^{d_1};\check{\mathscr{E}}_{\overline{B}^{A_2}(0,2^n)}(\mathbb{R}^{d_2};X)).$$

In particular, as in (2.13) we find that

$$||f_n(x_1,z_2)||_X \lesssim (2^n R_1)^{\mathrm{tr}(A_1)/r_1} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1,[d;1]}(||f_n||_X))(y_1,z_2) \tag{2.18}$$

for all $x_1, y_1 \in B^{A_1}(0,R_1)$ and $z_1 \in \mathbb{R}^{d_1}$, and

$$||f_n(z_1,x_2)||_X \lesssim (2^n R_2)^{\mathrm{tr}(A_2)/r_2} M^{A_2}_{r_2;[d;2]}(M^{A_2}_{r_2,[d;2]}(||f_n||_X))(z_1,y_2) \tag{2.19}$$

for all $x_2, y_2 \in B^{A_2}(0,R_2)$ and $z_2 \in \mathbb{R}^{d_2}$.

Then, for $z \in B^{\boldsymbol{A}}(0,\boldsymbol{R})$,

$$\begin{aligned}
&\int_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} ||f_n(x)||_X \,\mathrm{d}x \\
&\quad= \int_{B^{A_2}(0,R_2)} \int_{B^{A_1}(0,R_1)} ||f_n(x_1,x_2)||_X \,\mathrm{d}x_1 \mathrm{d}x_2 \\
&\quad\overset{(2.18)}{\lesssim} ((2^n R_1)^{\mathrm{tr}(A_1)/r_1})^{1-r_1} \int_{B^{A_2}(0,R_2)} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1;[d;1]}(||f_n(\,\cdot\,,x_2)||_X))(z_1)^{r_1-1} \\
&\qquad\quad \cdot \int_{B^{A_1}(0,R_1)} ||f_n(x_1,x_2)||_X^{r_1} \,\mathrm{d}x_1 \mathrm{d}x_2 \\
&\quad\lesssim 2^{n\mathrm{tr}(A_1)(1-r_1)/r_1} R_1^{\mathrm{tr}(A_1)/r_1} \int_{B^{A_2}(0,R_2)} M^{A_1}_{r_1;[d;1]}(M^{A_1}_{r_1;[d;1]}(||f_n(\,\cdot\,,x_2)||_X))(z_1)\mathrm{d}x_2 \\
&\quad\overset{(2.19)}{\lesssim} 2^{n\left(\mathrm{tr}(A_1)(1-r_1)/r_1 + \mathrm{tr}(A_2)(1-r_2)/r_2\right)} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \\
&\qquad\quad \cdot M^{A_1}_{r_1;[d;1]} M^{A_1}_{r_1;[d;1]} M^{A_2}_{r_2;[d;2]} M^{A_2}_{r_2;[d;2]}(||f_n||_X))(z_1,z_2)^{1-r_2} \\
&\qquad\quad \cdot M^{A_2}_{r_2;[d;1]} M^{A_2}_{r_2;[d;2]} M^{A_1}_{r_1;[d;2]} M^{A_1}_{r_1;[d;1]}(||f_n||_X))(z_1,z_2)^{r_2} \\
&\quad\le 2^{n(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})} \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} [M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(||f_n||_X)(z).
\end{aligned}$$

This implies that

$$1_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} \otimes \int_{B^{\boldsymbol{A}}(0,\boldsymbol{R})} \sum_{n=0}^{\infty} ||f_n(x)||_X \,\mathrm{d}x \lesssim \boldsymbol{R}^{\mathrm{tr}(\boldsymbol{A})\boldsymbol{r}^{-1}} \sum_{n=0}^{\infty} 2^{n(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})} [M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(||f_n||_X).$$

Since $\epsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})_+$, it follows that

$$\begin{aligned}
\Big|\Big| \sum_{n=0}^{\infty} ||f_n||_X \Big|\Big|_{E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge\boldsymbol{1}}}_{\boldsymbol{A}})]} &\overset{(2.7)}{\lesssim} ||([M^{\boldsymbol{A}}_{\boldsymbol{r}}]^4(||f_n||_X))_n||_E \\
&\lesssim ||(f_n)||_{E(X)}.
\end{aligned}$$

□

Proof of Theorem 2.3.17. We may without loss of generality assume that $\boldsymbol{r} \in (0,1]^{\ell}$.

As $L_0(S; B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$, the first inclusion in (2.16) follows from Lemma 2.3.18. So in (2.16) it remains to prove the second inclusion. To this end, let us first note that

$$\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{B}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}(X), X),\, \phi \mapsto \langle \cdot, \phi \rangle.$$

This induces

$$\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{B}(E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}(X)), E^{\boldsymbol{A}}_{\otimes}(X)),\, \phi \mapsto \langle \cdot, \phi \rangle.$$

Therefore, $f \mapsto [\phi \mapsto \langle f, \phi \rangle]$ is a continuous linear operator from $E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}(X))$ to $\mathscr{L}(\mathscr{S}(\mathbb{R}^d); E^{\boldsymbol{A}}_{\otimes}(X))$, which is a reformulation of the required inclusion.

As $L_0(S; B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}) \hookrightarrow L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$, the inclusion

$$Y^{\boldsymbol{A}}(E) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}})$$

follows from Lemma 2.3.18. We thus get a continuous bilinear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \times X^* \longrightarrow YL^{\boldsymbol{A}}(E) \hookrightarrow L_0(S; \mathscr{S}'(\mathbb{R}^d)),\, (f, x^*) \mapsto \langle f, x^* \rangle.$$

and a continuous linear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \longrightarrow L_0(S; \mathscr{S}'(\mathbb{R}^d; X^{**})),\, f \mapsto T_f, \tag{2.20}$$

defined by

$$\langle x^*, T_f(\phi) \rangle := \langle f, x^* \rangle(\phi), \qquad \phi \in \mathscr{S}(\mathbb{R}^d), x^* \in X^*.$$

Let us now show that $f \mapsto T_f$ (2.20) restricts to a bounded linear mapping

$$\widetilde{YL}^{\boldsymbol{A}}(E,X) \longrightarrow Y^{\boldsymbol{A}}(E; X^{**}),\, f \mapsto T_f. \tag{2.21}$$

To this end, let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$ and put $F := T_f$. Let $(g_n)_n$ and $(f_{x^*,n})_{(x^*,n)}$ be as in Definition 2.3.9 with $||(g_n)_n||_E \leq 2||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$. It will convenient to put $g_n := 0$ and $f_{x^*,n} := 0$ for $n \in \mathbb{Z}_{<0}$. By Lemma 2.3.18, as $(f_{x^*,n})_n \in E_{\boldsymbol{A}}$ and $B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}} \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$,

$$\langle f, x^* \rangle = \sum_{k=0}^{\infty} f_{x^*,k} \quad \text{in} \quad L_0(S; B^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}_{\boldsymbol{A}}) \hookrightarrow L_0(S; \mathscr{S}'(\mathbb{R}^d)), \qquad x^* \in X^*.$$

Now let $(S_n)_{n\in\mathbb{N}}$ be as in Proposition 2.3.14. There exists $h \in \mathbb{N}$ independent of f such that $S_n f_{x^*,k} = 0$ for all $x^* \in X^*$, $n \in \mathbb{N}$ and $k \in \mathbb{Z}_{<n-h}$. Let $x^* \in X^*$. Then

$$\begin{aligned}\langle x^*, S_n F \rangle = S_n \langle x^*, F \rangle = S_n \langle f, x^* \rangle = S_n \sum_{k=0}^{\infty} f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k} \\ = \sum_{k=n-h}^{\infty} S_n f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k+n-h}\end{aligned}$$

with convergence in $L_0(S;\mathscr{S}'(\mathbb{R}^d))$. Together with Corollary 2.A.6, this implies the pointwise estimates

$$|\langle x^*, S_n F\rangle| \le \sum_{k=0}^{\infty} |S_n f_{x^*,k+n-h}| \lesssim \sum_{k=0}^{\infty} 2^{(k-h)_+ \sum_{j=1}^{\ell} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)} M_{\boldsymbol{r}}^{\boldsymbol{A}}(f_{k+n-h,x^*})$$
$$\le \|x^*\| \sum_{k=0}^{\infty} 2^{(k-h)_+ \sum_{j=1}^{\ell} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)} M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h}).$$

Taking the supremum over $x^* \in X^*$ with $\|x^*\| \le 1$, we obtain

$$\|S_n F\|_{X^{**}} \le \sum_{k=0}^{\infty} 2^{(k-h)_+ \sum_{j=1}^{\ell} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)} M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h}).$$

Picking $\kappa > 0$ such that E has a κ-norm, we find that

$$\|(S_nF)_n\|_{E(X^{**})}^{\kappa} = \big\|(\|S_n f\|_{X^{**}})_n\big\|_E^{\kappa}$$
$$\lesssim \sum_{k=0}^{\infty} 2^{\kappa(k-h)_+ \sum_{j=1}^{\ell} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)} \big\|M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h})_n\big\|_E^{\kappa}$$

Since

$$\big\|M_{\boldsymbol{r}}^{\boldsymbol{A}}(g_{k+n-h})_n\big\|_E = \big\|(g_{k+n-h})_n\big\|_E \lesssim \begin{cases} \big\|(S_-)^{h-k}(g_n)_n\big\|_E, & k \le h, \\ \big\|(S_+)^{k-h}(g_{k+n-h})_n\big\|_E, & k \ge h, \end{cases}$$
$$\lesssim \left(2^{\varepsilon_-(h-k)_+} + 2^{-\varepsilon_+(k-h)_+}\right)\|(g_n)_n\|_E$$
$$\lesssim 2^{-\varepsilon_+(k-h)_+}\|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

for all $k \in \mathbb{N}$, it follows that

$$\|(S_nF)_n\|_{E(X^{**})}^{\kappa} \lesssim \sum_{k=0}^{\infty} 2^{\kappa(k-h)_+\left(\sum_{j=1}^{\ell} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)-\varepsilon_+\right)} \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}^{\kappa}.$$

As $\varepsilon_+ > \sum_{j=1}^{l} \mathrm{tr}(A_j)(\frac{1}{r_j}-1)$, we find that $\|(S_nF)_n\|_{E(X^{**})} \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ and thus that $F \in Y^{\boldsymbol{A}}(E;X^{**})$ with $\|F\|_{Y^{\boldsymbol{A}}(E;X^{**})} \lesssim \|f\|_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$ (see Proposition 2.3.14). So we obtain the desired (2.21).

Next we prove that

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow Y^{\boldsymbol{A}}(E;X). \tag{2.22}$$

So let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$. A combination of (2.21) and (2.16) gives that $F := T_f \in L_0(S;X^{**})$). Since $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ with $\langle x^*, F\rangle = \langle f, x^*\rangle$ for every $x^* \in X^*$, it follows that

$$f = F \in L_0(S;B_{\boldsymbol{A}}^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}(X^{**})) \cap L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)) \subset L_0(S;B_{\boldsymbol{A}}^{1,w_{\boldsymbol{A},\boldsymbol{r}\wedge 1}}(X)).$$

Therefore, by boundedness of (2.21),

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow \left\{g \in Y^{\boldsymbol{A}}(E;X^{**}) : g \in L_0(S;\mathscr{S}'(\mathbb{R}^d;X))\right\} = Y^{\boldsymbol{A}}(E;X). \qquad \square$$

For a quasi-Banach function space E on $\mathbb{R}^d \times \mathbb{N} \times S$ and a number $\sigma \in \mathbb{R}$ we define the quasi-Banach function space E^σ on $\mathbb{R}^d \times \mathbb{N} \times S$ by

$$||(f_n)_n||_{E^\sigma} := ||(2^{n\sigma} f_n)_n||_E, \qquad (f_n)_n \in L_0(\mathbb{R}^d \times \mathbb{N} \times S).$$

Note that $E^\sigma \in \mathscr{S}(\varepsilon_+ + \sigma, \varepsilon_- + \sigma, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ when $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$.

Proposition 2.3.19. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and $\sigma \in \mathbb{R}$. Let $\psi \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}^d)$ be such that $\psi(\xi) = \rho_{\boldsymbol{A}}(\xi)$ for $\rho_{\boldsymbol{A}}(\xi) \geq 1$ and $\psi(\xi) \neq 0$ for $\rho_{\boldsymbol{A}}(\xi) \leq 1$. Then $\phi(D) \in \mathscr{L}(L_0(S; \mathscr{S}'(\mathbb{R}^d; X)))$ restricts to an isomorphism*

$$\phi(D) : Y^{\boldsymbol{A}}(E^\sigma; X) \xrightarrow{\simeq} Y^{\boldsymbol{A}}(E; X).$$

Proof. Using Proposition 2.3.14 and Lemma 2.A.3, this can be proved as [156, Lemma 5.2.28] (also see [233, Theorem 2.3.8]). □

Lemma 2.3.20. *Let V be a quasi-normed space continuously embedded into a sequentially complete topological vector space W. Suppose that V has the* Fatou property *with respect to W, i.e. for all $(v_n)_{n \in \mathbb{N}} \subset V$ the following implication holds:*

$$\lim_{n\to\infty} v_n = v \text{ in } W, \liminf_{n\to\infty} ||v_n||_V < \infty \Longrightarrow v \in V, ||f||_V \leq \liminf_{n\to\infty} ||f_n||_V.$$

Then V is complete.

Proof. Suppose that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V. Then, on the one hand, $\liminf_{n\to\infty} ||v_n||_V \leq \sup_n ||v_n||_V < \infty$. On the other hand, $(v_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in the sequentially complete topological vector space W because of $V \hookrightarrow W$, whence converges to some v in W. By the Fatou property of V with respect to W, $v \in V$. To finish the proof we show that we also have convergence $v_n \stackrel{n\to\infty}{\longrightarrow} v$ with respect to the quasi-norm of V. To this end, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $||v_l - v_k||_V \leq \varepsilon$ for all $l, k \geq N$. Then, for all $k \geq N$, it holds that $v_l - v_k \in E$, $\liminf_{l\to\infty} ||v_l - v_k||_V \leq \varepsilon$ and $v_l - v_k \stackrel{l\to\infty}{\longrightarrow} v - v_k$ in W. So applying, for each $k \geq N$, the Fatou property of V (with respect to W) to the sequence of differences $(v_l - v_k)_{l\in\mathbb{N}}$ we obtain that $||v - v_k||_V \leq \varepsilon$ for all $k \geq N$. □

Proposition 2.3.21. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$. Then*

$$Y^{\boldsymbol{A}}(E; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; E^{\boldsymbol{A}}_{\otimes}(X)) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; L_0(S; X)) = L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$$

and $Y^{\boldsymbol{A}}(E; X)$, when equipped with an equivalent quasi-norm from Proposition 2.3.14, has the Fatou property with respect to $L_0(S; \mathscr{S}'(\mathbb{R}^d; X))$. As a consequence, $Y^{\boldsymbol{A}}(E; X)$ is a quasi-Banach space.

Proof. The chain of inclusions follow from a combination of Theorem 2.3.17 and Proposition 2.3.19.

In order to establish the Fatou property, suppose that $Y^{\boldsymbol{A}}(E;X)$ has been equipped with an equivalent quasi-norm from Proposition 2.3.14. Let $f_k \to f$ in $L_0(S;\mathscr{S}'(\mathbb{R}^d;X))$ with $\liminf_{k\to\infty} ||f_k||_{Y^{\boldsymbol{A}}(E;X)} < \infty$. Then

$$S_n f = \lim_{k\to\infty} S_n f_k \quad \text{in} \quad L_0(S;\mathscr{O}_M(\mathbb{R}^d;X)) \hookrightarrow L_0(S;L_{1,\mathrm{loc}}(\mathbb{R}^d;X)) \hookrightarrow L_0(\mathbb{R}^d \times S;X),$$

so that

$$(S_n f)_{n\in\mathbb{N}} = \lim_{k\to\infty} (S_n f_k)_{n\in\mathbb{N}} \quad \text{in} \quad L_0(\mathbb{R}^d \times S;X).$$

By passing to a suitable subsequence we may without loss of generality assume that $(S_n f_k)_{n\in\mathbb{N}} \to (S_n f)_{n\in\mathbb{N}}$ pointwise a.e. as $k\to\infty$. Using the Fatou property of E, we find

$$\begin{aligned} ||f||_{Y^{\boldsymbol{A}}(E;X)} &= \big|\big|(||S_n f||_X)_n\big|\big|_E = \big|\big|\liminf_{k\to\infty}(||S_n f_k||_X)_n\big|\big|_E \\ &\leq \liminf_{k\to\infty}\big|\big|(||S_n f_k||_X)_n\big|\big|_E = \liminf_{k\to\infty}||f_k||_{Y^{\boldsymbol{A}}(E;X)}. \end{aligned}$$ □

2.4. DIFFERENCE NORMS

In this section we several estimates for $YL^{\boldsymbol{A}}(E;X)$ and $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ involving differences... . The main interest lies in the estimates involving differences, as these form the basis for the intersection representation in Section 2.5.

2.4.1. Some notation

Let X be a Banach space. For each $M \in \mathbb{N}_{\geq 1}$ and $h \in \mathbb{R}^d$ we define difference operator Δ_h^M on $L_0(\mathbb{R}^d;X)$ by $\Delta_h^M := (L_h - I)^M = \sum_{i=0}^M (-1)^i \binom{M}{i} L_{(M-i)h}$, where L_h denotes the left translation by h:

$$\Delta_h^M f = \sum_{i=0}^{M} (-1)^i \binom{M}{i} f(\cdot + (M-i)h), \qquad f \in L_0(\mathbb{R}^d;X).$$

For $N \in \mathbb{N}$ we denote by $\mathscr{P}_N^d$ the space of polynomials of degree at most N on $\mathbb{R}^d$. We write $\mathscr{P}_N^d(\mathbb{Q}) \subset \mathscr{P}_N^d$ for the subset of polynomials having rational coefficients.

Let $M \in \mathbb{N}_{\geq 1}$. Let $F = L_{\boldsymbol{p},\mathscr{d}} = L_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d)$ with $\boldsymbol{p} \in (0,\infty)^\ell$. Let $B \subset \mathbb{R}^d$ be a bounded Borel set of non-zero measure. For $f \in L_0(\mathbb{R}^d)$ we define

$$\mathscr{E}_M(f,B,F) := \inf_{\pi\in\mathscr{P}_{M-1}^d} ||(f-\pi)1_B||_F = \inf_{\pi\in\mathscr{P}_{M-1}^d(\mathbb{Q})} ||(f-\pi)1_B||_F$$

and

$$\overline{\mathscr{E}}_M(f,B,F) := \frac{\mathscr{E}_M(f,B,F)}{\mathscr{E}_M(1,B,F)}.$$

We define the collection of dyadic anisotropic cubes $\{Q_{n,k}^{\boldsymbol{A}}\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$ by

$$Q_{n,k}^{\boldsymbol{A}} := \boldsymbol{A}_{2^{-n}}\left([0,1)^d + k\right).$$

For $b \in (0,\infty)$ we define $\{Q^{\boldsymbol{A}}_{n,k}(b)\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$ by

$$Q^{\boldsymbol{A}}_{n,k}(b) := \boldsymbol{A}_{2^{-n}}\left([0,1)^d(b)+k\right),$$

where $[0,1)^d(b)$ is the cube concentric to $[0,1)^d$ with sidelength b:

$$[0,1)^d(b) := \left[\frac{1-b}{2},\frac{1+b}{2}\right)^d.$$

We furthermore define the corresponding families of indicator functions $\{\chi^{\boldsymbol{A}}_{n,k}\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$ and $\{\chi^{\boldsymbol{A},b}_{n,k}\}_{(n,k)\in\mathbb{Z}\times\mathbb{Z}^d}$:

$$\chi^{\boldsymbol{A}}_{n,k} := 1_{Q^{\boldsymbol{A}}_{n,k}} \qquad \text{and} \qquad \chi^{\boldsymbol{A},b}_{n,k} := 1_{Q^{\boldsymbol{A}}_{n,k}(b)}.$$

Definition 2.4.1. Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. We define $y^{\boldsymbol{A}}(E)$ as the space of all $(s_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S)$ for which $(\sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k})_{n\in\mathbb{N}} \in E$. We equip $y^{\boldsymbol{A}}(E)$ with the quasi-norm

$$||(s_{n,k})_{(n,k)}||_{y^{\boldsymbol{A}}(E)} := \Big|\Big|\Big(\sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big|\Big|_E.$$

Definition 2.4.2. Let F be a quasi-Banach function space on the σ-finite measure space $(T,\mathscr{B},\nu)$. We define $\mathscr{F}_{\mathrm{M}}(X^*;F)$ as the space of all $\{F_{x^*}\}_{x^*\in X^*} \subset L_0(T)$ for which there exists $G \in F_+$ such that $|F_{x^*}| \leq ||x^*||G$. We equip $\mathscr{F}_{\mathrm{M}}(X^*;F)$ with the quasi-norm

$$||\{F_{x^*}\}_{x^*}||_{\mathscr{F}_{\mathrm{M}}(X^*;F)} := \inf||G||_F,$$

where the infimum is taken over all majorants G as above.

In the special case that $F = E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ in the above definition, it will be convenient to view $\mathscr{F}_{\mathrm{M}}(X^*;E)$ as the space of all $\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}} \subset L_0(S)$ for which there exists $(g_n)_n \in E_+$ such that $|g_{x^*,n}| \leq ||x^*||g_n$, equipped with the quasi-norm

$$||\{g_{x^*,n}\}_{(x^*,n)}||_{\mathscr{F}_{\mathrm{M}}(X^*;E)} := \inf||(g_n)_n||_E,$$

where the infimum is taken over all majorants $(g_n)_n$ as above.

Note that the corresponding properties from Definition 2.3.1 for $\mathscr{F}_{\mathrm{M}}(X^*;E)$ are inherited from E.

Definition 2.4.3. Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$. We define $\tilde{y}^{\boldsymbol{A}}(E;X)$ as the space of all $(s_{x^*,n,k})_{(x^*,n,k)\in X^*\times\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S)$ for which $(\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k})_{n\in\mathbb{N}} \in \mathscr{F}_{\mathrm{M}}(X^*;E)$. We equip $\tilde{y}^{\boldsymbol{A}}(E;X)$ with the quasi-norm

$$||(s_{x^*,n,k})_{(n,k)}||_{\tilde{y}^{\boldsymbol{A}}(E;X)} := \Big|\Big|\Big(\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big|\Big|_{\mathscr{F}_{\mathrm{M}}(X^*;E)}.$$

2.4.2. Statements of the results

Theorem 2.4.4. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ and suppose that $\varepsilon_+,\varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty)^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{p}^{-1})$ and $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. Given $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d;X))$, consider the following statements:*

(i) $f \in YL^{\boldsymbol{A}}(E;X)$.

(ii) There exist $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ and $(b_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S;C^M_c([-1,2]^d))$ with $\|b_{n,k}\|_{C^M_b} \leq 1$ such that, setting $a_{n,k} := b_{n,k}(\boldsymbol{A}_{2^n}\cdot -k)$, f has the representation

$$f = \sum_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} s_{n,k}a_{n,k} \qquad \text{in} \qquad L_0(S;L_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d;X)). \tag{2.23}$$

(iii) $f \in E_0(X) \cap L_0(S;L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ *and* $(d^{\boldsymbol{A},\boldsymbol{p}}_M(f)_n)_{n\geq 1} \in E(\mathbb{N}_{\geq 1})$, *where*

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left\| z \mapsto \Delta^M_z f\right\|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n});X)}, \qquad n \in \mathbb{N}.$$

Then ⇒ (i) ⇔ (ii) ⇒ (iii). *Moreover, there are the following estimates:*

$$\|f\|_{E_0(X)} + \|(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}\|_{E(\mathbb{N}_{\geq 1})} \lesssim \|f\|_{YL^{\boldsymbol{A}}(E;X)} \eqsim \|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.$$

Theorem 2.4.4 is partial extension of [119, Theorem 1.1.14], which is concerned with $YL(E)$ with $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,I,\boldsymbol{r})$. That result actually extends completely to the anisotropic scalar-valued setting $YL^{\boldsymbol{A}}(E)$ with $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r})$. However, in the general Banach space-valued case there arises a difficulty due to the unavailability of the Whitney inequality [119, (1.2.2)/Theorem A.1] (see [246, 247]) and the derived Lemma 2.4.9. We overcome this issue in Theorem 2.4.5 by extending [119, Theorem 1.1.14] to $\widetilde{YL}^{\boldsymbol{A}}(E;X)$. This was actually the motivation for introducing the space $\widetilde{YL}^{\boldsymbol{A}}(E;X)$, which is connected to $YL^{\boldsymbol{A}}(E;X)$ and $Y^{\boldsymbol{A}}(E;X)$ through Theorem 2.3.17.

Theorem 2.4.5. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ and suppose that $\varepsilon_+,\varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty)^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{p}^{-1})$ and $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. Given $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d;X))$, consider the following statements:*

(I) $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$.

(II) There exist $(s_{x^*,n,k})_{(n,k)} \in \tilde{y}^{\boldsymbol{A}}(E;X)$ and $(b_{x^*,n,k})_{(x^*,n,k)\in X^*\times\mathbb{N}\times\mathbb{Z}^d} \subset L_0(S;C^M_c([-1,2]^d))$ with $\|b_{x^*,n,k}\|_{C^M_b} \leq 1$ such that, setting $a_{x^*,n,k} := b_{x^*n,k}(\boldsymbol{A}_{2^n}\cdot -k)$, for all $x^* \in X^*$, $\langle f,x^*\rangle$ has the representation

$$\langle f,x^*\rangle = \sum_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} s_{x^*,n,k}a_{x^*,n,k} \qquad \text{in} \qquad L_0(S;L_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d)).$$

(III) $f \in E_0(X) \cap L_0(S;L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ *and*

$$\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}} \in \mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1})),$$

where

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\big|\big|z \mapsto \Delta^M_z\langle f,x^*\rangle\big|\big|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}, \qquad n\in\mathbb{N}.$$

(IV) $f\in E_0(X)\cap L_0(S;L_{\boldsymbol{p},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X))$ *and*

$$\{\mathscr{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}\in\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1})),$$

where

$$\mathscr{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)(x) := \overline{\mathscr{E}}_M(\langle f,x^*\rangle, B^{\boldsymbol{A}}(x,2^{-n}),L_{\boldsymbol{p},\mathpzc{d}}), \qquad x^*\in X^*, n\in\mathbb{N}.$$

(V) $f\in E_0(X)$ *and there is* $\{\pi_{x^*,n,k}\}_{(x^*,n,k)\in X^*\times\mathbb{N}_{\geq1}\times\mathbb{Z}}\in\mathscr{P}^d_{M-1}$ *such that*

$$g_{x^*,n} := \sum_{k\in\mathbb{Z}^d}|\langle f,x^*\rangle-\pi_{x^*,n,k}|1_{Q^{\boldsymbol{A}}_{n,k}(3)}, \qquad n\geq1,$$

satisfies $\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}\in\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1}))$.

For $f\in L_0(S;L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d;X))$ *it holds that* (V) $\Rightarrow$ (I) $\Leftrightarrow$ (II) $\Rightarrow$ (III) & (IV) *with corresponding estimates*

$$\begin{aligned}||f||_{E_0(X)} + ||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f))_{(x^*,n)}||_{\mathscr{F}_{\mathrm{M}}(X^*;E)} + ||\mathscr{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}||_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1}))}\\ \lesssim ||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} \eqsim ||(s_{x^*,n,k})_{(x^*,n,k)}||_{\tilde{y}^{\boldsymbol{A}}(E)}\\ \lesssim ||f||_{E_0(X)} + \big|\big|\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1}))}.\end{aligned}$$

Moreover, for f *of the form* $f=\sum_{i\in I}1_{S_i}\otimes f^{[i]}$ *with* $(S_i)_{i\in I}\subset\mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I}\in L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X)$*, it holds that* (I), (III), (IV), (II) *and* (II) *are equivalent statements and there are the corresponding estimates*

$$\begin{aligned}||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} &\eqsim ||(s_{x^*,n,k})_{(x^*,n,k)}||_{\tilde{y}^{\boldsymbol{A}}(E)}\\ &\eqsim ||f||_{E_0(X)} + \big|\big|\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1}))}\\ &\eqsim ||f||_{E_0(X)} + \big|\big|(\mathscr{E}^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_n\big|\big|_E\\ &\eqsim ||f||_{E_0(X)} + \big|\big|\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq1}))}.\end{aligned}$$

Corollary 2.4.6. *Let* $E\in\mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ *and suppose that* $\varepsilon_+>\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{1})_+$*. Let* $\boldsymbol{p}\in(0,\infty]^\ell$ *and* $M\in\mathbb{N}$ *satisfy* $\varepsilon_+>\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{p}^{-1})$ *and* $M\lambda^{\boldsymbol{A}}_{\min}>\varepsilon_-$*. Then, for each* $f\in L_0(S;L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d;X))$ *of the form* $f=\sum_{i\in I}1_{S_i}\otimes f^{[i]}$ *with* $(S_i)_{i\in I}\subset\mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I}\in L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X)$*,*

$$||f||_{Y^{\boldsymbol{A}}(E;X)} \eqsim ||f||_{YL^{\boldsymbol{A}}(E;X)} \eqsim ||f||_{E_0(X)} + ||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq1}||_{E(\mathbb{N}_{\geq1})}.$$

Theorem 2.4.7. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \mathbf{1}, (S, \mathscr{A}, \mu))$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $\boldsymbol{p} \in [1,\infty]^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\mathbf{1} - \boldsymbol{p}^{-1})$ and $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. Write*

$$I^{\boldsymbol{A}}_{M,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})} \int_{B^{\boldsymbol{A}}(0,2^{-n})} \Delta^M_z f \, dz, \qquad f \in L_0(S; L_{1,\mathrm{loc}}(\mathbb{R}^d; X)).$$

Then

$$\begin{aligned} ||f||_{Y^{\boldsymbol{A}}(E;X)} &\eqsim ||f||_{YL^{\boldsymbol{A}}(E;X)} \eqsim ||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} \\ &\eqsim ||f||_{E_0(X)} + ||(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} \\ &\eqsim ||f||_{E_0(X)} + ||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} \end{aligned}$$

for all $f \in E_0(X) \hookrightarrow E_i \hookrightarrow E^{\boldsymbol{A}}_{\otimes}[B^{\boldsymbol{r},w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}](X)$ (see Remark 2.3.7).

Proposition 2.4.8. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $c \in \mathbb{R}$. Let $\boldsymbol{p} \in (0,\infty]^\ell$ and $M \in \mathbb{N}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$ and $M > \varepsilon_-$. Then*

$$||\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,,n}(f)\}_n||_{E(X)} \lesssim ||f||_{YL^{\boldsymbol{A}}(E;X)}, \qquad f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X)),$$

and

$$||\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,x^*,n}(f)\}_{(x^*,n)}||_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \lesssim ||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}, \qquad f \in L_0(S; L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d; X)),$$

where

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{p}^{-1}} \big|\big|z \mapsto L_{cz}\Delta^M_z f\big|\big|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n};X))}$$

and

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,x^*,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{p}^{-1}} \big|\big|z \mapsto L_{cz}\Delta^M_z \langle f, x^*\rangle\big|\big|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}.$$

2.4.3. Some lemmas

Lemma 2.4.9. *Let $p \in (0,\infty]$ and $M \in \mathbb{N}_{\geq 1}$. Then there is a constant $C = C_{M,p,d}$ such that: if $f \in L_{p,\mathrm{loc}}(\mathbb{R}^d)$ and $Q = \boldsymbol{A}_\lambda([0,1)^d + b)$ with $\lambda \in (0,\infty)$ and $b \in \mathbb{R}^d$, then there is $\pi \in \mathscr{P}^d_{M-1}$ satisfying (with the usual modification if $p = \infty$):*

$$\begin{aligned} |f - \pi| 1_Q \leq C \left(\fint_{B^{\boldsymbol{A}}(0,\lambda)} |\Delta^M_z f|^p \, dz \right)^{1/p} \\ + C \left(\fint_{B^{\boldsymbol{A}}(0,\lambda)} \fint_{Q(2)} |\Delta^M_z f|^p \, dy \, dz \right)^{1/p}. \end{aligned}$$

Proof. The case $\lambda = 1$ is contained in [119, Lemma 1.2.1], from which the general case can be obtained by a scaling argument. □

From Lemma 2.4.10 to Corollary 2.4.12 we will actually only use Corollary 2.4.12 in the scalar-valued case in the proof of Theorem 2.4.5. However, although the scalar-valued case is easier, we have decided to present it in this way as it could be useful for potential extensions of Theorem 2.4.4 along these lines. In the latter the main obstacle is Lemma 2.4.9.

We write $\mathscr{P}^d_N(X) \simeq X^{M_{N,d}}$, where $M_{N,d} := \#\{\alpha \in \mathbb{N}^d : |\alpha| \leq M\}$, for the space of X-valued polynomials of degree at most N on $\mathbb{R}^d$.

Lemma 2.4.10. *Let $(T,\mathscr{B},\nu)$ a measure space, $\mathbb{F} \subset L_2(T)$ a finite dimensional subspace, $\mathbb{E} \subset L_0(T;X)$ a topological vector space with $\mathbb{F} \otimes X \subset \mathbb{E}$ such that*

$$\mathbb{F} \times X \longrightarrow \mathbb{E}, (p,f) \mapsto f \otimes x,$$

and

$$\mathbb{F} \times \mathbb{E} \longrightarrow L_1(T;X), (f,g) \mapsto fg,$$

are well-defined bilinear mappings that are continuous with respect to the second variable. Then $\mathbb{F} \otimes X$ is a complemented subspace of $\mathbb{E}$.

Proof. Choose an orthogonal basis $b_1,\ldots,b_n$ of the finite dimensional subspace $\mathbb{F}$ of $L_2(T)$. Then

$$\pi : \mathbb{E} \longrightarrow \mathbb{E}, g \mapsto \sum_{i=1}^{n} \left[\int_T b_i(t)g(t)\mathrm{d}\nu(t) \right] \otimes b_i,$$

is a well-defined continuous linear mapping on $\mathbb{E}$, which is a projection onto the linear subspace $\mathbb{F} \otimes X \subset \mathbb{E}$. □

Corollary 2.4.11. *If $\mathbb{E}$ in Lemma 2.4.10 is an F-space, then so is $(\mathbb{F} \otimes X, \tau_{\mathbb{E}})$. As a consequence, if τ is a topological vector space topology on $\mathbb{F} \otimes X$ with $(\mathbb{F} \otimes X, \tau_{\mathbb{E}}) \hookrightarrow (\mathbb{F} \otimes X, \tau)$, then the latter is in fact a topological isomorphism.*

Corollary 2.4.12. *Let $B = [-1,2]^d$, $N \in \mathbb{N}$ and $q \in [1,\infty)$. Set $B_{n,k} := \boldsymbol{A}_{2^{-n}}(B+k)$ for $(n,k) \in \mathbb{N} \times \mathbb{Z}^d$. Then*

$$||\pi(\boldsymbol{A}_{2^{-n}} \cdot +k)||_{C^N_b(B;X)} \lesssim 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})/q}||\pi||_{L_q(B_{n,k};X)}, \qquad \pi \in \mathscr{P}^d_N(X), (n,k) \in \mathbb{N} \times \mathbb{Z}^d.$$

Proof. Let us first note that a substitution gives

$$||\pi(\boldsymbol{A}_{2^{-n}} \cdot +k)||_{L_q(B;X)} = 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})/q}||\pi||_{L_q(B_{n,k};X)},$$

while $\pi(\boldsymbol{A}_{2^{-n}} \cdot +k) \in \mathscr{P}^d_N(X)$. Applying Corollary 2.4.11 to $\mathbb{F} = \mathscr{P}^d_N$, viewed as finite dimensional subspace of $L_2(B)$, and $\mathbb{E} = C^N_n(B;X)$ and τ the topology on $\mathscr{P}_N(X) = \mathbb{F} \otimes X$ induced from $L_q(B;X)$, we obtain the desired result. □

Lemma 2.4.13. *Let $q,p \in (0,\infty)$, $q \leq p$, $b \in (0,\infty)$ and $M \in \mathbb{N}_{\geq 1}$. Let $f \in L_{p,\mathrm{loc}}(\mathbb{R}^d)$ and let $\{\pi_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset \mathscr{P}^d_{M-1}$ such that*

$$||f - \pi_{n,k}||_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))} \leq 2\mathscr{E}_M(f, Q^{\boldsymbol{A}}_{n,k}(b), L_q),$$

and let $\{\phi_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{Z}^d} \subset L_\infty(\mathbb{R}^d)$ *be such that* $\operatorname{supp}\phi_{n,k} \subset Q^{\boldsymbol{A}}_{n,k}(b)$, $\sum_{k\in\mathbb{Z}^d}\phi_{n,k} \equiv 1$, *and* $\|\phi_{n,k}\|_{L_\infty} \leq 1$. *Then, for* $(f_n)_{n\in\mathbb{N}} \subset L_0(S)$ *defined by*

$$f_n := \sum_{k\in\mathbb{Z}^d} \pi_{n,k}\phi_{n,k},$$

there is the convergence $f = \lim_{n\to\infty} f_n$ *almost everywhere and in* $L_{p,\mathrm{loc}}$.

Proof. This can be proved as in [119, Lemma 1.2.3]. □

Lemma 2.4.14. *Let* $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$, $b\in(0,\infty)$ *and suppose that* $\varepsilon_+,\varepsilon_- > 0$. *Let* $\boldsymbol{p}\in(0,\infty]^\ell$ *satisfy* $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{r}^{-1}-\boldsymbol{p}^{-1})$. *Define the sublinear operator*

$$T^{\boldsymbol{A}}_{\boldsymbol{p}} : L_0(S)^{\mathbb{N}\times\mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}, \qquad (s_{n,k})_{(n,k)} \mapsto (t_{n,k})_{(n,k)},$$

by

$$t_{n,k} := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \Big\| \sum_{m,l} |s_{m,l}|\chi^{\boldsymbol{A}}_{m,l} \Big\|_{L_{\boldsymbol{p},\mathscr{d}}}$$

and the sum is taken over all indices $(m,l)\in\mathbb{N}\times\mathbb{Z}^d$ *such that* $Q^{\boldsymbol{A}}_{m,l}\subset Q^{\boldsymbol{A}}_{n,k}(b)$ *and* $m\geq n$. *Then* $T^{\boldsymbol{A}}_{\boldsymbol{p}}$ *restricts to a bounded sublinear operator on* $y^{\boldsymbol{A}}(E)$.

Proof. Let $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ and $(t_{n,k})_{(n,k)} = T^{\boldsymbol{A}}_{\boldsymbol{p}}[(s_{n,k})_{(n,k)}] \in L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}$. We need to show that $\|(t_{n,k})\|_{y^{\boldsymbol{A}}(E)} \lesssim \|(s_{n,k})\|_{y^{\boldsymbol{A}}(E)}$. Here we may without loss of generality assume that $s_{n,k}\geq 0$ for all (n,k).

Set

$$\delta := \frac{1}{2}\left(\varepsilon_+ - \mathrm{tr}(A)\cdot(\boldsymbol{r}^{-1}-\boldsymbol{p}^{-1})\right) \in (0,\infty).$$

Define

$$g_m := \sum_{l\in\mathbb{Z}^d} s_{m,l}\chi^{\boldsymbol{A}}_{m,l} \in L_0(S), \qquad m\in\mathbb{N}.$$

Then

$$t_{n,k} \leq 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \Big\| \sum_{m=n}^{\infty} g_m \Big\|_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}. \tag{2.24}$$

As the the right-hand side is increasing in $\boldsymbol{p}$ by Hölder's inequality, it suffices to consider the case $\boldsymbol{p}\geq\boldsymbol{r}$.

Several applications of the elementary embedding

$$\ell^{s_0}_{q_0}(\mathbb{N}) \hookrightarrow \ell^{s_1}_{q_1}(\mathbb{N}), \qquad s_0 > s_1, q_0, q_1 \in (0,\infty],$$

in combination with Fubini yield that

$$\Big\| \sum_{m=n}^{\infty} g_m \Big\|_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))} \lesssim \sum_{m=n}^{\infty} 2^{(m-n)\delta}\|g_m\|_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}. \tag{2.25}$$

In order to estimate the summands on the right-hand side of (2.24), we will use the following fact. Let $(T_1,\mathscr{B}_1,\nu_1),\ldots,(T_\ell,\mathscr{B}_\ell,\nu_\ell)$ be σ-finite measure spaces and let

$I_1,\ldots,I_\ell$ be countable sets. Put $T = T_1\times\ldots\times T_\ell$ and $I = I_1\times\ldots\times I_\ell$. Let $(c_{\boldsymbol{i}})_{\boldsymbol{i}\in I}\subset\mathbb{C}$ and, for each $j\in\{1,\ldots,\ell\}$, let $(A^{(j)}_{i_j\in I_j})\subset\mathscr{B}_j$ be a sequence of mutually disjoint sets. Then

$$\Big|\Big|\sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i}}\mathbb{1}_{A^{(1)}_{i_1}\times\ldots\times A^{(\ell)}_{i_\ell}}\Big|\Big|_{L_{\boldsymbol{p}}(T)} \le \left(\sup_{\boldsymbol{i}\in I}\prod_{j=1}^{\ell}|A^{(j)}_{i_j}|^{\frac{1}{p_j}-\frac{1}{r_j}}\right)\Big|\Big|\sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i}}\mathbb{1}_{A^{(1)}_{i_1}\times\ldots\times A^{(\ell)}_{i_\ell}}\Big|\Big|_{L_{\boldsymbol{r}}(T)}. \tag{2.26}$$

Indeed,

$$\begin{aligned}
&\Big|\Big|\sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i}}\mathbb{1}_{A^{(1)}_{i_1}\times\ldots\times A^{(\ell)}_{i_\ell}}\Big|\Big|_{L_{\boldsymbol{p}}(T)}\\
&= \left(\sum_{i_\ell\in I_\ell}|A^{(\ell)}_{i_\ell}|\left(\ldots\left(\sum_{i_1\in I_1}|A^{(1)}_{i_1}||c_{\boldsymbol{i}}|^{p_1}\right)^{p_2/p_1}\ldots\right)^{p_\ell/p_{\ell-1}}\right)^{1/p_\ell}\\
&\overset{\boldsymbol{p}\ge\boldsymbol{r}}{\le} \left(\sum_{i_\ell\in I_\ell}|A^{(\ell)}_{i_\ell}|^{r_\ell/p_\ell}\left(\ldots\left(\sum_{i_1\in I_1}|A^{(1)}_{i_1}|^{r_1/p_1}|c_{\boldsymbol{i}}|^{r_1}\right)^{r_2/r_1}\ldots\right)^{r_\ell/r_{\ell-1}}\right)^{1/r_\ell}\\
&\le \left(\sup_{\boldsymbol{i}\in I}\prod_{j=1}^{\ell}|A^{(j)}_{i_j}|^{\frac{1}{p_j}-\frac{1}{r_j}}\right)\left(\sum_{i_\ell\in I_\ell}|A^{(\ell)}_{i_\ell}|\left(\ldots\left(\sum_{i_1\in I_1}|A^{(1)}_{i_1}||c_{\boldsymbol{i}}|^{r_1}\right)^{r_2/r_1}\ldots\right)^{r_\ell/r_{\ell-1}}\right)^{1/r_\ell}\\
&= \left(\sup_{\boldsymbol{i}\in I}\prod_{j=1}^{\ell}|A^{(j)}_{i_j}|^{\frac{1}{p_j}-\frac{1}{r_j}}\right)\Big|\Big|\sum_{\boldsymbol{i}\in I} c_{\boldsymbol{i}}\mathbb{1}_{A^{(1)}_{i_1}\times\ldots\times A^{(\ell)}_{i_\ell}}\Big|\Big|_{L_{\boldsymbol{r}}(T)}.
\end{aligned}$$

Let us now use the above fact to estimate $||g_m||_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))}$:

$$\begin{aligned}
||g_m||_{L_{\boldsymbol{p},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b))} &\le \Big|\Big|\sum_{l\in\mathbb{Z}^d:Q^{\boldsymbol{A}}_{m,l}\cap Q^{\boldsymbol{A}}_{n,k}(b)\neq\varnothing} s_{m,l}\chi^{\boldsymbol{A}}_{m,l}\Big|\Big|_{L_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d)}\\
&\overset{(2.26)}{\le} 2^{-m\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{p}^{-1}-\boldsymbol{r}^{-1})}\Big|\Big|\sum_{l\in\mathbb{Z}^d:Q^{\boldsymbol{A}}_{m,l}\cap Q^{\boldsymbol{A}}_{n,k}(b)\neq\varnothing} s_{m,l}\chi^{\boldsymbol{A}}_{m,l}\Big|\Big|_{L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d)}\\
&\le 2^{-m\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{p}^{-1}-\boldsymbol{r}^{-1})}||g_m||_{L_{\boldsymbol{r},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2)))}\\
&= 2^{(m-n)((\varepsilon_+-2\delta))-n\mathrm{tr}(\boldsymbol{A})\cdot(\boldsymbol{p}^{-1}-\boldsymbol{r}^{-1})}||g_m||_{L_{\boldsymbol{r},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2)))}
\end{aligned}\tag{2.27}$$

Putting (2.24), (2.25) and (2.27) together, we obtain

$$\begin{aligned}
t_{n,k}\chi^{\boldsymbol{A}}_{n,k} &\le \sum_{m=n}^{\infty}2^{(m-n)((\varepsilon_+-\delta))+n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}||g_m||_{L_{\boldsymbol{r},\mathscr{d}}(Q^{\boldsymbol{A}}_{n,k}(b+2)))}\chi^{\boldsymbol{A}}_{n,k}\\
&\lesssim_{b,\boldsymbol{A},\boldsymbol{r}} \sum_{m=n}^{\infty}2^{(m-n)(\varepsilon_+-\delta)}M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m).
\end{aligned}\tag{2.28}$$

Since

$$\left(\sum_{m=n}^{\infty}2^{(m-n)(\varepsilon_+-\delta)}M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m)\right)_{n\in\mathbb{N}} = \sum_{i=0}^{\infty}2^{i(\varepsilon_+-\delta)}(S_+)^i M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[(g_n)_{n\in\mathbb{N}}\right],$$

it follows that $(t_{n,k}) \in y^{\boldsymbol{A}}(E)$ with

$$\begin{aligned} ||(t_{n,k})||^{\kappa}_{y^{\boldsymbol{A}}(E)} &= \Big|\Big|\Big(\sum_{k \in \mathbb{Z}^d} t_{n,k}\chi^{\boldsymbol{A}}_{n,k}\Big)_n\Big|\Big|^{\kappa}_E \\ &\lesssim \sum_{i=0}^{\infty} 2^{\kappa i((\varepsilon_+ - \delta))} ||(S_+)^i M^{\boldsymbol{A}}_{\boldsymbol{r}}\left[(g_n)_n\right]||^{\kappa}_E \\ &\lesssim \sum_{i=0}^{\infty} 2^{-\kappa i \delta))} ||(g_n)_n||^{\kappa}_E \lesssim ||(g_n)_n||^{\kappa}_E \\ &= ||(s_{n,k})||^{\kappa}_{y^{\boldsymbol{A}}(E)}, \end{aligned} \tag{2.29}$$

where κ is such that E has a κ-norm. □

Corollary 2.4.15. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ and suppose that $\varepsilon_+,\varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty]^{\ell}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$. Given $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$, set $g_n = \sum_{k \in \mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}$. Then $\sum_{n=0}^{\infty} |g_n|$ in $L_0(S; L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ and the series $\sum_{n=0}^{\infty} g_n$ converges almost everywhere, and in $L_0(S; L_{\boldsymbol{p},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d))$ (when $\boldsymbol{p} \in (0,\infty)^{\ell}$).*

Proof. This follows from (2.29), see [119, Corollary 1.2.5] for more details. □

Lemma 2.4.16. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$, $b \in (0,\infty)$ and $\lambda \in (\varepsilon_-,\infty)$. Define the sublinear operator*

$$T_\lambda : L_0(S)^{\mathbb{N}\times\mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{\mathbb{N}\times\mathbb{Z}^d}, \qquad (s_{n,k})_{(n,k)} \mapsto (t_{n,k})_{(n,k)},$$

by

$$t_{n,k} := \sum_{m,l} 2^{\lambda(n-m)}|s_{m,l}|,$$

the sum being taken over all indices $(m,l) \in \mathbb{N}\times\mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l}(b) \supset Q^{\boldsymbol{A}}_{n,k}$ and $m < n$. Then T_λ restricts to a bounded sublinear operator from $y^{\boldsymbol{A}}(E)$ to $y^{\boldsymbol{A}}(E)$.

Proof. This can be proved in the same way as [119, Lemma 1.2.6]. □

Lemma 2.4.17. *Let $\boldsymbol{r} \in (0,1]^{\ell}$ and $\rho \in (0,1)$ satisfy $\rho < \boldsymbol{r}_{\min}$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^d$ satisfying*

$$0 \leq \gamma_n(x) \lesssim (1 + 2^n \rho_{\boldsymbol{A}}(x))^{-d/\rho}.$$

If $(s_{n,k})_{(n,k)} \in L_0(S)^{\mathbb{N}\times\mathbb{Z}^d}$, $g_n = \sum_{k\in\mathbb{Z}^d} s_{n,k}\chi^{\boldsymbol{A}}_{n,k}$ and $h_n = \sum_{k\in\mathbb{Z}^d} |s_{n,k}|\gamma_n(\cdot - \boldsymbol{A}_{2^{-n}}k)$, then

$$h_n \lesssim M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_n), \qquad n \in \mathbb{N}.$$

Proof. We may of course without loss of generality assume that $\boldsymbol{r} = (r,\ldots,r)$ with $r \in (0,1]$. Now the statement can be established as in [119, Lemma 1.2.7]. □

Lemma 2.4.18. *Let $M \in \mathbb{N}$, $\lambda \in (0,\infty)$ and $\Phi \in C^M(\mathbb{R}^d; X)$ be such that*

$$(1+\rho_{\boldsymbol{A}}(x))^{\lambda} ||D^{\beta}\Phi(x)||_X \lesssim 1, \qquad x \in \mathbb{R}^d, |\beta| \leq M,$$

and let $\Psi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\Psi \perp \mathscr{P}^d_{M-1}$. Set $\Psi_t := t^{-\mathrm{tr}(\boldsymbol{A}^{\oplus})}\Psi(\boldsymbol{A}_{t^{-1}}\cdot)$ for $t \in (0,\infty)$. Then

$$||\Phi * \Psi_t(x)||_X \lesssim \frac{t^{\lambda^{\boldsymbol{A}}_{\min} M}}{(1+\rho_{\boldsymbol{A}}(x))^{\lambda}}, \qquad x \in \mathbb{R}^d, t \in (0,1].$$

Proof. As Ψ is a Schwartz function, there in particularly exists $C \in (0,\infty)$ such that

$$|\Psi(x)| \leq C(1+\rho_{\boldsymbol{A}}(x))^{-\lambda}(1+|x|)^{-(d+M+1)}, \qquad x \in \mathbb{R}^d.$$

The desired inequality can now be obtained as in [119, Lemma 1.2.8]. □

Lemmas 2.4.19 and 2.4.20 are the corresponding versions of Lemmas 2.4.14 and 2.4.16, respectively, for $\widetilde{y}^{\boldsymbol{A}}(E;X)$ instead of $y^{\boldsymbol{A}}(E;X)$.

Lemma 2.4.19. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S,\mathscr{A},\mu))$, $b \in (0,\infty)$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $\boldsymbol{p} \in (0,\infty]^{\ell}$ satisfy $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \boldsymbol{p}^{-1})$. Define the sublinear operator*

$$T^{\boldsymbol{A}}_{\boldsymbol{p}} : L_0(S)^{X^* \times \mathbb{N} \times \mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}, \qquad (s_{x^*,n,k})_{(x^*,n,k)} \mapsto (t_{x^*,n,k})_{(x^*,n,k)},$$

by

$$t_{x^*,n,k} := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot \boldsymbol{p}^{-1}} \Big|\Big| \sum_{m,l} |s_{x^*,m,l}| \chi^{\boldsymbol{A}}_{m,l} \Big|\Big|_{L_{\boldsymbol{p},\boldsymbol{d}}}$$

and the sum is taken over all indices $(m,l) \in \mathbb{N} \times \mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l} \subset Q^{\boldsymbol{A}}_{n,k}(b)$ and $m \geq n$. Then $T^{\boldsymbol{A}}_{\boldsymbol{p}}$ restricts to a bounded sublinear operator on $\widetilde{y}^{\boldsymbol{A}}(E)$.

Proof. Let $\delta \in (0,\infty)$ be as in the proof of Lemma 2.4.14. Let $(s_{n,k})_{(x^*,n,k)} \in \widetilde{y}^{\boldsymbol{A}}(E)$ and $(t_{x^*,n,k})_{(n,k)} = T^{\boldsymbol{A}}_{\boldsymbol{p}}[(s_{x^*,n,k})_{(x^*,n,k)}] \in L_0(S;[0,\infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}$. Define

$$g_{x^*,m} := \sum_{l \in \mathbb{Z}^d} s_{x^*,m,l} \chi^{\boldsymbol{A}}_{m,l} \in L_0(S), \qquad m \in \mathbb{N}.$$

Then $(g_{x^*,m})_{(x^*,m)} \in \mathscr{F}_{\mathrm{M}}(X^*;E)$ with $||(g_{x^*,m})_{(x^*,m)}||_{\mathscr{F}_{\mathrm{M}}(X^*;E)} = ||(s_{x^*,n,k})_{(x^*,n,k)}||_{\widetilde{y}^{\boldsymbol{A}}(E)}$. So there exists $(g_m)_m \in E_+$ with $||(g_m)_m|| \leq 2||(s_{x^*,n,k})_{(x^*,n,k)}||_{\widetilde{y}^{\boldsymbol{A}}(E)}$ such that $|g_{x^*,m}| \leq ||x^*||g_m$. By (2.28) from the proof of Lemma 2.4.14,

$$\begin{aligned} t_{x^*,n,k}\chi^{\boldsymbol{A}}_{n,k} &\lesssim_{b,\boldsymbol{A},\boldsymbol{r}} \sum_{m=n}^{\infty} 2^{(m-n)((\varepsilon_+-\delta))} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{x^*,m}) \\ &\leq ||x^*|| \sum_{m=n}^{\infty} 2^{(m-n)((\varepsilon_+-\delta))} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m). \end{aligned}$$

As (2.29) in proof of Lemma 2.4.14, we find that $(t_{x^*,n,k})_{(x^*,n,k)} \in \widetilde{y}^{\boldsymbol{A}}(E;X)$ with

$$||(t_{x^*,n,k})_{(x^*,n,k)}||_{\widetilde{y}^{\boldsymbol{A}}(E;X)} \lesssim ||(g_m)_m|| \leq 2||(s_{x^*,n,k})_{(x^*,n,k)}||_{\widetilde{y}^{\boldsymbol{A}}(E)}.$$ □

Lemma 2.4.20. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$, $b \in (0,\infty)$ and $\lambda \in (\varepsilon_-, \infty)$. Define the sublinear operator*

$$T_\lambda : L_0(S)^{X^* \times \mathbb{N} \times \mathbb{Z}^d} \longrightarrow L_0(S;[0,\infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}, \qquad (s_{x,*,n,k})_{(x^*,n,k)} \mapsto (t_{x^*,n,k})_{(x^*,n,k)},$$

by

$$t_{x^*,n,k} := \sum_{m,l} 2^{\lambda(n-m)} |s_{x^*,m,l}|,$$

the sum being taken over all indices $(m,l) \in \mathbb{N} \times \mathbb{Z}^d$ such that $Q^{\boldsymbol{A}}_{m,l}(b) \supset Q^{\boldsymbol{A}}_{n,k}$ and $m < n$. Then T_λ restricts to a bounded sublinear operator on $\widetilde{y}^{\boldsymbol{A}}(E;X)$.

Proof. This can be proved in the same way as [119, Lemma 1.2.6]. □

Lemma 2.4.21. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{1}, (S, \mathscr{A}, \mu))$ and let $k \in L_{1,c}(\mathbb{R}^d)$ fulfill the Tauberian condition*

$$|\hat{k}(\xi)| > 0, \qquad \xi \in \mathbb{R}^d, \frac{\varepsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\varepsilon,$$

for some $\varepsilon \in (0,\infty)$. Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\psi} \subset \{\xi : \varepsilon \le \rho_{\boldsymbol{A}}(\xi) \geq B\}$ for some $B \in (\varepsilon,\infty)$. Define $(k_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ by $k_n := 2^{n\operatorname{tr}(\boldsymbol{A}^{\oplus})} k(\boldsymbol{A}_{2^n}\,\cdot)$ and $\psi_n := 2^{n\operatorname{tr}(\boldsymbol{A}^{\oplus})}\psi(\boldsymbol{A}_{2^n}\,\cdot)$. Then

$$\|(\psi_n * f_n)_n\|_{E(X)} \lesssim \|(k_n * f_n)_n\|_{E(X)}, \qquad f \in L_0(S; L_{1,\mathrm{loc}}(\mathbb{R}^d;X)).$$

Proof. Pick $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\eta \subset B^{\boldsymbol{A}}(0,2\varepsilon)$ and $\eta(\xi) = 1$ for $\rho_{\boldsymbol{A}}(\xi) \le \frac{3\varepsilon}{2}$. Define $m \in \mathscr{S}(\mathbb{R}^d)$ by $m(\xi) := [\eta(\xi) - \eta(\boldsymbol{A}_2\xi)]\hat{k}(\xi)^{-1}$ if $\frac{\varepsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\varepsilon$ and $m(\xi) := 0$ otherwise; note that this gives a well-defined Schwartz function on $\mathbb{R}^d$ because $\eta - \eta(\boldsymbol{A}_2\,\cdot)$ is a smooth function supported in the set $\{\xi : \frac{\varepsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\varepsilon\}$ on which the function $\hat{k} \in BUC^\infty(\mathbb{R}^d)$ does not vanish. Define $(m_n)_{n\in\mathbb{N}}$ by $m_n := m(\boldsymbol{A}_{2^{-n}}\,\cdot)$. Then, by construction,

$$\sum_{l=n}^{n+N} m_l \hat{k}_l(\xi) = \eta(\boldsymbol{A}_{2^{-(n+N)}}\xi) - \eta(\boldsymbol{A}_{2^{-n+1}}\xi) = 1$$

for $2^n\varepsilon \le \rho_{\boldsymbol{A}}(\xi) \le 2^{n+N-1}3\varepsilon$, $n \in \mathbb{N}$, $N \in \mathbb{N}$. Since $\operatorname{supp}\hat{\psi}_n \subset \{\xi : 2^n\varepsilon \le \rho_{\boldsymbol{A}}(\xi) < 2^nB\}$ for every $n \in \mathbb{N}$, there thus exists $N \in \mathbb{N}$ such that $\sum_{l=n}^{n+N} m_l\hat{k}_l \equiv 1$ on $\operatorname{supp}\hat{\varphi}_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we consequently have

$$\psi_n = \psi_n * \left(\sum_{l=n}^{n+N} \check{m}_l * k_l\right) = \sum_{l=n}^{n+N} \psi_n * \check{m}_l * k_l = \sum_{l=0}^{N} \psi_n * \check{m}_{n+l} * k_{n+l}.$$

As $\psi, m \in \mathscr{S}(\mathbb{R}^d)$, we obtain the pointwise estimate

$$\|\psi_n * f\|_X \le \sum_{l=0}^{N} \|\psi_n * \check{m}_{n+l} * k_{n+l} * f\|_X \lesssim \sum_{l=0}^{N} M^{\boldsymbol{A}}(M^{\boldsymbol{A}}(\|k_{n+l} * f\|_X)).$$

It follows that

$$\|(\psi_n * f)_n\|_{E(X)} \lesssim \sum_{l=0}^{N} \left\|\left(M^{\boldsymbol{A}}(M^{\boldsymbol{A}}(\|k_{n+l} * f\|_X)\right)_n\right\|_E$$

$$\lesssim \sum_{l=0}^{N} ||(k_{n+l} * f)_n||_{E(X)} \lesssim \sum_{l=0}^{N} 2^{-\varepsilon_+ l} ||(k_n * f)_n||_{E(X)}$$
$$\lesssim ||(k_n * f)_n||_{E(X)}. \qquad \square$$

2.4.4. Proofs of the results in Section 2.4.2

Proof of Theorem 2.4.4. (i) ⇒ (ii): Fix $\omega \in C_c^\infty((-1,2)^d)$ with the property that

$$\sum_{k \in \mathbb{Z}^d} \omega(x-k) = 1, \qquad x \in \mathbb{R}^d.$$

Let $(f_n)_n$ be as in Definition 2.3.8 with $||(f_n)_n||_{E(X)} \leq 2||f||_{YL^{\boldsymbol{A}}(E;X)}$. For each $(n,k) \in \mathbb{N} \times \mathbb{Z}^d$, we put

$$\widetilde{a}_{n,k} := \omega(\boldsymbol{A}_{2^n}(\cdot - \boldsymbol{A}_{2^{-n}}k)) f_n, \qquad s_{n,k} := ||\widetilde{a}_{n,k}(\boldsymbol{A}_{2^{-n}} \cdot)||_{C_b^M(\mathbb{R}^d;X)},$$

and

$$a_{n,k} := \frac{\widetilde{a}_{n,k}}{s_{n,k}} 1_{\{s_{n,k} \neq 0\}}.$$

Note that

$$\begin{aligned}
|s_{n,k}| &= ||\widetilde{a}_{n,k}(\boldsymbol{A}_{2^{-n}} \cdot)||_{C_b^M(\mathbb{R}^d;X)} = ||\omega(\cdot - k) f_n(\boldsymbol{A}_{2^{-n}} \cdot)||_{C_b^M(\mathbb{R}^d;X)} \\
&\lesssim ||\omega(\cdot - k)||_{C_b^M(\mathbb{R}^d)} ||f_n(\boldsymbol{A}_{2^{-n}} \cdot)||_{C_b^M([-1,2]^d + k;X)} \\
&\lesssim \sup_{|\alpha| \leq M} \sup_{y \in [-1,2]^d + k} ||D^\alpha [f_n(\boldsymbol{A}_{2^{-n}} \cdot)](y)||_X
\end{aligned}$$

Given $x \in Q_{n,k}^{\boldsymbol{A}}$ and $\widetilde{x} = \boldsymbol{A}_{2^n} x \in [0,1)^d + k$, for $y \in [-1,2]^d + k$ we can write $y = \widetilde{x} + z$ with

$$z = y - \widetilde{x} = (y-k) - (\widetilde{x} - k) \in [-1,2]^d - [0,1)^d, \quad \text{so, in particularly, } \rho_{\boldsymbol{A}}(z) \leq C_d.$$

Combining the above and subsequently applying Lemma 2.A.1 to $f_n(\boldsymbol{A}_{2^{-n}} \cdot)$, whose Fourier support satisfies $\operatorname{supp} \mathscr{F}[f_n(\boldsymbol{A}_{2^{-n}} \cdot)] \subset B^{\boldsymbol{A}}(0,2)$, we find

$$\begin{aligned}
|s_{n,k}| &\lesssim \sup_{|\alpha| \leq M} \sup_{\rho_{\boldsymbol{A}}(z) \leq C_d} ||D^\alpha [f_n(\boldsymbol{A}_{2^{-n}} \cdot)](\widetilde{x} + z)||_X \\
&\lesssim M_{\boldsymbol{r}}^{\boldsymbol{A}} [||f_n(\boldsymbol{A}_{2^{-n}} \cdot)||_X] (\boldsymbol{A}_{2^n} x) = M_{\boldsymbol{r}}^{\boldsymbol{A}}(||f_n||_X)(x)
\end{aligned}$$

for $x \in Q_{n,k}^{\boldsymbol{A}}$. Therefore, $(s_{n,k})_{(n,k)} \in y^{\boldsymbol{A}}(E)$ with

$$||(s_{n,k})_{(n,k)}||_{y^{\boldsymbol{A}}(E)} \lesssim \left|\left| (M_{\boldsymbol{r}}^{\boldsymbol{A}}(||f_n||_X))_n \right|\right|_E \lesssim ||(f_n)_n||_{E(X)} \leq 2||f||_{YL^{\boldsymbol{A}}(E;X)}.$$

Finally, the convergence (2.23) follows from Corollary 2.4.15 and the observation that

$$f = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} s_{n,k} a_{n,k} \quad \text{in} \quad L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d; X)).$$

(ii) ⇒ (i): Set $g_n := \sum_{k\in\mathbb{Z}^d} |s_{n,k}|\chi^{\boldsymbol{A}}_{n,k}$ for $n \in \mathbb{N}$. For $n \in \mathbb{Z}_{<0}$, set $f_n := 0$ and $g_n := 0$. Pick $\kappa \in (0,\infty)$ such that E has a κ-norm. Pick $\lambda \in (0,\infty)$ such that $d/\lambda < \boldsymbol{r}_{\min} \wedge 1$. Pick $\psi = (\psi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\hat{\psi}_0 \subset B^{\boldsymbol{A}}(0,2), \qquad \operatorname{supp}\hat{\psi}_n \subset B^{\boldsymbol{A}}(0,2^{n+1}) \setminus B^{\boldsymbol{A}}(0,2^{n-1}), \quad n \geq 1,$$

and set $\Psi_n := 2^{n\operatorname{tr}(\boldsymbol{A}^{\oplus})}\psi_0(\boldsymbol{A}_{2^n}\,\cdot)$ for each $n \in \mathbb{N}$. Note that

$$a_{n,k} * \Psi_n = [b_{n,k} * \Psi](\boldsymbol{A}_{2^n}\cdot - k)$$

and

$$a_{n,k} * \psi_m = [b_{n,k} * \psi_{m-n}](\boldsymbol{A}_{2^n}\cdot - k), \qquad n < m.$$

An application of Lemma 2.4.18 thus yields that

$$\|a_{n,k} * \Psi_n(x)\|_X \lesssim \frac{1}{(1+2^n\rho_{\boldsymbol{A}}(x-\boldsymbol{A}_{2^{-n}}k))^{\lambda}} \tag{2.30}$$

and

$$\|a_{n,k} * \psi_m(x)\|_X \lesssim \frac{2^{-(m-n)\lambda^{\boldsymbol{A}}_{\min}M}}{(1+2^n\rho_{\boldsymbol{A}}(x-\boldsymbol{A}_{2^{-n}}k))^{\lambda}}, \qquad n < m. \tag{2.31}$$

Now put

$$\tilde{a}_{n,k,m} := \begin{cases} a_{n,k} * \Psi_n, & n = m, \\ a_{n,k} * \psi_m, & n < m. \end{cases}$$

Let us define

$$f_{n,m} := \sum_{k\in\mathbb{Z}^d} s_{n,k}\tilde{a}_{n,k,m}, \qquad n,m \in \mathbb{N}, m \geq n.$$

By construction (also see [119, Theorem 1.1.14(iv)⇒(i)]),

$$f = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} f_{n,m} = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} f_{m-l,m}.$$

By a combination of (2.30), (2.31) and Lemma 2.4.17,

$$\sum_{k\in\mathbb{Z}^d} |s_{m-l,k}|\,\|\tilde{a}_{m-l,k,m}\|_X \lesssim 2^{-l\lambda^{\boldsymbol{A}}_{\min}M} M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{m-l}).$$

Therefore,

$$\begin{aligned}
\|(f_{m-l,m})_{m\geq l}\|_{E(\mathbb{N}_{\geq l};X)} &\lesssim 2^{-l\lambda^{\boldsymbol{A}}_{\min}M}\|(M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{m-l}))_{m\geq l}\|_{E(\mathbb{N}_{\geq l})} \\
&= 2^{-l\lambda^{\boldsymbol{A}}_{\min}M}\left\|(S_-)^l(M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_m))_{m\in\mathbb{N}}\right\|_E \\
&\lesssim 2^{-l(\lambda^{\boldsymbol{A}}_{\min}M-\varepsilon_-)}\|(g_m)_{m\in\mathbb{N}}\|_E \\
&= 2^{-l(\lambda^{\boldsymbol{A}}_{\min}M-\varepsilon_-)}\|(s_{n,k})_{(n,k)}\|_{y^{\boldsymbol{A}}(E)}.
\end{aligned}$$

Since

$$\operatorname{supp}\hat{f}_{m-l,m} \subset \operatorname{supp}\hat{\psi}_m \subset B^{\boldsymbol{A}}(0,2^{m+1}),$$

it follows that $F_l := \sum_{m=l}^{\infty} f_{m-l,m}$ defines an element of $YL^{\boldsymbol{A}}(E;X)$ with

$$||F_l||_{YL^{\boldsymbol{A}}(E;X)} \lesssim 2^{-l(\lambda^{\boldsymbol{A}}_{\min}M-\varepsilon_-)}||(s_{n,k})_{(n,k)}||_{y^{\boldsymbol{A}}(E)}.$$

As $\lambda^{\boldsymbol{A}}_{\min}M > \varepsilon_-$, we find that $f = \sum_{l=0} F_l \in YL^{\boldsymbol{A}}(E;X)$ with

$$||f||_{YL^{\boldsymbol{A}}(E;X)} \lesssim ||(s_{n,k})_{(n,k)}||_{y^{\boldsymbol{A}}(E)}.$$

(ii) $\Rightarrow$ (iii): We will write down the proof in such a way that the proof of Proposition 2.4.8 only requires a slight modification. Combining the estimate corresponding to (ii) $\Rightarrow$ (i) with $YL^{\boldsymbol{A}}(E;X) \overset{(2.7)}{\hookrightarrow} E_0(X)$, we find

$$||f||_{E_0(X)} \lesssim ||(s_{n,k})_{(n,k)}||_{y^{\boldsymbol{A}}(E)}.$$

So let us focus on the remaining part of the required inequality. To this end, fix $c \in \mathbb{R}$ and set $R := (|c|+M)^{1/\lambda^{\boldsymbol{A}}_{\min}}$. Then (note $|c|+M \geq 1$)

$$\rho_{\boldsymbol{A}}(tz) \leq R\rho_{\boldsymbol{A}}(z), \qquad z \in \mathbb{R}^d, t \in [0,|c|+M].$$

Put

$$d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f) := 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|z \mapsto L_{cz}\Delta^M_z f\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n});X)}, \qquad n \in \mathbb{N}.$$

Now let f has a representation as in (ii) and write $h_n := \sum_{k\in\mathbb{Z}^d} s_{n,k}a_{n,k}$. Then

$$\begin{aligned} d^{\boldsymbol{A},\boldsymbol{p}}_{M,c,n}(f)(x) \lesssim\ & 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|z \mapsto \sum_{m=0}^{n-1}||L_{cz}\Delta^M_z h_m(x)||_X\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &+\ 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|z \mapsto \sum_{m=n}^{\infty}||L_{cz}\Delta^M_z h_m(x)||_X\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}. \end{aligned} \tag{2.32}$$

We use the identity

$$L_{cz}\Delta^M_z h_m(x) = \sum_{l=0}^{M}(-1)^{M-l}\binom{M}{l}h_j(x+(c+l)z)$$

to estimate the second term in (2.32) as follows

$$\begin{aligned} & 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|z \mapsto \sum_{m=n}^{\infty}||L_{cz}\Delta^M_z h_m(x)||_X\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim \sum_{l=0}^{M} 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|z \mapsto \sum_{m=n}^{\infty}||h_m(x+(c+l)z))||_X\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|\sum_{m=n}^{\infty}||h_m||_X\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(x,R2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|\sum_{m=n}^{\infty}\sum_{k\in\mathbb{Z}^d}||s_{m,k}||_X 1_{Q^{\boldsymbol{A}}_{m,k}(3)}\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}(B^{\boldsymbol{A}}(x,R2^{-n}))} \\ &\lesssim 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\left|\left|\sum_{m,l}||s_{m,l}||_X 1_{Q^{\boldsymbol{A}}_{m,l}(3)}\right|\right|_{L_{\boldsymbol{p},\mathpzc{d}}}, \end{aligned}$$

where the last sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3)$ intersects $B^{\boldsymbol{A}}(x,R2^{-n}))$ and $m \geq n$. From this it follows that

$$
\begin{aligned}
&2^{n\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\Big|\Big|z\mapsto\sum_{m=n}^{\infty}||L_{cz}\Delta_z^M h_m||_X\Big|\Big|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}\\
&\qquad\lesssim\sum_{k\in\mathbb{Z}^d}2^{n\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\Big|\Big|\sum_{m,l}||s_{m,l}||_X 1_{Q^{\boldsymbol{A}}_{m,l}(3)}\Big|\Big|_{L_{\boldsymbol{p},\mathscr{d}}}1_{Q^{\boldsymbol{A}}_{n,k}(3R)},
\end{aligned}
\tag{2.33}
$$

where the sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3)\subset Q^{\boldsymbol{A}}_{n,k}(3R)$ and $m\geq n$.

In order to estimate the first term in (2.32), note that

$$\Delta_z^M h_m(x)=\int_{[0,1]^M}D^M h_m(x+(t_1+\ldots+t_M)z)(z,\ldots,z)\,d(t_1,\ldots,t_M)$$

and thus that

$$
\begin{aligned}
||\Delta_z^M h_m(x)||_X &\leq \sup_{t\in[0,M]}||D^M h_m(x+tz)(z,\ldots,z)||_X\\
&=\sup_{t\in[0,M]}\big|\big|D^M[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}x+t\boldsymbol{A}_{2^m}z)(\boldsymbol{A}_{2^m}z,\ldots,\boldsymbol{A}_{2^m}z)\big|\big|_X\\
&\lesssim\sup_{t\in[0,M]}\sup_{|\alpha|\leq M}\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}x+t\boldsymbol{A}_{2^m}z)\big|\big|_X|\boldsymbol{A}_{2^m}z|^M,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
||L_{cz}\Delta_z^M h_m(x)||_X &\lesssim \sup_{t\in[0,M]}\sup_{|\alpha|\leq M}\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}x+(c+t)\boldsymbol{A}_{2^m}z)\big|\big|_X|\boldsymbol{A}_{2^m}z|^M\\
&\leq\sup_{y\in B^{\boldsymbol{A}}(0,R\rho_{\boldsymbol{A}}(z))}\sup_{|\alpha|\leq M}\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}[x+y])\big|\big|_X|\boldsymbol{A}_{2^m}z|^M.
\end{aligned}
$$

For $m\in\{0,\ldots,n-1\}$ and $z\in B^{\boldsymbol{A}}(0,2^{-n})$ this gives

$$
\begin{aligned}
||L_{cz}\Delta_z^M h_m(x)||_X &\lesssim \sup_{y\in B^{\boldsymbol{A}}(0,R2^{-n})}\sup_{|\alpha|\leq M}\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}[x+y])\big|\big|_X\rho_{\boldsymbol{A}}(\boldsymbol{A}_{2^m}z)^{\lambda^{\boldsymbol{A}}_{\min}M}\\
&\lesssim\sup_{y\in B^{\boldsymbol{A}}(0,R2^{-n})}\sup_{|\alpha|\leq M}\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}[x+y])\big|\big|_X2^{\lambda^{\boldsymbol{A}}_{\min}M(m-n)}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\big|\big|D^\alpha[h_m\circ\boldsymbol{A}_{2^{-m}}](\boldsymbol{A}_{2^m}[x+y])\big|\big|_X &\leq\sum_{l\in\mathbb{Z}^d}||s_{m,l}||_X 1_{[-1,2]^d+l}(\boldsymbol{A}_{2^m}[x+y])\\
&\leq\sum_{l\in\mathbb{Z}^d}||s_{m,l}||_X 1_{Q^{\boldsymbol{A}}_{m,l}(3)}(x+y),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&2^{n\operatorname{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}}\Big|\Big|z\mapsto\sum_{m=0}^{n-1}||L_{cz}\Delta_z^M h_m(x)||_X\Big|\Big|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))}\\
&\qquad\lesssim\sum_{m=0}^{n-1}\sup_{z\in B^{\boldsymbol{A}}(0,2^{-n})}||L_{cz}\Delta_z^M h_m(x)||_X2^{\lambda^{\boldsymbol{A}}_{\min}M(m-n)}
\end{aligned}
$$

$$\lesssim \sum_{m,l} 2^{\lambda^{\boldsymbol{A}}_{\min} M(m-n)} ||s_{m,l}||_X,$$

where the last sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3)$ intersects $B^{\boldsymbol{A}}(x,R2^{-n}))$ and $m<n$. From this it follows that

$$2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{p}^{-1}} \left|\left| z \mapsto \sum_{m=0}^{n-1} ||L_{cz}\Delta^M_z h_m(x)||_X \right|\right|_{L_{\boldsymbol{p},\mathscr{d}}(B^{\boldsymbol{A}}(0,2^{-n}))} \lesssim \sum_{m,l} ||s_{m,l}||_X, \tag{2.34}$$

where the last sum is taken over all (m,l) such that $Q^{\boldsymbol{A}}_{m,l}(3) \supset$ and $m<n$.

A combination of (2.32), (2.33), Lemma 2.4.14, (2.34) and Lemma 2.4.16 give the desired result.

□

Proof of Theorem 2.4.5. The chain of implications (I) ⇔ (II) ⇒ (III) with corresponding estimates for $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d;X))$ can be obtained in the same way as Theorem 2.4.4 with some natural modifications; in particular, Lemmas 2.4.14 and 2.4.16 need to need be replaced with Lemmas 2.4.19 and 2.4.20, respectively. Furthermore, (II) ⇒ (IV) can be done in the same way as [119, Theorem 1.1.14], similarly to the implication (II) ⇒ (III) (see the proof of (ii) ⇒ (iii) in Theorem 2.4.4).

Fix $q \in (0,\infty)$ with $q \leq \boldsymbol{r}_{\min} \wedge \boldsymbol{p}_{\min}(III)^*_q$ and let (IV)$^*_q$ be the statements (III) and (IV), respectively, in which $\boldsymbol{p}$ gets replaced by $\boldsymbol{q} := (q,\dots,q) \in (0,\infty)^\ell$. Then, clearly, (III) ⇒ (III)$^*_q$ and (IV) ⇒ (IV)$^*_q$.

To finish this proof, it suffices to establish the implication (V) ⇒ (IV)$^*_q$ for $f \in L_0(S;L_{\boldsymbol{r},\mathscr{d}}(\mathbb{R}^d;X))$ and the implications (III)$^*_q$ ⇒ (V) and (IV)$^*_q$ ⇒ (II) for f of the form $f=\sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i\in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)$.

(V) ⇒ (IV)$^*_q$: For this implication we just observe that, for $x \in Q^{\boldsymbol{A}}_{n,k}$ and $n \geq 1$,

$$\mathscr{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f))(x) \lesssim \overline{\mathscr{E}}_M(\langle f,x^*\rangle, Q^{\boldsymbol{A}}_{n,k}(3), L_q) \lesssim M^{\boldsymbol{A}}_{\boldsymbol{q}}(g_{x^*,n})(x) \leq M^{\boldsymbol{A}}_{\boldsymbol{r}}(g_{x^*,n})(x).$$

(III)$^*_q$ ⇒ (V) *for f of the form $f=\sum_{i\in I} 1_{S_i}\otimes f^{[i]}$ with $(S_i)_{i\in I}\subset\mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)$:* By Lemma 2.4.9, for each $i \in I$ and $(x^*,n,k) \in X^* \times \mathbb{N}_{\geq 1} \times \mathbb{Z}^d$ there exists a $\pi^{[i]}_{x^*,n,k} \in \mathscr{P}^d_{M-1}$ such that

$$\left|\langle f^{[i]},x^*\rangle - \pi^{[i]}_{x^*,n,k}\right| 1_{Q^{\boldsymbol{A}}_{n,k}(3)} \lesssim d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f^{[i]}) + \left(\fint_{Q^{\boldsymbol{A}}_{n,k}(6)} d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f^{[i]})(y)^q \, dy \right)^{1/q}.$$

Defining $\pi_{x^*,n,k} \in L_0(S;\mathscr{P}^d_{M-1})$ by $\pi_{x^*,n,k} := \sum_{i\in I} 1_{S_i} \otimes \pi^{[i]}_{x^*,n,k}$, we obtain

$$\left|\langle f,x^*\rangle - \pi_{x^*,n,k}\right| 1_{Q^{\boldsymbol{A}}_{n,k}(3)} \lesssim d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f) + M^{\boldsymbol{A}}_{\boldsymbol{q}}(d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)) \leq 2M^{\boldsymbol{A}}_{\boldsymbol{r}}(d^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)).$$

Since

$$\#\{k \in \mathbb{Z}^d : x \in Q^{\boldsymbol{A}}_{n,k}(3)\} \lesssim 1, \qquad x \in \mathbb{R}^d, n \in \mathbb{N},$$

it follows that

$$\begin{aligned}\big|\big|\{g_{x^*,n}\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1}))} &\lesssim \big|\big|\{M^{\boldsymbol{A}}_{\boldsymbol{r}}[d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)]\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1}))} \\ &\lesssim \big|\big|\{d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq 1}}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1}))}.\end{aligned}$$

(IV)$^*_q$ ⇒ (II) *for f of the form $f=\sum_{i\in I}1_{S_i}\otimes f^{[i]}$ with $(S_i)_{i\in I}\subset\mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i\in I}\in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)$:* Let $\omega\in C^\infty_c([-1,2]^d)$ be such that

$$\sum_{k\in\mathbb{Z}^d}\omega(x-k)=1, \qquad x\in\mathbb{R}^d,$$

and put $\omega_{n,k}:=\omega(\boldsymbol{A}_{2^n}\cdot-k)$ and $Q^\omega_{n,k}:=\boldsymbol{A}_{2^{-n}}([-1,2]^d+k)$ for $(n,k)\in\mathbb{N}\times\mathbb{Z}^d$; so $\mathrm{supp}\,(\omega_{n,k})\subset Q^\omega_{n,k}$. Define

$$I_{n,k}:=\big\{l\in\mathbb{Z}^d: Q^\omega_{n,k}\cap Q^\omega_{n-1,l}\neq\varnothing\big\}, \qquad (n,k)\in\mathbb{N}_{\geq 1}\times\mathbb{Z}^d.$$

Then $\#I_{n,k}\lesssim 1$ and there exists $b\in(1,\infty)$ such that

$$Q^\omega_{n,k}\subset Q^{\boldsymbol{A}}_{n,k}(b)\cap Q^{\boldsymbol{A}}_{n-1,l}(b), \qquad l\in I_{n,k},\,(n,k)\in\mathbb{N}_{\geq 1}\times\mathbb{Z}^d. \tag{2.35}$$

Furthermore, there exists $n_0\in\mathbb{N}_{\geq 1}$ such that

$$Q^{\boldsymbol{A}}_{n,k}(b)\cup Q^{\boldsymbol{A}}_{n-1,l}(b)\subset B^{\boldsymbol{A}}(x,2^{-(n-n_0)}), \qquad x\in Q^\omega_{n,k},\,(n,k)\in\mathbb{N}\times\mathbb{Z}^d. \tag{2.36}$$

For each $i\in I$, let us pick $(\pi^{[i]}_{x^*,n,k})_{(x^*,n,k)\in X^*\times\mathbb{N}\times\mathbb{Z}^d}\subset\mathscr{P}^d_{M-1}$ with the property that

$$||\langle f^{[i]},x^*\rangle-\pi_{x^*,n,k}||_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))}\leq 2\mathscr{E}_M(\langle f^{[i]},x^*\rangle,Q^{\boldsymbol{A}}_{n,k}(b),L_q) \tag{2.37}$$

and put $\pi_{x^*,n,k}:=\sum_{i\in I}1_{S_i}\otimes\pi^{[i]}_{x^*,n,k}\in L_0(S;\mathscr{P}^d_{M-1})$. Define

$$u_{x^*,n,k}:=\begin{cases}\omega_{n,k}\sum_{l\in\mathbb{Z}^d}\omega_{n-1,l}[\pi_{x^*,n,k}-\pi_{x^*,n-1,l}], & n>n_0,\\ \omega_{n,k}\pi_{x^*,n,k}, & n=n_0,\\ 0, & n<n_0.\end{cases}$$

Let $x^*\in X^*$ and $(n,k)\in\mathbb{N}_{\geq n_0+1}\times\mathbb{Z}^d$. Let $l\in I_{n,k}$. For $x\in Q^\omega_{n,k}$ we can estimate

$$\begin{aligned}||\pi_{x^*,n,k}-\pi_{x^*,n-1,l}||_{L_q(Q^\omega_{n,k})} &\overset{(2.35)}{\lesssim} ||\langle f,x^*\rangle-\pi_{x^*,n,k}||_{L_q(Q^{\boldsymbol{A}}_{n,k}(b))} \\ &\quad+||\langle f,x^*\rangle-\pi_{x^*,n-1,l}||_{L_q(Q^{\boldsymbol{A}}_{n-1,l}(b))} \\ &\overset{(2.36),(2.37)}{\leq} 4\mathscr{E}_M(\langle f,x^*\rangle,B^{\boldsymbol{A}}(x,2^{-(n-n_0)}),L_q),\end{aligned}$$

implying

$$
\begin{aligned}
||(\pi_{x^*,n,k} - \pi_{x^*,n-1,l})(\boldsymbol{A}_{2^{-n}} \cdot +k)||_{C_b^M([-1,2]^M)}& \\
\lesssim 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})/q} \mathscr{E}_M(\langle f, x^*\rangle, B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), L_q)&
\end{aligned}
$$

in view of Corollary 2.4.12. Since $\# I_{n,k} \lesssim 1$, it follows that

$$
\begin{aligned}
||u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot +k)||_{C_b^M([-1,2]^M)} &\lesssim \overline{\mathscr{E}}_M(\langle f, x^*\rangle, B^{\boldsymbol{A}}(x, 2^{-(n-n_0)}), L_q) \\
&= \mathscr{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n-n_0}(f)(x), \qquad x \in Q^{\omega}_{n,k}. \qquad (2.38)
\end{aligned}
$$

For $n = n_0$ we similarly have

$$
\begin{aligned}
||u_{x^*,n_0,k}(\boldsymbol{A}_{2^{-n_0}} \cdot +k)||_{C_b^M([-1,2]^M)} &\lesssim ||\langle f, x^*\rangle||_{L_{\boldsymbol{q},d}(B^{\boldsymbol{A}}(x,1))} \\
&\lesssim ||x^*||\, M^{\boldsymbol{A}}_{\boldsymbol{q}}(||f||_X)(x) \\
&\leq ||x^*||\, M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f||_X)(x), \qquad x \in Q^{\omega}_{n_0,k}. \qquad (2.39)
\end{aligned}
$$

Define $s_{x^*,n,k} := ||u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot +k)||_{C_b^M([-1,2]^M)}$,

$$
a_{x^*,n,k} := \begin{cases} \frac{u_{x^*,n,k}}{s_{x^*,n,k}}, & s_{x^*,n,k} \neq 0, \\ 0, & s_{x^*,n,k} = 0, \end{cases}
$$

and $b_{x^*,n,k} := u_{x^*,n,k}(\boldsymbol{A}_{2^{-n}} \cdot +k)$. Then $b_{x^*,n,k} \in C_c^M([-1,2]^d)$ with $||b_{x^*,n,k}||_{C_b^M} \leq 1$ and $(s_{x^*,n,k})_{(x^*,n,k)} \in \widetilde{y}^{\boldsymbol{A}}(E;X)$ with

$$
\begin{aligned}
||(s_{x^* n,k})_{(x^*,n,k)}||_{\widetilde{y}^{\boldsymbol{A}}(E;X)} &\overset{(2.38),(2.39)}{\lesssim} ||M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f||_X)||_{E_0} \\
&\quad + \left|\left|\{\mathscr{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n-n_0}(f)\}_{(x^*,n)\in X^*\times\mathbb{N}_{\geq n_0}}\right|\right|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq n_0+1}))} \\
&\lesssim ||f||_{E_0(X)} + 2^{\varepsilon - n_0}\left|\left|\{\mathscr{E}^{\boldsymbol{A},\boldsymbol{q}}_{M,x^*,n}(f)\}_{(x^*,n)}\right|\right|_{\mathscr{F}_{\mathrm{M}}(X^*;E(\mathbb{N}_{\geq 1}))}.
\end{aligned}
$$

Note that, for $n \geq n_0 + 1$,

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}^d} s_{x^*,n,k} a_{x^*,n,k} &= \sum_{k\in\mathbb{Z}^d} u_{x^*,n,k} \\
&= \sum_{k\in\mathbb{Z}^d} \pi_{x^*,n,k}\omega_{x^*,n,k} \sum_{l\in\mathbb{Z}^d} \omega_{n-1,l} - \sum_{k\in\mathbb{Z}^d} \omega_{n,k} \sum_{l\in\mathbb{Z}^d} \pi_{x^*,n-1,l}\omega_{n-1,l} \\
&= \sum_{k\in\mathbb{Z}^d} \pi_{x^*,n,k}\omega_{n,k} - \sum_{l\in\mathbb{Z}^d} \pi_{x^*,n-1,l}\omega_{n-1,l}.
\end{aligned}
$$

In combination with Lemma 2.4.13 and an alternating sum argument, this implies that

$$
\langle f, x^*\rangle = \sum_{n=0}^{\infty} \sum_{k\in\mathbb{Z}^d} s_{x^*,n,k} a_{x^*,n,k} \qquad \text{in } L_0(S; L_{q,\mathrm{loc}}(\mathbb{R}^d)).
$$

The required convergence finally follows from this with an argument as in (the last part of) the proof of the implication (i) ⇒ (ii) in Theorem 2.4.4. □

Proof of Corollary 2.4.6. This is an immediate consequence of Theorems 2.3.17, 2.4.4, 2.4.5 and the observation that

$$||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,x^*,n}(f))_{(x^*,n)}||_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \leq ||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1})}. \qquad \square$$

Proof of Theorem 2.4.7. The estimates

$$||f||_{Y^{\boldsymbol{A}}(E;X)} \eqsim ||f||_{YL^{\boldsymbol{A}}(E;X)} \eqsim ||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$$

follow from Theorem 2.3.17. Combining the inclusion

$$YL^{\boldsymbol{A}}(E;X) \overset{(2.7)}{\hookrightarrow} E_0(X)$$

with the estimate corresponding to the implication (i)⇒(iii) in Theorem 2.4.4 gives

$$||f||_{E_0(X)} + ||(d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} \lesssim ||f||_{YL^{\boldsymbol{A}}(E;X)}.$$

As it clearly holds that

$$||I^{\boldsymbol{A}}_{M,n}(f)||_X \leq d^{\boldsymbol{A},\boldsymbol{p}}_{M,n}(f), \qquad n \in \mathbb{N},$$

it remains to be shown that

$$||f||_{Y^{\boldsymbol{A}}(E;X)} \lesssim ||f||_{E_0(X)} + ||(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)}. \tag{2.40}$$

Put $K := 1_{B^{\boldsymbol{A}}}(0,1)$ and $K^{\Delta^M} := \sum_{l=0}^{M-1}(-1)^l\binom{M}{l}\widetilde{K}_{[M-l]^{-1}}$, where $\widetilde{K}_t := t^d K(-t\,\cdot)$ for $t \in (0,\infty)$. Furthermore, put

$$K^{\boldsymbol{A}}_M(t,f) := t^{-\mathrm{tr}(\boldsymbol{A}^{\oplus})}K^{\Delta^M}(\boldsymbol{A}_{t^{-1}}\,\cdot) * f + (-1)^M\hat{K}(0)f, \qquad t \in (0,\infty).$$

Note that

$$I^{\boldsymbol{A}}_{M,n}(f) = K^{\boldsymbol{A}}_M(2^{-n},f), \qquad n \in \mathbb{N}. \tag{2.41}$$

As $\widehat{K^{\Delta^M}}(0) = \sum_{l=0}^{M-1}(-1)^l\binom{M}{l}\hat{K}(0) = (-1)^{M+1}\hat{K}(0) \neq 0$, we can pick $\varepsilon, c \in (0,\infty)$ such that K^{Δ^m} fulfills the Tauberian condition

$$|\mathscr{F}K^{\Delta^m}(\xi)| \geq c, \qquad \xi \in \mathbb{R}^d, \frac{\varepsilon}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\varepsilon.$$

So there exists $N \in \mathbb{N}$ such that $k := 2^{N\mathrm{tr}(\boldsymbol{A}^{\oplus})}K^{\Delta^m}(\boldsymbol{A}_{2^N}\,\cdot) - K^{\Delta^m} \in L_{1,\mathrm{c}}(\mathbb{R}^d)$ satisfies

$$|\hat{k}(\xi)| \geq \frac{c}{2} > 0, \qquad \xi \in \mathbb{R}^d, \frac{\delta}{2} < \rho_{\boldsymbol{A}}(\xi) < 2\delta,$$

for $\delta := 2^N\varepsilon > 0$. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{\boldsymbol{A}}(\mathbb{R}^d)$ be such that $\mathrm{supp}\,\hat{\varphi}_1 \subset \{\xi : 2\varepsilon \leq \rho_{\boldsymbol{A}}(\xi)\}$ (see Definition 2.3.13). Let $(k_n)_{n\in\mathbb{N}}$ be defined by $k_n := 2^{n\mathrm{tr}(\boldsymbol{A}^{\oplus})}k(\boldsymbol{A}_{2^n}\,\cdot)$. Then, by construction,

$$k_n * f = K^{\boldsymbol{A}}_M(2^{-(n+N)},f) - K^{\boldsymbol{A}}_M(2^{-n},f) \overset{(2.41)}{=} I^{\boldsymbol{A}}_{M,n+N}(f) - I^{\boldsymbol{A}}_{M,n}(f), \qquad n \in \mathbb{N}.$$

An application of Lemma 2.4.21 thus yields that

$$\begin{aligned}||(\varphi_n * f)_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} &\lesssim ||(k_n * f)_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} \\ &\lesssim ||(I^{\boldsymbol{A}}_{M,n+N}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} + ||(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)} \\ &\lesssim (2^{-\varepsilon_+ N}+1)||(I^{\boldsymbol{A}}_{M,n}(f))_{n\geq 1}||_{E(\mathbb{N}_{\geq 1};X)}. \end{aligned} \tag{2.42}$$

As $||\varphi_0 * f||_X \lesssim M^{\boldsymbol{A}}(||f||_X)$, it furthermore holds that

$$||\varphi_0 * f||_{E_0(X)} \lesssim ||f||_{E_0(X)}. \tag{2.43}$$

A combination of Proposition 2.3.14, (2.42) and (2.43) finally gives (2.40). □

Proof of Proposition 2.4.8. Using the the estimate corresponding to the implication (i) ⇒ (ii) in Theorem 2.4.4, the first estimate can be obtained as in the proof of the implication (ii) ⇒ (iii) in Theorem 2.4.4. The second estimate can be obained similarly, replacing Theorem 2.4.4 by Theorem 2.4.5. □

2.5. AN INTERSECTION REPRESENTATION

Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ with $\varepsilon_+,\varepsilon_- > 0$. Let J be a nonempty subset of $\{1,\dots,\ell\}$, say $J = \{j_1,\dots,j_k\}$ with $1 \leq j_1 \leq \dots \leq j_k \leq \ell$. Put $\mathpzc{d}_J = (\mathpzc{d}_{j_1},\dots,\mathpzc{d}_{j_k})$, $d_J := |\mathpzc{d}_J|_1$ $\boldsymbol{A}_J := (A_{j_1},\dots,A_{j_k})$, $\boldsymbol{r}_J := (r_{j_1},\dots,r_{j_k})$ and

$$(S_J,\mathscr{A}_J,\mu_J) := (\mathbb{R}^{d-d_J},\mathscr{B}(\mathbb{R}^{d-d_J}),\lambda^{d-d_J}) \otimes (S,\mathscr{A},\mu)$$

Furthermore, define $E_{[\mathpzc{d};J]}$ as the quasi-Banach space E viewed as quasi-Banach function space on the measure space $\mathbb{R}^{d_J} \times \mathbb{N} \times S_J$. Then

$$E_{[\mathpzc{d};J]} \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A}_J,\boldsymbol{r}_J,(S_J,\mathscr{A}_J,\mu_J))$$

By Remark 2.3.10,

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S;L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X)).$$

In the same way,

$$\widetilde{YL}^{\boldsymbol{A}_J}(E_{[\mathpzc{d};J]};X) \hookrightarrow E^{\boldsymbol{A}}_{\otimes}(B^{1,w_{\boldsymbol{A},\boldsymbol{r}}}_{\boldsymbol{A}}(X)) \hookrightarrow L_0(S;L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X)),$$

In particular, it makes sense to compare $\widetilde{YL}^{\boldsymbol{A}_J}(E_{[\mathpzc{d};J]};X)$ with $\widetilde{YL}^{\boldsymbol{A}}(E;X)$.

Theorem 2.5.1. *Let* $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\boldsymbol{r},(S,\mathscr{A},\mu))$ *with* $\varepsilon_+,\varepsilon_- > 0$. *Let* $\{J_1,\dots,J_L\}$ *be a partition of* $\{1,\dots,\ell\}$.

(i) There is the estimate

$$||f||_{\widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[\mathpzc{d};J_l]};X)} \leq ||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}, \qquad l \in \{1,\dots,L\},$$

for all $f \in L_0(S;L_{\boldsymbol{r},\mathpzc{d},\mathrm{loc}}(\mathbb{R}^d;X))$.

(ii) There is the estimate

$$||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)} \lesssim \sum_{l=1}^{L} ||f||_{\widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]};X)}$$

for all $f \in L_0(S; L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X))$ *of the form* $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ *with* $(S_i)_{i\in I} \subset \mathscr{A}$ *a countable family of mutually disjoint sets and* $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)$.

In particular, in case $(S,\mathscr{A},\mu)$ *is atomic,*

$$\widetilde{YL}^{\boldsymbol{A}}(E;X) = \bigcap_{l=1}^{L} \widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]};X)$$

with an equivalence of quasi-norms.

Proof. Let us start with (i). Fix $l \in \{1,\dots,L\}$ and write $J := J_l$. Let $f \in \widetilde{YL}^{\boldsymbol{A}}(E;X)$. Let $\varepsilon > 0$. Choose $(g_n)_n$ and $(f_{x^*,n})_{(x^*,n)}$ as in Definition 2.3.9 with $||(g_n)_n||_E \le (1+\varepsilon)||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}$. As $f_{x^*,n} \in L_0(S;\mathscr{S}'(\mathbb{R}^d))$ with $\operatorname{supp} \hat{f}_{x^*,n} \subset B^{\boldsymbol{A}}(0,2^{n+1})$, we can naturally view $f_{x^*,n}$ as an element of $L_0(S_J;\mathscr{S}'(\mathbb{R}^{d-d_J}))$ with $\operatorname{supp} \hat{f}_{x^*,n} \subset B^{\boldsymbol{A}_J}(0,2^{n+1})$. Since

$$L_0(S;L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d)) \hookrightarrow L_0(S_J;L_{\boldsymbol{r}_J,d_J,\mathrm{loc}}(\mathbb{R}^{d_J})),$$

it follows that $f \in \widetilde{YL}^{\boldsymbol{A}_J}(E_{[d;J]};X)$ with

$$||f||_{\widetilde{YL}^{\boldsymbol{A}_J}(E_{[d;J]};X)} \lesssim ||(g_n)_n||_{E_{[d;J]}} = ||(g_n)_n||_E \le (1+\varepsilon)||f||_{\widetilde{YL}^{\boldsymbol{A}}(E;X)}.$$

Let us next treat (ii). We may without loss of generality assume that $L=\ell$ and that $J_l = \{l\}$ for each $l \in \{1,\dots,\ell\}$. We will write $E_{[d;j]} = E_{[d;\{j\}]}$.

Let $f \in \bigcap_{j=1}^{\ell} \widetilde{YL}^{A_j}(E_{[d;j]};X)$ be of the form $f = \sum_{i\in I} 1_{S_i} \otimes f^{[i]}$ with $(S_i)_{i\in I} \subset \mathscr{A}$ a countable family of mutually disjoint sets and $(f^{[i]})_{i\in I} \in L_{\boldsymbol{r},\mathscr{d},\mathrm{loc}}(\mathbb{R}^d;X)$. In order to establish the desired inequality, we will combine the estimate corresponding to the implication (III) ⇒ (I) from Theorem 2.4.5 for the space $\widetilde{YL}^{\boldsymbol{A}}(E;X)$ with the estimates from Proposition 2.4.8 for each of the spaces $\widetilde{YL}^{A_j}(E_{[d;j]};X)$. To this end, pick $M \in \mathbb{N}$ with $M\lambda^{\boldsymbol{A}}_{\min} > \varepsilon_-$. Now, let us define $(g_{x^*,n})_{(x^*,n)\in X^*\times\mathbb{N}}$ and $(g_{c,x^*,n,j})_{(x^*,n)\in X^*\times\mathbb{N}}$, with $j \in \{1,\dots,\ell\}$ and $c \in \mathbb{R}$, by

$$g_{x^*,n} := \begin{cases} d^{\boldsymbol{A},\boldsymbol{r}}_{0,x^*,0}(f), & n=0, \\ d^{\boldsymbol{A},\boldsymbol{r}}_{\ell M,x^*,n}(f), & n \ge 1, \end{cases}$$

and

$$g_{c,x^*,n,j} := \begin{cases} d^{[d;j],A_j,r_j}_{0,x^*,0}(f), & n=0, \\ d^{[d;j],A_j,r_j}_{M,c,x^*,n}(f), & n \ge 1, \end{cases}$$

where the notation is as in Theorem 2.4.5 and Proposition 2.4.8.

For $n=0$ we have

$$g_{x^*,0} = d^{\boldsymbol{A},\boldsymbol{r}}_{0,x^*,0}(f) \lesssim \left[\bigcirc_{i=2}^{\ell} M^{[d;i],A_i}_{r_i}\right]\left(d^{[d;1],A_1,r_1}_{0,x^*,0}(f)\right)$$

$$\leq M_{\boldsymbol{r}}^{\boldsymbol{A}}\big[d_{0,x^*,0}^{[d;1],A_1,r_1}(f)\big] = M_{\boldsymbol{r}}^{\boldsymbol{A}}\big[g_{c,x^*,0,1}\big], \qquad c \in \mathbb{R}. \tag{2.44}$$

Now let $n \geq 1$. We will use the following elementary fact (cf. [236, 4.16]): there exist $C \in (0,\infty)$, $K \in \mathbb{N}$ and $\{c_j^{[k]}\}_{j=1,\ldots,\ell;k=0,\ldots,K} \subset \mathbb{R}$ such that

$$|\Delta_z^{\ell M} h(x)| \leq C \sum_{k=0}^{K}\sum_{j=1}^{\ell}\Big|\Delta^M_{\iota_{[d;j]}z_j} h(x + \sum_{i=1}^{\ell} c_i^{[k]}\iota_{[d;i]}z_i)\Big|$$

for all $h \in L_0(\mathbb{R}^d)$. Applying this pointwise in S to $\langle f, x^*\rangle$, we find that

$$\begin{aligned}
g_{x^*,n} = d^{\boldsymbol{A},\boldsymbol{r}}_{\ell M,x^*,n}(f) &= 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}\big|\big|z \mapsto \Delta_z^{\ell M}\langle f,x^*\rangle\big|\big|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2^{-n}))} \\
&\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} 2^{n\mathrm{tr}(\boldsymbol{A})\cdot\boldsymbol{r}^{-1}}\Big|\Big|z \mapsto \Big[\prod_{i=1}^{\ell} L_{c_i^{[k]}\iota_{[d;i]}z_i}\Big]\Delta^M_{\iota_{[d;j]}z_j}\langle f,x^*\rangle\Big|\Big|_{L_{\boldsymbol{r},d}(B^{\boldsymbol{A}}(0,2^{-n}))} \\
&\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} 2^{n\mathrm{tr}(A_j)/r_j}\big[\bigcirc_{i\neq j} M_{r_i}^{[d;i],A_i}\big]\Big[\Big|\Big|z_j \mapsto L_{c_j^{[k]}\iota_{[d;j]}z_j}\Delta^M_{\iota_{[d;j]}z_j}\langle f,x^*\rangle\Big|\Big|_{L_{r_j}(B^{A_j}(0,2^{-n}))}\Big] \\
&\leq \sum_{k=0}^{K}\sum_{j=1}^{\ell} M_{\boldsymbol{r}}^{\boldsymbol{A}}\Big[2^{n\mathrm{tr}(A_j)/r_j}\Big|\Big|z_j \mapsto L_{c_j^{[k]}\iota_{[d;j]}z_j}\Delta^M_{\iota_{[d;j]}z_j}\langle f,x^*\rangle\Big|\Big|_{L_{r_j}(B^{A_j}(0,2^{-n}))}\Big] \\
&= \sum_{k=0}^{K}\sum_{j=1}^{\ell} M_{\boldsymbol{r}}^{\boldsymbol{A}}\Big[d^{[d;j],A_j,r_j}_{M,c_j^{[k]},x^*,n}(f)\Big] = \sum_{k=0}^{K}\sum_{j=1}^{\ell} M_{\boldsymbol{r}}^{\boldsymbol{A}}\Big[g_{c_j^{[k]},x^*,n,j}\Big].
\end{aligned} \tag{2.45}$$

A combination of (2.44) and (2.45) gives

$$g_{x^*,n} \lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell} M_{\boldsymbol{r}}^{\boldsymbol{A}}\Big[d^{[d;j],A_j,r_j}_{M,c_j^{[k]},x^*,n}(f)\Big] = \sum_{k=0}^{K}\sum_{j=1}^{\ell} M_{\boldsymbol{r}}^{\boldsymbol{A}}\Big[g_{c_j^{[k]},x^*,n,j}\Big]$$

for all $(x^*, n) \in X^* \times \mathbb{N}$. Therefore,

$$\begin{aligned}
\big|\big|\{g_{x^*,n}\}_{(x^*,n)}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} &\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell}\Big|\Big|\{M_{\boldsymbol{r}}^{\boldsymbol{A}}[g_{c_j^{[k]},x^*,n,j}]\}_{(x^*,n)}\Big|\Big|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \\
&\lesssim \sum_{k=0}^{K}\sum_{j=1}^{\ell}\big|\big|\{g_{c_j^{[k]},x^*,n,j}\}_{(x^*,n)}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E)} \\
&= \sum_{k=0}^{K}\sum_{j=1}^{\ell}\big|\big|\{g_{c_j^{[k]},x^*,n,j}\}_{(x^*,n)}\big|\big|_{\mathscr{F}_{\mathrm{M}}(X^*;E_{[d;j]})}.
\end{aligned}$$

The desired result now follows from a combination of Theorem 2.4.5 and Proposition 2.4.8.

□

As an immediate corollary to Theorems 2.3.17 and 2.5.1 we have:

Corollary 2.5.2. *Let $E \in \mathscr{S}(\varepsilon_+, \varepsilon_-, \boldsymbol{A}, \boldsymbol{r}, (S, \mathscr{A}, \mu))$ with $\varepsilon_+, \varepsilon_- > 0$ and $(S, \mathscr{A}, \mu)$ atomic. Let $\{J_1, \ldots, J_L\}$ be a partition of $\{1, \ldots, \ell\}$. If $\varepsilon_+ > \mathrm{tr}(\boldsymbol{A}) \cdot (\boldsymbol{r}^{-1} - \mathbf{1})_+$, then*

$$Y^{\boldsymbol{A}}(E;X) = YL^{\boldsymbol{A}}(E;X) = \widetilde{YL}^{\boldsymbol{A}}(E;X) = \bigcap_{l=1}^{L}\widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[d;J_l]};X)$$

$$= \bigcap_{l=1}^{L} YL^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X)$$

with an equivalence of quasi-norms.

Theorem 2.5.3. *Let $E \in \mathscr{S}(\varepsilon_+,\varepsilon_-,\boldsymbol{A},\mathbf{1},(S,\mathscr{A},\mu))$ with $\varepsilon_+,\varepsilon_- > 0$. Let $\{J_1,\ldots,J_L\}$ be a partition of $\{1,\ldots,\ell\}$. Then*

$$Y^{\boldsymbol{A}}(E;X) = YL^{\boldsymbol{A}}(E;X) = \widetilde{YL}^{\boldsymbol{A}}(E;X) = \bigcap_{l=1}^{L} \widetilde{YL}^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X)$$
$$= \bigcap_{l=1}^{L} YL^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X)$$

with an equivalence of quasi-norms.

Proof. In view of Theorem 2.3.17, this can be proved in exactly the same way as Theorem 2.5.1, using Theorem 2.4.7 instead of Theorem 2.4.5. □

Example 2.5.4. In light of Example 2.3.15, the intersection representation

$$Y^{\boldsymbol{A}}(E;X) = \bigcap_{l=1}^{L} Y^{\boldsymbol{A}_{J_l}}(E_{[\mathscr{d};J_l]};X)$$

from Corollary 2.5.2 and Theorem 2.5.3 extends the well-known Fubini property for the classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ (see [236, Section 4] and the references given therein). It also covers Theorem 2.1.1 and thereby (2.4), the intersection representation from [63, Proposition 3.23]. The intersection representation [156, Proposition 5.2.38] for anisotropic weighted mixed-norm Triebel-Lizorkin is a special case as well. Furthermore, it suggests an operator sum theorem for generalized Triebel-Lizorkin spaces in the sense of [148].

2.A. SOME MAXIMAL FUNCTION INEQUALITIES

Suppose that $\mathbb{R}^d$ is $\mathscr{d}$-decomposed with $\mathscr{d} \in (\mathbb{Z}_{\geq 1})^\ell$ and let $\boldsymbol{A} = (A_1,\ldots,A_\ell)$ be a $\mathscr{d}$-anisotropy.

Lemma 2.A.1 (Anisotropic Peetre's inequality)**.** *Let X be a Banach space, $\boldsymbol{r} \in (0,\infty)^\ell$, $K \subset \mathbb{R}^d$ a compact set and $N \in \mathbb{N}$. For all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq N$ and $f \in \mathscr{S}'(\mathbb{R}^d;X)$ with* supp $(\hat{f}) \subset K$*, there is the pointwise estimate*

$$\sup_{z\in\mathbb{R}^d} \frac{||D^\alpha f(x+z)||_X}{\prod_{j=1}^{\ell}(1+\rho_{A_j}(z_j))^{\mathrm{tr}(A_j)/r_j}} \lesssim \sup_{z\in\mathbb{R}^d} \frac{||f(x+z)||_X}{\prod_{j=1}^{\ell}(1+\rho_{A_j}(z_j))^{\mathrm{tr}(A_j)/r_j}}$$
$$\lesssim \left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(||f||_X)\right](x), \qquad x \in \mathbb{R}^d.$$

Proof. This can be obtained by combining the proof of [131, Proposition 3.11] (which is actually only a reference to [221, Theorem 1.6.4], the two-dimensional case that easily extends to arbitrary dimensions) for the case $\mathscr{d} = \mathbf{1}$ with the proof of [35, Lemma 3.4] for the case $\ell = 1$. □

For $f \in \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^d;X)$, $\boldsymbol{r} \in (0,\infty)^\ell$, $\boldsymbol{R} \in (0,\infty)^\ell$ we define the *maximal fuction of Peetre-Fefferman-Stein type* $f^*(\boldsymbol{A},\boldsymbol{r},\boldsymbol{R};\cdot)$ by

$$f^*(\boldsymbol{A},\boldsymbol{r},\boldsymbol{R};x) := \sup_{z\in\mathbb{R}^d} \frac{\|f(x+z)\|_X}{\prod_{j=1}^{\ell}(1+R_j\rho_{A_j}(z_j))^{\mathrm{tr}(A_j)/r_j}}.$$

Corollary 2.A.2. *Let X be a Banach space and $\boldsymbol{r} \in (0,\infty)^\ell$. For all $f \in \mathscr{S}'(\mathbb{R}^d;X)$ and $\boldsymbol{R} \in (0,\infty)^\ell$ with* $\operatorname{supp}(\hat{f}) \subset B^{\boldsymbol{A}}(0,\boldsymbol{R})$*, there is the pointwise estimate*

$$f^*(\boldsymbol{A},\boldsymbol{r},\boldsymbol{R};x) \lesssim_{\boldsymbol{A},\boldsymbol{r}} \left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|f\|_X)\right](x), \qquad x \in \mathbb{R}^d.$$

Proof. By a dilation argument it suffices to consider the case $\boldsymbol{R} = \boldsymbol{1}$, which is contained in Lemma 2.A.1. □

Lemma 2.A.3. *Let X and Y be Banach spaces. For all $(M_n)_{n\in\mathbb{N}} \subset \mathscr{F}L^1(\mathbb{R}^d;\mathscr{B}(X,Y))$, $(\boldsymbol{R}^{(n)})_{n\in\mathbb{N}} \subset (0,\infty)^\ell$, $c \in [1,\infty)$ and $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^d;X)$, there is the pointwise estimate*

$$\begin{aligned}
&\left\|[\mathscr{F}(M_n\hat{f}_n)](x)\right\|_Y \\
&\quad\lesssim c^{\sum_{j=1}^{\ell}\lambda_j} \sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} \|\check{M}_n(\boldsymbol{A}_{\boldsymbol{R}^{(n)}}y)\|_{\mathscr{B}(X,Y)} \prod_{j=1}^{\ell}(1+\rho_{A_j}(y_j))^{\lambda_j}\,dy \\
&\qquad\cdot \sup_{z\in\mathbb{R}^d} \frac{\|f_n(x+z)\|_X}{\prod_{j=1}^{\ell}(1+cR_j^{(n)}\rho_{A_j}(y_j))^{\lambda_j}}.
\end{aligned}$$

Proof. This can be shown as the pointwise estimate in the proof of [156, Proposition 3.4.8], which was in turn based on [182, Proposition 2.4]. □

The following proposition is an extension of [131, Proposition 3.13] to our setting, which is in turn a version of the pointwise estimate of pseudo-differential operators due to Marschall [170]. In order to state it, we first need to introduce the anisotropic mixed-norm homogeneous Besov space $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)$.

Let Z be a Banach space, $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Fix $(\phi_k)_{k\in\mathbb{Z}} \subset \mathscr{S}(\mathbb{R}^d)$ that satisfies $\hat{\phi}_k = \hat{\psi}(\boldsymbol{A}_{2^{-k}}\cdot) - \hat{\psi}(\boldsymbol{A}_{2^{-(k+)}}\cdot)$ for some $\psi \in \mathscr{F}C^\infty_c(\mathbb{R}^d)$ with $\hat{\psi} \equiv 1$ on a neighbourhood of 0. Then $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)$ is defined as the space of all $f \in [\mathscr{S}'/\mathscr{P}](\mathbb{R}^d;Z)$ for which

$$\|f\|_{\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}(\mathbb{R}^d;Z)} := \left\|(2^{sk}\phi_k * f)_{k\in\mathbb{Z}}\right\|_{\ell_q(\mathbb{Z})[L_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d)](Z)} < \infty.$$

Proposition 2.A.4. *Let X and Y be Banach spaces and $\boldsymbol{r} \in (0,1]^\ell$. Put $\tau := \boldsymbol{r}_{\min} \in (0,1]$. For all $b \in \mathscr{S}(\mathbb{R}^d;\mathscr{B}(X,Y))$, $u \in \mathscr{S}'(\mathbb{R}^d;X)$, $c \in (0,\infty)$ and $R \in [1,\infty)$ with* $\operatorname{supp}(b) \subset B^{\boldsymbol{A}}(0,c)$ *and* $\operatorname{supp}(\hat{u}) \subset B^{\boldsymbol{A}}(0,cR)$*, there is the pointwise estimate*

$$\|b(D)u(x)\|_Y \lesssim_{\boldsymbol{A},\boldsymbol{r}} (cR)^{\sum_{j=1}^{\ell}\mathrm{tr}(A_j)(\frac{1}{r_j}-1)} \|b\|_{\dot{B}^{\sum_{j=1}^{\ell}\mathrm{tr}(A_j)\frac{1}{r_j},\boldsymbol{A}}_{1,\tau}(\mathbb{R}^d;\mathscr{B}(X,Y))} \left[M^{\boldsymbol{A}}_{\boldsymbol{r}}(\|u\|_X)\right](x)$$

for each $x \in \mathbb{R}^d$.

In the proof of Proposition 2.A.4 we will use the following lemma.

Lemma 2.A.5. *Let X be a Banach space and $\boldsymbol{p},\boldsymbol{q} \in (0,\infty)^\ell$ with $\boldsymbol{p} \leq \boldsymbol{q}$. For every $f \in \mathscr{S}'(\mathbb{R}^d;X)$ and $\boldsymbol{R} \in (0,\infty)^\ell$ with* $\operatorname{supp}(\hat{f}) \subset B^{\boldsymbol{A}}(0,\boldsymbol{R})$,

$$||f||_{L_{\boldsymbol{q},\mathpzc{d}}(\mathbb{R}^d)} \lesssim_{\boldsymbol{p},\boldsymbol{q},\mathpzc{d}} \prod_{j=1}^{\ell} R_j^{\operatorname{tr}(A_j)(\frac{1}{p_j}-\frac{1}{q_j};X)} ||f||_{L_{\boldsymbol{p},\mathpzc{d}}(\mathbb{R}^d;X)}$$

Proof. By a scaling argument we may restrict ourselves to the case $\boldsymbol{R} = \mathbf{1}$. Now pick $\phi \in \mathscr{S}(\mathbb{R}^d)$ with $\hat{\phi} \equiv 1$ on $B^{\boldsymbol{A}}(0,\mathbf{1})$. Then $f = \phi * f$ and the desired inequality follows from an iterated use of Young's inequality for convolutions. □

Proof of Proposition 2.A.4. It holds that

$$b(D)u(x) = \int_{\mathbb{R}^d} \check{b}(y)u(x-y)\,dy, \qquad x \in \mathbb{R}^d.$$

For fixed $x \in \mathbb{R}^d$, by the quasi-triangle inequality for $\rho_{\boldsymbol{A}}$ (with constant $c_{\boldsymbol{A}}$),

$$\operatorname{supp}(\mathscr{F}[y \mapsto \check{b}(y)u(x-y)]) \subset B_{\boldsymbol{A}}(0,c) + B_{\boldsymbol{A}}(0,cR) \subset B_{\boldsymbol{A}}(0,c_{\boldsymbol{A}}(R+1)c).$$

Therefore,

$$\begin{aligned} ||b(D)u(x)||_Y &\leq ||y \mapsto \check{b}(y)u(x-y)||_{L_1(\mathbb{R}^d)} \\ &\lesssim (c_{\boldsymbol{A}}(R+1)c)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)} ||y \mapsto \check{b}(y)u(x-y)||_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)} \\ &\lesssim (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)} ||y \mapsto \check{b}(y)u(x-y)||_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)}, \end{aligned} \tag{2.46}$$

where we used Lemma 2.A.5 for the second estimate.

Let $(\phi_k)_{k\in\mathbb{Z}}$ be as in the definition of the anisotropic homogeneous Besov space $\dot{B}^{s,\boldsymbol{A}}_{\boldsymbol{p},q}$ as given preceding the proposition. Then $\sum_{k=-\infty}^{\infty}\hat{\phi}_k(-\,\cdot) = 1$ on $\mathbb{R}^d \setminus \{0\}$, so that

$$||\check{b}\,u(x-\cdot)||_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)} \leq \left(\sum_{k\in\mathbb{Z}} ||\hat{\phi}_k(-\,\cdot)\,\check{b}\,u(x-\cdot)||^{\tau}_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)}\right)^{1/\tau}. \tag{2.47}$$

Since

$$\begin{aligned} \sup_{y\in\mathbb{R}^d} ||\hat{\phi}_k(-y)\check{b}(y)||_{\mathscr{B}(X,Y)} &\leq ||\mathscr{F}^{-1}[\hat{\phi}_k(-\,\cdot)\,\check{b}]||_{L_1(\mathbb{R}^d;\mathscr{B}(X,Y))} \\ &= (2\pi)^{-d}||\mathscr{F}^{-1}[\hat{\phi}_k\hat{b}]||_{L_1(\mathbb{R}^d;\mathscr{B}(X,Y))} \end{aligned}$$

and $\operatorname{supp}(\hat{\phi}_k) \subset B^{\boldsymbol{A}}(0,2^{k+1})$, it follows from a combination of (2.46) and (2.47) that

$$||b(D)u(x)||_Y \lesssim (Rc)^{\sum_{j=1}^{\ell}\operatorname{tr}(A_j)(\frac{1}{r_j}-1)} \left(\sum_{k\in\mathbb{Z}} ||\hat{\phi}_k(-\,\cdot)\,\check{b}\,u(x-\,\cdot)||^{\tau}_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)}\right)^{1/\tau}$$

$$\lesssim (Rc)^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)(\frac{1}{r_j}-1)} \left(\sum_{k \in \mathbb{Z}} \left[2^{k \sum_{j=1}^{\ell} \operatorname{tr}(A_j) \frac{1}{r_j}} ||\mathscr{F}^{-1}[\hat{\phi}_k \hat{b}]||_{L_1} \right]^{\tau} \right)^{1/\tau}$$

$$\sup_{k \in \mathbb{Z}} 2^{-(k+1)\operatorname{tr}(A_j)\frac{1}{r_j}} ||1_{B^{A}(0,2^{k+1})} u(x-\cdot)||_{L_{\boldsymbol{r},\mathpzc{d}}(\mathbb{R}^d)}$$

$$\leq (Rc)^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)(\frac{1}{r_j}-1)} ||b||_{\dot{B}_{1,\tau}^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)\frac{1}{r_j}, \boldsymbol{A}}(\mathbb{R}^d;\mathscr{B}(X,Y))} \left[M_{\boldsymbol{r}}^{\boldsymbol{A}}(||u||_X) \right](x).$$

□

Corollary 2.A.6. *Let X and Y be Banach spaces, $\boldsymbol{r} \in (0,1]^{\ell}$ and $\psi \in C_c^{\infty}(\mathbb{R}^d;\mathscr{B}(X,Y))$. Put $\psi_k := \psi(\boldsymbol{A}_{2^{-k}} \cdot)$ for each $k \in \mathbb{N}$. Then, for all $(f_k)_{k\in\mathbb{N}} \subset \mathscr{S}'(\mathbb{R}^d;X)$ with* supp $\hat{f}_k \subset B^{\boldsymbol{A}}(0, r2^k)$ *for some $r \in [1,\infty)$, there is the pointwise estimate*

$$||\psi_k(D) f_k(x)||_Y \lesssim r^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)(\frac{1}{r_j}-1)} \left[M_{\boldsymbol{r}}^{\boldsymbol{A}}(||f_k||_X) \right](x), \qquad x \in \mathbb{R}^d.$$

Proof. Let $c \in [1,\infty)$ be such that supp $(\psi) \subset B^{\boldsymbol{A}}(0,c)$. Applying Proposition 2.A.4 to $b = \psi_k$, $u = f_k$ and $R = r2^k$, we find that

$$||\psi_k(D) f_k(x)||_Y \lesssim (cr2^k)^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)(\frac{1}{r_j}-1)} ||\psi_k||_{\dot{B}_{1,\tau}^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)\frac{1}{r_j}, \boldsymbol{A}}(\mathbb{R}^d)} \left[M_{\boldsymbol{r}}^{\boldsymbol{A}}(||f_k||_X) \right](x).$$

Observing that

$$||\psi_k||_{\dot{B}_{1,\tau}^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)\frac{1}{r_j}, \boldsymbol{A}}(\mathbb{R}^d)} = 2^{-k\sum_{j=1}^{\ell} \operatorname{tr}(A_j)(\frac{1}{r_j}-1)} ||\psi||_{\dot{B}_{1,\tau}^{\sum_{j=1}^{\ell} \operatorname{tr}(A_j)\frac{1}{r_j}, \boldsymbol{A}}(\mathbb{R}^d)},$$

we obtain the desired estimate. □

3

DIFFERENCE NORMS FOR VECTOR-VALUED BESSEL POTENTIAL SPACES

This chapter is based on the paper:

> *N. Lindemulder. Difference norms for vector-valued Bessel potential spaces with applications to pointwise multipliers. J. Funct. Anal., 272(4): 1435–1476, 2017.*

In this chapter we prove a randomized difference norm characterization for Bessel potential spaces with values in UMD Banach spaces. The main ingredients are $\mathscr{R}$-boundedness results for Fourier multiplier operators, which are of independent interest. As an application we characterize the pointwise multiplier property of the indicator function of the half-space on these spaces. All results are proved in the setting of weighted spaces.

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3.1. INTRODUCTION

Vector-valued Sobolev and Bessel potential spaces are important in the L^p-approach to abstract evolution and integral equations, both in the deterministic setting (see e.g. [5, 195, 251]) and in the stochastic setting (see e.g. [69, 191, 192]). Here a central role is played by the Banach spaces that have the so-called UMD property (unconditionality of martingale differences); see Section 3.2.1 and the remarks below. The class of Banach spaces that have UMD includes all Hilbert spaces, L^p-spaces with $p \in (1,\infty)$ and the reflexive Sobolev spaces, Triebel-Lizorkin spaces, Besov spaces and Orlicz spaces.

Let X be a Banach space, $s \in \mathbb{R}$ and $p \in (1,\infty)$. The Bessel potential space $H^s_p(\mathbb{R}^d;X)$ is defined in the usual Fourier analytic way via the Bessel potential operator $\mathscr{J}_s = (I-\Delta)^{s/2}$ based on the Lebesgue-Bochner space $L^p(\mathbb{R}^d;X)$; see Section 3.2.3. If X has UMD and $k \in \mathbb{N}$, then we have $H^k_p(\mathbb{R}^d;X) = W^k(\mathbb{R}^d;X)$, where $W^k_p(\mathbb{R}^d;X)$ denotes the k-th order X-valued Sobolev space on $\mathbb{R}^d$ with integrability parameter p; see [126], which also contains some converse results in this direction. Furthermore, if X has UMD and $s = k+\theta$ with $k \in \mathbb{N}$ and $\theta \in [0,1)$, then $H^s_p(\mathbb{R}^d;X)$ can be realized as the complex interpolation space

$$H^s_p(\mathbb{R}^d;X) = [W^k_p(\mathbb{R}^d;X), W^{k+1}_p(\mathbb{R}^d;X)]_\theta.$$

In the scalar-valued case $X = \mathbb{C}$, Strichartz [230] characterized the Bessel potential space $H^s_p(\mathbb{R}^d) = H^s_p(\mathbb{R}^d;\mathbb{C})$, with $s \in (0,1)$ and $p \in (1,\infty)$, by means of differences. The characterization says that, for every $f \in L^p(\mathbb{R}^d;\mathbb{C})$, there is the equivalence of extended norms

$$||f||_{H^s_p(\mathbb{R}^d;\mathbb{C})} \eqsim ||f||_{L^p(\mathbb{R}^d;\mathbb{C})} + \left|\left|\left(\int_0^\infty t^{-2s}\Big[t^{-d}\int_{B(0,t)} ||\Delta_h f||_{\mathbb{C}}\, dh\Big]^2 \frac{dt}{t}\right)^{1/2}\right|\right|_{L^p(\mathbb{R}^d)}, \tag{3.1}$$

where $\Delta_h f = f(\cdot+h)-f$ for each $h \in \mathbb{R}^d$. This extends to Hilbert spaces [242, Section 6.1]. In fact, given a Banach space X, the X-valued version of (3.1) is valid if and only if X is isomorphic to a Hilbert space. Indeed, the X-valued version of the right-hand side of (3.1) defines an extended norm on $L^p(\mathbb{R}^d;X)$ which characterizes the Triebel-Lizorkin space $F^s_{p,2}(\mathbb{R}^d;X)$ [220, Section 2.3]. But the identity

$$H^s_p(\mathbb{R}^d;X) = F^s_{p,2}(\mathbb{R}^d;X), \tag{3.2}$$

i.e. the classical Littlewood-Paley decomposition for Bessel potential spaces, holds true if and only if X is isomorphic to a Hilbert space [112, 212]. However, if X is a Banach space with UMD, then one can replace (3.2) with a randomized Littlewood-Paley decomposition [187] (see (3.13)), an idea which for the case $s = 0$ originally goes back to Bourgain [30] and McConnell [174]. In [187] this was used to investigate the pointwise multiplier property of the indicator function of the half-space on UMD-valued Bessel potential spaces. The randomized Littlewood-Paley decomposition will also play a crucial role in this paper to obtain a randomized difference norm characterization for UMD-valued Bessel potential spaces; see Theorem 3.1.1.

Since the early 1980's, randomization and martingale techniques have played a fundamental role in Banach space-valued analysis (cf. e.g. [43, 48, 59, 124, 126, 127, 135, 149, 190, 211]). In particular, in Banach space-valued harmonic analysis and Banach space-valued stochastic analysis, a central role is played by the UMD spaces. Indeed, many classical Hilbert space-valued results from both areas have been extended to the UMD-valued case, and many of these extensions in fact characterize the UMD property. In vector-valued harmonic analysis, (one of) the first major breakthrough(s) is the deep result due to Bourgain [28] and Burkholder [41] that a Banach space X has UMD if and only if it is of class $\mathscr{HT}$, i.e. the Hilbert transform has a bounded extension to $L^p(\mathbb{R};X)$ for some/all $p \in (1,\infty)$. As another major breakthrough we would like to mention the work of Weis [244] on operator-valued Fourier multipliers on UMD-valued L^p-spaces ($p \in (1,\infty)$) with an application to the maximal L^p-regularity problem for abstract parabolic evolution equations. A central notion in this work is the $\mathscr{R}$-boundedness of a set of bounded linear operators on a Banach space, which is a randomized boundedness condition stronger than uniform boundedness; see Section 3.2.1. In Hilbert spaces it coincides with uniform boundedness and in L^p-spaces ($p \in [1,\infty)$), or more generally in Banach function spaces with finite cotype, it coincides with so-called ℓ^2-boundedness. It follows from the work of Rubio de Francia (see [208–210] and [95]) that ℓ^2-boundedness in $L^p(\mathbb{R}^d)$ ($p \in (1,\infty)$) is closely related to weighted norm inequalities; also see [92].

Randomization techniques also play an important role in this paper. As already mentioned above, we work with a randomized substitute of (3.2). This approach naturally leads to the problem of determining the $\mathscr{R}$-boundedness of a sequence of Fourier multiplier operators. The latter forms a substantial part of this paper, which is also of independent interest; see Section 3.3.

The results in this paper are proved in the setting of weighted spaces, which includes the unweighted case. We consider weights from the so-called Muckenhoupt class A_p. This is a class of weights for which many harmonic analytic tools from the unweighted setting remain valid; see Section 3.2.2. An important example of an A_p-weight is the power weight w_γ, given by

$$w_\gamma(x_1,x') = |x_1|^\gamma, \qquad (x_1,x') \in \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}, \tag{3.3}$$

for the parameter $\gamma \in (-1, p-1)$. In the maximal L^p-regularity approach to parabolic evolution equations these power weights yield flexibility in the optimal regularity of the initial data (cf. e.g. [179, 180, 186, 197]).

The following theorem is our main result. Before we can state it, we first need to explain some notation. We denote by $\{\varepsilon_j\}_{j\in\mathbb{N}}$ a Rademacher sequence on some probability space $(\Omega,\mathscr{F},\mathbb{P})$, i.e. a sequence of independent symmetric $\{-1,1\}$-valued random variables on $(\Omega,\mathscr{F},\mathbb{P})$. For a natural number $m \geq 1$ and a function f on $\mathbb{R}^d$ with values in some vector space X, we write

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x+(m-j)h), \qquad x \in \mathbb{R}^d, h \in \mathbb{R}^d.$$

Theorem 3.1.1. *Let X be a UMD Banach space, $s>0$, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ and $m \in \mathbb{N}$, $m > s$. Suppose that*

- *$K = 1_{[-1,1]^d}$ in the unweighted case $w = 1$; or*
- *$K \in \mathscr{S}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}} K(y)\,dy \neq 0$ in the general weighted case.*

For all $f \in L^p(\mathbb{R}^d, w; X)$ we then have the equivalence of extended norms

$$\|f\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \|f\|_{L^p(\mathbb{R}^d,w;X)} + \sup_{J\in\mathbb{N}} \Big\| \sum_{j=1}^{J} \varepsilon_j 2^{js} \int_{\mathbb{R}^d} K(h)\Delta^m_{2^{-j}h} f\,dh \Big\|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}. \tag{3.4}$$

Remark 3.1.2. If $f \in H^s_p(\mathbb{R}^d, w; X)$, then the finiteness of the supremum on the RHS of (3.4) actually implies the convergence of the sum $\sum_{j=1}^{\infty} \varepsilon_j 2^{js} \int_{\mathbb{R}^d} K(h)\Delta^m_{2^{-j}h} f\,dh$ in $L^p(\Omega; L^p(\mathbb{R}^d, w; X))$. Moreover, (3.4) then takes the form

$$\|f\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \|f\|_{L^p(\mathbb{R}^d,w;X)} + \Big\| \sum_{j=1}^{\infty} \varepsilon_j 2^{js} \int_{\mathbb{R}^d} K(h)\Delta^m_{2^{-j}h} f\,dh \Big\|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}.$$

This follows from the convergence result [154, Theorem 9.29] together with the fact that $L^p(\mathbb{R}^d, w; X)$ (as a UMD space) does not contain a copy c_0.

Remark 3.1.3. We will in fact prove a slightly more general difference norm characterization for $H^s_p(\mathbb{R}^d, w; X)$, namely Theorem 3.4.1, where we consider kernels K satisfying certain integrability conditions plus an $\mathscr{R}$-boundedness condition. Here the $\mathscr{R}$-boundedness condition is only needed for the inequality '$\gtrsim$'. In the case $m = 1$ it corresponds to the $\mathscr{R}$-boundedness of the convolution operators $\{f \mapsto K_t * f : t = 2^j, j \geq 1\}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$, where $K_t = t^d K(t\,\cdot)$. For more information we refer to Section 3.4.2.

To the best of our knowledge, Theorem 3.1.1 is the first difference norm characterization for (non-Hilbertian) Banach space-valued Bessel potential spaces available in the literature. In the special case when X is a UMD Banach function space, the norm equivalence from this theorem takes (with possibly different implicit constants), by the Khinthchine-Maurey theorem, the square function form

$$\|f\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \|f\|_{L^p(\mathbb{R}^d,w;X)} + \Big\| \Big(\sum_{j=1}^{\infty} \Big| 2^{js} \int_{\mathbb{R}^d} K(h)\Delta^m_{2^{-j}h} f\,dh \Big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d,w;X)};$$

see Section 3.4.4. In the unweighted scalar-valued case $X = \mathbb{C}$, this a discrete version for the case $q = 2$ of the characterization [234, Theorem 2.6.3] of the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$ by weighted means of differences (recall (3.2)). Furthermore, in the unweighted scalar-valued case $X = \mathbb{C}$, one can also think of it as a discrete analogue of Strichartz's characterization (3.1).

As an application of Theorem 3.1.1, we characterize the boundedness of the indicator function $1_{\mathbb{R}^d_+}$ of the half-space $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ as a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ in terms of a continuous inclusion of the corresponding scalar-valued Bessel potential space $H^s_p(\mathbb{R}^d, w)$ into a certain weighted L^p-space; see Theorem 4.4.1. The importance of the pointwise multiplier property of $1_{\mathbb{R}^d_+}$ lies in the fact that it served as one of the main ingredients of Seeley's result [224] on the characterization of complex interpolation spaces of Sobolev spaces with boundary conditions. As an application of an extension of Seeley's characterization to the weighted vector-valued case one could, for example, characterize the fractional power domains of the time derivative with zero initial conditions on $L^p(\mathbb{R}_+, w_\gamma; X)$.

Theorem 3.1.4. *Let $X \neq \{0\}$ be a UMD space, $s \in (0,1)$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $w_{s,p}$ be the weight on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ given by $w_{s,p}(x_1, x') := |x_1|^{-sp} w(x_1, x')$ if $|x_1| \leq 1$ and $w_{s,p}(x_1, x') := w(x_1, x')$ if $|x_1| > 1$. Then $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if there is the inclusion*

$$H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p}). \tag{3.5}$$

In Section 3.5.2 we will take a closer look at the inclusion (3.5). Based on embedding results from [185], we will give explicit conditions (in terms of the weight and the parameters) for which this inclusion holds true. The important class of power weights (3.3) is considered in Example 3.5.5.

In the situation of the above theorem, let $\overline{w}_{s,p}$ be the weight on $\mathbb{R} \times \mathbb{R}^{d-1}$ defined by $\overline{w}_{s,p}(x_1, x') := |x_1|^{-sp} w(x_1, x')$. Note that, in view of the inclusion $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w)$, the inclusion (3.5) is equivalent to the inclusion

$$H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, \overline{w}_{s,p}).$$

In the unweighted scalar-valued case, the above theorem thus corresponds to a result of Triebel [233, Section 2.8.6] with $q = 2$, which states that the multiplier property for $F^s_{p,q}(\mathbb{R}^d)$ (recall (3.2)) is equivalent to the inequality

$$||x \mapsto |x_1|^s f(x)||_{L^p(\mathbb{R}^d)} \lesssim ||f||_{F^s_{p,q}(\mathbb{R}^d)}, \qquad f \in F^s_{p,q}(\mathbb{R}^d).$$

Similarly to Strichartz [230], who used (3.1) to prove that $1_{\mathbb{R}^d_+}$ acts a pointwise multiplier on $H^s_p(\mathbb{R}^d)$ in the parameter range

$$-\frac{1}{p'} < s < \frac{1}{p}, \qquad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1,^3$$

[3] This result is originally due to Shamir [226]. However, Strichartz [230] in fact obtained this result as a corollary to a more general pointwise multiplication result (in combination with a Fubini type theorem for Bessel potential spaces).

Triebel used a difference norm characterization in his proof. Our proof is closely related to the proof of Triebel [233, Section 2.8.6].

An alternative approach to pointwise multiplication is via the paraproduct technique (cf. e.g. the monograph of Runst and Sickel [215] for the unweighted scalar-valued setting). Based on a randomized Littlewood-Paley decomposition, Meyries and Veraar [187] followed such an approach to extend the classical result of Shamir [226] and Strichartz [230] to the weighted vector-valued case. They in fact proved a more general pointwise multiplication result for the important class of power weights w_γ (3.3), $\gamma \in (-1, p-1)$, in the UMD setting, from which the case of the characteristic function $1_{\mathbb{R}^d_+}$ can be derived. Their main result [187, Theorem 1.1] says that, given a UMD Banach space X, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$, $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w_\gamma; X)$ in the parameter range

$$-\frac{1+\gamma'}{p'} < s < \frac{1+\gamma}{p}, \qquad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1, \gamma' = -\frac{\gamma}{p-1}.$$

For positive smoothness $s \geq 0$ this pointwise multiplication result is contained in Example 3.5.5, from which the case of negative smoothness $s \leq 0$ can be derived via duality.

The paper is organized as follows. Section 3.2 is devoted to the necessary preliminaries. In Section 3.3 we treat $\mathscr{R}$-boundedness results for Fourier multiplier operators on $L^p(\mathbb{R}^d, w; X)$. The results from this section form (together with a randomized Littlewood-Paley decomposition) the main tools for this paper, but are also of independent interest. In Section 3.4 we state and prove the main result of this paper, Theorem 3.4.1, from which Theorem 3.1.1 can be obtained as a consequence. Finally, in Section 3.5 we use difference norms to prove the pointwise multiplier Theorem 4.4.1, and we also take a closer look at the inclusion (3.5) from this theorem.

Notations and conventions. All vector spaces are over the field of complex scalars $\mathbb{C}$. $|A|$ denotes the Lebesgue measure of Borel set $A \subset \mathbb{R}^d$. Given a measure space $(X, \mathscr{A}, \mu)$, for $A \in \mathscr{A}$ with $\mu(A) \in (0,\infty)$ we write

$$⨍_A d\mu = \frac{1}{\mu(A)} \int_A d\mu.$$

For a function $f : \mathbb{R}^d \longrightarrow X$, with X some vector space, we write $\widetilde{f}(x) = f(-x)$ and, unless otherwise stated, $f_t(x) = t^d f(tx)$ for every $x \in \mathbb{R}^d$ and $t > 0$. Given a Banach space X, we denote by $L^0(\mathbb{R}^d; X)$ the space of equivalence classes of Lebesgue strongly measurable X-valued functions on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r > 0$ we write $Q[x, r] = x + [-r, r]^d$ for the cube centered at x with side length $2r$.

3.2. PREREQUISITES

3.2.1. UMD Spaces and Randomization

The general references for this subsection are [126, 127, 149].

A Banach space X is called a UMD space if for any probability space $(\Omega,\mathscr{F},\mathbb{P})$ and $p\in(1,\infty)$ it holds true that martingale differences are unconditional in $L^p(\Omega;X)$ (see [43, 211] for a survey on the subject). It is a deep result due to Bourgain and Burkholder that a Banach space X has UMD if and only if it is of class $\mathscr{HT}$, i.e. the Hilbert transform has a bounded extension to $L^p(\mathbb{R};X)$ for any/some $p\in(1,\infty)$. Examples of Banach spaces with the UMD property include all Hilbert spaces and all L^q-spaces with $q\in(1,\infty)$.

Throughout this paper, we fix a *Rademacher sequence* $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ on some probability space $(\Omega,\mathscr{F},\mathbb{P})$, i.e. a sequence of independent symmetric $\{-1,1\}$-valued random variables on $(\Omega,\mathscr{F},\mathbb{P})$. If necessary, we denote by $\{\varepsilon'_j\}_{j\in\mathbb{Z}}$ a second Rademacher sequence on some probability space $(\Omega',\mathscr{F}',\mathbb{P}')$ which is independent of the first.

Let X be a Banach function space with finite cotype and let $p\in[1,\infty)$.[4] The Khinthchine-Maurey theorem says that, for all $x_0,\dots,x_n\in X$,

$$\Big\|\Big(\sum_{j=0}^{n}|x_j|^2\Big)^{1/2}\Big\|_X\eqsim\Big\|\sum_{j=0}^{n}\varepsilon_j x_j\Big\|_{L^p(\Omega;X)}. \tag{3.6}$$

In the special case $E=L^q(S)$ ($q\in[1,\infty)$) this easily follows from a combination of Fubini and the Kahane-Khintchine inequality. Morally, (3.6) means that square function estimates are equivalent to estimates for Rademacher sums.

The classical Littlewood-Paley inequality gives a two-sided estimate for the L^p-norm of a scalar-valued function by the L^p-norm of the square function corresponding to its dyadic spectral decomposition. This classical inequality has a UMD Banach space-valued version, due to Bourgain [30] and McConnell [174], in which the square function is replaced by a Rademacher sum (as in (3.6); see the survey paper [124]). One of the main ingredients of this paper is a similar inequality for Bessel potential spaces, namely the randomized Littlewood-Paley decomposition (3.13).

Let X be a Banach space and $p\in[1,\infty]$. As a special case of the (Kahane) contraction principle, for all $x_0,\dots,x_n\in X$ and $a_0,\dots,a_n\in\mathbb{C}$ it holds that

$$\Big\|\sum_{j=0}^{n}a_j\varepsilon_j x_j\Big\|_{L^p(\Omega;X)}\le 2|a|_\infty\Big\|\sum_{j=0}^{n}\varepsilon_j x_j\Big\|_{L^p(\Omega;X)}. \tag{3.7}$$

A family of operators $\mathscr{T}\subset\mathscr{B}(X)$ on a Banach space X is called *$\mathscr{R}$-bounded* if there exists a constant $C\ge0$ such that for all $T_0,\dots,T_N\in\mathscr{T}$ and $x_0,\dots,x_N\in X$ it holds that

$$\Big\|\sum_{j=0}^{N}\varepsilon_j T_j x_j\Big\|_{L^2(\Omega;X)}\le C\Big\|\sum_{j=0}^{N}\varepsilon_j x_j\Big\|_{L^2(\Omega;X)}. \tag{3.8}$$

The moments of order 2 above may be replaced by moments of any order p. The resulting least admissible constant is denoted by $\mathscr{R}_p(\mathscr{T})$. In the definition of $\mathscr{R}$-boundedness it actually suffices to check (3.8) for distinct operators $T_0,\dots,T_N\in\mathscr{T}$.

[4]A Banach space X has cotype $q\in[2,\infty]$ if $\left(\sum_{j=0}^n\|x_j\|^q\right)^{1/q}\lesssim\|\sum_{j=0}^n\varepsilon_j x_j\|_{L^2(\Omega;X)}$ for all $x_0,\dots,x_n\in X$. We say that X has finite cotype if it has cotype $q\in[2,\infty)$. The cotype of L^p is the maximum of 2 and p. Every UMD space has finite cotype.

A Banach space X is said to have *Pisier's contraction property* or *property* (α) if the contraction principle holds true for double Rademacher sums (for some extra fixed multiplicative constant); see [149, Definition 4.9] for the precise definition. Every space L^p with $p \in [1,\infty)$ enjoys property (α). Further examples are UMD Banach function spaces. However, the Schatten von Neumann class $\mathscr{S}_p$ enjoys property (α) if and only if $p = 2$.

A Banach space X is said to have the *triangular contraction property* or *property* (Δ) if there exists a constant $C \geq 0$ such that for all $\{x_{i,j}\}_{i,j=0}^n \subset X$

$$\Big\|\sum_{0\leq j\leq i\leq n} \varepsilon_i \varepsilon'_j x_{i,j}\Big\|_{L^2(\Omega\times\Omega';X)} \leq C\Big\|\sum_{i,j=0}^{n} \varepsilon_i \varepsilon'_j x_{i,j}\Big\|_{L^2(\Omega\times\Omega';X)};$$

see [134]. The moments of order 2 above may be replaced by moments of any order p. The resulting least admissible constant is denoted by $\Delta_{p,X}$. Every space with Pisier's contraction property trivially has the triangular contraction property. For vector-valued L^p-spaces we have $\Delta_{p,L^p(S;X)} = \Delta_{p,X}$. Furthermore, every UMD space has the triangular contraction property.

Let X be a Banach space. The space $\mathrm{Rad}(X)$ is the linear space consisting of all sequences $\{x_j\}_j \subset X$ for which $\sum_{j\in\mathbb{N}} \varepsilon_j x_j$ defines a convergent series in $L^2(\Omega;X)$. It becomes a Banach space under the norm $||\{x_j\}_j||_{\mathrm{Rad}(X)} := ||\sum_{j\in\mathbb{N}} \varepsilon_j x_j||_{L^2(\Omega;X)}$; see [127, 132, 149].

3.2.2. Muckenhoupt Weights

In this subsection the general reference is [103].

A *weight* is a positive measurable function on $\mathbb{R}^d$ that takes it values almost everywhere in $(0,\infty)$. Let w be a weight on $\mathbb{R}^d$. We write $w(A) = \int_A w(x)\,dx$ when A is Borel measurable set in $\mathbb{R}^d$. Furthermore, given a Banach space X and $p \in [1,\infty)$, we define the weighted Lebesgue-Bochner space $L^p(\mathbb{R}^d, w; X)$ as the Banach space of all $f \in L^0(\mathbb{R}^d; X)$ for which

$$||f||_{L^p(\mathbb{R}^d,w;X)} := \left(\int_{\mathbb{R}^d} ||f(x)||_X^p\, w(x)\,dx\right)^{1/p} < \infty.$$

For $p \in [1,\infty]$ we denote by $A_p = A_p(\mathbb{R}^d)$ the class of all Muckenhoupt A_p-weights, which are all the locally integrable weights for which the A_p-characteristic $[w]_{A_p} \in [1,\infty]$ is finite; see [103, Chapter 9] for more details. Let us recall the following facts:

- $A_\infty = \bigcup_{p\in(1,\infty)} A_p$, which often also taken as definition;
- For $p \in (1,\infty)$ and a weight w on $\mathbb{R}^d$: $w \in A_p$ if and only if $w^{-\frac{1}{p-1}} \in A_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$;
- For a weight w on $\mathbb{R}^d$ and $\lambda > 0$: $[w(\lambda\,\cdot)]_{A_p} = [w]_{A_p}$;
- For $p \in [1,\infty)$ and $w \in A_\infty(\mathbb{R}^d)$: $\mathscr{S}(\mathbb{R}^d) \overset{d}{\hookrightarrow} L^p(\mathbb{R}^d, w)$;

- The Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^d, w)$ if (and only if) $w \in A_p$.

An example of an A_∞-weight is the power weight w_γ (3.3) for $\gamma > -1$. Given $p \in (1,\infty)$, we have $w_\gamma \in A_p$ if and only if $\gamma \in (-1, p-1)$. Also see (3.48) for a slight variation.

A function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is called *radially decreasing* if it is of the form $f(x) = g(|x|)$ for some decreasing function $g : \mathbb{R} \longrightarrow \mathbb{R}$. We define $\mathscr{K}(\mathbb{R}^d)$ as the space of all $k \in L^1(\mathbb{R}^d)$ having a radially decreasing integrable majorant, i.e., all $k \in L^1(\mathbb{R}^d)$ for which there exists a radially decreasing $\psi \in L^1(\mathbb{R}^d)^+$ with $|k| \leq \psi$. Equipped with the norm

$$||k||_{\mathscr{K}(\mathbb{R}^d)} := \inf\left\{||\psi||_{L^1(\mathbb{R}^d)} : \psi \in L^1(\mathbb{R}^d)^+ \text{ radially decreasing}, |k| \leq \psi\right\},$$

$\mathscr{K}(\mathbb{R}^d)$ becomes a Banach space. Note that, given $k \in \mathscr{K}(\mathbb{R}^d)$ and $t > 0$, we have $k_t = t^d k(t\,\cdot) \in \mathscr{K}(\mathbb{R}^d)$ with $||k_t||_{\mathscr{K}(\mathbb{R}^d)} = ||k||_{\mathscr{K}(\mathbb{R}^d)}$.

Let X be a Banach space. For $k \in \mathscr{K}(\mathbb{R}^d)$ we have the pointwise estimate

$$\int_{\mathbb{R}^d} |k(x-y)|\,||f(y)||_X\,dy \leq ||k||_{\mathscr{K}(\mathbb{R}^d)} M(||f||_X)(x), \qquad f \in L^1_{loc}(\mathbb{R}^d; X), x \in \mathbb{R}^d.$$

As a consequence, if $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$, then k gives rise to a well-defined bounded convolution operator $k* : f \mapsto k * f$ on $L^p(\mathbb{R}^d, w; X)$, given by the formula

$$k * f(x) = \int_{\mathbb{R}^d} k(x-y) f(y)\,dy, \qquad x \in \mathbb{R}^d,$$

for which we have the norm estimate $||k*||_{\mathscr{B}(L^p(\mathbb{R}^d,w;X))} \lesssim_{p,d,w} ||k||_{\mathscr{K}(\mathbb{R}^d)}$.

3.2.3. Function Spaces

As general reference to the theory of vector-valued distributions we mention [6] (and [5, Section III.4]). For vector-valued function spaces we refer to [126, 220] (unweighted setting) and [187] (weighted setting) and the references given therein.

Let X be a Banach space. The space of X-valued tempered distributions $\mathscr{S}'(\mathbb{R}^d; X)$ is defined as $\mathscr{S}'(\mathbb{R}^d; X) := \mathscr{L}(\mathscr{S}(\mathbb{R}^d), X)$, the space of continuous linear operators from $\mathscr{S}(\mathbb{R}^d)$ to X, equipped with the locally convex topology of bounded convergence. Standard operators (derivative operators, Fourier transform, convolution, etc.) on $\mathscr{S}'(\mathbb{R}^d; X)$ can be defined as in the scalar-case, cf. [5, Section III.4].

Let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Then $w^{1-p'} = w^{-\frac{1}{p-1}} \in A_{p'}$, so that $\mathscr{S}(\mathbb{R}^d) \overset{d}{\hookrightarrow} L^{p'}(\mathbb{R}^d, w^{1-p'})$. By Hölder's inequality we find that $L^p(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$ in the natural way. For each $s \in \mathbb{R}$ we can thus define the Bessel potential space $H^s_p(\mathbb{R}^d, w; X)$ as the space of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which $\mathscr{J}_s f \in L^p(\mathbb{R}^d, w; X)$, equipped with the norm $||f||_{H^s_p(\mathbb{R}^d,w;X)} := ||\mathscr{J}_s f||_{L^p(\mathbb{R}^d,w;X)}$; here $\mathscr{J}_s \in \mathscr{L}(\mathscr{S}'(\mathbb{R}^d; X))$ is the Bessel potential operator given by

$$\mathscr{J}_s f := \mathscr{F}^{-1}[(1+|\cdot|^2)^{s/2}\hat{f}], \qquad f \in \mathscr{S}'(\mathbb{R}^d; X).$$

Furthermore, for each $n \in \mathbb{N}$ we can define the Sobolev space $W^n_p(\mathbb{R}^d, w; X)$ as the space of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which $\partial^\alpha f \in L^p(\mathbb{R}^d, w; X)$ for every $|\alpha| \leq n$, equipped with the norm $\|f\|_{W^n_p(\mathbb{R}^d,w;X)} := \sum_{|\alpha|\leq n} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d,w;X)}$. Note that $H^0_p(\mathbb{R}^d, w; X) = L^p(\mathbb{R}^d, w; X) = W^0_p(\mathbb{R}^d, w; X)$. If X is a UMD space, then we have $H^n_p(\mathbb{R}^d, w; X) = W^n_p(\mathbb{R}^d, w; X)$. In the reverse direction we have that if $H^1_p(\mathbb{R}; X) = W^1_p(\mathbb{R}; X)$, then X is a UMD space (see [126]).

For $0 < A < B < \infty$ we define $\Phi_{A,B}(\mathbb{R}^d)$ as the set of all sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^d; X)$ which can be constructed in the following way: given $\varphi_0 \in \mathscr{S}(\mathbb{R}^d)$ with

$$0 \leq \hat{\varphi} \leq 1,\ \hat{\varphi}(\xi) = 1 \text{ if } |\xi| \leq A,\ \hat{\varphi}(\xi) = 0 \text{ if } |\xi| \geq B,$$

$(\varphi_n)_{n\geq 1}$ is determined by

$$\hat{\varphi}_n = \hat{\varphi}_1(2^{-n+1}\,\cdot) = \hat{\varphi}_0(2^{-n}\,\cdot) - \hat{\varphi}_0(2^{-n+1}\,\cdot), \qquad n \geq 1.$$

Observe that

$$\operatorname{supp}\hat{\varphi}_0 \subset \{\xi : |\xi| \leq B\} \quad \text{and} \quad \operatorname{supp}\hat{\varphi}_n \subset \{\xi : 2^{n-1}A \leq |\xi| \leq 2^n B\},\ n \geq 1. \tag{3.9}$$

We furthermore put $\Phi(\mathbb{R}^d) := \bigcup_{0<A<B<\infty} \Phi_{A,B}(\mathbb{R}^d)$.

Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi(\mathbb{R}^d)$. We define the operators $\{S_n\}_{n\in\mathbb{N}} \subset \mathscr{L}(\mathscr{S}'(\mathbb{R}^d; X), \mathscr{O}_M(\mathbb{R}^d; X))$ by

$$S_n f := \varphi_n * f = \mathscr{F}^{-1}[\hat{\varphi}_n \hat{f}], \qquad f \in \mathscr{S}'(\mathbb{R}^d; X),$$

where $\mathscr{O}_M(\mathbb{R}^d; X)$ stands for the space of all X-valued slowly increasing smooth functions on $\mathbb{R}^d$. Given $s \in \mathbb{R}$, $p \in [1,\infty)$, $q \in [1,\infty]$ and $w \in A_\infty(\mathbb{R}^d)$, the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d, w; X)$ is defined as the space of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d,w;X)} := \|(2^{sn} S_n f)_{n\in\mathbb{N}}\|_{L^p(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} < \infty.$$

Each choice of $\varphi \in \Phi(\mathbb{R}^d)$ leads to an equivalent extended norm on $\mathscr{S}'(\mathbb{R}^d; X)$.

The H-spaces are related to the F-spaces as follows. In the scalar-valued case $X = \mathbb{C}$, we have

$$H^s_p(\mathbb{R}^d, w) = F^s_{p,2}(\mathbb{R}^d, w), \qquad p \in (1,\infty), w \in A_p. \tag{3.10}$$

In the unweighted vector-valued case, this identity is valid if and only if X is isomorphic to a Hilbert space. For general Banach spaces X we still have (see [182, Proposition 3.12])

$$F^s_{p,1}(\mathbb{R}^d, w; X) \hookrightarrow H^s_p(\mathbb{R}^d, w; X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d, w; X), \qquad p \in (1,\infty), w \in A_p(\mathbb{R}^d), \tag{3.11}$$

and

$$\left(\mathscr{S}(\mathbb{R}^d; X), \|\cdot\|_{F^s_{p,1}(\mathbb{R}^d,w;X)}\right) \hookrightarrow L^p(\mathbb{R}^d, w; X), \qquad p \in [1,\infty), w \in A_\infty. \tag{3.12}$$

For UMD spaces X there is a suitable randomized substitute for (6.31): if $p \in (1,\infty)$ and $w \in A_p$, then (see [187, Proposition 3.2])

$$\|f\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \sup_{N\in\mathbb{N}} \left\| \sum_{n=0}^N \varepsilon_n 2^{ns} S_n f \right\|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}, \qquad f \in \mathscr{S}'(\mathbb{R}^d; X). \tag{3.13}$$

Moreover, the implicit constants in (3.13) can be taken of the form $C = C_{X,p,d,s}([w]_{A_p})$ for some increasing function $C_{X,p,d,s} : [1,\infty) \longrightarrow (0,\infty)$ only depending on X, p, d and s.

3.2.4. Fourier Multipliers

Let X be a Banach space. We write $\widehat{L^1}(\mathbb{R}^d;X) := \mathscr{F}^{-1}L^1(\mathbb{R}^d;X) \subset \mathscr{S}'(\mathbb{R}^d;X)$. For a symbol $m \in L^\infty(\mathbb{R}^d)$ we define the operator T_m by

$$T_m : \widehat{L^1}(\mathbb{R}^d;X) \longrightarrow \widehat{L^1}(\mathbb{R}^d;X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}].$$

Given $p \in [1,\infty)$ and $w \in A_\infty(\mathbb{R}^d)$, we call m a *Fourier multiplier* on $L^p(\mathbb{R}^d,w;X)$ if T_m restricts to an operator on $\widehat{L^1}(\mathbb{R}^d;X) \cap L^p(\mathbb{R}^d,w;X)$ which is bounded with respect to the $L^p(\mathbb{R}^d,w;X)$-norm. In this case T_m has a unique extension to a bounded linear operator on $L^p(\mathbb{R}^d,w;X)$ due to the denseness of $\mathscr{S}(\mathbb{R}^d;X)$ in $L^p(\mathbb{R}^d,w;X)$, which we still denote by T_m. We denote by $\mathscr{M}_{p,w}(X)$ the set of all Fourier multipliers $m \in L^\infty(\mathbb{R}^d)$ on $L^p(\mathbb{R}^d,w;X)$. Equipped with the norm $||m||_{\mathscr{M}_{p,w}(X)} := ||T_m||_{\mathscr{B}(L^p(\mathbb{R}^d,w;X))}$, $\mathscr{M}_{p,w}(X)$ becomes a Banach algebra (under the natural pointwise operations) for which the natural inclusion $\mathscr{M}_{p,w}(X) \hookrightarrow \mathscr{B}(L^p(\mathbb{R}^d,w;X)$ is an isometric Banach algebra homomorphism; see [149] for the unweighted setting.

For each $N \in \mathbb{N}$ we define $\mathscr{M}_N(\mathbb{R}^d)$ as the space of all $m \in C^N(\mathbb{R}^d \setminus \{0\})$ for which

$$||m||_{\mathscr{M}_N} = ||m||_{\mathscr{M}_N(\mathbb{R}^d)} := \sup_{|\alpha|\leq N} \sup_{\xi\neq 0} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty.$$

If X is a UMD Banach space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$, then we have $\mathscr{M}_{d+2}(\mathbb{R}^d) \hookrightarrow \mathscr{M}_{p,w}(X)$ with norm $\leq C_{X,p,d}([w]_{A_p})$, where $C_{X,p,d} : [1,\infty) \longrightarrow (0,\infty)$ is some increasing function only depending on X, d and p; see [187, Proposition 3.1].

3.3. $\mathscr{R}$-BOUNDEDNESS OF FOURIER MULTIPLIERS

At several points in the proof of the randomized difference norm characterization from Theorem 3.1.1 we need the $\mathscr{R}$-boundedness of a sequence of Fourier multiplier operators on $L^p(\mathbb{R}^d,w;X)$. In this section we provide the necessary $\mathscr{R}$-boundedness results.

In many situations, the $\mathscr{R}$-boundedness of a family of operators is proved under the assumption of property (α) (see e.g. [48, 98, 149, 240]). Concerning operator families on $L^p(\mathbb{T}^d;X)$ or $L^p(\mathbb{R}^d;X)$, the necessity of property (α) for a number of conclusions of this kind is proved in [128]. For example, in the the setting of Fourier multipliers it holds true that every uniform set of Marcinkiewicz multipliers on $\mathbb{R}^d$ is $\mathscr{R}$-bounded on $L^p(\mathbb{R}^d;X)$ if and only if X is a UMD space with property (α). In particular, given a UMD space X, in the one-dimensional case $d=1$ one has that $\mathscr{M}_1(\mathbb{R}) \hookrightarrow \mathscr{M}_{p,\mathbf{1}_\mathbb{R}}(X)$ maps bounded sets to $\mathscr{R}$-bounded sets if and only if X has property (α). Regarding the sufficiency of property (α) for the $\mathscr{R}$-boundedness of Fourier multipliers, in the weighted setting we have:

Proposition 3.3.1. *Let X be a UMD space with property (α) and $p \in (1,\infty)$.*

(i) For all weights $w \in A_p(\mathbb{R}^d)$, $\mathscr{M}_{d+2}(\mathbb{R}^d) \hookrightarrow \mathscr{M}_{p,w}(X)$ maps bounded sets to $\mathscr{R}$-bounded sets.

(ii) Let $w \in A_p^{rec}(\mathbb{R}^d)$, i.e. w is a locally integrable weight on $\mathbb{R}^d$ which is uniformly A_p in each of the coordinates separately; see [150]. Write $\mathbb{R}^d_* = [\mathbb{R} \setminus \{0\}]^d$. If $\mathscr{M} \subset L^\infty(\mathbb{R}^d) \cap C^d(\mathbb{R}^d_*)$ satisfies

$$C_{\mathscr{M}} := \sup_{M \in \mathscr{M}} \sup_{\alpha \leq \mathbf{1}} \sup_{\xi \in \mathbb{R}^d_*} |\xi^\alpha||D^\alpha m(\xi)| < \infty,$$

then $\mathscr{M}$ defines an $\mathscr{R}$-bounded collection of Fourier multiplier operators $\mathscr{T}_{\mathscr{M}} = \{T_M : M \in \mathscr{M}\}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$ with $\mathscr{R}(\mathscr{T}_{\mathscr{M}}) \lesssim_{X,p,d,w} C_{\mathscr{M}}$.

Proof. (i) Let $w \in A_p$. For each $N \in \mathbb{N}$ we define $\mathscr{R}\mathscr{M}_N(\mathbb{R}^d; \mathscr{B}(X))$ as the space of all operator-valued symbols $m \in C^N(\mathbb{R}^d \setminus \{0\}; \mathscr{B}(X))$ for which

$$||m||_{\mathscr{R}\mathscr{M}_N} = ||m||_{\mathscr{R}\mathscr{M}_N(\mathbb{R}^d;\mathscr{B}(X))} := \mathscr{R}\left\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \neq 0, |\alpha| \leq N\right\} < \infty.$$

If Y is a UMD space, then $\mathscr{R}\mathscr{M}_{d+2}(\mathbb{R}^d; \mathscr{B}(Y)) \hookrightarrow \mathscr{M}_{p,w}(Y)$ (as remarked before [187, Proposition 3.1]). Using this for $Y = \mathrm{Rad}(X)$, the desired result follows in the same spirit as in [98, Section 3] (also see [124, 149]).

(ii) Put $I_j := [-2^j, -2^{j-1}) \cup (2^{j-1}, 2^j]$ for each $j \in \mathbb{Z}$. For each $k \in \{1, \ldots, d\}$ it can be shown that $\{1_{\mathbb{R}^k \times I_j \times \mathbb{R}^{d-k}}\}_{j \in \mathbb{Z}} \subset \mathscr{M}_{p,w}(X)$ and that the associated sequence of Fourier multiplier operators $\{\Delta_k[I_j]\}_{j \in \mathbb{Z}}$ defines an unconditional Schauder decomposition of $L^p(\mathbb{R}^d, w; X)$; see e.g. [156, Chapter 4]. Since $\{\Delta_k[I_j]\}_{j \in \mathbb{Z}}$ and $\{\Delta_l[I_j]\}_{j \in \mathbb{Z}}$ commute for $k, l \in \{1, \ldots, d\}$ and since X is assumed to have property (α), it follows (see [248, Remark 2.5.2]) that the product decomposition $\{\prod_{i=1}^d \Delta_k[I_j]\}$ is an unconditional Schauder decomposition of $L^p(\mathbb{R}^d, w; X)$. One can now proceed as in the unweighted case; see e.g. [149, Theorem 4.13&Example 5.2]. □

As we will see below, for general UMD spaces it is still possible to give criteria for the $\mathscr{R}$-boundedness of a sequence of Fourier multipliers. Before we go to the Fourier analytic setting, we start with a general proposition which serves as the main tool for the $\mathscr{R}$-boundedness of Fourier multipliers below. In order to state the proposition, we first need to introduce some notation.

Let Y be a Banach space. For a sequence $\{T_j\}_{j \in \mathbb{N}} \subset \mathscr{B}(Y)$ we write

$$||\{T_j\}_{j \in \mathbb{N}}||_{Y \to \mathrm{Rad}(Y)} := \inf\left\{C : \left|\left| \sum_{j=0}^n \varepsilon_j T_j y \right|\right|_{L^2(\Omega;Y)} \leq C||y||_Y, y \in Y\right\}$$

and

$$||\{T_j\}_{j \in \mathbb{N}}||_{\mathrm{Rad}(Y) \to Y} := \inf\left\{C : \left|\left| \sum_{j=0}^n T_j y_j \right|\right|_Y \leq C \left|\left| \sum_{j=0}^n \varepsilon_j y_j \right|\right|_{L^2(\Omega;Y)}, n \in \mathbb{N}, y_0, \ldots, y_n \in Y\right\}.$$

In the following remark we provide an interpretation of these quantities in terms of the space $\mathrm{Rad}(Y)$, which gives a motivation for the chosen notation.

Remark 3.3.2. Identifying $\{T_j\}_{j\in\mathbb{N}}$ with the linear operator $\mathbf{T} : Y \longrightarrow \ell^0(\mathbb{N}; X), y \mapsto (T_j y)_{j\in\mathbb{N}}$, we have

$$||\{T_j\}||_{Y\to\mathrm{Rad}(Y)} = ||\{T_j\}||_{\mathscr{B}(Y,\mathrm{Rad}(Y))} = ||\mathbf{T}||_{\mathscr{B}(Y,\mathrm{Rad}(Y))},$$

where $||\cdot||_{\mathscr{B}(Y,\mathrm{Rad}(Y))}$ is, in the natural way, viewed as an extended norm on $L(Y, \ell^0(\mathbb{N}; X))$, the space of linear operators from Y to $\ell^0(\mathbb{N}; X)$. Similarly, identifying $\{T_j\}_{j\in\mathbb{N}}$ with the linear operator $\mathbf{T}^{\mathrm{t}} : c_{00}(\mathbb{N}; X) \longrightarrow Y, (y_j)_{j\in\mathbb{N}} \mapsto \sum_{j\in\mathbb{N}} T_j y_j$, we have

$$||\{T_j\}||_{\mathrm{Rad}(Y)\to Y} = ||\{T_j\}||_{\mathscr{B}(\mathrm{Rad}(Y),Y)} = ||\mathbf{T}^{\mathrm{t}}||_{\mathscr{B}(\mathrm{Rad}(Y),Y)},$$

where $||\cdot||_{\mathscr{B}(\mathrm{Rad}(Y),Y)}$ is viewed, in the natural way, as an extended norm on $L(c_{00}(Y), Y)$.

Using that the natural map $i : \mathrm{Rad}(Y^*) \longrightarrow \mathrm{Rad}(Y)^*$ is a contraction (see [127]), we find that

$$\begin{aligned} ||\{T_j\}||_{\mathrm{Rad}(Y)\to Y} &= ||\mathbf{T}^{\mathrm{t}}||_{\mathscr{B}(\mathrm{Rad}(Y),Y)} = ||(\mathbf{T}^t)^*||_{\mathscr{B}(Y^*,\mathrm{Rad}(Y)^*)} = ||i\circ(\{T_j^*\})^t||_{\mathscr{B}(Y^*,\mathrm{Rad}(Y)^*)} \\ &\leq ||(\{T_j^*\})^t||_{\mathscr{B}(Y^*,\mathrm{Rad}(Y^*))} = ||\{T_j^*\}||_{Y^*\to\mathrm{Rad}(Y^*)}. \end{aligned}$$

If X is K-convex with K-convexity constant K_X,[5] then i is an isomorphism of Banach spaces with $||i^{-1}|| \leq K_X$ (see [127]), so that

$$\begin{aligned} ||\{T_j\}||_{Y\to\mathrm{Rad}(Y)} &= ||\mathbf{T}||_{\mathscr{B}(Y,\mathrm{Rad}(Y))} = ||\mathbf{T}^*||_{\mathscr{B}(\mathrm{Rad}(Y)^*,Y^*)} = ||\{T_j^*\}\circ i^{-1}||_{\mathscr{B}(\mathrm{Rad}(Y)^*,Y^*)} \\ &\leq K_X||\{T_j^*\}||_{\mathscr{B}(\mathrm{Rad}(Y^*),Y^*)} = K_X||\{T_j^*\}||_{\mathrm{Rad}(Y^*)\to Y^*}. \end{aligned}$$

Proposition 3.3.3. *Let Y be a Banach space and let $\{U_j\}_{j\in\mathbb{N}}$ and $\{V_j\}_{j\in\mathbb{N}}$ be two sequences of operators in $\mathscr{B}(Y)$.*

(i) The following inequalities hold true:

$$\mathscr{R}(\{U_j\}) \leq ||\{U_j\}||_{\mathrm{Rad}(Y)\to Y} \leq ||\{U_j\}||_{\mathrm{Rad}(\mathscr{B}(Y))} \leq \sup_n \sup_{\varepsilon_j=\pm1} \Big|\Big| \sum_{j=0}^{n} \varepsilon_j U_j \Big|\Big|, \tag{3.14}$$

$$\mathscr{R}(\{U_j\}) \leq ||\{U_j\}||_{Y\to\mathrm{Rad}(Y)} \leq ||\{U_j\}||_{\mathrm{Rad}(\mathscr{B}(Y))} \leq \sup_n \sup_{\varepsilon_j=\pm1} \Big|\Big| \sum_{j=0}^{n} \varepsilon_j U_j \Big|\Big| \tag{3.15}$$

and

$$||\{U_j V_j\}_{j\in\mathbb{N}}||_{\mathrm{Rad}(Y)\to Y} \leq ||\{U_j\}_{j\in\mathbb{N}}||_{\mathrm{Rad}(\mathscr{B}(Y))} \mathscr{R}(\{V_j\}_{j\in\mathbb{N}}). \tag{3.16}$$

(ii) Suppose that E has property (Δ). If

$$C_1 := ||\{U_j\}_{j\in\mathbb{N}}||_{\mathrm{Rad}(Y)\to Y} < \infty \quad \textit{and} \quad C_2 := ||\{V_j\}_{j\in\mathbb{N}}||_{Y\to\mathrm{Rad}(Y)} < \infty,$$

then $\{\sum_{j=0}^{n} U_k V_k\}$ is $\mathscr{R}$-bounded with $\mathscr{R}$-bound $\leq \Delta_E C_1 C_2$.

[5] For the definition of K-convexity we refer to [126, 172]. All UMD spaces are K-convex.

Proof. Except for (3.15), where we follow the estimates from the proof of [187, Lemma 4.1], the proposition follows easily by inspection of the proof of [134, Theorem 3.3]. Let us provide the details for the convenience of the reader.

(i) The third inequality in (3.14) is trivial and the second inequality in (3.14) is just the inequality (3.16) with $V_j = I$ for all j. For the first inequality in (3.14), let $y_0, \dots, y_n \in Y$. For every $\{\varepsilon_j\}_{j\in\mathbb{N}} \in \{-1,1\}^{n+1}$ we have

$$\Big\|\sum_{j=0}^{n} \varepsilon_j U_j y_j\Big\|_Y \leq \|\{U_j\}_{j\in\mathbb{N}}\|_{\mathrm{Rad}(Y)\to Y}\Big\|\sum_{j=0}^{n} \varepsilon_j y_j\Big\|_{L^2(\Omega;Y)}$$

because $\{\varepsilon_j\}_{j=0}^n$ and $\{\varepsilon_j\varepsilon_j\}_{j=0}^n$ are identically distributed. Plugging in $\varepsilon_j = \varepsilon_j(\omega)$ and taking L^2-norms with respect to $\omega \in \Omega$, the desired inequality follows.

In (3.15) we only need to prove the first inequality; the other two inequalities are trivial. For this we use the fact [99, Lemma 3.12] that for any $\{y_{j,k}\}_{j,k=0}^n \subset Y$ one has the inequality

$$\Big\|\sum_{j=0}^{n} \varepsilon_j y_{j,j}\Big\|_{L^2(\Omega;Y)} \leq \Big\|\sum_{j,k=0}^{n} \varepsilon_j\varepsilon'_k y_{j,k}\Big\|_{L^2(\Omega\times\Omega';Y)}. \tag{3.17}$$

Now let $y_0, \dots, y_n \in Y$. Denote by $\{\widetilde{U}_j\} \subset \mathscr{B}(L^2(\Omega;Y))$ the sequence of operators pointwise induced by $\{U_j\}$. Using Fubini one easily sees that $\|\{\widetilde{U}_j\}\|_{L^2(\Omega;Y)\to\mathrm{Rad}(L^2(\Omega;Y))} \leq \|\{U_j\}\|_{Y\to\mathrm{Rad}(Y)}$. Invoking (3.17) with $y_{j,k} = U_k y_j$, we thus find

$$\begin{aligned}
\Big\|\sum_{j=0}^{n} \varepsilon_j U_j y_j\Big\|_{L^2(\Omega;Y)} &\leq \Big\|\sum_{j,k=0}^{n} \varepsilon_j\varepsilon'_k U_k y_j\Big\|_{L^2(\Omega\times\Omega';Y)} \\
&= \Big\|\sum_{k=0}^{n} \varepsilon'_k \widetilde{U}_k\Big(\sum_{j=0}^{n} \varepsilon_j y_j\Big)\Big\|_{L^2(\Omega';L^2(\Omega;Y))} \\
&\leq \|\{U_j\}\|_{Y\to\mathrm{Rad}(Y)}\Big\|\sum_{j=0}^{n} \varepsilon_j y_j\Big\|_{L^2(\Omega;Y))}.
\end{aligned}$$

For (3.16) note that if $y_0, \dots, y_n \in Y$, then

$$\begin{aligned}
\Big\|\sum_{j=0}^{n} U_j V_j y_j\Big\|_Y &= \Big\|\mathbb{E}\Big[\Big(\sum_{j=0}^{n} \varepsilon_j U_j\Big)\Big(\sum_{j=0}^{n} \varepsilon_j V_j y_j\Big)\Big]\Big\|_Y \\
&\leq \Big\|\sum_{j=0}^{n} \varepsilon_j U_j\Big\|_{L^2(\Omega;\mathscr{B}(Y))}\Big\|\sum_{j=0}^{n} \varepsilon_j V_j y_j\Big\|_{L^2(\Omega;Y)} \\
&\leq \|\{U_j\}\|_{\mathrm{Rad}(\mathscr{B}(Y))}\mathscr{R}(\{V_j\})\Big\|\sum_{j=0}^{n} \varepsilon_j y_j\Big\|_{L^2(\Omega;Y)}.
\end{aligned}$$

(ii) Write $S_k := \sum_{j=0}^k U_j V_j$ for each $k \in \mathbb{N}$. For all $y_0, \dots, y_n \in Y$ we have

$$\Big\|\sum_{k=0}^{n} \varepsilon_k S_k y_k\Big\|_{L^2(\Omega;Y)} = \Big\|\sum_{j=0}^{n} U_j \sum_{k=j}^{n} \varepsilon_k V_j y_k\Big\|_{L^2(\Omega;Y)}$$

$$\begin{aligned} &\le C_1\Big|\Big|\sum_{j=0}^{n}\varepsilon_j'\sum_{k=j}^{n}\varepsilon_k V_j y_k\Big|\Big|_{L^2(\Omega;L^2(\Omega';Y))} \\ &\le \Delta_Y C_1\Big|\Big|\sum_{j=0}^{n}\varepsilon_j' V_j\sum_{k=0}^{n}\varepsilon_k y_k\Big|\Big|_{L^2(\Omega;L^2(\Omega';Y))} \\ &\le \Delta_Y C_1 C_2\Big|\Big|\sum_{k=0}^{n}\varepsilon_k y_k\Big|\Big|_{L^2(\Omega;;Y)}, \end{aligned}$$

which proves the required $\mathscr{R}$-bound. □

For later reference it will be convenient to record the following immediate corollary to the estimates (3.14) and (3.15) in (i) of the above proposition:

Corollary 3.3.4. *Let X be a Banach space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $\{m_j\}_{j\in\mathbb{N}} \subset \mathscr{M}_{p,w}(X)$ be a sequence of symbols such that*

$$K := \sup_n \sup_{\varepsilon_j=\pm1}\Big|\Big|\sum_{j=0}^{n}\varepsilon_j m_j\Big|\Big|_{\mathscr{M}_{p,w}(X)} < \infty. \tag{3.18}$$

Then $\{m_j\}_{j\in\mathbb{N}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{m_j}\}_{j\in\mathbb{N}}$ on $Y = L^p(\mathbb{R}^d, w; X)$ with $\mathscr{R}$-bound

$$\mathscr{R}(\{T_{m_j}\}) \le ||\{T_j\}||_{\mathrm{Rad}(Y)\to Y} \vee ||\{T_j\}||_{Y\to\mathrm{Rad}(Y)} \le ||\{T_j\}||_{\mathrm{Rad}(\mathscr{B}(Y))} \le K.$$

If X is a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$, then we have $\mathscr{M}_{d+2}(\mathbb{R}^d) \hookrightarrow \mathscr{M}_{p,w}(X)$. So the number K from (3.18) can be explicitly bounded via the Mihlin condition defining $\mathscr{M}_{d+2}(\mathbb{R}^d)$. In particular, for a bounded sequence in $\mathscr{M}_{d+2}(\mathbb{R}^d)$ which is locally finite in a uniform way we find:

Corollary 3.3.5. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $\{m_j\}_{j\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ be a sequence of symbols such that:*

(a) There exists $N \in \mathbb{N}$ such that every $\xi \in \mathbb{R}^d \setminus \{0\}$ possesses an open neighborhood $U \subset \mathbb{R}^d \setminus \{0\}$ with the property that $\#\{j : m_j|_U \ne 0\} \le N$.

(b) $\{m_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathscr{M}_{d+2}(\mathbb{R}^d)$.

Then $\{m_j\}_{j\in\mathbb{N}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{m_j}\}_{j\in\mathbb{N}}$ on $L^p(\mathbb{R}^d, w; X)$ with $\mathscr{R}$-bound

$$\mathscr{R}(\{T_{m_j}\}) \le \sup_n \sup_{\varepsilon_j=\pm1}\Big|\Big|\sum_{j=0}^{n}\varepsilon_j m_j\Big|\Big|_{\mathscr{M}_{p,w}(X)} \lesssim C_{X,p,d}([w]_{A_p})N\sup_{j\in\mathbb{N}}||m_j||_{\mathscr{M}_{d+2}},$$

where $C_{X,p,d} : [1,\infty) \longrightarrow (0,\infty)$ is some increasing function only depending on X, p and d.

An example for the 'uniform locally finiteness condition' (a) from the above corollary is a kind of dyadic corona condition on the supports of the symbols:

Example 3.3.6. Suppose that $\{m_j\}_{j\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ satisfies the support condition

$$\operatorname{supp} m_0 \subset \{\xi : |\xi| \le c\} \quad \text{and} \quad \operatorname{supp} m_j \subset \{\xi : c3^{-1}2^{j-J+1} \le |\xi| \le c2^j\},\ j \ge 1, \tag{3.19}$$

for some $c > 0$ and $J \in \mathbb{Z}_{>0}$. Then $\operatorname{supp} m_j \cap \operatorname{supp} m_k = \varnothing$ for all $j,k \in \mathbb{N}$ with $|j-k| \ge J+1$. In particular, condition (a) of Corollary 3.3.5 is satisfied with $N = J$.

Example 3.3.7. Suppose that $m_0 \in C_c^{d+2}(\mathbb{R}^d)$ and $m_1 \in C_c^{d+2}(\mathbb{R}^d \setminus \{0\})$. Set $m_j := m(2^{-j}\,\cdot)$ for each $j \ge 2$. Then $\{m_j\}_{j\in\mathbb{N}}$ fulfills the conditions (a) and (b) of Corollary 3.3.5, where (a) follows from Example 3.3.6 and (b) from the dilation invariance of the Mihlin condition defining $\mathscr{M}_{d+2}(\mathbb{R}^d)$. In particular, given $\varphi = \{\varphi_j\}_{j\in\mathbb{N}} \in \Phi(\mathbb{R}^d)$, Corollary 3.3.5 can be applied to the sequence of symbols $\{m_j\}_{j\in\mathbb{N}} = \{\hat{\varphi}_j\}_{j\in\mathbb{N}}$, whose associated sequence of Fourier multiplier operators is $\{S_j\}_{j\in\mathbb{N}}$.

Up to now we have only exploited Proposition 3.3.3(i) in order to get $\mathscr{R}$-boundedness of a sequence of Fourier multipliers. However, in many situations the condition (3.18) is too strong. It is for example not fulfilled by the sequence $\{m_j = m(2^{-j}\,\cdot)\}_{j\in\mathbb{N}}$, where $m \in C_c^\infty(\mathbb{R}^d)$ is a given symbol which is non-zero in the origin; this follows from the fact that $\mathscr{M}_{p,w}(X) \hookrightarrow L^\infty(\mathbb{R}^d)$. The case that m is constant on a neighborhood of the origin can be handled by the following proposition (see Corollary 3.3.10), of which the main ingredient is Proposition 3.3.3(ii):

Proposition 3.3.8. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $\{m_j\}_{j\in\mathbb{N}} \subset \mathscr{M}_{p,w}(X)$ be a sequence of Fourier multiplier symbols which satisfies the support condition* (3.19) *for some $c > 0$ and $J \in \mathbb{N}$. Write $T_j = T_{m_j}$ for the Fourier multiplier operator on $Y = L^p(\mathbb{R}^d, w; X)$ associated with m_j for each $j \in \mathbb{N}$. If*

$$K := \|\{T_j\}\|_{\mathrm{Rad}(Y)\to Y} \wedge \|\{T_j\}\|_{Y\to\mathrm{Rad}(Y)} < \infty, \tag{3.20}$$

then the collection of partial sums $\{\sum_{j=0}^n T_j : n \in \mathbb{N}\}$ is $\mathscr{R}$-bounded with $\mathscr{R}$-bound $\le (2J+1)C_{X,p,d}([w]_{A_p})\,K$ for some increasing function $C_{X,p,d} : [1,\infty) \longrightarrow (0,\infty)$ only depending on X, p and d.

Proof. Due to scaling invariance of the A_p-characteristic, we may without loss of generality assume that $c = \frac{3}{2}$. Fix $\varphi = (\varphi_j)_{j\in\mathbb{N}} \in \Phi_{1,\frac{3}{2}}(\mathbb{R}^d)$ and denote by $\{S_j\}_{j\in\mathbb{N}}$ the corresponding convolution operators. For convenience of notation we put $\varphi_j := 0$ and $S_j := 0$ for every $j \in \mathbb{Z}_{<0}$. For each $j \in \mathbb{N}$ we define $R_j := \sum_{\ell=-J}^{J} S_{j+\ell}$. By Example 3.3.7 (and Corollary 3.3.5), there exists an increasing function $\widetilde{C}_{X,p,d} : [1,\infty) \longrightarrow (0,\infty)$, only depending on X, p and d, such that

$$\|\{S_j\}\|_{\mathrm{Rad}(Y)\to Y} \vee \|\{S_j\}\|_{Y\to\mathrm{Rad}(Y)} \overset{(3.14),(3.15)}{\le} \widetilde{C}_{X,p,d}([w]_{A_p}),$$

and thus

$$\|\{R_j\}\|_{\mathrm{Rad}(Y)\to Y} \vee \|\{R_j\}\|_{Y\to\mathrm{Rad}(Y)} \le (2J+1)\widetilde{C}_{X,p,d}([w]_{A_p}). \tag{3.21}$$

As a consequence of the support condition (3.19) and the fact that

$$\sum_{\ell=-J}^{J} \hat{\varphi}_\ell(\xi) = 1 \text{ for } |\xi| \leq \frac{3}{2} \quad \text{and} \quad \sum_{\ell=-J}^{J} \hat{\varphi}_{j+\ell}(\xi) = 1 \text{ for } 2^{j-J} \leq |\xi| \leq \frac{3}{2}2^j,\ j \geq 1,$$

we have $T_j R_j = R_j T_j = T_j$ for every $j \in \mathbb{N}$. Since $\{T_j\}$ and $\{R_j\}$ are commuting and since $\Delta_Y \eqsim_p \Delta_{Y,p} = \Delta_{X,p} < \infty$ (X being a UMD space), the required $\mathscr{R}$-bound follows from an application of Proposition 3.3.3(ii) with either $U_j = T_j$ and $V_j = R_j$ or $U_j = R_j$ and $V_j = T_j$. □

Remark 3.3.9. The condition (3.20) in Proposition 3.3.8 may be replaced by the condition that $\{T_j\}$ is $\mathscr{R}$-bounded with $\mathscr{R}$-bound K: under this modification, it can be shown that the collection of partial sums is $\mathscr{R}$-bounded with $\mathscr{R}$-bound $\leq (2J+1)^2 C_{X,p,d}([w]_{A_p}) K$ for some increasing function $C_{X,p,d} : [1,\infty) \longrightarrow (0,\infty)$ only depending on X, p and d. Indeed, in the notation of the proof above, we have

$$\begin{aligned} \|\{T_j\}\|_{\mathrm{Rad}(Y)\to Y} &= \|\{R_j T_j\}\|_{\mathrm{Rad}(Y)\to Y} \overset{(3.16)}{\leq} \|\{R_j\}\|_{\mathrm{Rad}(\mathscr{B}(Y))}\mathscr{R}(\{T_j\}) \\ &\overset{(3.21)}{\leq} (2J+1)\widetilde{C}_{X,p,d}([w]_{A_p})\mathscr{R}(\{T_j\}). \end{aligned}$$

An alternative approach for the $\mathscr{R}$-boundedness condition would be to modify the proof of [48, Theorem 3.9] (or [248, Theorem 2.4.3]), which is a generalization of the vector-valued Stein inequality to the setting of unconditional Schauder decompositions. Via this approach one would get linear dependence on J instead of quadratic.

Corollary 3.3.10. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Suppose that $M \in C_c^{d+2}(\mathbb{R}^d)$ is constant on a neighborhood of* 0 *and put $M_j := M(2^{-j}\cdot)$ for each $j \in \mathbb{Z}$. Then $\{M_j\}_{j\in\mathbb{Z}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_j}\}_{j\in\mathbb{Z}}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$ with $\mathscr{R}$-bound $\lesssim_M C_{X,p,d}([w]_{A_p})$, where $C_{X,p,d}$ is the function from Proposition* 3.3.8.

Proof. By the scaling invariance of the A_p-characteristic, it suffices to prove the $\mathscr{R}$-boundedness statement for $\{M_j\}_{j\in\mathbb{N}}$ instead of $\{M_j\}_{j\in\mathbb{Z}}$. Indeed, for each $K \in \mathbb{Z}_{<0}$ we then in particular have that $\{M_j\}_{j\in\mathbb{N}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_j}\}_{j\in\mathbb{N}}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w(2^{-K}\cdot); X))$ with $\mathscr{R}$-bound $\lesssim_M C_{X,p,d}([w]_{A_p})$, or equivalently, that $\{M_j\}_{j\geq K}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_j}\}_{j\geq K}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w(2^K\cdot); X))$ with $\mathscr{R}$-bound $\lesssim_M C_{X,p,d}([w]_{A_p})$.

Define the sequence of symbols $\{m_j\}_{j\in\mathbb{N}}$ by $m_0 := M$, $m_1 := m_0(2^{-1}\cdot) - m_0$, and $m_j := m_1(2^{-j+1}\cdot)$ for $j \geq 2$. Then $\{m_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathscr{M}_{d+2}$ which satisfies the support condition (3.19). By a combination of Corollary 3.3.5, Example 3.3.6 and Proposition 3.3.8, the collection of partial sums $\{T_{M_i} : i \in \mathbb{N}\} = \{\sum_{k=0}^{i} T_{m_k} : i \in \mathbb{N}\}$ is $\mathscr{R}$-bounded in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$ (with the required dependence of the $\mathscr{R}$-bound). □

With the following theorem we can in particular treat dilations of symbols M belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ without any further restrictions. Note that this would be immediate from Proposition 3.3.1(i) in case of property (α).

Theorem 3.3.11. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $M \in C(\mathbb{R}^d) \cap C^{d+2}(\mathbb{R}^d \setminus \{0\})$ and set $M_j := M(2^{-j}\cdot)$ for each $j \in \mathbb{Z}$. Suppose that there exist $\delta_0, \delta_\infty > 0$ such that*

$$C_0 := \sup_{0<|\xi|\leq 1} |\xi|^{-\delta_0}|M(\xi) - M(0)| \vee \sup_{1\leq|\alpha|\leq d+2}\sup_{0<|\xi|\leq 1} |\xi|^{|\alpha|-\delta_0}\left|D^\alpha M(\xi)\right| < \infty \tag{3.22}$$

and

$$C_\infty := \sup_{|\alpha|\leq d+2}\sup_{|\xi|\geq 1} |\xi|^{|\alpha|+\delta_\infty}|D^\alpha M(\xi)| < \infty. \tag{3.23}$$

Then $\{M_j\}_{j\in\mathbb{Z}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_j}\}_{j\in\mathbb{Z}}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$ with $\mathscr{R}$-bound $\leq C_{X,d,p,\delta_0,\delta_\infty}([w]_{A_p})[||M||_\infty \vee C_0 \vee C_\infty]$, where $C_{X,d,p,\delta_0,\delta_\infty} : [1,\infty) \longrightarrow (0,\infty)$ is some increasing function only depending on X, p, d, δ_0 and δ_∞.

Remark 3.3.12. In the proof of Theorem 3.3.11 we use the Mihlin multiplier theorem $\mathscr{M}_{d+2} \hookrightarrow \mathscr{M}_{p,w}(X)$. The availability of better multiplier theorems would lead to weaker conditions on M. For example, using the classical Mihlin multiplier condition $|D^\alpha m| \lesssim |\xi|^{|\alpha|}$, $\alpha \in \{0,1\}^d$, we could treat symbols $M \in C(\mathbb{R}^d) \cap C^d(\mathbb{R}^d \setminus \{0\})$ satisfying (3.22) and (3.23) with the suprema taken over $\alpha \in \{0,1\}^d$ instead of $|\alpha| \leq d+2$; as in the unweighted case, for $w \in A_p^{rec}(\mathbb{R}^d)$ it can be shown that this classical Mihlin condition is sufficient for m to be a Fourier multiplier on $L^p(\mathbb{R}^d, w; X)$ (see [156, Chapter 4]). In the unweighted case one could even use multiplier theorems which incorporate information of the Banach space under consideration [99, 123]. In Theorem 3.3.14 (and Corollary 3.3.15) we will actually use the Mihlin-Hölder condition from [123, Theorem 3.1] (which is weaker than the Mihlin-Hörmander condition) for the one-dimensional case $d = 1$.

Proof. As in the proof of Corollary 3.3.10, it is enough to establish the $\mathscr{R}$-boundedness of $\{M_j\}_{j\in\mathbb{N}}$. Put $C := ||M||_\infty \vee C_0 \vee C_\infty$. Pick $\zeta \in C_c^\infty(R^d)$ with the property that $\chi(\xi) = 1$ if $|\xi| \leq 1$ and $\zeta(\xi) = 0$ if $|\xi| \geq 3/2$. Then

$$M := M(0)\zeta + \zeta(M - M(0)\zeta) + (1-\zeta)(M - M(0)\zeta) =: M^{[1]} + M^{[2]} + M^{[3]}.$$

For each $i \in \{1,2,3\}$ we define $\{M_j^{[i]}\}_{j\in\mathbb{N}}$ by $M_j^{[i]} := M^i(2^{-j}\cdot)$. By Corollary 3.3.10, $\{M_j^{[1]}\}_{j\in\mathbb{N}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$ with $\mathscr{R}$-bound $\lesssim_{X,d,p,w,\zeta} |M(0)| \leq C$. In order to get $\mathscr{R}$-boundedness for $i = 2,3$ we use Corollary 3.3.4 (in combination with $\mathscr{M}_{d+2} \hookrightarrow \mathscr{M}_{p,w}(X)$). To this end, let $\varepsilon = \{\varepsilon_j\}_{j=0}^N \in \{-1,1\}^{N+1}$, $N \in \mathbb{N}$, and put $M_\varepsilon^{[i]} := \sum_{j=0}^N \varepsilon_j M_j^{[i]}$ for each $i \in \{2,3\}$. In order to obtain a uniform bound for $M_\varepsilon^{[i]}$ in $\mathscr{M}_{d+2}$, we note that:

- $M^{[2]} \in C(\mathbb{R}^d) \cap C^{d+2}(\mathbb{R}^d \setminus \{0\})$ with $\operatorname{supp} M^{[2]} \subset B(0,2)$ and

$$C^{[2]} := \sup_{|\alpha|\leq d+2}\sup_{\xi\neq 0} |\xi|^{|\alpha|-\delta_0}|D^\alpha M^{[2]}(\xi)| \lesssim_{\zeta,\delta_0,\delta_\infty} C;$$

- $M^{[3]} \in C^{d+2}(\mathbb{R}^d)$ with $M^{[3]}(\xi) = 0$ for $|\xi| \le 1$ and

$$C^{[3]} := \sup_{|\alpha| \le d+2} \sup_{\xi \neq 0} |\xi|^{|\alpha|+\delta_\infty} |D^\alpha M^{[2]}(\xi)| \lesssim_{\zeta,\delta_0,\delta_\infty} C.$$

For notational convenience, for each $j \ge N+1$ we write $\varepsilon_j = 0$.

The case $i = 2$: Let $|\alpha| \le d+2$. For $\xi \in \overline{B}(0,2)$ we have

$$\begin{aligned}
|\xi|^{|\alpha|} |D^\alpha M_\varepsilon^{[2]}(\xi)| &\le \sum_{j=0}^{\infty} |\xi|^{|\alpha|} |D^\alpha M_j^{[2]}(\xi)| = \sum_{j=0}^{\infty} |2^{-j}\xi|^{|\alpha|} |D^\alpha M^{[2]}(2^{-j}\xi)| \\
&\le C^{[2]} \sum_{j=0}^{\infty} |2^{-j}\xi|^{\delta_0} = C^{[2]} \left(\sum_{j=0}^{\infty} 2^{-j\delta_0} \right) |\xi|^{\delta_0} \\
&\le C^{[2]} \frac{2^{\delta_0}}{1-2^{-\delta_0}}
\end{aligned}$$

and for $\xi \in B(0,2^{l+1}) \setminus \overline{B}(0,2^l)$, $l \in \mathbb{N}$, we similarly have, now using the support condition $\operatorname{supp} M^{[2]} \subset B(0,2)$,

$$\begin{aligned}
|\xi|^{|\alpha|} |D^\alpha M_\varepsilon^{[2]}(\xi)| &\le \sum_{j=0}^{\infty} |\xi|^{|\alpha|} |D^\alpha M_j^{[2]}(\xi)| = \sum_{j=0}^{\infty} |2^{-j}\xi|^{|\alpha|} |D^\alpha M^{[2]}(2^{-j}\xi)| \\
&= \sum_{j=l}^{\infty} |2^{-j}\xi|^{|\alpha|} |D^\alpha M_j^{[2]}(2^{-j}\xi)| \le C^{[2]} \sum_{j=l}^{\infty} |2^{-j}\xi|^{\delta_0} \\
&= C^{[2]} \left(\sum_{j=l}^{\infty} 2^{-j\delta_0} \right) |\xi|^{\delta_0} \le C^{[2]} \frac{2^{\delta_0}}{1-2^{-\delta_0}}.
\end{aligned}$$

Hence, $||M_\varepsilon^{[2]}||_{\mathscr{M}_{d+2}} \le C^{[2]} 2^{\delta_0} (1-2^{-\delta_0})^{-1}$.

The case $i = 3$: Fix $l \in \mathbb{N}$. Since $M^{[3]} \equiv 0$ on $B(0,1)$, we have

$$M_\varepsilon^{[3]}(\xi) = \sum_{j=0}^{l} \varepsilon_j M_j^{[3]}(\xi), \qquad \xi \in B(0,2^l) \setminus \overline{B}(0,2^{l-1}).$$

For all $|\alpha| \le d+2$ and $\xi \in B(0,2^l) \setminus \overline{B}(0,2^{l-1})$ we thus find

$$\begin{aligned}
|\xi|^{|\alpha|} |D^\alpha M_\varepsilon^{[3]}(\xi)| &= |\xi|^{|\alpha|} \left| \sum_{j=0}^{l} \varepsilon_j D^\alpha M_j^{[3]}(\xi) \right| \le \sum_{j=0}^{l} |\xi|^{|\alpha|} |D^\alpha M^{[3]}(\xi)| \\
&= \sum_{j=0}^{l} |2^{-j}\xi|^{|\alpha|} |D^\alpha M^{[3]}(2^{-j}\xi)| \le C^{[3]} \sum_{j=0}^{l} |2^{-j}\xi|^{-\delta_\infty} \\
&\le C^{[3]} \sum_{j=0}^{l} (2^{-j+l-1})^{-\delta_\infty} = C^{[3]} 2^{\delta_\infty} \sum_{j=0}^{l} 2^{-\delta_\infty(l-j)} \\
&= C^{[3]} 2^{\delta_\infty} \sum_{k=0}^{l} 2^{-\delta_\infty k} \le C^{[3]} \frac{2^{\delta_\infty}}{1-2^{-\delta_\infty}}.
\end{aligned}$$

As $l \in \mathbb{N}$ was arbitrary and $M_\varepsilon^{[3]} \equiv 0$ on $B(0,1)$, this shows that $||M_\varepsilon^{[3]}||_{\mathscr{M}_{d+2}} \le C^{[3]} 2^{\delta_\infty} (1-2^{-\delta_\infty})^{-1}$. □

Note that Theorem 3.3.11 does not cover the symbol $M(\xi) = \prod_{j=1}^{d} \operatorname{sinc}(\xi_j)$, where sinc is the function given by $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$ for $t \neq 0$ and $\operatorname{sinc}(0) = 1$; see the end of Section 3.4.2 for the relevance of this symbol, which is the Fourier transform of $2^{-d}1_{[-1,1]^d}$. However, as already mentioned in Remark 3.3.12, in the unweighted one-dimensional case we can use the Mihlin-Hölder multiplier theorem [125, Theorem 3.1] in order to relax the conditions from Theorem 3.3.11. This will lead to a criterium (Corollary 3.3.15) which covers the symbol $M = \operatorname{sinc}$; see Example 3.4.5.

For each $k \in \mathbb{Z}$ and $j \in \{-1,1\}$ we define $I_{k,j} := j\,[2^{k-2}, 2^{k+2}]$. For $\gamma \in (0,1)$ and $M \in C_b(\mathbb{R} \setminus \{0\})$ we put

$$[M]_\gamma := \sup_{k\in\mathbb{Z}, j=\pm1} 2^{k\gamma}[M|_{I_{k,j}}]_{C^\gamma(I_{k,j})} \quad \text{and} \quad |||M|||_\gamma := ||M||_\infty + [M]_\gamma.$$

Since

$$|M(\xi) - M(\xi - h)| \leq 4[M]_\gamma |h|^\gamma |\xi|^{-\gamma}, \quad |\xi| > 2|h|,$$

the following lemma is a direct corollary of the vector-valued Mihlin-Hölder multiplier theorem [125, Theorem 3.1]:

Lemma 3.3.13. *Let X be a UMD space and $p \in (1,\infty)$. Then there exists $\gamma_X \in (0,1)$, only depending on X, such that the following holds true: if $\gamma \in (\gamma_X, 1)$ and if $M \in C_b(\mathbb{R} \setminus \{0\})$ satisfies $|||M|||_\gamma < \infty$, then M defines a Fourier multiplier operator T_M on $L^p(\mathbb{R}; X)$ of norm $||T_M||_{\mathscr{B}(L^p(\mathbb{R};X))} \lesssim_{X,p,\gamma} |||M|||_\gamma$.*[6]

Using this lemma, we find the following variant of Theorem 3.3.11:

Theorem 3.3.14. *Let X be a UMD space $p \in (1,\infty)$. Let $\gamma \in (\gamma_X, 1)$, where $\gamma_X \in (0,1)$ is from Lemma 3.3.13. Let $M \in C_b(\mathbb{R})$ and set $M_n := M(2^{-n}\cdot)$ for each $n \in \mathbb{Z}$. Suppose that there exist $\delta_0, \delta_\infty > 0$ such that*

$$C_0 := \sup_{0<|\xi|\leq 1} |\xi|^{-\delta_0}|M(\xi) - M(0)| \vee \sup_{k\leq -1, j=\pm1} 2^{k(\gamma-\delta_0)}[M|_{I_{k,j}}]_{C^\gamma(I_{k,j})} < \infty$$

and

$$C_\infty := \sup_{|\xi|\geq 1} |\xi|^{\delta_\infty}|M(\xi)| \vee \sup_{k\geq 0, j=\pm1} 2^{k(\gamma+\delta_\infty)}[M|_{I_{k,j}}]_{C^\gamma(I_{k,j})} < \infty.$$

Then $\{M_n\}_{n\in\mathbb{Z}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_n}\}_{n\in\mathbb{Z}}$ in $\mathscr{B}(L^p(\mathbb{R}; X))$ with $\mathscr{R}$-bound $\lesssim_{X,p,\tau,q,\gamma,\delta_0,\delta_\infty} [||M||_\infty \vee C_0 \vee C_\infty]$.

Proof. This can be shown in a similar fashion as Theorem 3.3.11, now using the (Mihlin-Hölder multiplier theorem in the form of) Lemma 3.3.13 to treat the cases $i = 2,3$. □

[6] One can take $\gamma_X = \tau \vee q'$, where $\tau \in (1,2]$ and $q \in [2,\infty)$ denote the type and cotype of X, respectively. Here one needs the fact that X, as a UMD space, has non-trivial type and finite cotype; see [126].

Corollary 3.3.15. *Let X be a UMD space $p \in (1,\infty)$. Let $\gamma \in (\gamma_X, 1)$, where $\gamma_X \in (0,1)$ is from Lemma 3.3.13. Let $M \in C_b(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and set $M_n := M(2^{-n}\cdot)$ for each $n \in \mathbb{Z}$. Suppose that there exist $\delta_0, \delta_\infty > 0$ and $\theta \in [0,1]$ such that*

$$C_0 := \sup_{0<|\xi|\leq 1} |\xi|^{-\delta_0}|M(\xi)-M(0)| \vee \sup_{|\xi|\leq 1}|\xi|^{1-\delta_0}|M'(\xi)| < \infty \tag{3.24}$$

and

$$C_\infty := \sup_{|\xi|\geq 1}|\xi|^{\max\left\{\delta_\infty,(\gamma+\delta_\infty)\frac{1-\theta}{1-\gamma}\right\}}|M(\xi)| \vee \sup_{|\xi|\geq 1}|\xi|^{(\gamma+\delta_\infty)\frac{\theta}{\gamma}}|M'(\xi)| < \infty. \tag{3.25}$$

Then $\{M_n\}_{n\in\mathbb{Z}}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{M_n}\}_{n\in\mathbb{Z}}$ in $\mathscr{B}(L^p(\mathbb{R};X))$ with $\mathscr{R}$-bound $\lesssim_{X,p,\tau,q,\gamma,\delta_0,\delta_\infty} [||M||_\infty \vee C_0 \vee C_\infty]$.

Proof. For every $k \in \mathbb{Z}$ and $j \in \{-1,1\}$ we have

$$2^{k(\gamma-\delta_0)}[M|_{I_{k,j}}]_{C^\gamma(I_{k,j})} \lesssim_\gamma 2^{k(\gamma-\delta_0)}2^{k(1-\gamma)}||M'|_{I_{k,j}}||_\infty \eqsim \sup_{\xi\in I_{k,j}}|\xi|^{1-\delta_0}|M'(\xi)|$$

and

$$\begin{aligned} 2^{k(\gamma+\delta_\infty)}[M|_{I_{k,j}}]_{C^\gamma(I_{k,j})} &\leq 2^{k(\gamma+\delta_\infty)}2^{1-\gamma}||M|_{I_{k,j}}||_\infty^{1-\gamma}||M'|_{I_{k,j}}||_\infty^{\gamma} \\ &\lesssim_\gamma 2^{k(\gamma+\delta_\infty)\frac{1-\theta}{1-\gamma}}||M|_{I_{k,j}}||_\infty + 2^{k(\gamma+\delta_\infty)\frac{\theta}{\gamma}}||M'|_{I_{k,j}}||_\infty \\ &\eqsim \sup_{\xi\in I_{k,j}}|\xi|^{(\gamma+\delta_\infty)\frac{1-\theta}{1-\gamma}}|M(\xi)| + \sup_{\xi\in I_{k,j}}|\xi|^{(\gamma+\delta_\infty)\frac{\theta}{\gamma}}|M'(\xi)|. \end{aligned}$$

The result now easily follows from Theorem 3.3.14. □

3.4. DIFFERENCE NORMS

3.4.1. Notation

Let X be a Banach space. For each $m \in \mathbb{Z}_{\geq 1}$ and $h \in \mathbb{R}^d$ we define difference operator Δ_h^m on $L^0(\mathbb{R}^d;X)$ by $\Delta_h^m := (L_h - I)^m = \sum_{j=0}^m (-1)^j \binom{m}{j} L_{(m-j)h}$, where L_h denotes the left translation by h:

$$\Delta_h^m f(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} f(x+(m-j)h), \qquad f \in L^0(\mathbb{R}^d;X), x \in \mathbb{R}^d.$$

Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$, $m \in \mathbb{Z}_{\geq 1}$, and $K \in \mathscr{K}(\mathbb{R}^d)$. For every $c > 0$, $\widetilde{K}_c = c^d K(-c\,\cdot) \in \mathscr{K}(\mathbb{R}^d)$ gives rise to a (well-defined) bounded convolution operator $f \mapsto \widetilde{K}_c * f$ on $L^p(\mathbb{R}^d,w;X)$ of norm $\lesssim_{p,d,w} ||\widetilde{K}_c||_{\mathscr{K}(\mathbb{R}^d)} = ||K||_{\mathscr{K}(\mathbb{R}^d)}$, which is given by the formula

$$\widetilde{K}_c * f(x) = \int_{\mathbb{R}^d}\widetilde{K}_c(x-y)f(y)\,dy = \int_{\mathbb{R}^d}K(h)L_{c^{-1}h}f(x)\,dh, \qquad x \in \mathbb{R}^d;$$

see the last part of Section 3.2.2. Defining $K^{\Delta^m} := \sum_{j=0}^{m-1}(-1)^j\binom{m}{j}\widetilde{K}_{[(m-j)]^{-1}} \in \mathscr{K}(\mathbb{R}^d)$, for each $t>0$ the operator

$$f \mapsto K_m(t,f) := K^{\Delta^m}_{t^{-1}} * f + (-1)^m \hat{K}(0) f = \sum_{j=0}^{m-1}(-1)^j \binom{m}{j} \widetilde{K}_{[(m-j)t]^{-1}} * f + (-1)^m \hat{K}(0) f$$

is bounded on $L^p(\mathbb{R}^d, w; X)$ of norm $\lesssim_{p,d,w,m} ||K||_{\mathscr{K}(\mathbb{R}^d)}$, and the following identity holds

$$K_m(t,f)(x) = \int_{\mathbb{R}^d} K(h)\Delta^m_{th} f(x)\,dh, \qquad x \in \mathbb{R}^d.$$

Given $f \in L^p(\mathbb{R}^d, w; X)$, the functions $K_m(t,f)$ may be interpreted as weighted means of differences of f.

For $f \in L^p(\mathbb{R}^d, w; X)$ we set

$$[f]^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)} := \sup_{J\in\mathbb{N}} \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} K_m(2^{-j}, f) \Big|\Big|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}.$$

and

$$|||f|||^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)} := ||f||_{L^p(\mathbb{R}^d,w;X)} + [f]^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}.$$

3.4.2. Statement of the Main Result

The following theorem is the main result of this paper. As already announced in the introduction, it is (indeed) a more general version of Theorem 3.1.1 thanks to the $\mathscr{R}$-boundedness results Theorem 3.3.11 and Corollary 3.3.15; see Examples 3.4.4 and 3.4.5.

Theorem 3.4.1. *Let X be a UMD Banach space, $s>0$, $p\in(1,\infty)$, $w\in A_p(\mathbb{R}^d)$, $m\in\mathbb{Z}_{\geq 1}$ and $K\in\mathscr{K}(\mathbb{R}^d)$.*

(i) Suppose that $K \in L^1(\mathbb{R}^d, (1+|\cdot|)^{d+2})$ and that K^{Δ^m} fulfills the Tauberian condition

$$|\mathscr{F}K^{\Delta^m}(\xi)| \geq c, \qquad \xi\in\mathbb{R}^d, \frac{\varepsilon}{2} < |\xi| < 2\varepsilon, \tag{3.26}$$

for some $\varepsilon, c > 0$. Then we have the estimate

$$||f||_{H^s_p(\mathbb{R}^d,w;X)} \lesssim |||f|||^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}, \qquad f\in L^p(\mathbb{R}^d,w;X). \tag{3.27}$$

(ii) Suppose that $m>s$, $K\in L^1(\mathbb{R}^d,(1+|\cdot|)^{(d+3)m})$, and that $\{f\mapsto K_m(2^{-j},f) : j\in\mathbb{Z}_{\geq 1}\} \subset \mathscr{B}(L^p(\mathbb{R}^d,w;X))$ is $\mathscr{R}$-bounded. Then we have the estimate

$$|||f|||^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)} \lesssim ||f||_{H^s_p(\mathbb{R}^d,w;X)}, \qquad f\in L^p(\mathbb{R}^d,w;X). \tag{3.28}$$

Remark 3.4.2. The $\mathscr{R}$-boundedness condition in (ii) of the above theorem may be replaced by the (at first sight) weaker condition that

$$\Big|\Big| \sum_{j=1}^{N} \varepsilon_j K_m(2^{-j}, g_j) \Big|\Big|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))} \lesssim \Big|\Big| \sum_{j=1}^{N} \varepsilon_j g_j \Big|\Big|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}, \qquad N \in \mathbb{N},$$

for all $\{g_j\}_{j\geq 1} \subset L^p(\mathbb{R}^d, w; X)$ with Fourier support $\operatorname{supp} \hat{g}_j \subset \{\xi : |\xi| \geq c2^j\}$, where $c > 0$ is some fixed number. But the $\mathscr{R}$-boundedness condition in (ii) is in fact implied by this condition. Indeed, this condition implies the $\mathscr{R}$-boundedness of the sequence of Fourier multiplier operators associated with the the sequence of symbols $\big\{[(1-\zeta)\widehat{K^{\Delta^m}}](2^{-j}\,\cdot)\big\}_{j\geq 1}$, where $\zeta \in C_c^\infty(\mathbb{R}^d)$ is a bump function which is 1 on a neighborhood of the set $\{\xi : |\xi| \geq c\}$. On the other hand, we have $\zeta\widehat{K^{\Delta^m}} \in C_c^{d+2}(\mathbb{R}^d)$ in view of $\widehat{K^{\Delta^m}} \subset \mathscr{F}L^1(\mathbb{R}^d, (1+|\cdot|)^{d+2}) \subset C_b^{d+2}(\mathbb{R}^d)$, so that we can apply Theorem 3.3.11 to the symbol $\zeta\widehat{K^{\Delta^m}}$. We thus find that the sequence of symbols $\big\{\widehat{K_{2^j}^{\Delta^m}} = \widehat{K^{\Delta^m}}(2^{-j}\,\cdot)\big\}_{j\geq 1}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators on $L^p(\mathbb{R}^d, w; X)$, which is of course equivalent to the $\mathscr{R}$-boundedness condition in (ii).

Remark 3.4.3. Let X be a Banach space, $s > 0$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. For each $f \in L^p(\mathbb{R}^d, w; X)$ we put

$$[f]^{(m,K);\mathbb{Z}}_{H^s_p(\mathbb{R}^d,w;X)} := \sup_{J\in\mathbb{N}} \Big|\Big| \sum_{j=-J}^{J} \varepsilon_j 2^{js} \varepsilon_j 2^{js} K_m(2^{-j}, f) \Big|\Big|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))}.$$

On the one hand, $[\,\cdot\,]^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)} \leq [\,\cdot\,]^{(m,K);\mathbb{Z}}_{H^s_p(\mathbb{R}^d,w;X)}$ thanks to the contraction principle (3.7). On the other hand, $[\,\cdot\,]^{(m,K);\mathbb{Z}}_{H^s_p(\mathbb{R}^d,w;X)} \lesssim ||\cdot||_{L^p(\mathbb{R}^d,w;X)} + [\,\cdot\,]^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}$ because $s > 0$ and $\{f \mapsto K_m(2^{-j}, f) : j \in \mathbb{Z}\}$ is a uniformly bounded family in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$. In Theorem 3.4.1 we may thus replace $|||\cdot|||^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}$ by $||\cdot||_{L^p(\mathbb{R}^d,w;X)} + [\,\cdot\,]^{(m,K);\mathbb{Z}}_{H^s_p(\mathbb{R}^d,w;X)}$.

Example 3.4.4. Let $K \in \mathscr{K}(\mathbb{R}^d)$ and $m \in \mathbb{Z}_{\geq 1}$.

(i) Note that $\mathscr{F}K^{\Delta^m} \in C_b(\mathbb{R}^d)$ with $\mathscr{F}K^{\Delta^m}(0) = \sum_{j=0}^{m-1}(-1)^j\binom{m}{j}\hat{K}(0) = (-1)^{m+1}\hat{K}(0)$. So for K^{Δ^m} to fulfill the Tauberian condition (3.26) for some $\varepsilon, c > 0$ it is sufficient that $\hat{K}(0) \neq 0$.

(ii) Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Note that the $\mathscr{R}$-boundedness conditition in Theorem 3.4.1(ii) is equivalent to the $\mathscr{R}$-boundedness of the convolution operators $\{f \mapsto K_{2^j}^{\Delta^m} * f : j \in \mathbb{Z}_{\geq 1}\} \subset \mathscr{B}(L^p(\mathbb{R}^d, w; X))$. By Theorem 3.3.11, for the latter it is sufficient that $K \in L^1(\mathbb{R}^d, (1+|\cdot|)^{d+2}) \subset \mathscr{F}^{-1}C_b^{d+2}(\mathbb{R}^d)$ fulfills the condition

$$\sup_{|\alpha|\leq d+2} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^{|\alpha|+\delta} |D^\alpha \hat{K}(\xi)| < \infty \tag{3.29}$$

for some $\delta > 0$; in particular, it is sufficient that $K \in \mathscr{S}(\mathbb{R}^d)$.

Under the availability of better multiplier theorems than $\mathscr{M}_{d+2}(\mathbb{R}^d) \hookrightarrow \mathscr{M}_{p,w}(X)$, the condition (3.29) can be weakened; see Remark 3.3.12. For example, in the one-dimensional case $d = 1$ we can use $\mathscr{M}_1(\mathbb{R}) \hookrightarrow \mathscr{M}_{p,w}(X)$, resulting in the weaker condition that

$$\sup_{k=0,1} (1+|\xi|)^{k+\delta} |\hat{K}^{(k)}(\xi)| < \infty$$

for some $\delta > 0$. However, this condition is still to strong to handle the kernel $K = 2^{-1} 1_{[-1,1]} \in L^\infty_c(\mathbb{R}^d) \subset \mathscr{K}(\mathbb{R}^d) \cap \mathscr{F}^{-1} C^\infty_0(\mathbb{R}^d)$ with Fourier transform $\hat{K} = \operatorname{sinc}$, where $\operatorname{sinc}(t) = \sin(t)/t$ for $t \neq 0$ and $\operatorname{sinc}(0) = 1$. As already announced, in the unweighted case this K can be handled by Corollary 3.3.15:

Example 3.4.5. Let X be a UMD Banach space, $p \in (1,\infty)$ and $K = 2^{-d} 1_{Q[0,1]}$. For every $m \in \mathbb{Z}_{\geq 1}$ it holds that $\{f \mapsto K_m(2^{-j}, f) : j \in \mathbb{Z}\} \subset \mathscr{B}(L^p(\mathbb{R}^d; X))$ is $\mathscr{R}$-bounded.

Proof. It is enough to show that $\{T_{\hat{K}(\ell 2^{-j}\cdot)} : j \in \mathbb{Z}, \ell \in \{1,\ldots,m\}\} = \{f \mapsto K_{\ell^{-1}2^j} * f : j \in \mathbb{Z}, \ell \in \{1,\ldots,m\}\}$ is $\mathscr{R}$-bounded in $\mathscr{B}(L^p(\mathbb{R}^d; X))$. By the product structure of K it suffices to consider the case $d = 1$. So we only need to check that $M := \operatorname{sinc} = \mathscr{F}\frac{1}{2}1_{[-1,1]} \in C^\infty_0(\mathbb{R})$ satisfies the conditions from Corollary 3.3.15. In the notation of Corollary 3.3.15, let $\gamma \in (\gamma_X, 1)$ be fixed. The condition (3.24) is fulfilled for $\delta_0 = 1$ because sinc is a C^1-function on $[-1,1]$. Furthermore, the condition (3.25) is fulfilled for any $\delta_\infty \in (0, 1-\gamma)$ and $\theta = \gamma$. □

Still consider $K = 2^{-d} 1_{Q[0,1]} \in L^\infty_c(\mathbb{R}^d) \subset \mathscr{K}(\mathbb{R}^d)$. The $\mathscr{R}$-boundedness condition from Theorem 3.4.1(ii) is fulfilled provided that, for each $\ell \in \{1,\ldots,m\}$, the set of convolution operators $\{f \mapsto K_t * f : t = \ell^{-1}2^j, j \in \mathbb{Z}_{\geq 1}\} \subset \mathscr{B}(L^p(\mathbb{R}^d, w; X))$ is $\mathscr{R}$-bounded. A nice way to look at the convolution operator $f \mapsto K_{r^{-1}} * f$, $r > 0$, is as the averaging operator $A_r \in \mathscr{B}(L^p(\mathbb{R}^d, w; X))$ given by

$$A_r f(x) := \fint_{Q[x,r]} f(y)\, dy, \qquad f \in L^p(\mathbb{R}^d, w; X), x \in \mathbb{R}^d.$$

This leads to the following natural question:

Question 3.4.6. *Given a UMD space X, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ and $c > 0$, is the set of averaging operators $\{A_r : r = c2^{-j}, j \in \mathbb{Z}_{\geq 1}\}$ $\mathscr{R}$-bounded in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$?*

Three cases in which we can give a positive answer to this question are:

(i) X is a UMD space, $p \in (1,\infty)$ and $w = 1$;

(ii) X is a UMD space with property (α), $p \in (1,\infty)$ and $w \in A^{rec}_p(\mathbb{R}^d)$;[7]

(iii) X is a UMD Banach function space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$;

[7] Recall that A^{rec}_p is the class of weights on $\mathbb{R}^d$ which are uniformly A_p in each of the coordinates separately.

Here case (i) follows similarly to the proof of Example 3.4.5, case (ii) follows from an application of Proposition 3.3.1(ii), and case (iii) can be treated via the Banach lattice version of the Hardy-Littlewood maximal function by using the fact that $\mathscr{R}$-boundedness coincides with ℓ^2-boundedness in this situation (see Proposition 3.4.11 for a more general result in this direction). Note that in the cases (ii) and (iii) one in fact has $\mathscr{R}$-boundedness of $\{A_r : r > 0\}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$

3.4.3. Proof of the Main Result

Below we will use the following notation:

$$X_{p,w} := L^p(\Omega; L^p(\mathbb{R}^d, w; X)) = L^p(\mathbb{R}^d, w; L^p(\Omega; X)).$$

$$X_{p,w}(\mathbb{R}^d_\pm) := L^p(\Omega; L^p(\mathbb{R}^d_\pm, w; X)) = L^p(\mathbb{R}^d_\pm, w; L^p(\Omega; X)).$$

Proof of Theorem 3.4.1(i).

Lemma 3.4.7. *Let X be a UMD space, $s \in \mathbb{R}$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Suppose that $k \in \mathscr{K}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|\cdot|)^{d+2})$ fulfills the Tauberian condition*

$$|\hat{k}(\xi)| > 0, \qquad \xi \in \mathbb{R}^d, \frac{\varepsilon}{2} < |\xi| < 2\varepsilon,$$

for some $\varepsilon > 0$. For $f \in L^p(\mathbb{R}^d, w; X)$ we can then estimate

$$||f||_{H^s_p(\mathbb{R}^d,w;X)} \lesssim ||f||_{L^p(\mathbb{R}^d,w;X)} + \sup_{J\in\mathbb{N}} \Big|\Big| \sum_{j=1}^J \varepsilon_j 2^{js} k_j * f \Big|\Big|_{X_{p,w}}. \tag{3.30}$$

Proof. Pick $\varphi = (\varphi_j)_{j\in\mathbb{N}} \in \Phi(\mathbb{R}^d)$ such that $\operatorname{supp}\hat{\varphi}_1 \subset \{\xi : |\xi| \geq 2\varepsilon\}$; see (3.9). Using (3.13) in combination with $S_0 \in \mathscr{B}(L^p(\mathbb{R}^d, w; X))$, we get

$$||f||_{H^s_p(\mathbb{R}^d,w;X)} \lesssim ||f||_{L^p(\mathbb{R}^d,w;X)} + \sup_{J\in\mathbb{N}} \Big|\Big| \sum_{j=1}^J \varepsilon_j 2^{js} S_j f \Big|\Big|_{X_{p,w}}.$$

In view of the contraction principle (3.7), it is thus enough to find an $N \in \mathbb{N}$ such that

$$\Big|\Big| \sum_{j=1}^J \varepsilon_j 2^{js} S_j f \Big|\Big|_{X_{p,w}} \lesssim \Big|\Big| \sum_{j=1}^{J+N} \varepsilon_j 2^{js} k_j * f \Big|\Big|_{X_{p,w}}, \qquad f \in L^p(\mathbb{R}^d, w; X), J \in \mathbb{N}. \tag{3.31}$$

In order to establish (3.31), pick $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\eta \subset B(0, 2\varepsilon)$ and $\eta(\xi) = 1$ for $|\xi| \leq \frac{3\varepsilon}{2}$. Define $m \in C_c^{d+2}(\mathbb{R}^d) \subset \mathscr{M}_{d+2}(\mathbb{R}^d)$ by $m(\xi) := [\eta(\xi) - \eta(2\xi)]\hat{k}(\xi)^{-1}$ if $\frac{\varepsilon}{2} < |\xi| < 2\varepsilon$ and $m(\xi) := 0$ otherwise; note that this gives a well-defined C^{d+2}-function on $\mathbb{R}^d$ because $\eta - \eta(2\cdot)$ is a smooth function supported in the set $\{\xi : \frac{\varepsilon}{2} < |\xi| < 2\varepsilon\}$ on which the function $\hat{k} \in C^{d+2}(\mathbb{R}^d)$ does not vanish, where the regularity $\hat{k} \in C^{d+2}(\mathbb{R}^d)$ is a consequence of the assumption that $k \in L^1(\mathbb{R}^d, (1+|\cdot|)^{d+2})$. By Example 3.3.7, the sequence of (dyadic) dilated symbols $\{m_j := m(2^{-j}\cdot)\}_{j\geq 1}$ defines an $\mathscr{R}$-bounded sequence of Fourier multiplier operators $\{T_{m_j}\}_{j\geq 1}$ on $L^p(\mathbb{R}^d, w; X)$. Furthermore, by construction we have

$$\sum_{l=j}^{j+N} m_l \hat{k}_l(\xi) = \eta(2^{-(j+N)}\xi) - \eta(2^{-j+1}\xi) = 1 \quad \text{for } 2^j\varepsilon \leq |\xi| \leq 2^{j+N-1}3\varepsilon,\, j\geq 1, N\in\mathbb{N}.$$

Since $\operatorname{supp}\hat{\varphi}_j \subset \{\xi : 2^j\varepsilon \leq |\xi| < 2^j B\}$ for every $j\geq 1$ for some $B > \varepsilon$, there thus exists $N\in\mathbb{N}$ such that $\sum_{l=j}^{j+N} m_l\hat{k}_l \equiv 1$ on $\operatorname{supp}\hat{\varphi}_j$ for all $j\geq 1$. For each $j\geq 1$ we consequently have

$$S_j = T_{\hat{\varphi}_j} = T_{\hat{\varphi}_j\left(\sum_{l=j}^{j+N} m_l\hat{k}_l\right)} = \sum_{l=j}^{j+N} T_{\hat{\varphi}_j} T_{m_l} T_{\hat{k}_l} = \sum_{l=0}^{N} S_j T_{m_{j+l}}[k_{j+l} * \cdot] \quad \text{in } \mathscr{B}(L^p(\mathbb{R}^d, w; X)).$$

Using this together with the $\mathscr{R}$-boundedness of $\{S_j\}_{j\in\mathbb{N}}$ and $\{T_{m_j}\}_{j\geq 1}$ (see Example 3.3.7), for each $f\in L^p(\mathbb{R}^d, w; X)$ we obtain the estimates

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}S_j f\Big|\Big|_{X_{p,w}} &\leq \sum_{l=0}^{N}\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}S_j T_{m_{j+l}}[k_{j+l}*f]\Big|\Big|_{X_{p,w}} \\
&\lesssim \sum_{l=0}^{N}\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}k_{j+l}*f\Big|\Big|_{X_{p,w}} \\
&\lesssim \Big|\Big|\sum_{j=1}^{J+N}\varepsilon_j 2^{js}k_j*f\Big|\Big|_{X_{p,w}}.
\end{aligned}$$

□

Proof of Theorem 3.4.1(i). In view of (3.26) and the fact that $\mathscr{F}K^{\Delta^m}\in C_0(\mathbb{R}^d)$, there exists $N\in\mathbb{N}$ such that the function $k\in\mathscr{K}(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|\cdot|)^{d+2})$ determined by $\hat{k} = \mathscr{F}K^{\Delta^m}(2^{-N}\cdot) - \mathscr{F}K^{\Delta^m}$ fulfills the Tauberian condition

$$|\hat{k}(\xi)| \geq \frac{c}{2} > 0, \qquad \xi\in\mathbb{R}^d, \frac{\delta}{2} < |\xi| < 2\delta,$$

for $\delta := 2^N\varepsilon > 0$. Since

$$\begin{aligned}
k_j * f &= [K^{\Delta^m}_{2^{-(j+N)}} * f + (-1)^m\hat{K}(0)f] - [K^{\Delta^m}_{2^{-j}} * f + (-1)^m\hat{K}(0)f] \\
&= K_m(2^{-(j+N)}, f) - K_m(2^{-j}, f), \qquad j\geq 1,
\end{aligned}$$

with Lemma 3.4.7 it follows that

$$\begin{aligned}
||f||_{H^s_p(\mathbb{R}^d,w;X)} &\lesssim ||f||_{L^p(\mathbb{R}^d,w;X)} + \sup_J\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}k_j*f\Big|\Big|_{X_{p,w}} \\
&\lesssim ||f||_{L^p(\mathbb{R}^d,w;X)} + \sup_J 2^{-Ns}\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{(j+N)s}K_m(2^{-(j+N)}, f)\Big|\Big|_{X_{p,w}} \\
&\quad + \sup_J\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_m(2^{-j}, f)\Big|\Big|_{X_{p,w}}
\end{aligned}$$

$$\overset{(3.7)}{\leq} \quad ||f||_{L^p(\mathbb{R}^d,w;X)} + (2^{-Ns}+1)[f]^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}.$$

□

Proof of Theorem 3.4.1(ii).

Lemma 3.4.8. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and $\eta \in C_c^\infty(\mathbb{R}^d)$. For each $n \in \mathbb{Z}_{\leq 0}$ and $h \in \mathbb{R}^d$ we define the sequence of symbols $\{M_j^{h,n}\}_{j\in\mathbb{Z}} \subset L^\infty(\mathbb{R}^d)$ by*

$$M_j^{h,n}(\xi) := \begin{cases} (e^{\imath 2^{-j} h\cdot\xi} - 1)\chi(2^{-(n+j)}\xi), & n+j \geq 1 \\ (e^{\imath 2^{-j} h\cdot\xi} - 1)\eta(2^{-(n+j)}\xi), & n+j = 0 \\ 0, & n+j \leq -1 \end{cases}$$

Then each symbol $M_j^{h,n}$ defines a bounded Fourier multiplier operator $T_j^{h,n} = T_{M_j^{h,n}}$ on $L^p(\mathbb{R}^d, w; X)$ such that the following $\mathscr{R}$-bound is valid:

$$\mathscr{R}\{T_j^{h,n} : j \in \mathbb{Z}\} \lesssim 2^n(1+|h|)^{d+3}, \qquad h \in \mathbb{R}^d, n \in \mathbb{Z}_{\leq 0}. \tag{3.32}$$

Proof. By construction, $\{M_j^{h,n}\}_{j\in\mathbb{Z}} \subset C_c^\infty(\mathbb{R}^d)$ satisfies condition (a) of Corollary 3.3.5 for some $N \in \mathbb{N}$ independent of $n \in \mathbb{Z}_{\leq 0}$ and $h \in \mathbb{R}^d$. Therefore, it is enough to show that

$$||M_j^{h,n}||_{\mathscr{M}_{d+2}} \lesssim 2^n(1+|h|)^{d+3}, \qquad h \in \mathbb{R}^d, n \in \mathbb{Z}_{\leq 0}, j \in \mathbb{Z}. \tag{3.33}$$

We only consider the case $n+j \geq 1$ in (3.33), the case $n+j=0$ being comletely similar and the case $n+j \leq -1$ being trivial. Let $h \in \mathbb{R}^d$, $n \in \mathbb{Z}_{\leq 0}$ and $j \in \mathbb{Z}$ with $n+j \geq 1$ be given. Fix a multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d+2$. Using the Leibniz rule, we compute

$$\begin{aligned}
&|\xi|^{|\alpha|} D^\alpha M_j^{h,n}(\xi) \\
&= |\xi|^{|\alpha|} D_\xi^\alpha \left(\imath h\cdot\xi \int_0^{2^{-j}} e^{\imath s h\cdot\xi}\, ds\, \chi(2^{-(n+j)}\xi) \right) \\
&= \imath \sum_{\beta+\gamma\leq\alpha} c^\alpha_{\beta,\gamma} |\xi|^{|\beta|} D_\xi^\beta (h\cdot\xi)\, |\xi|^{|\gamma|} D_\xi^\gamma \left(\int_0^{2^{-j}} e^{\imath s h\cdot\xi}\, ds \right) |\xi|^{|\alpha|-|\beta|-|\gamma|} D_\xi^{\alpha-\beta-\gamma}[\chi(2^{-(n+j)}\xi)] \\
&= \imath \sum_{\gamma\leq\alpha} c^\alpha_{0,\gamma}\, h\cdot\xi\, |\xi|^{|\gamma|} \int_0^{2^{-j}} (\imath s h)^\gamma e^{\imath s h\cdot\xi}\, ds\, |2^{-(n+j)}\xi|^{|\alpha|-|\gamma|} [D^{\alpha-\gamma}\chi](2^{-(n+j)}\xi) \\
&+ \imath \sum_{\beta+\gamma\leq\alpha; |\beta|=1} c^\alpha_{\beta,\gamma} |\xi| h^\beta\, |\xi|^{|\gamma|} \int_0^{2^{-j}} (\imath s h)^\gamma e^{\imath s h\cdot\xi}\, ds\, |2^{-(n+j)}\xi|^{|\alpha|-|\beta|-|\gamma|} [D^{\alpha-\beta-\gamma}\chi](2^{-(n+j)}\xi).
\end{aligned}$$

Picking $R > 0$ such that $\operatorname{supp}\chi \subset B(0,R)$, we can estimate

$$|\xi|^{|\alpha|} |D^\alpha M_j^{h,n}(\xi)| \quad \lesssim \quad \sum_{\gamma\leq\alpha} |h|^{|\gamma|+1} 2^{-j(|\gamma|+1)}\, 1_{B(0,R)}(2^{-(n+j)}\xi)\, |\xi|^{|\gamma|+1}\, ||\chi||_{\mathscr{M}_{d+2}}$$

$$
\begin{aligned}
& + \sum_{\beta+\gamma\leq\alpha;|\beta|=1} |h|^{|\gamma|+1} 2^{-j(|\gamma|+1)} 1_{B(0,R)}(2^{-(n+j)}\xi)\,|\xi|^{|\gamma|+1}\,||\chi||_{\mathscr{M}_{d+2}} \\
\leq\quad & 2||\chi||_{\mathscr{M}_{d+2}} \sum_{\gamma\leq\alpha} |h|^{|\gamma|+1} 2^{n(|\gamma|+1)} R^{|\gamma|+1} \\
\overset{n\leq 0}{\lesssim}\quad & 2^{n}(1+|h|)^{d+3}.
\end{aligned}
$$

This proves the required estimate (3.33). □

Proof of Theorem 3.4.1(ii). Given $f \in L^p(\mathbb{R}^d, w; X)$, write $f_n := S_n f$ for $n \in \mathbb{N}$ and $f_n := 0$ for $n \in \mathbb{Z}_{<0}$. For each $j \in \mathbb{Z}_{>0}$ we then have $f = \sum_{n\in\mathbb{Z}} f_{n+j}$ in $L^p(\mathbb{R}^d, w; X)$, from which it follows that

$$
\Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} K_m(2^{-j}, f)\Big|\Big|_{X_{p,w}} \leq \sum_{n\in\mathbb{Z}} \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} K_m(2^{-j}, f_{n+j})\Big|\Big|_{X_{p,w}}. \tag{3.34}
$$

We first estimate the sum over $n \in \mathbb{Z}_{>0}$ in (3.34). Using the $\mathscr{R}$-boundedness of $\{f \mapsto K_m(2^{-j}, f) : j \geq 1\}$, we find

$$
\Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} K_m(2^{-j}, f_{n+j})\Big|\Big|_{X_{p,w}} \lesssim 2^{-ns} \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{(n+j)s} f_{n+j}\Big|\Big|_{X_{p,w}} \leq 2^{-ns} ||f||_{H^s_p(\mathbb{R}^d,w;X)}.
$$

Since $s > 0$, it follows that the sum over $n \in \mathbb{Z}_{>0}$ in (3.34) can be estimated from above by $C||f||_{H^s_p(\mathbb{R}^d,w;X)}$ for some constant C independent of f and J.

Next we estimate the sum over $n \in \mathbb{Z}_{\leq 0}$ in (3.34). To this end, let $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and $\eta \in C_c^\infty$ be such that $\chi \equiv 1$ on $\frac{1}{2}\operatorname{supp}\hat{\varphi}_1$ and $\eta \equiv 1$ on $\operatorname{supp}\hat{\varphi}_0$. For every $\lambda \in \mathbb{C}$ we define the function $e_\lambda : \mathbb{R}^d \to \mathbb{C}$ by $e_\lambda(\xi) := e^{\lambda\cdot\xi}$. For each $n \leq 0$, $h \in \mathbb{R}^d$ and $j \geq 1$, we then have

$$
\begin{aligned}
\Delta^m_{2^{-j}h} f_{n+j} &= \mathscr{F}^{-1}[(e_{\imath 2^{-j}h} - 1)^m \hat{f}_{n+j}] \\
&= \begin{cases} \mathscr{F}^{-1}\left[\left(e_{\imath 2^{-j}h} - 1)\chi(2^{-(n+j)}\,\cdot)\right)^m \hat{f}_{n+j}\right], & n+j \geq 1; \\ \mathscr{F}^{-1}\left[\left(e_{\imath 2^{-j}h} - 1)\eta(2^{-(n+j)}\,\cdot)\right)^m \hat{f}_{n+j}\right], & n+j = 0; \\ 0, & n+j \leq -1. \end{cases} \\
&= T^m_{M_j^{h,n}} f_{n+j},
\end{aligned}
$$

where $M_j^{h,n}$ is the Fourier multiplier symbol from Lemma 3.4.8. For each $n \leq 0$ we thus get

$$
\begin{aligned}
\Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} K_m(2^{-j}, f_{n+j})\Big|\Big|_{X_{p,w}} &\leq \int_{\mathbb{R}^d} |K(h)| \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} \Delta^m_{2^{-j}h} f_{n+j}(\cdot)\Big|\Big|_{X_{p,w}} dh \\
&= \int_{\mathbb{R}^d} |K(h)| \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{js} T^m_{M_j^{h,n}} f_{n+j}\Big|\Big|_{X_{p,w}} dh \\
&\overset{(3.32)}{\lesssim} 2^{n(m-s)} \int_{\mathbb{R}^d} |K(h)|(1+|h|)^{(d+3)m} dh
\end{aligned}
$$

$$\cdot \Big|\Big| \sum_{j=1}^{J} \varepsilon_j 2^{(n+j)s} f_{n+j} \Big|\Big|_{X_{p,w}}$$

$$\overset{(3.13)}{\lesssim} \quad 2^{n(m-s)} ||f||_{H^s_p(\mathbb{R}^d,w;X)}.$$

Since $m-s>0$, it follows that the sum over $n \in \mathbb{Z}_{\leq 0}$ in (3.34) can be estimated from above by $C||f||_{H^s_p(\mathbb{R}^d,w;X)}$ for some constant C independent of f and J. □

The idea to do the estimate (3.34) and to treat the sum over $n \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\leq 0}$ separately is taken from the proof of [220, Proposition 6], which is concerned with a difference norm characterization for $F^s_{p,q}(\mathbb{R}^d;X)$.

3.4.4. The Special Case of a Banach Function Space

In the special case that X is a Banach function space, we obtain the following corollary from the main result Theorem 3.4.1:

Corollary 3.4.9. *Let X be a UMD Banach function space, $s>0$, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ and $m \in \mathbb{N}$, $m>s$. Suppose that $K \in \mathscr{K}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d,(1+|\cdot|)^{(d+3)m})$ satisfies the Tauberian condition* (3.26) *for some $c,\varepsilon>0$. For all $f \in L^p(\mathbb{R}^d,w;X)$ we then have the equivalence of extended norms*

$$||f||_{H^s_p(\mathbb{R}^d,w;X)} \eqsim ||f||_{L^p(\mathbb{R}^d,w;X)} + \Big|\Big|\Big(\sum_{j=1}^{\infty} |2^{js}K_m(2^{-j},f)|^2 \Big)^{1/2} \Big|\Big|_{L^p(\mathbb{R}^d,w;X)}. \tag{3.35}$$

Proof. By the Khintchine-Maurey theorem, the right-hand side (RHS) of (3.35) defines an extended norm on $L^p(\mathbb{R}^d,w;X)$ which is equivalent to $|||\cdot|||^{(m,K)}_{H^s_p(\mathbb{R}^d,w;X)}$. Therefore, we only need to check the $\mathscr{R}$-boundedness condition in Theorem 3.4.1(ii). But this follows from Proposition 3.4.11 below (and the discussion after it). □

Remark 3.4.10. Let X be a UMD Banach function space, $s>0$, $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ and $m \in \mathbb{N}$, $m>s$. Suppose $K \in \mathscr{K}(\mathbb{R}^d)^+ \setminus \{0\}$. Then it is a natural question whether we can replace $K_m(2^{-j},f)$ by $d^m_K(2^{-j},f)$ in the RHS of (3.35), where

$$d^m_K(t,f)(x) := \int_{\mathbb{R}^d} K(h)|\Delta^m_h f(x)|\,dh, \qquad t>0, x \in \mathbb{R}^d.$$

In view of the domination $|K_m(t,f)| \leq d^m_K(t,f)$, this is certainly true for the inequality '$\lesssim$' in (3.35). For the reverse inequality '$\lesssim$' one could slightly modify the difference norm characterization that is obtained from a combination of Theorem 2.4.7 and Example 2.3.15.(iii). (b).

Proposition 3.4.11. *Let X be a UMD Banach function space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Then $\mathscr{K}(\mathbb{R}^d) \hookrightarrow \mathscr{B}(L^p(\mathbb{R}^d,w;X))$ maps bounded sets to $\mathscr{R}$-bounded sets.*

Proof. In the unweighted case $w = 1$ this can be found in [193, Section 4]. However, the Banach lattice version of the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{R}^d, w; X(\ell^2))$ for general $w \in A_p$, which which is implicitly contained [94]; also see [232]. Hence, the results from [193, Section 4] remain valid for general $w \in A_p$. □

Recall that, given $k \in \mathscr{K}(\mathbb{R}^d)$, for all $t > 0$ we have $k_t = t^d k(t\,\cdot) \in \mathscr{K}(\mathbb{R}^d)$ with $||k_t||_{\mathscr{K}(\mathbb{R}^d)} = ||k||_{\mathscr{K}(\mathbb{R}^d)}$. So, under the assumptions of the above proposition,

$$\mathscr{R}\{f \mapsto k_t * f : t > 0\} \lesssim_{X,p,d,w} ||k||_{\mathscr{K}(\mathbb{R}^d)} \quad \text{in} \quad \mathscr{B}(L^p(\mathbb{R}^d, w; X)).$$

In particular, if $m \in \mathbb{Z}_{\geq 1}$ and $K \in \mathscr{K}(\mathbb{R}^d)$, then the choice $k = K^{\Delta^m}$ leads to the $\mathscr{R}$-boundedness of $\{f \mapsto K_m(t, f) : t > 0\}$ in $\mathscr{B}(L^p(\mathbb{R}^d, w; X))$.

3.5. $1_{\mathbb{R}^d_+}$ AS POINTWISE MULTIPLIER

3.5.1. Proof of Theorem 4.4.1

Besides Theorem 3.1.1 (or Theorem 3.4.1), we need two lemmas for the proof of Theorem 4.4.1. The first lemma says that the inclusion (3.5) automatically implies its vector-valued version.

Lemma 3.5.1. *Let $s \geq 0$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $w_{s,p}$ be the weight from Theorem 4.4.1. If $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$, then there also is the inclusion*

$$H^s_p(\mathbb{R}^d, w; X) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p}; X) \tag{3.36}$$

for any Banach space X.

Proof. This can be shown as in [185, Proof of Theorem 1.3,pg. 8], which is based on the fact that the Bessel potential operator $\mathscr{J}_{-s}$ ($s \geq 0$) is positive as an operator from $L^p(\mathbb{R}^d, w)$ to $H^s_p(\mathbb{R}^d, w)$ (in the sense that $\mathscr{J}_{-s} f \geq 0$ whenever $f \geq 0$). □

The second lemma is very similar to Theorem 3.4.1(ii) and may be thought of as an $\mathbb{R}^d_+$-version for the case $m = 1$.

Lemma 3.5.2. *Let X be a UMD Banach space, $s \in (0,1)$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Let $K \in \mathscr{K}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1 + |\cdot|)^{d+3})$. For each $f \in L^p(\mathbb{R}^d, w; X)$ we define*

$$[f]^{\#}_{H^s_p(\mathbb{R}^d_+, w; X)} = [f]^{(K)}_{H^s_p(\mathbb{R}^d_+, w; X)} := \sup_{J \in \mathbb{N}} \Big|\Big| \sum_{j=-J}^{J} \varepsilon_j 2^{js} K_{\mathbb{R}^d_+}(2^{-j}, f) \Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)},$$

where we use the notation

$$K_{\mathbb{R}^d_+}(t, f)(x) := \int_{\{h_1 \geq -x_1 t^{-1}\}} K(h) \Delta_{th} f(x)\, dh, \qquad t > 0, x \in \mathbb{R}^d_+.$$

If $\{f \mapsto \widetilde{K}_t * f : t = 2^{-j}, j \in \mathbb{Z}_{\geq 1}\} \subset \mathscr{B}(L^p(\mathbb{R}^d, w; X))$ is $\mathscr{R}$-bounded, then we have the estimate

$$[f]^{\#}_{H^s_p(\mathbb{R}^d_+, w; X)} \lesssim ||f||_{H^s_p(\mathbb{R}^d, w; X)}, \qquad f \in L^p(\mathbb{R}^d, w; X).$$

Proof. Note that, for each $t>0$, $f \mapsto K_{\mathbb{R}^d_+}(t,f)$ is a well-defined bounded linear operator on $L^p(\mathbb{R}^d,w;X)$ of norm $\lesssim_{p,d,w} ||K||_{\mathscr{K}(\mathbb{R}^d)}$. Using that $s>0$, for $f \in L^p(\mathbb{R}^d,w;X)$ we can thus estimate

$$\Big|\Big|\sum_{j=-J}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f)\Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \lesssim ||f||_{L^p(\mathbb{R}^d,w;X)} + \Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f)\Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)}.$$

Now fix $f \in L^p(\mathbb{R}^d,w;X)$ and write $f_n := S_n f$ for $n \in \mathbb{N}$ and $f_n := 0$ for $n \in \mathbb{Z}_{<0}$. Then

$$\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f)\Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \leq \sum_{n\in\mathbb{Z}}\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f_{n+j})\Big|\Big|_{X_{p,w}} \tag{3.37}$$

We first estimate the sum over $n \in \mathbb{Z}_{>0}$ in (3.37). Since

$$K_{\mathbb{R}^d_+}(2^{-j},f_{n+j})(x) = \widetilde{K}_{2^{-j}} * (1_{\mathbb{R}^d_+}f)(x) + \left(\int_{\{h_1\geq -x_1 2^j\}} K(h)\,dh\right) f_{n+j}(x),$$

we can estimate

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f_{n+j})\Big|\Big|_{X_{p,w}} &\leq \Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}\widetilde{K}_{2^{-j}} * (1_{\mathbb{R}^d_+}f_{n+j})\Big|\Big|_{X_{p,w}} \\
&\quad + \Big|\Big|x\mapsto \sum_{j=1}^{J}\varepsilon_j 2^{js}\left(\int_{\{h_1\geq -x_1 2^j\}} K(h)\,dh\right) f_{n+j}(x)\Big|\Big|_{X_{p,w}}.
\end{aligned}$$

For the first term we can use the assumed $\mathscr{R}$-boundedness of the involved convolution operators and for the second term we can use the contraction principle, to obtain

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f_{n+j})\Big|\Big|_{X_{p,w}} &\lesssim \Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}1_{\mathbb{R}^d_+}f_{n+j}\Big|\Big|_{L^p(\Omega;L^p(\mathbb{R}^d,w;X))} + \Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}f_{n+j}\Big|\Big|_{X_{p,w}} \\
&\leq 2 2^{-ns}||\sum_{j=1}^{J}\varepsilon_j 2^{(n+j)s}f_{n+j}||_{X_{p,w}} \\
&\lesssim 2^{-ns}||f||_{H^s_p(\mathbb{R}^d,w;X)}.
\end{aligned}$$

Since $s>0$, it follows that the sum over $n \in \mathbb{Z}_{>0}$ in (3.37) can be estimated from above by $C||f||_{H^s_p(\mathbb{R}^d,w;X)}$ for some constant C independent of f and J.

We next estimate the sum over $n \in \mathbb{Z}_{\leq 0}$ in (3.37). For each $n \leq 0$ we have

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{J}\varepsilon_j 2^{js}K_{\mathbb{R}^d_+}(2^{-j},f_{n+j})\Big|\Big|_{X_{p,w}} &= \Big|\Big|x\mapsto \sum_{j=1}^{J}\varepsilon_j 2^{js}\int_{\mathbb{R}^d} 1_{[-2^{-j}h_1,\infty)}(x_1)K(h)\Delta_{2^{-j}h}f_{n+j}(x)\,dh\Big|\Big|_{X_{p,w}} \\
&\leq \int_{\mathbb{R}^d}|K(h)|\Big|\Big|x\mapsto \sum_{j=1}^{J}\varepsilon_j 2^{js}1_{[-2^{-j}h_1,\infty)}(x_1)\Delta_{2^{-j}h}f_{n+j}(x)\Big|\Big|_{X_{p,w}}\,dh \\
&\leq \int_{\mathbb{R}^d}|K(h)|\Big|\Big|x\mapsto \sum_{j=1}^{J}\varepsilon_j 2^{js}\Delta_{2^{-j}h}f_{n+j}(x)\Big|\Big|_{X_{p,w}}\,dh,
\end{aligned}$$

where we used the contraction principle (3.7) in the last step. We can now proceed as in the proof of Theorem 3.4.1(ii) to estimate the sum over $n \in \mathbb{Z}_{\leq 0}$ in (3.37) by $C\|f\|_{H^s_p(\mathbb{R}^d,w;X)}$ for some constant C independent of f and J. □

Proof of Theorem 4.4.1. In view of Lemma 3.5.1, we need to show that $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if there is the continuous inclusion (3.36). Defining $\overline{w}_{s,p}$ as the weight on $\mathbb{R} \times \mathbb{R}^{d-1}$ given by $\overline{w}_{s,p}(x_1, x') := |x_1|^{-sp} w(x_1, x')$, the inclusion (3.36) is equivalent to the inclusion

$$H^s_p(\mathbb{R}^d, w; X) \hookrightarrow L^p(\mathbb{R}^d, \overline{w}_{s,p}; X) \tag{3.38}$$

because $H^s_p(\mathbb{R}^d, w; X) \hookrightarrow L^p(\mathbb{R}^d, w; X)$. So we must show that $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if there is the continuous inclusion (3.38).

Step I. *Let $K \in \mathscr{S}(\mathbb{R}^d)$ satisfy $\hat{K}(0) \neq 0$. For a function g on $\mathbb{R}^d$ we write g^ϱ for the reflection in the hyperplane $\{0\} \times \mathbb{R}^{d-1}$, i.e. $g^\varrho(x) := g(-x)$. Then $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if*

$$\Big\| x \mapsto \Big(\sum_{j\in\mathbb{Z}} \Big| 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} k(h)\, dh \Big|^2 \Big)^{1/2} \|f(x)\|_X \Big\|_{L^p(\mathbb{R}^d_+, v)} \lesssim \|f\|_{H^s_p(\mathbb{R}^d, v; X)} \tag{3.39}$$

for $f \in L^p(\mathbb{R}^d, v; X)$, $v \in \{w, w^\varrho\}$, $k \in \{K, K^\varrho\}$.

Step I.(a) $1_{\mathbb{R}^d_+}$ *is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$ if and only if*

$$[1_{\mathbb{R}^d_\pm} f]_{H^s_p(\mathbb{R}^d_\pm, w; X)} \lesssim \|f\|_{H^s_p(\mathbb{R}^d, w; X)}, \qquad f \in L^p(\mathbb{R}^d, w; X), \tag{3.40}$$

where

$$[f]_{H^s_p(\mathbb{R}^d_\pm, w; X)} := \sup_{J\in\mathbb{N}} \Big\| \sum_{j=-J}^{J} \varepsilon_j 2^{js} K_1(2^{-j}, f) \Big\|_{L^p(\Omega; L^p(\mathbb{R}^d_\pm, w; X))}.^{8}$$

Since $[g]^{(1,K);\mathbb{Z}}_{H^s_p(\mathbb{R}^d,w;X)} = \big([g]^p_{H^s_p(\mathbb{R}^d_-,w;X)} + [g]^p_{H^s_p(\mathbb{R}^d_+,w;X)}\big)^{1/p} \eqsim [g]_{H^s_p(\mathbb{R}^d_-,w;X)} + [g]_{H^s_p(\mathbb{R}^d_+,w;X)}$ for $g \in L^p(\mathbb{R}^d, w; X)$, it follows from Theorem 3.1.1 (and Remark 3.4.3) that

$$\|g\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \|g\|_{L^p(\mathbb{R}^d,w;X)} + [g]_{H^s_p(\mathbb{R}^d_-,w;X)} + [g]_{H^s_p(\mathbb{R}^d_+,w;X)}, \qquad g \in L^p(\mathbb{R}^d, w; X). \tag{3.41}$$

First we assume that (3.40) holds true. For all $f \in L^p(\mathbb{R}^d, w; X)$ we can then estimate

$$\begin{aligned}
\|1_{\mathbb{R}^d_+} f\|_{H^s_p(\mathbb{R}^d,w;X)} &\overset{(3.41)}{\lesssim} \|1_{\mathbb{R}^d_+} f\|_{L^p(\mathbb{R}^d,w;X)} + [1_{\mathbb{R}^d_+} f]_{\mathbb{R}^d_-} + [1_{\mathbb{R}^d_+} f]_{\mathbb{R}^d_+} \\
&\leq \|f\|_{L^p(\mathbb{R}^d,w;X)} + [f]_{\mathbb{R}^d_-} + [1_{\mathbb{R}^d_-} f]_{\mathbb{R}^d_-} + [1_{\mathbb{R}^d_+} f]_{\mathbb{R}^d_+} \\
&\overset{(3.40),(3.41)}{\lesssim} \|f\|_{H^s_p(\mathbb{R}^d,w;X)}.
\end{aligned}$$

[8] Recall from Section 3.4.1 that $K_1(t,f)(x) = \int_{\mathbb{R}^d} K(h) \Delta_{th} f(x)\, dh$.

Next we assume that $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d, w; X)$. Then the inequality in (3.40) for $\mathbb{R}^d_+$ follows directly from (3.41). Since $1_{\mathbb{R}^d_-} = 1 - 1_{\mathbb{R}^d_+}$, the inequality in (3.40) for $\mathbb{R}^d_-$ follows as well.

Step I.(b) (3.39) ⇔ (3.40). We only show that the inequality in (3.40) for $\mathbb{R}^d_+$ is equivalent to the inequality in (3.39) with $v = w$ and $k = K$, the equivalence of the other inequalities being completely similar. We claim that the inequality in (3.40) for $\mathbb{R}^d_+$ is equivalent to the estimate

$$\sup_{J\in\mathbb{N}} \Big|\Big| x \mapsto \sum_{j=-J}^{J} \varepsilon_j 2^{js} \int_{h_1 \leq -x_1 2^j} K(h)\, dh\, f(x) \Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \lesssim ||f||_{H^s_p(\mathbb{R}^d, w; X)}, \qquad f \in L^p(\mathbb{R}^d, w; X). \tag{3.42}$$

Let us prove the claim. Note that, in view of the identity

$$K_1(2^{-j}, 1_{\mathbb{R}^d_+} f)(x) = K_{\mathbb{R}^d_+}(2^{-j}, f)(x) + \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh\, f(x),$$

we have the inequalities

$$[1_{\mathbb{R}^d_+} f]_{H^s_p(\mathbb{R}^d_+, w; X)} \leq [f]^{\#}_{H^s_p(\mathbb{R}^d_+, w; X)} + \sup_{J\in\mathbb{N}} \Big|\Big| x \mapsto \sum_{j=-J}^{J} \varepsilon_j 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh\, f(x) \Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \tag{3.43}$$

and

$$\sup_{J\in\mathbb{N}} \Big|\Big| x \mapsto \sum_{j=-J}^{J} \varepsilon_j 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh\, f(x) \Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \leq [1_{\mathbb{R}^d_+} f]_{H^s_p(\mathbb{R}^d_+, w; X)} + [f]^{\#}_{H^s_p(\mathbb{R}^d_+, w; X)}. \tag{3.44}$$

Furthermore, note that the $\mathscr{R}$-boundedness condition from Lemma 3.5.2 is fulfilled since $K \in \mathscr{S}(\mathbb{R}^d)$; see Example 3.4.4. Plugging the estimate from Lemma 3.5.2 into (3.43), we see that (3.42) implies the inequality in (3.40) for $\mathbb{R}^d_+$. The reverse implication is obtained by plugging the estimate from Lemma 3.5.2 into (3.44).

Using the claim, this step is now completed by the observation that

$$\begin{aligned} &\Big|\Big| x \mapsto \sum_{j=-J}^{J} \varepsilon_j 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh\, f(x) \Big|\Big|_{X_{p,w}(\mathbb{R}^d_+)} \\ &\qquad = \Big|\Big| x \mapsto \Big|\Big| \sum_{j=-J}^{J} \varepsilon_j 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh \Big|\Big|_{L^p(\Omega)} ||f(x)||_X \Big|\Big|_{L^p(\mathbb{R}^d_+, w)} \\ &\qquad = \Big|\Big| x \mapsto \Big(\sum_{j\in -J}^{J} \Big| 2^{js} \int_{\{h_1 \leq -x_1 2^j\}} K(h)\, dh \Big|^2 \Big)^{1/2} ||f(x)||_X \Big|\Big|_{L^p(\mathbb{R}^d_+, w)}. \end{aligned}$$

Step II. *Let $K = K^{[1]} \otimes K^{[2]} \in C^\infty_c(\mathbb{R}^d)$, where $K^{[1]} \in C^\infty_c(\mathbb{R})$ and $K^{[2]} \in C^\infty_c(\mathbb{R}^{d-1})$ satisfy $K^{[1]} = K^{[1]}(-\,\cdot)$, $1_{[-1,1]} \leq K^{[1]} \leq 1_{[-2,2]}$ and $\widehat{K^{[2]}}(0) = 1$. Then* (3.39) *is equivalent to* (3.38). In view of the reflection symmetry $K = K^{\varrho}$, we only need to show that

$$\Big(\sum_{j\in\mathbb{Z}} \Big| 2^{js} \int_{\{h_1 \leq -y 2^j\}} K(h)\, dh \Big|^2 \Big)^{1/2} \eqsim y^{-s}, \qquad y \in \mathbb{R}_+. \tag{3.45}$$

By the choice of K,

$$\big|[-(1\wedge y2^j), -y2^j]\big| \leq \int_{h_1 \leq -y2^j} K(h)\,dh \leq \big|[-(2\wedge y2^j), -y2^j]\big|, \qquad y \in \mathbb{R}_+.$$

For every $b > 0$ we have

$$\begin{aligned}
\Big(\sum_{j\in\mathbb{Z}} \Big[2^{js}\big|[-(b\wedge y2^j), -y2^j]\big|\Big]^2\Big)^{1/2} &\eqsim \Big(\int_0^\infty t^{-2s}\big|[-(b\wedge yt^{-1}), -yt^{-1}]\big|^2 \frac{dt}{t}\Big)^{1/2} \\
&= \Big(\int_{b^{-1}y}^\infty t^{-2s}(b-yt^{-1})^2 \frac{dt}{t}\Big)^{1/2} \\
&= b^{-(s+1)}y^{-s}\underbrace{\Big(\int_1^\infty \tau^{-2s-2}(\tau-1)^2 \frac{d\tau}{\tau}\Big)^{1/2}}_{<\infty}.
\end{aligned}$$

So we obtain (3.45) by taking $b = 1, 2$. □

3.5.2. A Closer Look at the Inclusion $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$

In this section we give explicit conditions, in terms of w, s and p, for which there is the continuous inclusion (3.5) from Theorem 4.4.1. These conditions will be obtained from the following embedding result.

Theorem 3.5.3. *([185, Theorem 1.2]) Let $w_0, w_1 \in A_\infty(\mathbb{R}^d)$, $s_0 > s_1$, $0 < p_0 \leq p_1 < \infty$, and $q_0, q_1 \in (0,\infty]$. Then there is the continuous inclusion*

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$$

if and only if

$$\sup_{\nu\in\mathbb{N}, m\in\mathbb{Z}^d} 2^{-\nu(s_0-s_1)} w_0(Q_{\nu,m})^{-1/p_0} w_1(Q_{\nu,m})^{1/p_1} < \infty,$$

where $Q_{\nu,m} = Q[2^{-\nu}m, 2^{-\nu-1}] \subset \mathbb{R}^d$ denotes for $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^d$ the d-dimensional cube with sides parallel to the coordinate axes, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$.

Proposition 3.5.4. *Let $s > 0$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. Suppose that $w_{s,p}(x_1, x') = |x_1|^{-sp}w(x_1, x')$ defines an A_∞-weight on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$. If*

$$\sup_{\nu\in\mathbb{N}, m\in\{0\}\times\mathbb{Z}^{d-1}} 2^{-\nu sp}\frac{1}{w(Q_{\nu,m})}\int_{Q_{\nu,m}} |x_1|^{-sp} w(x)\,dx < \infty, \tag{3.46}$$

then there is the continuous inclusion $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$. In case that $w_{s,p} \in A_p$, the converse holds true as well.

Proof. For the inclusion $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$ it is sufficient that $F^s_{p,2}(\mathbb{R}^d, w) \hookrightarrow F^0_{p,1}(\mathbb{R}^d, w_{s,p})$. This follows from the identity $F^s_{p,2}(\mathbb{R}^d, w) \overset{(6.31)}{=} H^s_p(\mathbb{R}^d, w)$, denseness of

$\mathscr{S}(\mathbb{R}^d)$ in $H^s_p(\mathbb{R}^d,w)$, the inclusion $(\mathscr{S}(\mathbb{R}^d),||\cdot||_{F^0_{p,1}(\mathbb{R}^d,w_{s,p})}) \overset{(3.12)}{\hookrightarrow} L^p(\mathbb{R}^d,w_{s,p})$ and the fact that $H^s_p(\mathbb{R}^d,w)$ and $L^p(\mathbb{R}^d,w_{s,p})$ are both continuously included in the Hausdorff topological space $L^0(\mathbb{R}^d)$. In the case that $w_{s,p}\in A_p$, there are the identities $F^s_{p,2}(\mathbb{R}^d,w) = H^s_p(\mathbb{R}^d,w)$ and $L^p(\mathbb{R}^d,w_{s,p}) = F^0_{p,2}(\mathbb{R}^d,w_{s,p})$ (see (6.31)), so the inclusion $H^s_p(\mathbb{R}^d,w)\hookrightarrow L^p(\mathbb{R}^d,w_{s,p})$ just becomes $F^s_{p,2}(\mathbb{R}^d,w)\hookrightarrow F^0_{p,2}(\mathbb{R}^d,w_{s,p})$. Therefore, in order to prove the proposition, it is enough to show that, for every $q\in[1,\infty]$, the inclusion

$$F^s_{p,2}(\mathbb{R}^d,w)\hookrightarrow F^0_{p,q}(\mathbb{R}^d,w_{s,p}) \tag{3.47}$$

is equivalent to the condition (3.46).

By Theorem 3.5.3, the inclusion (3.47) holds true if and only if

$$\sup_{\nu\in\mathbb{N},m\in\mathbb{Z}^d} 2^{-\nu s}||x\mapsto|x_1|^{-s}||_{L^p\left(Q_{\nu,m},\frac{1}{w(Q_{\nu,m})}w\right)} = \sup_{\nu\in\mathbb{N},m\in\mathbb{Z}^d} 2^{-\nu s}\left(\frac{w_{s,p}(Q_{\nu,m})}{w(Q_{\nu,m})}\right)^{1/p}<\infty.$$

But this condition is equivalent to (3.46). Indeed, for every $\nu\in\mathbb{N}$ and $m\in\mathbb{Z}^d$ with $m_1\neq 0$ we have

$$|x_1|\geq(|m_1|-1/2)2^{-\nu}\geq\frac{1}{2}|m_1|2^{-\nu},\qquad x\in Q_{\nu,m},$$

implying that

$$2^{-\nu s}||x\mapsto|x_1|^{-s}||_{L^p\left(Q_{\nu,m},\frac{1}{w(Q_{\nu,m})}w\right)}\leq 2^s|m_1|^{-s}\leq 2^s.$$

□

Let $d=n+k$ with $n,k\in\mathbb{N}$. For $\alpha,\beta>-n$ we define the weight $v_{\alpha,\beta}$ on $\mathbb{R}^d$ by

$$v_{\alpha,\beta}(x,y):=\begin{cases}|x|^\alpha & \text{if } |x|\leq 1,\\ |x|^\beta & \text{if } |x|>1,\end{cases}\qquad (x,y)\in\mathbb{R}^d=\mathbb{R}^n\times\mathbb{R}^k. \tag{3.48}$$

Given $p\in(1,\infty)$, we have $v_{\alpha,\beta}\in A_p$ if and only if $\alpha,\beta\in(-n,n(p-1))$; see [116, Proposition 2.6]. For $n=1$ and $k=d-1$, we have $v_{\gamma,\gamma}=w_\gamma$ (3.3) for every $\gamma>-1$.

Example 3.5.5. Let $s>0$ and $p\in(1,\infty)$.

(i) Suppose $w=w_1\otimes w_2$ with $w_1\in A_p(\mathbb{R})$ and $w_2\in A_p(\mathbb{R}^{d-1})$. Then (3.46) reduces to the corresponding 1-dimensional condition on w_1:

$$\sup_{\nu\in\mathbb{N}} 2^{-\nu sp}\frac{1}{w_1(Q_{\nu,0})}\int_{Q_{\nu,0}}|t|^{-sp}\,w_1(t)\,dt<\infty \tag{3.49}$$

(ii) Let $\alpha,\beta\in(-1,p-1)$. Consider the weight $w=v_{\alpha,\beta}$ from (3.48) for $n=1$ and $k=d-1$. There is the inclusion $H^s_p(\mathbb{R}^d,w)\hookrightarrow L^p(\mathbb{R}^d,w_{s,p})$ if and only if $s<\frac{1+\alpha}{p}$. Given a UMD space X, by Theorem 4.4.1 we thus have that $1_{\mathbb{R}^d_+}$ is a pointwise multiplier on $H^s_p(\mathbb{R}^d,v_{\alpha,\beta};X)$ if and only if $s<\frac{1+\alpha}{p}$. In the case $\alpha=\beta$ this is precisely [187, Theorem 1.1] restricted to positive smoothness; note that the general case $\alpha,\beta\in(-1,p-1)$ can be deduced from the case $\alpha=\beta\in(-1,p-1)$.

Proof of (ii). By (i) we may without loss of generality assume that $d = 1$. Note that $w_{s,p}$ is the the weight $v_{\alpha-sp,\beta}$ (3.48) for $n = 1$ and $k = 0$.

First assume that there is the inclusion $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$. Since $C^\infty_c(\mathbb{R}^d) \subset H^s_p(\mathbb{R}^d, w)$, it follows that $v_{\alpha-sp,\beta} = w_{s,p} \in L^1_{loc}(\mathbb{R}^d)$. Hence, $\alpha - sp > -1$.

Conversely, assume that $s < \frac{1+\alpha}{p}$. Then $\alpha - sp \in (-1, p-1)$, so that $w_{s,p} = v_{\alpha-sp,\beta} \in A_p$. Using that $s < \frac{1+\alpha}{p}$, a simple computation shows that (3.49) holds true for $w = v_{\alpha,\beta}$. By Proposition 3.5.4 we thus obtain that $H^s_p(\mathbb{R}^d, w) \hookrightarrow L^p(\mathbb{R}^d, w_{s,p})$. □

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4

COMPLEX INTERPOLATION WITH DIRICHLET BOUNDARY CONDITIONS ON THE HALF LINE

This chapter is based on the paper:

- N. Lindemulder, M. Meyries, and M.C. Veraar. Complex interpolation with dirichlet boundary conditions on the half line. *Mathematische Nachrichten*, 291(16):2435-2456.

In this chapter we prove results on the complex interpolation of the first order Sobolev space on the half line with Dirichlet boundary condition. Motivated by applications in evolution equations the results are presented for Banach space-valued Sobolev spaces with a weight. The proof is based on recent results on pointwise mutipliers in Bessel potential spaces, of which we provide a simpler proof as well. We apply the results to characterize the fractional domain spaces of the first derivative operator on the half line.

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4.1. INTRODUCTION

The main result of the present paper is the following. Let $W^{1,p}_0(\mathbb{R}_+;X)$ be the first order Sobolev space over the half line with values in a UMD Banach space X vanishing at $t=0$, where $p\in(1,\infty)$. Then for complex interpolation we have

$$[L^p(\mathbb{R}_+;X), W^{1,p}_0(\mathbb{R}_+;X)]_\theta = H^{\theta,p}_0(\mathbb{R}_+;X), \qquad \theta\in(0,1), \quad \theta\neq 1/p,$$

see Theorems 4.6.7 and (4.15). Here $H^{\theta,p}_0$ denotes the fractional order Bessel potential space with vanishing trace for $\theta>1/p$, and $H^{\theta,p}_0 = H^{\theta,p}$ for $\theta<1/p$. In more generality, we consider spaces with Muckenhoupt power weights $w_\gamma(t)=t^\gamma$, where the critical value $1/p$ is shifted accordingly.

In the scalar-valued case $X=\mathbb{C}$, the result is well-known and due to Seeley [224]. The vector-valued result was already used several times in the literature without proof. Seeley also considers the case $\theta=1/p$, which we ignore throughout for simplicity, and the case of domains $\Omega\subseteq\mathbb{R}^d$. The corresponding result for real interpolation is due to Grisvard [104] and more elementary to prove.

At the heart of complex interpolation theory with boundary conditions is the pointwise multiplier property of the characteristic function of the half-space $\mathbb{1}_{\mathbb{R}_+}$ on $H^{\theta,p}(\mathbb{R};X)$ for $0<\theta<1/p$. It is due to Shamir [226] and Strichartz [230] in the scalar-valued case. In [187] by the second and third author, a general theory of pointwise multiplication of weighted vector-valued functions was developed. As a main application the multiplier result was extended to the vector-valued and weighted setting. An alternative approach to this was found by the first author in Chapter 3 and is based on a new equivalent norm for vector-valued Bessel potential spaces. In Section 4.4 we present a new and simpler proof of the multiplier property of $\mathbb{1}_{\mathbb{R}_+}$, which is based on the representation of fractional powers of the negative Laplacian as a singular integral and the Hardy-Hilbert inequality.

For future reference and as it is only a minimal extra effort, we will formulate and prove some elementary assertions for the half space $\mathbb{R}^d_+$ for $d\geq 1$ or even domains, and general A_p weights w. In order to make the presentation as self-contained as possible, we further fully avoid the use of Triebel–Lizorkin spaces and Besov spaces, but we point out where they could be used. We will only use the UMD property of X through standard applications of the Mihlin multiplier theorem. Several results will be presented in such a way that the UMD property is not used. A detailed explanation of the theory of UMD spaces and their connection to harmonic analysis can be found in the monograph [126]. In their reflexive range, all standard function spaces are UMD spaces.

The complex interpolation result has applications in the theory of evolution equations, as it yields a characterization of the fractional power domains of the time derivative $D((d/dt)^\theta)$ and $D((-d/dt)^\theta)$ on $\mathbb{R}_+$. Here the half line usually stands for the time variable and X is a suitable function space for the space variable. For instance such spaces can be used in the theory of Volterra equations (see [195, 250, 251]), in evolution equations with form methods (see [70, 89]), in stochastic evolution equations (see [192]).

In order to deal with rough initial values it is useful to consider a power weights $w_\gamma(t) = t^\gamma$ in the time variable. Examples of papers in evolution equation where such weights are used include [11, 52, 141, 159, 186, 197, 200]. The monographs [168, 198] are an excellent source for applications of weighted spaces to evolution equations. In order to make our results available to this part of the literature as well, we present our interpolation results for weighted spaces. For the application to evolution equations it suffices to consider interpolation of vector-valued Sobolev spaces over $\mathbb{R}_+$ with Dirichlet boundary conditions and therefore we focus on this particular case. In a future paper we extend the results of [104] and [224] to weighted function spaces on more general domains $\Omega \subseteq \mathbb{R}^d$, in the scalar valued situation, where one of the advantages is that Bessel potential spaces have a simple square function characterization.

OVERVIEW

- In Section 4.2 we discuss some preliminaries from harmonic analysis.
- In Section 4.3 we introduce the weighted Sobolev spaces and Bessel potential spaces.
- In Section 4.4 we present an elementary proof of the pointwise multiplier theorem.
- In Section 4.5 we present some results on interpolation theory without boundary conditions.
- In Section 4.6 we present the main results on interpolation theory with boundary conditions and applications to fractional powers.

NOTATION

$\mathbb{R}^d_+ = (0,\infty) \times \mathbb{R}^{d-1}$ denotes the half space. We write $x = (x_1, \tilde{x}) \in \mathbb{R}^d$ with $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$ and define the weight w_γ by $w_\gamma(x_1, \tilde{x}) = |x_1|^\gamma$. Sometimes it will be convenient to also write $(t, x) \in \mathbb{R}^d$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^{d-1}$. The operator $\mathbb{F}$ denotes the Fourier transform. We write $A \lesssim_p B$ whenever $A \leq C_p B$ where C_p is a constant which depends on the parameter p. Similarly, we write $A \eqsim_p B$ if $A \lesssim_p B$ and $B \lesssim_p A$.

4.2. PRELIMINARIES

4.2.1. Weights

A locally integrable function $w : \mathbb{R}^d \to (0,\infty)$ will be called a *weight function*. Given a weight function w and a Banach space X we define $L^p(\mathbb{R}^d, w; X)$ as the space of all strongly measurable $f : \mathbb{R}^d \to X$ for which

$$\|f\|_{L^p(\mathbb{R}^d,w;X)} := \Big(\int \|f(x)\|^p \, w(x)\, dx\Big)^{\frac{1}{p}}$$

is finite. Here we identify functions which are a.e. equal.

Although we will be mainly interested in a special class of weights, it will be natural to formulate some of the result for the class of Muckenhoupt A_p-weights. For $p \in (1,\infty)$, we say that $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \frac{1}{|Q|}\int_Q w(x)\,dx \cdot \Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} < \infty.$$

Here the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes. For $p \in (1,\infty)$ and a weight $w : \mathbb{R}^d \to (0,\infty)$ one has $w \in A_p$ if and only the Hardy–Littlewood maximal function is bounded on $L^p(\mathbb{R}^d, w)$. We refer the reader to [101, Chapter 9] for standard properties of A_p-weights. For a fixed p and a weight $w \in A_p$, the weight $w' = w^{-1/(p-1)} \in A_{p'}$ is the p-dual weight. By Hölder's inequality one checks that

$$\int |f(x)||g(x)|\,dx \leq \|f\|_{L^p(\mathbb{R}^d,w)}\|g\|_{L^{p'}(\mathbb{R}^d,w')} \tag{4.1}$$

for $f \in L^p(\mathbb{R}^d, w)$ and $g \in L^{p'}(\mathbb{R}^d, w')$. Using this, for each $w \in A_p$ one can check that $L^p(\mathbb{R}^d, w; X) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$.

The following will be our main example.

Example 4.2.1. Let

$$w_\gamma(x_1, \widetilde{x}) = |x_1|^\gamma, \;\; x_1 \in \mathbb{R}, \widetilde{x} \in \mathbb{R}^{d-1}.$$

As in [101, Example 9.1.7]) one sees that $w_\gamma \in A_p$ if and only if $\gamma \in (-1, p-1)$.

Lemma 4.2.2. *Let $p \in (1,\infty)$ and $w \in A_p$. Assume $\phi \in L^1(\mathbb{R}^d)$ and $\int \phi\,dx = 1$. Let $\phi_n(x) = n^d\phi(nx)$. Assume ϕ satisfies any of the following conditions:*

1. *ϕ is bounded and compactly supported*
2. *There exists a radially decreasing function $\psi \in L^1(\mathbb{R}^d)$ such that $|\phi| \leq \psi$ a.e.*

Then for all $f \in L^p(\mathbb{R}^d; X)$, $\phi_n * f \to f$ in $L^p(\mathbb{R}^d, w; X)$ as $n \to \infty$. Moreover, there is a constant C only depending on ϕ such that $\|\phi_n * f\| \leq CMf$ almost everywhere.

Proof. For convenience of the reader we include a short proof. By [126, Theorem 2.40 and Corollary 2.41] $\phi_n * f \to f$ almost everywhere and $\|\phi_n * f\| \leq \|\psi\|_{L^1(\mathbb{R}^d)}Mf$ almost everywhere, where M denotes the Hardy–Littlewood maximal function. Therefore, the result follows from the dominated convergence theorem. □

4.2.2. Fourier multipliers and UMD spaces

Let $\mathscr{S}(\mathbb{R}^d; X)$ be the space of X-valued Schwartz functions and let $\mathscr{S}'(\mathbb{R}^d; X) = \mathscr{L}(\mathscr{S}(\mathbb{R}^d), X)$ be the space of X-valued tempered distributions. For $m \in L^\infty(\mathbb{R}^d)$ let $T_m : \mathscr{S}(\mathbb{R}^d; X) \to \mathscr{S}'(\mathbb{R}^d; X)$ be the Fourier multiplier operator defined by

$$T_m f = \mathbb{F}^{-1}(m\widehat{f}).$$

There are many known conditions under which T_m is a bounded linear operator on $L^p(\mathbb{R}^d; X)$. In the scalar-valued the set of all Fourier multiplier symbols on $L^2(\mathbb{R}^d)$ for instance coincides with $L^\infty(\mathbb{R}^d)$. In the case $p \in (1,\infty) \setminus \{2\}$ a large set of multipliers for which T_m is bounded is given by Mihlin's multiplier theorem. In the vector-valued case difficulties arise and geometric conditions on X are needed already if $d = 1$ and $m(\xi) = \operatorname{sign}(\xi)$; in fact, in [31, 42] it was shown that in this specific case the boundedness of T_m on $L^p(\mathbb{R}; X)$ characterizes the UMD property of X. Since the work of [31, 42, 174] it is well-known that the right class of Banach spaces for vector-valued harmonic analysis is the class of UMD Banach spaces, as many of the classical results in harmonic analysis, such as the classical Mihlin multiplier theorem, have been extended to this setting. We refer to [43, 126] for details on UMD spaces and Fourier multiplier theorems.

All UMD spaces are reflexive. Conversely, all spaces in the reflexive range of the classical function spaces have UMD: e.g.: L^p, Bessel potential spaces, Besov spaces, Triebel-Lizorkin spaces, Orlicz spaces.

The following result is a weighted version of the Mihlin multiplier theorem which can be found in [187, Proposition 3.1] and is a simple consequence of [113].

Proposition 4.2.3. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p$. Assume that $m \in C^{d+2}(\mathbb{R}^d \setminus \{0\})$ satisfies*

$$C_m := \sup_{|\alpha|\leq d+2} \sup_{\xi\neq 0} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty.$$

Then T_m is bounded on $L^p(\mathbb{R}^d, w; X)$ and has an operator norm that only depends $C_m, d, p, X, [w]_{A_p}$.

4.3. WEIGHTED FUNCTION SPACES

In this section we present several results on weighted function spaces, which do not require the UMD property of the underlying Banach space (except in Proposition 4.3.2).

4.3.1. Definitions and basic properties

For an open set $\Omega \subseteq \mathbb{R}^d$ let $\mathscr{D}(\Omega)$ denote the space compactly supported smooth functions on Ω equipped with its usual inductive limit topology. For a Banach space X, let $\mathscr{D}'(\Omega; X) = \mathscr{L}(\mathscr{D}(\Omega), X)$ be the space of X-valued distributions. For a distribution $u \in \mathscr{D}'(\Omega; X)$ and an open subset $\Omega_0 \subseteq \Omega$, we define the restriction $u|_{\Omega_0} \in \mathscr{D}'(\Omega_0; X)$ as $u|_{\Omega_0}(f) = u(f)$ for $f \in \mathscr{D}(\Omega_0)$.

For $p \in (1,\infty)$ and $w \in A_p$ let $W^{k,p}(\Omega, w; X) \subseteq \mathscr{D}'(\Omega; X)$ be the *Sobolev space* of all $f \in L^p(\Omega, w; X)$ with $D^\alpha f \in L^p(\Omega, w; X)$ for all $|\alpha| \leq k$ and set

$$\|f\|_{W^{k,p}(\Omega,w;X)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Omega,w;X)},$$

$$[f]_{W^{k,p}(\Omega,w;X)} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\Omega,w;X)}.$$

Here for $\alpha \in \mathbb{N}^d$, $D^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$.

Let $\mathscr{J}_s$ denote the Bessel potential operator of order $s \in \mathbb{R}$ defined by

$$\mathscr{J}_s f = (1-\Delta)^{s/2} f := \mathbb{F}^{-1}(1+|\cdot|^2)^{s/2}\widehat{f},$$

where $\widehat{f}$ denotes the Fourier transform of f and $\Delta = \sum_{j=1}^d \partial_j^2$. For $p \in (1,\infty)$, $s \in \mathbb{R}$ and $w \in A_p$ let $H^{s,p}(\mathbb{R}^d, w; X) \subseteq \mathscr{S}'(\mathbb{R}^d; X)$ denote the *Bessel potential space* of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which $\mathscr{J}_s f \in L^p(\mathbb{R}^d, w; X)$ and set

$$\|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} = \|\mathscr{J}_s f\|_{L^p(\mathbb{R}^d,w;X)}.$$

In the following lemma we collect some properties of the operators $\mathscr{J}_s$.

Lemma 4.3.1. *Fix $s > 0$. There exists a function $G_s : \mathbb{R}^d \to [0,\infty)$ such that $G_s \in L^1(\mathbb{R}^d)$ and $\mathscr{J}_{-s} f = G_s * f$ for all $f \in \mathscr{S}'(\mathbb{R}^d; X)$. Moreover, G_s has the following properties:*

1. *For all $|y| \geq 2$, $G_s(y) \lesssim_{s,d} e^{-\frac{|y|}{2}}$.*
2. *For $|x| \leq 2$,*

$$G_s(x) \lesssim_{s,d} \begin{cases} |x|^{s-d}, & s \in (0,d), \\ 1+\log(\frac{2}{|x|}), & s = d, \\ 1, & s > d, \end{cases}$$

3. *for all $s > k \geq 0$ and all $|\alpha| \leq k$, there exists a radially decreasing function $\phi \in L^1(\mathbb{R}^d)$ such that $|D^\alpha G_s| \leq \phi$ pointwise.*

In particular, if $d = 1$, $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $s > \frac{1+\gamma}{p}$, then $G_s \in L^{p'}(\mathbb{R}, w'_\gamma)$.

Proof. The fact that the positive function $G_s \in L^1(\mathbb{R}^d)$ exists, together with (1) and (2), follows from [101, Section 6.1.b].

To prove (3), we use the following representation of G_s (see [101, Section 6.1.b]):

$$G_s(x) = C_{s,d}\int_0^\infty e^{-t} e^{-\frac{|x|^2}{4t}} t^{\frac{s-d}{2}} \frac{dt}{t}.$$

By induction one sees that $D^\alpha G_s(x)$ is a linear combination of functions of the form $G_{s-2j}(x)|x|^\beta$ with $|\beta| \leq j \leq k$. Therefore, by (2) for $|x| \leq 2$, $|D^\alpha G_s(x)| \lesssim_{s,d,\alpha} |x|^{\varepsilon-d}$ for some $\varepsilon \in (0,d)$. On the other hand for $|x| \geq 2$, $|D^\alpha G_s(x)| \lesssim_{s,d,\alpha} |x|^\beta e^{-\frac{|x|}{2}} \lesssim_{d,s,k} e^{-\frac{|x|}{4}}$. Now the function $\phi(x) = C_1|x|^{\varepsilon-d}$ for $|x| \leq 2$ and $\phi(x) = C_2 e^{-\frac{|x|}{4}}$ for certain constants $C_1, C_2 > 0$. satisfies the required conditions.

To prove the final assertion for $d = 1$, note that the blow-up behaviour near 0 gets worse as s decreases. Therefore, without loss of generality we may assume that $s \in (\frac{1+\gamma}{p}, 1)$, in which case (2) yields

$$|G_s(x)|^{p'} w'_\gamma(x) \lesssim_{s,p,\gamma} |x|^{\frac{(s-1)p-\gamma}{p-1}} = |x|^{-1+\frac{p}{p-1}(s-\frac{1+\gamma}{p})} \quad \text{for } |x| \leq 2.$$

which is integrable. Integrability, for $|x| > 2$, is clear from (1). □

The following result is proved in [187, Proposition 3.2 and 3.7] by a direct application of Proposition 4.2.3.

Proposition 4.3.2. *Let X be a UMD space, $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $w \in A_p$. Then $H^{k,p}(\mathbb{R}^d, w; X) = W^{k,p}(\mathbb{R}^d, w; X)$ with norm equivalence only depending on d, X, p, k and $[w]_{A_p}$.*

The UMD property is necessary in Proposition 4.3.2 (see [126, Theorem 5.6.12]). Sometimes it can be avoided by instead using the following simple embedding result which holds for any Banach space. The sharper version $W^{k,p}(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X)$ if $s < k$ and $k \in \mathbb{N}_0$. can be obtained from [182, Propositions 3.11 and 3.12] but is more complicated.

Lemma 4.3.3. *Let X be a Banach space, $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $s \in (k,\infty)$ and $w \in A_p$. Then the following continuous embeddings hold*

$$W^{2k,p}(\mathbb{R}^d, w; X) \hookrightarrow H^{2k,p}(\mathbb{R}^d, w; X), \quad H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow W^{k,p}(\mathbb{R}^d, w; X),$$

with embedding constants which only depend on d, s, k and $[w]_{A_p}$.

Proof. The first embedding is immediate from $J_{2k} f = (1-\Delta)^k f$ and Leibniz' rule. For the second embedding let $f \in H^{s,p}(\mathbb{R}^d, w; X)$ and write $f_s = J_s f \in L^p(\mathbb{R}^d, w; X)$. By Lemma 4.3.1 (3) and Lemma 4.2.2, for all $|\alpha| \le k$,

$$\|D^\alpha f\|_X = \|D^\alpha G_s * f_s\|_X \le \phi * \|f_s\|_X \le C_\phi M(\|f_s\|_X),$$

where $\phi \in L^1(\mathbb{R}^d)$ is a radially decreasing function depending on α, k and s. Therefore, by the boundedness of the Hardy–Littlewood maximal function, we have $D^\alpha f \in L^p(\mathbb{R}^d, w; X)$ with

$$\|D^\alpha f\|_{L^p(\mathbb{R}^d, w; X)} \lesssim_{p,[w]_{A_p}} \|f_s\|_{L^p(\mathbb{R}^d, w; X)} = \|f\|_{H^{s,p}(\mathbb{R}^d, w; X)}.$$

Now the result follows by summation over all α. □

We proceed with two density results.

Lemma 4.3.4. *Let X be a Banach space, $p \in (1,\infty)$, $s \in \mathbb{R}$ and $w \in A_p$. Then $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$. Moreover, $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $H^{s,p}(\mathbb{R}^d, w; X)$.*

Proof. First we prove that $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X)$. It suffices to prove this in the case $s = 0$ by continuity of $\mathscr{J}_s = (1-\Delta)^{s/2}$ on $\mathscr{S}(\mathbb{R}^d; X)$. In the case $s = 0$, the continuity of the embedding follows from

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^d, w; X)} &\le \|(1+|x|^2)^{-n}\|_{L^p(\mathbb{R}^d, w)} \|(1+|x|^2)^n f\|_{L^\infty(\mathbb{R}^d; X)} \\ &\lesssim_{d,n,p,w} \sum_{|\alpha| \le 2n} \sup_{x \in \mathbb{R}^d} \|x^\alpha f(x)\| \end{aligned}$$

for $n \in \mathbb{N}$ with $n \ge dp$ (see [182, Lemma 4.5]).

To prove the density assertion note that $L^p(\mathbb{R}^d, w) \otimes X$ is dense in $L^p(\mathbb{R}^d, w; X)$ and $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, w)$ (see [101, Exercise 9.4.1]) it follows that $\mathscr{S}(\mathbb{R}^d) \otimes X$ is dense in $L^p(\mathbb{R}^d, w; X)$. Since J^{-s} leaves $\mathscr{S}(\mathbb{R}^d)$ invariant, also $\mathscr{S}(\mathbb{R}^d) \otimes X$ is dense in $H^{s,p}(\mathbb{R}^d, w; X)$. Combining this with $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X)$ and the fact that $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$ (see [79, Lemma 14.7]) we obtain the desired density assertion.

To prove the embedding $H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$ it suffices again to consider $s = 0$. In this case from (4.1) and $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^{p'}(\mathbb{R}^d, w')$ densely, we deduce

$$L^p(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{L}(L^{p'}(\mathbb{R}^d, w'), X) \hookrightarrow \mathscr{L}(\mathscr{S}(\mathbb{R}^d), X) = \mathscr{S}'(\mathbb{R}^d; X).$$

□

Lemma 4.3.5. *Let X be a Banach space, $p \in (1, \infty)$, $k \in \mathbb{N}$ and $w \in A_p$. Then $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow W^{k,p}(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$. Moreover, $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $W^{k,p}(\mathbb{R}^d, w; X)$.*

Proof. The case $k = 0$ follows from Lemma 4.3.4 and the case $k \geq 1$ follow by differentiation.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \phi \, dx = 1$ and define $\phi_n := n^d \phi(n\cdot)$ for every $n \in \mathbb{N}$. Then, by Lemma 4.2.2 and standard properties of convolutions, $f_n := \phi_n * f \to f$ in $W^{k,p}(\mathbb{R}^d, w; X)$ as $n \to \infty$ with $\phi_n * f \in W^{\infty,p}(\mathbb{R}^d, w; X) = \bigcap_{l \in \mathbb{N}} W^{l,p}(\mathbb{R}^d, w; X)$. In particular, $W^{2k+2,p}(\mathbb{R}^d, w; X)$ is dense in $W^{k,p}(\mathbb{R}^d, w; X)$. This yields $H^{k+1,p}(\mathbb{R}^d, w; X) \overset{d}{\hookrightarrow} W^{k,p}(\mathbb{R}^d, w; X)$ by Lemma 4.3.3. The density of $C_c^\infty(\mathbb{R}^d) \otimes X$ in $W^{k,p}(\mathbb{R}^d, w; X)$ now follows from Lemma 4.3.4. □

Lemma 4.3.6. *Let X be a Banach space, $p \in (1, \infty)$, $s \in \mathbb{R}$ and $w \in A_p$. Assume $\phi \in C_c^\infty(\mathbb{R})$ with $\int \phi \, dx = 1$. Let $\phi_n(x) = n^d \phi(nx)$. Then, for all $f \in H^{s,p}(\mathbb{R}^d, w; X)$,*

$$\|\phi_n * f\|_{H^{s,p}(\mathbb{R}^d, w; X)} \lesssim_{s,p,[w],d} \|f\|_{H^{s,p}(\mathbb{R}^d, w; X)}$$

with $\phi_n * f \to f$ in $H^{s,p}(\mathbb{R}^d, w; X)$ as $n \to \infty$ with $\phi_n * f \in H^{\infty,p}(\mathbb{R}^d, w; X) = \bigcap_{t \in \mathbb{R}} H^{t,p}(\mathbb{R}^d, w; X)$.

Proof. The first part of the statement follows from Lemma 4.2.2 and $\mathscr{J}_s(\phi_n * f) = \phi_n * \mathscr{J}_s f$. For the last part, note that $\phi_n * f = \mathscr{J}_{-s}[\phi_n * \mathscr{J}_s f] \in H^{\infty,p}(\mathbb{R}^d, w; X)$ by basic properties of convolutions in combination with Lemma 4.3.3. □

The following version of the Hardy inequality will be needed (see [182, Corolllary 1.4] for a related result). The result can be deduced from [185, Theorem 1.3 and Proposition 4.3] but for convenience we include an elementary proof.

Lemma 4.3.7 (Hardy inequality with power weights)**.** *Let $\gamma \in (-1, p-1)$ and $s \in (0,1)$. Let $w_\gamma(t, x) = |t|^\gamma$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^{d-1}$. Then $H^{s,p}(\mathbb{R}^d, w_\gamma; X) \hookrightarrow L^p(\mathbb{R}^d, w_{\gamma - sp}; X)$.*

Proof. It suffices to prove $\|G_s * f\|_{L^p(w_{\gamma-sp}; X)} \lesssim_{p,s,d,\gamma} \|f\|_{L^p(w_\gamma; X)}$, where G_s is as in Lemma 4.3.1 and $f \in L^p(w_\gamma; X)$. Since $G_s \geq 0$, by the triangle inequality it suffices to consider the case of scalar functions f with $f \geq 0$.

To prove the result we first apply Minkowski's and Young's inequality in $\mathbb{R}^{d-1}$:

$$\|G_s * f(t,\cdot)\|_{L^p(\mathbb{R}^{d-1})} \le \int_{\mathbb{R}} \|G_s(t-\tau,\cdot)\|_{L^1(\mathbb{R}^{d-1})} \|f(\tau,\cdot)\|_{L^p(\mathbb{R}^{d-1})} d\tau = g_s * \phi(\tau).$$

Here $g_s(t) = \|G_s(t,\cdot)\|_{L^1(\mathbb{R}^{d-1})}$ and $\phi(\tau) = \|f(\tau,\cdot)\|_{L^p(\mathbb{R}^{d-1})}$. Then for $|t| \le 2$, by Lemma 4.3.1 (1) and (2),

$$g_s(t) \lesssim_{s,d} \int_{\mathbb{R}^{d-1}} (|t|+|x|)^{s-d} dx = |t|^{s-1} \int_{\mathbb{R}^{d-1}} (1+|x|)^{s-d} dx = C|t|^{s-1},$$

where we used $s<1$. For $|t|>2$, by Lemma 4.3.1 (2) and $|(t,x)| \eqsim |t|+|x|$, we find

$$g_s(t) \lesssim_{s,d} e^{-\frac{|t|}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x|}{2}} dx \eqsim_d e^{-\frac{|t|}{2}}.$$

Finally by the weighted version of Young's inequality (see and [136, Theorem 3.4(3.7)]) in dimension one, we find that

$$\|G_s * f\|_{L^p(\mathbb{R}^d, w_{\gamma-sp})} \le \|g_s * \phi\|_{L^p(\mathbb{R}, w_{\gamma-sp})} \le C\|\phi\|_{L^p(\mathbb{R}, w_\gamma)} = C\|f\|_{L^p(\mathbb{R}^d, w_\gamma)},$$

where $C = \sup_{t\in\mathbb{R}} |t|^{1-s} g_s(t) < \infty$. □

We end this section with a weighted version of the classical Hardy–Hilbert inequality.

Lemma 4.3.8 (Hardy–Hilbert inequality with power weights)**.** *Let $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Let $w_\gamma(x_1,\tilde{x}) = |x_1|^\gamma$ and $k(x,y) = \frac{1}{((|x_1|+|y_1|)^2+|\tilde{x}-\tilde{y}|^2)^{d/2}}$, where $x=(x_1,\tilde{x})$ and $y=(y_1,\tilde{y})$. Then the formula*

$$I_k h(x) := \int_{\mathbb{R}^d} k(x,y)h(y)\,dy$$

yields a well-defined bounded linear operator I_k on $L^p(\mathbb{R}^d, w_\gamma)$.

Proof. It suffices to consider $h \ge 0$. Moreover, by symmetry it is enough to consider $x_1, y_1 > 0$. Thus we need to show that

$$\|x \mapsto \int_{\mathbb{R}^d_+} k(x,y)h(y)\,dy\|_{L^p(\mathbb{R}^d_+, w_\gamma)} \lesssim_{p,d,\gamma} \|h\|_{L^p(\mathbb{R}^d_+, w_\gamma)}, \qquad h \in L^p(\mathbb{R}^d_+, w_\gamma), h \ge 0.$$

Step I. The case $d=1$. Replacing k by

$$k_\beta(x,y) = \frac{w_\gamma(x)^{1/p} w_\gamma(y)^{-1/p}}{(|x|+|y|)} = \frac{|x|^\beta |y|^{-\beta}}{|x|+|y|},$$

with $\beta = \gamma/p$, it suffices to consider the unweighted case.

To prove the required result we apply Schur's test in the same way as in [96, Theorem 5.10.1]. Let $s(x) = t(x) = x^{-\frac{1}{pp'}}$. Then since $-1 < \beta - \frac{1}{p'} < 0$

$$\int_0^\infty s(x)^p k_\beta(x,y)\,dx = \int_0^\infty \frac{x^{\beta-\frac{1}{p'}} y^{-\beta}}{x+y}\,dx = t(y)^p \int_0^\infty \frac{z^{\beta-\frac{1}{p}}}{z+1}\,dz = C_{p,\beta} t(y)^p.$$

Similarly, since $-1 < -\beta - \frac{1}{p} < 0$

$$\int_0^\infty t(y)^{p'} k_\beta(x,y)\,dy = \int_0^\infty \frac{x^\beta y^{-\beta-\frac{1}{p}}}{x+y}\,dy = s(x)^{p'} \int_0^\infty \frac{z^{-\beta-\frac{1}{p}}}{1+z}\,dz = C_{p,\beta} s(x)^{p'}.$$

Step II. The general case. By Minkowski's inequality we find

$$\|I_k f(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1})} \leq \int_0^\infty \Big(\int_{\mathbb{R}^{d-1}} \Big(\int_{\mathbb{R}^{d-1}} \frac{f(y_1,\tilde{y})}{((x_1+y_1)^2 + |\tilde{x}-\tilde{y}|^2)^{d/2}}\,d\tilde{y}\Big)^p d\tilde{x}\Big)^{1/p} dy_1.$$

Fix $y_1 > 0$ and let $g_r(\tilde{y}) = f(y_1, r\tilde{y})$. Setting $r = x_1 + y_1$ and substituting $u := \tilde{x}/r$ and $v := \tilde{y}/r$ we can write

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \Big(\int_{\mathbb{R}^{d-1}} \frac{f(y_1,\tilde{y})}{(|x_1+y_1|^2 + |\tilde{x}-\tilde{y}|^2)^{d/2}}\,d\tilde{y}\Big)^p d\tilde{x} \\
&\quad = r^{-p+d-1} \int_{\mathbb{R}^{d-1}} \Big(\int_{\mathbb{R}^{d-1}} \frac{g_r(v)}{(1+|u-v|^2)^{d/2}}\,dv\Big)^p du \\
&\quad \leq r^{-p+d-1} \|g_r\|^p_{L^p(\mathbb{R}^{d-1})} \|(1+|\cdot|^2)^{-d/2}\|^p_{L^1(\mathbb{R}^{d-1})} = C_{d,p} r^{-p} \|g_1\|^p_{L^p(\mathbb{R}^{d-1})},
\end{aligned}$$

where we applied Young's inequality for convolutions. Therefore,

$$\|I_k f(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1})} \leq C_{d,p} \int_0^\infty \frac{\|f(y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1})}}{x_1+y_1}\,dy_1.$$

Taking $L^p((0,\infty), w_\gamma)$-norms in x_1 and applying Step I yields the required result. □

Remark 4.3.9. Actually, the kernel k of Lemma 4.3.8 is a standard Calderón–Zygmund kernel, because k is a.e. differentiable and

$$|\nabla_x k(x,y)| + |\nabla_y k(x,y)| \leq |x-y|^{-d-1}, \quad x \neq y.$$

Although we will not need it below let us note that [113, Corollary 2.10] implies that I_k is bounded on $L^p(\mathbb{R}^d, w)$ for any $w \in A_p$

4.4. POINTWISE MULTIPLICATION WITH $\mathbb{1}_{\mathbb{R}^d_+}$

In this section we prove the pointwise multiplier result, which is central in the characterization of the complex interpolation spaces of Sobolev spaces with boundary conditions in Section 4.6. Let $w_\gamma(x_1,\tilde{x}) = |x_1|^\gamma$, where $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$.

Theorem 4.4.1. *Let X be a UMD space, $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\gamma' = -\gamma/(p-1)$, and assume $-\frac{\gamma'+1}{p'} < s < \frac{\gamma+1}{p}$. Then for all $f \in H^{s,p}(\mathbb{R}^d, w_\gamma; X) \cap L^p(\mathbb{R}^d, w_\gamma; X)$, we have $\mathbb{1}_{\mathbb{R}^d_+} f \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ and*

$$\|\mathbb{1}_{\mathbb{R}^d_+} f\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} \lesssim_{X,p,\gamma,s} \|f\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)},$$

and therefore, pointwise multiplication by $\mathbb{1}_{\mathbb{R}^d_+}$ extends to a bounded linear operator on $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$.

To prove this the UMD property will only be used through the norm equivalence of Lemma 4.4.2 below.

Lemma 4.4.2. *Let X be a UMD space, $p \in (1,\infty)$, $s \in \mathbb{R}$, $\sigma \geq 0$, $w \in A_p$. Then*

$$(-\Delta)^{\sigma/2} : \mathscr{S}(\mathbb{R}^d;X) \longrightarrow \mathscr{S}'(\mathbb{R}^d;X),\ f \mapsto \mathscr{F}^{-1}[(\xi \mapsto |\xi|^{\sigma})\widehat{f}]$$

defines for each $r \in \mathbb{R}$ (by extension by density) a bounded linear operator from $H^{r+\sigma,p}(\mathbb{R}^d,w;X)$ to $H^{r,p}(\mathbb{R}^d,w;X)$, independent of r and w (in the sense of compatibility), which we still denote by $(-\Delta)^{\sigma/2}$. Moreover, $f \in H^{s+\sigma,p}(\mathbb{R}^d,w;X)$ if and only if $f, (-\Delta)^{\sigma/2}f \in H^{s,p}(\mathbb{R}^d,w;X)$, in which case

$$\|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} \eqsim_{s,p,w,d,\sigma,X} \|f\|_{H^{s-\sigma,p}(\mathbb{R}^d,w;X)} + \|(-\Delta)^{\sigma/2}f\|_{H^{s-\sigma,p}(\mathbb{R}^d,w;X)}.$$

Proof. All assertions follow from the fact that the symbols

$$\xi \mapsto \frac{|\xi|^{\sigma}}{(1+|\xi|^2)^{2/\sigma}},\quad \xi \mapsto \frac{1}{(1+|\xi|^2)^{2/\sigma}},\quad \xi \mapsto \frac{(1+|\xi|^2)^{2/\sigma}}{1+|\xi|^{\sigma}}$$

satisfy the conditions of Proposition 4.2.3. □

In the proof of Theorem 4.4.1 we will use the norm equivalence of the above lemma via (a variant of) a well known representation for $(-\Delta)^{\sigma/2}$ as a singular integral. For $f \in H^{\sigma,p}(\mathbb{R}^d)$ this representation reads as follows:

$$(-\Delta)^{\sigma/2}f = \lim_{r\to 0^+} C_{d,\sigma} \int_{\mathbb{R}^d\setminus B(0,r)} \frac{T_hf-f}{h}\,dh,$$

with limit in $L^p(\mathbb{R}^d)$ (see [151, Theorem 1.1(e)]); here T_h denotes the left translation and $C_{d,\sigma}$ is a constant only depending on d and σ.

In the proof we want to use a formula as above for f replaced by $1_{\mathbb{R}^d_+}f$, which in general is an irregular function even if f is smooth; in particular, a priori it is not clear that $1_{\mathbb{R}^d_+}f \in H^{\sigma,p}(\mathbb{R}^d)$. We overcome this technical obstacle by Proposition 4.4.4 below, which provides a (non sharp) representation formula for $(-\Delta)^{\sigma/2}$ in spaces of distributions.

For the proof of Proposition 4.4.4 we need the following simple identity.

Lemma 4.4.3. *For each $\sigma \in (0,1)$ there exists a constant $c_{d,\sigma} \in (-\infty,0)$ such that*

$$|\xi|^{\sigma} = c_{d,\sigma}\int_{\mathbb{R}^d} \frac{e^{\imath h\cdot\xi}-1}{|h|^{d+\sigma}}\,dh, \qquad \xi \in \mathbb{R}^d.$$

Moreover, for all $\phi \in \mathscr{S}(\mathbb{R}^d)$

$$\begin{aligned}[\xi \mapsto |\xi|^{\sigma}](\phi) := \int_{\mathbb{R}^d} |\xi|^{\sigma}\phi(\xi)\,d\xi &= c_{d,\sigma}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{e^{\imath h\xi}-1}{|h|^{d+\sigma}}\phi(\xi)\,d\xi\,dh \qquad (4.2)\\ &=: c_{d,\sigma}\int_{\mathbb{R}^d}\left[\xi \mapsto \frac{e^{\imath h\xi}-1}{|h|^{d+\sigma}}\right](\phi)\,dh.\end{aligned}$$

Proof. Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and choose $R \in \mathrm{O}(n)$ with $R\xi = |\xi| e_1$. Then $h\cdot\xi = Rh\cdot R\xi = |\xi| Rh\cdot e_1$ and the substitution $y = |\xi| Rh$ yields

$$\int_{\mathbb{R}^d} \frac{e^{\imath h\cdot\xi} - 1}{|h|^{d+\sigma}} = |\xi|^{\sigma} \int_{\mathbb{R}^d} \frac{e^{\imath y_1} - 1}{|y|^{d+\sigma}}\, dy.$$

Observing that the integral on the right is a number in $(-\infty, 0)$, the first identity follows.

Next we show (4.2). Given $\phi \in \mathscr{S}(\mathbb{R}^d)$, the first identity gives

$$[\xi \mapsto |\xi|^{\sigma}](\phi) = \int_{\mathbb{R}^d} |\xi|^{\sigma} \phi(\xi)\, d\xi = c_{d,\sigma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{e^{\imath h\xi} - 1}{|h|^{d+\sigma}}\, dh\, \phi(\xi)\, d\xi.$$

Since $\phi \in \mathscr{S}(\mathbb{R}^d)$ and

$$\frac{|e^{\imath h\xi} - 1|}{|h|^{d+\sigma}} \leq 1_{|h|\leq 1} h^{-(d-1+\sigma)} |\xi| + 2 \cdot 1_{|h|>1} |h|^{-(d+\sigma)},$$

we may invoke Fubini's theorem in order to get

$$[\xi \mapsto |\xi|^{\sigma}](\phi) = c_{d,\sigma} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \frac{e^{\imath h\xi} - 1}{|h|^{d+\sigma}} \phi(\xi)\, d\xi\, dh = c_{d,\sigma} \int_{\mathbb{R}^d} \left[\xi \mapsto \frac{e^{\imath h\xi} - 1}{|h|^{d+\sigma}}\right](\phi)\, dh,$$

as desired. □

For $f \in \mathscr{S}'(\mathbb{R}^d; X)$ let $\delta_h f = T_h f - f$, where T_h denotes the left translation by h. For $0 < r < R$ let $A(r,R) := \{x \in \mathbb{R}^d : r < |x| < R\}$ be an annulus.

Proposition 4.4.4 (Representation of $(-\Delta)^{\frac{\sigma}{2}}$)**.** *Let $p \in (1,\infty)$ and $\sigma \in (0,1)$. For all $s \geq 0$ and $f \in H^{s,p}(\mathbb{R}^d) \otimes X \subset L^p(\mathbb{R}^d; X)$ we have*

$$(-\Delta)^{\frac{\sigma}{2}} f = \frac{1}{c_{d,\sigma}} \lim_{r\searrow 0, R\nearrow\infty} \left[x \mapsto \int_{A(r,R)} \frac{\delta_h f(x)}{|h|^{d+\sigma}}\, dh\right] \qquad in \quad H^{s-2,p}(\mathbb{R}^d; X),$$

where $c_{d,\sigma}$ is the constant of Lemma 4.4.3.

The weights are left out on purpose, because translations are not well-behaved on weighted L^p-spaces. Moreover, no UMD is required in the result above.

Proof. We prove this proposition by proving the following three statements:

1. The linear operator
$$f \mapsto \left[h \mapsto \frac{\delta_h f}{|h|^{d+\sigma}}\right]$$
is bounded from $H^{s,p}(\mathbb{R}^d; X)$ to $L^1(\mathbb{R}^d; H^{s-2,p}(\mathbb{R}^d; X))$ for all $s \in \mathbb{R}$ and thus gives rise to a bounded linear operator
$$\mathscr{I}_{\sigma} : H^{s,p}(\mathbb{R}^d; X) \longrightarrow H^{s-2,p}(\mathbb{R}^d; X),\ f \mapsto \int_{\mathbb{R}^d} \frac{\delta_h f}{|h|^{d+\sigma}}\, dh,$$

2. For all $s \geq 0$ we have
$$\mathscr{I}_\sigma f = \lim_{r\searrow 0, R\nearrow\infty}\left[x \mapsto \int_{A(r,R)} \frac{\delta_h f(x)}{|h|^{d+\sigma}}\,dh\right] \qquad \text{in} \quad H^{s-2,p}(\mathbb{R}^d;X)$$
for every $f \in H^{s,p}(\mathbb{R}^d;X) \subset L^p(\mathbb{R}^d;X)$.

3. For all $f \in H^{-\infty,p}(\mathbb{R}^d)\otimes X$,
$$\mathscr{I}_\sigma f = c_{d,\sigma}(-\Delta)^{\frac{\sigma}{2}} f \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^d;X), \tag{4.3}$$
where $c_{d,\sigma}$ is the constant of Lemma 4.4.3. Here $H^{-\infty,p}(\mathbb{R}^d) = \cup_{s\in\mathbb{R}} H^{s,p}(\mathbb{R}^d)$.

(1): To prove this it is enough to establish the boundedness from $H^{s,p}(\mathbb{R}^d;X)$ to $L^1(\mathbb{R}^d;H^{s-2,p}(\mathbb{R}^d;X))$. As the Bessel potential operator $\mathscr{J}_s$ commutes with δ_h, we may restrict ourselves to the case $s=2$. Since by Lemma 4.3.3 $H^{2,p}(\mathbb{R}^d;X) \hookrightarrow W^{1,p}(\mathbb{R}^d;X)$, we only need to estimate
$$\int_{\mathbb{R}^d} \frac{\|\delta_h f\|_{L^p(\mathbb{R}^d;X)}}{|h|^{d+\sigma}}\,dh \lesssim_{d,\sigma,p} \|f\|_{W^{1,p}(\mathbb{R}^d;X)}, \qquad f \in W^{1,p}(\mathbb{R}^d;X). \tag{4.4}$$
To this end, let $f \in W^{1,p}(\mathbb{R}^d;X)$. Then
$$\frac{\delta_h f}{h^{1+\sigma}} = 1_{|h|\leq 1}|h|^{-(d-1+\sigma)}\int_0^1 T_{th}\left[\nabla f \cdot \frac{h}{|h|}\right]dt + 1_{|h|>1}|h|^{-(d+\sigma)}(T_h f - f),$$
where the integral is an $L^p(\mathbb{R}^d;X)$-valued Bochner integral. It follows that
$$\begin{aligned}\frac{\|\delta_h f\|_{L^p(\mathbb{R}^d;X)}}{|h|^{d+\sigma}} &\leq 1_{|h|\leq 1}|h|^{-(d-1+\sigma)}\int_0^1 \|T_{th}\|\nabla f\|_{X^d}\|_{L^p(\mathbb{R}^d)}\,dt \\ &\quad + 1_{|h|>1}|h|^{-(d+\sigma)}\left(\|T_h f\|_{L^p(\mathbb{R};X)} + \|f\|_{L^p(\mathbb{R};X)}\right) \\ &= 1_{|h|\leq 1}h^{-(d-1+\sigma)}\|\nabla f\|_{L^p(\mathbb{R};X^d)} + 2\cdot 1_{|h|>1}|h|^{-(d+\sigma)}\|f\|_{L^p(\mathbb{R};X)}.\end{aligned}$$
Integrating over h gives (4.4).

(2): Let $s \geq 0$ and $f \in H^{s,p}(\mathbb{R}^d;X) \subset L^p(\mathbb{R}^d;X)$. By the first assertion and the Lebesgue dominated convergence theorem,
$$\mathscr{I}_\sigma f = \lim_{r\searrow 0, R\nearrow\infty}\int_{A(r,R)} \frac{\delta_h f}{|h|^{d+\sigma}}\,dh \qquad \text{in} \quad H^{s-2,p}(\mathbb{R}^d;X), \tag{4.5}$$
where the integrals $\int_{A(r,R)} \frac{\delta_h f}{|h|^{d+\sigma}}\,dh$ are Bochner integrals in $H^{s-2,p}(\mathbb{R}^d;X)$. As $f \in L^p(\mathbb{R}^d;X)$, $h \mapsto \frac{\delta_h f}{|h|^{d+\sigma}}$ is in $L^1(A(r,R);L^p(\mathbb{R}^d;X))$ for every $0<r<R<\infty$. Since $L^p(\mathbb{R}^d;X), H^{s-2,p}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$, it follows that the integrals $\int_{A(r,R)} \frac{\delta_h f}{|h|^{d+\sigma}}\,dh$ in (4.5) can also be considered as Bochner integrals in $L^p(\mathbb{R}^d;X)$, implying that $\int_{A(r,R)} \frac{\delta_h f}{|h|^{d+\sigma}}\,dh = \left[x \mapsto \int_{A(r,R)} \frac{\delta_h f(x)}{|h|^{d+\sigma}}\,dh\right]$ (see [126, Proposition 1.2.25]).

(3) By linearity it suffices to consider the scalar case $f \in H^{s,p}(\mathbb{R}^d)$ for some $s \in \mathbb{R}$. By the density of $\mathscr{S}(\mathbb{R}^d) \subseteq H^{s,p}(\mathbb{R}^d)$ (see Lemma 4.3.4) it suffices to consider $f \in \mathscr{S}(\mathbb{R}^d)$. Indeed, this follows from the boundedness of $\mathscr{I}_\sigma$ and $(-\Delta)^{\sigma/2}$ (see (1). Now (4.3) follows from well-known results (see [151, Theorem 1.1(e)]). For convenience we include a direct proof. Using Lemma 4.4.3, for each $f \in \mathscr{S}(\mathbb{R}^d;X)$ we find

$$\begin{aligned}(-\Delta)^{\sigma/2} f = \mathscr{F}^{-1}[(\xi \mapsto |\xi|^\sigma)\widehat{f}] &= \mathscr{F}^{-1}\left[c_{d,\sigma}\int_{\mathbb{R}^d}\left[\xi \mapsto \frac{e^{\imath h\xi}-1}{|h|^{d+\sigma}}\widehat{f}(\xi)\right]dh\right]\\ &= c_{d,\sigma}\int_{\mathbb{R}^d}\mathscr{F}^{-1}\left[\xi \mapsto \frac{e^{\imath h\xi}-1}{|h|^{d+\sigma}}\widehat{f}(\xi)\right]dh = c_{d,\sigma}\int_{\mathbb{R}^d}\frac{\delta_h f}{|h|^{d+\sigma}}\,dh,\end{aligned}$$

where all integrals are in $\mathscr{S}'(\mathbb{R}^d;X)$. By (1), for every $f \in \mathscr{S}(\mathbb{R}^d;X) \subset H^{0,p}(\mathbb{R}^d;X)$ we have $\mathscr{I}_\sigma f = \int_{\mathbb{R}^d}\frac{\delta_h f}{|h|^{d+\sigma}}\,dh$, where the integral is taken in $H^{-1,p}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$. This proves (4.3), as desired. □

Finally we are in position to prove the pointwise multiplier result.

Proof of Theorem 4.4.1. We only consider $s \geq 0$. The case $s < 0$ follows from a duality argument using [187, Proposition 3.5].

By Lemma 4.3.4 it is enough to prove $\|\mathbb{1}_{\mathbb{R}^d_+} f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)} \lesssim_{s,p,d,\gamma,X} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)}$ for an arbitrary $f \in \mathscr{S}(\mathbb{R}^d)\otimes X$. Let $g := \mathbb{1}_{\mathbb{R}^d_+} f \in L^p(\mathbb{R}^d)\otimes X$. By Lemma 4.4.2, we have

$$\|g\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)} \lesssim_{s,p,d,\gamma,X} \|g\|_{L^p(\mathbb{R}^d,w_\gamma;X)} + \|(-\Delta)^{s/2} g\|_{L^p(\mathbb{R}^d,w_\gamma;X)}.$$

Clearly, $\|g\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \leq \|f\|_{L^p(\mathbb{R}^d,w_\gamma;X)}$ from which we see that it suffices to show

$$\|(-\Delta)^{s/2} g\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \lesssim_{s,p,d,\gamma} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)}. \tag{4.6}$$

By Proposition 4.4.4,

$$\mathscr{I}_{s,j} g := \left[x \mapsto \int_{A(\frac{1}{j},j)}\frac{\delta_h g(x)}{|h|^{d+s}}\,dh\right] \xrightarrow{j\to\infty} (-\Delta)^{s/2} g \qquad \text{in} \quad H^{s-2,p}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X).$$

In order to finish the proof, it is thus enough to show that $\mathscr{I}_{s,j} g$ converges in $L^p(\mathbb{R}^d,w_\gamma;X)+L^p(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$ to some G satisfying

$$\|G\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \lesssim_{s,p,d,\gamma,X} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)}. \tag{4.7}$$

Indeed, then $(-\Delta)^{s/2} g = G$ and (4.6) holds.

Defining

$$S := \left\{(y,z) \in \mathbb{R}^2 : [z < -y \text{ and } y > 0] \text{ or } [z > -y \text{ and } y < 0]\right\}$$

we have

$$\mathscr{I}_{s,j} g = G_{1,j} + G_{2,j}$$

$$:= \mathbb{1}_{\mathbb{R}^d_+}\mathscr{I}_{s,j}f + \left[x \mapsto -\mathrm{sgn}(x_1)\int_{A(\frac{1}{j},j)} \mathbb{1}_S(x_1,h_1)\frac{f(x+h)}{|h|^{d+s}}\,dh\right], \tag{4.8}$$

where $\mathscr{I}_{s,j}f$ is defined analogously to $\mathscr{I}_{s,j}g$:

$$\mathscr{I}_{s,j}f := \left[x \mapsto \int_{A(\frac{1}{j},j)} \frac{\delta_h f(x)}{|h|^{d+s}}\,dh\right].$$

We first consider $\{G_{1,j}\}_{j\in\mathbb{N}}$. Since $\mathscr{I}_{s,j}f \overset{j\to\infty}{\longrightarrow} (-\Delta)^{s/2}f$ in $L^p(\mathbb{R}^d;X)$ by Proposition 4.4.4, it follows that $G_1 := \mathbb{1}_{\mathbb{R}^d_+}(-\Delta)^{s/2}f = \lim_{j\to\infty} G_{1,j}$ in $L^p(\mathbb{R}^d;X)$. By Proposition Lemma 4.4.2,

$$\|G_1\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \leq \|(-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \lesssim_{s,p,d,\gamma,X} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)}.$$

We next consider $\{G_{2,j}\}_{j\in\mathbb{N}}$. Observing that

$$|h| = (|h_1|^2 + |\widetilde{h}|^2)^{1/2} = ((|t| + |h_1+t|)^2 + |\widetilde{h}|^2)^{1/2}$$

for all $h = (h_1,\widetilde{h}) \in \mathbb{R}^d$ and $t \in \mathbb{R}$ with $(t,h_1) \in S$, we find

$$\begin{aligned}\int_{A(\frac{1}{j},j)} \mathbb{1}_S(x_1,h_1)\frac{\|f(x+h)\|_X}{|h|^{d+s}}\,dh &\leq \int_{\mathbb{R}^d} \frac{\|f(x+h)\|_X}{((|x_1| + |h_1+x_1|)^2 + |\widetilde{h}|^2)^{\frac{d+s}{2}}}\,dh \\ &= \int_{\mathbb{R}^d} \frac{\|f(y)\|_X}{((|x_1| + |y_1|)^2 + |\widetilde{y}-\widetilde{x}|^2)^{\frac{d+s}{2}}}\,dy \\ &\leq \int_{\mathbb{R}^d} k(x,y)|y_1|^{-s}\|f(y)\|_X\,dy,\end{aligned}$$

where $k(x,y) = ((|x_1|+|y_1|)^2 + |\widetilde{y}-\widetilde{x}|^2)^{\frac{d}{2}}$. Applying Lemma 4.3.8 to the function $\phi(y) = |y_1|^{-s}\|f(y)\|_X$ we thus obtain

$$\begin{aligned}\left\|x \mapsto \int_{A(\frac{1}{j},j)} \mathbb{1}_S(x_1,h_1)\frac{\|f(x+h)\|_X}{|h|^{d+s}}\,dh\right\|_{L^p(\mathbb{R}^d,w_\gamma)} &\leq \|I_k\phi\|_{L^p(\mathbb{R}^d,w_\gamma)} \\ &\lesssim_{p,d,\gamma} \|\phi\|_{L^p(\mathbb{R}^d,w_\gamma)} \\ &= \|f\|_{L^p(\mathbb{R}^d,w_{\gamma-sp};X)}. \\ &\lesssim_{p,d,\gamma} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)},\end{aligned}$$

where in the last step we applied Lemma 4.3.7. It follows that the limit $G_2 := \lim_{j\to\infty} G_{2,j}$ exists in $L^p(\mathbb{R}^d,w_\gamma;X)$ and, moreover,

$$\|G_2\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \lesssim_{p,d,\gamma} \|f\|_{H^{s,p}(\mathbb{R}^d,w_\gamma;X)}.$$

Finally, combining the just obtained results for $\{G_{1,j}\}_{j\in\mathbb{N}}$ and $\{G_{2,j}\}_{j\in\mathbb{N}}$, we see that $G := G_1 + G_2 = \lim_{j\to\infty}\mathscr{I}_{s,j}g$ in $L^p(\mathbb{R}^d,w_\gamma;X) + L^p(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R};X)$ and (4.7) holds as desired. □

4.5. INTERPOLATION THEORY WITHOUT BOUNDARY CONDITIONS

For details on interpolation theory we refer the reader to [23, 235]. In this section we present some weighted and vector-valued versions of known results.

The following extension operator will allow us to reduce the half space case $\mathbb{R}^d_+$ to the full space $\mathbb{R}^d$.

Lemma 4.5.1 (Extension operator)**.** *Let X be a Banach space. Let $p \in (1,\infty)$, and $m \in \mathbb{N}_0$. Let $w \in A_p$ be such that $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ for $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$. Then there exists an operator $\mathscr{E}^m_+ : L^p(\mathbb{R}^d_+, w; X) \to L^p(\mathbb{R}^d, w; X)$ such that*

1. *For all $f \in L^p(\mathbb{R}^d_+, w; X)$, $(\mathscr{E}^m_+ f)|_{\mathbb{R}^d_+} = f$;*
2. *for all $k \in \{0, \dots, , m\}$, $\mathscr{E}^m_+ : W^{k,p}(\mathbb{R}^d_+, w; X) \to W^{k,p}(\mathbb{R}^d, w; X)$ is bounded,*

Moreover, if $f \in L^p(\mathbb{R}^d_+, w; X) \cap C^m(\mathbb{R}^d_+; X)$, then $\mathscr{E}^m_+ f$ is m-times continuous differentiable on $\mathbb{R}^d$.

By a reflection argument the same holds for $\mathbb{R}^d_-$. The corresponding operator will be denoted by $\mathscr{E}^m_-$.

Proof. The result is a simple extension of the classical construction given in [2, Theorem 5.19] to the weighted setting. The final assertion is clear from the construction of $\mathscr{E}^m_+$. □

To define Bessel potential spaces on domains, we proceed in an abstract way using factor spaces.

Definition 4.5.2. Let $\mathbb{F} \hookrightarrow \mathscr{D}'(\mathbb{R}^d; X)$ be a Banach space. Define the *restricted space/factor space* to an open set $\Omega \subseteq \mathbb{R}^d$ as

$$\mathbb{F}(\Omega) := \{f \in \mathscr{D}'(\mathbb{R}^d; X) : \exists g \in \mathbb{F}, f = g|_\Omega\}$$

and let

$$\|f\|_{\mathbb{F}(\Omega)} = \inf\{\|g\|_{\mathbb{F}} : g|_\Omega = f\}.$$

We say that $\mathscr{E}$ is an *extension operator for* $\mathbb{F}(\Omega)$ if

1. for all $f \in \mathbb{F}(\Omega)$, $(\mathscr{E} f)|_\Omega = f$;
2. $\mathscr{E} : \mathbb{F}(\Omega) \to \mathbb{F}$ is bounded.

For $p \in (1,\infty)$, $w \in A_p$ and an open set $\Omega \subset \mathbb{R}^d$, we define the Bessel potential space $H^{s,p}(\Omega, w; X)$ as the factor space

$$H^{s,p}(\Omega, w; X) := [H^{s,p}(\mathbb{R}^d, w; X)](\Omega).$$

By Lemma 4.5.1 and for w as stated there, we find that $W^{k,p}(\mathbb{R}^d_+, w; X)$ can be identified (up to an equivalent norm) with the factor space $[W^{k,p}(\mathbb{R}^d, w; X)](\mathbb{R}^d_+)$, where an extension operator can also be found. Indeed, let $W^{k,p}_{\text{factor}}(\mathbb{R}^d_+, w; X) = [W^{k,p}(\mathbb{R}^d, w; X)](\mathbb{R}^d_+)$

denote the factor space. For $f \in W^{k,p}_{\text{factor}}(\mathbb{R}^d, w; X)$ let $g \in W^{k,p}(\mathbb{R}^d, w; X)$ be such that $g|_{\mathbb{R}^d_+} = f$. Then

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+, w;X)} \leq \|g\|_{W^{k,p}(\mathbb{R}^d, w;X)}.$$

Taking the infimum over all of the above g, we find

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+, w;X)} \leq \|f\|_{W^{k,p}_{\text{factor}}(\mathbb{R}^d_+, w;X)}.$$

Next let $f \in W^{k,p}(\mathbb{R}^d_+, w; X)$. Then $\mathcal{E}_+ f \in W^{k,p}(\mathbb{R}^d, w; X)$ and

$$\|f\|_{W^{k,p}_{\text{factor}}(\mathbb{R}^d_+, w;X)} \leq \|\mathcal{E}_+ f\|_{W^{k,p}(\mathbb{R}^d, w;X)} \leq C\|f\|_{W^{k,p}(\mathbb{R}^d_+, w;X)}.$$

Next we present two abstract lemmas to identify factor spaces in the complex interpolation scale. The result is a straightforward consequence of [235, Theorem 1.2.4]. We include the short in order to be able to track the constants. For details on complex interpolation theory we refer to [235, Section 1.9.3].

Lemma 4.5.3. *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples and let $X_\theta = [X_0, X_1]_\theta$ and $Y_\theta = [Y_0, Y_1]_\theta$ for a given $\theta \in (0,1)$. Assume $R : X_0 + X_1 \to Y_0 + Y_1$ and $S : Y_0 + Y_1 \to X_0 + X_1$ are linear operators such that $S \in \mathcal{L}(Y_j, X_j)$, $R \in \mathcal{L}(X_j, Y_j)$ and RS is the identity operator on Y_j for $j \in \{0,1\}$. Then SR defines a projection on X_θ and R is an isomorphism from $SR(X_\theta)$ onto Y_θ with inverse S. Moreover, the following estimates hold:*

$$C_S^{-1}\|Sy\|_{X_\theta} \leq \|y\|_{Y_\theta} \leq C_R\|Sy\|_{X_\theta}, \quad y \in Y_\theta,$$
$$\|Rx\|_{Y_\theta} \leq C_R\|x\|_{X_\theta}, \quad x \in X_\theta,$$
$$\|x\|_{X_\theta} \leq C_S\|Rx\|_{Y_\theta}, \quad x \in SR(X_\theta),$$

where $C_R = \max_{j\in\{0,1\}} \|R\|_{\mathcal{L}(X_j,Y_j)}$ and $C_S = \max_{j\in\{0,1\}} \|S\|_{\mathcal{L}(X_j,Y_j)}$.

Proof. By complex interpolation we know

$$\|S\|_{\mathcal{L}(Y_\theta,X_\theta)} \leq C_S, \text{ and } \|R\|_{\mathcal{L}(X_\theta,Y_\theta)} \leq C_R$$

and RS is the identity operator on Y_θ. This proves the upper estimates for S and R. To see that SR is a projection note that $(SR)(SR) = SR$. The lower estimate for S follows from

$$\|y\|_{Y_\theta} = \|RSy\|_{Y_\theta} \leq C_R\|Sy\|_{X_\theta}, \quad y \in Y_\theta.$$

To prove the lower estimate for R note that for $x := SRu \in SR(X_\theta)$

$$\|x\|_{X_\theta} = \|SRSRu\|_{X_\theta} \leq C_S\|RSRu\|_{Y_\theta} = C_S\|Rx\|_{X_\theta}.$$

□

Lemma 4.5.4. *Let $\mathbb{F}^0, \mathbb{F}^1 \hookrightarrow \mathscr{D}'(\mathbb{R}^d; X)$ be two Banach spaces. For $\theta \in (0,1)$, let*

$$\mathbb{F}^\theta = [\mathbb{F}^0, \mathbb{F}^1]_\theta.$$

Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and define $\mathbb{F}^\theta(\Omega)$ as in Definition 4.5.2, and assume there is an extension operator $\mathscr{E}$ for $\mathbb{F}^s(\Omega)$ for $s \in \{0,1\}$. Then $[\mathbb{F}^0(\Omega), \mathbb{F}^1(\Omega)]_\theta = \mathbb{F}^\theta(\Omega)$ and

$$C^{-1}\|f\|_{\mathbb{F}^\theta(\Omega)} \leq \|f\|_{[\mathbb{F}^0(\Omega),\mathbb{F}^1(\Omega)]_\theta} \leq \|f\|_{\mathbb{F}^\theta(\Omega)}$$

where C only depends on the norms of the extension operator. Moreover, $\mathscr{E}$ is an extension operator for $\mathbb{F}^\theta(\Omega)$.

Proof. Define $R : \mathbb{F}^j \to \mathbb{F}^j(\Omega)$ by $Rf = f|_\Omega$ and $S : \mathbb{F}^j(\Omega) \to \mathbb{F}^j$ as $S = \mathscr{E}$. Then $\|R\| \leq 1, \|S\| \leq C$ and $RS = I$. From Lemma 4.5.3 we conclude that for all $f \in [\mathbb{F}^0(\Omega), \mathbb{F}^1(\Omega)]_\theta$

$$C^{-1}\|f\|_{\mathbb{F}^\theta(\Omega)} \leq C^{-1}\|\mathscr{E}f\|_{\mathbb{F}^\theta} \leq \|f\|_{[\mathbb{F}^0(\Omega),\mathbb{F}^1(\Omega)]_\theta}.$$

Conversely, let $f \in \mathbb{F}^\theta(\Omega)$. Choose, $g \in \mathbb{F}^\theta$ such that $Rg = g|_\Omega = f$. Since $\|R\| \leq 1$, by complex interpolation we find

$$\|f\|_{[\mathbb{F}^0(\Omega),\mathbb{F}^1(\Omega)]_\theta} \leq \|g\|_{[\mathbb{F}^0,\mathbb{F}^1]_\theta} = \|g\|_{\mathbb{F}^\theta}$$

Taking the infimum over all g as above, the result follows.

To show the final assertion, note that $\mathscr{E} \in \mathscr{L}(\mathbb{F}^\theta(\Omega), \mathbb{F}^\theta)$ by the above. Moreover, for $f \in \mathbb{F}^0(\Omega) \cap \mathbb{F}^1(\Omega)$, $(\mathscr{E}f)|_\Omega = f$. By density (see [235, Theorem 1.9.3]) this extends to all $f \in \mathbb{F}^\theta(\Omega)$. □

Proposition 4.5.5. *Let X be a UMD space, $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and assume $w \in A_p$ is such that $w(x_1, \tilde{x}) = w(-x_1, \tilde{x})$ for $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$. Then $H^{k,p}(\mathbb{R}^d_+, w; X) = W^{k,p}(\mathbb{R}^d_+, w; X)$*

Proof. This is immediate from Proposition 4.3.2 and the fact that $W^{k,p}(\mathbb{R}^d_+, w; X)$ coincides with the factor space $[W^{k,p}(\mathbb{R}^d, w; X)](\mathbb{R}^d_+)$. □

Next we identify the complex interpolation spaces of $H^{s,p}(\Omega, w; X)$. Here the UMD property is needed to obtain bounded imaginary powers of $-\Delta$.

Proposition 4.5.6. *Let X be a UMD space and $p \in (1,\infty)$. Let $w \in A_p$ be such that $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ for all $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$.*

(1) *Let $\theta \in [0,1]$ and $s_0, s_1, s \in \mathbb{R}$ be such that $s = s_0(1-\theta) + s_1\theta$. Then for $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ one has*

$$[H^{s_0,p}(\Omega, w; X), H^{s_1,p}(\Omega, w; X)]_\theta = H^{s,p}(\Omega, w; X)$$

(2) *For each $m \in \mathbb{N}_0$ there exists an $\mathscr{E}^m_+ \in \mathscr{L}(H^{-m,p}(\mathbb{R}^d_+, w; X), H^{-m,p}(\mathbb{R}^d, w; X))$ such that*

- *for all $|s| \leq m$, $\mathscr{E}_+ \in \mathscr{L}(H^{s,p}(\mathbb{R}^d_+, w; X), H^{s,p}(\mathbb{R}^d, w; X))$,*
- *for all $|s| \leq m$, $f \mapsto (\mathscr{E}_+ f)|_{\mathbb{R}^d_+}$ equals the identity operator on $H^{s,p}(\mathbb{R}_+, w; X)$.*

Moreover, if $f \in L^p(\mathbb{R}^d_+, w; X) \cap C^m(\overline{\mathbb{R}^d_+}; X)$*, then* $\mathscr{E}^m_+ f \in C^m(\mathbb{R}^d; X)$*.*

By a reflection argument the same holds for $\mathbb{R}^d_-$. The corresponding operator will be denoted by $\mathscr{E}^m_-$.

Proof. (1): For $\Omega = \mathbb{R}^d$, the result follows from [187, Proposition 3.2 and 3.7] (see [126, Theorem 5.6.9] for the unweighed case).

(2): Fix $m \in \mathbb{N}$. We first construct $\widetilde{\mathscr{E}}^m_+ \in \mathscr{L}(H^{-m,p}(\mathbb{R}^d, w; X))$ such that

(i) $\widetilde{\mathscr{E}}^m_+ \in \mathscr{L}(H^{s,p}(\mathbb{R}^d, w; X))$ for all $|s| \leq m$;

(ii) $\widetilde{\mathscr{E}}^m_+ f|_{\mathbb{R}^d_+} = f|_{\mathbb{R}^d_+}$;

(iii) $\widetilde{\mathscr{E}}^m_+ f = 0$ if $f|_{\mathbb{R}^d_+} = 0$;

Given $\widetilde{\mathscr{E}}^m_+$ we can define $\mathscr{E}^m_+ : H^{s,p}(\mathbb{R}^d_+, w; X) \to H^{s,p}(\mathbb{R}^d, w; X)$ by $\mathscr{E}^m_+ f = \widetilde{\mathscr{E}}^m_+ \widetilde{f}$ where $\widetilde{f} \in H^{s,p}(\mathbb{R}^d, w; X)$ satisfies $\widetilde{f}|_{\mathbb{R}^d_+} = f$. This is well-defined by (iii).

In order to construct $\widetilde{\mathscr{E}}^m_+$ let $0 < \lambda_1 < \ldots < \lambda_{2m+2} < \infty$ and $b_1, \ldots, b_{2m+2} \in \mathbb{R}$ be as in [235, 2.9.3]. For $\lambda \in \mathbb{R} \setminus \{0\}$ we write $T_\lambda f(x) = f(-\lambda x_1, \widetilde{x})$. Let $\widetilde{\mathscr{E}}^m_+ \in \mathscr{L}(L^p(\mathbb{R}^d, w; X))$ and $\widetilde{E}^m_+ \in \mathscr{L}(L^{p'}(\mathbb{R}^d, w'; X^*))$ be defined by

$$\widetilde{\mathscr{E}}^m_+ f = \mathbb{1}_{\mathbb{R}^d_+} f + \mathbb{1}_{\mathbb{R}^d_-} \sum_{j=1}^{2m+2} b_j T_{\lambda_j} f, \qquad \widetilde{E}^m_+ g = \mathbb{1}_{\mathbb{R}^d_+} \Big(g + \sum_{j=1}^{2m+2} b_j \lambda_j^{-1} T_{\lambda_j^{-1}} g \Big).$$

Then one can check that

$$\langle \widetilde{\mathscr{E}}^m_+ f, g \rangle = \langle f, \widetilde{E}^m_+ g \rangle, \quad f \in L^p(\mathbb{R}^d, w; X), \;\; g \in L^{p'}(\mathbb{R}^d, w'; X^*). \tag{4.9}$$

Moreover, by the special choice of $b_1, \ldots, b_{2m+2}$ it is standard to check that $\widetilde{\mathscr{E}}^m_+ \in \mathscr{L}(W^{m,p}(\mathbb{R}^d, w; X))$ and $\widetilde{E}^m_+ \in \mathscr{L}(W^{m,p'}(\mathbb{R}^d, w'; X^*))$. In view of (2) for $\Omega = \mathbb{R}^d$ and Proposition 4.3.2, complex interpolation gives $\widetilde{\mathscr{E}}^m_+ \in \mathscr{L}(H^{s,p}(\mathbb{R}^d, w; X))$ and $\widetilde{E}^m_+ \in \mathscr{L}(H^{s,p'}(\mathbb{R}^d, w'; X^*))$ for all $0 \leq s \leq m$.

Recall that $H^{s,p}(\mathbb{R}^d, w; X) = (H^{-s,p'}(\mathbb{R}^d, w'; X^*))^*$ (see [187, Proposition 3.5]), X being reflexive as a UMD space (see [126, Theorem 4.3.3]). By the duality relation (4.9) we find that $\widetilde{\mathscr{E}}^m_+$ extends to a bounded linear operator on $H^{s,p}(\mathbb{R}^d, w; X)$ for each $s \in [-m, 0]$. Therefore, (i) follows and moreover (ii) follows by a density argument. To check (iii) let $f \in H^{-m,p}(\mathbb{R}^d, w; X)$ with $f|_{\mathbb{R}^d_+} = 0$ be given. Let $\phi \in C^\infty_c(\mathbb{R}^d_-)$ be such that $\int \phi \, dx = 1$ and set $\phi_n := n^{-d} \phi(n \cdot)$ for $n \in \mathbb{N}$. Then, by Lemma 4.3.6, $\phi_n * f \to f$ in $H^{-m,p}(\mathbb{R}^d, w; X)$ and $\phi_n * f \in L^p(\mathbb{R}^d, w; X)$. Now since $\phi_n * f|_{\mathbb{R}^d_+} = 0$ it follows that $\widetilde{\mathscr{E}}^m_+ f|_{\mathbb{R}^d_+} = \lim_{n\to\infty} \widetilde{\mathscr{E}}^m_+ \phi_n * f|_{\mathbb{R}^d_+} = 0$.

Finally, note that for $f \in L^p(\mathbb{R}^d_+, w; X) \cap C^m(\overline{\mathbb{R}^d_+}; X)$, $\mathscr{E}^+_m f \in C^m(\overline{R}^d_-; X) \oplus C^m(\overline{R}^d_+; X)$ with

$$\mathscr{E}^+_m f|_{\mathbb{R}^d_+} = f \quad \text{and} \quad \mathscr{E}^+_m f|_{\mathbb{R}^d_-} = \sum_{j=1}^{2m+2} b_j T_{\lambda_j} f$$

and by the special choice of $b_1, \ldots, b_{2m+2}$, one can check that $f \in C^m(\mathbb{R}^d; X)$.

Now (1) for $\Omega = \mathbb{R}^d_+$ follows from Lemma 4.5.4 and (2). □

For an open set $\Omega \subseteq \mathbb{R}^d$, and $s \in \mathbb{R}$ let $H^{s,p}_{\overline{\Omega}}(\mathbb{R}^d, w_\gamma; X)$ be the closed subspace of $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ of functions with support in $\overline{\Omega}$.

Proposition 4.5.7. *Let X be a UMD space, $p \in (1,\infty)$, $k \in \mathbb{N}$, $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ for all $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$. Let $\theta \in [0,1]$ and $s_0, s_1, s \in \mathbb{R}$ be such that $s = s_0(1-\theta) + s_1\theta$. Then the following identity holds with equivalence of norms*

$$[H^{s_0,p}_{\mathbb{R}^d_\pm}(\mathbb{R}^d, w; X), H^{s_1,p}_{\mathbb{R}^d_\pm}(\mathbb{R}^d, w; X)]_\theta = H^{s,p}_{\mathbb{R}^d_\pm}(\mathbb{R}^d, w; X).$$

Proof. To show this we consider the case of $\mathbb{R}^d_+$. The other case can be proved in the same way. Let $\mathcal{E}^m_-$ be the (reflected) extension operator of Proposition 6.3.7 with m the least integer above $\max\{|s_0|, |s_1|\}$. Define $R : H^{s_0\wedge s_1,p}(\mathbb{R}^d, w; X) \to H^{s_0\wedge s_1,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X)$ by

$$Rf := f - \mathcal{E}^m_-(f|_{\mathbb{R}^d_-})$$

and let $S : H^{s_0\wedge s_1,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X) \to H^{s_0\wedge s_1,p}(\mathbb{R}^d, w; X)$ be the inclusion operator. For each $t \in [s_0 \wedge s_1, m]$, R and S restrict to bounded linear operators $R : H^{t,p}(\mathbb{R}^d, w; X) \to H^{t,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X)$ and $S : H^{t,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X) \to H^{t,p}(\mathbb{R}^d, w; X)$ with the property that $SR(H^{t,p}(\mathbb{R}^d, w; X)) = H^{t,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X)$. Using Lemma 4.5.3 in combination with Proposition 6.3.7 we find that R restricts to an isomorphism from $H^{s,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X) = SR(H^{s,p}(\mathbb{R}^d, w; X))$ to $[H^{s_0,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X), H^{s_1,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X)]_\theta$. Since $Rf = f$ for all $f \in H^{s,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w; X)$, this proves the required identity for the interpolation space. The norm equivalence follows from the estimates in Lemma 4.5.3 as well. □

To end this section we present a variation of a classical interpolation inequality. The result can be deduced from the weighted Gagliardo-Nirenberg type inequality [182, Proposition 5.1]. We provide a more direct proof which also yields additional information. The unweighted and scalar-valued case can be found in [145, Theorem 1.5.1]. However, the proof given there does not extend to the weighted setting. The lemma can also be deduced from Proposition 4.2.3, but this would require X to be a UMD space (cf. the proof of [93, Corollary 5.3]).

Lemma 4.5.8 (Gagliardo-Nirenberg inequality)**.** *Let X be a Banach space and $k \in \mathbb{N}$. Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Let $w \in A_p$ be such that $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ if $\Omega = \mathbb{R}^d_+$. Then for all $u \in W^{k,p}(\Omega, w; X)$ and $j \in \{1, \dots, k-1\}$,*

$$[u]_{W^{j,p}(\Omega,w;X)} \lesssim_{p,k,[w]_{A_p}} \|u\|^{1-\frac{j}{k}}_{L^p(\Omega,w;X)} [u]^{\frac{j}{k}}_{W^{k,p}(\Omega,w;X)}.$$

Proof. By an iteration argument one sees that it suffices to consider $j = 1$ and $k = 2$ (see [145, Exercise 1.5.6]).

First consider the case $\Omega = \mathbb{R}^d$. For $u \in W^{2,p}(\mathbb{R}^d, w; X)$, it follows from Lemma 4.3.3 that

$$[u]_{W^{1,p}(\mathbb{R}^d,w;X)} \le \|u\|_{W^{1,p}(\mathbb{R}^d,w;X)}$$

$$\lesssim_{p,[w]_{A_p}} \|u\|_{H^{2,p}(\mathbb{R}^d,w;X)} \leq \|u\|_{L^p(\mathbb{R}^d,w;X)} + [u]_{W^{2,p}(\mathbb{R}^d,w;X)}.$$

For $\lambda > 0$ let $u_\lambda(x) = u(\lambda x)$ and $w_\lambda = w(\lambda x)$ and note that $[w]_{A_p} = [w_\lambda]_{A_p}$. Then applying the estimate to u_λ and the weight w_λ, a substitution yields

$$[u]_{W^{1,p}(\mathbb{R}^d,w;X)} \lesssim_{p,n,[w]_{A_p}} \lambda^{-1}\|u\|_{L^p(\mathbb{R}^d,w;X)} + \lambda[u]_{W^{2,p}(\mathbb{R}^d,w;X)}.$$

Minimizing over $\lambda > 0$ the result follows.

In the case $\Omega = \mathbb{R}^d_+$ we use a standard extension argument. Let $\mathscr{E}^2_+$ be the extension operator from Lemma 4.5.1. Then by [2, Theorem 5.19], $\mathscr{E}^2_+$ has the following additional property: for all $|\alpha| \leq 2$, $D^\alpha \mathscr{E}^2_+ = E_\alpha D^\alpha$, where E_α is an extension operator for $W^{2-|\alpha|}(\mathbb{R}^d_+, w; X)$. Therefore, from the case $\Omega = \mathbb{R}^d$ applied to $\mathscr{E}^2_+ u$ and the boundedness of the extension operators we find that

$$[u]_{W^{1,p}(\mathbb{R}^d_+,w;X)} \leq [\mathscr{E}^n_+ u]_{W^{1,p}(\mathbb{R}^d,w;X)} \lesssim_{p,d,[w]_{A_p}} \|\mathscr{E}^n_+ u\|^{1/2}_{L^p(\mathbb{R}^d,w;X)} [\mathscr{E}^n_+ u]^{1/2}_{W^{2,p}(\mathbb{R}^d,w;X)}$$

Clearly, $\|\mathscr{E}^n_+ u\|_{L^p(\mathbb{R}^d,w;X)} \leq \|u\|_{L^p(\mathbb{R}^d_+,w;X)}$. Moreover, since $D^\alpha \mathscr{E}^2_+ = E_0 D^\alpha$,

$$[\mathscr{E}^n_+ u]_{W^{2,p}(\mathbb{R}^d,w;X)} = \sum_{|\alpha|=2} \|E_0 D^\alpha u\|_{L^p(\mathbb{R}^d,w;X)} \leq [u]_{W^{2,p}(\mathbb{R}^d_+,w;X)}.$$

Therefore, the result follows if we combine the two estimates. □

4.6. APPLICATION TO INTERPOLATION THEORY AND THE FIRST DERIVATIVE

For $p \in (1,\infty)$, $s \in \mathbb{R}$ and a weight $w \in A_p$, let $H^{s,p}_0(\mathbb{R}, w; X)$ denote the closure of $C^\infty_c(\mathbb{R} \setminus \{0\}; X)$ in $H^{s,p}_0(\mathbb{R}, w; X)$. In this section we characterize the complex interpolation space $[L^p(\mathbb{R}_+, w_\gamma; X), H^{1,p}_0(\mathbb{R}_+, w_\gamma; X)]_\theta$. Moreover, we use this to characterize the domains of fractional powers of the first derivative.

4.6.1. Results on the whole real line

For $k \in \mathbb{N}_0$ let

$$W^{k+1,1}_{\mathrm{loc},0}(\mathbb{R}; X) := \{f \in W^{k+1,1}_{\mathrm{loc}}(\mathbb{R}; X) : f(0) = \ldots = f^{(k)}(0) = 0\}.$$

Since $f(y) - f(x) = \int_x^y f'(t)\,dt$, it follows that f has a version which is uniformly continuous on bounded intervals, and hence $f^{(j)}(0)$ for $j \in \{0,\ldots,k\}$ is defined in a pointwise sense

We will need the following simple lemma.

Lemma 4.6.1. *Let X be a Banach space and $k \in \mathbb{N}_0$. If $f \in W^{k+1,1}_{\mathrm{loc}}(\mathbb{R}; X)$ satisfies $f(0) = \ldots = f^{(k)}(0) = 0$, then $1_{\mathbb{R}_+} f \in W^{k+1,1}_{\mathrm{loc}}(\mathbb{R}; X)$ with*

$$(1_{\mathbb{R}_+} f)^{(j)} = 1_{\mathbb{R}_+} f^{(j)}, \qquad j \in \{1,\ldots,k+1\}.$$

Proof. Using an inductive argument we may reduce to the case $k = 0$. So suppose $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}; X)$ satisfies $f(0) = 0$. Then $f(x) = \int_0^x f'(t)\,dt$ for all $x \in \mathbb{R}$, from which it follows that

$$\mathbb{1}_{\mathbb{R}_+} f(x) = \int_0^x \mathbb{1}_{\mathbb{R}_+} f'(t)\,dt, \qquad x \in \mathbb{R}.$$

This shows $\mathbb{1}_{\mathbb{R}_+} f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}; X)$ with $(1_{\mathbb{R}_+} f)' = 1_{\mathbb{R}_+} f'$. □

Proposition 4.6.2. *Let X be a UMD Banach space, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Assume $s > \frac{1+\gamma}{p} - 1$ and $k \in \mathbb{N}_0$ are such that $\frac{1+\gamma}{p} - 1 + k < s < \frac{1+\gamma}{p} + k$. For all $f \in H^{s,p}(\mathbb{R}, w_\gamma; X) \cap W^{k+1,1}_{\mathrm{loc},0}(\mathbb{R}; X)$ we then have*

$$\|\mathbb{1}_{\mathbb{R}_+} f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)} \lesssim_{s,p,\gamma,X} \|f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)}.$$

As a consequence, $\mathbb{1}_{\mathbb{R}_+}$ is a pointwise multiplier on $H^{s,p}_0(\mathbb{R}, w_\gamma; X)$. Moreover, for all $f \in H^{s,p}_0(\mathbb{R}, w_\gamma; X)$ it holds that

$$(\mathbb{1}_{\mathbb{R}_+} f)^{(j)} = \mathbb{1}_{\mathbb{R}_+} f^{(j)}, \qquad j \in \{0,\dots k\}. \tag{4.10}$$

Proof. As in [187, Proposition 3.4] one checks the following equivalence of extended norms on $\mathscr{S}'(\mathbb{R}; X)$:

$$\begin{aligned} \|f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)} &\eqsim_{s,\gamma,p,X} \|f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} + \|D^k f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} \\ &\eqsim_{s,\gamma,p,X} \textstyle\sum_{j=0}^k \|D^j f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)}. \end{aligned} \tag{4.11}$$

Let $f \in H^{s,p}(\mathbb{R}, w_\gamma; X) \cap W^{k+1,1}_{\mathrm{loc},0}(\mathbb{R}; X)$ Using (4.11), Lemma 4.6.1 and Theorem 4.4.1 we find

$$\begin{aligned} \|\mathbb{1}_{\mathbb{R}_+} f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)} &\lesssim_{s,p,\gamma,X} \|\mathbb{1}_{\mathbb{R}_+} f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} + \|D^k(\mathbb{1}_{\mathbb{R}_+} f)\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} \\ &= \|\mathbb{1}_{\mathbb{R}_+} f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} + \|\mathbb{1}_{\mathbb{R}_+} D^k f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} \\ &\lesssim_{s,p,\gamma,X} \|f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} + \|D^k f\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} \\ &\lesssim_{s,p,\gamma,X} \|f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)}. \end{aligned}$$

By a density argument we find that $\mathbb{1}_{\mathbb{R}_+}$ is a pointwise multiplier on $H^{s,p}_0(\mathbb{R}, w_\gamma; X)$.

Finally, to check that (4.10) holds for $f \in H^{s,p}_0(\mathbb{R}, w_\gamma; X)$, observe that for $0 \le j \le k$, by (4.11) and the above estimate

$$\|D^j(\mathbb{1}_{\mathbb{R}_+} f)\|_{H^{s-k,p}(\mathbb{R},w_\gamma;X)} \le C\|\mathbb{1}_{\mathbb{R}_+} f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)} \le C\|f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)}.$$

Therefore, if $f \in H^{s,p}_0(\mathbb{R}, w_\gamma; X)$, then letting $f_n \in C^\infty_c(\mathbb{R} \setminus \{0\}; X)$ be such that $f_n \to f$ in $H^{s,p}_0(\mathbb{R}, w_\gamma; X)$, we find that $D^j(\mathbb{1}_{\mathbb{R}_+} f_n) \to D^j(\mathbb{1}_{\mathbb{R}_+} f)$ in $H^{s-k,p}(\mathbb{R}, w_\gamma; X)$. Since $D^j f_n \to D^j f$ in $H^{s-k,p}(\mathbb{R}, w_\gamma; X)$, by Theorem 4.4.1 also $\mathbb{1}_{\mathbb{R}_+} D^j f_n \to \mathbb{1}_{\mathbb{R}_+} D^j f$ in $H^{s-k,p}(\mathbb{R}, w_\gamma; X)$. The validity of (4.10) for functions from $C^\infty_c(\mathbb{R}\setminus\{0\})$ and uniqueness of limits in $H^{s-k,p}(\mathbb{R}, w_\gamma; X)$ yields (4.10) for general $f \in H^{s,p}_0(\mathbb{R}, w_\gamma; X)$. □

Proposition 4.6.3. *Let* $\gamma \in (-1, p-1)$ *and* $s \in \mathbb{R}$. *Assume* $k \in \mathbb{N}_0$ *satisfies* $k + \frac{1+\gamma}{p} < s$. *Then the following assertions hold:*

(1) $Tr_k : H^{s,p}(\mathbb{R}, w_\gamma; X) \cap C^k(\mathbb{R}; X) \to X^k$ *given by* $Tr_k f = (f(0), f'(0), \ldots, f^{(k)}(0))$ *uniquely extends to a bounded linear mapping* $Tr_k : H^{s,p}(\mathbb{R}, w_\gamma; X) \to X^{k+1}$.

(2) *If* $f \in H^{s,p}(\mathbb{R}, w_\gamma; X)$ *satisfies* $f|_{(0,\delta)} = 0$ *or* $f|_{(-\delta,0)} = 0$ *for some* $\delta > 0$, *then* $Tr_k f = 0$.

(3) *There exists a bounded mapping* $\mathrm{ext}_k : X^{k+1} \to H^{s,p}(\mathbb{R}, w_\gamma; X)$ *such that* $Tr_k(\mathrm{ext}_k)$ *is the identity on* X^{k+1}.

Proof. We first prove (1). By Lemma 4.3.4, it is enough to establish boundedness of

$$\mathrm{Tr}_k : (H^{s,p}(\mathbb{R}, w_\gamma; X) \cap C^k(\mathbb{R}; X), \|\cdot\|_{H^{s,p}(\mathbb{R}, w_\gamma; X)}) \to X^{k+1}.$$

Choosing $x_j^* \in X^*$ with $\|x_j^*\| = 1$ and $\|f^{(j)}(0)\| = \langle f^{(j)}(0), x_j^* \rangle$ for each $j \in \{0, \ldots, k\}$ we have $\langle f, x_j^* \rangle \in H^{s,p}(\mathbb{R}, w_\gamma) \cap C^k(\mathbb{R})$ with

$$\|f^{(j)}(0)\| = |\langle f^{(j)}(0), x_j^* \rangle| = |\langle f, x_j^* \rangle^{(j)}(0)|, \quad \|\langle f, x_j^* \rangle\|_{H^{s,p}(\mathbb{R}, w_\gamma)} \le \|f\|_{H^{s,p}(\mathbb{R}, w_\gamma; X)}.$$

So we may restrict ourselves to the case $X = \mathbb{C}$. Recall from [187, Proposition 3.4] that d/dt is a bounded linear operator from $H^{\sigma,p}(\mathbb{R}, w_\gamma)$ to $H^{\sigma-1,p}(\mathbb{R}, w_\gamma)$ for every $\sigma \in \mathbb{R}$. By differentiation it thus suffices to prove that, given $\theta \in (\frac{1+\gamma}{p}, \frac{1+\gamma}{p} + 1)$, the following estimate holds

$$|f(0)| \lesssim_{\theta,\gamma,p} \|f\|_{H^{\theta,p}(\mathbb{R}, w_\gamma)}, \qquad f \in H^{\theta,p}(\mathbb{R}, w_\gamma) \cap C(\mathbb{R}).$$

Here we actually only need to consider $f \in H^{\theta,p}(\mathbb{R}, w_\gamma) \cap C_c(\mathbb{R})$; indeed, given $\eta \in C_c^\infty(\mathbb{R})$ with $\eta(0) = 1$, $f \mapsto \eta f$ defines by complex interpolation (see Proposition 6.3.7) a bounded linear operator on $H^{\theta,p}(\mathbb{R}, w_\gamma)$ and we may consider ηf instead of f. Using Lemma 4.3.6 together with [100, Theorem 1.2.19] one can check that $C_c^\infty(\mathbb{R})$ is dense in $H^{\theta,p}(\mathbb{R}, w_\gamma) \cap C_c(\mathbb{R})$, where $C_c(\mathbb{R})$ has been equipped with the supremum norm. It thus is enough to estimate

$$|f(0)| \lesssim_{\theta,\gamma,p} \|f\|_{H^{\theta,p}(\mathbb{R}, w_\gamma; X)}, \qquad f \in C_c^\infty(\mathbb{R}).$$

To this end, let $f \in C_c^\infty(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ and put $g := (1-\Delta)^{\theta/2} f \in \mathscr{S}(\mathbb{R})$. Then, letting $G_\theta \in L^1(\mathbb{R})$ be the kernel Lemma 4.3.1, we find

$$f(0) = (1-\Delta)^{-\theta/2} g(0) = G_\theta * g(0) = \int_{\mathbb{R}} G_\theta(x) g(-x)\, dx.$$

By Lemma 4.3.1 we find

$$|f(0)| \le \int_{\mathbb{R}} |G_\theta(x)||g(-x)|\, dx \le \|G_\theta\|_{L^{p'}(\mathbb{R}, w'_\gamma)} \|g\|_{L^p(\mathbb{R}, w_\gamma)} \lesssim_{\theta,\gamma,p} \|f\|_{H^{\theta,p}(\mathbb{R}, w_\gamma)}.$$

To prove (2) consider the case that $f = 0$ on $(0,\delta)$. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\int \phi(x)\, dx = 1$ and ϕ is supported on $(-2,-1)$ and put $\phi_n(x) := n\phi(nx)$. By Lemma 4.3.6,

$\|\phi_n * f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)} \lesssim_{p,\gamma} \|f\|_{H^{s,p}(\mathbb{R},w_\gamma;X)}$ with $\phi_n * f \to f$ in $H^{s,p}(\mathbb{R}, w_\gamma; X)$. Clearly, $\phi_n * f \in C^\infty(\mathbb{R}; X)$ and by the support conditions one sees that $\phi_n * f(0) = 0$ for all $n > 2\delta^{-1}$. Therefore, $\mathrm{Tr}_k(\phi_n * f) = 0$ and the result follows by letting $n \to \infty$ and using the continuity of Tr_k.

To prove (3) choose $\phi_0, \dots, \phi_k \in C_c^\infty(\mathbb{R})$ such that $\phi_j^{(n)}(0) = \delta_{jn}$ for all $0 \le j \le k$ and $0 \le n \le k$ and let $\mathrm{ext}_k(x_j)_{j=1}^k = \sum_{j=0}^k \phi_j x_j$. This clearly satisfies the required properties. □

We can now give a characterization of $H_0^{s,p}(\mathbb{R}, w_\gamma; X)$ in terms of traces. For it will be convenient to say that the statement $\mathrm{Tr}_k f = 0$ for $k \le -1$ is empty.

Proposition 4.6.4. *Let X be a Banach space, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Let $s \in \mathbb{R}$ be such that $k + \frac{1+\gamma}{p} < s < k + 1 + \frac{1+\gamma}{p}$ with $k \in \mathbb{Z}, k \ge -1$. Then*

$$H_0^{s,p}(\mathbb{R}, w_\gamma; X) = \{f \in H^{s,p}(\mathbb{R}, w_\gamma; X) : \mathit{Tr}_k f = 0\}.$$

Note that $\mathrm{Tr}_k f$ is well defined by Proposition 4.6.3.

Proof. Clearly, $\mathrm{Tr}_k f = 0$ for every $f \in C_c^\infty(\mathbb{R} \setminus \{0\}; X)$. By continuity this extends to every $f \in H_0^{s,p}(\mathbb{R}, w_\gamma; X)$ (see Proposition 4.6.3) and hence "⊆" follows. To prove the converse, let $f \in H^{s,p}(\mathbb{R}, w_\gamma; X)$ be such that $\mathrm{Tr}_k f = 0$. By Lemma 4.3.4 we can find $\{g_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}) \otimes X$ such that $g_n \to f$ in $H^{s,p}(\mathbb{R}, w_\gamma; X)$ as $n \to \infty$. Let ext_k be as constructed in the proof of Proposition 4.6.3 and put $h_n := g_n - \mathrm{ext}_k(g_n^{(j)}(0))_{j=0}^k$ for each $n \in \mathbb{N}$. Then $h_n \in \{h \in C_c^\infty(\mathbb{R}) : \mathrm{Tr}_k h = 0\} \otimes X$ and, by Proposition 4.6.3, $h_n \to f - \mathrm{ext}_k(0)_{j=0}^k = f$ in $H^{s,p}(\mathbb{R}, w_\gamma; X)$ as $n \to \infty$.

It remains to show that we can approximate a function $h \in C_c^\infty(\mathbb{R})$ satisfying $\mathrm{Tr}_k h = 0$ by a function in $C_c^\infty(\mathbb{R} \setminus \{0\})$ with respect to the norm of $H^{s,p}(\mathbb{R}, w_\gamma)$. Writing $h = \mathbb{1}_{\mathbb{R}_+} h + \mathbb{1}_{\mathbb{R}_-} h =: h_0 + h_1$, it follows from Proposition 4.6.2 that $h_0, h_1 \in H^{s,p}(\mathbb{R}, w_\gamma; X)$ and hence it suffices to approximate each of the terms h_0 and h_1. Fix $\phi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \phi\, dx = 1$ and $\operatorname{supp}\phi \subseteq [1,\infty)$ and define $\phi_n := n\phi(n\cdot)$ for each $n \in \mathbb{N}$. Then $\phi_n * h_0 \in C_c^\infty(\mathbb{R} \setminus \{0\})$ with $\phi_n * h_0 \to h_0$ in $H^{s,p}(\mathbb{R}, w_\gamma)$ as $n \to \infty$ by Lemma 4.3.6. A similar argument can be used for h_1. □

We can now prove the main result of this section:

Theorem 4.6.5. *Let X be a UMD space and $\gamma \in (-1, p-1)$. Let $\theta \in (0,1)$ and $s_0, s_1 > -1 + \frac{\gamma+1}{p}$. Let $s = s_0(1-\theta) + s_1\theta$. If $s_0, s_1, s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$, then*

$$[H_0^{s_0,p}(\mathbb{R}, w_\gamma; X), H_0^{s_1,p}(\mathbb{R}, w_\gamma; X)]_\theta = H_0^{s,p}(\mathbb{R}, w_\gamma; X). \tag{4.12}$$

Proof. Assume $s_0, s_1, s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$ and let $E_{\mathrm{prod}}^{\sigma,p} := H_{\mathbb{R}_+}^{\sigma,p}(\mathbb{R}, w_\gamma; X) \times H_{\mathbb{R}_-}^{\sigma,p}(\mathbb{R}, w_\gamma; X)$, $\sigma \in \mathbb{R}$, for shorthand notation.

Let $\sigma > -1 + \frac{\gamma+1}{p}$ with $\sigma \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$. By Proposition 4.6.3 Tr_k vanishes on $H_{\mathbb{R}_\pm}^{\sigma,p}(\mathbb{R}, w_\gamma; X)$ for integers $k \in [0, \sigma - \frac{\gamma+1}{p})$. Thus, in view of Proposition 4.6.4, the map

$$R : E_{\mathrm{prod}}^{\sigma,p} \to H_0^{\sigma,p}(\mathbb{R}, w_\gamma; X), \qquad R(g,h) := g + h,$$

is a well-defined contraction. That the map

$$S : H_0^{\sigma,p}(\mathbb{R}, w_\gamma; X) \to E^{\sigma,p}_{\text{prod}}, \qquad Sf := (\mathbb{1}_{\mathbb{R}^d_+} f, \mathbb{1}_{\mathbb{R}^d_-} f),$$

is well-defined and continuous follows from Propositions 4.6.2 and 4.6.4. Since $R^{-1} = S$, the result follows from Proposition 4.5.7. □

4.6.2. Results on the positive half line

Let $\gamma \in (-1, p-1)$ and $s \in \mathbb{R}$. Assume $k \in \mathbb{N}_0$ satisfies $k + \frac{1+\gamma}{p} < s$. By Proposition 4.6.3, if $\widetilde{f}_1, \widetilde{f}_2 \in H^{s,p}(\mathbb{R}, w_\gamma; X)$ satisfy $\widetilde{f}_{1|\mathbb{R}_+} = \widetilde{f}_{2|\mathbb{R}_+}$, then $\mathrm{Tr}_k \widetilde{f}_1 = \mathrm{Tr}_k \widetilde{f}_2$. Therefore, $\mathrm{Tr}_k : H^{s,p}(\mathbb{R}, w_\gamma; X) \to X^{k+1}$ gives rise to a well-defined bounded linear operator $\mathrm{Tr}_{k,+} : H^{s,p}(\mathbb{R}_+, w_\gamma; X) \to X^{k+1}$ given by $\mathrm{Tr}_{k,+} f = \mathrm{Tr}_k \widetilde{f}$ whenever $\widetilde{f}_{|\mathbb{R}_+} = f$. After reducing to the scalar-valued case, Proposition 6.3.7 shows that

$$\mathrm{Tr}_{k,+} f = (f(0), f'(0), \dots, f^{(k)}(0)), \qquad f \in H^{s,p}(\mathbb{R}_+, w_\gamma; X) \cap C^k([0,\infty); X); \tag{4.13}$$

in the case $X = \mathbb{C}$ we simply pick the least integer $m \geq |s|$ and observe that $\mathrm{Tr}_{k,+} = \mathrm{Tr}_k \circ \mathscr{E}^m_+$.

Let $H_0^{s,p}(\mathbb{R}_+, w_\gamma; X)$ denote the closure of $C_c^\infty((0,\infty); X)$ in $H^{s,p}(\mathbb{R}_+, w_\gamma; X)$.

Proposition 4.6.6. *Let X be a Banach space, $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $s \in \mathbb{R}$. Assume $k \in \mathbb{N}_0$ satisfies $k + \frac{1+\gamma}{p} < s < k + 1 + \frac{1+\gamma}{p}$. Then*

$$H_0^{s,p}(\mathbb{R}_+, w_\gamma; X) = \{f \in H^{s,p}(\mathbb{R}_+, w_\gamma; X) : Tr_{k,+} f = 0\}.$$

Proof. Clearly, $\subseteq$ holds. To prove the converse let $f \in H^{s,p}(\mathbb{R}_+, w_\gamma; X)$ be such that $\mathrm{Tr}_{k,+} f = 0$. Pick $\widetilde{f} \in H^{s,p}(\mathbb{R}, w_\gamma; X)$ with $\widetilde{f}_{|\mathbb{R}_+} = f$. Then $\mathrm{Tr}_k \widetilde{f} = \mathrm{Tr}_{k,+} f = 0$. By Proposition 4.6.4 we thus get $\widetilde{f} = \lim_{n\to\infty} \widetilde{f}_n$ in $H^{s,p}(\mathbb{R}, w_\gamma; X)$ for some sequence $(\widetilde{f}_n)_{n\in\mathbb{N}}$ from $C_c^\infty(\mathbb{R} \setminus \{0\}; X)$. Now $f_n := \widetilde{f}_{n|\mathbb{R}_+} \in C_c^\infty((0,\infty); X)$ with $f_n \to f$ in $H^{s,p}(\mathbb{R}_+, w_\gamma; X)$ as $n \to \infty$. □

Theorem 4.6.7. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Let $\theta \in (0,1)$ and $s_0, s_1 > -1 + \frac{\gamma+1}{p}$. Let $s = s_0(1-\theta) + s_1\theta$. If $s_0, s_1, s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$, then*

$$[H_0^{s_0,p}(\mathbb{R}_+, w_\gamma; X), H_0^{s_1,p}(\mathbb{R}_+, w_\gamma; X)]_\theta = H_0^{s,p}(\mathbb{R}_+, w_\gamma; X). \tag{4.14}$$

Proof. Let m be the least integer such that $m \geq \max\{|s_0|, |s_1|\}$. For each $\sigma > -1 + \frac{\gamma+1}{p}$ with $|\sigma| \leq m$ and $\sigma \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$,

$$S : H_0^{\sigma,p}(\mathbb{R}_+, w_\gamma; X) \to H_0^{\sigma,p}(\mathbb{R}, w_\gamma; X), \quad Sf := \mathscr{E}^m_+ f,$$

is a well-defined bounded linear operator thanks to Propositions 4.6.4 and 4.6.6. For each $\sigma \in \mathbb{R}$, let $R : H_0^{\sigma,p}(\mathbb{R}, w_\gamma; X) \to H_0^{\sigma,p}(\mathbb{R}_+, w_\gamma; X)$ denote the restriction operator. Using Theorem 4.6.5, the proof can now be completed as in Proposition 4.5.7 (2). □

4.6.3. Fractional domain spaces

For $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$ let

$$W_0^{k,p}(\mathbb{R}_+, w_\gamma; X) = \{f \in W^{k,p}(\mathbb{R}_+, w_\gamma; X) : f(0) = f^{(1)}(0) = \ldots = f^{(k-1)}(0) = 0\}.$$

If X is a UMD space, then it follows from Propositions 4.5.5, 4.6.6 and (4.13) that

$$W_0^{k,p}(\mathbb{R}_+, w_\gamma; X) = H_0^{k,p}(\mathbb{R}_+, w_\gamma; X). \tag{4.15}$$

Let us now briefly recall the H^∞-calculus for sectorial operators, for which there are several conventions in the literature. For a survey and an extensive treatment of the subject we refer the reader to [245] and [110, 127, 149], respectively.

For each $\theta \in (0,\pi)$ we define the sector

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \theta\}.$$

A closed densely defined linear operator $(A, D(A))$ on X is said to be *sectorial of type* $\sigma \in (0,\pi)$ if it is injective and has dense range, $\Sigma_{\pi-\sigma} \subset \rho(-A)$, and for all $\sigma' \in (\sigma, \pi)$

$$\sup\{\|\lambda(\lambda + A)^{-1}\| : \lambda \in \Sigma_{\pi-\sigma'}\} < \infty.$$

The infimum of all $\sigma \in (0,\pi)$ such that A is sectorial of type σ is called the *sectoriality angle* of A and is denoted by ϕ_A.

Let $H^\infty(\Sigma_\theta)$ denote the Banach space of all bounded analytic functions $f : \Sigma_\theta \to \mathbb{C}$, endowed with the supremum norm. Let $H_0^\infty(\Sigma_\theta)$ denote its linear subspace of all f for which there exists $\varepsilon > 0$ and $C \geq 0$ such that

$$|f(z)| \leq \frac{C|z|^\varepsilon}{(1+|z|)^{2\varepsilon}}, z \in \Sigma_\theta.$$

If A is sectorial of type $\sigma_0 \in (0,\pi)$, then for all $\sigma \in (\sigma_0, \pi)$ and $f \in H_0^\infty(\Sigma_\sigma)$ we define the bounded linear operator $f(A)$ by

$$f(A) := \frac{1}{2\pi\imath} \int_{\partial\Sigma_\sigma} f(z)(z + A)^{-1}\, dz.$$

A sectorial operator A of type $\sigma_0 \in (0,\pi)$ is said to have a *bounded* $H^\infty(\Sigma_\sigma)$*-calculus* for $\sigma \in (\sigma_0, \pi)$ if there exists a $C \in [0,\infty)$ such that

$$\|f(A)\| \leq \|f\|_{H^\infty(\Sigma_\sigma)}, \qquad f \in H_0^\infty(\Sigma_\sigma).$$

In this case the mapping $f \mapsto f(A)$ extends to a bounded algebra homomorphism from $H^\infty(\Sigma_\sigma)$ to $\mathscr{B}(X)$ of norm $\leq C$. The H^∞-angle of A is defined as the infimum of all σ for which A has a bounded $H^\infty(\Sigma_\sigma)$-calculus and is denoted by ϕ_A^∞.

Below we will make use of the following fact. Let A be an operator on a reflexive Banach space X. If A is a sectorial operator having a bounded H^∞-calculus, then so is A^* with $\phi_A^\infty = \phi_{A^*}^\infty$.

Theorem 4.6.8. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$.*

1. *The realization of ∂_t on $L^p(\mathbb{R}_+, w_\gamma; X)$ with domain $W_0^{1,p}(\mathbb{R}_+, w_\gamma; X)$ has a bounded H^∞-calculus of angle $\pi/2$ with $\mathrm{D}(\partial_t^s) = H_0^{s,p}(\mathbb{R}_+, w_\gamma; X)$ for every $s > 0$ with $s \notin \frac{1+\gamma}{p} + \mathbb{N}_0$.*

2. *The realization of $-\partial_t$ on $L^p(\mathbb{R}_+, w_\gamma; X)$ with domain $W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ has a bounded H^∞-calculus of angle $\pi/2$ with $\mathrm{D}((-\partial_t)^s) = H^{s,p}(\mathbb{R}_+, w_\gamma; X)$ for every $s > 0$.*

For $\gamma \in [0, p-1)$ the case $\frac{d}{dt}$ follows from [197, Theorem 4.5]. For $\gamma \in [0, p-1)$ the case $-\frac{d}{dt}$ follows from [179, Theorem 2.7]. Below we present a proof that works for all $\gamma \in (-1, p-1)$, in which (1) is derived from (2) by a simple duality argument.

Proof. Let us first establish the assertions regarding the H^∞-calculus. We start with (2), from which we will derive (1) by duality.

For (2) we denote by A the realization of $-\partial_t$ on $L^p(\mathbb{R}_+, w_\gamma; X)$ with domain $W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ and by $\widetilde{A}$ the realization of $-\partial_t$ on $L^p(\mathbb{R}, w_\gamma; X)$ with domain $W^{1,p}(\mathbb{R}, w_\gamma; X)$. As in [149, Example 10.2], using Proposition 4.2.3, one can show that $\widetilde{A}$ has a bounded H^∞-calculus of angle $\pi/2$. So it is enough to show that $\mathbb{C}_+ \subset \rho(-A)$ with

$$(\lambda + A)^{-1} f = R(\lambda + \widetilde{A})^{-1} E f =: S(\lambda) f, \qquad \lambda \in \mathbb{C}_+, f \in L^p(\mathbb{R}_+, w_\gamma; X),$$

where $E \in \mathscr{B}(L^p(\mathbb{R}_+, w_\gamma; X), L^p(\mathbb{R}, w_\gamma; X))$ is the extension by zero operator, and R denotes the operator of restriction from $\mathbb{R}$ to $\mathbb{R}_+$. For each $\lambda \in \mathbb{C}_+$, $S(\lambda)$ defines a linear operator from $L^p(\mathbb{R}_+, w_\gamma; X)$ to $W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ with the property that $(\lambda + A)S(\lambda) = I$. So, fixing $\lambda \in \mathbb{C}_+$, we only need to show that $\ker(\lambda + A) = \{0\}$. To this end, let $u \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ satisfy $(\lambda - \partial_t)u = 0$. By basic distribution theory (cf. [79, Theorem 9.4]) we find that u is a classical solution in the sense that $u \in C^\infty(\mathbb{R}_+; X)$ with $u' = \lambda u$, implying that $u = c\exp(\lambda\cdot)$ for some $c \in X$. Since $\exp(\lambda\cdot) \notin L^p(\mathbb{R}_+, w_\gamma)$, it follows that $u = 0$.

For (1) we denote by A the realization of ∂_t on $L^p(\mathbb{R}_+, w_\gamma; X)$ with domain $W_0^{1,p}(\mathbb{R}_+, w_\gamma; X)$ and by B the realization of $-\partial_t$ on $L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*)$ with domain $W^{1,p'}(\mathbb{R}_+, w_{\gamma'}; X^*)$. Recall that $[L^p(\mathbb{R}_+, w_\gamma; X)]^* = L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*)$ with respect to the natural pairing (see [187, Proposition 3.5]), X being reflexive as a UMD space (see [126, Theorem 4.3.3]). Integration by parts (see Lemma 4.6.9 below) yields $A \subset B^*$. By (2) (and the fact that duals of UMD spaces are again UMD) it is enough to establish the reverse. By [85, Exercise 1.21(4)], for the latter it suffices that $\lambda + A$ is surjective and $\lambda + B^*$ is injective for some $\lambda \in \mathbb{C}$. To this end, let us establish this for some fixed $\lambda \in \mathbb{C}_+$. Then $\lambda \in \rho(-B) = \rho(-B^*)$ by (2); in particular, $\lambda + B^*$ is injective. As in (2) we can find a linear operator $S(\lambda) : L^p(\mathbb{R}_+, w_\gamma; X) \to W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ such that $(\lambda + A)S(\lambda) = I$. Then the operator $T(\lambda) : L^p(\mathbb{R}_+, w_\gamma; X) \to W_0^{1,p}(\mathbb{R}_+, w_\gamma; X)$ given by

$$T(\lambda) f := S(\lambda) f - [S(\lambda) f](0) \exp(-\lambda\cdot),$$

satisfies $(\lambda + A)T(\lambda) = I$, which shows that $\lambda + A$ is surjective.

Finally we will identify the fractional domain spaces. From the definitions it follows that $D(\partial_t^k) = W_0^{k,p}(\mathbb{R}_+, w_\gamma; X)$ and $D((-\partial_t)^k) = W^{k,p}(\mathbb{R}_+, w_\gamma; X)$ as sets for every $k \in \mathbb{N}$. Moreover, it follows from Lemma 4.5.8 and Young's inequality for products that there is also an equivalence of norms. The assertions concerning the fractional domain spaces subsequently follow from [110, Theorem 6.6.9], Proposition 4.5.5 and Theorem 4.6.7. □

Lemma 4.6.9 (Integration by parts)**.** *Let X be a Banach space, $p \in (1,\infty)$ and $w \in A_p$. For all $u \in W^{1,p}(\mathbb{R}_+, w; X)$ and $v \in W^{1,p'}(\mathbb{R}_+, w'; X^*)$, where $w' = w^{-\frac{1}{p-1}}$ is the p-dual weight of w, there holds the integration by parts identity*

$$\langle u', v\rangle_{\langle L^p(\mathbb{R}_+,w;X), L^{p'}(\mathbb{R}_+,w';X)\rangle} = -u(0)\,v(0) - \langle u, v'\rangle_{\langle L^p(\mathbb{R}_+,w;X), L^{p'}(\mathbb{R}_+,w';X)\rangle}.$$

Proof. By the remark preceding this lemma and Lemma 4.3.5, $C_c^\infty(\overline{\mathbb{R}}_+) \otimes X$ is dense in $W^{1,p}(\mathbb{R}_+, w; X)$ and $C_c^\infty(\overline{\mathbb{R}}_+) \otimes X^*$ is dense in $W^{1,p'}(\mathbb{R}_+, w'; X^*)$. The desired result thus follows from integration by parts for functions from $C_c^\infty(\overline{\mathbb{R}}_+)$. □

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II

Boundary Value Problems

5

THE HEAT EQUATION SUBJECT TO THE DIRICHLET BOUNDARY CONDITION

This chapter is based on the paper:

> *N. Lindemulder and M.C. Veraar. The heat equation with rough boundary conditions and holomorphic functional calculus.*

In this chapter we consider the Laplace operator with Dirichlet boundary conditions on a smooth domain. We prove that it has a bounded H^∞-calculus on weighted L^p-spaces for power weights which fall outside the classical class of A_p-weights. Furthermore, we characterize the domain of the operator and derive several consequences on elliptic and parabolic regularity. In particular, we obtain a new maximal regularity result for the heat equation with rough inhomogeneous boundary data.

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5.1. INTRODUCTION

Often solutions to PDEs can have blow-up behavior near the boundary of an underlying domain $\mathscr{O} \subseteq \mathbb{R}^d$. Using weighted spaces with weights of the form $w_\gamma^{\mathscr{O}}(x) := \operatorname{dist}(x, \partial\mathscr{O})^\gamma$ for appropriate values of γ, one can create additional flexibility and even obtain well-posedness for problems which appear ill-posed at first sight. PDEs in weighted spaces have been considered by many authors (see e.g. [75, 139, 143, 144]). Moreover, the H^∞-functional calculus properties of differential operators on weighted space have been treated in several papers as well (see e.g. [14, 18, 19, 155, 171]).

The development of the H^∞-calculus was motivated by the Kato square root problem (see [175] for a survey) which was eventually solved in [17]. An H^∞-calculus approach to the solution was obtained later in [21]. Since the work [134] it has turned out that the H^∞-calculus is an extremely efficient tool in the L^p-theory of partial differential equations (see the monographs [63, 198] and references therein).

In this paper we study the boundedness of the H^∞-calculus of the Laplace operator with Dirichlet boundary conditions Δ_{Dir} for bounded C^2-domains $\mathscr{O}$. This operator and its generalizations have been studied in many papers (see [58, 59, 149]. Our contribution is that we study Δ_{Dir} and its functional calculus on weighted spaces which do not fall into the classical setting, but which are useful for certain partial differential equations. In particular, we prove the following result.

Theorem 5.1.1. *Let $\mathscr{O}$ be a bounded C^2-domain. Let $p \in (1,\infty)$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and set $w_\gamma^{\mathscr{O}}(x) = \operatorname{dist}(x, \partial\mathscr{O})^\gamma$. Then the operator $-\Delta_{\mathrm{Dir}}$ on $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$ with $D(\Delta_{\mathrm{Dir}}) = W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}})$, has a bounded H^∞-calculus of angle zero. In particular, Δ_{Dir} generates an analytic C_0-semigroup on $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$.*

A similar result holds on the half space $\mathbb{R}^d_+$ or small deformations of the half space. The range $\gamma \in (p-1, 2p-1)$ falls outside the classical A_p-setting and Theorem 5.1.1 is new in this range. The range $\gamma \in (-1, p-1)$ can be treated by classical methods, and it can be derived from the general A_p-case which will be considered in Section 5.4.

The boundedness of the H^∞-calculus has many interesting consequences for the operator Δ_{Dir} on $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$. Loosely speaking, the boundedness of the H^∞-calculus can be used as a black box to ensure existence of certain singular integrals. In particular, the boundedness of the H^∞-calculus implies:

- Continuous and discrete square function estimates (see [127, Theorems 10.4.4 & 10.4.23]), which are closely related to the classical Littlewood–Paley inequalities.

- Well-posedness and maximal regularity of the Laplace equation and the heat equation on $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$ (see Corollaries 5.5.8, 5.5.10, 5.6.3).

- Maximal regularity for the stochastic heat equation on $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$ (see [192, Theorem 1.1]).

On bounded domains we analyse the spectrum of Δ_{Dir} and in particular we show that the analytic semigroup generated by Δ_{Dir} is exponentially stable. Additionally we use the functional calculus to characterize several of the fractional domain spaces.

The main difficulty in the proof of Theorem 5.1.1 in the non-A_p setting is that standard tools from harmonic analysis are not available. For instance, the boundedness of the Hilbert transform, the boundedness of the Hardy-Littlewood maximal function operator, and the Littlewood–Paley decomposition all hold on $L^p(\mathbb{R}^d, w_\gamma^{\mathcal{O}})$ if and only if $\gamma \in (-1, p-1)$ (see [101, Chapter 9] and [218]). Here one also needs to use the fact that the A_p-condition holds if and only if $\gamma \in (-1, p-1)$. As a consequence, we have to find a new approach to obtain the domain characterizations, sectoriality estimates and the boundedness of the functional calculus.

We have already mentioned that Theorem 5.1.1 implies maximal regularity results. As a further application we will derive a maximal regularity result for the heat equation on weighted spaces with *rough inhomogeneous boundary conditions*. The main reason we can allow much rougher boundary data than in previous works is that we allow $\gamma \in (p-1, 2p-1)$. Maximal regularity results can be used to study nonlinear equations in an effective way (see e.g. [199] and references therein). The result below is a special case of Theorem 5.7.16. In order to make the result transparent without losing the main innovative part of the result, we state the result in the special case $u_0 = 0$, $f = 0$ and $p = q$ and without weights in time.

Theorem 5.1.2. *Let $\mathcal{O}$ be a bounded C^2-domain. Let $\lambda \geq 0$. Let $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1, 2p-3\}$ and set $\delta = 1 - \frac{1+\gamma}{2p}$. Assume*

$$g \in B^{\delta}_{p,p}(\mathbb{R}_+; L^p(\partial\mathcal{O})) \cap L^p(\mathbb{R}_+; B^{2\delta}_{p,p}(\partial\mathcal{O})),$$

with $g(0,\cdot) = 0$ in the case $\gamma \in (-1, 2p-3)$. Then there exists a unique $u \in W^{1,p}(\mathbb{R}_+; L^p(\mathcal{O}, w_\gamma^{\mathcal{O}})) \cap L^p(\mathbb{R}_+; W^{2,p}(\mathcal{O}, w_\gamma^{\mathcal{O}}))$ such that

$$\begin{cases} u' + (\lambda - \Delta)u &= 0, \quad \text{on } \mathbb{R}_+ \times \mathcal{O}, \\ \operatorname{tr}_{\partial\mathcal{O}} u &= g, \quad \text{on } \mathbb{R}_+ \times \partial\mathcal{O}, \\ u(0) &= 0, \quad \text{on } \mathcal{O}. \end{cases}$$

Conversely, the conditions on g are necessary in order for u to be in the intersection space. Note that $\delta \in (0,1)$ can be taken arbitrarily close to zero by taking γ arbitrarily close to $2p-1$. Moreover, if $\gamma \in (2p-3, 2p-1)$ then the compatibility condition $g(0,\cdot) = 0$ also vanishes.

Theorem 5.1.2 was proved in [61] and [243] for $\gamma = 0$, and in this case the smoothness parameter equals $\delta = 1 - \frac{1}{2p}$. In [61] actually the general setting of higher order operators A with boundary conditions of Lopatinskii-Shapiro was consider. In [159] the first author extended the latter result to the weighted situation with $\gamma \in (-1, p-1)$, in which case $\delta \in (\frac{1}{2}, 1)$ can only be taken arbitrarily close to $\frac{1}{2}$ by taking γ close to $p-1$. It would be interesting to investigate if one can extend special cases of [159] to other values of γ. In ours proofs the main technical reason that we can extend the range of γ's in the Dirichlet setting is that the heat kernel on a half space has a zero of order one at the boundary. The heat kernel in the case of Neumann boundary conditions does not have this property. Moreover, the Neumann trace operator is not well-defined for $\gamma \in (p-1, 2p-1)$. It is a natural question to ask for which kernels associated to higher order elliptic operators with different boundary conditions one has similar behavior at the boundary. In such cases one might be able to allow for rougher boundary data as well.

There exist several theories of elliptic and parabolic boundary value problems on other classes of function spaces than the $L^q(L^p)$-framework of the above. The case that L^p is replaced by a weighted Besov or Triebel-Lizorkin space is considered by the first named author in [163] in the elliptic setting and in [162] in the parabolic setting. The advantage in that setting is that one can use Fourier multiplier theorems for A_∞-weights. The results in [162, 163] are independent from the results presented here since in the non-A_p setting Triebel-Lizorkin spaces do not coincide with Sobolev spaces. For results in the framework of tent spaces have been obtained in [10, 15, 20] for elliptic equations and in [16] for parabolic equations. Here in some cases the boundary data is allowed to be in L^p or L^2.

The paper is organized as follows. In Section 5.3 we present some results on traces, Hardy inequalities and interpolation inequalities which will be needed. In Section 5.4 we consider the half space case with A_p-weights. In Section 5.5 we consider the half space case for non-A_p-weights. We extend the results to bounded domains in Section 5.6, where Theorem 5.1.1 can be derived from Corollary 5.6.2. In Section 5.7 we consider the heat equation with inhomogeneous boundary conditions and, in particular, we will derive Theorem 5.1.2. In many of our considerations we consider the vector-valued situation. This is mainly because it can be convenient to write Sobolev spaces as the intersection of several simpler vector-valued Sobolev spaces.

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NOTATION

$\mathbb{R}^d_+ = (0,\infty) \times \mathbb{R}^{d-1}$ denotes the half space. We write $x = (x_1, \tilde{x}) \in \mathbb{R}^d$ with $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$. The following shorthand notation will be used throughout the paper

$$w_\gamma(x) = |x_1|^\gamma \text{ and } w_\gamma^{\mathscr{O}}(x) = \operatorname{dist}(x, \partial\mathscr{O})^\gamma.$$

For two topological vector spaces X and Y (usually Banach spaces), $\mathscr{L}(X,Y)$ denotes the space of continuous linear operators. We write $A \lesssim_p B$ whenever $A \leq C_p B$ where C_p is a constant which depends on the parameter p. Similarly, we write $A \eqsim_p B$ if $A \lesssim_p B$

and $B \lesssim_p A$. Unless stated otherwise in the rest of the paper X is assumed to be a Banach space.

5.2. PRELIMINARIES

5.2.1. Function spaces and weights

Let X be a Banach space. For an open set $\mathscr{O} \subseteq \mathbb{R}^d$ let $\mathscr{D}(\mathscr{O};X)$ denote the space of compactly supported smooth functions from $\mathscr{O}$ into X equipped with its usual inductive limit topology. Let $\mathscr{D}'(\mathscr{O};X) = \mathscr{L}(\mathscr{D}(\mathscr{O}),X)$ be the space of X-valued distributions. Let $C_c^\infty(\overline{\mathscr{O}};X)$ be the space of infinite differentiable functions which vanish outside a compact set $K \subseteq \overline{\mathscr{O}}$. Furthermore, $\mathscr{S}(\mathbb{R}^d;X)$ denotes the space of Schwartz functions and $\mathscr{S}'(\mathbb{R}^d;X) = \mathscr{L}(\mathscr{S}(\mathbb{R}^d),X)$ is the space of X-valued tempered distributions. We refer to [6, 9] for introductions to the theory of vector-valued distribution.

A locally integrable function $w : \mathscr{O} \to (0,\infty)$ is called a weight. A weight w will be called *even* if $w(-x_1,\tilde{x}) = w(x_1,\tilde{x})$ for $x_1 > 0$ and $\tilde{x} \in \mathbb{R}^{d-1}$.

Although we will be mainly interested in a special class of weights, it will be natural to formulate some of the result for the class of Muckenhoupt A_p-weights. For $p \in (1,\infty)$ and a weight $w : \mathbb{R}^d \to (0,\infty)$, we say that $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \frac{1}{|Q|}\int_Q w(x)\,dx \cdot \Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} < \infty.$$

Here the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes. For $p \in (1,\infty)$ and a weight $w : \mathbb{R}^d \to (0,\infty)$ one has $w \in A_p$ if and only the Hardy–Littlewood maximal function is bounded on $L^p(\mathbb{R}^d,w)$. We refer the reader to [101, Chapter 9] for standard properties of A_p-weights. For a fixed p and a weight $w \in A_p$, the weight $w' = w^{-1/(p-1)} \in A_{p'}$ is the p-dual weight. Define $A_\infty = \bigcup_{p>1} A_p$. Recall that $w_\gamma(x) := |x_1|^\gamma$ is in A_p if and only if $\gamma \in (-1,p-1)$.

For a weight $w : \mathscr{O} \to (0,\infty)$ and $p \in [1,\infty)$, Let $L^p(\mathscr{O},w;X)$ denote the Bochner space of all strongly measurable functions $f : \mathscr{O} \to X$ such that

$$\|f\|_{L^p(\mathscr{O},w;X)} = \Big(\int_{\mathscr{O}} |f(x)|^p\,w(x)\,dx\Big)^{1/p} < \infty.$$

For a set $\Omega \subseteq \mathbb{R}^d$ with nonempty interior and $w : \Omega \to (0,\infty)$ let $L^1_{\rm loc}(\Omega;X)$ denote the set of all functions such that for all bounded open sets $\Omega_0 \subseteq \Omega$, we have $f|_{\Omega_0} \in L^1(\Omega_0,w;X)$. In this case f is called *locally integrable* on Ω. If the p-dual weight $w' = w^{-1/(p-1)}$ ($w' = 1$ when $p = 1$) is locally integrable on $\mathscr{O}$, then $L^p(\mathscr{O},w;X) \hookrightarrow \mathscr{D}'(\mathscr{O};X)$.

For $p \in (1,\infty)$, an integer $k \geq 0$ and a weight w with $w' = w^{-1/(p-1)} \in L^1_{\rm loc}(\mathscr{O})$, let $W^{k,p}(\mathscr{O},w;X) \subseteq \mathscr{D}'(\mathscr{O};X)$ be the *Sobolev space* of all $f \in L^p(\mathscr{O},w;X)$ with $D^\alpha f \in L^p(\mathscr{O},w;X)$ for all $|\alpha| \leq k$ and set

$$\|f\|_{W^{k,p}(\mathscr{O},w;X)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\mathscr{O},w;X)},$$

$$[f]_{W^{k,p}(\mathscr{O},w;X)} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\mathscr{O},w;X)}.$$

$W^{k,p}(\mathscr{O},w;X)$ is a Banach space. We refer to [146, 147] for a detailed study of weighted Sobolev spaces. Finally, for a set $\Omega \subseteq \mathbb{R}^d$ with nonempty interior we let $W^{k,1}_{\mathrm{loc}}(\Omega,w;X)$ denote the space of functions such that $D^\alpha f \in L^1_{\mathrm{loc}}(\Omega,w;X)$ for all $|\alpha| \leq k$.

Let us mention that density of $C^\infty_c(\overline{\mathscr{O}};X)$ in $W^{1,p}(\mathscr{O},w;X)$ is not true in general, not even for $w \in A_\infty$. A sufficient condition class is $w \in A_p$ (see [238, Corollary 2.1.6]). Further examples and counterexamples can be found in [146, Chapter 7 & 11] and [252].

We further would like to point out that in general $W^{k,p}(\mathscr{O},w)$ does not coincide with a Triebel-Lizorkin space $F^k_{p,2}(\mathscr{O},w)$ if $w \notin A_p$. Moreover, in the X-valued setting this is even wrong for $w = 1$ unless X is isomorphic to a Hilbert space (see [112]).

5.2.2. Localization and C^k-domains

Definition 5.2.1. Let $\mathscr{O} \subset \mathbb{R}^d$ be a domain and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathscr{O}$ is called a *special C^k-domain* when, after rotation and translation, it is of the form

$$\mathscr{O} = \{x = (y,x') \in \mathbb{R}^d : y > h(x')\} \tag{5.1}$$

for some C^k-function $h : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$. If h can be chosen with compact support, then $\mathscr{O}$ is called a *special C^k_c-domain.*

For later it will be convenient to define, given a special C^k_c-domain $\mathscr{O}$ with $k \in \mathbb{N}_0$, the numbers

$$[\mathscr{O}]_{C^k} := \inf_h ||h||_{C^k_b(\mathbb{R}^{d-1})} \tag{5.2}$$

where the infimum is taken over all $h \in C^k_c(\mathbb{R}^{d-1};\mathbb{R})$ for which $\mathscr{O}$, after rotation and translation, can be represented as (5.1).

Definition 5.2.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A domain $\mathscr{O} \subset \mathbb{R}^d$ is said to be a *C^k-domain* when every boundary point $x \in \partial\mathscr{O}$ admits an open neighborhood V with the property that

$$\mathscr{O} \cap V = W \cap V \quad \text{and} \quad \partial\mathscr{O} \cap V = \partial W \cap V$$

for some special C^k-domain $W \subset \mathbb{R}^d$.

Note that, in the above definition, V may be replaced by any smaller open neighborhood of x. Hence, we may without loss of generality assume that W is a C^k_c-domain. Moreover, if $k \in \mathbb{N}_0$ then for any $\varepsilon > 0$ we can arrange that $[W]_{C^k} < \varepsilon$.

If $U, V \subseteq \mathbb{R}^d$ are open and $\Phi : U \to V$ is a C^1-diffeomorphism, then we define $\Phi_* : L^1_{\mathrm{loc}}(U) \to L^1_{\mathrm{loc}}(V)$ by

$$\langle \Phi_* f, g\rangle := \langle f, j_\Phi g \circ \Phi\rangle, \quad f \in L^1_{\mathrm{loc}}(U), g \in C_c(V),$$

where $j_\Phi = \det(\nabla\Phi)$ denotes the Jacobian. In this way $\Phi_* f = f \circ \Phi^{-1}$.

Now assume $h \in C_c^k(\mathbb{R}^{d-1})$ with $k \geq 1$ and

$$\mathscr{O} = \{(x_1, \tilde{x}) : \tilde{x} \in \mathbb{R}^{d-1}, x_1 > h(\tilde{x})\}. \tag{5.3}$$

Define a C^k-diffeomorphism $\Phi : \mathscr{O} \to \mathbb{R}^d_+$ by

$$\Phi(x) = (x_1 - h(\tilde{x}), \tilde{x}). \tag{5.4}$$

Obviously, $\det(\nabla\Phi) = 1$. For a weight $w : \mathbb{R}^d \to (0,\infty)$, let $w_\Phi : \mathscr{O} \to (0,\infty)$ be defined by $w_\Phi(x) = w(\Phi(x))$. In the important case that $w(x) = |x_1|^\gamma$, we have

$$w_\Phi(x) = |x_1 - h(\tilde{x})|^\gamma \eqsim \operatorname{dist}(x, \partial\mathscr{O})^\gamma, \quad x \in \mathscr{O}.$$

In this way for $k \in \mathbb{N}_0$, the mapping Φ_* defines a bounded isomorphism

$$\Phi_* : W^{k,p}(\mathscr{O}, w_\Phi) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma)$$

with inverse $(\Phi^{-1})_*$.

In the paper we will often use a standard localization procedure. We will usually leave out the details as they are standard. In the localization argument for the functional calculus (see Theorem 5.6.1) we do give the full details as a precise reference with weighted spaces seems unavailable.

Given a bounded C^k-domain $\mathscr{O}$ with $k \geq 1$, then we can find $\eta_0 \in C_c^\infty(\mathscr{O})$ and $\{\eta_n\}_{n=1}^N \subset C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\eta_n) \subset V_n$ for each $n \in \{1,\dots,N\}$ and $\sum_{n=0}^N \eta_n^2 = 1$ (see [145, Ch.8, Section 4]). These functions can be used to decompose the space $E_k := W^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}}; X)$ as

$$F_k := W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W^{k,p}(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X)$$

The mappings $\mathscr{I} : E_k \longrightarrow F_k$ and $\mathscr{P} : F_k \longrightarrow E_k$ given by

$$\mathscr{I} f = (\eta_n f)_{n=0}^N \qquad \text{and} \qquad \mathscr{P}(f_n)_{n=0}^N = \sum_{n=0}^N \eta_n f_n. \tag{5.5}$$

satisfy $\mathscr{P}\mathscr{I} = I$, thus $\mathscr{P}$ is a retraction with coretraction $\mathscr{I}$.

5.2.3. Functional calculus

Let $\Sigma_\varphi = \{z \in \mathbb{C} : |\arg(z)| < \varphi\}$. We say that an unbounded operator A on a Banach space X is a *sectorial operator* if A is injective, closed, has dense range and there exists a $\varphi \in (0,\pi)$ such that $\sigma(A) \subseteq \overline{\Sigma_\varphi}$ and

$$\sup_{\lambda \in \mathbb{C}\setminus\Sigma_\phi} \|\lambda R(\lambda, A)\| < \infty.$$

The infimum over all possible φ is called the *angle of sectoriality* and denoted by $\omega(A)$. In this case we also say that A is *sectorial of angle* $\omega(A)$. The condition that A has dense range is automatically fulfilled if X is reflexive (see [127, Proposition 10.1.9]).

Let $H^\infty(\Sigma_\omega)$ denote the space of all bounded holomorphic functions $f:\Sigma_\omega\to\mathbb{C}$ and let $\|f\|_{H^\infty(\Sigma_\omega)}=\sup_{z\in\Sigma_\omega}|f(z)|$. Let $H^\infty_0(\Sigma_\omega)\subseteq H^\infty(\Sigma_\omega)$ be the set of all f for which there exists an $\varepsilon>0$ and $C>0$ such that $|f(z)|\le C\frac{|z|^\varepsilon}{1+|z|^{2\varepsilon}}$.

If A is sectorial, $\omega(A)<\nu<\omega$, and $f\in H^\infty_0(\Sigma_\omega)$ we let

$$f(A)=\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)R(\lambda,A)\,d\lambda,$$

where the spectrum is assumed to be on the right of the integration path. The operator A is said to have a *bounded $H^\infty(\Sigma_\omega)$-calculus* if there exists a constant C such that for all $f\in H^\infty_0(\Sigma_\omega)$

$$\|f(A)\|\le C\|f\|_{H^\infty(\Sigma_\omega)}.$$

The infimum over all possible $\omega>\omega(A)$ is called the angle of the H^∞-calculus and is denoted by $\omega_{H^\infty}(A)$. In this case we also say that A has a bounded H^∞-calculus of angle $\omega_{H^\infty}(A)$.

For details on the H^∞-functional calculus we refer the reader to [110] and [127].

The following well-known result on the domains of fractional powers and complex interpolation will be used frequently. For the definitions of the powers A^α with $\alpha\in\mathbb{C}$ we refer to [110, Chapter 3]. For details on complex interpolation we refer to [23, 126, 235].

We say that A has *BIP (bounded imaginary powers)* if for every $s\in\mathbb{R}$, A^{is} extends to a bounded operator on X. In this case one can show that there exists $M,\sigma\ge 0$ such that (see [110, Corollary 3.5.7])

$$\|A^{is}\|\le Me^{\sigma s},\qquad s\in\mathbb{R}. \tag{5.6}$$

Let $\omega_{\mathrm{BIP}}(A)=\inf\{\omega\in\mathbb{R}:\exists M>0$ such that for all $s\in\mathbb{R}$ $\|A^{is}\|\le Me^{\omega|s|}\}$. One can easily check that $\omega_{\mathrm{BIP}}(A)\le\omega_{H^\infty}(A)$.

The next result can be found in [110, Theorem 6.6.9] and [235, Theorem 1.15.3].

Proposition 5.2.3. *Assume A is a sectorial operator such that A has BIP. Then for all $\theta\in(0,1)$ and $0\le\alpha<\beta$ we have*

$$[D(A^\alpha),D(A^\beta)]_\theta=D(A^{(1-\theta)\alpha+\theta\beta}),$$

where the constant in the norm equivalence depends α,β,θ, the sectoriality constants and on the constant M and σ in (5.6).

For two closed operators $(A,D(A))$ and $(B,D(A))$ on X we define $D(A+B):=D(A)\cap D(B)$ and $(A+B)u=Au+Bu$. Often it is a difficult to determine whether $A+B$ with the above domain is a closed operator. Sufficient conditions are given in the following theorem which will be used several times throughout this paper (see [78, 201]).

Theorem 5.2.4 (Dore–Venni)**.** *Let X be a UMD space. Assume A and B are sectorial operators on X with commuting resolvents and assume A and B both have BIP with $\omega_{\mathrm{BIP}}(A)+\omega_{\mathrm{BIP}}(B)<\pi$. Then the following assertions hold:*

(1) $A+B$ is a closed sectorial operator with with $\omega(A+B) \le \max\{\omega_{\mathrm{BIP}}(A), \omega_{\mathrm{BIP}}(B)\}$

(2) There exists a constant $C \ge 0$ such that for all $x \in D(A) \cap D(B)$,

$$\|Ax\| + \|Bx\| \le C\|Ax + Bx\|,$$

and if $0 \in \rho(A)$ or $0 \in \rho(B)$, then $0 \in \rho(A+B)$.

The following can be used to obtain boundedness of the H^∞-calculus for translated operators ((1) is straightforward and (2) follows from [133, Proposition 6.10]):

Remark 5.2.5. Let $\sigma \in (0,\pi)$ and assume A is a sectorial operator of angle $\le \sigma$

(1) If A has a bounded H^∞-calculus of angle $\le \sigma$, then for all $\lambda \ge 0$, $A+\lambda$ has a bounded H^∞-calculus of angle $\le \sigma$.

(2) If there exists a $\widetilde{\lambda} > 0$ such that $A+\widetilde{\lambda}$ has a bounded H^∞-calculus of angle $\le \sigma$, then for all $\lambda > 0$, $A+\lambda$ has a bounded H^∞-calculus of angle $\le \sigma$.

5.2.4. UMD spaces and Fourier multipliers

Below the geometric condition UMD will often be needed for X. UMD stands for unconditional martingale differences. One can show that a Banach space X is a UMD space if and only the Hilbert transform is bounded if and only if the vector-valued analogue of the Mihlin multiplier theorem holds. For details we refer to [126, Chapter 5]. Here we recall the important examples for our considerations.

- Every Hilbert space is a UMD space;
- If X is a UMD space, (S,Σ,μ) is σ-finite and $p \in (1,\infty)$, then $L^p(S;X)$ is a UMD space.
- UMD spaces are reflexive.

For $m \in L^\infty(\mathbb{R}^d)$ define

$$T_m : \mathscr{S}(\mathbb{R}^d;X) \to \mathscr{S}'(\mathbb{R}^d;X), \qquad T_m f = \mathbb{F}^{-1}(m\widehat{f}).$$

For $p \in [1,\infty)$ and $w \in A_\infty$ the Schwartz class $\mathscr{S}(\mathbb{R}^d;X)$ is dense in $L^p(\mathbb{R}^d, w;X)$ (see Lemma 5.3.5).

The following is a weighted version of Mihlin's type multiplier theorem and can be found in [187, Proposition 3.1]

Proposition 5.2.6. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p$. Assume that $m \in C^{d+2}(\mathbb{R}^d \setminus \{0\})$ satisfies*

$$C_m = \sup_{|\alpha|\le d+2} \sup_{\xi\neq 0} |\xi|^{|\alpha|}|D^\alpha m(\xi)| < \infty. \tag{5.7}$$

Then T_m extends to a bounded operator on $L^p(\mathbb{R}^d, w;X)$, and its operator norm only depends on d, X, p, w and C_m.

Proposition 5.2.7. *Let X be a UMD space. Let $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$. Then the following assertions hold:*

1. *The operator $\frac{d}{dt}$ with $D(\frac{d}{dt}) = W^{1,q}(\mathbb{R}, v; X)$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\frac{d}{dt}) \leq \frac{\pi}{2}$.*

2. *The operator $\frac{d}{dt}$ with $D(\frac{d}{dt}) = W_0^{1,q}(\mathbb{R}_+, v; X)$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\frac{d}{dt}) \leq \frac{\pi}{2}$.*

Here $W_0^{1,q}(\mathbb{R}_+, v; X)$ denotes the closed subspace of $W^{1,q}(\mathbb{R}_+, v; X)$ of functions which are zero at $t = 0$.

Proof. (1) follows from Proposition 5.2.6 and [127, Theorem 10.2.25]. (2) can be derived as a consequence by repeating part of the proof of Theorem 4.6.8 where the case $v(t) = |t|^\gamma$ was considered. □

For $p \in (1,\infty)$, $w \in A_p$ and $s \in \mathbb{R}$, we define the *Bessel potential space* $H^{s,p}(\mathbb{R}^d, w; X)$ as the space of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which $\mathbb{F}^{-1}[(1+|\cdot|^2)^{s/2}\widehat{f} \in L^p(\mathbb{R}^d, w; X)$. This is a Banach space when equipped with the norm

$$\|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} = \|\mathbb{F}^{-1}[(1+|\cdot|^2)^{s/2}\widehat{f}]\|_{L^p(\mathbb{R}^d,w;X)}.$$

For an open subset $\mathscr{O} \subseteq \mathbb{R}^d$ the space $H^{s,p}(\mathscr{O}, w; X)$ is defined as all restriction $f|_{\mathscr{O}}$ where $f \in H^{s,p}(\mathscr{O}, w; X)$. This is a Banach space when equipped with the norm

$$\|f\|_{H^{s,p}(\mathscr{O},w;X)} = \inf\{\|g\|_{H^{s,p}(\mathbb{R}^d,w;X)} : g|_{\mathscr{O}} = f\}.$$

The next result can be found in [187, Propositions 3.2 & 3.5].

Proposition 5.2.8. *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p$. Then*

$$H^{m,p}(\mathbb{R}^d, w; X) = W^{m,p}(\mathbb{R}^d, w; X) \qquad \text{for all } m \in \mathbb{N}_0.$$

Moreover, for all $s \in \mathbb{R}$, one has $[H^{s,p}(\mathbb{R}^d, w; X)]^* = H^{-s,p'}(\mathbb{R}^d, w'; X^*)$.

The UMD condition is also necessary in the above result (see [126, Theorem 5.6.12]).

Proposition 5.2.9 (Intersection representation)**.** *Let $d, d_1, d_2, n \geq 1$ be integers such that $d_1 + d_2 = d$. Let $w \in A_p(\mathbb{R}^{d_1})$. Then*

$$W^{n,p}(\mathbb{R}^d, w; X) = W^{n,p}(\mathbb{R}^{d_1}, w; L^p(\mathbb{R}^{d_2}; X)) \cap L^p(\mathbb{R}^{d_1}, w; W^{n,p}(\mathbb{R}^{d_2}; X)).$$

In the above we use the convention that w is extended in a constant way in the remaining d_2 coordinates. In this way $w \in A_p(\mathbb{R}^d)$ as well.

Proof. $\hookrightarrow$ is obvious. To prove the converse Let α be a multiindex with $k := |\alpha| \le n$. It suffices to prove $\|D^\alpha u\|_{L^p(w;X)} \le C(\|u\|_{L^p(w;X)} + \sum_{j=1}^d \|D_j^k u\|_{L^p(w;X)})$. This follows by using the Fourier multiplier m:

$$m(\xi) = \frac{(2\pi\xi)^\alpha}{1 + \sum_{j=1}^d (2\pi\rho(\xi_j)\xi_j)^k}.$$

Here $\rho \in C^\infty(\mathbb{R})$ is an odd function with $\rho = 0$ on $[0,1/2]$ and $\rho = 1$ on $[1,\infty]$. Now using Proposition 5.2.6 one can argue in a similar way as in [126, Theorem 5.6.11]. □

5.3. HARDY'S INEQUALITY, TRACES, DENSITY AND INTERPOLATION

In this section we will prove some elementary estimates of Hardy and Sobolev type and obtain some density and interpolation results. We will present the results in the X-valued setting, and later on apply this in the special case $X = L^p(\mathbb{R}^{d-1})$ to obtain extensions to higher dimensions in Theorem 5.5.7.

Details on traces in weighted Sobolev spaces can be found in [137] and [159]. We will need some simple existence results in one dimension.

5.3.1. Hardy's inequality and related results

Lemma 5.3.1. *Let $p \in [1,\infty)$ and let w be a weight such that $\|w^{-\frac{1}{p-1}}\|_{L^1(0,t)} < \infty$ for all $t \in (0,\infty)$. Then $W^{1,p}(\mathbb{R}_+, w; X) \hookrightarrow C([0,\infty); X)$ and for all $u \in W^{1,p}(\mathbb{R}_+, w; X)$,*

$$\sup_{x\in[0,t]} \|u(x)\| \le C_{t,p,w}\|u\|_{W^{1,p}(\mathbb{R}_+,w;X)}, \quad t \in [0,\infty)$$

Moreover, the following results hold in the special case that $w(x) = w_\gamma(x) = |x_1|^\gamma$:

(1) *If $\gamma \in [0, p-1)$, then $u(x) \to 0$ as $x \to \infty$ and for all $u \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)$,*

$$\sup_{x\ge 0} \|u(x)\| \le C_{p,\gamma}\|u\|_{W^{1,p}(\mathbb{R}_+,w_\gamma;X)}.$$

(2) *If $\gamma < -1$, then for all $u \in L^p(\mathbb{R}_+, w_\gamma; X) \cap C([0,\infty); X)$, $u(0) = 0$.*

Note that the local L^1-condition on w holds in particular for $w \in A_p$.

Proof. Let $u \in W^{1,p})\mathbb{R}_+, w; X)$. By Hölder's inequality and the assumption on w we have $L^p((0,t), w; X) \hookrightarrow L^1(0,t;X)$. In particular u and u' are locally integrable on $[0,\infty)$. Let

$$v(s) = \int_0^s u'(x)\,dx, \quad s \in (0,t).$$

Then v is continuous on $[0,t]$ and moreover $v' = u'$ on $(0,t)$ (see [126, Lemma 2.5.8]). It follows that there is a $z \in X$ such that $u = z + v$ for all $s \in (0,t)$. In particular, u has a continuous extension $\overline{u}$ to $[0,t]$ given by $\overline{u} = z + v$.

To prove the required estimates we just write u instead of $\overline{u}$. Let $x \in [0,\infty)$. Define ζ as $\zeta(x) = 1-x$ for $x \in [0,1]$ and $\zeta = 0$ on $[1,\infty)$. Then for $x \in [0,t]$, we have

$$u(x) = \int_0^1 \frac{d}{ds}(u(s+x)\zeta(s))\,ds = \underbrace{\int_0^1 u'(s+x)\zeta(s)\,ds}_{T_1} + \underbrace{\int_0^1 u(s+x)\zeta'(s)\,ds}_{T_2}$$

Then by Hölder's inequality

$$\begin{aligned}\|T_1\| &\le \Big(\int_0^1 \|u'(s+x)\|^p w(s+x)\,ds\Big)^{1/p} \|s \mapsto w(s+x)^{-1/(p-1)}\|_{L^1(0,1)}^{1/p'} \\ &\le C_{w,t,p}\|u'\|_{L^p(\mathbb{R}_+,w;X)},\end{aligned}$$

where $C_{w,t,p}^{p'} = \|w^{-1/(p-1)}\|_{L^1(0,t+1)}$. Similarly, $\|T_2\| \le C_{w,t,p}\|u\|_{L^p(\mathbb{R}_+,w;X)}$. Therefore, the required estimate for $\sup_{x\in[0,t]}\|u(x)\|$ follows.

The estimate in (1) follows from

$$\int_0^1 w_\gamma(s+x)^{-1/(p-1)}\,ds \le \int_0^1 w_\gamma(s)^{-1/(p-1)}\,ds =: C_{p,\gamma}.$$

Moreover, $u(x) \to 0$ as $x \to \infty$ because $\int_0^1 w_\gamma(s+x)^{-1/(p-1)}\,ds \to 0$ as $x \to \infty$.

To prove (2) note that

$$\|u(0)\| = \lim_{t\to\infty} \frac{1}{t}\int_0^t \|u(s)\|\,ds.$$

Now by Hölder's inequality we have

$$\frac{1}{t}\int_0^t \|u(s)\|\,ds \le \frac{1}{t}\|u\|_{L^p(\mathbb{R}_+,w_\gamma)}\Big(\int_0^t s^{-\gamma p'}\,ds\Big)^{1/p'} \le C\|u\|_{L^p(\mathbb{R}_+,w_\gamma)}\,t^{-\frac{\gamma+1}{p}}$$

and the latter tends to zero as $t \to 0$. □

Next we state two well-known consequences of Hardy's inequality (see [96, Theorem 10.3.1] and [146, Section 5]).

Lemma 5.3.2. *Assume $p \in [1,\infty)$. Let $u \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)$. Then*

$$\|u\|_{L^p(\mathbb{R}_+,w_{\gamma-p};X)} \le C_{p,\gamma}\|u'\|_{L^p(\mathbb{R}_+,w_\gamma;X)}.$$

if ($\gamma < p-1$ and $u(0) = 0$) or $\gamma > p-1$.

In the above result, by Lemma 5.3.1, $u \in C([0,\infty);X)$ if $\gamma < p-1$.

Proof. First consider $\gamma < p-1$. Writing $u(t) = \int_0^t u'(s)ds$, it follows that

$$\|u(t)\|_X \le \int_0^t \|u'(s)\|_X ds.$$

Now the result follows from Hardy's inequality (see [96, Theorem 10.3.1]). The case $\gamma > p-1$ follows similarly by writing $u(t) = \int_t^\infty u'(s)ds$. Here we use the fact that, by approximation, it suffices to consider the case where $u = 0$ on $[n,\infty)$. □

For other exponents γ than the ones considered in Lemma 5.3.1 another embedding result follows. Note that this falls outside the class of A_p-weights.

Lemma 5.3.3. *Let $p \in [1,\infty)$ and $\gamma \in (p-1, 2p-1)$. Then $W^{2,p}(\mathbb{R}_+, w_\gamma; X) \hookrightarrow C_b([0,\infty); X)$ and for all $u \in W^{2,p}(\mathbb{R}_+, w_\gamma; X)$, $u(x) \to 0$ as $x \to \infty$ and*

$$\sup_{x\geq 0} \|u(x)\| \leq C_{t,p,\gamma} \|u\|_{W^{2,p}(\mathbb{R}_+, w_\gamma; X)}.$$

Proof. By Lemma 5.3.2 $\|u^{(k)}\|_{L^p(\mathbb{R}_+, w_{\gamma-p}; X)} \leq C_{p,\gamma} \|u^{(k+1)}\|_{W^{2,p}(\mathbb{R}_+, w_\gamma; X)}$ for $k \in \{0,1\}$. Therefore, $u \in W^{1,p}(\mathbb{R}_+, w_{\gamma-p}; X)$. Now the required continuity and estimate of $\|u(x)\|$ for $x \in [0,1]$ follows from Lemma 5.3.1. To prove the estimate for $x \in [1,\infty)$, we can repeat the argument used in Lemma 5.3.1 (1). Indeed, for $x \geq 1$,

$$\int_0^1 w_\gamma(s+x)^{-1/(p-1)}\, ds \leq \int_0^1 w_\gamma(s+1)^{-1/(p-1)}\, ds =: C_{p,\gamma}. \qquad \square$$

5.3.2. Traces and Sobolev embedding

For $u \in W^{1,1}_{\text{loc}}(\overline{\mathbb{R}^d_+}; X)$ we say that $\text{tr}(u) = 0$ if $\text{tr}(\varphi u) = 0$ for every $\varphi \in C^\infty$ with bounded support in $\overline{\mathbb{R}^d_+}$. Note that $\varphi u \in W^{1,1}([0,\infty); L^1(\mathbb{R}^{d-1}; X))$ whenever, $u \in W^{1,p}(\mathbb{R}^d_+, w; X)$. Thus the existence of the trace of φu follows from Lemma 5.3.1.

For integers $k \in \mathbb{N}_0$, $p \in (1,\infty)$ and $w \in A_p$, we let

$$\begin{aligned} W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X) &= \{u \in W^{k,p}(\mathbb{R}^d_+, w; X) : \text{tr}(u) = 0\}, \\ W^{k,p}_0(\mathbb{R}^d_+, w; X) &= \{u \in W^{k,p}(\mathbb{R}^d_+, w; X) : \text{tr}(D^\alpha u) = 0 \text{ for all } |\alpha| < k\}. \end{aligned} \tag{5.8}$$

The traces in the above formulas exists since $W^{k,p}(\mathbb{R}^d_+, w; X) \hookrightarrow W^{k,1}_{\text{loc}}(\overline{\mathbb{R}^d_+}; X)$.

We extend the definitions of the above spaces to the non-A_p-setting. For $p \in [1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $k \in \mathbb{N}_0$ let

$$W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = \left\{u \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \text{tr}(u) = 0 \text{ if } k > \frac{\gamma+1}{p}\right\}.$$
$$W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) = \left\{u \in W^{k,p}(\mathbb{R}^d_+, w; X) : \text{tr}(D^\alpha u) = 0 \text{ if } k - |\alpha| > \frac{\gamma+1}{p}\right\}.$$

Here the trace exists if $j := k - \alpha > \frac{\gamma+1}{p}$ since then $j \geq 2$ and, by Lemmas 5.3.1 and 5.3.3,

$$W^{j,p}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{j,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \hookrightarrow C([0,\infty); L^p(\mathbb{R}^{d-1}; X)).$$

For $\gamma \in (-\infty, -1)$ and $k \in \mathbb{N}_0$ we further let

$$W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X).$$

This notation is suitable since for $k \in \mathbb{N}_1$, by Lemma 5.3.1,

$$W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{k,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))$$

$$\subseteq \{u \in C([0,\infty;X);L^p(\mathbb{R}^{d-1})) : u(0)=0\}.$$

Using the C^k-diffeomorphisms Φ of Subsection 5.2.2 and localization one can extend the definitions of the traces and function spaces $W^{k,p}_{\mathrm{Dir}}(\mathscr{O}, w_\Phi; X)$ and $W^{k,p}_0(\mathscr{O}, w_\Phi; X)$ to special C^k_c-domains $\mathscr{O}$ and bounded C^k-domains.

The following Sobolev embeddings are a direct consequence of Lemma 5.3.2 and a localization argument (also see [146, Theorem 8.2 & 8.4]).

Corollary 5.3.4. *Let $p \in [1,\infty)$, $k \in \mathbb{N}_1$ and $\gamma \in \mathbb{R}$. Let $\mathscr{O}$ be a bounded C^k-domain or a special C^k_c-domain. Then*

$$\begin{aligned}
W_0^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}};X) &\hookrightarrow W^{k-1,p}(\mathscr{O}, w_{\gamma-p}^{\mathscr{O}};X), && \text{if } \gamma < p-1,\\
W^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}};X) &\hookrightarrow W^{k-1,p}(\mathscr{O}, w_{\gamma-p}^{\mathscr{O}};X), && \text{if } \gamma > p-1,\\
W_0^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}};X) &\hookrightarrow W_0^{k-1,p}(\mathscr{O}, w_{\gamma-p}^{\mathscr{O}};X), && \text{if } \gamma \notin \{jp-1 : j \in \mathbb{N}_1\}.
\end{aligned}$$

5.3.3. Density results

Lemma 5.3.5. *Let $w \in A_\infty$ and $p \in [1,\infty)$. Let $\mathscr{O}$ be an open subset of $\mathbb{R}^d$. Then $C_c^\infty(\mathscr{O}) \otimes X$ is dense in $L^p(\mathscr{O}, w; X)$.*

Proof. Since $L^p(\mathscr{O}, w) \otimes X$ is dense in $L^p(\mathscr{O}, w; X)$ it suffices to setting the scalar setting. We claim that it furthermore suffices to approximate functions which are compactly supported in $\mathscr{O}$.

To prove the claim, let $f \in L^p(\mathscr{O}, w)$ and let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion by compact sets of $\mathscr{O}$. Observe that $f\mathbb{1}_{K_n} \to f$ by the dominated convergence theorem. Therefore, it suffices to consider functions f with compact support in $\mathscr{O}$. Extending such functions f by zero to $\mathbb{R}^d$, the claim follows.

Let $q \in (p,\infty)$ be such that $w \in A_q$. Then for all functions $f \in L^p(\mathbb{R}^d, w)$ with compact support $K \subseteq \mathscr{O}$, by Hölder's inequality one has

$$\|f\|_{L^p(\mathbb{R}^d,w)} \le \|f\|_{L^q(\mathbb{R}^d,w)}\, w(K)^{\frac{q-p}{q}}.$$

Therefore, it suffices to approximate such functions f in the $L^q(\mathbb{R}^d, w)$ norm. To do so one can use a standard argument (see Lemma 4.2.2) by using a mollifier with compact support. □

Lemma 5.3.6. *Let $p \in (1,\infty)$, $w \in A_p$ and $k \in \mathbb{N}_0$. Let $\mathscr{O} = \mathbb{R}^d$ or a bounded C^k-domain or a special C^k_c-domain with $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $C^k_c(\overline{\mathscr{O}}) \otimes X$ is dense in $W^{k,p}(\mathscr{O}, w; X)$.*

Proof. The case $\mathscr{O} = \mathbb{R}^d$ follows from Lemma 4.3.5. In all other situations, by localization, it suffices to consider $\mathscr{O} = \mathbb{R}^d_+$. This case can be proved by combining the argument of Lemma 4.3.5 with [145, Theorem 1.8.5]. □

The density result [146, Theorem 7.2] can be extended to the vector-valued setting:

Lemma 5.3.7. *Let $p \in (1,\infty)$ and $\gamma \geq 0$. Let $\mathscr{O}$ be a bounded C^0-domain or a special C^0_c-domain. Then $C_c^\infty(\overline{\mathscr{O}};X)$ is dense in $W^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}};X)$.*

Next we will prove a density result for power weights of arbitrary order using functions with compact support in Ω.

Proposition 5.3.8. *Let $\gamma \in \mathbb{R}\setminus\{jp-1 : j \in \mathbb{N}_1\}$. Let $\mathscr{O}$ be a bounded C^k-domain or a special C^k_c-domain with $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $C_c^k(\mathscr{O};X)$ is dense in $W_0^{k,p}(\mathscr{O}, w_\gamma;X)$.*

Proof. By a standard localization argument it suffices to consider $\mathscr{O} = \mathbb{R}^d_+$. To this end, let $u \in W_0^{k,p}(\mathbb{R}^d_+, w_\gamma;X)$. By a simple truncation argument we may assume that u is compactly supported on $\overline{\mathbb{R}^d_+}$. To prove the required result we will truncate u near the plane $x_1 = 0$. For this let $\phi \in C^\infty([0,\infty))$ be such that $\phi = 0$ on $[0,1/2]$ and $\phi = 1$ on $[1,\infty)$. Let $\phi_n(x_1) = \phi(nx_1)$ and define $u_n(x) = \phi_n(x_1)u(x)$. We claim that $u_n \to u$ in $W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)$. This will be proved below. Using the claim the proof can be finished as follows. It remains to show that each $u \in W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)$ with compact support can be approximated by functions in $C_c^\infty(\mathbb{R}^d_+;X)$. For each $v \in W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)$ with compact support K it holds that

$$\|v\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)} \eqsim_{K,\gamma} \|v\|_{W^{k,p}(\mathbb{R}^d_+;X)}. \tag{5.9}$$

Therefore, it suffices to approximate u in the $W^{k,p}(\mathbb{R}^d_+;X)$-norm. This can be done by extension by zero on $\overline{\mathbb{R}^d_-}$ followed by a standard mollifier argument (see Lemma 4.2.2).

To prove the claim for convenience we will only consider $d = 1$. Since ϕ_n does not depend on $\tilde{x}$ the general case is similar. Fix $m \in \{0,\dots,k\}$. By Leibniz formula one has $(\phi_n u)^{(m)} = \sum_{i=0}^m c_{i,m}\phi_n^{(m-i)}u^{(i)}$. By the dominated convergence theorem $\phi_n u^{(m)} \to u^{(m)}$ in $L^p(\mathbb{R}^d_+, w_\gamma;X)$. It remains to prove that $\phi_n^{(m-i)}u^{(i)} \to 0$ for $i \in \{0,\dots,m-1\}$. By Corollary 5.3.4

$$u^{(i)} \in W_0^{m-i,p}(\mathbb{R}_+, w_\gamma;X) \hookrightarrow L^p(\mathbb{R}_+, w_{\gamma-(m-i)p};X).$$

Now we find

$$\begin{aligned}\|\phi_n^{(m-i)}u^{(i)}\|_{L^p(\mathbb{R}_+,w_\gamma;X)} &= \int_0^{1/n} n^{p(m-i)}|\phi^{(m-i)}(nx)|^p\|u^{(i)}(x)\|^p|x|^\gamma\,dx\\ &\leq \|\phi^{(m-i)}\|_\infty^p \int_0^{1/n} \|u^{(i)}(x)\|^p|x|^{\gamma-(m-i)p}\,dx.\end{aligned}$$

The latter tends to zero as $n \to \infty$ by the dominated convergence theorem. □

In the next result we prove a density result in real and complex interpolation spaces. It will be used as a technical ingredient in the proofs of Lemma 5.3.14 and Proposition 5.3.17.

Lemma 5.3.9. *Let $p \in (1,\infty)$, $\gamma \in \mathbb{R}\setminus\{jp-1 : j \in \mathbb{N}_0\}$, $q \in [1,\infty)$ and $k \in \mathbb{N}\setminus\{0\}$. Let $\mathscr{O}$ be a bounded C^k-domain or a special C^k_c-domain with integer $k \geq 2$ or $k = \infty$ and let $\ell \in \{0,\dots,k\}$. If $\theta \in (0,1)$ satisfies $k\theta < \frac{\gamma+1}{p}$ then the space $C_c^k(\mathscr{O};X)$ is dense in both $(L^p(\mathscr{O}, w_\gamma;X), W^{\ell,p}(\mathscr{O}, w_\gamma;X))_{\theta,q}$ as $[L^p(\mathscr{O}, w_\gamma;X), W^{\ell,p}(\mathscr{O}, w_\gamma;X)]_\theta$.*

Proof. First consider the real interpolation space. In the case $\gamma < -1$ the result follows from $W^{\ell,p}(\mathscr{O}, w_\gamma; X) = W^{\ell,p}_0(\mathscr{O}, w_\gamma; X)$, Proposition 5.3.8 and [235, Theorem 1.6.2].

In the case $\gamma \in \mathbb{R} \setminus \{jp-1 : j \in \mathbb{N}_0\}$, it suffices to consider $\mathscr{O} = \mathbb{R}^d_+$ by a localization argument. Write $Y_j = W^{j,p}(\mathbb{R}^d_+, w_\gamma; X)$ for $j \in \mathbb{N}_0$. Since $Y_\ell \overset{d}{\hookrightarrow} (Y_0, Y_\ell)_{\theta,q}$ (see [235, Theorem 1.6.2]), by Lemma's 5.3.6 and 5.3.7 it suffices to consider $u \in C^\infty_c(\overline{\mathbb{R}^d_+}; X)$ and to approximate it by functions in $C^\infty_c(\mathbb{R}^d_+; X)$ in the $(Y_0, Y_\ell)_{\theta,q}$-norm. Moreover, note that

$$\|v\|_{(Y_0,Y_\ell)_{\theta,q}} \leq C\|v\|^{1-\theta}_{Y_0}\|v\|^\theta_{Y_\ell}$$

for all $v \in Y_\ell$ (see [235, Theorem 1.3.3]). Therefore, it suffices to construct $v_n \in C^\infty_c(\mathbb{R}^d_+; X)$ such that $\|v_n - u\|^{1-\theta}_{Y_0}\|v_n - u\|^\theta_{Y_\ell} \to 0$ as $n \to \infty$. As in Proposition 5.3.8, letting $u_n = \phi_n u$, it suffices to show that $\|u_n - u\|^{1-\theta}_{Y_0}\|u_n - u\|^\theta_{Y_\ell} \to 0$ as $n \to \infty$. Note that, for example in the case $d = 1$, for one of the terms

$$\|(\phi_n - 1)u\|_{L^p(\mathbb{R}_+, w_\gamma; X)} \leq \|\phi\|_\infty\|u\|_\infty\Big(\int_0^{1/n} |x|^\gamma \, dx\Big)^{1/p} \leq \|\phi\|_\infty C_{\gamma,p} n^{-\frac{\gamma+1}{p}}$$

and similarly,

$$\|\phi^{(\ell)}_n u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \leq n^{\ell - \frac{\gamma+1}{p}}\|\phi''\|_\infty\|u\|_\infty.$$

Now we obtain that there is a constant C independent of n such that

$$\|(\phi_n - 1)u\|^{1-\theta}_{L^p(\mathbb{R}_+, w_\gamma; X)}\|\phi^{(\ell)}_n u\|^\theta_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \leq C n^{-\frac{\gamma+1}{p} + \ell\theta}.$$

The latter tends to zero by the assumptions. The other terms can be treated with similar arguments. Finally one can approximate each u_n by using (5.9) and the arguments given there.

The density in the complex case follows from

$$(L^p(\mathscr{O}, w_\gamma; X), W^{\ell,p}(\mathscr{O}, w_\gamma; X))_{\theta,1} \overset{d}{\hookrightarrow} [L^p(\mathscr{O}, w_\gamma; X), W^{\ell,p}(\mathscr{O}, w_\gamma; X)]_\theta$$

(see [235, Theorems 1.9.3 (c) & 1.10.3]). □

The next standard lemma gives a sufficient condition for a function to be in $W^{1,1}_{\text{loc}}(\mathbb{R}^d; X)$ when it consists of two $W^{1,1}_{\text{loc}}$-functions which are glued together. To prove the result one can reduce to the one-dimensional setting and use the formula $u(t) - u(0) = \int_0^t u(s)\, ds$. We leave the details to the reader.

Lemma 5.3.10. *Let $u \in L^1_{\text{loc}}(\mathbb{R}^d; X)$ be such that $u_+ := u|_{\mathbb{R}^d_+} \in W^{1,1}_{\text{loc}}(\overline{\mathbb{R}^d_+}; X)$ and $u_- := u|_{\mathbb{R}^d_-} \in W^{1,1}_{\text{loc}}(\overline{\mathbb{R}^d_-}; X)$. If $tr(u_+) = tr(u_-)$. Then $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^d; X)$ and*

$$D_j u = \begin{cases} D_j(u_+), & \text{on } \mathbb{R}^d_+; \\ D_j(u_-), & \text{on } \mathbb{R}^d_-; \end{cases}$$

Finally we will need the following simple density result in the A_p-case.

Lemma 5.3.11. *Let $\mathscr{O}$ be a bounded C^k-domain or a special C^k_c-domain with $k \in \mathbb{N}_0 \cup \{\infty\}$. If $p \in (1,\infty)$, $w \in A_p$ and $k \in \mathbb{N}_0$, then $E_0 : W^{k,p}_0(\mathscr{O}, w; X) \to W^{k,p}(\mathbb{R}^d, w; X)$ given by the extension by zero defines a bounded linear operator. Moreover, $W^{k,p}_0(\mathscr{O}, w; X) = \overline{C^k_c(\mathscr{O}; X)}$.*

Proof. By localization it suffices to consider $\mathscr{O} = \mathbb{R}^d_+$. If $u \in W^{k,p}_0(\mathbb{R}^d_+, w; X)$, then, by Lemma 5.3.10, $E_0 u \in W^{k,1}_{\mathrm{loc}}(\mathbb{R}^d_+; X)$ and

$$D^\alpha E_0 u = E_0 D^\alpha u, \qquad |\alpha| \le k.$$

In particular, this shows that E_0 is bounded.

For the final assertion let $u \in W^{k,p}_0(\mathbb{R}^d_+, w)$. By a truncation we may assume u has bounded support. Take $\zeta \in C^\infty_c(\mathbb{R}^d_-)$ such that $\int \zeta \, dx = 1$ and set $\zeta_n(x) = n^d \zeta(nx)$. Then $\zeta_n * E_0 u \to E_0 u$ in $W^{k,p}(\mathbb{R}^d, w; X)$ (see Lemma 4.2.2). Since $\zeta_n * E_0 u \in C^\infty_c(\mathbb{R}^d_+; X)$, the result follows. □

5.3.4. Interpolation

We continue with two interpolation inequalities. The first one is Lemma 4.5.8.

Lemma 5.3.12. *Let $p \in (1,\infty)$ and let $w \in A_p$ be even. Let $\mathscr{O} = \mathbb{R}^d$ or $\mathscr{O} = \mathbb{R}^d_+$. Then for every $k \in \mathbb{N} \setminus \{0,1\}$, $j \in \{1, \dots, k-1\}$ and $u \in W^{k,p}(\mathscr{O}, w; X)$ we have*

$$[u]_{W^{j,p}(\mathscr{O},w;X)} \le C_{p,[w]_{A_p}} \|u\|^{1-\frac{j}{k}}_{L^p(\mathscr{O},w;X)} [u]^{\frac{j}{k}}_{W^{k,p}(\mathscr{O},w;X)}.$$

The above result holds on smooth domains as well provided we replace the homogeneous norms $[\cdot]_{W^{k,p}}$ by $\|\cdot\|_{W^{k,p}}$. In order to extend this interpolation inequality to a class of non-A_p-weights, we will use the following pointwise multiplication mappings M and M^{-1}.

Let $M : C^\infty_c(\mathbb{R}^d_+; X) \to C^\infty_c(\mathbb{R}^d_+; X)$ be given by $Mu(x) = x_1 u(x)$. By duality we obtain a mapping $M : \mathscr{D}'(\mathbb{R}^d_+; X) \to \mathscr{D}'(\mathbb{R}^d_+; X)$ as well. Similarly, we define M^{-1} on $C^\infty_c(\mathbb{R}^d_+; X)$ and $\mathscr{D}'(\mathbb{R}^d_+; X)$.

Lemma 5.3.13. *Let $p \in (1,\infty)$, $\gamma \in (-1, 2p-1)$ and $k \in \{0,1,2\}$. Then $M : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to W^{k,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)$ is bounded. Moreover, $M : W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) \to W^{k,p}_0(\mathbb{R}^d_+, w_{\gamma-p}; X)$ is an isomorphism.*

Proof. Since the derivatives with respect to x_i with $i \neq 1$ commute with M, we only prove the result in the case $d = 1$. Observe that $\|Mu\|_{L^p(\mathbb{R}_+, w_{\gamma-p}; X)} = \|u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}$. Moreover, by the product rule, we have $(Mu)^{(j)} = j u^{(j-1)} + M u^{(j)}$ for $j \in \{0,1,2\}$. Therefore,

$$\|Mu\|_{W^{k,p}(\mathbb{R}_+, w_{\gamma-p}; X)} = \|Mu\|_{L^p(\mathbb{R}_+, w_{\gamma-p}; X)} + \sum_{j=1}^{k} \|(Mu)^{(j)}\|_{L^p(\mathbb{R}_+, w_{\gamma-p}; X)}$$

$$
\begin{aligned}
&\leq \|u\|_{L^p(\mathbb{R}_+,w_\gamma;X)} + \sum_{j=1}^{k} \|ju^{(j-1)}\|_{L^p(\mathbb{R}_+,w_{\gamma-p};X)} + \|Mu^{(j)}\|_{L^p(\mathbb{R}_+,w_{\gamma-p};X)} \\
&\leq \|u\|_{L^p(\mathbb{R}_+,w_\gamma;X)} + C\sum_{j=1}^{k} \|u^{(j)}\|_{L^p(\mathbb{R}_+,w_\gamma;X)} \\
&\leq (C+1)\|u\|_{W^{k,p}(\mathbb{R}_+,w_\gamma;X)},
\end{aligned}
$$

where we applied Lemma 5.3.2. This proves the required boundedness of M.

By density of $C_c^\infty(\mathbb{R}_+;X)$ in $W_0^{k,p}(\mathbb{R}_+,w_\gamma;X)$ (see Proposition 5.3.8 and Lemma 5.3.11) it follows that $M: W_0^{k,p}(\mathbb{R}_+,w_\gamma;X) \to W_0^{k,p}(\mathbb{R}_+,w_{\gamma-p};X)$ is bounded. It remains to prove boundedness of $M^{-1}: W_0^{k,p}(\mathbb{R}_+,w_{\gamma-p};X) \to W_0^{k,p}(\mathbb{R}_+,w_\gamma;X)$. By Proposition 5.3.8 and Lemma 5.3.11 it suffices to prove the required estimate for $u \in C_c^\infty(\mathbb{R}_+;X)$. By the product rule, we have $(M^{-1}u)^{(j)} = \sum_{i=0}^{j} c_{i,j} M^{-1+i-j} u^{(i)}$. Therefore,

$$
\begin{aligned}
\|M^{-1}u\|_{W^{k,p}(\mathbb{R}_+,w_\gamma;X)} &= \sum_{j=0}^{k} \|(M^{-1}u)^{(j)}\|_{L^p(\mathbb{R}_+,w_\gamma;X)} \\
&\leq C\sum_{j=0}^{k}\sum_{i=0}^{j} \|M^{-1-i}u^{(j-i)}\|_{L^p(\mathbb{R}_+,w_\gamma;X)} \\
&\leq C\|u\|_{W^{k,p}(\mathbb{R}_+,w_{\gamma-p};X)} + \sum_{j=0}^{k}\sum_{i=1}^{j} \|u^{(j-i)}\|_{L^p(\mathbb{R}_+,w_{\gamma-(i+1)p};X)}.
\end{aligned}
$$

Now it remains to observe that by Lemma 5.3.2 (applied i times)

$$
\|u^{(j-i)}\|_{L^p(\mathbb{R}_+,w_{\gamma-(i+1)p};X)} \leq C\|u^{(j)}\|_{L^p(\mathbb{R}_+,w_{\gamma-p};X)} \leq C\|u\|_{W^{k,p}(\mathbb{R}_+,w_{\gamma-p};X)}.
$$

□

Lemma 5.3.14. *Let $p \in (1,\infty)$ and $\gamma \in (-p-1, 2p-1) \setminus \{-1, p-1\}$. Then for every $k \in \mathbb{N} \setminus \{0,1\}$, $j \in \{1,\dots,k-1\}$ and $u \in W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)$ we have*

$$
[u]_{W^{j,p}(\mathbb{R}^d_+,w_\gamma;X)} \leq C_{\gamma,p,k}\|u\|^{1-\frac{j}{k}}_{L^p(\mathbb{R}^d_+,w_\gamma;X)} [u]^{\frac{j}{k}}_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)}.
$$

Proof. By an iteration argument as in [145, Exercise 1.5.6], it suffices to consider $k = 2$ and $j = 1$. Moreover, by a scaling involving $u(\lambda\cdot)$ it suffices to show that

$$
\|u\|_{W^{1,p}(\mathbb{R}^d_+,w_\gamma;X)} \leq C_{\gamma,p}\|u\|^{1/2}_{L^p(\mathbb{R}^d_+,w_\gamma;X)}\|u\|^{1/2}_{W^{2,p}(\mathbb{R}^d_+,w_\gamma;X)}. \tag{5.10}
$$

The case $\gamma \in (-1, p-1)$ is contained in Lemma 5.3.12, where we actually do not need to proceed through (5.10). So it remains to treat the case $\gamma \in (-p-1,-1) \cup (p-1, 2p-1)$. By standard arguments (see e.g. [235, Lemma 1.10.1]), it suffices to show that

$$
(L^p(\mathbb{R}^d_+, w_\gamma;X), W^{2,p}(\mathbb{R}^d_+, w_\gamma;X))_{\frac{1}{2},1} \hookrightarrow W^{1,p}(\mathbb{R}^d_+, w_\gamma;X).
$$

We first assume that $\gamma \in (p-1, 2p-1)$. Using Lemma 5.3.13 and real interpolation of operators, we see that M is bounded as an operator

$$(L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{1}{2},1} \longrightarrow (L^p(\mathbb{R}^d_+, w_{\gamma-p}; X), W^{2,p}(\mathbb{R}^d_+, w_{\gamma-p}; X))_{\frac{1}{2},1}.$$

By a combination of [235, Lemma 1.10.1] and (5.10) for the case $\gamma \in (-1, p-1)$, the space on the right hand side is continuously embedded into $W^{1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)$. Therefore, M is a bounded operator

$$M : (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{1}{2},1} \longrightarrow W^{1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X). \tag{5.11}$$

From Lemma 5.3.9 and the fact that $MC_c^\infty(\mathbb{R}^d_+; X) \subset C_c^\infty(\mathbb{R}^d_+; X) \subset W_0^{1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)$, it follows that M is a bounded operator

$$M : (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{1}{2},1} \longrightarrow W_0^{1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X).$$

Combining this with Lemma 5.3.13 we obtain (5.10).

Next we assume $\gamma \in (-p-1, -1)$. As (5.11) in the previous case, M^{-1} is a bounded operator

$$M^{-1} : (L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{2,p}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{1}{2},1} \longrightarrow W_0^{1,p}(\mathbb{R}^d_+, w_{\gamma+p}; X).$$

Combining this with $W^{n,p}(\mathbb{R}^d_+, w_\gamma; X) = W_0^{n,p}(\mathbb{R}^d_+, w_\gamma; X)$ $(n \in \mathbb{N})$ and Lemma 5.3.13 we obtain (5.10). □

Proposition 5.3.15. *Let X be a UMD space, $p \in (1, \infty)$ and $\gamma \in (-1, p-1)$. Let $\mathscr{O}$ be a bounded C^k-domain or a special C_c^k-domain with $k \in \mathbb{N}_0 \cup \{\infty\}$. Then for every $j \in \{0, \ldots, k\}$ the following holds:*

$$[L^p(\mathscr{O}, w_\gamma^{\mathscr{O}}; X), W_0^{k,p}(\mathscr{O}, w_\gamma^{\mathscr{O}}; X)]_{\frac{j}{k}} = W_0^{j,p}(\mathscr{O}, w_\gamma^{\mathscr{O}}; X).$$

Proof. By a localization argument it suffices to consider the case $\mathscr{O} = \mathbb{R}^d_+$. The operator ∂_1 on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ with domain $D(\partial_1) = W_0^{1,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\partial_1) = \frac{\pi}{2}$ by Theorem 4.6.8. Moreover, $D((\partial_1)^n) = W_0^{n,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))$ for all $n \in \mathbb{N}$. For the operator Δ_{d-1} on $L^p(\mathbb{R}^d_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))$, defined by

$$D(\Delta_{d-1}) := L^p(\mathbb{R}_+, w_\gamma, W^{2,p}(\mathbb{R}^{d-1}; X)), \quad \Delta_{d-1} u := \sum_{k=2}^d \partial_k^2 u,$$

it holds that $-\Delta_{d-1}$ a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta_{d-1}) = 0$. Moreover, $D((-\Delta_{d-1})^{n/2}) = L^p(\mathbb{R}_+, w_\gamma; W^{n,p}(\mathbb{R}^{d-1}; X))$ for all $n \in \mathbb{N}$. It follows that $(1+\partial_t)^k$ with $D((1+\partial_1)^k) = W_0^{k,p}(\mathbb{R}_+, w_\gamma; X)$ is sectorial having bounded imaginary powers with angle $\leq \pi/2$ and that $(1-\Delta_{d-1})^{k/2}$ with $D((1-\Delta_{d-1})^{k/2}) = L^p(\mathbb{R}_+, w_\gamma; W^{k,p}(\mathbb{R}^{d-1}; X))$ is sectorial having bounded imaginary powers with angle 0. By a combination of Proposition 5.2.3 and [86, Lemma 9.5],

$$[L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{k,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{k,p}(\mathbb{R}^{d-1}; X))]_{\frac{j}{k}}$$

$$
\begin{aligned}
&= [L^p(\mathbb{R}^d_+, w_\gamma; X), D((1+\partial_1)^k) \cap D((1-\Delta_{d-1})^{k/2})]_{\frac{j}{k}} \\
&= D((1+\partial_1)^j) \cap D((1-\Delta_{d-1})^{j/2}) \\
&= W^{j,p}_0(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{j,p}(\mathbb{R}^{d-1}; X)).
\end{aligned}
$$

Now the result follows from the following intersection representation for $n \in \mathbb{N}$:

$$
W^{n,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{n,p}_0(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{n,p}(\mathbb{R}^{d-1}; X)).
$$

Here $\hookrightarrow$ is clear. To prove the converse let u be in the intersection space. We first claim that $u \in W^{n,p}(\mathbb{R}^d_+, w_\gamma; X)$. Using a suitable extension operator it suffices to show the result with $\mathbb{R}_+$ and $\mathbb{R}^d_+$ replaced by $\mathbb{R}$ and $\mathbb{R}^d$ respectively. Now the claim follows from Proposition 5.2.9. To prove $u \in W^{n,p}_0(\mathbb{R}^d_+, w_\gamma; X)$ let $|\alpha| \le n-1$ and write $\alpha = (\alpha_1, \widetilde{\alpha})$. It remains to show $\operatorname{tr}(D^\alpha u) = 0$. By assumption and the claim $D^{\alpha_1} u \in W^{1,p}_0(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))$ and $D^{\alpha_1} u \in W^{1,p}(\mathbb{R}_+, w_\gamma; W^{n-1-\alpha_1}(\mathbb{R}^{d-1}))$. It follows that $D^{\alpha_1} u \in W^{1,p}_0(\mathbb{R}_+, w_\gamma; W^{n-1-\alpha_1}(\mathbb{R}^{d-1}))$ and therefore, we obtain $D^\alpha u \in W^{1,p}_0(\mathbb{R}_+; L^p(\mathbb{R}^{d-1}))$ as required. □

Now we extend the last identity to the non-A_p setting for $j = 1$ and $k = 2$.

Proposition 5.3.16. *Let X be a UMD space, $p \in (1,\infty)$ $\gamma \in (-p-1, 2p-1) \setminus \{-1, p-1\}$. Let $\mathscr{O}$ be a bounded C^2-domain or a special C^2_c-domain. Then the complex interpolation space satisfies*

$$
[L^p(\mathscr{O}, w_\gamma; X), W^{2,p}_0(\mathscr{O}, w_\gamma; X)]_{\frac{1}{2}} = W^{1,p}_0(\mathscr{O}, w_\gamma; X).
$$

Proof. The case $\gamma \in (-1, p-1)$ is contained in Proposition 5.3.15. For the case $\gamma \in (jp-1, (j+1)p-1)$ with $j = 1$ or $j = -1$ we reduce to the previous case. By a localization argument it suffices to consider $\mathscr{O} = \mathbb{R}^d_+$. By Lemma 5.3.13 and since the complex interpolation method is exact we deduce

$$
\begin{aligned}
[L^p(\mathbb{R}^d_+, w_\gamma; X), &W^{2,p}_0(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{1}{2}} \\
&= [M^{-j} L^p(\mathbb{R}^d_+, w_{\gamma-jp}; X), M^{-j} W^{2,p}_0(\mathbb{R}^d_+, w_{\gamma-jp}; X)]_{\frac{1}{2}} \\
&= M^{-j} [L^p(\mathbb{R}^d_+, w_{\gamma-jp}; X), W^{2,p}_0(\mathbb{R}^d_+, w_{\gamma-jp}; X)]_{\frac{1}{2}} \\
&= M^{-j} W^{1,p}_0(\mathbb{R}^d_+, w_{\gamma-jp}; X) = W^{1,p}_0(\mathbb{R}^d_+, w_\gamma; X).
\end{aligned}
$$
□

Next we prove a version of Proposition 5.3.16 without boundary conditions by reducing to the case with boundary conditions.

Proposition 5.3.17. *Let X be a UMD space, $p \in (1,\infty)$, $\gamma \in (-p-1, 2p-1) \setminus \{-1, p-1\}$. Let $\mathscr{O}$ be bounded C^2-domain or a special C^2_c-domain. Then the complex interpolation space satisfies*

$$
[L^p(\mathscr{O}, w_\gamma; X), W^{2,p}(\mathscr{O}, w_\gamma; X)]_{\frac{1}{2}} = W^{1,p}(\mathscr{O}, w_\gamma; X).
$$

Proof. By a localization argument it suffices to consider $\mathcal{O} = \mathbb{R}^d_+$. The case $\gamma \in (-1, p-1)$ follows from Propositions 4.5.5 and 6.3.7 and the case $\gamma \in (-p-1, -1)$ follows from Proposition 5.3.16.

It remains to establish the case $\gamma \in (p-1, 2p-1)$. The inclusion $\hookleftarrow$ follows from Proposition 5.3.16 and $W^{1,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)$. To prove $\hookrightarrow$, by Lemma 5.3.9 it suffices to show that

$$\|u\|_{W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)} \le C\|u\|_{[L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{1}{2}}}, \quad u \in C^\infty_c(\mathbb{R}^d_+; X).$$

Since $W^{1,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)$, using Lemma 5.3.13 twice and the result for the A_p-case already proved, we obtain

$$\begin{aligned}\|u\|_{W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)} \lesssim \|Mu\|_{W^{1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)} &\lesssim \|Mu\|_{[L^p(\mathbb{R}^d_+, w_{\gamma-p}; X), W^{2,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)]_{\frac{1}{2}}} \\ &\lesssim \|u\|_{[L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{1}{2}}}.\end{aligned}$$

□

Next we turn to a different type of interpolation result. It unifies and extends several existing results in the literature. The case $p_0 = p_1$ and $w_0 = w_1$ can be found in [187, Proposition 3.7].

Theorem 5.3.18. *Let X_j be a UMD space, $p_j \in (1, \infty)$, $w_j \in A_{p_j}$ and $s_j \in \mathbb{R}$ for $j \in \{0, 1\}$. Let $\theta \in (0, 1)$ and set $X_\theta = [X_0, X_1]_\theta$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $w = w_0^{(1-\theta)p/p_0} w_1^{\theta p/p_1}$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[H^{s_0,p_0}(\mathbb{R}^d, w_0; X_0), H^{s_1,p_1}(\mathbb{R}^d, w_1; X_1)]_\theta = H^{s,p}(\mathbb{R}^d, w; X_\theta).$$

Observe that $w \in A_p$ by [101, Exercise 9.1.5]. The proof of the theorem will be given below.

As a corollary of Proposition 5.3.17 and Theorem 5.3.18 we obtain (using the identification from Proposition 5.2.8) the following mixed-derivative theorem:

Corollary 5.3.19. *Let X be a UMD space, $p \in (1, \infty)$, $\gamma \in (-p-1, 2p-1) \setminus \{-1, p-1\}$ and $d \ge 2$. Then*

$$\begin{aligned}W^{2,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) \cap L^p(\mathbb{R}^{d-1}; W^{2,p}(\mathbb{R}_+, w_\gamma; X)) \\ \hookrightarrow W^{1,p}(\mathbb{R}^{d-1}, W^{1,p}(\mathbb{R}_+, w_\gamma; X)).\end{aligned}$$

Proof. By Proposition 5.2.8, Theorem 5.3.18, and Proposition 5.3.17,

$$\begin{aligned}&L^p(\mathbb{R}^{d-1}; W^{2,p}(\mathbb{R}_+, w_\gamma; X)) \cap W^{2,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) \\ &\quad = H^{0,p}(\mathbb{R}^{d-1}; W^{2,p}(\mathbb{R}_+, w_\gamma; X)) \cap H^{2,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) \\ &\quad \hookrightarrow [H^{0,p}(\mathbb{R}^{d-1}; W^{2,p}(\mathbb{R}_+, w_\gamma; X)), H^{2,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X))]_{\frac{1}{2}} \\ &\quad = H^{1,p}(\mathbb{R}^{d-1}; [W^{2,p}(\mathbb{R}_+, w_\gamma; X), L^p(\mathbb{R}_+, w_\gamma; X)]_{\frac{1}{2}}) \\ &\quad = W^{1,p}(\mathbb{R}^{d-1}; W^{1,p}(\mathbb{R}_+, w_\gamma; X)).\end{aligned}$$

□

For the proof of Theorem 5.3.18 we need two preliminary results. The first result follows as in [235, Theorems 1.18.4 & 1.18.5].

Proposition 5.3.20. *Let $(A,\mathscr{A},\mu)$ be a measure space. Let X_j be a Banach space, $p_j \in (1,\infty)$ and $w_j : S \to (0,\infty)$ measurable for $j \in \{0,1\}$. Let $\theta \in (0,1)$ and set $X_\theta = [X_0,X_1]_\theta$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $w = w_0^{(1-\theta)p/p_0} w_1^{\theta p/p_1}$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[L^{p_0}(A,w_0;X_0), L^{p_1}(A,w_1;X_1)]_\theta = L^p(\mathbb{R}^d,w;X_\theta).$$

For the next result we need to introduce some notation. Let $(\varepsilon_k)_{k\geq 0}$ be a Rademacher sequence on a probability space Ω. Let $\sigma : \mathbb{N} \to (0,\infty)$ be a weight function, $p \in (1,\infty)$ and let $\mathrm{Rad}^{\sigma,p}(X)$ denote the space of all sequences $(x_k)_{k\geq 0}$ in X for which

$$\left\|(x_k)_{k\geq 0}\right\|_{\mathrm{Rad}^{\sigma,p}(X)} := \sup_{n\geq 1} \left\| \sum_{k=0}^{n} \varepsilon_k \sigma(k) x_k \right\|_{L^p(\Omega;X)} < \infty.$$

The above space is p-independent and the norms for different values of p are equivalent (see [127, Proposition 6.3.1]). If $\sigma \equiv 1$, we write $\mathrm{Rad}^p(X) := \mathrm{Rad}^{\sigma,p}(X)$. Clearly $(x_k)_{k\geq 0} \mapsto (\sigma(k)x_k)_{k\geq 0}$ defines an isometric isomorphism from $\mathrm{Rad}^{\sigma,p}(X)$ onto $\mathrm{Rad}^p(X)$. By [127, Corollary 6.4.12], if X does not contain a copy isomorphic to c_0 (which is the case for UMD spaces), then $(x_k)_{k\geq 0}$ in $\mathrm{Rad}^{\sigma,p}(X)$ implies that $\sum_{k\geq 0} \varepsilon_k \sigma(k) x_k$ converges in $L^p(\Omega;X)$ and in this case

$$\left\|(x_k)_{k\geq 0}\right\|_{\mathrm{Rad}^{\sigma,p}(X)} = \left\| \sum_{k\geq 0} \varepsilon_k \sigma(k) x_k \right\|_{L^p(\Omega;X)}.$$

Interpolation of the unweighted spaces

$$[\mathrm{Rad}^{p_0}(X_0), \mathrm{Rad}^{p_1}(X_1)]_\theta = \mathrm{Rad}^p(X_\theta) \tag{5.12}$$

holds if X_0 and X_1 are K-convex spaces (see [127, Theorem 7.4.16] for details). In particular, UMD spaces are K-convex (see [126, Proposition 4.3.10]). We need the following weighted version of complex interpolation of Rad-spaces.

Proposition 5.3.21. *Let X_j be a K-convex space, $\sigma_j : \mathbb{N} \to (0,\infty)$ and let $p_j \in (1,\infty)$ for $j \in \{0,1\}$. Let $\theta \in (0,1)$ and set $X_\theta = [X_0,X_1]_\theta$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\sigma = \sigma_0^{1-\theta}\sigma_1^\theta$. Then*

$$[\mathrm{Rad}^{\sigma_0,p_0}(X_0), \mathrm{Rad}^{\sigma_1,p_1}(X_1)]_\theta = \mathrm{Rad}^{\sigma,p}(X_\theta).$$

Proof. We use the same method as in [235, 1.18.5]. Let

$$T : \mathbb{F}_-(\mathrm{Rad}^{\sigma_0,p_0}(X_0), \mathrm{Rad}^{\sigma_1,p_1}(X_1), 0) \to \mathbb{F}_-(\mathrm{Rad}^{p_0}(X_0), \mathrm{Rad}^{p_1}(X_1), 0)$$

be defined by

$$Tf(k,z) = \sigma_0(k)^{1-z}\sigma_1(k)^z f(k,z).$$

Then $f \mapsto Tf(\cdot,\theta)$ is an isomorphism

$$[\mathrm{Rad}^{\sigma_0,p_0}(X_0), \mathrm{Rad}^{\sigma_1,p_1}(X_1)]_\theta \longrightarrow [\mathrm{Rad}^{p_0}(X_0), \mathrm{Rad}^{p_1}(X_1)]_\theta = \mathrm{Rad}^p(X_\theta),$$

where we used (5.12) in the last step. $\square$

Proof of Theorem 5.3.18. Set $Y_j = L^{p_j}(\mathbb{R}^d, w_j, X_j))$ for $j \in \{0,1\}$ and let $Y_\theta = L^p(\mathbb{R}^d, w, X_\theta))$. Then by Proposition 5.3.20 $Y_\theta = [Y_0, Y_1]_\theta$. Let $\sigma_j(n) = 2^{s_j n}$ and let $(\varphi_k)_{k\geq 0}$ be a smooth Littlewood-Paley sequence as in [187, Section 2.2] and let $\phi_{-1} = 0$. By [187, Proposition 3.2] and [127, Theorem 6.2.4] we have $f \in H^{s_j,p_j}(\mathbb{R}^d, w_j; X_j)$ if and only if $(\varphi_k * f)_{k\geq 0} \in \mathrm{Rad}^{\sigma_j,p_j}(Y_j)$ and in this case

$$\|(\varphi_k * f)_{k\geq 0}\|_{\mathrm{Rad}^{\sigma_j,p_j}(Y_j)} \eqsim \|f\|_{H^{s_j,p_j}(\mathbb{R}^d,w_j;X_j)} \tag{5.13}$$

with implicit constants only depending on $p_j, X_j, s_j, [w_j]_{A_{p_j}}$. Now to reduce the statement to Proposition 5.3.21 we use a retraction-coretraction argument (see [235, Theorem 1.2.4] and Lemma 4.5.3. Let $\psi_n = \sum_{k=n-1}^{n+1} \phi_k$ for $n \geq 0$, and let $\psi_{-1} = 0$. Then $\widehat{\psi}_k = 1$ on $\mathrm{supp}\,(\widehat{\phi}_k)$ for all $k \geq 0$, and $\mathrm{supp}\,(\widehat{\psi}_0) \subseteq \{\xi : |\xi| \leq 2\}$ and $\mathrm{supp}\,(\widehat{\psi}_k) \subseteq \{\xi : 2^{k-2} \leq |\xi| \leq 2^{k+1}\}$ for $k \geq 1$. Let $R : \mathrm{Rad}^{\sigma_j,p_j}(Y_j) \to H^{s_j,p_j}(\mathbb{R}^d, w_j; X_j)$ be defined by $R(f_\ell)_{\ell\geq 0} = \sum_{\ell\geq 0} \psi_\ell * f_\ell$ and let $S : H^{s_j,p_j}(\mathbb{R}^d, w_j; X_j) \to \mathrm{Rad}^{p_j,\sigma_j}(Y_j)$ be given by $Sf = (\varphi_k * f)_{k\geq 0}$. The boundedness of S follows from (5.13). We claim that R is bounded and this will be explained below. By the special choice of ψ_k we have $RS = I$. Therefore, the retraction-coretraction argument applies and the interpolation result follows.

To prove claim let $E_j = L^{p_j}(\Omega; Y_j))$. Due to (5.13) and by density it suffices to show that, for all finitely-nonzero sequences $(f_\ell)_{\ell\geq 0}$ in Y_j and all $n \geq 0$,

$$\Big\| \sum_{k=0}^{n} \varepsilon_k 2^{s_j k} \varphi_k * \sum_{\ell\geq 0} \psi_\ell * f_\ell \Big\|_{E_j} \leq C \Big\| \sum_{k\geq 0} \varepsilon_k 2^{s_j k} f_k \Big\|_{E_j}. \tag{5.14}$$

Below, for convenience of notation, we view sequences on $\mathbb{N}$ as sequences on $\mathbb{Z}$ through extension by zero. Under this convention, by the Fourier support properties of $(\varphi_k)_k$ and the R-boundedness of $\{\varphi_k * : k \geq 0\}$ (see [187, Lemma 4.1]) and the implied R-boundedness of $\{\psi_k * : k \geq 0\}$, we have

$$\begin{aligned}
\Big\| \sum_{k=0}^{n} \varepsilon_k 2^{s_j k} \varphi_k * \sum_{\ell\geq 0} \psi_\ell * f_\ell \Big\|_{E_j} &\leq \sum_{j=-2}^{2} \Big\| \sum_{k=0}^{n} \varepsilon_k 2^{s_j k} \varphi_k * \psi_{k+j} * f_{k+j} \Big\|_{E_j} \\
&\lesssim \sum_{j=-2}^{2} \Big\| \sum_{k=0}^{n} \varepsilon_k 2^{s_j k} f_{k+j} \Big\|_{E_j} \\
&\lesssim \Big\| \sum_{k\geq 0} \varepsilon_k 2^{s_j k} f_k \Big\|_{E_j},
\end{aligned}$$

where in the last step we used the contraction principle (see [126, Proposition 3.24]). □

5.4. Δ_{Dir} ON $\mathbb{R}^d_+$ IN THE A_p-SETTING

Let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d)$. We consider the Dirichlet Laplacian Δ_{Dir} on $L^p(\mathbb{R}^d_+, w; X)$, defined by

$$D(\Delta_{\mathrm{Dir}}) := W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X), \qquad \Delta_{\mathrm{Dir}} u := \sum_{j=1}^{d} \partial_j^2 u.$$

Let $G_z : \mathbb{R}^d \to \mathbb{R}$ denote the standard heat kernel on $\mathbb{R}^d$:

$$G_z(x) = \frac{1}{(4\pi z)^{d/2}} e^{-|x|^2/(4z)}, \qquad z \in C_+.$$

It is well-known that $|G_z * f| \le \cos^{-d/2}(\arg(z)) Mf$, where M denotes the Hardy-Littlewood maximal function (see [127, Section 8.2]). Therefore, $f \mapsto G_z * f$ is bounded on $L^p(\mathbb{R}^d, w; X)$ for any $w \in A_p$.

Define $T(z) : L^p(\mathbb{R}^d_+, w; X) \to L^p(\mathbb{R}^d_+, w; X)$ by

$$T(z)f(x) := H_z * f(x) := \int_{\mathbb{R}^d_+} H_z(x,y) f(y)\, dy = \int_{\mathbb{R}^d} G_z(x-y)\overline{f}(y)\, dy, \;\; z \in \mathbb{C}_+, \tag{5.15}$$

with $\overline{f}(y) = \operatorname{sign}(y_1) f(|y_1|, \tilde{y})$ and

$$H_z(x,y) = G_z(x_1 - y_1, \tilde{x} - \tilde{y}) - G_z(x_1 + y_1, \tilde{x} - \tilde{y}), \;\; x, y \in \mathbb{R}^d_+. \tag{5.16}$$

By the properties of G_z, the operator $T(z)$ is bounded on $L^p(\mathbb{R}^d_+, w; X)$ for any $w \in A_p$ with $\|T(z)\| \le \|M\|_{\mathscr{B}(L^p(w))} \cos^{-d/2}(\arg(z))$. In Theorem 5.4.1 we will show that $T(z)$ is an analytic C_0-semigroup with generator Δ_{Dir}. Moreover, in case X is a UMD space we characterize $D(\Delta_{\mathrm{Dir}})$ and prove that Δ_{Dir} is a sectorial operator with a bounded H^∞-calculus of angle zero.

Recall that a weight w is called even if $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ for $x_1 > 0$ and $\tilde{x} \in \mathbb{R}^{d-1}$.

The next result is the main result of this section on the functional calculus of $-\Delta_{\mathrm{Dir}}$ on L^p-spaces with A_p-weights. The result on the whole of $\mathbb{R}^d$ is well-known to experts, but seems not to have appeared anywhere. By a standard reflection argument we deduce the result on $\mathbb{R}^d_+$. It can be seen as a warm-up for Theorem 5.5.7 where weights outside the A_p-class are considered.

Theorem 5.4.1. *Let X be a UMD space. Let $p \in (1,\infty)$ and let $w \in A_p$ be even. Then the following assertions hold:*

(1) $-\Delta_{\mathrm{Dir}}$ *is a sectorial operator with $\omega(-\Delta_{\mathrm{Dir}}) = 0$, $D(\Delta_{\mathrm{Dir}}) = W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$ with equivalent norms, the analytic C_0-semigroup $(e^{z\Delta_{\mathrm{Dir}}})_{z\in\mathbb{C}_+}$ is uniformly bounded on any sector Σ_ω with $\omega \in (0,\pi/2)$ and*

$$e^{z\Delta_{\mathrm{Dir}}} f = T(z) f, \quad z \in \mathbb{C}_+.$$

(2) *For all $\lambda \ge 0$, $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.*

Moreover, all the implicit constants only depend on X, p, d and $[w]_{A_p}$.

For the proof we use a simple lemma on odd extensions. For $u \in L^p(\mathbb{R}^d, w; X)$, the functions $\overline{u}$ and $E_{\mathrm{odd}} u$ denote the *odd extension of* u:

$$\overline{u}(-x_1, \tilde{x}) = E_{\mathrm{odd}} u(-x_1, \tilde{x}) = -u(x_1, \tilde{x}) \text{ for } x_1 > 0 \text{ and } \tilde{x} \in \mathbb{R}^{d-1}.$$

For $k \in \mathbb{N}_0$ let $W^{k,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X)$ denote the closed subspace of all even functions in $W^{k,p}(\mathbb{R}^d, w; X)$.

Lemma 5.4.2. *Let $p \in (1,\infty)$ and let $w \in A_p$ be even. Let $k \in \{0,1,2\}$. Then $E_{\text{odd}} : W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X) \to W^{k,p}_{\text{odd}}(\mathbb{R}^d, w; X)$ is an isomorphism and*

$$\|u\|_{W^{k,p}(\mathbb{R}^d,w;X)} \le \|E_{\text{odd}} u\|_{W^{k,p}(\mathbb{R}^d,w;X)} \le 2^{1/p}\|u\|_{W^{k,p}(\mathbb{R}^d,w;X)}.$$

Moreover, $\{u \in C^\infty_c(\overline{\mathbb{R}^d_+}) : u(0,\cdot) = 0\} \otimes X$ is dense in $W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X)$.

Proof. The case $k = 0$ is easy, so let us assume $k \in \{1,2\}$. For $u \in W^{k,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X)$ one has

$$D^\alpha \overline{u}(x) = (\operatorname{sign}(x_1))^{|\alpha_1|+1}(D^\alpha u)(|x_1|, \tilde{x}), \quad |\alpha| \le k. \tag{5.17}$$

Indeed, this follows from Lemmas 5.3.10 and 5.3.1.

From (5.17) we find that $\overline{u} \in W^{2,p}(\mathbb{R}^d, w; X)$ and that the stated estimates hold.

If $\overline{u} \in W^{k,p}(\mathbb{R}^d, w)$, then by Lemma 5.3.6 we can find $u_n \in C^\infty_c(\mathbb{R}^d) \otimes X$ such that $u_n \to \overline{u}$ in $W^{2,p}(\mathbb{R}^d, w; X)$. Then also $u_n(-\cdot,\cdot) \to \overline{u}$ in $W^{2,p}(\mathbb{R}^d, w; X)$. Now $v_n := (u_n + u_n(-\cdot,\cdot))/2$ satisfies $v_n \in C^\infty_c(\mathbb{R}^d; X)$ and $v_n(0,\cdot) = 0$ and $v_n \to u$ in $W^{2,p}(\mathbb{R}^d_+, w; X)$. Since $\operatorname{tr}(v_n) = 0$ the continuity of the trace implies $\operatorname{tr}(u) = 0$ as well. This part of the prove also implies the desired density result. □

Proof of Theorem 5.4.1. Let us first consider the result on $\mathbb{R}^d$. Then $-\Delta$ with $D(\Delta) = W^{2,p}(\mathbb{R}^d, w; X)$ is a closed operator which is sectorial of angle zero (see [93, Theorem 5.1] and Proposition 5.2.6). Moreover, by Proposition 5.2.6 and [127, Theorem 10.2.25]), one can has that $-\Delta$ has a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta) = 0$. Moreover, by Remark 5.2.5 the same holds for $\lambda - \Delta$. Now the half space case follows by a well-known reflection argument, which we partly include here for completeness.

Since $E_{\text{odd}}(\Delta_{\text{Dir}} f) = (\Delta E_{\text{odd}} f)$, $E_{\text{odd}} : W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X) \to W^{2,p}_{\text{odd}}(\mathbb{R}^d, w; X)$ is an isomorphism (see Lemma 5.4.2), $\Delta : W^{2,p}_{\text{odd}}(\mathbb{R}^d, w; X) \to L^p_{\text{odd}}(\mathbb{R}^d, w; X)$, and $D(\Delta|_{L^p_{\text{odd}}(\mathbb{R}^d,w;X)}) = W^{2,p}_{\text{odd}}(\mathbb{R}^d, w; X)$, one has

$$\rho(\Delta) \subseteq \rho(\Delta_{\text{Dir}}), \;\; R(\lambda, \Delta_{\text{Dir}})f = (R(\lambda,\Delta)E_{\text{odd}} f)|_{\mathbb{R}^d_+} \text{ and } D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X).$$

All the statements now follow. □

Corollary 5.4.3 (Laplace equation)**.** *Let X be a UMD space. Let $p \in (1,\infty)$ and let $w \in A_p$ be even. For all $u \in W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X)$ there holds the estimates*

$$[u]_{W^{2,p}(\mathbb{R}^d_+,w;X)} \eqsim_{X,p,d,w} \|\Delta u\|_{L^p(\mathbb{R}^d_+,w;X)}. \tag{5.18}$$

Furthermore, for every $f \in L^p(\mathbb{R}^d_+, w; X)$ and $\lambda > 0$ there exists a unique $u \in W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w; X)$ such that $\lambda u - \Delta_{\text{Dir}} u = f$ and

$$\sum_{|\alpha|\le 2} |\lambda|^{1-\frac{1}{2}|\alpha|}\|D^\alpha u\|_{L^p(\mathbb{R}^d_+,w;X)} \lesssim_{X,p,d,w} \|f\|_{L^p(\mathbb{R}^d_+,w;X)}. \tag{5.19}$$

Proof. We first prove (5.18). Let $u \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$. For $r > 0$ we put $u_r := u(r\cdot)$ and $w_r := w(r\cdot)$. Then $w_r \in A_p$ with $[w_r]_{A_p} = [w]_{A_p}$. So we can apply Theorem 5.4.1 with w_r instead of w to obtain

$$\begin{aligned}\sum_{|\alpha|\leq 2} r^{|\alpha|-\frac{d}{p}} \|\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w; X)} &= \|u_r\|_{W^{2,p}(\mathbb{R}^d_+, w_r; X)} \\ &\eqsim_{X,p,d,w} \|u_r\|_{L^p(\mathbb{R}^d_+, w_r; X)} + \|\Delta u_r\|_{L^p(\mathbb{R}^d_+, w_r; X)} \\ &= r^{-\frac{d}{p}} \|u\|_{L^p(\mathbb{R}^d_+, w; X)} + r^{2-\frac{d}{p}} \|\Delta u\|_{W^{2,p}(\mathbb{R}^d_+, w_r; X)}.\end{aligned}$$

Dividing by $r^{2-\frac{d}{p}}$ and taking the limit $r \to \infty$ gives (5.18).

The existence and uniqueness in the second claim follow from the sectoriality in Theorem 5.4.1. Moreover, together with (5.18), the sectoriality yields the estimates for $|\alpha| = 0$ and $|\alpha| = 2$ in (5.19). The case $|\alpha| = 1$ subsequently follows from Lemma 5.3.12. □

Corollary 5.4.4 (Heat equation)**.** *Let X be a UMD space. Let $p, q \in (1,\infty)$, $v \in A_q(\mathbb{R})$, $w \in A_p(\mathbb{R}^d)$ and assume w is even. Let $J \in \{\mathbb{R}, \mathbb{R}_+\}$. Then the following assertions hold: For all $\lambda > 0$ and $f \in L^q(J, v; L^p(\mathbb{R}^d_+, w; X))$ there exists a unique $u \in W^{1,q}(J, v; L^p(\mathbb{R}^d_+, w; X)) \cap L^q(J, v; W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X))$ such that $u' + (\lambda - \Delta_{\mathrm{Dir}})u = f$, $u(0) = 0$ in case $J = \mathbb{R}_+$. Moreover, the following estimate holds*

$$\begin{aligned}\|u'\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))} + \sum_{|\alpha|\leq 2} \lambda^{1-\frac{1}{2}|\alpha|} \|D^\alpha u\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))} \\ \lesssim_{p,q,v,w,d} \|f\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))}.\end{aligned}$$

Proof. Since $L^p(\mathbb{R}^d_+, w; X)$ is a UMD space, by Proposition 5.2.7, d/dt had a bounded H^∞-calculus on $L^q(J, v; L^p(\mathbb{R}^d_+, w; X))$. Therefore, from Theorem 5.4.1, Remark 5.2.5 (1), and Theorem 5.2.4

$$\|u'\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))} + \|(\lambda - \Delta)u\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))} \lesssim_{p,q,v,w,d} \|f\|_{L^q(J,v;L^p(\mathbb{R}^d_+, w;X))}.$$

Now the result follows from Corollary 5.4.3 applied pointwise in t. □

Remark 5.4.5.

(i) The same result as in Corollary 5.4.4 holds for Δ on the whole of $\mathbb{R}^d$. For results on elliptic and parabolic equations with A_p-weights in space we refer to [111].

(ii) Due to Calderón-Zygmund extrapolation theory one can add A_q-weights in time after considering the unweighted case (see [44]).

(iii) It would be interesting to extend Corollary 5.4.4 to spaces of the form $L^p(\mathbb{R} \times \mathbb{R}^d_+, w; X)$ where w depends on time and space. For some result in this direction concerning the maximal regularity estimate we refer to [74].

(iv) The estimate in Corollaries 5.4.3 and 5.4.4 also hold for $\lambda = 0$. However, solvability does not hold for general f.

5.5. Δ_{Dir} ON $\mathbb{R}^d_+$ IN THE NON-A_p-SETTING

In this section we will extend the results of Section 5.4 to weighted L^p-spaces with $w_\gamma(x) = |x_1|^\gamma$ where $\gamma \in (p-1, 2p-1)$. This case is not included in the A_p-weight and is therefore not accessible through classical harmonic analysis. The reflection argument cannot be applied since the weight is not locally integrable in $\mathbb{R}^d$.

5.5.1. The heat semigroup

Let $T(z) : L^p(\mathbb{R}^d_+; X) \to L^p(\mathbb{R}^d_+; X)$ be defined by (5.15). We first show that $T(z)$ is also bounded on $L^p(\mathbb{R}^d_+, w_\gamma)$ with $w_\gamma(x) = |x_1|^\gamma$ for $\gamma \in (-p-1, 2p-1)$ and that this range is optimal. Note that $w_\gamma \in A_p(\mathbb{R}^d)$ if and only if $\gamma \in (-1, p-1)$.

Proposition 5.5.1. *Let $p \in [1,\infty)$ and $\gamma \in (-1-p, 2p-1)$. For every $|\phi| < \pi/2$, $(T(z))_{z\in\Sigma_\phi}$ defines a bounded analytic C_0-semigroup on $L^p(\mathbb{R}^d_+, w_\gamma; X)$.*

Proof. First we consider $p \in (1,\infty)$. The result for $\gamma \in (-1, p-1)$ follows from Theorem 5.4.1. In the remaining cases by duality it suffices to consider $\gamma \in [p-1, 2p-1)$.

Let $|\delta| < \phi$ and write $z = te^{i\delta}$ for $t > 0$.

Step 1: Reduction to an estimate in the case $X = \mathbb{C}$. In this step we show that it is enough to prove the estimate

$$\| |H_z| * \|f\| \|_{L^p(\mathbb{R}^d_+, w_\gamma)} \lesssim_{\phi,\gamma,p} \|f\|_{L^p(\mathbb{R}^d_+, w_\gamma)} \tag{5.20}$$

for all $f \in C_c(\mathbb{R}^d_+)$. Having this estimate, we get

$$\|T(z)f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \le \| |H_z| * \|f\|_X \|_{L^p(\mathbb{R}^d_+, w_\gamma)} \le C_{\phi,\gamma,p} \|f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}$$

for all $f \in C_c(\mathbb{R}^d_+) \otimes X$, from which the analyticity and strong continuity follow. Indeed, note that for $g \in C_c(\mathbb{R}_+) \otimes X$, $z \mapsto \langle T(z)f, g\rangle$ is analytic on Σ_ϕ and continuous on $\overline{\Sigma_\phi}$ by Theorem 5.4.1 with $w = 1$. Therefore, in case $X = \mathbb{C}$, the weak continuity of T on $\overline{\Sigma_\phi}$ follows by density in the case $p \in (1,\infty)$ and by weak*-sequential density of C_c in L^∞ in case $p = 1$ (see [213, Corollary 2.24]). This in turn implies strong continuity by [85, Theorem I.5.8]. For general X, the continuity of $T(z)f$ for $f \in C_c(\mathbb{R}^d_+) \otimes X$ is clear from the scalar case, yielding the case of general $f \in L^p(\mathbb{R}^d_+; X)$ by density. The analyticity of T on Σ_ϕ follows from [12, Theorem A.7].

Step 2: Reduction to the case $d = 1$. Writing H^1_z for the kernel of $T(z)$ in case $d = 1$ and G^{d-1}_z for the standard heat kernel in dimension $d-1$, we have

$$|H_z| * |f|(x_1, \tilde{x}) = \int_0^\infty |H^1_z(x_1, y_1)| \int_{\mathbb{R}^{d-1}} G^{d-1}_z(\tilde{x} - \tilde{y}) |f(y_1, \tilde{y})| \, d\tilde{y} \, dy_1.$$

Taking $L^p(\mathbb{R}^{d-1})$-norms for fixed $x_1 \in \mathbb{R}_+$, and using Minkowski's inequality and Young's inequality, we obtain

$$\| |H_z| * |f|(x_1, \cdot) \|_{L^p(\mathbb{R}^{d-1})}$$

$$\leq \int_0^\infty |H_z^1(x_1,y_1)| \Big\| \widetilde{x} \mapsto \int_{\mathbb{R}^{d-1}} G_z^{d-1}(\widetilde{x}-\widetilde{y})|f(y_1,\widetilde{y})|d\widetilde{y} \Big\|_{L^p(\mathbb{R}^{d-1})} dy_1$$
$$\leq C_\phi \int_0^\infty |H_z^1(x_1,y_1)| \|f(y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1})} dy_1,$$

where $C_\phi = \sup_{z\in\Sigma_\phi} \|G_z^{d-1}\|_{L^1(\mathbb{R}^d)} < \infty$. Therefore, it remains to prove (5.20) in the case $d=1$.

Step 3: The case $d=1$. Setting $g(x) := x^{\frac{\gamma+1}{p}}|f(x)|$, $k_z(s,y) := y(x/y)^{\frac{\gamma+1}{p}}|H_z(x,y)|$ and

$$h_z(x) := \int_0^\infty k_z(x,y) g(y) \frac{dy}{y},$$

we see that (5.20) holds if and only if $\|h_z\|_{L^p(\mathbb{R}_+,\frac{dx}{x})} \lesssim_\phi \|g\|_{L^p(\mathbb{R}_+,\frac{dx}{x})}$. To prove this, by Schur's test (see [96, Theorem 5.9.2]) it is enough to show

$$\sup_{x>0} \int_0^\infty k_z(x,y) \frac{dy}{y} \leq A, \tag{5.21}$$
$$\sup_{y>0} \int_0^\infty k_z(x,y) \frac{dx}{x} \leq B. \tag{5.22}$$

In order to prove these estimates, observe that with $z = te^{i\delta}$,

$$\begin{aligned}(4\pi t)^{1/2}|H_z(x,y)| &= \left| e^{\frac{-|x-y|^2 e^{-i\delta}}{4t}} - e^{\frac{-|x+y|^2 e^{-i\delta}}{4t}} \right| \\ &= e^{\frac{-|x-y|^2\cos(\delta)}{4t}} \left|1 - e^{-\frac{xye^{-i\delta}}{t}}\right| \\ &\leq e^{\frac{-|x-y|^2\cos(\delta)}{4t}} \int_0^{xy/t} e^{-s\cos(\delta)}\, ds \\ &= (4\pi t)^{1/2}\cos(\delta)^{-1} H_{t\cos(\delta)}(x,y).\end{aligned} \tag{5.23}$$

Therefore, by replacing x and y by $(4t/\cos(\delta))^{1/2}x$ and $(4t/\cos(\delta))^{1/2}y$, respectively, in (5.21) and (5.22) it suffices to consider $t = 1/4$ and $\delta = 0$.

From now on we write

$$k(x,y) := y(x/y)^{\frac{\gamma+1}{p}}(e^{-|x-y|^2} - e^{-|x+y|^2}) = y(x/y)^{\frac{\gamma+1}{p}} e^{|x-y|^2}\left|1-e^{-4xy}\right|.$$

One can check that $|1-e^{-4xy}| \leq \min\{1, 4xy\}$. Therefore, k satisfies

$$k(x,y) \leq y(x/y)^{\frac{\gamma+1}{p}} e^{-|x-y|^2} \min\{1,4xy\}$$

It follows that

$$\int_0^\infty k(x,y)\frac{dy}{y} \leq \int_0^\infty (x/y)^{\frac{\gamma+1}{p}} e^{-|x-y|^2}\min\{1,4xy\} dy$$
$$\leq \int_0^{x/2} (x/y)^{\frac{\gamma+1}{p}} e^{-|x-y|^2} 4xy dy + \int_{x/2}^\infty (x/y)^{\frac{\gamma+1}{p}} e^{-|x-y|^2} dy$$

$$= T_1 + T_2.$$

The first term satisfies

$$T_1 \le 4\int_0^{x/2} x^{\frac{\gamma+1}{p}+1} y^{1-\frac{\gamma+1}{p}} e^{-x^2/4}\,dy = C_1 x^3 e^{-x^2/4} \le A_1,$$

where we used $1-\frac{\gamma+1}{p} > -1$. Since $\gamma > -1$, the second term satisfies

$$T_2 \le C_2 \int_{-\infty}^{\infty} e^{-|x-y|^2}\,dy = C_2 \int_{-\infty}^{\infty} e^{-|y|^2}\,dy = A_2$$

Next we estimate the integral over the x-variable. For $y \in (0,1)$, we can write

$$\begin{aligned}\int_0^\infty k(x,y)\frac{dx}{x} &\le \int_0^\infty (x/y)^{\frac{\gamma+1}{p}-1} e^{-|x-y|^2} 4xy\,dx\\ &\le 4\int_0^\infty x^{\frac{\gamma+1}{p}} y^{2-\frac{\gamma+1}{p}} e^{-|x-y|^2}\,dx\\ &= \int_{-y}^\infty (x+y)^{\frac{\gamma+1}{p}} y^{2-\frac{\gamma+1}{p}} e^{-|x|^2}\,dx\\ &= \int_{-\infty}^\infty (x+1)^{\frac{\gamma+1}{p}} e^{-|x|^2}\,dx \le B_1,\end{aligned}$$

where we used $2-\frac{\gamma+1}{p} \ge 0$ and $\gamma+1 \ge 0$. For $y \ge 1$, since $\frac{\gamma+1}{p} \ge 1$ we have

$$\begin{aligned}\int_0^\infty k(x,y)\frac{dx}{x} &\le \int_0^\infty (x/y)^{\frac{\gamma+1}{p}-1} e^{-|x-y|^2}\,dx\\ &= \int_{-y}^\infty \left(\tfrac{x}{y}+1\right)^{\frac{\gamma+1}{p}-1} e^{-x^2}\,dx\\ &\le \int_{-\infty}^\infty \big(|x|+1\big)^{\frac{\gamma+1}{p}-1} e^{-x^2}\,dx \le B_2.\end{aligned}$$

Step 4: The case $p=1$: One can still reduce to the case $d=1$ by Fubini's theorem. Moreover, instead of using Schur's lemma, by Fubini's theorem it suffices to show that

$$\sup_{y>0}\int_0^\infty k(x,y)\frac{dx}{x} < \infty.$$

The case $\gamma \in [0,1)$ can be treated in the same way as in the above proof. In case $\gamma \in (-2,0)$ we argue as follows:

$$\int_{y/2}^\infty k(x,y)\frac{dx}{x} \le \int_{y/2}^\infty (x/y)^\gamma e^{-|x-y|^2}\,dx \le 2^\gamma \int_{-\infty}^\infty e^{-|x-y|^2}\,dx = C.$$

On the other hand, since $\gamma+2>0$, we have

$$\begin{aligned}\int_0^{y/2} k(x,y)\frac{dx}{x} &\le \int_0^{y/2} (x/y)^\gamma e^{-|x-y|^2} 4xy\\ &\le 4e^{-y^2/4} y^{-\gamma+1}\int_0^{y/2} x^{\gamma+1}\,dx = 4ye^{-y^2/4}.\end{aligned}$$

□

In the next example we show that the range for γ in Proposition 5.5.1 is optimal.

Example 5.5.2. Let $p \in (1,\infty)$ and $\gamma \notin (-p-1, 2p-1)$. We give an example of a function $f \in L^p(\mathbb{R}^d_+, w_\gamma)$ such that for all $t > 0$, $T(t)f \notin L^p(\mathbb{R}^d_+, w_\gamma)$. Here $T(t)f$ is defined by (5.15). By duality we only need to consider $\gamma \geq 2p-1$. Let $\beta \in (1/p, 1)$ and set $f(x) = x_1^{-2}|\log(x_1)|^{-\beta}\mathbb{1}_Q(x)$, where $Q = [0,1]^d$. Then, on the one hand, $f \in L^p(\mathbb{R}^d_+; w_\gamma)$. On the other hand, for $x \in Q^d$,

$$\begin{aligned} T(t)f(x) &= c_{t,d}\int_Q e^{-\frac{|x-y|^2}{4t}}[1-e^{-\frac{-x_1y_1}{t}}]y_1^{-2}|\log(y_1)|^{-\beta}\,dy \\ &= \widetilde{c}_{t,d}\int_0^1 y_1^{-1}|\log(y_1)|^{-\beta}\,dy_1 = \infty; \end{aligned}$$

in particular, $T(t)f \notin L^p(\mathbb{R}^d_+, w_\gamma)$.

Let $-A$ denote the generator of the semigroup $(T(z))_z$ of Proposition 5.5.1. Then by standard results of analytic semigroups we see that A is sectorial with $\omega(A) = 0$.

In the case of a X is a UMD space, $-A$ even has a bounded H^∞-calculus:

Proposition 5.5.3. *Let X be a UMD space. Let $-A$ be the generator of the heat semigroup on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ given in Proposition 5.5.1 with $p \in (1,\infty)$ and $\gamma \in (-1-p, 2p-1)$. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = 0$.*

Proof. The case $\gamma \in (-1, p-1)$ follows from Theorem 5.4.1. For the other values of γ we use a classical perturbation argument (see [142]).

Step 1: Let $0 < \sigma < \omega < \pi/2$ Let $\phi \in H^\infty_0(\Sigma_\omega)$ with $\omega \in (0, \pi/2)$ satisfy $\|\phi\|_\infty \leq 1$ and let $\Gamma = \partial\Sigma_\sigma$. By definition we have

$$\phi(A) = \frac{1}{2\pi i}\int_\Gamma \phi(\lambda)R(\lambda, A)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_+\cup\Gamma_-}\phi(\lambda)R(\lambda, A)\,d\lambda, \tag{5.24}$$

where $\Gamma_\pm = \{te^{\pm\sigma i} : t \in (0,\infty)\}$.

Fix $f \in C_c(\mathbb{R}^d_+; X))$ and let $g = w_\gamma^{\frac{1}{p}} f$ and $\psi(x, y) = \Big(\frac{w_\gamma^{\frac{1}{p}}(x_1)}{w_\gamma^{\frac{1}{p}}(y_1)} - 1\Big)$. Then for $x \in \mathbb{R}^d_+$

$$\begin{aligned} \phi(A)f(x) &= w_\gamma^{-\frac{1}{p}}(x_1)\phi(A)(w_\gamma^{\frac{1}{p}} f)(x) + w_\gamma^{-\frac{1}{p}}(x_1)\phi(A)\big(w_\gamma^{\frac{1}{p}}(x_1) - w_\gamma^{\frac{1}{p}})f\big)(x) \\ &= w_\gamma^{-\frac{1}{p}}(x_1)\phi(A)(g)(x) + w_\gamma^{-\frac{1}{p}}(x_1)\phi(A)(\psi(x,\cdot)g)(x). \end{aligned}$$

Therefore,

$$\|\phi(A)f\|_{L^p(\mathbb{R}_+, w_\gamma; X)} \leq \|\phi(A)g\|_{L^p(\mathbb{R}_+; X)} + \|x \mapsto \phi(A)(\psi(x,\cdot)g)(x)\|_{L^p(\mathbb{R}_+; X)}.$$

The first term on the right-hand side can be estimated by the boundedness of the H^∞-calculus in the unweighted case (see Theorem 5.4.1):

$$\|\phi(A)g\|_{L^p(\mathbb{R}_+; X)} \leq C\|g\|_{L^p(\mathbb{R}_+; X)} = C\|f\|_{L^p(\mathbb{R}_+, w_\gamma; X)}.$$

Therefore, it remains to show

$$\Big\| x \mapsto \phi(A)(\psi(x,\cdot)g)(x)\Big\|_{L^p(\mathbb{R}_+;X)} \le C\|g\|_{L^p(\mathbb{R}_+;X)} = C\|f\|_{L^p(\mathbb{R}_+,w_\gamma;X)}. \tag{5.25}$$

Step 2: To prove (5.25) we estimate the integrals over $\Gamma_\pm$ in (5.24) separately. By symmetry it suffices to consider Γ_+. Let $\delta = (\pi-\sigma)/2$. For $\lambda = re^{i\sigma}$ with $r>0$ and $h \in L^p(\mathbb{R}_+, w_\gamma; X)$, we have the following Laplace transform representation for the resolvent (see [85]):

$$\begin{aligned} R(\lambda, A)h &= (\lambda - A)^{-1}h = -(re^{i(\sigma-\pi)} + A)^{-1}h \\ &= -e^{i\delta}(re^{-i\delta} + e^{i\delta}A)^{-1}h = e^{i\delta}\int_0^\infty e^{-tre^{-i\delta}}e^{-te^{i\delta}A}h\,dt. \end{aligned}$$

Observe that by (5.23) we can write

$$\begin{aligned} &\Big\|\int_{\Gamma_+}\phi(\lambda)R(\lambda,A)(\psi(x,\cdot)g)(x)\,d\lambda\Big\|_X \\ &\le \int_0^\infty \|R(re^{i\sigma},A)(\psi(x,\cdot)g)(x)\|\,dr \\ &\le \int_0^\infty\int_0^\infty \|e^{-te^{i\delta}A}e^{-tre^{-i\delta}}(\psi(x,\cdot)g)(x)\|_X\,dt\,dr \\ &\le \frac{1}{\cos(\delta)}\int_0^\infty\int_0^\infty e^{-t\cos(\delta)A}\|\psi(x,\cdot)g\|(x)e^{-tr\cos(\delta)}\,dt\,dr \\ &= \frac{1}{\cos^2(\delta)}\int_0^\infty e^{-t\cos(\delta)A}\|\psi(x,\cdot)g\|(x)\frac{dt}{t}. \end{aligned}$$

Below we will write $x = (x_1, \tilde{x})$ and $y = (y_1, \tilde{y})$. Using the kernel representation of the semigroup we can write

$$\begin{aligned} &\int_0^\infty e^{-t\cos(\delta)A}\|\psi(x,\cdot)g\|(x)\frac{dt}{t} \\ &= \int_0^\infty e^{-tA}\|\psi(x,\cdot)g\|(x)\frac{dt}{t} \\ &= \int_0^\infty\int_{\mathbb{R}^d_+}(G_t(x_1-y_1,\tilde{x}-\tilde{y}) - G_t(x_1+y_1,\tilde{x}-\tilde{y}))|\psi(x,y)|\,\|g(y)\|\,dy\frac{dt}{t} \\ &= \int_{\mathbb{R}^d_+}\int_0^\infty(G_t(x_1-y_1,\tilde{y}) - G_t(x_1+y_1,\tilde{y}))\frac{dt}{t}|\psi(x_1,y_1)|\,\|g(y_1,\tilde{x}-\tilde{y})\|\,dy \\ &= C_1\int_{\mathbb{R}^d_+}\Big(\frac{1}{|(x_1/y_1-1,\tilde{y}/y_1)|^d} - \frac{1}{|(x_1/y_1+1,\tilde{y}/y_1)|^d}\Big)|(x_1/y_1)^{\frac{\gamma}{p}}-1\Big|\|g(y)\|\,\frac{dy}{y_1^d} \\ &=: C_1\int_{\mathbb{R}^d_+}\ell(x_1/y_1,\tilde{y}/y_1)\,\|g(y_1,\tilde{x}-\tilde{y})\|\frac{dy}{y_1^d} \\ &= C_1\int_{\mathbb{R}^d_+}\ell(x_1/y_1,\tilde{y})\,\|g(y_1,\tilde{x}-y_1\tilde{y})\|\frac{dy}{y_1}, \end{aligned}$$

where we used

$$\int_0^\infty G_t(x)\,\frac{dt}{t} = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}\,\frac{dt}{t} = \int_0^\infty (4\pi)^{-d/2} s^{d/2} e^{-\frac{s}{4}}\,\frac{ds}{s}|x|^{-d} = C_1|x|^{-d}.$$

Now,

$$\begin{aligned}
&\Big\|x\mapsto \int_{\mathbb{R}^d_+} \ell(x_1/y_1,\tilde{y})\,\|g(y_1,\tilde{x}-y_1\tilde{y})\|\,\frac{dy}{y_1}\Big\|_{L^p(\mathbb{R}^d_+)}\\
&\le \Big\|x_1\mapsto \int_{\mathbb{R}^d_+} \ell(x_1/y_1,\tilde{y})\,\|g(y_1,\cdot-y_1\tilde{y})\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{dy}{y_1}\Big\|_{L^p(\mathbb{R}_+)}\\
&= \Big\|x_1\mapsto \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(x_1/y_1,\tilde{y})\,\|g(y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{dy_1}{y_1}\,d\tilde{y}\Big\|_{L^p(\mathbb{R}_+)}\\
&= \Big\|x_1\mapsto \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(y_1,\tilde{y})\,\|g(x_1/y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{dy_1}{y_1}\,d\tilde{y}\Big\|_{L^p(\mathbb{R}_+)}\\
&= \Big\|x_1\mapsto \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(y_1,\tilde{y})y_1^p(x_1/y_1)^p\,\|g(x_1/y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{dy_1}{y_1}\,d\tilde{y}\Big\|_{L^p(\mathbb{R}_+,\frac{dx_1}{x_1})}\\
&\le \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(y_1,\tilde{y})y_1^p\,\Big\|x_1\mapsto (x_1/y_1)^p\|g(x_1/y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\Big\|_{L^p(\mathbb{R}_+,\frac{dx_1}{x_1})}\frac{dy_1}{y_1}\,d\tilde{y}\\
&= \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(y_1,\tilde{y})y_1^p\,\Big\|x_1\mapsto x_1^p\|g(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\Big\|_{L^p(\mathbb{R}_+,\frac{dx_1}{x_1})}\frac{dy_1}{y_1}\,d\tilde{y}\\
&= C_2\|g\|_{L^p(\mathbb{R}^d_+;X)}.
\end{aligned}$$

Here we use $-1-p<\gamma<2p-1$ to obtain

$$\begin{aligned}
C_2 &:= \int_{\mathbb{R}^{d-1}}\int_0^\infty \ell(y_1,\tilde{y})y_1^p\,\frac{dy_1}{y_1}\,d\tilde{y}\\
&= \int_0^\infty\int_{\mathbb{R}^{d-1}}\Big(\frac{1}{|(y_1-1,\tilde{y})|^d}-\frac{1}{|(y_1+1,\tilde{y})|^d}\Big)\,d\tilde{y}\,|y_1^{\frac{\gamma}{p}}-1|y_1^p\,\frac{dy_1}{y_1}\\
&= C_3\int_0^\infty\Big(\frac{1}{|y_1-1|}-\frac{1}{|y_1+1|}\Big)|y_1^{\frac{\gamma}{p}}-1|y_1^p\,\frac{dy_1}{y_1}<\infty,
\end{aligned}$$

where $C_3=\int_{\mathbb{R}^{d-1}}(1+|\tilde{y}|)^{-d}\,d\tilde{y}$ if $d\ge 2$ and $C_3=1$ otherwise. Combining the above estimates we obtain the required estimate

$$\Big\|x\mapsto \int_{\Gamma_+}\phi(\lambda)R(\lambda,A)(\psi(x,\cdot)g)(x)\,d\lambda\Big\|_{L^p(\mathbb{R}_+;X)} \le \frac{C\|g\|_{L^p(\mathbb{R}_+;X)}}{\cos^2(\delta)}.$$

□

5.5.2. The Dirichlet Laplacian on $\mathbb{R}_+$

Proposition 5.5.4. *Let $p\in(1,\infty)$ and $\gamma\in(p-1,2p-1)$. Then Δ_{Dir}, defined as*

$$D(\Delta_{\mathrm{Dir}}) := W^{2,p}_{\mathrm{Dir}}(\mathbb{R}_+,w_\gamma;X),\quad \Delta_{\mathrm{Dir}}u := u'',$$

is the generator of the heat semigroup on $L^p(\mathbb{R}_+,w_\gamma;X)$ given in Proposition 5.5.1.

For the case $\gamma \in (-p-1,-1)$ we refer the reader to Section 5.5.5.

Proof. Let $-A$ denote the generator of the heat semigroup T of Proposition 5.5.1. We first show that $\Delta_{\rm Dir} \subseteq -A$, that is, $W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X) \subseteq D(A)$ and for $u \in W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X)$ one has $-Au = \Delta_{\rm Dir} u$. From Theorem 5.4.1 we see that for $u \in C_c^\infty(\mathbb{R}_+; X)$,

$$T(t)u - u = \int_0^t T(s)\Delta_{\rm Dir} u\,ds.$$

Therefore, $\frac{1}{t}(T(t)u-u) \to \Delta_{\rm Dir} u$ in $L^p(\mathbb{R}_+, w_\gamma; X)$ by strong continuity of $(T(s))_{s\geq 0}$. Therefore, $u \in D(A)$ with $-Au = \Delta_{\rm Dir} u$. Now for $u \in W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X)$, using Proposition 5.3.8, we can find a sequence $(u_n)_{n\geq 1}$ in $C_c^\infty(\mathbb{R}_+; X)$ such that $u_n \to u$ in $W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X)$. Then $-Au_n = \Delta_{\rm Dir} u_n \to \Delta_{\rm Dir} u$ in $L^p(\mathbb{R}_+, w; X)$. Therefore, the closedness of A yields that $u \in D(A)$ and $-Au = \Delta_{\rm Dir} u$.

Next we show $-A \subseteq \Delta_{\rm Dir}$. Using $\Delta_{\rm Dir} \subseteq -A$, for this it is enough that $1 + A$ is injective and $1 - \Delta_{\rm Dir}$ is surjective. Being the generator of a bounded analytic semigroup (see Proposition 5.5.1), A is sectorial, implying that $1 + A$ is injective. For the surjectivity of $1 - \Delta_{\rm Dir}$ we consider the equation $u - \Delta_{\rm Dir} u = f$, for $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$.

Let us first consider $f \in C_c^\infty(\mathbb{R}_+; X)$. Let $\overline{f}$ denote the odd extension of f. Clearly, $\overline{f} \in C_c^\infty(\mathbb{R}; X) \subset \mathscr{S}(\mathbb{R}; X)$. So we can define $\overline{u} \in \mathscr{S}(\mathbb{R}; X)$ by $\overline{u} := \mathbb{F}^{-1}[\xi \mapsto \frac{\mathbb{F}\overline{f}(\xi)}{1+\xi^2}]$, yielding a solution of the equation $\overline{u} - \overline{u}'' = \overline{f}$. Since $\overline{u}$ is odd, it also satisfies the Dirichlet condition $u(0) = 0$. By restriction to $\mathbb{R}_+$ we obtain a solution $u := \overline{u}_{|\mathbb{R}_+} \in W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X)$ of the equation $(1 - \Delta_{\rm Dir})u = f$. As $W^{2,p}_{\rm Dir}(\mathbb{R}^d, w_\gamma; X)$ is complete and $C_c^\infty(\mathbb{R}_+; X)$ is dense in $L^p(\mathbb{R}^d_+, w_\gamma; X)$ (see Proposition 5.3.8), it suffices to prove the estimate $\|u\|_{W^{2,p}(\mathbb{R}^d, w_\gamma; X)} \lesssim \|f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}$.

To finish, we prove this estimate. As $\Delta_{\rm Dir} \subset A$, we have $u \in D(A)$ with $(1 - A)u = f$, so $u = R(1, A)f$. It follows that $\|u\|_{L^p(\mathbb{R}_+, w_\gamma; X)} \lesssim \|f\|_{L^p(\mathbb{R}_+, w_\gamma; X)}$. Since $u'' = u - f$ we find that $\|u''\|_{L^p(\mathbb{R}_+, w_\gamma; X)} \lesssim \|f\|_{L^p(\mathbb{R}_+, w_\gamma; X)}$. By interpolation the same estimate holds for u' (see Lemma 5.3.14). □

Corollary 5.5.5. *Let $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$. For all $\lambda > 0$ and $f \in L^p(\mathbb{R}_+; w_\gamma; X)$ there exists a unique $u \in W^{2,p}_{\rm Dir}(\mathbb{R}_+, w_\gamma; X)$ such that $\lambda u - u'' = f$ and*

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \|D^j u\|_{L^p(\mathbb{R}_+, w_\gamma; X)} \lesssim_{p,\gamma} \|f\|_{L^p(\mathbb{R}_+; w_\gamma; X)}. \tag{5.26}$$

Proof. This can be done in the same way as the second statement in Corollary 5.4.3. □

Combining Propositions and 5.5.4 and 5.5.3, we find the following result in the one-dimensional case:

Corollary 5.5.6. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$. Then $-\Delta_{\rm Dir}$ has a bounded H^∞-calculus on $L^p(\mathbb{R}_+, w_\gamma; X)$ with $\omega_{H^\infty}(-\Delta_{\rm Dir}) = 0$.*

5.5.3. The Dirichlet Laplacian on $\mathbb{R}^d_+$

The main result of this section is the following theorem. Note that the case $\gamma \in (-1, p-1)$ was already considered in Theorem 5.4.1. See Section 5.5.5 for the case $\gamma \in (-p-1,-1)$.

Before we state the theorem, let us first define the Dirichlet Laplacian Δ_{Dir} on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ with $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$:

$$D(\Delta_{\text{Dir}}) := W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w_\gamma; X), \qquad \Delta_{\text{Dir}} u := \Delta u.$$

Theorem 5.5.7. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$. Then the following assertions hold:*

1. *Δ_{Dir} is the generator of the heat semigroup from Proposition 5.5.1.*
2. *Δ_{Dir} is a closed and densely defined linear operator on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ with*

$$\begin{aligned} D(\Delta_{\text{Dir}}) &= W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w_\gamma; X) \\ &= W^{2,p}_{\text{Dir}}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{2,p}(\mathbb{R}^{d-1}; X)) \end{aligned}$$

   with an equivalence of norms only depending on X, p, d, γ.

3. *For all $\lambda \geq 0$, $\lambda - \Delta_{\text{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta_{\text{Dir}}) = 0$.*

Proof. Note that (3) follows from (1) by Proposition 5.5.3 and Remark 5.2.5. So we only need to prove (1) and (2).

Below we will frequently use and Fubini's theorem in the form of the identification

$$L^p(\mathbb{R}^d_+, w_\gamma; X) = L^p(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) = L^p(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)),$$

and that UMD-valued L^p-spaces have UMD again. By Corollary 5.5.6, for the operator $\Delta_{1,\text{Dir}}$ on $L^p(\mathbb{R}^d_+, w_\gamma; X)$, defined by

$$D(\Delta_{1,\text{Dir}}) := W^{2,p}_{\text{Dir}}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))), \quad \Delta_{1,\text{Dir}} u := \partial_1^2 u,$$

it holds that $-\Delta_{1,\text{Dir}}$ a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta_{1,\text{Dir}}) = 0$. By [127, Theorem 10.2.25], for the operator Δ_{d-1} on $L^p(\mathbb{R}^d_+, w_\gamma; X)$, defined by

$$D(\Delta_{d-1}) := L^p(\mathbb{R}_+, w_\gamma, W^{2,p}(\mathbb{R}^{d-1}; X)), \quad \Delta_{d-1} u := \sum_{k=2}^{d} \partial_k^2 u,$$

it holds that $-\Delta_{d-1}$ a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta_{d-1}) = 0$. The operators $\Delta_{1,\text{Dir}}$ and $D(\Delta_{d-1})$ are clearly resolvent commuting. Therefore, by Theorem 5.2.4 for the operator sum $\Delta^\Sigma_{\text{Dir}} := \Delta_{1,\text{Dir}} + \Delta_{d-1}$ with $D(\Delta^\Sigma_{\text{Dir}}) = D(\Delta_{1,\text{Dir}}) \cap D(\Delta_{d-1})$ it holds that $-\Delta^\Sigma_{\text{Dir}}$ is a sectorial operator with angle $\omega(-\Delta^\Sigma_{\text{Dir}}) = 0$. Moreover,

$$e^{t\Delta^\Sigma_{\text{Dir}}} = e^{t\Delta_{1,\text{Dir}}} e^{t\Delta_{d-1}}, \quad t \geq 0. \tag{5.27}$$

Writing H^1_t for the kernel in (5.16) in dimension 1 and G^{d-1}_t for the standard heat kernel in dimension $d-1$, (5.27) and Proposition 5.5.4 give

$$\begin{aligned}[][e^{t\Delta^{\Sigma}_{\mathrm{Dir}}}f](x) &= \int_0^\infty H^1_t(x_1,y_1)\int_{\mathbb{R}^{d-1}} G^{d-1}_t(\tilde{x}-\tilde{y})f(y_1,\tilde{y})\,d\tilde{y}\,dy_1\\ &= \int_{\mathbb{R}^d_+} H_t(x,y)f(y)\,dy\end{aligned}$$

for all $f\in L^p(\mathbb{R}^d_+,w_\gamma;X)$. Therefore, $\Delta^{\Sigma}_{\mathrm{Dir}}$ is the generator of the heat semigroup from Proposition 5.5.1.

We now show that

$$\begin{aligned}D(\Delta^{\Sigma}_{\mathrm{Dir}}) &= W^{2,p}_{\mathrm{Dir}}(\mathbb{R}_+,w_\gamma;L^p(\mathbb{R}^{d-1};X))\cap L^p(\mathbb{R}_+,w_\gamma;W^{2,p}(\mathbb{R}^{d-1};X))\\ &= W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_\gamma;X)\end{aligned}$$

with an equivalence of norms. Note that then $\Delta_{\mathrm{Dir}}=\Delta^{\Sigma}_{\mathrm{Dir}}$ and the assertions (1), (2) follow. Since $\Delta^{\Sigma}_{\mathrm{Dir}}=\Delta_{1,\mathrm{Dir}}+\Delta_{d-1}$ with $D(\Delta^{\Sigma}_{\mathrm{Dir}})=D(\Delta_{1,\mathrm{Dir}})\cap D(\Delta_{d-1})$, the first identity follows from the domain descriptions of $\Delta_{1,\mathrm{Dir}}$ and Δ_{d-1}. The second identity follows from Corollary 5.3.19. □

Corollary 5.5.8. *Let X be a UMD space, $p\in(1,\infty)$ and $\gamma\in(p-1,2p-1)$. For all $u\in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_\gamma;X)$ there holds the estimates*

$$[u]_{W^{2,p}(\mathbb{R}^d_+,w_\gamma;X)} \eqsim_{X,p,d,\gamma} \|\Delta u\|_{L^p(\mathbb{R}^d_+,w_\gamma;X)}. \tag{5.28}$$

Furthermore, for every $f\in L^p(\mathbb{R}^d_+,w_\gamma;X)$ and $\lambda>0$ there exists a unique $u\in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_\gamma;X)$ such that $\lambda u-\Delta_{\mathrm{Dir}}u=f$ and

$$\sum_{|\alpha|\le 2}|\lambda|^{1-\frac{1}{2}|\alpha|}\|D^\alpha u\|_{L^p(\mathbb{R}^d_+,w_\gamma;X)} \lesssim_{X,p,d,\gamma} \|f\|_{L^p(\mathbb{R}^d_+,w_\gamma;X)}. \tag{5.29}$$

Proof. This can be done in the same way as Corollary 5.4.3, now using the explicit formula $w_\gamma(r\,\cdot)=r^\gamma w_\gamma$ in the scaling argument. □

Remark 5.5.9. The second statement in Corollary 5.5.8 also follows from [143, Theorem 4.1 & Remark 4.2]. In our setting it follows from operator sum methods involving bounded imaginary powers (obtained through the H^∞-calculus).

Now using Theorem 5.5.7, as in Corollary 5.4.4 we obtain the following maximal regularity result for the weights w_γ with $\gamma\in(p-1,2p-1)$. The case $\gamma\in(-1,p-1)$ was already considered in Corollary 5.4.4.

Corollary 5.5.10 (Heat equation). *Let X be a UMD space. Let $p,q\in(1,\infty)$, $v\in A_q(\mathbb{R})$, $\gamma\in(p-1,2p-1)$. Let $J\in\{\mathbb{R}_+,\mathbb{R}\}$. Then the following assertions hold:*

(1) *$\frac{d}{dt}-\Delta_{\mathrm{Dir}}$ is a closed sectorial operator on $L^q(J,v;L^p(\mathbb{R}^d_+,w_\gamma;X))$ which has a bounded H^∞-calculus with $\omega_{H^\infty}(\frac{d}{dt}-\Delta_{\mathrm{Dir}})\le\frac{\pi}{2}$.*

(2) *For all $\lambda > 0$ and $f \in L^q(J, v; L^p(\mathbb{R}^d_+, w_\gamma; X))$ there exists a unique $u \in W^{1,q}(J, v; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(J, v; W^{2,p}_{\rm Dir}(\mathbb{R}^d_+, w_\gamma; X))$ such that $u' + (\lambda - \Delta_{\rm Dir})u = f$, $u(0) = 0$ in case $J = \mathbb{R}_+$. Moreover, the following estimate holds*

$$\|u'\|_{L^q(J,v;L^p(\mathbb{R}^d_+,w_\gamma;X))} + \sum_{|\alpha|\le 2} \lambda^{1-\frac{1}{2}|\alpha|} \|D^\alpha u\|_{L^q(J,v;L^p(\mathbb{R}^d_+,w_\gamma;X))} \lesssim_{p,q,v,\gamma,d} \|f\|_{L^q(J,v;L^p(\mathbb{R}^d_+,w_\gamma;X))}.$$

Remark 5.5.11. In the case $v = 1$, Corollary 5.5.10 (2) reduces to [144, Theorem 0.1], where it was deduced using completely different methods. Let us mention here that in [144, Theorem 0.1] and [75, Theorem 2.1] more general elliptic operators with time and space-dependent coefficients have been considered.

Problem 5.5.12. *Let $p \in (1, \infty)$.*

1. *Characterize those weights w for which $e^{t\Delta_{\rm Dir}}$ extends to a bounded analytic semigroup on $L^p(\mathbb{R}^d_+, w)$.*
2. *Characterize those weights w for which $\Delta_{\rm Dir}$ has a bounded H^∞-calculus on $L^p(\mathbb{R}^d_+, w)$.*
3. *Characterize those weights w for which $\Delta_{\rm Dir}$ on $L^p(\mathbb{R}^d_+, w)$ is a closed operator with $D(\Delta_{\rm Dir}) = W^{2,p}_{\rm Dir}(\mathbb{R}^d_+, w)$.*

Given the results of Sections 5.4 and 5.5 it would be natural to conjecture that all weights of the form $w(x) = v_0(x) + x_1 v_1(x)$ with $v_0, v_1 \in A_p$ are included.

5.5.4. Extrapolation of functional calculus

As soon as one knows the boundedness of the functional calculus of a generator on a space $L^2(\mathbb{R}^d_+, d\mu)$ for some doubling measure μ, then, if the heat kernel satisfies Gaussian estimates with respect to μ, one can extrapolate the boundedness of the functional calculus to $L^p(\mathbb{R}^d_+, w\,d\mu)$ for $p \in (1, \infty)$ and $w \in A_p(\mu)$. Here $A_p(\mu)$ is the weight class associated to the measure μ on $\mathbb{R}^d_+$. The above is presented in the setting of homogeneous spaces in [83] in the unweighted setting and in [171, Theorem 7.3] in the weighted setting. Extension to the setting without kernel bounds can be found in [19, 27].

In order to apply [171, Theorem 7.3] to our setting, we set $d\mu(x) = x_1\,dx$. The reason to take this measure is that the kernel $H_z(x, y)$ as defined in (5.15) has a zero of order one at $x_1 = 0$. Then μ is doubling and one can check that $w_\alpha(x) := x_1^\alpha$ is in $A_p(\mu)$ if and only if $\alpha \in (-2, 2p-2)$. From Theorem 5.5.7 we know that on $L^2(\mathbb{R}^d_+, \mu)$ one has $-\Delta_{\rm Dir} \in \mathcal{H}^\infty$ with $\omega_{H^\infty}(-\Delta_{\rm Dir}) = 0$. So in order to extrapolate the latter to $L^p(\mathbb{R}^d_+, w\,d\mu)$ for $p \in (1, \infty)$ and $w \in A_p(\mu)$ it suffices to check the kernel condition of [171, Theorem 7.3]. For this (due to (5.23)) it suffices to show that there exist constant $C, c > 0$ such that

$$\frac{H_t(x,y)}{x_1} \le \frac{Ce^{-c|x-y|^2/t}}{\mu(B(x,t^{1/2}))}, \quad x, y \in \mathbb{R}^d_+, t > 0. \tag{5.30}$$

Here the nominator x_1 is due to the choice of the measure μ. First consider $x_1 > t^{1/2}$. After renormalization the condition (5.30) is equivalent to

$$\frac{e^{-|x_1-y_1|^2} - e^{-|x_1+y_1|^2}}{x_1} e^{-|\tilde{x}-\tilde{y}|^2} \le Ce^{-c|x-y|^2}.$$

Since $\frac{1-e^{-4x_1y_1}}{x_1} \le \min\{1, 4y_1\}$, we find

$$\frac{e^{-|x_1-y_1|^2} - e^{-|x_1+y_1|^2}}{x_1} e^{-|\tilde{x}-\tilde{y}|^2} = e^{-|x-y|^2} \frac{1-e^{-4xy}}{x} \le e^{-|x-y|^2}$$

as required. The case $x_1 \in (0, t^{1/2})$ can be proved by similar argument. As a consequence we obtain the following result.

Theorem 5.5.13. *Let $d\mu = x_1\,dx$, $p \in (1,\infty)$ and $w \in A_p(\mu)$. Then the heat semigroup given by* (5.15) *extends to an analytic semigroup on $L^p(\mathbb{R}^d_+, w)$ and its generator $-A$ has the property that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = 0$.*

Note that this does not directly imply the same for $-\Delta_{\text{Dir}}$ because it is unclear whether $A = -\Delta_{\text{Dir}}$ in the above setting, because we do not know whether the domains coincide. Note that the approach presented in Theorem 5.5.7 also works for weights of the form $w(x) := x_1^\gamma v(\tilde{x})$ with $v \in A_p$.

Instead of applying Theorem 5.5.7 in the above situation one could also apply the simpler Theorem 5.4.1 with $d\mu(x) = x_1^\beta\,dx$ with $\beta \in (0,1)$. Indeed, then $w_\alpha \in A_p(\mu)$ if and only if $-1 < \alpha + \beta < \beta p + p - 1$. Again one can check condition (5.30) with left-hand side $\frac{1}{x_1^\beta}|H_z(x,y)|$ and for the new measure μ. Therefore, choosing β arbitrary close to 1, we obtain $-\Delta_{\text{Dir}} \in \mathscr{H}^\infty$ on $L^p(\mathbb{R}^d_+, w_\gamma)$ for $\gamma \in (-1, 2p-1)$. Finally, let us remark that some work needs to be done in order to obtain Theorem 5.5.13 in the vector-valued setting using the above approach.

5.5.5. Some comments on the case $\gamma \in (-p-1, -1)$

In Theorem 5.4.1, Proposition 5.5.4 and Theorem 5.5.7 we have characterized the generator of the heat semigroup from Proposition 5.5.1 for the case $\gamma \in (-1, p-1) \cup (p-1, 2p-1)$ as the Dirichlet Laplacian Δ_{Dir} with domain $D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathbb{R}^d_+, w_\gamma; X)$. In this subsection we will discuss the failure of this domain description for the case $\gamma \in (-p-1, -1)$.

Let us start with the one-dimensional case. The point where the proof of Proposition 5.5.4 does not work for the case $\gamma \in (-p-1, -1)$ is the fact that $\mathscr{S}_{\text{odd}}(\mathbb{R}_+; X) \not\subseteq W^{2,p}(\mathbb{R}_+, w_\gamma; X)$ in that case, which is illustrated by the following example.

Example 5.5.14. Let $p \in [1,\infty)$ and $\gamma \in (-p-1, -1)$. Suppose $u \in \mathscr{S}(\mathbb{R}_+; X)$ satisfies $u(0) = u''(0) = 0$. Then $u, u'' \in L^p(\mathbb{R}_+, w_\gamma; X)$, but

$$u \in W^{2,p}(\mathbb{R}_+, w_\gamma; X) \iff u'(0) = 0.$$

Proof. Note that $u, u'' \in W^{1,p}_0(\mathbb{R}_+, w_{\gamma+p}; X)$. So $u, u'' \in L^p(\mathbb{R}_+, w_\gamma; X)$ by Lemma 5.3.2 (or Corollary 5.3.4). In the same way, $u' \in L^p(\mathbb{R}_+, w_\gamma; X)$ if $u'(0) = 0$. On the other hand, $u' \in L^p(\mathbb{R}_+, w_\gamma; X)$ only if $u'(0) = 0$ by (the proof of) Lemma 5.3.1 (2). □

As a consequence of the above example,

$$W^{2,p}(\mathbb{R}_+, w_\gamma; X) \subsetneq \{u \in L^p(\mathbb{R}_+, w_\gamma; X) : u'' \in L^p(\mathbb{R}_+, w_\gamma; X)\} \tag{5.31}$$

for $p \in [1,\infty)$ and $\gamma \in (-p-1,-1)$, despite of the interpolation inequality from Lemma 5.3.14. Note that here $W^{2,p}(\mathbb{R}_+, w_\gamma; X) = W^{2,p}_{\mathrm{Dir}}(\mathbb{R}_+, w_\gamma; X)$.

A duality argument yields that the right-hand side space in (5.31) actually is the "correct" the domain for the Dirichlet Laplacian Δ_{Dir} on $L^p(\mathbb{R}_+, w_\gamma; X)$ when $\gamma \in (-p-1,-1)$:

Proposition 5.5.15. *Let $p \in (1,\infty)$ and $\gamma \in (-p-1,-1)$. Then Δ_{Dir}, defined as*

$$D(\Delta_{\mathrm{Dir}}) := \{u \in L^p(\mathbb{R}_+, w_\gamma; X) : u'' \in L^p(\mathbb{R}_+, w_\gamma; X)\}, \quad \Delta_{\mathrm{Dir}} u := u'',$$

is the generator of the heat semigroup on $L^p(\mathbb{R}_+, w_\gamma; X)$ given in Proposition 5.5.1.

Proof. Let $\gamma' = \frac{-\gamma}{p-1} \in (p'-1, 2p'-1)$ be the p-dual exponent of γ and let Δ'_{Dir} be the Dirichlet Laplacian on $L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*)$:

$$D(\Delta'_{\mathrm{Dir}}) := W^{2,p'}_{\mathrm{Dir}}(\mathbb{R}_+, w_{\gamma'}; X^*), \quad \Delta'_{\mathrm{Dir}} u := u''.$$

Then, viewing $L^p(\mathbb{R}_+, w_\gamma; X)$ as closed subspace of $[L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*)]^*$, we have that Δ_{Dir} coincides with the realization of $[\Delta'_{\mathrm{Dir}}]^*$ in $L^p(\mathbb{R}_+, w_\gamma; X)$. To see this, denote the latter operator by A. Given $v \in D(\Delta_{\mathrm{Dir}})$, we have, for all u in the dense subspace $C^\infty_c(\mathbb{R}_+) \otimes X^*$ of $D(\Delta'_{\mathrm{Dir}}) = W^{2,p'}_{\mathrm{Dir}}(\mathbb{R}_+, w_{\gamma'}; X^*) = W^{2,p'}_0(\mathbb{R}_+, w_{\gamma'}; X^*)$ (see Proposition 5.3.8),

$$\begin{aligned}
\langle \Delta'_{\mathrm{Dir}} u, v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} &= \langle u'', v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} \\
&= \langle u'', v\rangle_{\langle \mathscr{D}(\mathbb{R}_+; X^*), \mathscr{D}'(\mathbb{R}_+; X)\rangle} \\
&= \langle u, v''\rangle_{\langle \mathscr{D}(\mathbb{R}_+; X^*), \mathscr{D}'(\mathbb{R}_+; X)\rangle} \\
&= \langle u, v''\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle},
\end{aligned}$$

showing that $\Delta_{\mathrm{Dir}} \subset [\Delta'_{\mathrm{Dir}}]^*$, and hence $\Delta_{\mathrm{Dir}} \subset A$. Given $v \in D(A)$, we have, for all $u \in C^\infty_c(\mathbb{R}_+) \otimes X^* \subset D(\Delta'_{\mathrm{Dir}})$,

$$\begin{aligned}
\langle u, Av\rangle_{\langle \mathscr{D}(\mathbb{R}_+; X^*), \mathscr{D}'(\mathbb{R}_+; X)\rangle} &= \langle u, Av\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} \\
&= \langle \Delta'_{\mathrm{Dir}} u, v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} \\
&= \langle u'', v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} \\
&= \langle u'', v\rangle_{\langle \mathscr{D}(\mathbb{R}_+; X^*), \mathscr{D}'(\mathbb{R}_+; X)\rangle} \\
&= \langle u, v''\rangle_{\langle \mathscr{D}(\mathbb{R}_+; X^*), \mathscr{D}'(\mathbb{R}_+; X)\rangle},
\end{aligned}$$

and thus $Av = v''$, showing that $A \subset \Delta_{\mathrm{Dir}}$. Since the heat semigroup on $L^p(\mathbb{R}_+, w_\gamma; X)$ from Proposition 5.5.1 is the restriction to $L^p(\mathbb{R}_+, w_\gamma; X)$ of the strongly continuous adjoint (in the sense of [239, page 6]) of the heat semigroup on $L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*)$ from Proposition 5.5.1, the required result follows Proposition 5.5.4 and [239, Theorem 1.3.3]. □

Let us next turn to the d-dimensional case.

Proposition 5.5.16. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (-p-1,-1)$. Then Δ_{Dir}, defined as*

$$D(\Delta_{\mathrm{Dir}}) := \{u \in L^p(\mathbb{R}^d_+, w_\gamma; X) : \Delta u \in L^p(\mathbb{R}^d_+, w_\gamma; X)\}, \quad \Delta_{\mathrm{Dir}} u := \Delta u,$$

is the generator of the heat semigroup on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ given in Proposition 5.5.1. Moreover,

$$D(\Delta_{\mathrm{Dir}}) = \left\{u \in L^p(\mathbb{R}_+, w_\gamma; W^{2,p}(\mathbb{R}^{d-1}; X)) : \partial_1^2 u \in L^p(\mathbb{R}^d_+, w_\gamma; X)\right\}.$$

with an equivalence of norms only depending on X, p, d, γ.

Proof. The first statement can be proved in the same way as Proposition 5.5.15, using Theorem 5.5.7 (1) instead of Proposition 5.5.4. The second statement can be proved using the operator sum method as in Theorem 5.5.7, using Proposition 5.5.15 instead of Proposition 5.5.4. □

5.6. Δ_{Dir} ON BOUNDED DOMAINS

In this section we will use standard localization arguments to obtain versions of Theorems 5.4.1 and 5.5.7 for bounded C^2-domains $\mathscr{O} \subseteq \mathbb{R}^d$. In particular it will be shown that the Dirichlet Laplacian Δ_{Dir} on $L^p(\mathscr{O}, w_\gamma)$ with domain $W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma)$ is a closed and densely defined linear operator for which $-\Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle zero. Moreover, $(e^{z\Delta_{\mathrm{Dir}}})_{z\in\mathbb{C}_+}$ is an exponentially stable analytic C_0-semigroup.

5.6.1. Main results

Let the Dirichlet Laplacian Δ_{Dir} on $L^p(\mathscr{O}, w^{\mathscr{O}}_\gamma; X)$ be defined by

$$D(\Delta_{\mathrm{Dir}}) := W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\mathscr{O}}_\gamma; X), \qquad \Delta_{\mathrm{Dir}} u := \Delta u.$$

Here, $w^{\mathscr{O}}_\gamma(x) = \operatorname{dist}(x, \partial\mathscr{O})^\gamma$.

The main result of this section is the following version of Theorems 5.4.1 and 5.5.7 for bounded C^2-domains.

Theorem 5.6.1. *Let $\mathscr{O}$ be a bounded C^2-domain, X a UMD space, $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Then*

(1) Δ_{Dir} *is the generator of an analytic C_0-semigroup on $L^p(\mathscr{O}, w^{\mathscr{O}}_\gamma; X)$.*

(2) Δ_{Dir} *is a closed and densely defined linear operator on* $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}}; X)$ *with*

$$D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}}; X)$$

with an equivalence of norms only depending on X, p, d, γ *and* $\mathscr{O}$.

(3) *For every* $\varphi > 0$ *there exists a* $\widetilde{\lambda} \in \mathbb{R}$ *such that for all* $\lambda \geq \widetilde{\lambda}$ *the operator* $\lambda - \Delta_{\text{Dir}}$ *has a bounded* H^∞*-calculus with* $\omega_{H^\infty}(\lambda - \Delta_{\text{Dir}}) \leq \varphi$.

In the scalar case Theorem 5.6.1 implies the following result where we obtain additional information on the value of $\widetilde{\lambda}$.

Corollary 5.6.2. *Let* $\mathscr{O}$ *be a bounded* C^2*-domain,* $p \in (1,\infty)$ *and* $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. *Then the following assertions hold:*

(1) $\sigma(-\Delta_{\text{Dir}}) = \{\lambda_i : i \in \mathbb{N}_0\}$, *where* $\delta_{\mathscr{O}} > 0$ *and* $\lambda_i \geq \delta_{\mathscr{O}}$ *are not depending on* $p \in (1,\infty)$ *and* $\gamma \in (-1, 2p-1) \setminus \{p-1\}$.

(2) *For all* $\lambda > -\delta_{\mathscr{O}}$, $\lambda - \Delta_{\text{Dir}}$ *has a bounded* H^∞*-calculus of angle zero.*

(3) Δ_{Dir} *is a closed and densely defined operator on* $L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}})$ *for which there is an equivalence of norms in* $D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathscr{O}, w^{\mathscr{O}})$ *and* Δ_{Dir} *generates an exponentially stable analytic* C_0*-semigroup on* $L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$.

(4) *For every* $\lambda \geq 0$ *and* $f \in L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})$ *there exists a unique* $u \in W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}})$ *such that* $\lambda u - \Delta_{\text{Dir}} u = f$, *and there exists a constant* $C_{p,\gamma,\mathscr{O}}$ *such that*

$$\sum_{|\alpha|\leq 2} (\lambda+1)^{1-\frac{1}{2}|\alpha|} \|D^\alpha u\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})} \leq C_{p,\gamma,\mathscr{O}} \|f\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})}.$$

Proof. (3): All assertions follow from Theorem 5.6.1 except the exponential stability. The latter will follow from (2).

(2): Fix $\phi > 0$. Then, by Theorem 5.6.1, for $\lambda > 0$ large enough, $\lambda - \Delta \in \mathscr{H}^\infty$ with $\omega_{H^\infty}(\lambda - \Delta) \leq \phi$. Next we will show that this holds for small values of λ as well. For this we first prove (1). Note that

$$D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}}) \hookrightarrow W^{1,p}(\mathscr{O}, w_\gamma^{\mathscr{O}}) \overset{\text{compact}}{\hookrightarrow} L^p(\mathscr{O}, w_\gamma^{\mathscr{O}}),$$

where the compactness follows from [194, Theorem 8.8]. We obtain that $(\lambda - \Delta_{\text{Dir}})^{-1}$ is compact for $\lambda \in \rho(\Delta_{\text{Dir}})$. By Riesz' theory of compact operators (see [214, Chapter 4]), we obtain that $(\lambda - \Delta_{\text{Dir}})^{-1}$ has a discrete countable spectrum $\{\mu_i : i \geq 0\}$ and for every $\mu_i \neq 0$, μ_i is an eigenvalue of $(\lambda - \Delta_{\text{Dir}})^{-1}$. Moreover, 0 is in the spectrum of $(\lambda - \Delta_{\text{Dir}})^{-1}$ and is the only accumulation point of the spectrum. We find that $\sigma(-\Delta_{\text{Dir}}) = \{\mu_i^{-1} - \lambda : i \geq 0 \text{ with } \mu_i \neq 0\}$. In the case $p = 2$ and $\gamma = 0$, it is standard that the spectrum has the required form as stated in (1) (see e.g. [87, Theorem 6.5.1]). Now arguing as in [57, Corollary 1.6.2] one sees that the spectrum is independent of $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and $p \in (1, \infty)$.

By the analyticity of $z \mapsto (z - \Delta)^{-1}$ for $\mathbb{C} \setminus (-\infty, -\delta]$ and the sectoriality of $\mu - \Delta$ with angle $\le \phi$, it follows that for any $\lambda > -\delta_{\mathscr{O}}$ and any $\phi' > \phi$, the operator $\lambda - \Delta$ is sectorial of angle $\le \phi'$. Therefore, Remark 5.2.5 implies that for any $\lambda > -\delta_{\mathscr{O}}$, $\lambda - \Delta \in \mathscr{H}^\infty$ with $\omega_{H^\infty}(\lambda - \Delta) \le \phi'$. Finally, since ϕ is arbitrary (2) follows.

(4): By the sectoriality of $-\frac{1}{2}\delta_{\mathscr{O}} - \Delta_{\text{Dir}}$, we have

$$(\lambda + \tfrac{1}{2}\delta_{\mathscr{O}})\|u\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})} \le C\|f\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})}$$

for all $\lambda \ge 0$. On the other hand,

$$\|\Delta_{\text{Dir}} u\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})} \le (C+1)\|f\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})}.$$

Therefore, since $D(\Delta_{\text{Dir}}) = W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}})$ and Δ_{Dir} is invertible we can deduce

$$\|u\|_{W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}})} \lesssim \|f\|_{L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})}.$$

Finally, the estimates for the first order terms follow from Lemma 5.6.10 below. □

As in Corollaries 5.4.4 and 5.5.10, Corollary 5.6.2 has the following consequence. This time we can allow $\lambda = 0$ since the semigroup is exponentially stable. A similar maximal regularity consequence can be deduced from Theorem 5.6.1 in the X-valued case, but this time with additional conditions on λ.

Corollary 5.6.3 (Heat equation)**.** *Let $p, q \in (1, \infty)$, $v \in A_q(\mathbb{R})$ and let $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Let $J \in \{\mathbb{R}_+, \mathbb{R}\}$. Then for all $\lambda \ge 0$ and $f \in L^q(J, v; L^p(\mathscr{O}, w_\gamma^{\mathscr{O}}))$ there exists a unique $u \in W^{1,q}(J, v; L^p(\mathscr{O}, w_\gamma^{\mathscr{O}})) \cap L^q(J, v; W^{2,p}_{\text{Dir}}(\mathscr{O}, w_\gamma))$ such that $u' + (\lambda - \Delta_{\text{Dir}})u = f$, $u(0) = 0$ in the case $J = \mathbb{R}_+$. Moreover, the following estimates hold*

$$\|u\|_{W^{1,q}(J,v;L^p(\mathscr{O},w_\gamma^{\mathscr{O}}))} + \|u\|_{L^q(J,v;W^{2,p}_{\text{Dir}}(\mathscr{O},w_\gamma^{\mathscr{O}}))} \lesssim_{p,q,v,\gamma,d,X} \|f\|_{L^q(J,v;L^p(\mathscr{O},w_\gamma^{\mathscr{O}}))},$$

and

$$\sum_{|\alpha| \le 1} (\lambda+1)^{1-\frac{1}{2}|\alpha|}\|D^\alpha u\|_{L^q(J,v;L^p(\mathscr{O},w_\gamma^{\mathscr{O}}))} \lesssim_{p,q,v,\gamma,d,X} \|f\|_{L^q(J,v;L^p(\mathscr{O},w_\gamma^{\mathscr{O}}))}.$$

Remark 5.6.4. Maximal regularity results have been obtained in [140, Theorem 2.10], [139] and [138, Theorem 3.13] for the case $\gamma \in (p-1, 2p-1)$ for very general elliptic operators A with time-dependent coefficient on bounded C^1-domains. The boundedness of the H^∞-calculus in the weighted case seems to be new for all $\gamma \in (-1, 2p-1)$.

5.6.2. The adjoint operator $[\Delta_{\text{Dir}}]^*$

Recall that every UMD space is reflexive. Let X be a reflexive Banach space, $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$. Then $L^p(\mathscr{O}, w_\gamma; X)$ is a reflexive Banach space with $[L^p(\mathscr{O}, w_\gamma; X)]^* = L^{p'}(\mathscr{O}, w_{\gamma'}; X^*)$ (see [126, Corollary 1.3.22]). Here $\gamma' = \frac{-\gamma}{p-1}$ and we use the unweighted pairing

$$\langle f, g\rangle = \int_{\mathscr{O}} \langle f(x), g(x)\rangle\, dx.$$

Proposition 5.6.5. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Let Δ_{Dir} be the Dirichlet Laplacian on $L^p(\mathscr{O}, w_\gamma; X)$ and let Δ'_{Dir} be the Dirichlet Laplacian on $L^{p'}(\mathscr{O}, w_{\gamma'}; X^*)$. Then $[\Delta_{\mathrm{Dir}}]^* = \Delta'_{\mathrm{Dir}}$.*

Proof. Integration by parts yields that

$$\langle \Delta'_{\mathrm{Dir}} u, v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle} = \langle u, \Delta v\rangle_{\langle L^{p'}(\mathbb{R}_+, w_{\gamma'}; X^*), L^p(\mathbb{R}_+, w_\gamma; X)\rangle}$$

for all $u \in D(\Delta'_{\mathrm{Dir}})$ and $v \in D(\Delta_{\mathrm{Dir}})$, showing that $\Delta_{\mathrm{Dir}} \subset [\Delta'_{\mathrm{Dir}}]^*$ and $\Delta'_{\mathrm{Dir}} \subset [\Delta_{\mathrm{Dir}}]^*$. The first inclusion gives $\Delta'_{\mathrm{Dir}} = [\Delta'_{\mathrm{Dir}}]^{**} \subset [\Delta_{\mathrm{Dir}}]^*$. Hence, $[\Delta_{\mathrm{Dir}}]^* = \Delta'_{\mathrm{Dir}}$. □

Proposition 5.6.6. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$. Let Δ_{Dir} be the Dirichlet Laplacian on $L^p(\mathscr{O}, w_\gamma; X)$. Then*

$$D([\Delta_{\mathrm{Dir}}]^*) = \Big\{u \in L^{p'}(\mathscr{O}, w_{\gamma'}; X^*) : \Delta u \in L^{p'}(\mathscr{O}, w_{\gamma'}; X^*)\Big\}, \qquad [\Delta_{\mathrm{Dir}}]^* u = \Delta u.$$

Proof. This can be shown in the same way as in the proof of Proposition 5.5.15. □

5.6.3. Intermezzo: identification of $D((-\Delta_{\mathrm{Dir}})^{\frac{k}{2}})$

In order to transfer the results of the previous sections to smooth domains (and in particular to prove Theorem 5.6.1) we will use standard argument. However, in order to use perturbation arguments we need to identify several fractional domain spaces and interpolation spaces. In principle this topic is covered by the literature as well. However, the weighted setting is not available for the class of weights we consider and requires additional arguments.

We start with a simple interpolation result for general A_p-weights. In the next result we extend the definition of (5.8) to all $k \in \mathbb{N}_0$ in the following way

$$W^{k,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X) = \{u \in W^{k,p}(\mathbb{R}^d_+, w; X) : \mathrm{tr}(\Delta^j u) = 0 \ \forall j < k/2\}.$$

Proposition 5.6.7. *Let X be a UMD space. Let $p \in (1,\infty)$ and let $w \in A_p$ be even. Then for any $k \in \mathbb{N}_1$ and $j \in \{0, \dots, k\}$ the following holds:*

$$[L^p(\mathbb{R}^d_+, w; X), W^{k,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X)]_{\frac{j}{k}} = W^{j,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X).$$

In particular, for any $k \in \mathbb{N}_0$, $D((-\Delta_{\mathrm{Dir}})^{k/2}) = W^{k,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X)$.

Proof. To identity the complex interpolation spaces recall from Lemma 5.4.2 that $E_{\mathrm{odd}} : W^{k,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X) \to W^{k,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X)$ is an isomorphism for $k \in \{0,1,2\}$. Moreover, from (5.17) we see that Δ_{Dir} commutes with E_{odd}. Therefore, the above isomorphism extends to all $k \in \mathbb{N}_0$.

Therefore, by a standard retraction-coretraction argument (see [235, Theorem 1.2.4] and see Lemma 4.5.3 for explicit estimates), it is sufficient to prove

$$[L^p_{\mathrm{odd}}(\mathbb{R}^d, w; X), W^{k,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X)]_{\frac{j}{k}} = W^{j,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X).$$

Define $R : W^{m,p}(\mathbb{R}^d, w; X) \to W^{m,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X)$ by $Rf(x) = (f(x_1,\tilde{x}) - f(-x_1,\tilde{x}))/2$ and let $S : W^{m,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X) \to W^{m,p}(\mathbb{R}^d, w; X)$ denote the injection. By the symmetry of w, R is bounded. Moreover, RS equals the identity operator, and since by Proposition 5.2.8 and Theorem 5.3.18 we have $[L^p(\mathbb{R}^d, w; X), W^{k,p}(\mathbb{R}^d, w; X)]_{\frac{j}{k}} = W^{j,p}(\mathbb{R}^d, w; X)$, the required identity follows from the retraction-coretraction argument again.

The final assertion is clear for even k. For odd $k = 2\ell + 1$ with $\ell \in \mathbb{N}_0$ by Proposition 5.2.3, Theorem 5.4.1 and the result in the even case we can write

$$\begin{aligned} D((-\Delta_{\mathrm{Dir}})^{k/2}) &= [L^p(\mathbb{R}^d_+, w; X), D((-\Delta_{\mathrm{Dir}})^{\ell})]_{\frac{k}{2\ell}} \\ &= [L^p(\mathbb{R}^d_+, w; X), W^{2\ell,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X)]_{\frac{k}{2\ell}} = W^{k,p}_{(\Delta,\mathrm{Dir})}(\mathbb{R}^d_+, w; X). \end{aligned}$$

□

We can now prove the two main results of this section.

Theorem 5.6.8. *Let X be a UMD space. Let $p \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$. Then*

$$\begin{aligned} D((-\Delta_{\mathrm{Dir}})^{1/2}) &= [L^p(\mathbb{R}^d_+, w_\gamma; X), D(\Delta_{\mathrm{Dir}})]_{\frac{1}{2}} = W^{1,p}(\mathbb{R}^d_+, w_\gamma; X). \\ D((-\Delta^{3/2}_{\mathrm{Dir}})) &= [L^p(\mathbb{R}^d_+, w_\gamma; X), D(\Delta^2_{\mathrm{Dir}})]_{\frac{3}{4}} = \{u \in W^{3,p}(\mathbb{R}^d_+, w_\gamma; X) : tr(u) = 0\}. \end{aligned}$$

Proof. By Theorem 5.4.1 $-\Delta_{\mathrm{Dir}}$ has bounded imaginary powers. Therefore, by Proposition 5.2.3 $D((-\Delta_{\mathrm{Dir}})^{j/k}) = [L^p(\mathbb{R}_+, w_\gamma; X), D(\Delta^k_{\mathrm{Dir}})]_{\frac{j}{k}}$ for all integers $0 \le j \le k$. It remains to identity the complex interpolation spaces. For $d = 1$ we can use Proposition 5.3.16 and the fact that $W^{2,p}_0(\mathbb{R}^d_+, w_\gamma; X) = D(\Delta_{\mathrm{Dir}})$, and $W^{1,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)$ for $\gamma > p-1$. For $d \ge 2$ we can use the $d = 1$ case and standard results about Δ_{d-1} combined with [86, Lemma 9.5] to obtain

$$\begin{aligned} D((-\Delta_{\mathrm{Dir}})^{1/2}) &= D((2-\Delta_{\mathrm{Dir}})^{1/2}) = D((1-\Delta_{\mathrm{Dir},1})^{1/2}) \cap D((1-\Delta_{d-1})^{1/2}) \\ &= W^{1,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{1,p}(\mathbb{R}^{d-1}; X)) \\ &= W^{1,p}(\mathbb{R}^d_+, w_\gamma; X). \end{aligned}$$

To identity $D((-\Delta_{\mathrm{Dir}})^{3/2})$ in the case $\gamma > p-1$ we first consider $d = 1$. By Theorem 5.5.7 and the previous case one has

$$\begin{aligned} D((-\Delta_{\mathrm{Dir}})^{3/2}) &= \{u \in D(\Delta_{\mathrm{Dir}}) : \Delta_{\mathrm{Dir}} u \in D((-\Delta_{\mathrm{Dir}})^{1/2})\} \\ &= \{u \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}_+, w_\gamma; X) : \mathrm{tr}\, u = 0, u'' \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)\} \\ &= \{u \in W^{3,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{tr}(u) = 0\}. \end{aligned}$$

If $d \ge 2$, then

$$\begin{aligned} D((-\Delta_{\mathrm{Dir}})^{3/2}) &= D((1-\Delta_{\mathrm{Dir}})^{3/2}) \\ &= \{u \in L^p(\mathbb{R}^d_+, w_\gamma) : (1-\Delta_{\mathrm{Dir}})u \in D((2-\Delta_{\mathrm{Dir}})^{1/2}), \mathrm{tr}(u) = 0\}\} \end{aligned}$$

$$= \{u \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) : (1-\Delta_{\mathrm{Dir}})u \in W^{1,p}(\mathbb{R}^d_+, \gamma; X)\}\}.$$

Observe that

$$W^{1,p}(\mathbb{R}^d_+, w_\gamma; X) = W^{1,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap L^p(\mathbb{R}_+, w_\gamma; W^{1,p}(\mathbb{R}^{d-1}; X)).$$

Thus by the $d = 1$ case, the boundedness of $\Delta_{\mathrm{Dir},1}(1-\Delta_{\mathrm{Dir}})^{-1}$ and $\Delta_{d-1}(1-\Delta_{\mathrm{Dir}})^{-1}$ (see Corollary 5.5.8), we obtain that for $u \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X)$, we have $(1-\Delta_{\mathrm{Dir}})u \in W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)$ if and only if

$$\begin{aligned} u \in\ & W^{3,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X)) \cap W^{2,p}(\mathbb{R}_+, w_\gamma; W^{1,p}(\mathbb{R}^{d-1}; X)) \cap \\ & \cap L^p(\mathbb{R}_+, w_\gamma; W^{3,p}(\mathbb{R}^{d-1}; X)) \cap W^{1,p}(\mathbb{R}_+, w_\gamma; W^{2^p}(\mathbb{R}^{d-1}; X)) \\ =\ & W^{3,p}(\mathbb{R}^d_+, w_\gamma; X), \end{aligned}$$

with the required norm estimate. Therefore, the required identity for $D((-\Delta_{\mathrm{Dir}})^{3/2})$ follows. □

5.6.4. Localization: the proof of Theorem 5.6.1

As a first step in the localization we prove the following result for Δ_{Dir} on small deformations of half-spaces.

Lemma 5.6.9. *Let X be a UMD space. Let $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. For all $\varphi > 0$ there exists an $\varepsilon > 0$ and $\lambda > 0$ such that if $\mathscr{O}$ is a special C^2_c-domain with $[\mathscr{O}]_{C^1} < \varepsilon$ (see (5.2.1) and (5.2)), then the following assertions hold for Δ_{Dir} on $L^p(\mathscr{O}, w^{\mathscr{O}}; X)$:*

(1) *$\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) \le \varphi$.*

(2) *Δ_{Dir} is a closed and densely defined operator on $L^p(\mathbb{R}^d_+, w^{\mathscr{O}}_\gamma; X)$ for which there is an equivalence of norms in $D(\Delta_{\mathrm{Dir}}) = W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\mathscr{O}}; X)$.*

Proof. Let $\mathscr{O}$ be a special C^2_c-domain with $[\mathscr{O}]_{C^1} < \varepsilon$. Then we can choose $h \in C^2_c(\mathbb{R}^{d-1})$ as in (5.3) with $\|h\|_{C^1_b(\mathbb{R}^{d-1})} \le \varepsilon$.

Let Φ be as in (5.4). Let $\Delta^\Phi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+; X) \to L^2_{\mathrm{loc}}(\mathbb{R}^d_+; X)$ be defined by

$$\Delta^\Phi = \Phi_* \Delta (\Phi^{-1})_*,$$

where Φ is as below (5.3). Let $\Delta^\Phi_{\mathrm{Dir}}$ denote the restriction of Δ^Φ to $D(\Delta^\Phi_{\mathrm{Dir}}) = W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w^{\mathscr{O}}_\gamma; X)$. By the above transformations, it suffices to prove the result for Δ^Φ on $L^p(\mathbb{R}^d_+, w^{\mathscr{O}}_\gamma; X)$. For this we use the perturbation theorem [58, Theorem 3.2].

Without loss of generality we can take $\varepsilon \in (0,1)$. A simple calculation shows that

$$\Delta^\Phi = \Delta + \underbrace{|\nabla h|^2 \partial_1^2 - 2\partial_1(\nabla h \cdot \nabla_{d-1})}_{=:A} \underbrace{-(\Delta h)\partial_1}_{=:B} \tag{5.32}$$

We first apply perturbation theory to obtain a bounded H^∞-calculus for $\Delta_{\mathrm{Dir}} + A$. By the assumption we have

$$\|Au\|_{L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)} \le C\varepsilon \|u\|_{W^{2,p}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)} \le C'\varepsilon \|(1-\Delta)u\|_{L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)},$$

where in the last step we used Corollary 5.5.8. This proves one of the required conditions for the perturbation theorem. In particular, this part is enough to obtain that for any $\varphi > 0$ and for ε small enough $D(\Delta_{\mathrm{Dir}} + A) = D(\Delta_{\mathrm{Dir}})$ and $1 - \Delta_{\mathrm{Dir}} - A$ is sectorial of angle $\le \varphi$ (see [168, Proposition 2.4.2]).

In order to apply [58, Theorem 3.2] it remains to show $AD((1-\Delta_{\mathrm{Dir}})^{1+\alpha}) \subseteq D((1-\Delta_{\mathrm{Dir}})^{\alpha})$ and

$$\|(1-\Delta_{\mathrm{Dir}})^{\alpha} Au\| \le C\|(1-\Delta_{\mathrm{Dir}})^{1+\alpha} u\|, \quad u \in D((1-\Delta_{\mathrm{Dir}})^{1+\alpha}). \tag{5.33}$$

for some $\alpha \in (0,1)$. We will check this for $\alpha = 1/2$. For any $u \in W^{3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)$ we have

$$\|Au\|_{W^{1,p}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)} \le C\|h\|^2_{C^2_b} \|u\|_{W^{3,p}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)}.$$

Therefore, by Proposition 5.6.7 and Theorem 5.6.8, condition (5.33) follows. Here we used the standard fact $D((-\Delta_{\mathrm{Dir}})^{\alpha}) = D((1-\Delta_{\mathrm{Dir}})^{\alpha})$, which is true for any sectorial operator and $\alpha > 0$. We can conclude that for $\varepsilon \in (0,1)$ small enough, $1 - \Delta_{\mathrm{Dir}} - A$ has a bounded H^∞-calculus of angle $\le \varphi$.

To obtain the same result for $\lambda - \Delta_\Phi$ for $\lambda > 0$ large enough it remains to apply a lower order perturbation result (see [149, Proposition 13.1]). For this observe

$$\|Bu\|_{L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)} \le \|h\|_{C^2_b} \|u\|_{W^{1,p}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)} \le C\|h\|_{C^2_b}, \quad u \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X).$$

The required estimate follows since by Proposition 5.6.7 and Theorem 5.6.8,

$$\begin{aligned} W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X) &= [L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X)]_{\frac12} \\ &= [L^p(\mathbb{R}^d_+, w_\gamma^{\mathscr{O}};X), D(1-\Delta_{\mathrm{Dir}} - A)]_{\frac12} = D((1-\Delta_{\mathrm{Dir}} - A)^{1/2}), \end{aligned}$$

where in the last step we applied Proposition 5.2.3.

The two perturbation arguments give $\lambda > 0$ such that $\lambda - \Delta^{\Phi}_{\mathrm{Dir}})$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta^{\Phi}_{\mathrm{Dir}}) \le \phi$. Moreover, there is an equivalence of norms in $D(\Delta^{\Phi}_{\mathrm{Dir}}) = D(\Delta_{\mathrm{Dir}}) = W^{2,p}(\mathbb{R}^d_+, w_\gamma;X)$. The desired results follow. □

The following lemma follows from Proposition 5.6.7 and Theorem 5.6.8 under a change of coordinates according to the C^2-diffeomorphism Φ from (5.4) and a standard retration-coretraction argument using (5.5).

Lemma 5.6.10. *Let X be a UMD space. Let $\mathscr{O}$ be a bounded C^2-domain or a special C^2_c-domain, $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Then*

$$[L^p(\mathscr{O}, w_\gamma^{\mathscr{O}};X), W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}};X)]_{\frac12} = W^{1,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}};X).$$

The next step in the proof of the above theorem is a localization argument. This localization argument is a modification of the one in [58, Section 8] combined with the one in [145, Ch. 8, Sections 4 & 5] and results in the next lemma. On an abstract level the localization argument takes the following form.

Lemma 5.6.11. *Let A be a linear operator on a Banach space X, $\widetilde{A}$ a densely defined closed linear operator on a Banach space Y such that $\widetilde{A} \in \mathcal{H}^\infty$. Assume there exists bounded linear mapping $\mathscr{P} : Y \to X$ and $\mathscr{I} : X \to Y$ such that the following conditions hold:*

(1) $\mathscr{P}\mathscr{I} = I$.

(2) $\mathscr{I}D(A) \subseteq D(\widetilde{A})$ *and* $\mathscr{P}D(\widetilde{A}) \subseteq D(A)$.

(3) $\widetilde{B} := (\mathscr{I}A - \widetilde{A}\mathscr{I})\mathscr{P} : D(\widetilde{A}) \longrightarrow Y$ *and* $\widetilde{C} := \mathscr{I}(A\mathscr{P} - \mathscr{P}\widetilde{A}) : D(\widetilde{A}) \longrightarrow Y$ *both extend to bounded linear operators* $[Y, D(\widetilde{A})]_\theta \longrightarrow Y$ *for some* $\theta \in (0,1)$.

Then A is a closed and densely defined operator and for every $\phi > \omega_{H^\infty}(\widetilde{A})$ there exists $\mu > 0$ such that $A + \mu \in \mathcal{H}^\infty$ with $\omega_{H^\infty}(A+\mu) \leq \phi$.

Proof. Let $\phi > \omega_{H^\infty}(\widetilde{A})$. By a lower order perturbation result (see [149, Proposition 13.1]), there exist $\widetilde{\mu} > 0$ such that $\widetilde{A} + \widetilde{B} + \widetilde{\mu} \in \mathcal{H}^\infty$ with $\omega_{H^\infty}(\widetilde{A} + \widetilde{B} + \widetilde{\mu}) \leq \phi$. From the definition of B one sees

$$\mathscr{I}A = (\widetilde{A} + \widetilde{B})\mathscr{I} \quad \text{on } D(A).$$

Since $\widetilde{A} + \widetilde{B}$ is closed, the injectivity of $\mathscr{I}$ implies that A is closed. Since $\mathscr{P}$ is surjective, we have

$$X = \mathscr{P}Y = \mathscr{P}\overline{D(\widetilde{A})} \subseteq \overline{\mathscr{P}D(\widetilde{A})} \subseteq \overline{D(A)}$$

Therefore, A is densely defined. Now we will transfer the functional calculus properties of $\widetilde{A}+\widetilde{B}$ to A. For this we claim that for μ large enough and $\lambda \in \mathbb{C} \setminus \Sigma_\phi$ we have $\lambda \in \rho(A+\mu)$ and

$$R(\lambda, A+\mu) = \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B} + \mu)\mathscr{I}.$$

This clearly yields that $A + \mu$ has a bounded H^∞-calculus of angle $\leq \phi$.

In order to prove the claim we first show that given $\lambda \in \rho(\widetilde{A} + \widetilde{B})$, for $u \in D(A)$ and $f \in X$ it holds that

$$(\lambda - A)u = f \quad \Longrightarrow \quad u = \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B})\mathscr{I}f. \tag{5.34}$$

Indeed, if $(\lambda - A)u = f$, then since $\mathscr{I}(\lambda - A) = (\lambda - \widetilde{A} - \widetilde{B})\mathscr{I}$ on $D(A)$, we obtain $(\lambda - \widetilde{A} - \widetilde{B})\mathscr{I}u = \mathscr{I}f$ and hence the required identity for u follows. We next prove that if $\mathscr{P}R(\lambda, \widetilde{A}+\widetilde{B})\mathscr{I} : X \longrightarrow D(A)$ is injective, then (5.34) becomes an equivalence. and in this case $\lambda \in \rho(A)$ and

$$R(\lambda, A) = \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B})\mathscr{I} \tag{5.35}$$

To prove the implication $\Longleftarrow$, define $u = \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B})\mathscr{I}f$ and $g = (\lambda - A)u$. Then by the implication $\Longrightarrow$ we find $u = \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B})\mathscr{I}g$ and thus by injectivity $f = g$ as required and additionally (5.35) holds.

Next we prove that there exists $\mu \geq \tilde{\mu} > 0$ with the property that for all $\lambda \in \mathbb{C} \setminus \Sigma_\phi$, $\mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B} + \mu)\mathscr{I}$ is injective. Let $f \in X$ be such that $\mathscr{P}\tilde{u} := \mathscr{P}R(\lambda, \widetilde{A} + \widetilde{B} + \mu)\mathscr{I} f = 0$. Observing that $\widetilde{B} = \widetilde{B}\mathscr{I}\mathscr{P}$, we get $\widetilde{B}\tilde{u} = 0$. So $(\widetilde{A} + \mu - \lambda)\tilde{u} = \mathscr{I} f - \widetilde{B}\tilde{u} = \mathscr{I} f$, or equivalently, $\tilde{u} = R(\lambda, \widetilde{A} + \mu)\mathscr{I} f$. It follows that

$$\begin{aligned} 0 &= \mathscr{I}(A + \mu - \lambda)\mathscr{P}\tilde{u} = (\mathscr{I}\mathscr{P}(\widetilde{A} + \mu - \lambda) + \widetilde{C})R(\lambda, \widetilde{A} + \mu)\mathscr{I} f \\ &= \mathscr{I} f + \widetilde{C}R(\lambda, \widetilde{A} + \mu)\mathscr{I} f. \end{aligned}$$

Estimating

$$\begin{aligned} \|\widetilde{C}R(\lambda, \widetilde{A} + \mu)\mathscr{I} f\|_Y &\lesssim \|R(\lambda, \widetilde{A} + \mu)\mathscr{I} f\|_{[Y, D(\widetilde{A})]_\theta} \\ &\leq \|R(\lambda, \widetilde{A} + \mu)\mathscr{I} f\|_Y^{1-\theta} \|R(\lambda, \widetilde{A} + \mu)\mathscr{I} f\|_{D(\widetilde{A})}^\theta \\ &\lesssim |\lambda - \mu|^{\theta - 1}\|\mathscr{I} f\|_Y \lesssim_\phi |\mu|^{\theta-1}\|\mathscr{I} f\|_Y, \end{aligned}$$

we see that $\widetilde{C}R(\lambda, \widetilde{A} + \mu)\mathscr{I}$ is a contraction from X to Y when μ is sufficiently large, in which case $\mathscr{I} f = -\widetilde{C}R(\lambda, \widetilde{A} + \mu)\mathscr{I} f$ implies that $\mathscr{I} f = 0$ and hence $f = 0$. This yields the required injectivity. □

Proof of Theorem 5.6.1. In this proof we let $A = \Delta_{\mathrm{Dir}}$ on $\mathscr{O}$. Let $\varepsilon > 0$ be as in Lemma 5.6.9. Choose a finite open cover $\{V_n\}_{n=1}^N$ of $\partial\mathscr{O}$ together with special C_c^2-domains $\{\mathscr{O}_n\}_{n=1}^N$ such that

$$\mathscr{O} \cap V_n = \mathscr{O}_n \cap V_n \quad \text{and} \quad \partial\mathscr{O} \cap V_n = \partial\mathscr{O}_n \cap V_n, \qquad n = 1, \ldots, N,$$

and $[\mathscr{O}_n]_{C^2} \leq \varepsilon$ for $n = 1, \ldots, N$. Let $\{\eta_n\}_{n=1}^N \subset C_c^\infty(\mathbb{R}^d)$, Y, $\mathscr{P}$ and $\mathscr{I}$ be the objects associated to the above sets as in Subsection 5.2.2. Define the linear operator $\widetilde{A} : D(\widetilde{A}) \subset Y \longrightarrow Y$ as the direct sum $\widetilde{A} := \bigoplus_{n=0}^N \widetilde{A}_n$, where $\widetilde{A}_0 : D(\widetilde{A}_0) \subset L^p(\mathbb{R}^d; X) \longrightarrow L^p(\mathbb{R}^d; X)$ is defined by

$$D(\widetilde{A}_0) := W^{2,p}(\mathbb{R}^d) \quad \text{and} \quad \widetilde{A}_0 u := \Delta u$$

and where, for each $n \in \{1, \ldots, N\}$, $\widetilde{A}_n : D(\widetilde{A}_n) \subset L^p(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X) \longrightarrow L^p(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X)$ is defined by

$$D(\widetilde{A}_n) := W^{2,p}_{\mathrm{Dir}}(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X) \quad \text{and} \quad \widetilde{A}_n u := \Delta u.$$

Furthermore, we define $B : D(A) \longrightarrow Y$ by $Bu := ([\Delta, \eta_n]u)_{n=0}^N$ and $C : D(\widetilde{A}) \longrightarrow X$ by $C\tilde{u} := \sum_{n=0}^N [\Delta, \eta_n]\tilde{u}$.

By Lemma 5.6.9, there exists $\mu > 0$ such that $\mu - \widetilde{A}_n \in \mathscr{H}^\infty$ with $\omega_{H^\infty}(\mu - \widetilde{A}_n) \leq \phi$ for $n = 1, \ldots, N$. Since $-A_0 \in \mathscr{H}^\infty$ with $\omega_{H^\infty}(-\widetilde{A}_0) = 0$, it follows that $\widetilde{A} - \mu \in \mathscr{H}^\infty$ with $\omega_{H^\infty}(\widetilde{A} - \mu) \leq \phi$ (see [149, Example 10.2]). Moreover, by a combination of Lemmas 5.6.9 and 5.6.10, $[L^p(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X), D(\widetilde{A}_n)]_{\frac{1}{2}} = W^{1,p}_{\mathrm{Dir}}(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X)$. Since $[L^p(\mathbb{R}^{d;X}), D(\widetilde{A}_0)]_{\frac{1}{2}} = W^{1,p}(\mathbb{R}^d; X)$ by [126, Theorems 5.6.9 and 5.6.11], it follows that

$$[Y, D(\widetilde{A})]_{\frac{1}{2}} = [L^p(\mathbb{R}^d; X), D(\widetilde{A}_0)]_{\frac{1}{2}} \oplus \bigoplus_{n=1}^N [L^p(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n}; X), D(\widetilde{A}_n)]_{\frac{1}{2}}$$

$$= W^{1,p}(\mathbb{R}^d;X) \oplus \bigoplus_{n=1}^{N} W^{1,p}_{\mathrm{Dir}}(\mathscr{O}_n, w_\gamma^{\mathscr{O}_n};X). \tag{5.36}$$

Note that $\mathscr{I}$ maps $D(A)$ into $D(\widetilde{A})$ and that $\mathscr{I}Au = \widetilde{A}\mathscr{I}u + Bu$ for every $u \in D(A)$. Also note that $\mathscr{P}$ maps $D(\widetilde{A})$ to $D(A)$ and that $A\mathscr{P}\tilde{u} = \mathscr{P}\widetilde{A}\tilde{u} + C\tilde{u}$ for every $\tilde{u} \in D(\widetilde{A})$. Since each commutator $[\Delta, \eta_n]$ is a first order partial differential operator with C_c^∞-coefficients, it follows that $\mathscr{I}A - \widetilde{A}\mathscr{I}$ extends to a bounded linear operator from $W^{1,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}};X)$ to Y. Since $\mathscr{P}$ is a bounded linear operator from $[Y, D(\widetilde{A})]_{\frac{1}{2}}$ to $W^{1,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\mathscr{O}};X)$ in view of (5.36), it follows that $(\mathscr{I}A - \widetilde{A}\mathscr{I})\mathscr{P}$ extends to a bounded linear operator from $[Y, D(\widetilde{A})]_{\frac{1}{2}}$ to Y. Similarly we see that $\mathscr{I}(A\mathscr{P} - \mathscr{P}\widetilde{A})$ extends to a bounded linear operator from $[Y, D(\widetilde{A})]_{\frac{1}{2}}$ to Y. An application of Lemma 5.6.11 finishes the proof. □

5.7. THE HEAT EQUATION WITH INHOMOGENEOUS BOUNDARY CONDITIONS

In this section we will consider the heat equation on a smooth domain $\mathscr{O} \subseteq \mathbb{R}^d$ with inhomogeneous boundary conditions of Dirichlet type. In particular, Theorem 5.1.2 is a special case of Theorem 5.7.16 below. The main novelty is that we consider weights of the form $w_\gamma^{\mathscr{O}}(x) = \mathrm{dist}(x, \partial\mathscr{O})^\gamma$ with $\gamma \in (p-1, 2p-1)$, which allows us to treat the heat equation with very rough boundary data.

5.7.1. Identification of the spatial trace space

We begin with an extension of a trace result from [159] to the range $\gamma \in (p-1, 2p-1)$.

Theorem 5.7.1 (Spatial trace space). *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^k-domain. Let X be a UMD space, $\ell \in \mathbb{N}_1$, $k \in \mathbb{N}_2$, $p, q \in (1,\infty)$, $v \in A_q(\mathbb{R})$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Put $w_\gamma = w_\gamma^{\mathscr{O}}$. Then $\mathrm{tr}_{\mathscr{O}}$ is a retraction from*

$$W^{\ell,q}(\mathbb{R}, v; L^p(\mathscr{O}, w_\gamma;X)) \cap L^q(\mathbb{R}, v; W^{k,p}(\mathscr{O}, w_\gamma;X))$$

to

$$F^{\ell - \frac{\ell}{k}\frac{1+\gamma}{p}}_{p,q}(\mathbb{R}, v; L^p(\partial\mathscr{O};X)) \cap L^q(\mathbb{R}, v; B^{k-\frac{1+\gamma}{p}}_{p,p}(\partial\mathscr{O};X)).$$

In order to prove this we need a preliminary result. On the compact C^2-boundary $\partial\mathscr{O}$, we define the Besov spaces $B^s_{p,q}(\partial\mathscr{O};X)$, $p \in (1,\infty)$, $q \in [1,\infty]$ $s \in (0,2) \setminus \{1\}$, by real interpolation:

$$B^s_{p,q}(\partial\mathscr{O};X) := (W^{n,p}(\partial\mathscr{O};X), W^{n+1,p}(\partial\mathscr{O};X))_{\theta,q}, \quad s = \theta + n, \theta \in (0,1), n \in \{0,1\}.$$

In the proof of this theorem we use weighted mixed-norm anisotropic Triebel-Lizorkin spaces as considered in [159, Section 2.4]/Section 6.3.4 (see [156] for more details); for definitions and notations we simply refer the reader to these references.

As in the standard isotropic case (see [163]), we have:

Lemma 5.7.2. *Let $\ell, k \in \mathbb{N}_1$, $p, q \in (1,\infty)$, $v \in A_q(\mathbb{R})$ and $\gamma \in (-1,\infty)$. Then*

$$F^{1,(\frac{1}{k},\frac{1}{\ell})}_{(p,q),1,(d,1)}(\mathbb{R}^d_+ \times \mathbb{R},(w_\gamma,v);X) \hookrightarrow W^{l,q}(\mathbb{R},v;L^p(\mathbb{R}^d_+,w_\gamma;X)) \cap L^q(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)).$$

Proof. We cannot reduce to the $\mathbb{R}^d$-case directly since $L^p(\mathbb{R}^d,w_\gamma;X) \not\hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^d;X)$ for $\gamma \geq p-1$ and therefore cannot be seen as a subspace of the distributions on $\mathbb{R}^d$. However, we can proceed as follows. An easy direct argument (see [182, Remark 3.13] or [156, Proposition 5.2.31]) shows that

$$\|f\|_{L^q(\mathbb{R},v;L^p(\mathbb{R}^d,w_\gamma;X))} \lesssim \|f\|_{F^{0,(\frac{1}{k},\frac{1}{\ell})}_{(p,q),1,(d,1)}(\mathbb{R}^d\times\mathbb{R},(w_\gamma,v);X)}$$

for all $f \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R};X)$. Using

$$L^q(\mathbb{R},v;L^p(\mathbb{R}^d_+,w_\gamma;X)) \hookrightarrow \mathscr{D}'(\mathbb{R}^d_+ \times \mathbb{R};X)$$

and using density of $\mathscr{S}(\mathbb{R}^d \times \mathbb{R};X)$ in $F^{0,(\frac{1}{k},\frac{1}{\ell})}_{(p,q),1,(d,1)}(\mathbb{R}^d \times \mathbb{R},(w_\gamma,v)$, we find that the restriction operator

$$R : \mathscr{D}'(\mathbb{R}^d \times \mathbb{R};X) \to \mathscr{D}'(\mathbb{R}^d_+ \times \mathbb{R};X),\ f \mapsto f_{|\mathbb{R}^d_+\times\mathbb{R}},$$

restricts to a bounded linear operator

$$R : F^{0,(\frac{1}{k},\frac{1}{\ell})}_{(p,q),1,(d,1)}(\mathbb{R}^d \times \mathbb{R},(w_\gamma,v);X) \longrightarrow L^q(\mathbb{R},v;L^p(\mathbb{R}^d_+,w_\gamma;X)).$$

By [159, Section 2.4] (see [156, Proposition 5.2.29]), this implies that R is also bounded as an operator

$$R : F^{1,(\frac{1}{k},\frac{1}{\ell})}_{(p,q),1,(d,1)}(\mathbb{R}^d \times \mathbb{R},(w_\gamma,v);X) \longrightarrow W^{l,q}(\mathbb{R},v;L^p(\mathbb{R}^d_+,w_\gamma;X)) \cap L^q(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)).$$

The desired inclusion now follows. □

With a similar argument as in the above proof one can show the following embedding for an arbitrary open set $\mathscr{O} \subseteq \mathbb{R}^d$:

$$B^k_{p,1}(\mathscr{O},w_\gamma;X) \hookrightarrow F^k_{p,1}(\mathscr{O},w_\gamma;X) \hookrightarrow W^{k,p}(\mathscr{O},w_\gamma;X), \tag{5.37}$$

where for $k \in \mathbb{N}_0$, $\gamma > -1$ and $p \in [1,\infty)$.

In the proof of Theorem 5.7.1 we will furthermore use the following Sobolev embedding, which is a partial extension of Corollary 5.3.4 to the case $k = 0$, obtained by dualizing Corollary 5.3.4.

Proposition 5.7.3. *Let X be a UMD space, $p \in (1,\infty)$ and $\gamma \in (p-1,2p-1)$. Let $\mathscr{O}$ be a bounded C^1-domain or a special C^1_c-domain. Then*

$$L^p(\mathscr{O},w_\gamma;X) \hookrightarrow H^{-1,p}(\mathscr{O},w_{\gamma-p};X).$$

To prove this embedding we need a simple lemma.

Lemma 5.7.4. *Let X be a UMD space, $p \in (1,\infty)$ and let $w \in A_p$ be even. Let $\mathscr{O} \subseteq \mathbb{R}^d$ be a bounded C^1-domain or a special C^1_c-domain. Then $H^{-1,p}(\mathscr{O}, w; X)$ is reflexive and*

$$\mathscr{D}(\mathscr{O}; X) \overset{d}{\hookrightarrow} H^{-1,p}(\mathscr{O}, w; X) \overset{d}{\hookrightarrow} \mathscr{D}'(\mathscr{O}; X), \tag{5.38}$$

Under the natural pairing, we have

$$\mathscr{D}(\mathscr{O}; X^*) \overset{d}{\hookrightarrow} [H^{-1,p}(\mathscr{O}, w; X)]^* \overset{d}{\hookrightarrow} \mathscr{D}'(\mathscr{O}; X^*), \tag{5.39}$$

$$[H^{-1,p}(\mathscr{O}, w; X)]^* = W^{1,p'}_{\mathrm{Dir}}(\mathscr{O}, w'; X^*). \tag{5.40}$$

Proof. The reflexivity of $H^{-1,p}(\mathscr{O}, w; X)$ follows from Proposition 5.2.8. The continuity of the inclusions in (5.38) are obvious. The density in the first embedding of (5.38) holds because of Lemma 5.3.5 and $L^p(\mathscr{O}, w; X) \overset{d}{\hookrightarrow} H^{-1,p}(\mathscr{O}, w; X)$. The density of the second embedding in (5.38) follows from the density of $\mathscr{D}(\mathscr{O}; X)$ in $\mathscr{D}'(\mathscr{O}; X)$. The dense embeddings (5.39) follow from (5.38), $\mathscr{D}(\mathscr{O}; X)^* = \mathscr{D}'(\mathscr{O}; X^*)$ and $\mathscr{D}'(\mathscr{O}; X)^* = \mathscr{D}(\mathscr{O}; X^*)$ and the reflexivity of $H^{-1,p}(\mathscr{O}, w; X)$. To prove (5.40), by density (see Lemma 5.3.11) it suffices to prove

$$\|f\|_{[H^{-1,p}(\mathscr{O},w;X)]^*} \eqsim \|f\|_{W^{1,p'}(\mathbb{R}^d,w';X^*)}, \quad f \in \mathscr{D}(\mathscr{O}; X^*). \tag{5.41}$$

Let $f \in \mathscr{D}(\mathscr{O}; X^*)$. Then, by Proposition 5.2.8, for all $g \in \mathscr{D}(\mathscr{O}; X)$,

$$|\langle f, g\rangle| \le \|f\|_{W^{1,p'}(\mathbb{R}^d,w';X^*)} \|g\|_{H^{-1,p}(\mathbb{R}^d,w;X)}.$$

Taking the infimum over all such g and using (5.38), the estimate $\lesssim$ in (5.41) follows. For the converse we use Proposition 5.2.8 to obtain

$$\|f\|_{W^{1,p'}(\mathbb{R}^d,w';X^*)} \eqsim \|f\|_{H^{1,p'}(\mathbb{R}^d,w';X^*)} = \|f\|_{[H^{-1,p}(\mathbb{R}^d,w;X)]^*}.$$

For an appropriate $g \in H^{-1,p}(\mathbb{R}^d, w; X)$ of norm ≤ 1 we obtain

$$\|f\|_{W^{1,p'}(\mathbb{R}^d,w';X^*)} \lesssim |\langle f, g\rangle| = |\langle f, g_{|\mathbb{R}_+}\rangle| \le \|f\|_{[H^{-1,p}(\mathscr{O},w;X)]^*},$$

where we used $\|g_{|\mathbb{R}_+}\|_{H^{-1,p}(\mathscr{O},w;X)} \le 1$. ☐

Proof of Proposition 5.7.3. Let us first note that X is reflexive as a UMD space. Put $\gamma' := -\frac{\gamma}{p-1} \in (-p'-1, -1)$. Then $[w_\gamma]' = w_{\gamma'}$ and $[w_{\gamma-p}]' = w_{\gamma'+p'}$, the p-duals weights of w_γ and $w_{\gamma-p}$, respectively. Note that $\gamma - p \in (-1, p-1)$ and $\gamma' + p' \in (-1, p'-1)$, so $w_{\gamma-p} \in A_p$ and $w_{\gamma'+p'} \in A_{p'}$. By Corollary 5.3.4 and Lemma 5.3.5,

$$W^{1,p'}_{\mathrm{Dir}}(\mathscr{O}, w_{\gamma'+p'}; X^*) \overset{d}{\hookrightarrow} L^{p'}(\mathscr{O}, w_{\gamma'}; X^*).$$

Therefore, Proposition 5.2.8s and Lemma 5.7.4 give that

$$\begin{aligned} L^p(\mathscr{O}, w_\gamma; X) = [L^{p'}(\mathscr{O}, w_{\gamma'}; X^*)]^* &\hookrightarrow [W^{1,p'}_{\mathrm{Dir}}(\mathscr{O}, w_{\gamma'+p'}; X^*)]^* \\ &= [H^{-1,p}(\mathscr{O}, w_{\gamma-p}; X)]^{**} = H^{-1,p}(\mathscr{O}, w_{\gamma-p}; X), \end{aligned}$$

where we again used reflexivity of X. ☐

Proof of Theorem 5.7.1. By a standard localization argument it suffices to consider the case $\mathscr{O} = \mathbb{R}^d_+$. The case $\gamma \in (-1, p-1)$ is already considered in [159, Theorem 2.1 & Corollary 4.9] (also see [159, Theorem 4.4]), so from now on we will assume $\gamma \in (p-1, 2p-1)$.

Let us write

$$\mathbb{M} := W^{\ell,q}(\mathbb{R}, v; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_\gamma; X))$$

and

$$\mathbb{B} := F_{p,q}^{\ell - \frac{\ell}{k}\frac{1+\gamma}{p}}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k-\frac{1+\gamma}{p}}(\mathbb{R}^{d-1}; X)).$$

By Theorem 5.3.18, Proposition 5.7.3, Corollary 5.3.4 and Propositions 4.5.5 and 6.3.7,

$$\begin{aligned}
\mathbb{M} &\hookrightarrow H^{\ell(1-\frac{1}{k}),q}(\mathbb{R}; v; [L^p(\mathbb{R}^d_+, w_\gamma; X), W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{1}{k}}) \\
&\hookrightarrow H^{\ell(1-\frac{1}{k}),q}(\mathbb{R}; v; [H^{-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X), W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)]_{\frac{1}{k}}) \\
&= H^{\ell(1-\frac{1}{k}),q}(\mathbb{R}; v; L^p(\mathbb{R}^d_+, w_{\gamma-p}; X)).
\end{aligned}$$

Therefore, once applying Corollary 5.3.4,

$$\mathbb{M} \hookrightarrow H^{\ell(1-\frac{1}{k}),q}(\mathbb{R}; v; L^p(\mathbb{R}^d_+, w_{\gamma-p}; X)) \cap L^q(\mathbb{R}, v; W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)), \tag{5.42}$$

which reduces the problem to the A_p-weight setting. By [159, Theorem 2.1 & Corollary 4.9] (also see [159, Theorem 4.4]), $\mathrm{tr}_{\partial\mathbb{R}^d_+}$ is bounded from the last space to

$$F_{p,q}^{\ell(1-\frac{1}{k}) - \frac{\ell(1-\frac{1}{k})}{k-1}\frac{1+\gamma-p}{p}}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k-1-\frac{1+\gamma-p}{p}}(\mathbb{R}^{d-1}; X)) = \mathbb{B}.$$

Finally, that there is a coretraction $\mathrm{ext}_{\partial\mathbb{R}^d_+}$ corresponding to $\mathrm{tr}_{\partial\mathbb{R}^d_+}$ simply follows from a combination of [159, Theorems 2.1 & 4.6 & Remark 4.7] and Lemma 5.7.2. □

5.7.2. Identification of the temporal trace space

For $p \in (1,\infty)$, $q \in [1,\infty]$, $\gamma \in (-1, 2p-1)$ and $s \in (0,2)$ we use the following notation:

$$W^s_{p,q}(\mathscr{O}, w_\gamma; X) := (L^p(\mathscr{O}, w_\gamma; X), W^{2,p}(\mathscr{O}, w_\gamma; X))_{\frac{s}{2},q}. \tag{5.43}$$

In the case $\gamma \in (-1, p-1)$ (with general A_p-weight) these spaces can be identified with Besov spaces (see [182, Proposition 6.1]). In the case $\gamma \in (p-1, 2p-1)$ we only have embedding result (see Lemma 5.7.9 below).

In the next result we identity the temporal trace space.

Theorem 5.7.5 (Temporal trace space)**.** *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain and let J be either $\mathbb{R}$ or $(0,T)$ with $T \in (0,\infty]$. Let X be a UMD space, $p, q \in (1,\infty)$, $\mu \in (-1, q-1)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. If $1 - \frac{1+\mu}{q} \neq \frac{1}{2}\frac{1+\gamma}{p}$, then the temporal trace operator $\mathrm{tr}_{t=0}$: $u \mapsto u(0)$ is a retraction*

$$W^{1,q}(J, v_\mu; L^p(\mathscr{O}, w_\gamma; X)) \cap L^q(J, v_\mu; W^{2,p}(\mathscr{O}, w_\gamma; X)) \longrightarrow W_{p,q}^{2(1-\frac{1+\mu}{q})}(\mathscr{O}, w_\gamma; X). \tag{5.44}$$

It follows from the trace method (see [169, Section 1.2] or [235, Section 1.8]) that $\mathrm{tr}_{t=0}$ is a quotient mapping (5.44). The nontrivial fact in the above theorem is to show that it is a retraction. In order to show this we want to apply [186, Theorem 1.1]/[197, Theorem 3.4.8]. However, these results can only be applied directly in the special case that the boundary condition vanishes in the real interpolation space. In the case $\gamma \in (-1, p-1)$ this difficulty does not arise because by using a suitable extension operator one can reduce to the case $\mathscr{O} = \mathbb{R}^d$. To cover the remaining cases we have found a workaround which requires some preparations. The first result is the characterization of the spatial trace of the spaces defined in (5.43). The result will be proved further below.

Proposition 5.7.6. *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain. Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2)$. If $s > \frac{1+\gamma}{p}$, then $\mathrm{tr}_{\partial\mathscr{O}}$ extends to a retraction*

$$W^s_{p,q}(\mathscr{O}, w_\gamma; X) \longrightarrow B^{s-\frac{1+\gamma}{p}}_{p,q}(\partial\mathscr{O}).$$

In the setting of the above proposition we define, for $s \in (0,2) \setminus \{\frac{1+\gamma}{p}\}$,

$$W^s_{p,q,\mathrm{Dir}}(\mathscr{O}, w_\gamma; X) := \begin{cases} W^s_{p,q}(\mathscr{O}, w_\gamma; X), & s < \frac{1+\gamma}{p}, \\ \{u \in W^s_{p,q}(\mathscr{O}, w_\gamma; X) : \mathrm{tr}_{\partial\mathscr{O}} u = 0\}, & s > \frac{1+\gamma}{p}. \end{cases}$$

For these spaces we have the following result which will be proved below as well.

Proposition 5.7.7. *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain. Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2) \setminus \{\frac{1+\gamma}{p}\}$. Then*

$$W^s_{p,q,\mathrm{Dir}}(\mathscr{O}, w_\gamma; X) = (L^p(\mathscr{O}, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma; X))_{\frac{s}{2},q}.$$

From Proposition 5.3.17 and reiteration (see [235, Theorem 1.10.2]) we immediately obtain the following.

Lemma 5.7.8. *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain. Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty]$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2) \setminus \{1\}$. If $s = \theta + n$ with $\theta \in (0,1)$ and $n \in \{0,1\}$, then*

$$W^s_{p,q}(\mathscr{O}, w_\gamma; X) = (W^{n,p}(\mathscr{O}, w_\gamma; X), W^{n+1,p}(\mathscr{O}, w_\gamma; X))_{\theta,q}.$$

Lemma 5.7.9. *Let $\mathscr{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain. Let $p \in (1,\infty)$, $q \in [1,\infty]$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2)$. Then*

$$B^s_{p,q}(\mathscr{O}, w_\gamma; X) \hookrightarrow W^s_{p,q}(\mathscr{O}, w_\gamma; X). \tag{5.45}$$

The inclusion is dense if $q < \infty$.

For γ in the A_p-range $(-1, p-1)$ it holds that $B^s_{p,q}(\mathscr{O}, w_\gamma; X) = W^s_{p,q}(\mathscr{O}, w_\gamma; X)$ (which can be obtained from [182, Proposition 6.1]). However, the reverse inclusion to (5.45) does not hold for $\gamma \in (p-1, 2p-1)$, see Remark 5.7.14 below.

Proof. By [38, Theorem 3.5] and a retraction-coretraction argument using Rychkov's extension operator (see [163, 216]),

$$B^s_{p,q}(\mathcal{O}, w_\gamma; X) = (F^0_{p,1}(\mathcal{O}, w_\gamma; X), F^2_{p,1}(\mathcal{O}, w_\gamma; X))_{\frac{s}{2},q};$$

these references are actually in the scalar-valued setting, but the arguments remain valid in the vector-valued setting. The inclusion now follows from (5.37). Density follows from Lemma 5.3.7, [235, Theorem 1.6.2] and the fact that $C^\infty_c(\overline{\mathcal{O}}; X) \subset B^s_{p,q}(\mathcal{O}, w_\gamma; X)$. □

Lemma 5.7.10. *Let $\mathcal{O}$ be either $\mathbb{R}^d_+$ or a bounded C^2-domain. Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty]$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2)$. Then*

$$W^s_{p,q}(\mathcal{O}, w_\gamma; X) \hookrightarrow B^{s-1}_{p,q}(\mathcal{O}, w_{\gamma-p}; X).$$

Proof. By Corollary 5.3.4 and Proposition 5.7.3

$$\begin{aligned} W^s_{p,q}(\mathcal{O}, w_\gamma; X) &= ((L^p(\mathcal{O}, w_\gamma); X), W^{2,p}(\mathcal{O}, w_\gamma; X))_{\frac{s}{2},q} \\ &\hookrightarrow ((H^{-1,p}(\mathcal{O}, w_{\gamma-p}); X), W^{1,p}(\mathcal{O}, w_{\gamma-p}; X))_{\frac{s}{2},q} \\ &= B^{s-1}_{p,q}(\mathcal{O}, w_{\gamma-p}; X), \end{aligned}$$

where the last identity follows from [182, Proposition 6.1] and a retraction-coretraction argument. □

Before we proceed, we recall some trace theory for weighted B-spaces, for which we refer to [165]. Let $s \in \mathbb{R}$, $p \in (1,\infty)$, $q \in [1,\infty)$ and $\gamma \in (-1, p-1)$. If $s = \frac{1+\gamma}{p} + k + \theta$ with $k \in \mathbb{N}_0$ and $\theta \in (0,1)$, then $\mathrm{tr}_k : u \mapsto (\mathrm{tr}\, u, \dots, \mathrm{tr}\,\partial_1^k u)$ is well-defined on $B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$. For such s we put

$$B^s_{p,q,0}(\mathbb{R}^d_+, w_\gamma; X) := \{u \in B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{tr}_k u = 0\}.$$

For $s < \frac{1+\gamma}{p}$ we put $B^s_{p,q,0}(\mathbb{R}^d_+, w_\gamma; X) := B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$.

The following result follows from [165].

Lemma 5.7.11. *Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty]$ and $\gamma \in (-1, p-1)$. Let $k \in \mathbb{N}$ and $\theta \in (0,1)$ be such that $k\theta \notin \mathbb{N}_0 + \frac{1+\gamma}{p}$. Then*

$$B^{k\theta}_{p,q,0}(\mathbb{R}^d_+, w_\gamma; X) = (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X))_{\theta,q}.$$

Let ${}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ be defined as the closure of $\{u \in C^\infty_c(\overline{\mathbb{R}^d_+}; X) : u_{|\partial\mathbb{R}^d_+} = 0\}$ in $W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$. The following identities hold for the real interpolation spaces.

Lemma 5.7.12. *Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $s \in (0,2) \setminus \{\frac{1+\gamma}{p}\}$. Then*

$${}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) = (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{s}{2},q}$$

and the map M, defined above Lemma 5.3.13, is an isomorphism

$$M : {}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow B^s_{p,q,0}(\mathbb{R}^d_+, w_{\gamma-p}; X).$$

As a consequence of Lemmas 5.3.9 and 5.7.12 for $p \in (1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $s \in (0, \frac{\gamma+1}{p})$ we have

$$ {}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) = W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \tag{5.46} $$

Proof. We first show that

$$ (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{s}{2},q} \hookrightarrow {}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X). \tag{5.47} $$

By Proposition 5.3.8, $C^\infty_c(\mathbb{R}^d_+; X) \overset{d}{\subset} W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X)$. Therefore, $C^\infty_c(\mathbb{R}^d_+; X)$ is dense in $(L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{s}{2},q}$ (see [235, Theorem 1.6.2]). As

$$ (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{s}{2},q} \hookrightarrow W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) $$

clearly holds, (5.47) follows.

Next we show that M is a bounded operator

$$ M : {}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow B^s_{p,q,0}(\mathbb{R}^d_+, w_{\gamma-p}; X). \tag{5.48} $$

Lemma 5.3.13 and real interpolation yield that M is a bounded operator

$$ M : W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow B^s_{p,q}(\mathbb{R}^d_+, w_{\gamma-p}; X). $$

Since

$$ \begin{aligned} M\{u \in C^\infty_c(\overline{\mathbb{R}^d_+}; X) : u_{|\partial\mathbb{R}^d_+} = 0\} &\subset \{u \in C^\infty_c(\overline{\mathbb{R}^d_+}; X) : u_{|\partial\mathbb{R}^d_+} = (\partial_1 u)_{|\partial\mathbb{R}^d_+} = 0\} \\ &\subset B^s_{p,q,0}(\mathbb{R}^d_+, w_{\gamma-p}; X), \end{aligned} $$

(5.48) follows.

From a combination of Lemma 5.3.13 and Lemma 5.7.11 it follows that M^{-1} is a bounded operator

$$ M^{-1} : B^s_{p,q,0}(\mathbb{R}^d_+, w_{\gamma-p}; X) \longrightarrow (L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X))_{\frac{s}{2},q}. $$

Combining this with (5.47) and (5.48) finishes the proof. □

Lemma 5.7.13. *Let X be a UMD space, $p \in (1,\infty)$, $q \in [1,\infty)$, $\gamma \in (p-1, 2p-1)$ and $s \in (\frac{1+\gamma}{p}, 2)$. Then $tr_{\partial\mathbb{R}^d_+}$ extends to a retraction*

$$ W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow B^{s-\frac{1+\gamma}{p}}_{p,q}(\mathbb{R}^{d-1}; X) $$

with

$$ {}_0W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) = \{u \in W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) : tr_{\partial\mathbb{R}^d_+} u = 0\}. \tag{5.49} $$

Moreover, there exists a coretraction E corresponding to $tr_{\partial\mathbb{R}^d_+}$ such that

$$ \|u\|_{W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)} \eqsim \|u\|_{B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)}, \qquad u \in \ker(I - E \circ tr_{\partial\mathbb{R}^d_+}). \tag{5.50} $$

Proof. By trace theory of weighted B-spaces (see [165] and see [159, Section 4.1] for the anistropic setting) and Lemmas 5.7.9 and 5.7.10, there is the commutative diagram

$$\begin{array}{ccccc} B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) & \overset{d}{\hookrightarrow} & W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) & \hookrightarrow & B^{s-1}_{p,q}(\mathbb{R}^d_+, w_{\gamma-p}; X) \\ & \nwarrow^{E} & & & \downarrow \mathrm{tr}_{\partial\mathbb{R}^d_+} \\ & & B^{s-\frac{1+\gamma}{p}}_{p,q}(\mathbb{R}^{d-1}; X) & = & B^{s-1-\frac{1+\gamma-p}{p}}_{p,q}(\mathbb{R}^{d-1}; X) \end{array}$$

for some extension operator E. All statements different from (5.49) directly follow from this. Next we claim that

$$\{u \in B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{tr}_{\partial\mathbb{R}^d_+} u = 0\} \overset{d}{\hookrightarrow} \{u \in W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{tr}_{\partial\mathbb{R}^d_+} u = 0\}.$$

Indeed, if $u \in W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ satisfies $\mathrm{tr}_{\partial\mathbb{R}^d_+} u = 0$, then we can find $u_n \in B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ such that $u_n \to u$ in $W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$. Now it remains to set $v_n = u_n - E\mathrm{tr}_{\partial\mathbb{R}^d_+} u_n \in B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ and observe that $\mathrm{tr}_{\partial\mathbb{R}^d_+} v_n = 0$ and $E\mathrm{tr}_{\partial\mathbb{R}^d_+} u_n \to 0$ in $W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$.

Since $\{u \in C^\infty_c(\overline{\mathbb{R}^d_+}; X) : u_{\partial\mathbb{R}^d_+} = 0\}$ is dense in the space on the left hand side by [165], it follows from the claim that it is also dense in the space on the right hand side. This density implies (5.49). □

Proof of Proposition 5.7.6. The statement simply follows from Lemma 5.7.13 by a standard localization argument. □

Proof of Proposition 5.7.7. A combination of (5.46) and Lemma 5.7.13 gives the desired statement for the case $\mathscr{O} = \mathbb{R}^d_+$, from the general case follows by a standard localization argument. □

Proof of Theorem 5.7.5. Let us first establish the asserted boundedness of $\mathrm{tr}_{t=0}$. It suffices to consider the case $J = \mathbb{R}_+$, where the boundedness statement follows from [169, Proposition 1.2.2] or [235, Section 1.8].

In order to show that there is a coretraction corresponding to $\mathrm{tr}_{t=0}$, it suffices to consider the case $\mathscr{O} = \mathbb{R}^d_+$ and $J = \mathbb{R}$. The case $\gamma \in (-1, p-1)$ follows from [159, Equation (38)], and therefore it remains to consider $\gamma \in (p-1, 2p-1)$.

Let $\delta = 2(1 - \frac{1+\mu}{q})$. In view of Theorem 5.5.7, we can apply [186, Theorem 1.1] or [197, Theorem 3.4.8] to $-\Delta_{\mathrm{Dir}}$ on $L^p(\mathbb{R}^d_+, w_\gamma; X)$, which by Proposition 5.7.7 gives an extension operator

$$\begin{aligned} \mathscr{E}_{\mathrm{Dir}} : W^\delta_{p,q,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow & W^{1,q}(\mathbb{R}, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \\ & \cap L^q(\mathbb{R}, v_\mu; W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X)). \end{aligned}$$

If $\delta < \frac{1+\gamma}{p}$, then $W^\delta_{p,q,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = W^\delta_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ and we can simply take $\mathscr{E}_{\mathrm{Dir}}$ as the required coretraction.

Finally, let us consider the case $\delta > \frac{1+\gamma}{p}$. In the notation of Lemma 5.7.13, put $\pi := E \circ \mathrm{tr}_{\partial\mathbb{R}^d_+}$. Then

$$W^{\delta}_{p,q}(\mathbb{R}^d_+, w_\gamma; X) = \ker(I-\pi) \oplus W^{\delta}_{p,q,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) \tag{5.51}$$

under the projection π with the norm equivalence (5.50) on $\ker(I-\pi)$. In view of [163], we can apply [186, Theorem 1.1] or [197, Theorem 3.4.8] to the realization of $I-\Delta$ in $B^0_{p,1}(\mathbb{R}^d, w_\gamma; X)$ with domain $B^2_{p,1}(\mathbb{R}^d, w_\gamma; X)$, which by real interpolation of weighted B-spaces (see [38, Theorem 3.5]) gives an extension operator

$$\mathscr{E}_{\mathbb{R}^d} : B^{\delta}_{p,q}(\mathbb{R}^d, w_\gamma; X) \to W^{1,q}(\mathbb{R}, v_\mu; B^0_{p,1}(\mathbb{R}^d, w_\gamma; X)) \cap L^q(\mathbb{R}, v_\mu; B^2_{p,1}(\mathbb{R}^d, w_\gamma; X)).$$

By extension (using for instance Rychkov's extension operator [216]) and restriction we obtain an extension operator $\mathscr{E}_{\mathbb{R}^d_+}$ which maps $B^{\delta}_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ into

$$\begin{aligned} &W^{1,q}(\mathbb{R}, v_\mu; B^0_{p,1}(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}, v_\mu; B^2_{p,1}(\mathbb{R}^d_+, w_\gamma; X)) \\ &\quad \hookrightarrow W^{1,q}(\mathbb{R}, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}, v_\mu; W^{2,p}(\mathbb{R}^d_+, w_\gamma; X)), \end{aligned}$$

where the embedding follows from (5.37). By (5.51) and (5.50), $\mathscr{E} := \mathscr{E}_{\mathbb{R}^d_+}\pi + \mathscr{E}_{\mathrm{Dir}}(I-\pi)$ defines a coretraction corresponding to $\mathrm{tr}_{\partial\mathscr{O}}$. □

Remark 5.7.14. Let $p \in (1,\infty)$, $q \in [1,\infty)$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and $s \in (0,2) \setminus \{\frac{1+\gamma}{p}\}$. Then

$$W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d_+, w_\gamma; X) \quad \Longrightarrow \quad \gamma \in (-1, p-1).$$

Proof. Assume there is the inclusion $W^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d_+, w_\gamma; X)$. Considering the linear mapping $u \mapsto u \otimes x$ for some $x \in X \setminus \{0\}$, we find $W^s_{p,q}(\mathbb{R}^d_+, w_\gamma) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d_+, w_\gamma)$. In particular,

$$W^s_{p,q,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma) \hookrightarrow B^s_{p,\infty,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma). \tag{5.52}$$

Consider the interpolation-extrapolation scale $\{E_\eta : \eta \in [-1,\infty)\}$ generated by the operator $(1-\Delta_{\mathrm{Dir}})$ on $L^p(\mathbb{R}^d_+, w_\gamma)$ and the complex interpolation functors $[\cdot,\cdot]_\theta$, $\theta \in (0,1)$, the interpolation-extrapolation scale $\{E_{\eta,q} : \eta \in [-1,\infty)\}$ generated by the operator $(1-\Delta_{\mathrm{Dir}})$ on $L^p(\mathbb{R}^d_+, w_\gamma)$ and the real interpolation functors $(\cdot,\cdot)_{\theta,q}$, $\theta \in (0,1)$, and the interpolation-extrapolation scale $\{F_{\eta,\infty} : \eta \in [-1,\infty)\}$ generated by the operator $(1-\Delta_{\mathrm{Dir}})$ on $B^0_{p,\infty}(\mathbb{R}^d_+, w_\gamma)$ and the complex interpolation functors $[\cdot,\cdot]_\theta$, $\theta \in (0,1)$ (see [5, Section V.1.5]); the operator $(1-\Delta_{\mathrm{Dir}})$ on $B^0_{p,\infty}(\mathbb{R}^d_+, w_\gamma)$ is considered in [163]. By Proposition 5.7.7 and [163],

$$E_{\eta,q} = W^{2\eta}_{p,q,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma), \qquad \eta \in (0,1) \setminus \left\{\frac{1+\gamma}{2p}\right\}. \tag{5.53}$$

and

$$F_{\eta,\infty} = B^{2\eta}_{p,\infty,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma), \qquad \eta \in \left(\frac{1+\gamma}{2p} - 1, \frac{1+\gamma}{2p} + 1\right) \setminus \left\{\frac{1+\gamma}{2p}\right\}, \tag{5.54}$$

respectively. Now (5.52), (5.53) and (5.54) imply $E_{\frac{s}{2},q} \hookrightarrow F_{\frac{s}{2},\infty}$ and by lifting we obtain

$$E_{\eta,q} \hookrightarrow F_{\eta,\infty}, \qquad \eta \in \left[\frac{s}{2} + \mathbb{Z}\right] \cap [-1,\infty). \tag{5.55}$$

By the reiteration property from [5, Theorem V.1.5.4], $E_0 = [E_{-1}, E_1]_{\frac{1}{2}}$. So, by [235, Theorem 1.10.3.1],

$$E_0 = [E_{-1}, E_1]_{\frac{1}{2}} \hookrightarrow (E_{-1}, E_1)_{\frac{1}{2},\infty}. \tag{5.56}$$

Doing a reiteration ([235, Theorem 1.10.2] and [5, Theorem V.1.5.4]), we find

$$\begin{aligned}
(E_{-1}, E_1)_{\frac{1}{2},\infty} &= ((E_{-1}, E_1)_{\frac{s}{4},q}, (E_{-1}, E_1)_{\frac{s}{4}+\frac{1}{2},q})_{1-\frac{s}{2},\infty} \\
&= ((E_{-1}, [E_{-1}, E_1]_{\frac{1}{2}})_{\frac{s}{2},q}, ([E_{-1}, E_1]_{\frac{1}{2}}, E_1)_{\frac{s}{2},q})_{1-\frac{s}{2},\infty} \\
&= ((E_{-1}, E_0)_{\frac{s}{2},q}, (E_0, E_1)_{\frac{s}{2},q})_{1-\frac{s}{2},\infty} \\
&= (E_{\frac{s}{2}-1,q}, E_{\frac{s}{2},q})_{1-\frac{s}{2},\infty} \overset{(5.55)}{\hookrightarrow} (F_{\frac{s}{2}-1,\infty}, F_{\frac{s}{2},\infty})_{1-\frac{s}{2},\infty} \\
&= ([F_{\frac{s}{2}-1,\infty}, F_{\frac{s}{2},\infty}]_{\eta_0-\frac{s}{2}+1}, [F_{\frac{s}{2}-1}, F_{\frac{s}{2}}]_{\eta_1-\frac{s}{2}+1})_{\lambda,\infty} \\
&= (F_{\eta_0,\infty}, F_{\eta_1,\infty})_{\lambda,\infty}
\end{aligned} \tag{5.57}$$

for any $\eta_0, \eta_1 \in \mathbb{R}$ and $\lambda \in (0,1)$ with $\frac{s}{2} - 1 < \eta_0 < \eta_1 < \frac{s}{2}$ and $0 = (1-\lambda)\eta_0 + \lambda\eta_1$. Now pick $\eta_0, \eta_1 \in \mathbb{R}$ and $\lambda \in (0,1)$ with $\frac{s}{2} - 1 < \eta_0 < \eta_1 < \frac{s}{2}$ and $0 = (1-\lambda)\eta_0 + \lambda\eta_1$ such that $\eta_0, \eta_1 \in (\frac{1+\gamma}{2p} - 1, \frac{1+\gamma}{2p})$. Then

$$(F_{\eta_0,\infty}, F_{\eta_1,\infty})_{\lambda,\infty} \overset{(5.54)}{=} (B^{2\eta_0}_{p,\infty}(\mathbb{R}^d_+, w_\gamma), B^{2\eta_1}_{p,\infty}(\mathbb{R}^d_+, w_\gamma))_{\lambda,\infty} = B^0_{p,\infty}(\mathbb{R}^d_+, w_\gamma) \tag{5.58}$$

by real interpolation of weighted B-spaces (see [38, Theorem 3.5]). Combining (5.56), (5.57) and (5.58) gives

$$L^p(\mathbb{R}^d_+, w_\gamma) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d_+, w_\gamma). \tag{5.59}$$

We finally show that the inclusion (5.59) implies $\gamma \in (-1, p-1)$. Taking odd extensions in (5.59) (see [163]) gives

$$(\mathscr{S}_{\mathrm{odd}}(\mathbb{R}^d), \|\cdot\|_{L^p(\mathbb{R}^d, w_\gamma)}) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d, w_\gamma).$$

Now a slight modification of the argument given in [182, Remark 3.13] gives $C, c > 0$ such that

$$\frac{1}{|Q|}\int_Q w_\gamma(x)\,dx \cdot \left(\frac{1}{|Q|}\int_Q w_\gamma(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^d_+$ with $|Q| \leq c$. A computation as in [101, Example 9.1.7] shows that $\gamma \in (-1, p-1)$. $\square$

5.7.3. Weighted L^q-L^p-maximal regularity

Let us first introduce some notation. Let $\mathscr{O}$ be a bounded C^2-domain. Let $p,q \in (1,\infty)$, $v \in A_q(\mathbb{R})$ and $\gamma \in (-1,2p-1)$. For an interval $J \subset \mathbb{R}$ we set $\mathbb{D}^{q,p}_{v,\gamma}(J) := L^q(J,v;L^p(\mathscr{O},w^{\mathscr{O}}_\gamma))$,

$$\mathbb{M}^{q,p}_{v,\gamma}(J) := W^{1,q}(J,v;L^p(\mathscr{O},w^{\mathscr{O}}_\gamma)) \cap L^q(J,v;W^{2,p}(\mathscr{O},w_\gamma))$$

and

$$\mathbb{B}^{q,p}_{v,\gamma}(J) := F^{1-\frac{1}{2}\frac{1+\gamma}{p}}_{q,p}(J,v;L^p(\partial\mathscr{O})) \cap L^q(J,v;B^{2-\frac{1+\gamma}{p}}_{p,p}(\partial\mathscr{O})).$$

For the power weight $v = v_\mu$, with $\mu \in (-1,q-1)$, we simply replace v by μ in the subscripts: $\mathbb{D}^{q,p}_{\mu,\gamma}(J) := \mathbb{D}^{q,p}_{v_\mu,\gamma}(J)$, $\mathbb{M}^{q,p}_{\mu,\gamma}(J) := \mathbb{M}^{q,p}_{v_\mu,\gamma}(J)$ and $\mathbb{B}^{q,p}_{\mu,\gamma}(J) := \mathbb{B}^{q,p}_{v_\mu,\gamma}(J)$.

Theorem 5.7.15 (Heat equation)**.** *Let $\mathscr{O}$ be a bounded C^2-domain. Let $p,q \in (1,\infty)$, $v \in A_q(\mathbb{R})$ and $\gamma \in (-1,2p-1)\setminus\{p-1\}$. For all $\lambda \geq 0$, $u \mapsto (u' + (\lambda-\Delta)u, tr_{\partial\mathscr{O}} u)$ defines an isomorphism of Banach spaces $\mathbb{M}^{q,p}_{v,\gamma}(\mathbb{R}) \longrightarrow \mathbb{D}^{q,p}_{v,\gamma}(\mathbb{R}) \oplus \mathbb{B}^{q,p}_{v,\gamma}(\mathbb{R})$; in particular, for all $\lambda \geq 0$, $f \in \mathbb{D}^{q,p}_{v,\gamma}(\mathbb{R})$ and $g \in \mathbb{B}^{q,p}_{v,\gamma}(\mathbb{R})$, there exists a unique solution $u \in \mathbb{M}^{q,p}_{v,\gamma}(\mathbb{R})$ of the parabolic boundary value problem*

$$\begin{cases} u' + (\lambda-\Delta)u &= f, \\ tr_{\partial\mathscr{O}} u &= g. \end{cases}$$

Moreover, there are the estimates

$$\|u\|_{\mathbb{M}^{q,p}_{v,\gamma}(\mathbb{R})} \eqsim_{p,q,v,\gamma,d,\lambda} \|f\|_{\mathbb{D}^{q,p}_{v,\gamma}(\mathbb{R})} + \|g\|_{\mathbb{B}^{q,p}_{v,\gamma}(\mathbb{R})}.$$

Proof. The required boundedness of the mapping $u \mapsto (u' + (\lambda-\Delta)u, \mathrm{tr}_{\partial\mathscr{O}} u)$ follows from Theorem 5.7.1 while the injectivity follows from Corollary 5.6.3. So it remains to be shown that it has a bounded right-inverse, i.e. there is a bounded solution operator to the associated parabolic boundary value problem. Using Theorem 5.7.1 we will reduce to the case $g = 0$. After this reduction, the desired result follows from Corollary 5.6.3. Finally, to give the reduction to $g = 0$, write $U = u - \mathrm{ext}_{\partial\mathbb{R}^d_+} g$, where $\mathrm{ext}_{\partial\mathbb{R}^d_+} : \mathbb{B}^{q,p}_{v,\gamma}(\mathbb{R}) \to \mathbb{M}^{q,p}_{v,\gamma}(\mathbb{R})$ is the coretraction of $\mathrm{tr}_{\partial\mathscr{O}}$ of Theorem 5.7.1. Then U satisfies $U' + (\lambda-\Delta)U = F$ and $\mathrm{tr}_{\partial\mathscr{O}} U = 0$ where

$$F = f - (\frac{d}{dt} + \lambda - \Delta)\mathrm{ext}_{\partial\mathbb{R}^d_+} g.$$

Now Corollary 5.6.3 gives

$$\|U\|_{\mathbb{M}^{q,p}_{v,\gamma}(\mathbb{R})} \leq C\|F\|_{\mathbb{D}^{q,p}_{v,\gamma}(\mathbb{R})} \leq C\|f\|_{\mathbb{D}^{q,p}_{v,\gamma}(\mathbb{R})} + \widetilde{C}\|g\|_{\mathbb{B}^{q,p}_{v,\gamma}(\mathbb{R})}.$$

The corresponding estimate for u follows from this. □

As a consequence of the above theorem we obtain the following corresponding result on time intervals $J = (0,T)$ with $T \in (0,\infty]$ in the case of the power weight $v = v_\mu$ (with $\mu \in (-1,q-1)$), where we need to take initial values into account.

For the initial data we need to introduce the space

$$\mathbb{I}^{q,p}_{\mu,\gamma} := W^{2(1-\frac{1+\mu}{q})}_{p,q}(\mathscr{O}, w_\gamma) \stackrel{(5.43)}{=} (L^p(\mathscr{O}, w_\gamma), W^{2,p}(\mathscr{O}, w_\gamma))_{1-\frac{1+\mu}{q},q}.$$

Recall from Lemma 5.7.9 that $B^s_{p,q}(\mathscr{O}, w_\gamma; X) \hookrightarrow \mathbb{I}^{q,p}_{\mu,\gamma}$ with equality if $\gamma \in (-1, p-1)$.

Concerning the compatability condition in the space of initial-boundary data $\mathbb{IB}^{q,p}_{\mu,\gamma}(J)$ below, let us note the following. Assume $1 - \frac{1+\mu}{q} > \frac{1}{2}\frac{1+\gamma}{p}$. Then, on the one hand, by Proposition 5.7.6, there is a well-defined trace operator $\mathrm{tr}_{\partial\mathscr{O}}$ on $\mathbb{I}^{q,p}_{\mu,\gamma}(J)$; in fact, $\mathrm{tr}_{\partial\mathscr{O}}$ is a retraction from $\mathbb{I}^{q,p}_{\mu,\gamma}$ to $B^{2(1-\frac{1+\mu}{q})-\frac{1+\gamma}{p}}_{p,q}(\partial\mathscr{O}; X)$. On the other hand, as a consequence of [186, Theorem 1.1], $\mathrm{tr}_{t=0} : g \mapsto g(0)$ is a well-defined retraction from $\mathbb{B}^{q,p}_{\mu,\gamma}(J)$ to $B^{2(1-\frac{1+\mu}{q})-\frac{1+\gamma}{p}}_{p,q}(\partial\mathscr{O}; X)$. Motivated by this we set

$$\mathbb{IB}^{q,p}_{\mu,\gamma}(J) := \left\{(g, u_0) \in \mathbb{B}^{q,p}_{\mu,\gamma}(J) \oplus \mathbb{I}^{q,p}_{\mu,\gamma} : g(0) = \mathrm{tr}_{\partial\mathscr{O}} u_0 \text{ when } 1 - \frac{1+\mu}{q} > \frac{1}{2}\frac{1+\gamma}{p}\right\}.$$

Now we can state the main result for the initial value problem with inhomogeneous boundary condition.

Theorem 5.7.16 (Heat equation)**.** *Let $\mathscr{O}$ be a bounded C^2-domain and let $J = (0, T)$ with $T \in (0, \infty]$. Let $p, q \in (1, \infty)$, $\mu \in (-1, q-1)$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ with $1 - \frac{1+\mu}{q} \neq \frac{1}{2}\frac{1+\gamma}{p}$. For all $\lambda \geq 0$,*

$$\mathbb{M}^{q,p}_{\mu,\gamma}(J) \longrightarrow \mathbb{D}^{q,p}_{\mu,\gamma}(J) \oplus \mathbb{IB}^{q,p}_{\mu,\gamma}(J),\ u \mapsto (u' + (\lambda - \Delta)u, tr_{\partial\mathscr{O}} u, u(0))$$

defines an isomorphism of Banach spaces; in particular, for all $\lambda \geq 0$, $f \in \mathbb{D}^{q,p}_{\mu,\gamma}$ and $g \in \mathbb{B}^{q,p}_{\mu,\gamma}$, there exists a unique solution $u \in \mathbb{M}^{q,p}_{\mu,\gamma}$ of the parabolic initial-boundary value problem

$$\begin{cases} u' + (\lambda - \Delta)u &= f, \\ tr_{\partial\mathscr{O}} u &= g, \\ u(0) &= u_0. \end{cases}$$

Moreover, there are the estimates

$$\|u\|_{\mathbb{M}^{q,p}_{\mu,\gamma}(J)} \eqsim_{p,q,\mu,\gamma,d,\lambda} \|f\|_{\mathbb{D}^{q,p}_{\mu,\gamma}(J)} + \|(g, u_0)\|_{\mathbb{IB}^{q,p}_{\mu,\gamma}(J)}.$$

In the proof of the theorem we will use the following notation:

$${}_0\mathbb{B}^{q,p}_{\mu,\gamma}(I) := \begin{cases} \mathbb{B}^{q,p}_{\mu,\gamma}(I), & 1 - \frac{1+\mu}{q} < \frac{1}{2}\frac{1+\gamma}{p}, \\ \{g \in \mathbb{B}^{q,p}_{\mu,\gamma}(I) : g(0) = 0\}, & 1 - \frac{1+\mu}{q} > \frac{1}{2}\frac{1+\gamma}{p}, \end{cases}$$

and ${}_0\mathbb{M}^{q,p}_{\mu,\gamma}(I) := \{u \in \mathbb{M}^{q,p}_{\mu,\gamma}(I) : u(0) = 0\}$, where $I \in \{\mathbb{R}_+, \mathbb{R}\}$. We will furthermore use the following lemma.

Lemma 5.7.17. *Let the notation and assumptions be as in Theorem 5.7.16. Then operator E_0 of extension by zero from $\mathbb{R}_+$ to $\mathbb{R}$ is a bounded linear operator from ${}_0\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}_+)$ to $\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R})$.*

Proof. It suffices to show that

$$E_0 \in \mathscr{B}({}_0F_{q,p}^{1-\frac{1}{2}\frac{1+\gamma}{p}}(\mathbb{R}_+, v_\mu; L^p(\partial\mathscr{O})), F_{q,p}^{1-\frac{1}{2}\frac{1+\gamma}{p}}(\mathbb{R}, v_\mu; L^p(\partial\mathscr{O}))).$$

Using [187, Theorem 1.3], which says that $1_{\mathbb{R}_+}$ is a pointwise multiplier on $F^s_{p,q}(\mathbb{R}, v_\mu; X)$ for $s \in (\frac{1+\mu}{q} - 1, \frac{1+\mu}{q})$ and a Banach space X, this can be shown as in [165]. We would like to remark that this pointwise multiplier result could also be proved through a difference norm characterization as in [233, Section 2.8.6, Proposition 1], using that $F^s_{q,p}(\mathbb{R}, v_\mu; X) \hookrightarrow L^q(\mathbb{R}, v_{\mu-sq}; X)$ for $s \in (0, \frac{1+\mu}{q})$ (see [182]). □

Proof of Theorem 5.7.16. That $u \mapsto (u' + (\lambda - \Delta)u, \mathrm{tr}_{\partial\mathscr{O}} u, u(0))$ is a bounded operator

$$\mathbb{M}^{q,p}_{\mu,\gamma}(J) \longrightarrow \mathbb{D}^{q,p}_{\mu,\gamma}(J) \oplus \mathbb{B}^{q,p}_{\mu,\gamma}(J) \oplus \mathbb{I}^{q,p}_{\mu,\gamma}$$

follows from a combination of Theorem 5.7.1 and Theorem 5.7.5. That it maps to $\mathbb{D}^{q,p}_{\mu,\gamma} \oplus \mathbb{IB}^{q,p}_{\mu,\gamma}(J)$ can be seen as follows. Of course, we only need to show that

$$\mathrm{tr}_{t=0}\mathrm{tr}_{\partial\mathscr{O}} u = \mathrm{tr}_{\partial\mathscr{O}}\mathrm{tr}_{t=0} u, \qquad u \in \mathbb{M}^{q,p}_{\mu,\gamma}(J), \tag{5.60}$$

when $1 - \frac{1+\mu}{q} > \frac{1}{2}\frac{1+\gamma}{p}$. So assume $1 - \frac{1+\mu}{q} > \frac{1}{2}\frac{1+\gamma}{p}$. By a standard convolution argument and an extension and restriction argument, we see that $W^1_q(J, v_\mu; W^2_{p,\gamma}(\mathscr{O}))$ is dense in $\mathbb{M}^{q,p}_{\mu,\gamma}(J)$, from which (6.62) follows.

Injectivity of $u \mapsto (u' + (\lambda - \Delta)u, \mathrm{tr}_{\partial\mathscr{O}} u, u(0))$ follows from the fact that Δ_{Dir} generates a strongly continuous semigroup (see [85]) by Theorem 5.6.2. So it remains to be shown that it has a bounded right-inverse, i.e. there is a bounded solution operator to the associated parabolic initial-boundary value problem. Using Theorem 5.7.5 followed by Theorem 5.7.1 and (6.62), we may restrict ourselves to the case $u_0 = 0$. Furthermore, by Corollary 5.6.3 we may restrict ourselves to the case $f = 0$. By extension and restriction it is enough to treat the resulting problem for $J = \mathbb{R}_+$. We must show that there is a bounded linear solution operator $\mathscr{S} : {}_0\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}_+) \to {}_0\mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R}_+)$, $g \mapsto u$ for the problem

$$\begin{cases} u' + (\lambda - \Delta)u &= 0, \\ \mathrm{tr}_{\partial\mathscr{O}} u &= g. \end{cases} \tag{5.61}$$

Let $E_0 \in \mathscr{B}({}_0\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}_+), \mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}))$ be the operator of extension by zero (see Lemma 5.7.17) and let $\mathscr{S}_{\mathbb{R}} : \mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}) \to \mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$, $g \mapsto u$ be the solution operator for the problem (5.61) on $\mathbb{R}$ from Theorem 5.7.15.

It suffices to show that $\mathscr{S}_{\mathbb{R}} \circ E_0$ maps to ${}_0\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}_+)$ to ${}_0\mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$; indeed, in that case $\mathscr{S} g := (\mathscr{S} E_0 g)_{|\mathbb{R}_+}$ is as desired. To do so we follow a modification of an argument given in [176, Lemma 2.2.7].

Let $g \in {}_0\mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}_+)$ and set $u := \mathscr{S}_{\mathbb{R}} E_0 g \in \mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$. Pick $\phi \in C^\infty_c(\mathbb{R}_+)$ with $\int_{\mathbb{R}} \phi(x)\,dx = 1$ and put $\phi_n(x) := n^d \phi(nx)$ for each $n \in \mathbb{N}_1$. Now consider $g_n := \phi_n * E_0 g \in \mathbb{B}^{q,p}_{\mu,\gamma}(\mathbb{R}) \cap$

$C^\infty(\mathbb{R};L^p(\partial\mathscr{O}))$ and $u_n := \phi_n * u \in W^{\infty,q}(\mathbb{R},v_\mu;W^{2,p}(\mathscr{O},w_\gamma)) \subset \mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R}) \cap C^\infty(\mathbb{R};L^p(\mathscr{O},w_\gamma))$. Then

$$u = \lim_{n\to\infty} u_n \qquad \text{in} \qquad \mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R}) \tag{5.62}$$

and

$$\begin{cases} u_n' + (\lambda-\Delta)u_n &= \phi_n * (u' + (\lambda-\Delta)u) = 0, \\ \operatorname{tr}_{\partial\mathscr{O}} u_n &= \phi_n * \operatorname{tr}_{\partial\mathscr{O}} u = \phi_n * E_0 g, \end{cases}$$

so that $u_n = \mathscr{S}_{\mathbb{R}} g_n$ by uniqueness of solutions. Furthermore, $g_n(0) = 0$, implying that $\operatorname{tr}_{\partial\mathscr{O}}[u_n(0)] = [\operatorname{tr}_{\partial\mathscr{O}} u_n](0) = g_n(0) = 0$, so that $u_n(0) \in W^{2,p}_{\mathrm{Dir}}(\mathscr{O},w_\gamma)$. Now, as $\lambda - \Delta_{\mathrm{Dir}}$ is exponentially stable, we may define $v_n \in {}_0\mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$ by

$$v_n(t) := \begin{cases} u_n(t) - e^{t(\lambda-\Delta_{\mathrm{Dir}})}u_n(0), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

But then v_n satisfies

$$\begin{cases} v_n' + (\lambda-\Delta)v_n &= 0, \\ \operatorname{tr}_{\partial\mathscr{O}} v_n &= g_n, \end{cases}$$

so that $v_n = \mathscr{S}_{\mathbb{R}} g_n = u_n$ by uniqueness of solutions. Therefore, $u_n \in {}_0\mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$. We may thus conclude that $u \in {}_0\mathbb{M}^{q,p}_{\mu,\gamma}(\mathbb{R})$ in view of (5.62). □

Remark 5.7.18. Theorems 5.7.15 and 5.7.16 also remain valid in the X-valued setting as long as X is a UMD space and $\lambda \geq \lambda_0$, where λ_0 depends on the geometry of X.

6

ELLIPTIC AND PARABOLIC BOUNDARY VALUE PROBLEMS SUBJECT TO LOPATINSKII-SHAPIRO BOUNDARY CONDITIONS

This chapter is based on the paper:

> *F.B. Hummel and N. Lindemulder. Elliptic and Parabolic Boundary Value Problems in Weighted Function Spaces. in preparation.*

In this paper we study elliptic and parabolic boundary value problems with inhomogeneous boundary conditions in weighted function spaces of Sobolev, Bessel potential, Besov and Triebel-Lizorkin type. As the main result, we solve the problem of weighted L_q*-maximal regularity in weighted Triebel-Lizorkin spaces for the parabolic case, where the spatial weight is a power weight in the Muckenhoupt* A_∞*-class. Going beyond the* A_p*-range, where* p *is the integrability parameter of the Triebel-Lizorkin space under consideration, yields extra flexibility in the sharp regularity of the boundary inhomogeneities. This extra flexibility allows us to treat rougher boundary data and provides a quantitative smoothing effect on the interior of the domain. The main ingredient is an analysis of anisotropic Poisson operators.*

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6.1. INTRODUCTION

6.2. INTRODUCTION

The idea to work in weighted function spaces equipped with temporal and/or spatial power weights of the type

$$v_\mu(t) = t^\mu \quad (t \in J) \qquad \text{and} \qquad w_\gamma^{\partial\mathscr{O}}(x) = \operatorname{dist}(\cdot, \partial\mathscr{O})^\gamma \quad (x \in \mathscr{O}), \tag{6.1}$$

has already proven to be very useful in several situations. In an abstract semigroup setting temporal weights were introduced by Clément & Simonett [52] and Prüss & Simonett [197], in the context of maximal continous regularity and maximal L_p-regularity, respectively. Other works on maximal temporally weighted L_p-regularity are [141, 153] for quasilinear parabolic evolution equations and [180] for parabolic problems with inhomogeneous boundary conditions. Concerning the use of spatial weights in applications to (S)PDES, we would like to mention [3, 36, 37, 46, 47, 75, 88, 139, 143, 144, 159, 162, 167, 173, 189] and Chapter 5.

An important feature of the power weights (6.1) is that they allow to treat "rougher" behaviour in the initial time and on the boundary by increasing the parameter μ and γ, respectively. In [159, 162, 180, 197] and Chapter 5 this is for instance reflected in the lower regularity of the initial/initial-boundary data that can be dealt with. In the L_p-approach to parabolic problems with Dirichlet boundary noise, where the noise is a source of roughness on the boundary, weights are even necessary to obtain function space-valued solution processes [3, 88, 167].

As in [159], in this paper we exploit this feature of the power weights (6.1) in the study vector-valued parabolic initial-boundary value problems of the form

$$\begin{aligned} \partial_t u(x,t) + \mathscr{A}(x,D,t)u(x,t) &= f(x,t), && x \in \mathscr{O}, && t \in J, \\ \mathscr{B}_j(x',D,t)u(x',t) &= g_j(x',t), && x' \in \partial\mathscr{O}, && t \in J, \quad j = 1,\dots,n, \\ u(x,0) &= u_0(x), && x \in \mathscr{O}. \end{aligned} \tag{6.2}$$

Here, J is a finite time interval, $\mathscr{O} \subset \mathbb{R}^n$ is a C^∞-domain with a compact boundary $\partial\mathscr{O}$ and the coefficients of the differential operator $\mathscr{A}$ and the boundary operators $\mathscr{B}_1,\dots,\mathscr{B}_m$ are $\mathscr{B}(X)$-valued, where X is a UMD Banach space. One could for instance take $X = \mathbb{C}^N$, describing a system of N initial-boundary value problems. Our structural assumptions on $\mathscr{A}, \mathscr{B}_1,\dots,\mathscr{B}_n$ are an ellipticity condition and a condition of Lopatinskii-Shapiro type. For homogeneous boundary data (i.e. $g_j = 0$, $j = 1,\dots,m$) these problems include linearizations of reaction-diffusion systems and of phase field models with Dirichlet, Neumann and Robin conditions. However, if one wants to use linearization techniques to treat such problems with non-linear boundary conditions, it is crucial to have a sharp theory for the fully inhomogeneous problem.

Maximal regularity provides sharp/optimal estimates for PDEs. Indeed, maximal regularity means that there is an isomorphism between the data and the solution of the problem in suitable function spaces. It is an important tool in the theory of nonlinear

PDEs: having established maximal regularity for the linearized problem, the nonlinear problem can be treated with tools as the contraction principle and the implicit function theorem (see [198]).

The main result of this paper is concerned with weighted L_q-maximal regularity in weighted Triebel-Lizorkin spaces for (6.2), where we use the weights (6.1). In order to elaborate on this, let us for reasons of exposition consider as a specific easy example of (6.2) the heat equation with the Dirichlet boundary condition

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{on} \quad J \times \mathscr{O}, \\ u_{|\partial\mathscr{O}} &= g \quad \text{on} \quad J \times \partial\mathscr{O}, \\ u(0) &= u_0 \quad \text{on} \quad \mathscr{O}, \end{cases} \tag{6.3}$$

where $J = (0,T)$ with $T \in (0,\infty)$ and where $\mathscr{O}$ is a smooth domain in $\mathbb{R}^n$ with a compact boundary $\partial\mathscr{O}$.

In order to introduce the weighted L_q-maximal regularity problem for (6.3) in an abstract setting, let $q \in (1,\infty)$, $\mu \in (-1, q-1)$ and $\mathbb{E} \subset \mathscr{D}'(\mathscr{O})$ a Banach space of distributions on $\mathscr{O}$ such that there exists a notion of trace on the associated second order space $\mathbb{E}^2 = \{u \in \mathscr{D}(\mathscr{O}) : D^\alpha u \in \mathbb{E}, |\alpha| \leq 2\}$ that is described by a bounded linear operator $\mathrm{Tr}_{\partial\mathscr{O}} : \mathbb{E}^2 \longrightarrow \mathbb{F}$ for some suitable Banach space.

In the $L_{q,\mu}$-$\mathbb{E}$-maximal regularity approach to (6.3) one is looking for solutions u in the *maximal regularity space*

$$W^1_q(J, v_\mu; \mathbb{E}) \cap L_q(J, v_\mu; \mathbb{E}^2), \tag{6.4}$$

where the boundary condition $u_{|\partial\mathscr{O}} = g$ has to be interpreted as $\mathrm{Tr}_{\partial\mathscr{O}} u = g$. The problem (6.3) is said to enjoy the property of *maximal $L_{q,\mu}$-$\mathbb{E}$-regularity* if there exists a (necessarily unique) space of initial-boundary data $\mathscr{D}_{i.b.} \subset L_q(J, v_\mu; \mathbb{F}) \times \mathbb{E}$ such that for every $f \in L_q(J, v_\mu; \mathbb{E})$ it holds that (6.3) has a unique solution u in (6.4) if and only if $(g, u_0) \in \mathscr{D}_{i.b.}$. In this situation there exists a Banach norm on $\mathscr{D}_{i.b.}$, unique up to equivalence, with

$$\mathscr{D}_{i.b.} \hookrightarrow L_q(J, v_\mu; \mathbb{F}) \oplus \mathbb{E},$$

which makes the associated solution operator a topological linear isomorphism between the data space $L_q(J, v_\mu; \mathbb{E})) \oplus \mathscr{D}_{i.b.}$ and the solution space $W^1_q(J, v_\mu; \mathbb{E}) \cap L_q(J, v_\mu; \mathbb{E}^2)$. The *maximal $L_{q,\mu}$-$\mathbb{E}$-regularity problem* for (6.3) consists of establishing maximal $L_{q,\mu}$-$\mathbb{E}$-regularity for (6.3) and explicitly determining the space $\mathscr{D}_{i.b.}$.

In the special case that $\mathbb{E} = L_p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma)$, $\mathbb{E}^2 = W^2_p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma)$ and $\mathbb{F} = L_p(\partial\mathscr{O})$ with $p \in (1,\infty)$ and $\gamma \in (-1, 2p-1)$, $L_{q,\mu}$-$\mathbb{E}$-maximal regularity is referred to as $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity.

The $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem for (6.3) has recently been solved (besides some exceptional parameter values) in Chapter 5. Here, the boundary datum g has to be in the intersection space

$$F^{\delta}_{q,p}(J, v_\mu; L_p(\partial\mathscr{O})) \cap L_p(J, v_\mu; B^{2\delta}_{p,p}(\partial\mathscr{O})) \tag{6.5}$$

with $\delta = \delta_{p,\gamma} := 1 - \frac{1+\gamma}{2p}$, which in the case $q = p$ coincides with $W^{\delta}_{p}(J, v_{\mu}; L_p(\partial\mathscr{O})) \cap L_p(J, v_{\mu}; W^{2\delta}_{p}(\partial\mathscr{O}))$; here $F^{s}_{q,p}$ denotes a Triebel-Lizorkin space and $W^{s}_{p} = B^{s}_{p,p}$ a non-integer order Sobolev-Slobodeckii space or Besov space.

Note that $\delta \in (0,1)$ can be taken arbitrarily close to 0 by choosing γ sufficiently close to $2p-1$. In [159] the maximal $L_{q,\mu}$-$L_{p,\gamma}$-regularity problem with $\gamma \in (-1, p-1)$ was solved for the more general (6.2), which in the special case (6.3) gives the restriction $\delta \in (\frac{1}{2}, 1)$.

The restriction $\gamma \in (-1, p-1)$ for the spatial weight $w^{\partial\mathscr{O}}_{\gamma}$ in [159] is a restriction of harmonic analytic nature. Indeed, $(-1, p-1)$ is the Muckenhoupt A_p-range for $w^{\partial\mathscr{O}}_{\gamma}$: given $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$, it holds that

$$w^{\partial\mathscr{O}}_{\gamma} = \mathrm{dist}(\cdot, \partial\mathscr{O})^{\gamma} \in A_p(\mathbb{R}^n) \quad \Longleftrightarrow \quad \gamma \in (-1, p-1). \tag{6.6}$$

The Muckenhoupt class $A_p(\mathbb{R}^n)$ ($p \in (1,\infty)$) is a class of weights for which many harmonic analytic tools from the unweighted setting, such as Mikhlin Fourier multiplier theorems and Littlewood-Paley decompositions, remain valid for the corresponding weighted L_p-spaces. For example, the Littlewood-Paley decomposition for $L_p(\mathbb{R}^n, w)$ with $w \in A_p(\mathbb{R}^n)$ and its variant for $W^{k}_{p}(\mathbb{R}^n, w)$, $k \in \mathbb{N}$, can be formulated by means of Triebel-Lizorkin spaces as

$$L_p(\mathbb{R}^n, w) = F^{0}_{p,2}(\mathbb{R}^n, w), \qquad W^{k}_{p}(\mathbb{R}^n, w) = F^{k}_{p,2}(\mathbb{R}^n, w). \tag{6.7}$$

The main difficulty in Chapter 5 in the non-A_p setting is that these standard tools are no longer available.

One way to avoid these difficulties is to work in weighted Triebel-Lizorkin spaces instead of $\mathbb{E} = L_p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma})$. The advantage of the scale of weighted Triebel-Lizorkin spaces is the strong harmonic analytic nature of these function spaces, leading the availability of many powerful tools (see e.g. [38–40, 115–118, 162, 162, 182, 185, 186, 228]). In particular, there is a Mikhlin-Hörmander Fourier multiplier theorem. That Mikhlin-Hörmander Fourier multiplier theorem

In the special case $\mathbb{E} = F^{s}_{p,r}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma})$, $\mathbb{E}^2 = F^{s+2}_{p,r}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma})$ and $\mathbb{F} = L_p(\partial\mathscr{O})$ with $p, r \in (1,\infty)$, $\gamma \in (-1,\infty)$ and $s \in (\frac{1+\gamma}{p} - 2, \frac{1+\gamma}{p})$, $L_{q,\mu}$-$\mathbb{E}$-maximal regularity is referred to as $L_{q,\mu}$-$F^{s}_{p,r,\gamma}$-maximal regularity.

The $L_{q,\mu}$-$F^{s}_{p,r,\gamma}$-maximal regularity problem for (6.3) has recently been solved (besides some exceptional parameter values) in [162]. Again, the boundary datum g has to be in the intersection space (6.5), but now with $\delta = \delta_{p,\gamma,s} := \frac{s}{2} + 1 - \frac{1+\gamma}{p}$.

As a consequence of (6.6) and (6.7), $L_{q,\mu}$-$F^{0}_{p,2,\gamma}$-maximal regularity coincides with $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity when $\gamma \in (-1, p-1)$. For other values of γ the two notions are independent. However, there still is a connection between the $L_{q,\mu}$-$F^{s}_{p,r,\gamma}$-maximal regularity problem and the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem provided by the following weakening of (6.7) to an elementary embedding combined with a Sobolev embedding:

$$F^{k+\frac{\nu-\gamma}{p}}_{p,r}(\mathscr{O}, w^{\partial\mathscr{O}}_{\nu}) \hookrightarrow F^{k}_{p,1}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma}) \hookrightarrow W^{k}_{p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma}), \qquad \nu > \gamma, r \in [1,\infty]. \tag{6.8}$$

Indeed, in view of (6.8) and the invariance

$$\delta = \delta_{p,\nu,s} = \delta_{p,\gamma}, \qquad s = \frac{\nu - \gamma}{p},$$

in connection with (6.5), in order to obtain a solution operator for (6.3) with $f = 0$, $u_0 = 0$ it suffices to treat the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-case.

As the main result of this paper (Theorem 6.6.2), we solve the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity problem for (6.2) with $\gamma \in (-1,\infty)$ and $s \in (\frac{1+\gamma}{p} + m_* - 2m, \frac{1+\gamma}{p})$, where $m = \frac{1}{2}\mathrm{ord}(\mathscr{A})$ and $m_* = \max\{\mathrm{ord}(\mathscr{B}_1),\dots,\mathrm{ord}(\mathscr{B}_m)\}$. Besides that the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity problem for (6.2) is already interesting on its own, it also contributes to the corresponding $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem through the above discussion, reducing that problem to the case $g_1 = \ldots = g_m = 0$. The latter can be treated in an abstract operator theoretic setting, leading to the problem of determining R-sectoriality or even a stronger bounded H^∞-calculus (see [198]). It would be very interesting to extend the boundedness of the H^∞-calculus for the Dirichlet Laplacian on $L_p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma)$ obtained in Chapter 5 to realizations of elliptic boundary value problems corresponding to (6.2) and thereby solve the $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity problem (at least for the case of trivial initial datum $u_0 = 0$).

Whereas, given $\gamma \in (-1, p-1)$, $L_{q,\mu}$-$F^0_{p,2,\gamma}$-maximal regularity coincides with $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity in the scalar-valued setting (or even the Hilbert space-valued setting), they are incomparable in the general Banach space-valued setting. However, the main result of the paper (Theorem 6.6.2) also contains a solution to the $L_{q,\mu}$-$H^s_{p,\gamma}$-maximal regularity problem for (6.2) with $\gamma \in (-1, p-1)$ and $s \in (\frac{1+\gamma}{p} + m_* - 2m, \frac{1+\gamma}{p})$, yielding $L_{q,\mu}$-$L_{p,\gamma}$-maximal regularity when $s = 0$. In the $L_{q,\mu}$-$L_{p,\gamma}$-case the proof even simplifies a bit on the function space theoretic side of the problem (see Remark 6.6.3), yielding this in particularly yields a simplification of the previous approaches ([61] ($\mu = 0$, $\gamma = 0$), [180] ($q = p$, $\mu \in [0, p-1)$, $\gamma = 0$) and [159]).

The main technical ingredient in is an analysis of anisotropic Poisson operators and their mapping properties on weighted mixed-norm anisotropic function spaces. The Poisson operators under consideration naturally occur as (or in) solution operators to the model problems

$$\begin{aligned} \partial_t u(x,t) + (1+\mathscr{A}(D))u(x,t) &= 0, && x \in \mathbb{R}^n_+, && t \in \mathbb{R}, \\ \mathscr{B}_j(D)u(x',t) &= g_j(x',t), && x' \in \mathbb{R}^{n-1}, && t \in \mathbb{R}, \quad j = 1,\dots,m, \end{aligned} \tag{6.9}$$

where $\mathscr{A}(D)$ and $\mathscr{B}_j(D)$ are homogeneous with constant coefficients. Moreover, they are operators K of the form

$$Kg(x_1,x',t) = (2\pi)^{-n}\int_{\mathbb{R}^{n-1}\times\mathbb{R}} e^{\imath(x',t)\cdot(\xi',\tau)}\widetilde{k}(x_1,\xi',\tau)\hat{g}(\xi',\tau)\,d(\xi,\tau), \qquad g \in \mathscr{S}(\mathbb{R}^{n-1}\times\mathbb{R}), \tag{6.10}$$

for some anisotropic Poisson symbol-kernel $\widetilde{k}$.

The anisotropic Poisson operator (6.10) is an anisotropic (x',t)-independent version of the classical Poisson operator from the Boutet the Monvel calculus. The Boutet the

Monvel calculus is pseudodifferential calculus that in some sense can be considered as a relatively small "algebra", containing the elliptic boundary value problems as well as their solution operators (or parametrices). The calculus was introduced by, as the name already suggests, Boutet de Monvel [32, 33], having its origin in the works of Vishik and Eskin [241], and was furhter developped in e.g. [105–107, 129, 206]; for an introduction to or an overview of the subject we refer the reader to [107, 108, 223].

A parameter-dependent version of the Boutet de Monvel calculus has been introduced and worked out by Grubb and collaborators (see [107] in the references given therein). This calculus contains the parameter-elliptic boundary value problems as well as their solution operators (or parametrices). In particular, resolvent analysis can be carried out in this calculus.

In the present paper we also consider a variant of the parameter-dependent Poisson operators from [107] in the x'-independent setting. Besides that this is one of the key ingredients in our treatment of the parabolic problems (6.2) through the anisotropic Poisson operators (6.10), it also forms the basis for our parameter-dependent estimates in weighted Besov, Triebel-Lizorkin and Bessel potential spaces for the elliptic boundary value problems

$$\begin{aligned} (\lambda+\mathscr{A}(x,D))u(x) &= f(x), \quad x\in\mathscr{O} \\ \mathscr{B}_j(x',D)u(x') &= g_j(x'), \quad x'\in\partial\mathscr{O}, \quad j=1,\dots,m. \end{aligned} \tag{6.11}$$

These parameter dependent estimates are an extension of [163] on second order elliptic boundary value problems subject to the Dirichlet boundary condtition, which was in turn in the spirit of [67, 109].

In the latter the scales of weighted $\mathscr{B}$- and $\mathscr{F}$-spaces, the dual scales to the scales of weighted B- and F-spaces, are also included. These scales naturally appear in duality theory and can for instance be used in the study of parabolic boundary value problems with multiplicative noise at the boundary in a setting of weighted L_p-spaces, see Remark 6.7.7.

Outline.

The outline of the paper is as follows.

- *Section 6.3:* Preliminaries from weighted (mixed-norm anisotropic) function spaces, distribution theory, UMD Banach spaces and L_q-maximal regularity, differential boundary value systems.
- *Section 6.4:* Sobolev embedding and trace results for mixed-norm anisotropic function spaces.
- *Section 6.5:* Introduction and basic properties of Poisson operators, solution operators to model problems and mapping properties.
- *Section 6.6:* $L_{q,\mu}$-maximal regularity for parabolic boundary value problems (6.2).

- *Section 6.7:* Parameter-dependent estimates elliptic boundary value problems (6.11).

Notation and convention.

$\mathbb{N}$, $\mathbb{N}_1$ $\Sigma_\phi = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \phi\}$.

Acknowledgement.

The authors would like to thank Mark Veraar for pointing out the Phragmen-Lindelöf Theorem (see [**?** , Corollary 6.4.4]) for the proof of Lemma 6.3.1.

6.3. PRELIMINARIES

6.3.1. Weighted Lebesgue Spaces

A reference for the general theory of Muckenhoupt weights is [102, Chapter 9].

A *weight* on a measure space $(S, \mathscr{A}, \mu)$ is a measurable function $w : S \longrightarrow [0, \infty]$ that takes it values almost everywhere in $(0, \infty)$. We denote by $\mathscr{W}(S)$ the sets of all weights on $(S, \mathscr{A}, \mu)$. For $w \in \mathscr{W}(S)$ and $p \in [1, \infty)$ we denote by $L_p(S, w)$ the space of all equivalence classes of measurable functions $f : S \longrightarrow \mathbb{C}$ with

$$||f||_{L^p(S,w)} := \left(\int_S |f(x)|^p \, w(x) \, d\mu(x) \right)^{1/p} < \infty.$$

If $p \in (1, \infty)$, then $w' = w'_p := w^{-\frac{1}{p-1}}$ is also a weight on S, called the p-dual weight of w. Furthermore, for $p \in (1, \infty)$ we have $[L_p(S, w)]^* = L_{p'}(S, w')$ isometrically with respect to the pairing

$$L_p(S, w) \times L_{p'}(S, w') \longrightarrow \mathbb{C}, \; (f, g) \mapsto \int_S f g \, d\mu. \tag{6.12}$$

Supppose $(S, \mathscr{A}, \mu) = \bigotimes_{j=1}^{l} (S_j, \mathscr{A}_j, \mu_j)$ is a product measure space. For $\boldsymbol{p} \in [1, \infty)^l$ and $\boldsymbol{w} \in \prod_{j=1}^{l} \mathscr{W}(S_j)$ we denote by $L_{\boldsymbol{p}}(S_1 \times \ldots \times S_l, \boldsymbol{w})$ the mixed-norm space

$$L_{\boldsymbol{p}}(S, \boldsymbol{w}) := L_{p_l}(S_l, w_l)[\ldots [L_{p_1}(S_1, w_1)] \ldots],$$

that is, $L_{\boldsymbol{p}}(S, \boldsymbol{w})$ is the space of all $f \in L_0(S)$ with

$$||f||_{L_{\boldsymbol{p}}(S,\boldsymbol{w})} := \left(\int_{S_l} \ldots \left(\int_{S_1} |f(x)|^{p_1} w_1(x_1) d\mu_1(x_1) \right)^{p_2/p_1} \ldots w_l(x_l) d\mu_l(x_l) \right)^{1/p_l} < \infty.$$

We equip $L_{\boldsymbol{p}}(S, \boldsymbol{w})$ with the norm $|| \cdot ||_{L_{\boldsymbol{p}}(S,\boldsymbol{w})}$, which turns it into a Banach space. As an extension (and in fact consequence) of (6.12), for $\boldsymbol{p} \in (1, \infty)$ we have $[L_{\boldsymbol{p}}(S, \boldsymbol{w})]^* = L_{\boldsymbol{p}'}(S, \boldsymbol{w}'_{\boldsymbol{p}})$ isometrically with respect to the pairing

$$L_{\boldsymbol{p}}(S, \boldsymbol{w}) \times L_{\boldsymbol{p}'}(S, \boldsymbol{w}'_{\boldsymbol{p}}) \longrightarrow \mathbb{C}, \; (f, g) \mapsto \int_S f g \, d\mu, \tag{6.13}$$

where $\boldsymbol{p}' = (p_1', \dots, p_l')$ and $\boldsymbol{w}_{\boldsymbol{p}}' = (w_{p_1}', \dots, w_{p_l}')$.

Given a Banach space X, we denote by $L_{\boldsymbol{p}}(S, \boldsymbol{w}; X)$ the associated Bochner space

$$L_{\boldsymbol{p}}(S, \boldsymbol{w}; X) := L_{\boldsymbol{p}}(S, \boldsymbol{w})[X] = \{f \in L_0(\mathbb{R}^n; X) : ||f||_X \in L_{\boldsymbol{p}}(S, \boldsymbol{w})\}.$$

For $p \in (1,\infty)$ we denote by $A_p = A_p(\mathbb{R}^n)$ the class of all Muckenhoupt A_p-weights, which are all the locally integrable weights for which the A_p-characteristic $[w]_{A_p} \in [1,\infty]$ is finite. We furthermore set $A_\infty := \bigcup_{p \in (1,\infty)} A_p$.

For $p \in (1,\infty)$ we denote by $A_p^{\mathrm{rec}} = A_p^{\mathrm{rec}}(\mathbb{R}^n)$ the class of all rectangular Muckenhoupt A_p-weights, which are all the locally integrable weights for which the A_p^{rec}-characteristic $[w]_{A_p^{\mathrm{rec}}} \in [1,\infty]$ is finite. Here $[w]_{A_p^{\mathrm{rec}}}$ is defined as $[w]_{A_p}$ by replacing cubes with sides parallel to the coordinate axes by rectangles with sides parallel to the coordinate axes in the definition.

The relevant weights for this paper are the power weights of the form $w = \mathrm{dist}(\cdot, \partial\mathscr{O})^\gamma$, where $\mathscr{O}$ is a C^∞-domain in $\mathbb{R}^n$ and where $\gamma \in (-1,\infty)$. If $\mathscr{O} \subset \mathbb{R}^n$ is a Lipschitz domain and $\gamma \in \mathbb{R}$, $p \in (1,\infty)$, then (see [91, Lemma 2.3] or [189, Lemma 2.3])

$$w_\gamma^{\mathscr{O}} := \mathrm{dist}(\cdot, \partial\mathscr{O})^\gamma \in A_p \quad \Longleftrightarrow \quad \gamma \in (-1, p-1); \tag{6.14}$$

in particular,

$$w_\gamma^{\mathscr{O}} = \mathrm{dist}(\cdot, \partial\mathscr{O})^\gamma \in A_\infty \quad \Longleftrightarrow \quad \gamma \in (-1,\infty). \tag{6.15}$$

For the important model problem case $\mathscr{O} = \mathbb{R}^n_+$ we simply write $w_\gamma := w_\gamma^{\mathbb{R}^n_+} = \mathrm{dist}(\cdot, \partial\mathbb{R}^n_+)^\gamma$.

Furthermore, in connection with the pairing (6.12), for $p \in (1,\infty)$ we have

$$w \in A_p \quad \Longleftrightarrow \quad w' \in A_{p'} \quad \Longleftrightarrow \quad w, w' \in A_\infty.$$

Let $p \in (1,\infty)$. We define $[A_\infty]_p' = [A_\infty]_p'(\mathbb{R}^n)$ as the set of all weights w on $\mathbb{R}^n$ for which $w_p' = w^{-\frac{1}{p-1}} \in A_\infty$. If $\mathscr{O} \subset \mathbb{R}^n$ is a Lipschitz domain and $\gamma \in \mathbb{R}$, $p \in (1,\infty)$, then

$$w_\gamma^{\mathscr{O}} \in [A_\infty]_p' \quad \Longleftrightarrow \quad \gamma_p' := -\frac{\gamma}{p-1} \in (-1,\infty) \quad \Longleftrightarrow \quad \gamma \in (-\infty, p-1) \tag{6.16}$$

in view of (6.15).

6.3.2. UMD Spaces and L_q-maximal Regularity

The general references for this subsection are [126, 127, 149].

The UMD property of Banach spaces is defined through the unconditionality of martingale differences, which is a primarily probabilistic notion. A deep result due to Bourgain and Burkholder gives a pure analytic characterization in terms of the Hilbert transform: a Banach space X has the UMD property if and only if it is of class HT, i.e. the Hilbert transform H has a bounded extension H_X to $L_p(\mathbb{R}; X)$ for any/some $p \in (1,\infty)$. A Banach space with the UMD property is called a UMD Banach space. Some facts:

- Every Hilbert space is a UMD space;

- If X is a UMD space, (S,Σ,μ) is σ-finite and $p \in (1,\infty)$, then $L_p(S;X)$ is a UMD space.

- UMD spaces are reflexives.

- Closed subspaces and quotients of UMD spaces are again UMD spaces.

In particular, weighted Besov and Triebel-Lizorkin spaces (see Section 6.3.4) are UMD spaces in the reflexive range.

Let A be a closed linear operator on a Banach space X. For $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$ we say that A enjoys the property of

- $L_q(v,\mathbb{R})$-maximal regularity if $\partial_t + A$ is invertible as an operator on $L_q(v,\mathbb{R};X)$ with domain $W^1_q(\mathbb{R},v;X) \cap L_q(\mathbb{R},v;D(A))$.

- $L_q(v,\mathbb{R}_+)$-maximal regularity if $\partial_t + A$ is invertible as an operator on $L_q(v,\mathbb{R}_+;X)$ with domain ${}_0W^1_q(\mathbb{R}_+,v;X) \cap L_q(\mathbb{R}_+,v;D(A))$, where

$$ {}_0W^1_q(\mathbb{R}_+,v;X) = \{u \in W^1_q(\mathbb{R}_+,v;X) : u(0) = 0\}. $$

In the specific case of the power weight $v = v_\mu$ with $q \in (-1,q-1)$, we speak of $L_{q,\mu}(\mathbb{R})$-maximal regularity and $L_{q,\mu}(\mathbb{R}_+)$-maximal regularity.

Note that $L_q(v,\mathbb{R})$-maximal regularity and $L_q(v,\mathbb{R}_+)$-maximal regularity can also be formulated in terms of evolution equations. For instance, A enjoys the property of $L_q(v,\mathbb{R}_+)$-maximal regularity if and only if, for each $f \in L_q(v,\mathbb{R}_+;X)$, there exists a unique solution $u \in W^1_q(\mathbb{R}_+,v;X) \cap L_q(\mathbb{R}_+,v;D(A))$ of

$$ u' + Au = f, \quad u(0) = 0. $$

References for $L_q(\mathbb{R})$-maximal regularity and $L_q(\mathbb{R}_+)$-maximal regularity include [13, 188] and [77, 149]. Works on $L_q(\mathbb{R}_+,v)$-maximal regularity include [44, 45, 90].

Lemma 6.3.1. *Let X be a Banach space, $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$. Let A be a linear operator on X and let $|||\cdot|||$ be a Banach norm on $D(A)$ with $(D(A),|||\cdot|||) \hookrightarrow D(A)$. If*

$$ \partial_t + A : W^1_q(\mathbb{R},v;X) \cap L_q(\mathbb{R},v;(D(A),|||\cdot|||)) \longrightarrow L_q(\mathbb{R},v;X) $$

is an isomorphism of Banach spaces, then $|||\cdot||| \eqsim ||\cdot||_{D(A)}$ and $\imath\mathbb{R} \subset \rho(-A)$ with

$$ ||(\imath\xi + A)^{-1}||_{\mathscr{B}(X)} \lesssim \frac{1}{1+|\xi|}, \qquad \xi \in \mathbb{R}. $$

In particular, A is a closed linear operator on X enjoying the property of $L_q(v,\mathbb{R})$-maximal regularity.

Proof. A slight modification of [188, Satz 2.2] gives a mapping $R:\mathbb{R}\to\mathscr{B}(X,(D(A),|||\cdot|||))$ with the property that

$$(\imath\xi + A)R(\xi) = I_X \quad\text{and}\quad ||R(\xi)||_{\mathscr{B}(X)} \lesssim \frac{1}{1+|\xi|}, \qquad \xi\in\mathbb{R}.$$

Similarly to [77, Theorem 4.1], using [77, Theorem 3.7] modified to the real line, it follows from the construction of $R(\xi)$ from [188, Satz 2.2] that also $R(\xi)(\imath\xi + A) = I_{D(A)}$ for each $\xi\in\mathbb{R}$. This shows that $\imath\mathbb{R}\subset\rho(-A)$ with $(\imath\xi + A)^{-1} = R(\xi)$. But then

$$|||x||| = |||R(0)Ax||| \lesssim ||Ax||_X \le ||x||_{D(A)}, \qquad x\in D(A). \qquad \square$$

Lemma 6.3.2. *Let X be a Banach space, $q\in(1,\infty)$ and $v\in A_q(\mathbb{R})$. Let A be a closed linear operator on X with $\mathbb{C}_+\subset\rho(-A)$ enjoying the property of $L_q(\mathbb{R},v)$-maximal regularity, where $\mathbb{C}_+ = \{z\in\mathbb{C} : \mathrm{Re}(z) > 0\}$. Suppose that $\lambda\mapsto\|\lambda + A\|_{\mathscr{B}(X)}$ is bounded on $\mathbb{C}_+$. Then $-A$ is the generator of an exponentially stable analytic semigroup on X and A also enjoys the property of $L_q(\mathbb{R}_+,v)$-maximal regularity.*

Proof. As A enjoys the property of $L_q(\mathbb{R},v)$-maximal regularity, Lemma 6.3.1 applies with $|||\cdot||| = ||\cdot||_{D(A)}$. Therefore, $\overline{\mathbb{C}_+}\subset\rho(-A)$ and $\lambda\mapsto(\lambda + A)^{-1}$ and $\lambda\mapsto\lambda(\lambda + A)^{-1}$ are well-defined analytic functions $\overline{\mathbb{C}_+}\to\mathscr{B}(X)$. Moreover, both mappings satisfy the assumptions of the Phragmen-Lindelöf Theorem (see [**?** , Corollary 6.4.4]) so that both mappings are bounded. Hence, it follows from Poisson's formula that

$$\sup_{\lambda\in\mathbb{C}_+}||(\lambda + A)^{-1}||_{\mathscr{B}(X)} \le \sup_{\theta\in\mathbb{R}}||(\imath\theta + A)^{-1}||_{\mathscr{B}(X)} < \infty$$

and

$$\sup_{\lambda\in\mathbb{C}_+}||\lambda(\lambda + A)^{-1}||_{\mathscr{B}(X)} \le \sup_{\theta\in\mathbb{R}}||\imath\theta(\imath\theta + A)^{-1}||_{\mathscr{B}(X)} < \infty.$$

It follows that $-A$ is the generator of an exponentially stable analytic semigroup $(e^{-tA})_{t\ge0}$ on X.

Finally, as $-A$ is the generator of an exponentially stable analytic semigroup on X, the variation of constants formula yields $L_q(\mathbb{R}_+,v)$-maximal regularity. Indeed, viewing ${}_0W^1_q(\mathbb{R}_+,v;X)\cap L_q(\mathbb{R}_+,v;D(A))$ and $L_q(\mathbb{R}_+,v;X)$ as closed subspaces of $W^1_q(\mathbb{R},v;X)\cap L_q(\mathbb{R},v;D(A))$ and $L_q(\mathbb{R},v;X)$, respectively, through extension by zero, the formula

$$[(\partial_t + A)^{-1}f](t) = \int_{-\infty}^{t} e^{-(t-s)A}f(s)\,ds, \qquad f\in L_q(\mathbb{R},v;X),\, t\in\mathbb{R},$$

shows that $(\partial_t + A)^{-1}$ maps $L_q(\mathbb{R}_+,v;X)$ to ${}_0W^1_q(\mathbb{R}_+,v;X)\cap L_q(\mathbb{R}_+,v;D(A))$. $\square$

As an application of its operator-valued Fourier multiplier theorem, Weis [244] characterized $L_q(\mathbb{R}_+)$-maximal regularity in terms of $\mathscr{R}$-sectoriality in the setting of UMD Banach spaces. The corresponding result for $L_q(\mathbb{R})$-maximal regularity involves R-bisectoriality, see [13]. Using [90, Theorem 3.5] and Theorem 6.A.1, these results carry over to the weighted setting.

Let us introduce the notion of R-boundedness. Let X be a Banach space. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a Rademacher sequence on some probability space $(\Omega,\mathscr{F},\mathbb{P})$, i.e. a sequence of independent random variables with $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$. A collection of operators $\mathscr{T} \subset \mathscr{B}(X)$ is called R-bounded if there exists a finite constant $C \geq 0$ such that, for all $T_0,\dots,T_K \in \mathscr{T}$ and $x_0,\dots,x_K \in X$,

$$|| \sum_{k=0}^{K} \varepsilon_k T_k x_k ||_{L_2(\Omega;X)} \leq C || \sum_{k=0}^{K} \varepsilon_k x_k ||_{L_2(\Omega;X)}.$$

The least such constant C is called the R-bound of $\mathscr{T}$ and is denoted by $\mathscr{R}(\mathscr{T})$.

The space $\mathrm{Rad}_p(\mathbb{N};X)$, where $p \in [1,\infty)$, is defined as the Banach space of sequence $(x_k)_{k\in\mathbb{N}}$ for which there is convergence of $\sum_{k=0}^{\infty} \varepsilon_k x_k$ in $L_p(\Omega;X)$, endowed with the norm

$$||(x_k)_{k\in\mathbb{N}}||_{\mathrm{Rad}_p(\mathbb{N};X)} := || \sum_{k=0}^{\infty} \varepsilon_k x_k ||_{L_p(\Omega;X)} = \sup_{K\geq 0} || \sum_{k=0}^{K} \sum_{k=0}^{K} \varepsilon_k x_k ||_{L_p(\Omega;X)}.$$

As a consequence of the Kahane-Khintchine inequalities, $\mathrm{Rad}_p(\mathbb{N};X) = \mathrm{Rad}_q(\mathbb{N};X)$ with an equivalence of norms. We put $\mathrm{Rad}(\mathbb{N};X) = \mathrm{Rad}_2(\mathbb{N};X)$. Note that collection of operators $\mathscr{T} \subset \mathscr{B}(X)$ is called R-bounded if and only if $\{\mathrm{diag}(T_0,\dots,T_K) : T_0,\dots,T_K \in \mathscr{T}\} \subset \mathscr{B}(\mathrm{Rad}(\mathbb{N};X))$ is a uniformly bounded, in which case the R-bound coincides with that uniform bound; here

$$\mathrm{diag}(T_0,\dots,T_K)(x_k)_{k\in\mathbb{N}} = (T_0x_0,\dots,T_Kx_K,0,0,0,\dots).$$

Furthermore, note that, as a consequence of the Kahane-Khintchine inequalities and Fubini, given $p \in [1,\infty)$ and a σ-finite measure space $(S,\mathscr{A},\mu)$, there is a natural isomorphism of Banach spaces

$$\mathrm{Rad}(\mathbb{N};L_p(S;X)) = L_p(S;\mathrm{Rad}(\mathbb{N};X)).$$

Having introduced the notion of R-boundedness, we can now give the definition of R-sectoriality, which is an R-boundedness version of sectoriality.

Recall that an unbounded operator A on a Banach space X is a *sectorial operator* if A is injective, closed, has dense range and there exists a $\phi \in (0,\pi)$ such that $\Sigma_{\pi-\phi} \subset \rho(-A)$ and

$$\sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\|_{\mathscr{B}(X)} < \infty.$$

The infimum over all possible ϕ is called the *angle of sectoriality* and is denoted by $\omega(A)$. In this case we also say that A is *sectorial of angle* $\omega(A)$. The condition that A has dense range is automatically fulfilled if X is reflexive (see [127, Proposition 10.1.9]).

We say that an unbounded operator A on a Banach space X is an *R-sectorial operator* if A is injective, closed, has dense range and there exists a $\phi \in (0,\pi)$ such that $\Sigma_{\pi-\phi} \subset \rho(-A)$ and

$$\mathscr{R}(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) < \infty \quad \text{in} \quad \mathscr{B}(X).$$

The infimum over all possible ϕ is called the *angle of R-sectoriality* and is denoted by $\omega_R(A)$. In this case we also say that A is *R-sectorial of angle* $\omega_R(A)$.

A way to approach L_q-maximal regularity is through operator sum methods, as initiated by Dore & Venni [78]. Using the Kalton–Weis operator sum theorem [134, Theorem 6.3] in combination with [**?** , Proposition 2.7], we obtain the following result:

Proposition 6.3.3. *Let X be a UMD space, $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$. If A is a closed linear operator on a Banach space X with $0 \in \rho(A)$ that is R-sectorial of angle $\omega_R(A) < \frac{\pi}{2}$, then A enjoys the properties of $L_q(v,\mathbb{R})$-maximal regularity and $L_q(v,\mathbb{R}_+)$-maximal regularity.*

6.3.3. Decomposition and Anisotropy

Let $d = |\mathpzc{d}|_1 = \mathpzc{d}_1 + \ldots + \mathpzc{d}_l$ with $\mathpzc{d} = (\mathpzc{d}_1, \ldots, \mathpzc{d}_l) \in (\mathbb{Z}_{\geq 1})^l$. The decomposition

$$\mathbb{R}^n = \mathbb{R}^{\mathpzc{d}_1} \times \ldots \times \mathbb{R}^{\mathpzc{d}_l}.$$

is called the *$\mathpzc{d}$-decomposition* of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we accordingly write $x = (x_1, \ldots, x_l)$ and $x_j = (x_{j,1}, \ldots, x_{j,\mathpzc{d}_j})$, where $x_j \in \mathbb{R}^{\mathpzc{d}_j}$ and $x_{j,i} \in \mathbb{R}$ $(j = 1, \ldots, l; i = 1, \ldots, \mathpzc{d}_j)$. We also say that we view $\mathbb{R}^n$ as being *$\mathpzc{d}$-decomposed.* Furthermore, for each $k \in \{1, \ldots, l\}$ we define the inclusion map

$$\iota_k = \iota_{[\mathpzc{d};k]} : \mathbb{R}^{\mathpzc{d}_k} \longrightarrow \mathbb{R}^n,\; x_k \mapsto (0, \ldots, 0, x_k, 0, \ldots, 0),$$

and the projection map

$$\pi_k = \pi_{[\mathpzc{d};k]} : \mathbb{R}^n \longrightarrow \mathbb{R}^{\mathpzc{d}_k},\; x = (x_1, \ldots, x_l) \mapsto x_k.$$

Given $\boldsymbol{a} \in (0,\infty)^l$, we define the $(\mathpzc{d}, \boldsymbol{a})$-anisotropic dilation $\delta_\lambda^{(\mathpzc{d},\boldsymbol{a})}$ on $\mathbb{R}^n$ by $\lambda > 0$ to be the mapping $\delta_\lambda^{(\mathpzc{d},\boldsymbol{a})}$ on $\mathbb{R}^n$ given by the formula

$$\delta_\lambda^{(\mathpzc{d},\boldsymbol{a})} x := (\lambda^{a_1} x_1, \ldots, \lambda^{a_l} x_l), \qquad x \in \mathbb{R}^n.$$

A $(\mathpzc{d}, \boldsymbol{a})$-anisotropic distance function on $\mathbb{R}^n$ is a function $u : \mathbb{R}^n \longrightarrow [0,\infty)$ satisfying

(i) $u(x) = 0$ if and only if $x = 0$.

(ii) $u(\delta_\lambda^{(\mathpzc{d},\boldsymbol{a})} x) = \lambda u(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$.

(iii) There exists a $c > 0$ such that $u(x + y) \leq c(u(x) + u(y))$ for all $x, y \in \mathbb{R}^n$.

All $(\mathpzc{d}, \boldsymbol{a})$-anisotropic distance functions on $\mathbb{R}^n$ are equivalent: Given two $(\mathpzc{d}, \boldsymbol{a})$-anisotropic distance functions u and v on $\mathbb{R}^n$, there exist constants $m, M > 0$ such that $mu(x) \leq v(x) \leq Mu(x)$ for all $x \in \mathbb{R}^n$

In this paper we will use the $(\mathpzc{d}, \boldsymbol{a})$-anisotropic distance function $|\cdot|_{\mathpzc{d},\boldsymbol{a}} : \mathbb{R}^n \longrightarrow [0,\infty)$ given by the formula

$$|x|_{\mathpzc{d},\boldsymbol{a}} := \left(\sum_{j=1}^{l} |x_j|^{2/a_j} \right)^{1/2} \qquad (x \in \mathbb{R}^n).$$

6.3.4. Distribution Theory and Function Spaces

[7] [38] [156]

DISTRIBUTION THEORY AND SOME GENERIC FUNCTION SPACE THEORY

Let X be a Banach space. The spaces of X-valued distributions and X-valued tempered distributions on $\mathbb{R}^n$ are defined as $\mathscr{D}'(\mathbb{R}^n;X) := \mathscr{L}(\mathscr{D}(\mathbb{R}^n;X))$ and $\mathscr{S}'(\mathbb{R}^n;X) := \mathscr{L}(\mathscr{S}(\mathbb{R}^n;X))$, respectively; for the theory of vector-valued distributions we refer to [7] (and [4, Section III.4]).

Let $\mathbb{E} \hookrightarrow \mathscr{D}'(U;X)$ be a Banach space of distributions on an open subset $U \subset \mathbb{R}^n$. Given an open subset $V \subset U$,

$$\mathbb{E}(V) := \{f \in \mathscr{D}'(V;X) : \exists g \in \mathbb{E}, g_{|V} = f\}$$

equipped with the norm

$$||f||_{\mathbb{E}(V)} := \inf\{||g||_{\mathbb{E}} : g \in \mathbb{E}, g_{|V} = f\}$$

is a Banach space with $\mathbb{E}(V) \hookrightarrow \mathscr{D}'(V;X)$. Note that $f \mapsto f_V$ defines defines a contraction $\mathbb{E} \to \mathbb{E}(V)$. Furthermore, note that, if $\mathbb{E} \hookrightarrow \mathbb{F} \hookrightarrow \mathscr{D}'(U;X)$, then $\mathbb{E}(V) \hookrightarrow \mathbb{F}(V)$. More generally, given Banach spaces $\mathbb{E} \hookrightarrow \mathscr{D}'(U_1;X_1)$ and $\mathbb{F} \hookrightarrow \mathscr{D}'(U_2;X_2)$, $T \in \mathscr{B}(\mathbb{E},\mathbb{F})$ and open subsets $V_1 \subset U_1$, $V_2 \subset U_2$ with the property that

$$\forall f,g \in \mathbb{E},\ f_{|V_1} = g_{|V_1} \Longrightarrow (Tf)_{|V_2} = (Tg)_{|V_2},$$

T induces an operator $\widetilde{T} \in \mathscr{B}(\mathbb{E}(V_1),\mathbb{F}(V_2))$ satisfying $(Tf)_{V_2} = \widetilde{T}(f_{|V_1})$ for all $f \in \mathbb{E}$.

Given a Banach space Z, $\mathscr{O}_M(\mathbb{R}^n;Z)$ denotes the space of slowly increasing Z-valued smooth functions on $\mathbb{R}^n$. Pointwise multiplication $(f,g) \mapsto fg$ yields separately continuous bilinear mappings

$$\begin{array}{ccccc} \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(X)) & \times & \mathscr{S}(\mathbb{R}^n;X) & \longrightarrow & \mathscr{S}(\mathbb{R}^n;X), \\ \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(X)) & \times & \mathscr{S}'(\mathbb{R}^n;X) & \longrightarrow & \mathscr{S}'(\mathbb{R}^n;X). \end{array} \tag{6.17}$$

As a consequence, $(m,f) \mapsto \mathscr{F}^{-1}[m\hat{f}]$ yields separately continuous bilinear mappings (6.17). We use the following notation:

$$T_m f = \mathrm{OP}[m]f = m(D)f := \mathscr{F}^{-1}[m\hat{g}].$$

Let $\mathbb{E} \hookrightarrow \mathscr{D}'(U;X)$ be a Banach space of distributions on an open subset $U \subset \mathbb{R}^n$. For a finite set of multi-indices $J \subset \mathbb{N}^d$ we define the Sobolev space $\mathscr{W}^J[\mathbb{E}]$ as the space of all $f \in \mathbb{E}$ with $D^\alpha f \in \mathbb{E}$ for every $\alpha \in J$, equipped with the norm

$$||f||_{\mathscr{W}^J[\mathbb{E}]} := \sum_{\alpha \in J} ||D^\alpha f||_{\mathbb{E}}.$$

Then $\mathscr{W}^J[\mathbb{E}]$ is a Banach space with $\mathscr{W}^J[\mathbb{E}] \hookrightarrow \mathbb{E} \hookrightarrow \mathscr{D}'(U;X)$. Note that if $\mathbb{F} \hookrightarrow \mathscr{D}'(U;X)$ is another Banach space, then

$$\mathbb{E} \hookrightarrow \mathbb{F} \quad \text{implies} \quad \mathscr{W}^J[\mathbb{E}] \hookrightarrow \mathscr{W}^J[\mathbb{F}].$$

Given $\boldsymbol{n} \in (\mathbb{N}_{\geq 1})^l$, we define $\mathscr{W}^{\boldsymbol{n}}_{\mathpzc{d}}[\mathbb{E}] := \mathscr{W}^{J_{\boldsymbol{n},\mathpzc{d}}}[\mathbb{E}]$, where

$$J_{\boldsymbol{n},\mathpzc{d}} := \left\{ \alpha \in \bigcup_{j=1}^{l} \iota_{[\mathpzc{d};j]}\mathbb{N}^{\mathpzc{d}_j} : |\alpha_j| \leq n_j \right\}.$$

Suppose $\mathbb{R}^n$ is $\mathpzc{d}$-decomposed as in Section 6.3.3. For a Banach space Z, $\boldsymbol{a} \in (0,\infty)^l$ and $N \in \mathbb{N}$ we define $\mathscr{M}^{(\mathpzc{d},\boldsymbol{a})}_N(Z)$ as the space of all $m \in C^N(\mathbb{R}^n;Z)$ for which

$$||m||_{\mathscr{M}^{(\mathpzc{d},\boldsymbol{a})}_N(Z)} := \sup_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^n} (1+|\xi|_{\mathpzc{d},\boldsymbol{a}})^{\boldsymbol{a}\cdot_{\mathpzc{d}}\alpha} ||D^\alpha m(\xi)||_Z < \infty.$$

When $\boldsymbol{a} = \mathbf{1}$ we simply write $\mathscr{M}_N(Z) = \mathscr{M}^{(\mathpzc{d},\mathbf{1})}_N(Z)$.

Let $\boldsymbol{a} \in (0,\infty)^l$. A normed space $\mathbb{E} \subset \mathscr{S}'(\mathbb{R}^n;X)$ is called $(\mathpzc{d},\boldsymbol{a})$-admissible if there exists an $N \in \mathbb{N}$ such that

$$m(D)f \in \mathbb{E} \quad \text{with} \quad ||m(D)f||_{\mathbb{E}} \lesssim ||m||_{\mathscr{M}^{(\mathpzc{d},\boldsymbol{a})}_N} ||f||_{\mathbb{E}}, \qquad (m,f) \in \mathscr{O}_M(\mathbb{R}^n) \times \mathbb{E}.$$

In case $\boldsymbol{a} = \mathbf{1}$ we simply speak of admissible.

To each $\sigma \in \mathbb{R}$ we associate the operators $\mathscr{J}^{[\mathpzc{d};j]}_\sigma \in \mathscr{L}(\mathscr{S}'(\mathbb{R}^n;X))$ and $\mathscr{J}^{\mathpzc{d},\boldsymbol{a}}_\sigma \in \mathscr{L}(\mathscr{S}'(\mathbb{R}^n;X))$ given by

$$\mathscr{J}^{[\mathpzc{d};j]}_\sigma f := \mathscr{F}^{-1}[(1+|\pi_{[\mathpzc{d};j]}|^2)^{\sigma/2}\hat{f}] \qquad \text{and} \qquad \mathscr{J}^{\mathpzc{d},\boldsymbol{a}}_\sigma f := \sum_{k=1}^{l} \mathscr{J}^{[\mathpzc{d};k]}_{\sigma/a_k} f.$$

We call $\mathscr{J}^{\mathpzc{d},\boldsymbol{a}}_\sigma$ the $(\mathpzc{d},\boldsymbol{a})$-anisotropic Bessel potential operator of order σ.

Let $\mathbb{E} \hookrightarrow \mathscr{S}'(\mathbb{R}^n;X)$ be a Banach space. Given $\boldsymbol{n} \in (\mathbb{N}_1)^l$, $\boldsymbol{s}, \boldsymbol{a} \in (0,\infty)^l$, and $s \in \mathbb{R}$, we define the Banach spaces $\mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}], \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}] \hookrightarrow \mathscr{S}'(\mathbb{R}^n;X)$ as follows:

$$\begin{aligned} \mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}] &:= \{f \in \mathscr{S}'(\mathbb{R}^n) : \mathscr{J}^{[\mathpzc{d};j]}_{s_j} f \in \mathbb{E}, j = 1,\dots,l\}, \\ \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}] &:= \{f \in \mathscr{S}'(\mathbb{R}^n) : \mathscr{J}^{\mathpzc{d},\boldsymbol{a}}_s f \in \mathbb{E}\}, \end{aligned}$$

with the norms

$$||f||_{\mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}]} = \sum_{j=1}^{l} ||\mathscr{J}^{[\mathpzc{d};j]}_{s_j} f||_{\mathbb{E}}, \qquad ||f||_{\mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}]} = ||\mathscr{J}^{\mathpzc{d},\boldsymbol{a}}_s f||_{\mathbb{E}}.$$

Note that $\mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}] \hookrightarrow \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}]$ contractively in case that $\boldsymbol{s} = (s/a_1,\dots,s/a_l)$. Furthermore, note that if $\mathbb{F} \hookrightarrow \mathscr{S}'(\mathbb{R}^n;X)$ is another Banach space, then

$$\mathbb{E} \hookrightarrow \mathbb{F} \quad \text{implies} \quad \mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}] \hookrightarrow \mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{F}], \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}] \hookrightarrow \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{F}]. \tag{6.18}$$

We write

$$\widetilde{J}_{\boldsymbol{n},\mathpzc{d}} := \{0\} \cup \left\{ \iota_{[\mathpzc{d};j]} e_i^{[d_j]} : j = 1,\dots,l, i = 1,\dots,d_i \right\}, \qquad \boldsymbol{n} \in (\mathbb{N}_{\geq 1})^l,$$

where $e_i^{[d_j]}$ is the standard i=th basis vector in $\mathbb{R}^{d_j}$. If $\mathbb{E} \hookrightarrow \mathscr{S}'(\mathbb{R}^n; X)$ is a $(\mathpzc{d}, \boldsymbol{a})$-admissible Banach space for a given $\boldsymbol{a} \in (0,\infty)^l$, then

$$\mathscr{W}^J[\mathbb{E}] = \mathscr{H}^{\boldsymbol{n}}_{\mathpzc{d}}[\mathbb{E}] = \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}], \qquad s \in \mathbb{R}, \boldsymbol{n} = s\boldsymbol{a}^{-1} \in (\mathbb{Z}_{\geq 1})^l, \widetilde{J}_{\boldsymbol{n},\mathpzc{d}} \subset J \subset J_{\boldsymbol{n},\mathpzc{d}}, \tag{6.19}$$

and

$$\mathscr{H}^{\boldsymbol{s}}_{\mathpzc{d}}[\mathbb{E}] = \mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}], \qquad s > 0, \boldsymbol{s} = s\boldsymbol{a}^{-1}. \tag{6.20}$$

Furthermore,

$$D^\alpha \in \mathscr{B}(\mathscr{H}^{s,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}], \mathscr{H}^{s-\boldsymbol{a}\cdot_{\mathpzc{d}}\alpha,\boldsymbol{a}}_{\mathpzc{d}}[\mathbb{E}]), \qquad s \in \mathbb{R}, \alpha \in \mathbb{N}^d. \tag{6.21}$$

FUNCTION SPACES

Anisotropic mixed-norm spaces Let X be a Banach space and suppose that $\mathbb{R}^n$ is $\mathpzc{d}$-decomposed as in Section 6.3.3.

Let $\mathscr{O} = \prod_{j=1}^l \mathscr{O}_j \subset \mathbb{R}^n$ with $\mathscr{O}_j$ an open subset of $\mathbb{R}^{d_j}$ for each j. For $\boldsymbol{p} \in (1,\infty)^l$ and a weight vector $\boldsymbol{w} \in \prod_{j=1}^l \mathscr{W}(\mathscr{O}_j)$ with $\boldsymbol{p}$-dual weight vector $\boldsymbol{w}'_{\boldsymbol{p}} \in \prod_{j=1}^l L_{1,\mathrm{loc}}(\mathscr{O}_j)$, there is the inclusion $L_{\boldsymbol{p}}(\mathscr{O}, \boldsymbol{w}; X) \hookrightarrow \mathscr{D}'(\mathscr{O}; X)$ (which can be seen through the pairing (6.13)). So we can define the associated Sobolev space of order $\boldsymbol{k} \in \mathbb{N}^l$

$$W^{\boldsymbol{k}}_{\boldsymbol{p}}(\mathscr{O}, \boldsymbol{w}; X) := \mathscr{W}^{(\boldsymbol{k},\mathpzc{d})}[L_{\boldsymbol{p}}(\mathscr{O}, \boldsymbol{w}; X)].$$

An example of a weight w on a C^∞−domain $\mathscr{O} \subset \mathbb{R}^n$ for which the p-dual weight $w'_p = w^{-\frac{1}{p-1}} \in L_{1,\mathrm{loc}}(\mathscr{O})$ is the power weight $w^{\partial\mathscr{O}}_\gamma = \mathrm{dist}(\cdot, \partial\mathscr{O})^\gamma$ with $\gamma \in \mathbb{R}$. Furthermore, note that $w'_p \in A_\infty(\mathbb{R}^n) \subset L_{1,\mathrm{loc}}(\mathbb{R}^n)$ for $w \in [A_\infty]'_p(\mathbb{R}^n) \supset A_p(\mathbb{R}^n)$.

Let $\boldsymbol{p} \in (1,\infty)^l$ and $\boldsymbol{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Then $\boldsymbol{w}'_{\boldsymbol{p}} \in A_{p'}$, so that $\mathscr{S}(\mathbb{R}^n) \overset{d}{\hookrightarrow} L_{\boldsymbol{p}'}(\mathbb{R}^n, \boldsymbol{w}'_{\boldsymbol{p}})$. Using the pairing (6.13), we find that $L_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^n; X)$ in the natural way. For $\boldsymbol{a} \in (0,\infty)^l$, $s \in \mathbb{R}$ and $\boldsymbol{s} \in (0,\infty)^l$ we can thus define the Bessel potential spaces

$$H^{\boldsymbol{s}}_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X) := \mathscr{H}^{(\boldsymbol{s},\mathpzc{d})}_{\mathpzc{d}}[L_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)], \qquad H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X) := \mathscr{H}^{s,(\boldsymbol{a},\mathpzc{d})}_{\mathpzc{d}}[L_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)].$$

If X is a UMD space and $\boldsymbol{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})^3$, then $L_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)$ is $(\boldsymbol{a}, \mathpzc{d})$-admissible (see [90]). In particular, if X is a UMD space, then (6.19), (6.20) and (6.21) hold true with $\mathbb{E} = L_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)$.

Let $\boldsymbol{a} \in (0,\infty)^l$. For $0 < A < B < \infty$ we define $\Phi^{\mathpzc{d},\boldsymbol{a}}_{A,B}(\mathbb{R}^n)$ as the set of all sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n)$ which are constructed in the following way: given a $\varphi_0 \in \mathscr{S}(\mathbb{R}^n)$ satisfying

$$0 \leq \hat{\varphi}_0 \leq 1, \; \hat{\varphi}_0(\xi) = 1 \text{ if } |\xi|_{\mathpzc{d},\boldsymbol{a}} \leq A, \; \hat{\varphi}_0(\xi) = 0 \text{ if } |\xi|_{\mathpzc{d},\boldsymbol{a}} \geq B,$$

[3] $\boldsymbol{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$ should already work, but this is not available in the literature and not needed in this paper anyway.

$(\varphi_n)_{n\geq 1} \subset \mathscr{S}(\mathbb{R}^n)$ is defined via the relations

$$\hat{\varphi}_n(\xi) = \hat{\varphi}_1(\delta^{(d,\boldsymbol{a})}_{2^{-n+1}}\xi) = \hat{\varphi}_0(\delta^{(d,\boldsymbol{a})}_{2^{-n}}\xi) - \hat{\varphi}_0(\delta^{(d,\boldsymbol{a})}_{2^{-n+1}}\xi), \quad \xi \in \mathbb{R}^n, n \geq 1.$$

Observe that

$$\operatorname{supp}\hat{\varphi}_0 \subset \{\xi \mid |\xi|_{d,\boldsymbol{a}} \leq B\} \quad \text{and} \quad \operatorname{supp}\hat{\varphi}_n \subset \{\xi \mid 2^{n-1}A \leq |\xi|_{d,\boldsymbol{a}} \leq 2^n B\}, \quad n \geq 1.$$

We put $\Phi^{d,\boldsymbol{a}}(\mathbb{R}^n) := \bigcup_{0<A<B<\infty}\Phi^{d,\boldsymbol{a}}_{A,B}(\mathbb{R}^n)$. In case $l=1$ we write $\Phi^a(\mathbb{R}^n) = \Phi^{d,a}(\mathbb{R}^n)$, $\Phi(\mathbb{R}^n) = \Phi^1(\mathbb{R}^n)$, $\Phi^a_{A,B}(\mathbb{R}^n) = \Phi^{d,a}_{A,B}(\mathbb{R}^n)$, and $\Phi_{A,B}(\mathbb{R}^n) = \Phi^1_{A,B}(\mathbb{R}^n)$.

To $\varphi \in \Phi^{d,a}(\mathbb{R}^n)$ we associate the family of convolution operators $(S_n)_{n\in\mathbb{N}} = (S^{\varphi}_n)_{n\in\mathbb{N}} \subset \mathscr{L}(\mathscr{S}'(\mathbb{R}^n;X), \mathscr{O}_M(\mathbb{R}^n;X)) \subset \mathscr{L}(\mathscr{S}'(\mathbb{R}^n;X))$ given by

$$S_n f = S^{\varphi}_n f := \varphi_n * f = \mathscr{F}^{-1}[\hat{\varphi}_n \hat{f}] \qquad (f \in \mathscr{S}'(\mathbb{R}^n;X)). \tag{6.22}$$

It holds that $f = \sum_{n=0}^{\infty} S_n f$ in $\mathscr{S}'(\mathbb{R}^n;X)$ respectively in $\mathscr{S}(\mathbb{R}^n;X)$ whenever $f \in \mathscr{S}'(\mathbb{R}^n;X)$ respectively $f \in \mathscr{S}(\mathbb{R}^n;X)$.

Given $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in [1,\infty)^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $\boldsymbol{w} \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$, the Besov space $B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)$ is defined as the Banach space of all $f \in \mathscr{S}'(\mathbb{R}^n;X)$ for which

$$||f||_{B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)} := ||(2^{ns}S^{\varphi}_n f)_{n\in\mathbb{N}}||_{\ell_q(\mathbb{N})[L_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w})](X)} < \infty$$

and the Triebel-Lizorkin space $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)$ is defined as the Banach space of all $f \in \mathscr{S}'(\mathbb{R}^n;X)$ for which

$$||f||_{F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)} := ||(2^{ns}S^{\varphi}_n f)_{n\in\mathbb{N}}||_{L_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w})[\ell_q(\mathbb{N})](X)} < \infty.$$

Up to an equivalence of extended norms on $\mathscr{S}'(\mathbb{R}^n;X)$, $||\cdot||_{B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)}$ and $||\cdot||_{F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)}$ do not depend on the particular choice of $\varphi \in \Phi^{d,\boldsymbol{a}}(\mathbb{R}^n)$.

Let us note some basic relations between these spaces. Monotonicity of ℓ^q-spaces yields that, for $1 \leq q_0 \leq q_1 \leq \infty$,

$$B^{s,\boldsymbol{a}}_{\boldsymbol{p},q_0,d}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow B^{s,\boldsymbol{a}}_{\boldsymbol{p},q_1,d}(\mathbb{R}^n,\boldsymbol{w};X), \qquad F^{s,\boldsymbol{a}}_{\boldsymbol{p},q_0,d}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow F^{s,\boldsymbol{a}}_{\boldsymbol{p},q_1,d}(\mathbb{R}^n,\boldsymbol{w};X). \tag{6.23}$$

For $\varepsilon > 0$ it holds that

$$B^{s,\boldsymbol{a}}_{\boldsymbol{p},\infty,d}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow B^{s-\varepsilon,\boldsymbol{a}}_{\boldsymbol{p},1,d}(\mathbb{R}^n,\boldsymbol{w};X). \tag{6.24}$$

Furthermore, Minkowksi's inequality gives

$$B^{s,\boldsymbol{a}}_{\boldsymbol{p},\min\{p_1,\dots,p_l,q\},d}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow B^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow B^{s,\boldsymbol{a}}_{\boldsymbol{p},\max\{p_1,\dots,p_l,q\},d}(\mathbb{R}^n,\boldsymbol{w};X). \tag{6.25}$$

The Besov space $B^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X)$ and the Triebel-Lizorkin space $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X)$ are examples of $(d,\boldsymbol{a})$-admissible Banach spaces. In fact (see [156, Proposition 5.2.26]), if $\mathbb{E} = B^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X)$ or $\mathbb{E} = F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X)$, then there exists an $N \in \mathbb{N}$, independent of X, such that

$$||m(D)f||_{\mathbb{E}} \lesssim_{\boldsymbol{p},q,\boldsymbol{a},\boldsymbol{w}} ||m||_{\mathscr{M}^{(d,\boldsymbol{a})}_N(\mathscr{B}(X))}||f||_{\mathbb{E}}, \qquad (m,f) \in \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(X)) \times \mathbb{E}. \tag{6.26}$$

Lemma 6.3.4. *Let X be a Banach space, $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in [1,\infty)^l$, $q \in [1,\infty)$, $\boldsymbol{w} \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$, $\mathscr{A} \in \{B,F\}$ and $s \in \mathbb{R}$. There exists $N \in \mathbb{N}$, only depending on $\boldsymbol{a},\boldsymbol{p},q,n,\boldsymbol{w}$, such that if $\mathscr{M} \subset \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(X))$ satisfies*

$$||\mathscr{M}||_{\mathscr{R}\mathscr{M}_N^{(\mathscr{d},\boldsymbol{a})}} := \sup_{|\alpha|\leq N} \mathscr{R}\left\{(1+|\xi|_{\mathscr{d},\boldsymbol{a}})^{\boldsymbol{a}\cdot_{\mathscr{d}}\alpha} D^\alpha m(\xi) : \xi \in \mathbb{R}^n, m \in \mathscr{M}\right\},$$

then

$$\mathscr{R}\{T_m : m \in \mathscr{M}\} \lesssim_{\boldsymbol{a},\boldsymbol{p},q,n,\boldsymbol{w}} ||\mathscr{M}||_{\mathscr{R}\mathscr{M}_N^{(\mathscr{d},\boldsymbol{a})}} \qquad in \qquad \mathscr{B}(\mathscr{A}^s_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X))).$$

Proof. For simplicity of notation we only treat the case $\mathscr{A} = F$. Let N be as in (6.26) for $\mathbb{E} = F^s_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};\mathrm{Rad}(\mathbb{N};X)))$. Now consider $\mathscr{M} \subset \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(X))$ satisfying $||\mathscr{M}||_{\mathscr{R}\mathscr{M}_N^{(\mathscr{d},\boldsymbol{a})}} < \infty$. Let $m_0,\dots,m_M \in \mathscr{M}$. Then

$$\boldsymbol{m}(\xi) := \mathrm{diag}(m_1(\xi),\dots,m_M(\xi))$$

defines a symbol $\boldsymbol{m} \in \mathscr{O}_M(\mathbb{R}^n;\mathscr{B}(\mathrm{Rad}(\mathbb{N};X)))$ with $||\boldsymbol{m}||_{\mathscr{M}_N(\mathscr{B}(\mathrm{Rad}(\mathbb{N};X)))} \leq ||\mathscr{M}||_{\mathscr{R}\mathscr{M}_N}$. So, by (6.26), $T_{\boldsymbol{m}} \in \mathscr{B}(F^s_{p,r}(\mathbb{R}^n,w;\mathrm{Rad}(\mathbb{N};X))))$ with

$$||T_{\boldsymbol{m}}||_{\mathscr{B}(F^s_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};\mathrm{Rad}(\mathbb{N};X))))} \lesssim_{\boldsymbol{a},\boldsymbol{p},q,n,\boldsymbol{w}} ||\mathscr{M}||_{\mathscr{R}\mathscr{M}_N}.$$

Now note that

$$F^s_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};\mathrm{Rad}(\mathbb{N};X))) = \mathrm{Rad}(\mathbb{N};F^s_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X))$$

as a consequence of the Kahane-Khintchine inequalities and Fubini. Finally, the observation that $T_{\boldsymbol{m}} = \mathrm{diag}(T_{m_0},\dots,T_{m_M})$ completes the proof. □

Let $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in [1,\infty)^l$, $q \in [1,\infty]$, and $\boldsymbol{w} \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. For $s, s_0 \in \mathbb{R}$ it holds that

$$B^{s+s_0,\boldsymbol{a}}_{\boldsymbol{p},q,\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X) = \mathscr{H}^{s,\boldsymbol{a}}_{\mathscr{d}}[B^{s_0,\boldsymbol{a}}_{\boldsymbol{p},q,\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X)], \quad F^{s+s_0,\boldsymbol{a}}_{\boldsymbol{p},q,\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X) = \mathscr{H}^{s,\boldsymbol{a}}_{\mathscr{d}}[F^{s_0,\boldsymbol{a}}_{\boldsymbol{p},q,\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X)]. \tag{6.27}$$

Let $\boldsymbol{p} \in (1,\infty)^l$ and $\boldsymbol{w} \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{\mathscr{d}})$ If

- $\mathbb{E} = W^{\boldsymbol{n}}_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X)$, $\boldsymbol{n} \in \mathbb{N}^l$, $\boldsymbol{n} = s\boldsymbol{a}^{-1}$; or
- $\mathbb{E} = H^{s,\boldsymbol{a}}_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X)$; or
- $\mathbb{E} = H^{\boldsymbol{a}}_{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^n,\boldsymbol{w};X)$, $\boldsymbol{a} \in (0,1)^l$, $\boldsymbol{a} = s\boldsymbol{a}^{-1}$,

then we have the inclusions

$$F^{s,\boldsymbol{a}}_{\boldsymbol{p},1}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow \mathbb{E} \hookrightarrow F^{s,\boldsymbol{a}}_{\boldsymbol{p},\infty}(\mathbb{R}^n,\boldsymbol{w};X). \tag{6.28}$$

The following result is a representation for anisotropic mixed-norm Triebel-Lizorkin spaces in terms of classical isotropic Triebel-Lizorkin spaces (see Paragraph 6.3.4).

Theorem 6.3.5 ([157])**.** *Let X be a Banach space, $l = 2$, $\boldsymbol{a} \in (0,\infty)^2$, $p, q \in (1,\infty)$, $s > 0$, and $\boldsymbol{w} \in A_p(\mathbb{R}^{d_1}) \times A_q(\mathbb{R}^{d_2})$. Then*

$$F^{s,\boldsymbol{a}}_{(p,q),p}(\mathbb{R}^n, \boldsymbol{w}; X) = F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X)) \cap L^q(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)) \quad (6.29)$$

with equivalence of norms.

This intersection representation is actually a corollary of a more general intersection representation in [157]. In the above form it can also be found in [156, Theorem 5.2.35]. For the case $X = \mathbb{C}$, $d_1 = 1$, $\boldsymbol{w} = \mathbf{1}$ we refer to [64, Proposition 3.23].

In the parameter range that we have defined the function spaces $H^{s,\boldsymbol{a}}_{\boldsymbol{p},d}(\mathbb{R}^n, \boldsymbol{w}; X)$, $H^{s,\boldsymbol{a}}_{\boldsymbol{p},d}(\mathbb{R}^n, \boldsymbol{w}; X)$, $B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n, \boldsymbol{w}; X)$ and $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n, \boldsymbol{w}; X)$ above, the corresponding versions on open subsets $\mathscr{O} \subset \mathbb{R}^n$ are defined by restriction:

$$\begin{aligned} H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathscr{O}, \boldsymbol{w}; X) &:= [H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)](\mathscr{O}), & H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathscr{O}, \boldsymbol{w}; X) &:= [H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathbb{R}^n, \boldsymbol{w}; X)][\mathscr{O}], \\ B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathscr{O}, \boldsymbol{w}; X) &:= B^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n, \boldsymbol{w}; X)[\mathscr{O}], & F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathscr{O}, \boldsymbol{w}; X) &:= F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n, \boldsymbol{w}; X)[\mathscr{O}]. \end{aligned}$$

Isotropic spaces

Parameter-independent spaces In the special case $l = 1$ and $\boldsymbol{a} = 1$, the anisotropic mixed-norm spaces introduced in Paragraph 6.3.4 reduce to classical isotropic Sobolev, Bessel potential, Besov and Triebel-Lizorkin spaces $W^k_p(\mathscr{O}, w; X)$, $H^s_p(\mathscr{O}, w; X)$, $B^s_{p,q}(\mathscr{O}, w; X)$, $F^s_{p,q}(\mathscr{O}, w; X)$, respectively. In the case that $\mathscr{O}$ is a C^∞-domain and $w = w^{\partial\mathscr{O}}_\gamma$, we use the notation:

$$\begin{aligned} W^k_{p,\gamma}(\mathscr{O}; X) &:= W^k_p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), & H^s_{p,\gamma}(\mathscr{O}; X) &:= H^s_p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), \\ B^s_{p,q,\gamma}(\mathscr{O}; X) &:= B^s_{p,q}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), & F^s_{p,q,\gamma}(\mathscr{O}; X) &:= F^s_{p,q}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X). \end{aligned}$$

If X is a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, then $L_p(\mathbb{R}^n, w; X)$ is an admissible Banach space of tempered distributions. By lifting, $H^s_p(\mathbb{R}^n, w; X)$ is admissible as well. In fact, there is an operator-valued Mikhlin theorem for $H^s_p(\mathbb{R}^n, w; X)$ (obtained by lifting from $L_p(\mathbb{R}^n, w; X)$):

Proposition 6.3.6. *Let X be UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$. If $m \in C^{d+2}(\mathbb{R}^n \setminus \{0\}; \mathscr{B}(X))$ satisfies*

$$||m||_{\mathscr{R}\mathscr{M}_{n+2}} = \sup_{|\alpha| \le n+2} \mathscr{R}\{|\xi|^\alpha D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\} < \infty,$$

then

$$T_m : \mathscr{S}(\mathbb{R}^n; X) \longrightarrow L_\infty(\mathbb{R}^n; X),\ m \mapsto \mathscr{F}^{-1}[m\hat{f}],$$

extends to a bounded linear operator on $H^s_p(\mathbb{R}^n, w; X)$ with

$$||T_m||_{\mathscr{B}(H^s_p(\mathbb{R}^n, w; X))} \lesssim_{X,p,w,n}$$

Proof. The case $s = 0$ can be obtained as in [187, Proposition 3.1], from which the case of general $s \in \mathbb{R}$ subsequently follows by lifting. □

If X is a UMD space, $n \in \mathbb{N}$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, then, as a consequence of the admissibility,

$$H^n_p(\mathbb{R}^n, w; X) = W^n_p(\mathbb{R}^n, w; X). \tag{6.30}$$

In the reverse direction we for instance have that, given a Banach space X, if $H^1_p(\mathbb{R}; X) = W^1_p(\mathbb{R}; X)$, then X is a UMD space (see [126]).

In the scalar-valued case $X = \mathbb{C}$, we have

$$H^s_p(\mathbb{R}^n, w) = F^s_{p,2}(\mathbb{R}^n, w), \qquad p \in (1,\infty), w \in A_p. \tag{6.31}$$

In the Banach space-valued case, this identity is valid if and only if X is isomorphic to a Hilbert space. For general Banach spaces X we still have (see [182, Proposition 3.12])

$$F^s_{p,1}(\mathbb{R}^n, w; X) \hookrightarrow H^s_p(\mathbb{R}^n, w; X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^n, w; X), \qquad p \in (1,\infty), w \in A_p(\mathbb{R}^n), \tag{6.32}$$

$$F^k_{p,1}(\mathbb{R}^n, w; X) \hookrightarrow W^k_p(\mathbb{R}^n, w; X) \hookrightarrow F^k_{p,\infty}(\mathbb{R}^n, w; X), \qquad p \in (1,\infty), w \in A_p(\mathbb{R}^n), \tag{6.33}$$

and (see [**?** , (7.1)])

$$F^k_{p,1,\gamma}(\mathscr{O}; X) \hookrightarrow W^k_{p,\gamma}(\mathscr{O}; X), \qquad k \in \mathbb{N}, p \in (1,\infty)\gamma \in (-1,\infty), \tag{6.34}$$

where $\mathscr{O} \subset \mathbb{R}^n$ is a C^∞-domain with compact boundary.

For UMD spaces X there is a suitable randomized substitute for (6.31) (see [187, Proposition 3.2])

Let $\mathscr{O} \subset \mathbb{R}^n$ be a Lipschitz domain, $p \in [1,\infty)$, $r_0, r_1 \in [1,\infty]$, $\gamma_0, \gamma_1 \in (-1,\infty)$ and $s_0, s_1 \in \mathbb{R}$. By [182, 185], if $\gamma_1 > \gamma_0$ and $s_0 = s_1 + \frac{\gamma_0 - \gamma_1}{p}$, then

$$F^{s_0}_{p,r_0}(\mathbb{R}^n, w^{\partial\mathscr{O}}_{\gamma_0}; X) \hookrightarrow F^{s_1}_{p,r_1}(\mathbb{R}^n, w^{\partial\mathscr{O}}_{\gamma_1}; X). \tag{6.35}$$

For the next result the reader is referred to [164, Propositions 5.5& 5.6].

Proposition 6.3.7. *Let X be a UMD space and $p \in (1,\infty)$. Let $w \in A_p$ be such that $w(-x_1, \tilde{x}) = w(x_1, \tilde{x})$ for all $x_1 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^{d-1}$.*

(1) $H^{k,p}(\mathbb{R}^d_+, w; X) = W^{k,p}(\mathbb{R}^d_+, w; X)$ *for all* $k \in \mathbb{N}$.

(2) *Let* $\theta \in [0,1]$ *and* $s_0, s_1, s \in \mathbb{R}$ *be such that* $s = s_0(1-\theta) + s_1\theta$. *Then for* $\mathscr{O} = \mathbb{R}^d$ *or* $\mathscr{O} = \mathbb{R}^d_+$ *one has*

$$[H^{s_0,p}(\mathscr{O}, w; X), H^{s_1,p}(\mathscr{O}, w; X)]_\theta = H^{s,p}(\mathscr{O}, w; X)$$

(3) *For each* $m \in \mathbb{N}$ *there exists an* $\mathscr{E}^m_+ \in \mathscr{B}(H^{-m,p}(\mathbb{R}^d_+, w; X), H^{-m,p}(\mathbb{R}^d, w; X))$ *such that*

- *for all* $|s| \le m$, $\mathscr{E}^m_+ \in \mathscr{B}(H^{s,p}(\mathbb{R}^d_+, w; X), H^{s,p}(\mathbb{R}^d, w; X))$,
- *for all* $|s| \le m$, $f \mapsto (\mathscr{E}^m_+ f)|_{\mathbb{R}^d_+}$ *equals the identity operator on* $H^{s,p}(\mathbb{R}_+, w; X)$.

Moreover, if $f \in L^p(\mathbb{R}^d_+, w; X) \cap C^m(\overline{\mathbb{R}^d_+}; X)$*, then* $\mathscr{E}^m_+ f \in C^m(\mathbb{R}^d; X)$.

Theorem 6.3.8 (Rychkov's extension operator [217])**.** *Let* $\mathscr{O}$ *be a special Lipschitz domain in* $\mathbb{R}^n$ *or a Lipschitz domain in* $\mathbb{R}^n$ *with a compact boundary and let* X *be a Banach space. Then there exists a linear operator*

$$\mathscr{E} : D(\mathscr{E}) \subset \mathscr{D}'(\mathscr{O}; X) \longrightarrow \mathscr{D}'(\mathbb{R}^n; X)$$

with the properties that

- $(\mathscr{E} f)_{|\mathscr{O}} = f$ *for all* $f \in D(\mathscr{E})$*;*
- $\mathscr{A}^s_{p,q}(\mathscr{O}, w; X) \subset D(\mathscr{E})$ *with* $\mathscr{E} \in \mathscr{B}(\mathscr{A}^s_{p,q}(\mathscr{O}, w; X), \mathscr{A}^s_{p,q}(\mathbb{R}^n, w; X))$ *whenever* $p \in [1,\infty)$*,* $q \in [1,\infty]$ *and* $w \in A_\infty(\mathbb{R}^n)$*. In particular,* $\mathscr{S}(\mathscr{O}; X) \subset D(\mathscr{E})$ *with* $\mathscr{E} \in \mathscr{B}(\mathscr{S}(\mathscr{O}; X), BC^\infty(\mathbb{R}^n; X))$*.*

Proof. The existence of such an operator for the unweighted scalar-valued variant was obtained in [217, Theorem 4.1]. However, the proof given there extends to the weighted Banach space-valued setting. □

Let $\mathscr{O}$ be either $\mathbb{R}^n_+$ or a C^∞-domain in $\mathbb{R}^n$ with a compact boundary $\partial\mathscr{O}$. Let X be Banach space, $p \in [1,\infty)$, $q \in [1,\infty]$, $\gamma \in (-1,\infty)$ and $s \in \mathbb{R}$. It will be convenient to define

$$\partial B^s_{p,q,\gamma}(\partial\mathscr{O}; X) := B^{s-\frac{1+\gamma}{p}}_{p,q}(\partial\mathscr{O}; X) \qquad \text{and} \qquad \partial F^s_{p,q,\gamma}(\partial\mathscr{O}; X) := F^{s-\frac{1+\gamma}{p}}_{p,p}(\partial\mathscr{O}; X).$$

If $\mathscr{A} \in \{B, F\}$ and $s > \frac{1+\gamma}{p}$, then we have retractions

$$\mathrm{tr}_{\partial\mathscr{O}} : \mathscr{A}^s_{p,q}(\mathbb{R}^n, w^{\partial\mathscr{O}}_\gamma; X) \longrightarrow \partial\mathscr{A}^s_{p,q,\gamma}(\partial\mathscr{O}; X)$$

and

$$\mathrm{Tr}_{\partial\mathscr{O}} : \mathscr{A}^s_{p,q,\gamma}(\mathscr{O}; X) \longrightarrow \partial\mathscr{A}^s_{p,q,\gamma}(\partial\mathscr{O}; X)$$

that are related by $\mathrm{tr}_{\partial\mathscr{O}} = \mathrm{Tr}_{\partial\mathscr{O}} \circ \mathscr{E}$, where $\mathscr{E}$ is any choice of Ryckov's extension operator (from Theorem 6.3.8). There is compatibility for both of the trace operators $\mathrm{tr}_{\partial\mathscr{O}}$ and $\mathrm{Tr}_{\partial\mathscr{O}}$ on the different function spaces that are allowed above.

Let us now introduce reflexive Banach space-valued versions of the $\mathscr{B}$- and $\mathscr{F}$-scales, the scales dual to the B- and F-scales, respectively, as considered in [163]. Let X be a reflexive Banach space, $p, q \in (1,\infty)$, $w \in [A_\infty]'_p(\mathbb{R}^n)$ and $s \in \mathbb{R}$. Recall that $w'_p \in A_\infty$ by definition of $[A_\infty]'_p(\mathbb{R}^n)$. For $\mathscr{A} \in \{B, F\}$, $\mathscr{A}^{-s}_{p',q'}(\mathbb{R}^n, w'_p; X^*)$ is a reflexive Banach space with

$$\mathscr{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} \mathscr{A}^{-s}_{p',q'}(\mathbb{R}^n, w'_p; X^*) \hookrightarrow \mathscr{S}'(\mathbb{R}^n; X),$$

so that

$$\mathscr{S}(\mathbb{R}^n; X) \overset{d}{\hookrightarrow} [\mathscr{A}^{-s}_{p',q'}(\mathbb{R}^n, w'_p; X^*)]^* \hookrightarrow \mathscr{S}'(\mathbb{R}^n; X)$$

under the natural identifications. We define

$$\mathscr{B}^s_{p,q}(\mathbb{R}^n, w; X) := [B^{-s}_{p',q'}(\mathbb{R}^n, w'_p; X^*)]^* \quad \text{and} \quad \mathscr{F}^s_{p,q}(\mathbb{R}^n, w; X) := [F^{-s}_{p',q'}(\mathbb{R}^n, w'_p; X^*)]^*.$$

For $w \in A_p$ we have

$$\mathscr{B}^s_{p,q}(\mathbb{R}^n, w; X) = B^s_{p,q}(\mathbb{R}^n, w; X) \quad \text{and} \quad \mathscr{F}^s_{p,q}(\mathbb{R}^n, w; X) = F^s_{p,q}(\mathbb{R}^n, w; X). \tag{6.36}$$

Notationally it will be convenient to define

$$\{B, F, \mathscr{B}, \mathscr{F}\} \longrightarrow \{B, F, \mathscr{B}, \mathscr{F}\},\ \mathscr{A} \mapsto \mathscr{A}^\bullet = \begin{cases} \mathscr{B}, & \mathscr{A} = B, \\ \mathscr{F}, & \mathscr{A} = F, \\ B, & \mathscr{A} = \mathscr{B}, \\ F, & \mathscr{A} = \mathscr{F}. \end{cases}$$

Let X be a reflexive Banach space, $p, q \in (1,\infty)$, $\gamma \in (-\infty, p-1)$ and $s \in \mathbb{R}$. We put

$$\partial\mathscr{B}^s_{p,q,\gamma}(\partial\mathscr{O}; X) := B^{s-\frac{1+\gamma}{p}}_{p,q}(\partial\mathscr{O}; X) \qquad \text{and} \qquad \partial\mathscr{F}^s_{p,q,\gamma}(\partial\mathscr{O}; X) := F^{s-\frac{1+\gamma}{p}}_{p,p}(\partial\mathscr{O}; X).$$

Parameter-dependent spaces We now present an extension to the reflexive Banach space-valued setting of the parameter-dependent function spaces discussed in [163, Section 6], which was in turn partly based on [109]. As the theory presented in [163, Section 6] carries over verbatim to this setting, we only state results without proofs. The reflexivity condition comes from duality arguments involving the dual scales that are needed outside the A_p-range. Although for the B- and F-scales duality is only used in Corollary 6.3.10, for simplicity we restrict ourselves to the setting of reflexive Banach spaces from the start.

For $\sigma \in \mathbb{R}$ and $\mu \in [0,\infty)$ we define $\Xi^\sigma_\mu \in \mathscr{L}(\mathscr{S}(\mathbb{R}^n; X)) \cap \mathscr{L}(\mathscr{S}'(\mathbb{R}^n; X))$ by

$$\Xi^\sigma_\mu f := \mathscr{F}^{-1}[\langle\cdot,\mu\rangle^\sigma \hat{f}], \qquad f \in \mathscr{S}'(\mathbb{R}^n; X),$$

where $\langle\xi,\mu\rangle = (1 + |\xi|^2 + \mu^2)^{1/2}$.

Let X be a reflexive Banach space and let either

(i) $p \in [1,\infty)$, $q \in [1,\infty]$, $w \in A_\infty(\mathbb{R}^n)$ and $\mathscr{A} \in \{B, F\}$; or

(ii) $p, q \in (1,\infty)$, $w \in [A_\infty]'_p(\mathbb{R}^n)$ and $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}\}$.

For $s, s_0 \in \mathbb{R}$ and $\mu \in [0,\infty)$ we define

$$||f||_{\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n,w;X)} := ||\Xi^{s-s_0}_\mu f||_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)}, \qquad f \in \mathscr{S}'(\mathbb{R}^n; X)$$

and denote by $\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X)$ the space $\{f \in \mathscr{S}'(\mathbb{R}^n; X) : ||f||_{\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n,w;X)} < \infty\}$ equipped with this norm. For the Bessel-potential scale we proceed in a similar way. Suppose that X is a UMD Banach space and let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$. We define

$$\|f\|_{H^{s,\mu,s_0}_p(\mathbb{R}^n,w;X)} := \|\Xi^{s-s_0}_\mu f\|_{H^{s_0}_p(\mathbb{R}^n,w;X)}$$

and write $H^{s,\mu,s_0}_p(\mathbb{R}^n, w; X)$ for the space $\{f \in \mathscr{S}'(\mathbb{R}^n; X) : ||f||_{H^{s,\mu,s_0}_p(\mathbb{R}^n,w;X)} < \infty\}$ endowed with this norm. In all the different cases we suppress s_0 when $s_0 = 0$.

It trivially holds that

$$
\begin{aligned}
\Xi^t_\mu &: \mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X) \xrightarrow{\simeq} \mathscr{A}^{s-t,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X), \quad \text{isometrically,}\\
\Xi^t_\mu &: H^{s,\mu,s_0}_p(\mathbb{R}^n, w; X) \xrightarrow{\simeq} H^{s-t,\mu,s_0}_p(\mathbb{R}^n, w; X), \quad \text{isometrically.}
\end{aligned}
\tag{6.37}
$$

It furthermore holds that $\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X) = \mathscr{A}^{s}_{p,q}(\mathbb{R}^n, w; X)$ as well as $H^{s,\mu,s_0}_p(\mathbb{R}^n, w; X) = H^s_p(\mathbb{R}^n, w; X)$, but with an equivalence of norms that is μ-dependent. If $s, s_0, \widetilde{s}_0 \in \mathbb{R}$ with $s_0 \leq \widetilde{s}_0$, then

$$
\begin{aligned}
\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X) &\hookrightarrow \mathscr{A}^{s,\mu,\widetilde{s}_0}_{p,q}(\mathbb{R}^n, w; X) \quad \text{uniformly in } \mu \in [0,\infty),\\
H^{s,\mu,s_0}_p(\mathbb{R}^n, w; X) &\hookrightarrow H^{s,\mu,\widetilde{s}_0}_p(\mathbb{R}^n, w; X) \quad \text{uniformly in } \mu \in [0,\infty).
\end{aligned}
$$

For an open subset $U \subset \mathbb{R}^n$ we put

$$
\mathscr{A}^{s,\mu,s_0}_{p,q}(U, w; X) := [\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X)](U), \quad H^{s,\mu,s_0}_p(U, w; X) := [H^{s,\mu,s_0}_p(\mathbb{R}^n, w; X)](U).
$$

If $s \geq s_0$ and $\mathscr{O}$ is either $\mathbb{R}^n$, $\mathbb{R}^n_+$ or a C^∞-domain in $\mathbb{R}^n$ with a compact boundary $\partial\mathscr{O}$, then it holds that

$$
\begin{aligned}
||f||_{\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathscr{O},w;X)} &\eqsim ||f||_{\mathscr{A}^{s}_{p,q}(\mathscr{O},w;X)} + \langle\mu\rangle^{s-s_0}||f||_{\mathscr{A}^{s_0}_{p,q}(\mathscr{O},w;X)},\\
||f||_{H^{s,\mu,s_0}_p(\mathscr{O},w;X)} &\eqsim ||f||_{H^{s}_p(\mathscr{O},w;X)} + \langle\mu\rangle^{s-s_0}||f||_{H^{s_0}_p(\mathscr{O},w;X)}, \qquad f \in \mathscr{S}'(\mathbb{R}^n), \mu \in [0,\infty)
\end{aligned}
\tag{6.38}
$$

Let X be a reflexive Banach space, $p, q \in (1,\infty)$, $(w, \mathscr{A}) \in A_\infty(\mathbb{R}^n) \times \{B, F\} \cup [A_\infty]'_p(\mathbb{R}^n) \times \{\mathscr{B}, \mathscr{F}\}$ and $\mathscr{B} = \mathscr{A}^\bullet$. For s, s_0 it holds that

$$
[\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n, w; X)]^* = \mathscr{B}^{-s,\mu,-s_0}_{p',q'}(\mathbb{R}^n, w'_p; X^*), \quad \text{uniformly in } \mu \in [0,\infty).
$$

Next we consider a vector-valued version of the parameter-dependent Besov spaces as introduced in [109], but in the notation of [163, Section 6]. Let X be a reflexive Banach space, $p \in [1,\infty)$, $q \in [1,\infty]$ and $s \in \mathbb{R}$. For each $\mu \in [0,\infty)$ the norm $||\cdot||_{\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n;X)}$ is defined by:

$$
||f||_{\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n;X)} := \langle\mu\rangle^{s-\frac{d}{p}}||M_\mu f||_{B^s_{p,q}(\mathbb{R}^n;X)}, \qquad f \in \mathscr{S}'(\mathbb{R}^n; X),
$$

where $M_\mu \in \mathscr{L}(\mathscr{S}(\mathbb{R}^n; X)) \cap \mathscr{L}(\mathscr{S}'(\mathbb{R}^n; X))$ denotes the operator of dilation by $\langle\mu\rangle^{-1}$. We furhtermore write $\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n; X)$ for the space $\{f \in \mathscr{S}'(\mathbb{R}^n; X) : ||f||_{\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n;X)} < \infty\}$ equipped with this norm. Then

$$
\Xi^t_\mu : \mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n; X) \xrightarrow{\simeq} \mathbb{B}^{s-t,\mu}_{p,q}(\mathbb{R}^n; X), \quad \text{uniformly in } \mu. \tag{6.39}
$$

If $s > 0$, then it holds that

$$
||f||_{\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n;X)} \eqsim ||f||_{B^s_{p,q}(\mathbb{R}^n;X)} + \langle\mu\rangle^s||f||_{L_p(\mathbb{R}^n;X)}, \qquad f \in \mathscr{S}'(\mathbb{R}^n; X), \mu \in [0,\infty). \tag{6.40}
$$

If $p \in (1,\infty)$ and $q \in [1,\infty)$, then

$$[\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n;X)]^* = \mathbb{B}^{-s,\mu}_{p',q'}(\mathbb{R}^n;X^*) \qquad \text{uniformly in } \mu \in [0,\infty). \tag{6.41}$$

For a compact smooth manifold M we define $\mathbb{B}^{s,\mu}_{p,q}(M)$ in terms of $\mathbb{B}^{s,\mu}_{p,q}(\mathbb{R}^n)$ in the standard way. Then the analogues of (6.40) and (6.41) for $\mathbb{B}^{s,\mu}_{p,q}(M)$ are valid.

It will be convenient to write

$$\partial\mathscr{A}^{s,\mu}_{p,q,\gamma}(\partial\mathscr{O};X) := \begin{cases} \mathbb{B}^{s-\frac{1+\gamma}{p},\mu}_{p,q}(\partial\mathscr{O};X), & \mathscr{A} \in \{B,\mathscr{B}\}, \\ \mathbb{B}^{s-\frac{1+\gamma}{p},\mu}_{p,p}(\partial\mathscr{O};X), & \mathscr{A} \in \{F,\mathscr{F}\} \end{cases}$$

as well as

$$\partial H^{s,\mu}_{p,q,\gamma}(\partial\mathscr{O};X) := \mathbb{B}^{s-\frac{1+\gamma}{p},\mu}_{p,p}(\partial\mathscr{O};X).$$

Proposition 6.3.9. *Let X be a reflexive Banach space, let $\mathscr{O}$ be either $\mathbb{R}^n_+$ or a C^∞-domain in $\mathbb{R}^n$ with a compact boundary $\partial\mathscr{O}$, let $U \in \{\mathbb{R}^n, \mathscr{O}\}$, let either*

(i) $p \in [1,\infty)$, $q \in [1,\infty]$, $\gamma \in (-1,\infty)$ and $\mathscr{A} \in \{B,F\}$; or

(ii) $p,q \in (1,\infty)$, $\gamma \in (-\infty, p-1)$ and $\mathscr{A} \in \{\mathscr{B},\mathscr{F}\}$,

and let $s \in (\frac{1+\gamma}{p},\infty)$ and $s_0 \in (-\infty, \frac{1+\gamma}{p})$. Then

$$\mathrm{tr}_{\partial\mathscr{O}} : \mathscr{A}^{s,\mu,s_0}_{p,q}(U, w^{\partial\mathscr{O}}_\gamma;X) \longrightarrow \partial\mathscr{A}^{s,\mu}_{p,q,\gamma}(\partial\mathscr{O};X) \qquad \textit{uniformly in } \mu \in [0,\infty),$$

that is,

$$||\mathrm{tr}_{\partial\mathscr{O}} f||_{\partial\mathscr{A}^{s,\mu}_{p,q,\gamma}(\partial\mathscr{O};X)} \lesssim ||f||_{\mathscr{A}^{s,\mu,s_0}_{p,q}(U,w^{\partial\mathscr{O}}_\gamma;X)}, \qquad f \in \mathscr{A}^{s,\mu,s_0}_{p,q}(U, w^{\partial\mathscr{O}}_\gamma;X), \mu \in [0,\infty).$$

The respective assertion also holds for the Bessel potential scale if X is a UMD Banach space and if $\gamma \in (-1, p-1)$.

Corollary 6.3.10. *Let X be a reflexive Banach space, $p,q \in (1,\infty)$, $(\gamma,\mathscr{A}) \in (-1,\infty) \times \{B,F\} \cup (-\infty,p-1) \times \{\mathscr{B},\mathscr{F}\}$, $s \in (-\infty, \frac{1+\gamma}{p}-1)$ and $s_0 \in (\frac{1+\gamma}{p}-1,\infty)$. Then*

$$||\delta_0 \otimes f||_{\mathscr{A}^{s,\mu,s_0}_{p,q}(\mathbb{R}^n,w_\gamma;X)} \lesssim ||f||_{\partial\mathscr{A}^{s+1,\mu}_{p,q,\gamma}(\mathbb{R}^{d-1};X)}, \qquad f \in \partial\mathscr{A}^{s+1,\mu}_{p,q,\gamma}(\mathbb{R}^{d-1};X), \mu \in [0,\infty).$$

6.3.5. *Differential Boundary Value Systems*

THE EQUATIONS

Here we introduce some of the notation and terminology that will be used in Sections 6.6 and 6.7 on parabolic and elliptic boundary value problems. In this thesis we for simplicity of presentation only formulate these problems for boundary value systems having BC^∞-coefficients; we refer the reader for more general coefficients to the paper [122] on which this chapter is based.

Let X be a Banach space, $\mathscr{O} \subset \mathbb{R}^n$ a C^∞-domain with a compact boundary $\partial\mathscr{O}$ and $J \subset \mathbb{R}$ an interval. Let $m \in \mathbb{N}_{\geq 1}$ and let $m_1, \ldots, m_m \in \mathbb{N}$ satisfy $m_i \leq 2m-1$ for each $i \in \{1, \ldots, m\}$.

Systems on $\mathscr{O}$: For each $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2m$, $i \in \{1, \ldots, m\}$ and $\beta \in \mathbb{N}^n$, $|\beta| \leq m_i$, let $a_\alpha \in BC^\infty(\mathscr{O}; \mathscr{B}(X))$ and $b_{i,\beta} \in BC^\infty(\partial\mathscr{O}; \mathscr{B}(X))$. Put

$$\mathscr{A}(D) := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha,$$
$$\mathscr{B}_i(D) := \sum_{|\beta| \leq m_i} b_{i,\beta} \mathrm{tr}_{\partial\mathscr{O}} D^\beta, \qquad i = 1, \ldots, n,$$

We call $(\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D))$ a $\mathscr{B}(X)$-valued BC^∞-differential boundary value system (of order $2m$) on $\mathscr{O}$.

Systems on $\mathscr{O} \times J$: For each $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2m$, $i \in \{1, \ldots, m\}$ and $\beta \in \mathbb{N}^n$, $|\beta| \leq m_i$, let $a_\alpha \in BC^\infty(\mathscr{O} \times J; \mathscr{B}(X))$ and $b_{i,\beta} \in BC^\infty(\partial\mathscr{O} \times J; \mathscr{B}(X))$. Put

$$\mathscr{A}(D) := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha,$$
$$\mathscr{B}_i(D) := \sum_{|\beta| \leq m_i} b_{i,\beta} \mathrm{tr}_{\partial\mathscr{O}} D^\beta, \qquad i = 1, \ldots, n,$$

We call $(\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D))$ a $\mathscr{B}(X)$-valued BC^∞-differential boundary value system (of order $2m$) on $\mathscr{O} \times J$.

ELLIPTICITY AND LOPATINSKII-SHAPIRO CONDITIONS

Let us now turn to the two structural assumptions on $\mathscr{A}, \mathscr{B}_1, \ldots, \mathscr{B}_m$. For each $\phi \in [0, \pi)$ we introduce the conditions $(\mathrm{E})_\phi$ and $(\mathrm{LS})_\phi$.

The condition $(\mathrm{E})_\phi$ is parameter ellipticity. In order to state it, we denote by the subscript # the principal part of a differential operator: given a differential operator $P(D) = \sum_{|\gamma| \leq k} p_\gamma D^\gamma$ of order $k \in \mathbb{N}$, $P_\#(D) = \sum_{|\gamma| = k} p_\gamma D^\gamma$.

$(\mathrm{E})_\phi$ For all $t \in \overline{J}$, $x \in \overline{\mathscr{O}}$ and $|\xi| = 1$ it holds that $\sigma(\mathscr{A}_\#(x, \xi, t)) \subset \Sigma_\phi$. If $\mathscr{O}$ is unbounded, then it in addition holds that $\sigma(\mathscr{A}_\#(\infty, \xi, t)) \subset \mathbb{C}_+$ for all $t \in \overline{J}$ and $|\xi| = 1$.

The condition $(\mathrm{LS})_\phi$ is a condition of Lopatinskii-Shapiro type. Before we can state it, we need to introduce some notation. For each $x \in \partial\mathscr{O}$ we fix an orthogonal matrix $O_{\nu(x)}$ that rotates the outer unit normal $\nu(x)$ of $\partial\mathscr{O}$ at x to $(0, \ldots, 0, -1) \in \mathbb{R}^n$, and define the rotated operators $(\mathscr{A}^\nu, \mathscr{B}^\nu)$ by

$$\mathscr{A}^\nu(x, D, t) := \mathscr{A}(x, O_{\nu(x)}^T D, t), \quad \mathscr{B}^\nu(x, D, t) := \mathscr{B}(x, O_{\nu(x)}^T D, t).$$

$(\mathrm{LS})_\phi$ For each $t \in \overline{J}$, $x \in \partial\mathscr{O}$, $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{d-1}$ with $(\lambda, \xi') \neq 0$ and all $h \in X^n$, the ordinary initial value problem

$$\begin{aligned} \lambda w(y) + \mathscr{A}_\#^\nu(x, \xi', D_y, t) w(y) &= 0, \qquad y > 0 \\ \mathscr{B}_{j,\#}^\nu(x, \xi', D_y, t) w(y)|_{y=0} &= h_j, \quad j = 1, \ldots, n. \end{aligned}$$

has a unique solution $w \in C^\infty([0, \infty); X)$ with $\lim_{y \to \infty} w(y) = 0$.

In the scalar-valued case, there are several equivalent characterizations for the Lopatinskii-Shapiro condition. It is a common approach to consider the polynomial

$$\mathscr{A}_{\#}^{\nu,+}(x,\xi,\tau,t) := \prod_{j=1}^{m}(\tau-\tau_j(x,\xi',t))$$

where $\tau_1(x,\xi',t),\dots,\tau_m(x,\xi',t)$ are the roots of the polynomial $\mathscr{A}_{\#}^{\nu}(x,\xi',\cdot,t)$ with positive imaginary part. If we write $\overline{\mathscr{B}_{j,\#}^{\nu}}(x,\xi',\tau,t)$ for the equivalence classes of $\mathscr{B}_{j,\#}^{\nu}(x,\xi',\tau,t)$ in $\mathbb{C}[\tau]/(\mathscr{A}_{\#}^{\nu,+}(x,\xi,\tau,t))$, then we can formulate the following result:

Proposition 6.3.11. *The Lopatinskii-Shapiro condition is satisfied if and only if* $\overline{\mathscr{B}_{j,\#}^{\nu}}(x,\xi',\tau,t)$ $(j=1,\dots,m)$ *are linearly independent in* $\mathbb{C}[\tau]/(\mathscr{A}_{\#}^{\nu,+}(x,\xi,\tau,t))$.

This condition is sometimes called covering condition. A proof for this statement can for example be found in Chapter 3.2 of [207]. A similar condition can be formulated using the so-called Lopatinskii matrix. If $\widetilde{\mathscr{B}_{j,\#}^{\nu}}(x,\xi',\tau,t)$ are the representatives of $\overline{\mathscr{B}_{j,\#}^{\nu}}(x,\xi',\tau,t)$ with minimal degree, then there degree is smaller than m. Hence, there is a unique matrix $L(x,\xi',t)\in\mathbb{C}^{m\times m}$ such that

$$\begin{pmatrix}\widetilde{\mathscr{B}_{1,\#}^{\nu}}(x,\xi',\tau,t)\\ \vdots\\ \widetilde{\mathscr{B}_{m,\#}^{\nu}}(x,\xi',\tau,t)\end{pmatrix} = L(x,\xi',t)\begin{pmatrix}\tau^0\\ \vdots\\ \tau^{m-1}.\end{pmatrix}$$

This matrix $L(x,\xi',t)$ is called Lopatinskii matrix. From Proposition 6.3.12 one can easily derive the following result:

Proposition 6.3.12. *The Lopatinskii-Shapiro condition is satisfied if and only if the Lopatinskii matrix* $L(x,\xi',t)$ *is invertible.*

Using Proposition 6.3.12 one can easily see that if $B_j(x,\xi',\tau,t)=\tau^{j-1}$, then the Lopatinskii-Shapiro condition is satisfied for all elliptic operators. In particular, this includes the usual Dirichlet boundary conditions for second order equations. Also Neumann boundary conditions satisfy the Lopatinskii-Shapiro condition. For further examples we refer to Section 11.2 in [249].

6.4. EMBEDDING AND TRACE RESULTS FOR MIXED-NORM ANISOTROPIC SPACES

6.4.1. Embedding Results

The following result is a partial extension of [183, Theorem 1.2] to the mixed-norm anisotropic setting.

Proposition 6.4.1. *Let X be a Banach space,* $\boldsymbol{p},\widetilde{\boldsymbol{p}}\in(1,\infty)^l$, $q,\widetilde{q}\in[1,\infty]$, $s,\widetilde{s}\in\mathbb{R}$, $\boldsymbol{a}\in(0,\infty)^l$, *and* $\boldsymbol{w},\widetilde{\boldsymbol{w}}\in\prod_{j=1}^{l}A_\infty(\mathbb{R}^{d_j})$. *Suppose that*

- $p_1 \leq \widetilde{p}_1$, $p_j = \widetilde{p}_j$ *and* $w_j = \widetilde{w}_j$ *for* $j \in \{2,\ldots,l\}$;
- $w_1(x_1) = |x_1|^{\gamma_1}$ *and* $\widetilde{w}_1(x_1) = |x_1|^{\widetilde{\gamma}_j}$ *for some* $\gamma_1, \widetilde{\gamma}_1 \in (-\mathpzc{d}_1,\infty)$ *satisfying*

$$\frac{\widetilde{\gamma}_1}{\widetilde{p}_1} \leq \frac{\gamma_1}{p_1} \quad \textit{and} \quad \frac{\mathpzc{d}_1+\widetilde{\gamma}_1}{\widetilde{p}_1} < \frac{\mathpzc{d}_1+\gamma_1}{p_1}.$$

If $s - a_1\frac{\mathpzc{d}_1+\gamma_1}{p_1} \geq \widetilde{s} - a_1\frac{\mathpzc{d}_1+\widetilde{\gamma}_1}{\widetilde{p}_1}$, *then*

$$F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow F^{\widetilde{s},\boldsymbol{a}}_{\widetilde{\boldsymbol{p}},\widetilde{q},\mathpzc{d}}(\mathbb{R}^n,\widetilde{\boldsymbol{w}};X).$$

Remark 6.4.2. In this paper we only apply Proposition 6.4.1 in the case that $\boldsymbol{p} = \widetilde{\boldsymbol{p}}$. In this case the embedding result takes the form: if $\gamma_1 > \widetilde{\gamma}_1$ and $s \geq \widetilde{s} + a_1\frac{\gamma_1-\widetilde{\gamma}_1}{p_1}$, then

$$F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X) \hookrightarrow F^{\widetilde{s},\boldsymbol{a}}_{\boldsymbol{p},\widetilde{q},\mathpzc{d}}(\mathbb{R}^n,\widetilde{\boldsymbol{w}};X).$$

One of the nice things about this embedding, which has already turned out to be a powerful technical tool in the isotropic case (see e.g. [162, 163, 167, 186]).) , is the *(inner) trace space invariance* in the sharp case $s = \widetilde{s} + a_1\frac{\gamma_1-\widetilde{\gamma}_1}{p_1}$, see Proposition 6.4.6 below. In the two other embedding results in this section, Lemmas 6.4.3 and 6.4.4 below, there also is such an invariance.

Proof of Proposition 6.4.1. The embedding can be proved in the same way as [183, Theorem 1.2 (2)⇒(1)], as follows. It suffices to consider the case $\widetilde{q} = 1$ and $s - a_1\frac{\mathpzc{d}_1+\gamma_1}{p_1} = \widetilde{s} - a_1\frac{\mathpzc{d}_1+\widetilde{\gamma}_1}{\widetilde{p}_1}$. Furthermore, in order to prove the norm estimate corresponding to the embedding we may restrict ourselves to $f \in \mathscr{S}(\mathbb{R}^n;X)$. Let $\theta \in (0,1)$ be such that

$$\nu := \frac{\widetilde{\gamma}_1/\widetilde{p}_1 - (1-\theta)\gamma_1/p_1}{\frac{1}{\widetilde{p}_1} - \frac{1-\theta}{p_1}} > -\mathpzc{d}_1,$$

let r be defined by $\frac{1}{\widetilde{p}_1} = \frac{1-\theta}{p_1} + \frac{\theta}{r}$ and let t be defined by $t - a_1\frac{\mathpzc{d}_1+\nu}{r} = \widetilde{s} - a_1\frac{\mathpzc{d}_1+\widetilde{\gamma}_1}{\widetilde{p}_1}$. Note that $r \in [\widetilde{p}_1,\infty)$, $t \in (-\infty,s)$, $\widetilde{s} = \theta t + (1-\theta)s$ and $\frac{\theta\widetilde{p}_1}{r}\nu + \frac{(1-\theta)\widetilde{p}_1}{p_1}\gamma_1 = \widetilde{\gamma}$. Therefore, as [183, Proposition 5.1] directly extends to the setting of mixed-norm anisotropic Triebel-Lizorkin spaces,

$$||f||_{F^{\widetilde{s},\boldsymbol{a}}_{\widetilde{\boldsymbol{p}},1,\mathpzc{d}}(\mathbb{R}^n,\widetilde{\boldsymbol{w}};X)} \lesssim ||f||^{1-\theta}_{F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X)} ||f||^{\theta}_{F^{t,\boldsymbol{a}}_{(r,\boldsymbol{p}'),r,\mathpzc{d}}(\mathbb{R}^n,(|\cdot|^{\nu},\widetilde{\boldsymbol{w}}');X)}. \tag{6.42}$$

Furthermore, as a consequence [183, Proposition 4.1], since

$$\frac{\widetilde{\gamma}_1}{\widetilde{p}_1} - \frac{\nu}{r} = \frac{1-\theta}{\theta}\left(\frac{\gamma_1}{p_1} - \frac{\widetilde{\gamma}_1}{\widetilde{p}_1}\right) \geq 0,$$

$$||f||_{F^{t,\boldsymbol{a}}_{(r,\boldsymbol{p}'),r,\mathpzc{d}}(\mathbb{R}^n,(|\cdot|^{\nu},\widetilde{\boldsymbol{w}}');X)} = ||(2^{tn}S^{(\mathpzc{d},\boldsymbol{a})}_n f)_{n\in\mathbb{N}}||_{L_{\boldsymbol{p}',\mathpzc{d}'}(\mathbb{R}^{d-\mathpzc{d}_1},\widetilde{\boldsymbol{w}}')[\ell_r(\mathbb{N})[L_r(\mathbb{R}^{\mathpzc{d}_1},|\cdot|^{\nu})]](X)}$$

$$\begin{aligned}&\lesssim \|(2^{\tilde{s}n}S_n^{(d,\boldsymbol{a})}f)_{n\in\mathbb{N}}\|_{L_{\boldsymbol{p}',d'}(\mathbb{R}^{d-d_1},\tilde{\boldsymbol{w}}')[\ell_{\tilde{p}_1}(\mathbb{N})[L_{\tilde{p}_1}(\mathbb{R}^{d_1},|\cdot|^{\tilde{\gamma}_1})]](X)}\\ &= \|f\|_{F^{\tilde{s},\boldsymbol{a}}_{\tilde{\boldsymbol{p}},1,d}(\mathbb{R}^n,\tilde{\boldsymbol{w}};X)};\end{aligned} \tag{6.43}$$

here we apply [183, Proposition 4.1] to $S_n^{(d,\boldsymbol{a})}f(\cdot,x')$ for each $x'\in\mathbb{R}^{d-d_1}$, which is a Schwartz function with Fourier support in $[-c2^{na_1},c2^{na_1}]^{d_1}$ (with c independent of f and n), to obtain

$$\begin{aligned}\|S_n^{(d,\boldsymbol{a})}f(\cdot,x')\|_{L_r(\mathbb{R}^{d_1},|\cdot|^{v};X)} &\lesssim (2^{na_1})^{\frac{d_1+v}{r}-\frac{d_1+\tilde{\gamma}_1}{\tilde{p}_1}}\|S_n^{(d,\boldsymbol{a})}f(\cdot,x')\|_{L_{\tilde{p}_1}(\mathbb{R}^{d_1},|\cdot|^{\tilde{\gamma}_1};X)}\\ &= 2^{n(t-\tilde{s})}\|S_n^{(d,\boldsymbol{a})}f(\cdot,x')\|_{L_{\tilde{p}_1}(\mathbb{R}^{d_1},|\cdot|^{\tilde{\gamma}_1};X)},\end{aligned}$$

from which '$\lesssim$' in (6.43) follows. A combination of (6.42) and (6.43) gives the desired estimate

$$\|f\|_{F^{\tilde{s},\boldsymbol{a}}_{\tilde{\boldsymbol{p}},1,d}(\mathbb{R}^n,\tilde{\boldsymbol{w}};X)} \lesssim \|f\|_{F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^n,\boldsymbol{w};X)}.$$ □

Lemma 6.4.3. *Let X be a UMD Banach space, $q,p,r\in(1,\infty)$, $v\in A_q(\mathbb{R})$, $\gamma\in(-1,\infty)$, $s\in\mathbb{R}$ and $\rho\in(0,\infty)$. Put $s_+:=\max\{s,0\}$, $\gamma_+:=\gamma+(s_+-s)p$ and $\sigma:=\frac{s_+}{\rho}+1$. Let $\delta\in(0,\infty)$ be such that $\gamma_+-\delta p\in(-1,p-1)$ and put $\eta:=\frac{1}{\sigma-1}\delta$. Then*

$$F^{\sigma+\frac{\eta}{\rho},(\frac{1}{\rho},1)}_{(p,q),1}(\mathbb{R}^n_+\times\mathbb{R},(w_{\gamma_++\eta p},v);X)\hookrightarrow W^1_q(\mathbb{R},v;F^s_{p,r}(\mathbb{R}^n_+,w_\gamma);X)\cap L_q(\mathbb{R},v;F^{s+\rho}_{p,r}(\mathbb{R}^n_+,w_\gamma);X). \tag{6.44}$$

Proof. By the Sobolev embedding from Proposition 6.4.1, (6.44) is equivalent to

$$\begin{aligned}F^{\sigma,(\frac{1}{\rho},1)}_{(p,q),1}(\mathbb{R}^n_+\times\mathbb{R},(w_{\gamma_+},v);X)\cap F^{\sigma+\frac{\eta}{\rho},(\frac{1}{\rho},1)}_{(p,q),1}(\mathbb{R}^n_+\times\mathbb{R},(w_{\gamma_++\eta p},v);X)\\ \hookrightarrow\quad W^1_q(\mathbb{R},v;F^s_{p,r}(\mathbb{R}^n_+,w_\gamma);X)\cap L_q(\mathbb{R},v;F^{s+\rho}_{p,r}(\mathbb{R}^n_+,w_\gamma);X).\end{aligned}$$

The latter embedding can be shown as in [162, Lemma 3.3] about the scalar-valued case. Let us elaborate a bit.

Taking the X-valued version of [162, Lemma 3.2] for granted, [162, (27)] is the only thing that needs an extra explanation. Its X-valued version reads as follows:

$$\mathbb{F}^{\sigma}_{q,1}(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma);X)\hookrightarrow H^{\sigma}_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;X)), \tag{6.45}$$

where

$$\begin{aligned}&\mathbb{F}^{\sigma}_{q,1}(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma);X)\\ &\quad=\left\{f\in\mathscr{S}'(\mathbb{R};L_p(\mathbb{R}^n_+,w_\gamma;X)):(2^{\sigma j}S_nf)_j\in L_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;\ell_1(\mathbb{N};X))\right\}.\end{aligned}$$

The desired embedding (6.45) follows from [184, Proposition 3.2] and

$$\begin{aligned}L_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;\ell_1(\mathbb{N};X))&\hookrightarrow L_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;\mathrm{Rad}(\mathbb{N};X))\\ &=\mathrm{Rad}(\mathbb{N};L_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;X))),\end{aligned}$$

where the space $\mathrm{Rad}(\mathbb{N};Z)$ is introduced in Section 6.3.2.

Concerning [162, Lemma 3.2], let us remark that X is reflexive as a UMD space, so that the duality arguments given there remain valid. □

Lemma 6.4.4. *Let X be a UMD Banach space, $q,p \in (1,\infty)$, $v \in A_q(\mathbb{R})$, $\gamma \in (-1,\infty)$, $s \in \mathbb{R}$ and $\rho \in (0,\infty)$. If $\theta \in [0,1]$ is such that $s+\theta\rho \in (0,\infty) \cap (\frac{1+\gamma}{p}-1, \frac{1+\gamma}{p})$, then*

$$\begin{aligned} &W^1_q(\mathbb{R}, v; F^s_{p,\infty}(\mathbb{R}^n_+, w_\gamma); X) \cap L_q(\mathbb{R}, v; F^{s+\rho}_{p,\infty}(\mathbb{R}^n_+, w_\gamma); X) \\ &\quad \hookrightarrow \quad H^{1-\theta}_q(\mathbb{R}, v; L_p(\mathbb{R}^n_+, w_{\gamma-(s+\theta\rho)p}); X) \cap L_q(\mathbb{R}, v; H^{(1-\theta)\rho}_p(\mathbb{R}^n_+, w_{\gamma-(s+\theta\rho)p}); X). \end{aligned} \tag{6.46}$$

Note that $s+\theta\rho \in (\frac{1+\gamma}{p}-1, \frac{1+\gamma}{p})$ is equivalent to $\gamma-(s+\theta\rho)p \in (-1,p-1)$, which is in turn equivalent to $w_{\gamma-(s+\theta\rho)p} \in A_p$.

Proof. The proof given in [162, Lemma 3.4] on the scalar-valued case carries over verbatim. □

6.4.2. Trace Results

Proposition 6.4.1 with $\boldsymbol{p} = \widetilde{\boldsymbol{p}}$ (see Remark 6.4.2) enables us to give an alternative proof of the trace theorem [160, Theorem 4.6] for anisotropic weighted mixed-norm Triebel-Lizorkin spaces. The special case $d_1 = 1$ in Proposition 6.4.6 actually yields [160, Theorem 4.6], which is the only case that is used in this paper.

For the statement of Proposition 6.4.6 we need some notation and terminology that we first introduce.

SOME NOTATION

We slightly modify the notation from [160, Sections 4.3.1 & 4.3.2] to our setting.

The working definition of the trace Let $\varphi \in \Phi^{d,a}(\mathbb{R}^n)$ with associated family of convolution operators $(S_k)_{k\in\mathbb{N}} \subset \mathcal{L}(\mathcal{S}'(\mathbb{R}^n;X))$ be fixed. In order to motivate the definition to be given in a moment, let us first recall that $f = \sum_{k=0}^{\infty} S_k f$ in $\mathcal{S}(\mathbb{R}^n;X)$ (respectively in $\mathcal{S}'(\mathbb{R}^n;X)$) whenever $f \in \mathcal{S}(\mathbb{R}^n;X)$ (respectively $f \in \mathcal{S}'(\mathbb{R}^n;X)$), from which it is easy to see that

$$f_{|\{0_{d_1}\}\times\mathbb{R}^{n-d_1}} = \sum_{k=0}^{\infty} (S_k f)_{|\{0_{d_1}\}\times\mathbb{R}^{n-d_1}} \text{ in } \mathcal{S}(\mathbb{R}^{n-d_1};X), \qquad f \in \mathcal{S}(\mathbb{R}^n;X).$$

Furthermore, given a general tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n;X)$, recall that $S_n f \in \mathcal{O}_M(\mathbb{R}^n;X)$; in particular, each $S_n f$ has a well defined classical trace with respect to $\{0_{d_1}\} \times \mathbb{R}^{n-d_1}$. This suggests to define the trace operator $\tau = \tau^\varphi : \mathcal{D}(\gamma^\varphi) \subset \mathcal{S}'(\mathbb{R}^n;X) \longrightarrow \mathcal{S}'(\mathbb{R}^{n-d_1};X)$ by

$$\tau^\varphi f := \sum_{n=0}^{\infty} (S_n f)_{|\{0_{d_1}\}\times\mathbb{R}^{n-d_1}} \tag{6.47}$$

on the domain $\mathscr{D}(\tau^{\varphi})$ consisting of all $f \in \mathscr{S}'(\mathbb{R}^n;X)$ for which this defining series converges in $\mathscr{S}'(\mathbb{R}^{n-d_1};X)$. Note that $\mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n;X)$ is a subspace of $\mathscr{D}(\tau^{\varphi})$ on which τ^{φ} coincides with the classical trace of continuous functions with respect to $\{0_{d_1}\}\times\mathbb{R}^{n-d_1}$; of course, for an f belonging to $\mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n;X)$ there are only finitely many S_nf non-zero.

The distributional trace operator Let us now introduce the concept of distributional trace operator. The reason for us to introduce it is the right inverse from Lemma 6.4.5.

The distributional trace operator r (with respect to the plane $\{0_{d_1}\}\times\mathbb{R}^{n-d_1}$) is defined as follows. Viewing $C(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X))$ as subspace of $\mathscr{D}'(\mathbb{R}^n;X)=\mathscr{D}'(\mathbb{R}^{d_1}\times\mathbb{R}^{n-d_1};X)$ via the canonical identification $\mathscr{D}'(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X))=\mathscr{D}'(\mathbb{R}^{d_1}\times\mathbb{R}^{n-d_1};X)$ (arising from the Schwartz kernel theorem),

$$C(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X)) \hookrightarrow \mathscr{D}'(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X)) = \mathscr{D}'(\mathbb{R}^{d_1}\times\mathbb{R}^{n-d_1};X),$$

we define $r \in \mathscr{L}(C(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X)),\mathscr{D}'(\mathbb{R}^{n-d_1};X))$ as the 'evaluation in 0 map'

$$r : C(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X)) \longrightarrow \mathscr{D}'(\mathbb{R}^{n-d_1};X),\ f \mapsto \mathrm{ev}_0 f.$$

Then, in view of

$$C(\mathbb{R}^n;X) = C(\mathbb{R}^{d_1}\times\mathbb{R}^{n-d_1};X) = C(\mathbb{R}^{d_1};C(\mathbb{R}^{n-d_1};X)) \hookrightarrow C(\mathbb{R}^{d_1};\mathscr{D}'(\mathbb{R}^{n-d_1};X)),$$

we have that the distributional trace operator r coincides on $C(\mathbb{R}^n;X)$ with the classical trace operator with respect to the plane $\{0_{d_1}\}\times\mathbb{R}^{n-d_1}$, i.e.,

$$r : C(\mathbb{R}^n;X) \longrightarrow C(\mathbb{R}^{n-d_1};X),\ f \mapsto f_{|\{0_{d_1}\}\times\mathbb{R}^{n-d_1}}.$$

The following lemma can be established as in [131, Section 4.2.1].

Lemma 6.4.5. *Let* $\rho \in \mathscr{S}(\mathbb{R}^{d_1})$ *such that* $\rho(0)=1$ *and* $\operatorname{supp}\hat{\rho} \subset [1,2]^{d_1}$, $a_1 \in \mathbb{R}^{d_1}$, $\widetilde{\mathpzc{d}} \in (\mathbb{Z}_{>0})^{l-d_1}$ *with* $\mathpzc{d}=(d_1,\widetilde{\mathpzc{d}})$, $\widetilde{\boldsymbol{a}} \in (0,\infty)^{l-d_1}$, *and* $(\phi_n)_{n\in\mathbb{N}} \in \Phi^{\widetilde{\mathpzc{d}},\widetilde{\boldsymbol{a}}}(\mathbb{R}^{d-d_1})$. *Then, for each* $g \in \mathscr{S}'(\mathbb{R}^{d-d_1};X)$,

$$\operatorname{ext} g := \sum_{n=0}^{\infty} \rho(2^{na_1}\,\cdot) \otimes [\phi_n * g] \tag{6.48}$$

defines a convergent series in $\mathscr{S}'(\mathbb{R}^n;X)$ *with*

$$\begin{aligned} &\operatorname{supp}\mathscr{F}[\rho\otimes[\phi_0 * g]] \subset \{\xi \mid |\xi|_{\mathpzc{d},a} \le c\} \\ &\operatorname{supp}\mathscr{F}[\rho(2^{na_1}\,\cdot)\otimes[\phi_n * g]] \subset \{\xi \mid c^{-1}2^n \le |\xi|_{\mathpzc{d},a} \le c2^n\}\ , n\ge 1, \end{aligned} \tag{6.49}$$

for some constant $c>0$ *independent of* g. *Moreover, the operator* ext *defined via this formula is a linear operator*

$$\operatorname{ext} : \mathscr{S}'(\mathbb{R}^{d-d_1};X) \longrightarrow C_b(\mathbb{R}^{d_1};\mathscr{S}'(\mathbb{R}^{d-d_1};X))$$

which acts as a right inverse of $r : C(\mathbb{R}^{d_1};\mathscr{S}'(\mathbb{R}^{d-d_1};X)) \longrightarrow \mathscr{S}'(\mathbb{R}^{d-d_1};X)$.

THE RESULTS

We will use the following notation. We write $\mathpzc{d}'' = (d_2,\ldots,d_l)$. Similarly, given $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in [1,\infty)^l$ and $\boldsymbol{w} \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$, we write $\boldsymbol{a}'' := (a_2,\ldots,a_l)$, $\boldsymbol{p} := (p_2,\ldots,p_l)$ and $\boldsymbol{w}'' := (w_2,\ldots,w_l)$.

Proposition 6.4.6. *Let X be a Banach space, $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in (1,\infty)^l$, $q \in [1,\infty]$, $\gamma \in (-d_1,\infty)$ and $s > \frac{a_1}{p_1}(d_1+\gamma)$. Let $\boldsymbol{w} \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$ be such that $w_1(x_1) = |x_1|^\gamma$ and $\boldsymbol{w}'' \in \prod_{j=2}^l A_{p_j/r_j}(\mathbb{R}^{d_j})$ for some $\boldsymbol{r}'' = (r_2,\ldots,r_l) \in (0,1)^{l-1}$ satisfying $s - \frac{a_1}{p_1}(d_1+\gamma) > \sum_{j=2}^l a_j d_j(\frac{1}{r_j}-1)$.*[4] *Then the trace operator $\tau = \tau^\varphi$ (6.47) is well-defined on $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X)$, where it is independent of φ, and restricts to a retraction*

$$\tau : F^{s,\boldsymbol{a}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),\boldsymbol{a}''}_{\boldsymbol{p}'',p_1}(\mathbb{R}^{n-d_1},\boldsymbol{w}'';X) \tag{6.50}$$

for which the extension operator ext *from Lemma 6.4.5 (with $\widetilde{\mathpzc{d}} = \mathpzc{d}''$ and $\widetilde{\boldsymbol{a}} = \boldsymbol{a}''$) restricts to a corresponding coretraction.*

Proof. Using the Sobolev embedding from Proposition 6.4.1 with $\boldsymbol{p} = \widetilde{\boldsymbol{p}}$ (see Remark 6.4.2) in combination with the invariance of the space on the right-hand side of (6.50) under this embedding, we may without loss of generality assume that $p_1 = q$. So

$$L_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w})[\ell_q(\mathbb{N})] = L_{\boldsymbol{p}'}(\mathbb{R}^{n-d_1},\boldsymbol{w}'')[\ell_q(\mathbb{N})[L_{p_1}(\mathbb{R}^{d_1},|\cdot|^\gamma)]].$$

Now the proof goes analogously to the proof of [156, Theorem 5.2.52]. □

Corollary 6.4.7. *Let X be a Banach space, $\boldsymbol{a} \in (0,\infty)^l$, $\boldsymbol{p} \in (1,\infty)^l$, $\gamma \in (-d_1, d_1(p_1-1))$ and $s > \frac{a_1}{p_1}(d_1+\gamma)$. Let $\boldsymbol{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$ be such that $w_1(x_1) = |x_1|^\gamma$. Suppose that either*

- *$\mathbb{E} = W^{\boldsymbol{n}}_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w};X)$, $\boldsymbol{n} \in (\mathbb{Z}_{\geq 1})^l$, $\boldsymbol{n} = s\boldsymbol{a}^{-1}$; or*
- *$\mathbb{E} = H^{s,\boldsymbol{a}}_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w};X)$; or*
- *$\mathbb{E} = H^{\boldsymbol{s}}_{\boldsymbol{p}}(\mathbb{R}^n,\boldsymbol{w};X)$, $\boldsymbol{s} \in (0,\infty)^l$, $\boldsymbol{a} = s\boldsymbol{a}^{-1}$.*

Then the trace operator $\tau = \tau^\varphi$ (6.47) is well-defined on $\mathbb{E}$, where it is independent of φ, and restricts to a retraction

$$\tau : \mathbb{E} \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),\boldsymbol{a}''}_{\boldsymbol{p}'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1},\boldsymbol{w}'';X)$$

for which the extension operator ext *from Lemma 6.4.5 (with $\widetilde{\mathpzc{d}} = \mathpzc{d}''$ and $\widetilde{\boldsymbol{a}} = \boldsymbol{a}''$) restricts to a corresponding coretraction.*

[4] This technical condition on $\boldsymbol{w}''$ is in particular satisfied when $\boldsymbol{w}'' \in \prod_{j=2}^l A_{p_j}(\mathbb{R}^{d_j})$.

Corollary 6.4.8. *Let X be a UMD Banach space, $q,p\in(1,\infty)$, $v\in A_q(\mathbb{R})$, $\gamma\in(-1,\infty)$, $s\in(-\infty,\frac{1+\gamma}{p})$, $\rho\in(0,\infty)$ and $\beta\in\mathbb{N}^n$. If $s+\rho-|\beta|>\frac{1+\gamma}{p}$, then $\operatorname{tr}_{\partial\mathbb{R}^n_+}\circ D^\beta$ is a bounded linear operator*

$$W^1_q(\mathbb{R},v;F^s_{p,\infty}(\mathbb{R}^n_+,w_\gamma;X))\cap L_q(\mathbb{R},v;F^{s+\rho}_{p,\infty}(\mathbb{R}^n_+,w_\gamma;X))\longrightarrow F^{\frac{1}{\rho}(s+\rho-|\beta|-\frac{1+\gamma}{p}),(\frac{1}{\rho},1)}_{(p,q),p}(\mathbb{R}^{n-1}\times\mathbb{R},(1,v);X)$$

Proof. Let $\theta\in[0,1]$ be such that $s+\theta\rho\in(0,\infty)\cap(\frac{1+\gamma}{p}-1,\frac{1+\gamma}{p})$. Such a θ exists because $s<\frac{1+\gamma}{p}$ and $s+\rho>s+\rho-|\beta|>\frac{1+\gamma}{p}$. Using Lemma 6.4.4

$$H^{1-\theta,(\frac{1}{\rho},1)}_{(p,q)}(\mathbb{R}^n_+\times\mathbb{R},(w_\gamma,v);X)$$

we find that D^β is a bounded linear operator from

$$W^1_q(\mathbb{R},v;F^s_{p,\infty}(\mathbb{R}^n_+,w_\gamma;X))\cap L_q(\mathbb{R},v;F^{s+\rho}_{p,\infty}(\mathbb{R}^n_+,w_\gamma;X))$$

to

$$H^{1-\theta-\frac{1}{\rho}|\beta|,(\frac{1}{\rho},1)}_{(p,q)}(\mathbb{R}^n_+\times\mathbb{R},(w_{\gamma-(s+\theta\rho)p},v);X).$$

The desired result now follows from Corollary 6.4.7/[160, Corollary 4.9] and the observation that

$$\frac{1}{\rho}(s+\rho-|\beta|-\frac{1+\gamma}{p})=(1-\theta-\frac{1}{\rho}|\beta|)-\frac{1+[\gamma-(s+\theta\rho)p]}{p}.\qquad\square$$

6.5. POISSON OPERATORS

6.5.1. Symbol Classes

In this subsection we give the definition and derive some properties of the symbol classes we want to work with. We will restrict ourselves to symbols with constant coefficients and infinite regularity in the parameter-dependent case. For the main results of this paper, treating the general symbol classes which are usually considered in the framework of the Boutet de Monvel calculus is not necessary. Nonetheless we will treat them in a forthcoming paper for the discussion of pseudo-differential boundary value problems.

In this section, our parameter-dependent symbols usually depend on a complex variable. If we say that the symbol is differentiable with respect to that variable, we interpret this complex variable as an element of $\mathbb{R}^2$ and mean that the symbol is differentiable in the real sense. Likewise, if there is a complex variable appearing in the Besselpotential, we treat it as a variable in $\mathbb{R}^2$.

Definition 6.5.1. Let Z be a Banach space, $d\in\mathbb{R}$ and $\Sigma\subset\mathbb{C}$. Let further $l\in\mathbb{Z}_{\geq1}$, $\mathpzc{d}=(\mathpzc{d}_1,\dots,\mathpzc{d}_l)\in(\mathbb{Z}_{\geq1})^l$ such that $|\mathpzc{d}|=n$ and $\boldsymbol{a}=(a_1,\dots,a_l)\in(0,\infty)^l$.

(a) The parameter-independent *Hörmander class* of order d with constant coefficients denoted by $S^d(\mathbb{R}^n; Z)$ is the space of all smooth functions $p \in C^\infty(\mathbb{R}^n; Z)$ with

$$\|p\|_k^{(d)} := \sup_{\substack{\xi \in \mathbb{R}^n \\ |\alpha| \le k}} \langle \xi \rangle^{-(d-|\alpha|)} ||D_\xi^\alpha p(\xi)||_Z < \infty$$

for all $\alpha \in \mathbb{N}^n$. Here, as usual the Besselpotential is defined by $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$.

(b) The anisotropic parameter-independent *Hörmander class* of order d with constant coefficients denoted by $S^d_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^n; Z)$ is the space of all smooth functions $p \in C^\infty(\mathbb{R}^n; Z)$ with

$$\|p\|^{(d)}_{\mathscr{d},\boldsymbol{a},k} := \sup_{\substack{\xi,\lambda \in \mathbb{R}^n \times \Sigma \\ |\alpha_1|+\ldots+|\alpha_l| \le k}} \langle \xi \rangle_{\mathscr{d},\boldsymbol{a}}^{-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|)} \|\partial_{\xi_1}^{\alpha_1} \ldots \partial_{\xi_l}^{\alpha_l} p(\xi)\|_Z < \infty$$

for all $\alpha_k \in (\mathbb{Z}_{\ge 1})^{d_k}$ $(k = 1, \ldots, l)$. Here, the anisotropic Besselpotential is defined by

$$\langle \xi \rangle_{\mathscr{d},a} := (1+|\xi_1|^{2/a_1} + \ldots + |\xi_l|^{2/a_l})^{1/2}.$$

Definition 6.5.2. Let Z be a Banach space, $d \in \mathbb{R}$ and $\Sigma \subset \mathbb{C}$. Let further $l \in \mathbb{Z}_{\ge 1}$, $\mathscr{d} = (\mathscr{d}_1, \ldots, \mathscr{d}_l) \in (\mathbb{Z}_{\ge 1})^l$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \ldots, a_{l+1}) \in (0,\infty)^{l+1}$.

(a) The isotropic parameter-dependent *Hörmander class* of order d and regularity ∞ with constant coefficients denoted by $S^{d,\infty}(\mathbb{R}^n \times \Sigma; Z)$ is the space of all smooth functions $p \in C^\infty(\mathbb{R}^n \times \Sigma; Z)$ with

$$\|p\|_k^{(d,\infty)} := \sup_{\substack{\xi \in \mathbb{R}^n \\ |\alpha|+j \le k}} (\langle \xi, \mu \rangle^{-(d-|\alpha|-j)} ||D_\xi^\alpha D_\mu^j p(\xi,\mu)||_Z < \infty$$

for all $\alpha \in \mathbb{N}^n$ and all $j \in \mathbb{N}^2$. Here, the parameter-dependent Besselpotential is defined by

$$\langle \xi \rangle_{\mathscr{d},a} := (1+|\xi|^2 + |\mu|^2)^{1/2}.$$

(b) The anisotropic parameter-dependent *Hörmander class* of order d and regularity ∞ with constant coefficients denoted by $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^n \times \Sigma; Z)$ is the space of all smooth functions $p \in C^\infty(\mathbb{R}^n \times \Sigma; Z)$ with

$$\|p\|^{(d,\infty)}_{\mathscr{d},\boldsymbol{a},k} := \sup_{\substack{\xi,\lambda \in \mathbb{R}^n \times \Sigma \\ |\alpha_1|+\ldots+|\alpha_l|+j \le k}} \langle \xi, \lambda \rangle_{\mathscr{d},\boldsymbol{a}}^{-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|-a_{l+1}j)} \|\partial_{\xi_1}^{\alpha_1} \ldots \partial_{\xi_l}^{\alpha_l} \partial_\lambda^j p(\xi,\lambda)\|_Z < \infty$$

for all $\alpha_k \in (\mathbb{Z}_{\ge 1})^{d_k}$ $(k = 1, \ldots, l)$, $\alpha_{l+1} \in \mathbb{N}$ and all $j \in \mathbb{N}^2$. Here, the anisotropic Besselpotential is defined by

$$\langle \xi, \lambda \rangle_{\mathscr{d},a} := (1+|\xi_1|^{2/a_1} + \ldots + |\xi_l|^{2/a_l} + |\lambda|^{2/a_{l+1}})^{1/2}.$$

In the special case $l = 1$ we also omit $\mathscr{d}$ in the notation and write $S^{d,\infty}_{\boldsymbol{a}}(\mathbb{R}^n \times \Sigma; Z)$ and $\|p\|^{(d,\infty)}_{\boldsymbol{a},k}$ instead.

Definition 6.5.3. Let Z be a Banach space, $d \in \mathbb{R}$ and $1 \leq p \leq \infty$. Let further $l \in \mathbb{Z}_{\geq 1}$, $\mathscr{d} = (\mathscr{d}_1, \dots, \mathscr{d}_l) \in (\mathbb{Z}_{\geq 1})^l$ with $\mathscr{d}_l = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \dots, a_l) \in (0, \infty)^l$.

(a) By $S^d(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ we denote the space of all smooth functions

$$\widetilde{k}\colon \mathbb{R}_+ \times \mathbb{R}^{n-1} \to Z, \ (x_1, \xi') \mapsto \widetilde{k}(x_1, \xi')$$

satisfying

$$\|\widetilde{k}\|_{S^d(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z)), \alpha', m, m'}$$
$$:= \sup_{\xi' \in \mathbb{R}^{n-1}, x_1 \in \mathbb{R}_+} \langle \xi' \rangle^{-(d-m+m'-|\alpha'|)-1+\frac{1}{p}} \| x_1 \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} \widetilde{k}(x_1, \xi') \|_{L^p(\mathbb{R}_+, Z)} < \infty$$

for all $\alpha \in \mathbb{N}^{n-1}$ and all $m, m' \in \mathbb{N}$. The elements of $S^d(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ will be called parameter-independent Poisson symbol-kernels of order $d+1$ or degree d.

(b) We denote by $S^d_{\mathscr{d}, \boldsymbol{a}}(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ the space of all smooth functions

$$\widetilde{k}: \mathbb{R}_+ \times \mathbb{R}^{n-1} \longrightarrow Z, \ (x_1, \xi') \mapsto \widetilde{k}(x_1, \xi')$$

satisfying

$$\|\widetilde{k}\|_{S^d_{\mathscr{d}, \boldsymbol{a}}(\mathscr{S}_{L_p}(\mathbb{R}_+; Z)), \alpha, m, m'}$$
$$:= \sup_{\xi', \lambda} \langle \xi' \rangle_{\mathscr{d}', \boldsymbol{a}'}^{-(d-(m-m')a_1 - a_2|\alpha_2| - \ldots - a_l|\alpha_l|) + a_1(\frac{1}{p}-1)} \| x_1 \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} \widetilde{k}(x_1, \xi') \|_{L_p(\mathbb{R}_+; Z)} < \infty$$

for every $\alpha' = (\alpha_2, \dots, \alpha_l) \in \mathbb{N}^{n-1}$ an all $m, m' \in \mathbb{N}$. The elements of $S^d_{\mathscr{d}, \boldsymbol{a}}(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ will be called anisotropic parameter-independent Poisson symbol-kernels of order $d + a_1$ or degree d. In the special case $a_1 = \ldots = a_l$ we omit $\mathscr{d}$ in the notation and write $S^d_{\boldsymbol{a}}(\mathbb{R}^{n-1}; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ and $\|\widetilde{k}\|_{S^d_{\boldsymbol{a}}(\mathscr{S}_{L_p}(\mathbb{R}_+; Z)), \alpha, m, m'}$ instead.

Definition 6.5.4. Let Z be a Banach space, $d \in \mathbb{R}$ and $1 \leq p \leq \infty$. Let further $l \in \mathbb{Z}_{\geq 1}$, $\mathscr{d} = (\mathscr{d}_1, \dots, \mathscr{d}_l) \in (\mathbb{Z}_{\geq 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \dots, a_{l+1}) \in (0, \infty)^{l+1}$.

(a) By $S^{d,\infty}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ we denote the space of all smooth functions

$$\widetilde{k}\colon \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \Sigma \to Z, \ (x_1, \xi', \mu) \mapsto \widetilde{k}(x_1, \xi', \mu)$$

satisfying

$$\|\widetilde{k}\|_{S^{d,\infty}(\mathbb{R}^{n-1} \times \Sigma_\theta; \mathscr{S}_{L_p}(\mathbb{R}_+; Z)), \alpha', m, m', \gamma}$$
$$:= \sup_{\xi' \in \mathbb{R}^{n-1}, x_1 \in \mathbb{R}_+} \langle \xi', \mu \rangle^{-d+m-m'+|\alpha'|+\gamma-1+\frac{1}{p}} \| x_1 \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} D_\mu^\gamma \widetilde{k}(x_1, \xi', \mu) \|_{L^p(\mathbb{R}_+, Z)} < \infty$$

for all $\alpha \in \mathbb{N}^{n-1}$, $\gamma \in \mathbb{N}^2$ and all $m, m' \in \mathbb{N}$. The elements of $S^{d,\infty}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ will be called parameter-dependent Poisson symbol-kernels of order $d+1$ or degree d and regularity ∞.

(b) We denote by $S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z))$ the space of all smooth functions

$$\widetilde{k}:\mathbb{R}_+\times\mathbb{R}^{n-1}\times\Sigma\longrightarrow Z,\ (x_1,\xi',\lambda)\mapsto\widetilde{k}(x_1,\xi',\lambda)$$

satisfying

$$\begin{aligned}&||\widetilde{k}||_{S^{d,\infty}_{d,\boldsymbol{a}}(\mathscr{S}_{L_p}(\mathbb{R}_+;Z)),\alpha',m,m',\gamma}\\ :=&\sup_{\substack{x_1\in\mathbb{R}_+,\xi'\in\mathbb{R}^{n-1},\\ \lambda\in\Sigma}}\langle\xi',\lambda\rangle_{d',\boldsymbol{a}'}^{-(d-(m-m')a_1-a_2|\alpha_2|-\ldots-a_l|\alpha_l|-|\gamma|a_{l+1})+a_1(\frac{1}{p}-1)}\\ &||x_1\mapsto x_1^mD^{m'}_{x_1}D^{\alpha'}_{\xi'}D^{\gamma}_{\lambda}\widetilde{k}(x_1,\xi',\lambda)||_{L_p(\mathbb{R}_+;Z)}<\infty\end{aligned}$$

for every $\alpha'=(\alpha_2,\ldots,\alpha_{l+1})\in\mathbb{N}^n$, $m,m'\in\mathbb{N}$ and $\gamma\in\mathbb{N}^2$. The elements of $S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z))$ will be called anisotropic parameter-dependent Poisson symbol-kernels of order $d+a_1$ or degree d and regularity ∞. In the special case $a_1=\ldots=a_l$ we omit d in the notation and write $S^{d,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z))$ and $||\widetilde{k}||_{S^{d,\infty}_{\boldsymbol{a}}(\mathscr{S}_{L_p}(\mathbb{R}_+;Z)),\alpha',m,m',\gamma}$ instead.

Lemma 6.5.5. *Let X be a Banach space. For $p\in[1,\infty]$, $m,m'\in\mathbb{N}$ let*

$$\|f\|_{\mathscr{S}_{L_p}(\mathbb{R}_+,X),m,m'}:=\|x\mapsto x^mD^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)}\quad(f\in\mathscr{S}(\mathbb{R}_+,X)).$$

We write $\mathscr{S}_{L_p}(\mathbb{R}_+,X)$ if we endow $\mathscr{S}(\mathbb{R}_+,X)$ with the norm generated by $\{\|\cdot\|_{\mathscr{S}_{L_p}(\mathbb{R}_+,X),m,m'}:m,m'\in\mathbb{N}\}$.

(a) The topology on $\mathscr{S}(\mathbb{R}_+,X)$ generated by the family $\{\|\cdot\|_{\mathscr{S}_{L_p}(\mathbb{R}_+,X),m,m'}:m,m'\in\mathbb{N}\}$ is independent of p.

(b) The symbol-kernel class $S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z))$ is independent on p. The respective assertion also holds in the isotropic or parameter-independent case.

Proof. (a): We simply show that $\mathscr{S}_{L_p}(\mathbb{R}_+,X)\hookrightarrow\mathscr{S}_{L_q}(\mathbb{R}_+,X)$ for all choices of $p,q\in[1,\infty]$. If $q<p$ we can use Hölder's inequality. Let $m,m'\in\mathbb{N}$, $r\in[1,\infty]$ such that $1/q=1/r+1/p$. Then, we have

$$\begin{aligned}\|x\mapsto x^mD^{m'}_xf(x)\|_{L_q(\mathbb{R}_+,X)}&\le\|x\mapsto\langle x\rangle^{-1}\|_{L^r}\|x\mapsto\langle x\rangle x^mD^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)}\\ &\lesssim\max\{\|x\mapsto x^mD^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)},\|x\mapsto x^{m+1}D^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)}\}\end{aligned}$$

If $q\ge p$ we use the embedding $W^1_p(\mathbb{R}_+,X)\hookrightarrow L_q(\mathbb{R}_+,X)$ (cf. Proposition 3.12 in combination with Proposition 7.2 in [182]). This embedding yields

$$\begin{aligned}&\|x\mapsto x^mD^{m'}_xf(x)\|_{L_q(\mathbb{R}_+,X)}\lesssim\|x\mapsto x^mD^{m'}_xf(x)\|_{W^1_p(\mathbb{R}_+,X)}\\ &\lesssim\max\{\|x\mapsto x^mD^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)},\|x\mapsto x^mD^{m'+1}_xf(x)\|_{L_p(\mathbb{R}_+,X)},\|x\mapsto x^{m-1}D^{m'}_xf(x)\|_{L_p(\mathbb{R}_+,X)}\}\end{aligned}$$

for all $m,m'\in\mathbb{N}$. Altogether, we obtain the assertion.

(b): We will derive this from (a) by a scaling argument. For simplicity of notation we restrict ourselves to the isotropic parameter-dependent case. Consider a smooth function

$$\widetilde{k}\colon \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \Sigma \to Z,\ (x_1,\xi',\mu) \mapsto \widetilde{k}(x_1,\xi',\mu)$$

Let $\alpha' \in \mathbb{N}^{n-1}$, $\gamma \in \mathbb{N}^2$ and put $\widetilde{k}_{\alpha',\gamma}(x_1,\xi',\mu) := D_{\xi'}^{\alpha'} D_\mu^\gamma \widetilde{k}(x_1,\xi',\mu)$ and $\widetilde{h}_{\alpha',\gamma}(t_1,\xi',\mu) := \widetilde{k}_{\alpha',\gamma}(\langle\xi',\mu\rangle^{-1} t_1,\xi',\mu)$. Then

$$\langle\xi',\mu\rangle^{\frac{1}{p}+m-m'} \|\widetilde{k}_{\alpha',\gamma}(\cdot,\xi',\mu)\|_{\mathscr{S}_{L_p},m,m'} = \|\widetilde{h}_{\alpha',\gamma}(\cdot,\xi',\mu)\|_{\mathscr{S}_{L_p},m,m'}, \qquad m,m' \in \mathbb{N}, p \in [1,\infty].$$

Applying the seminorm estimates associated with (a) to $\widetilde{h}_{\alpha',\gamma}(\cdot,\xi',\mu)$ the desired result follows.

□

Remark 6.5.6. (a) Occasionally, we will need the estimates in the definitions of the Poisson symbol-kernel classes with m being a non-negative real number instead of a natural number. But the respective estimates follow by using Young's inequality. Indeed, for example in the anisotropic parameter-dependent case we have for all $\theta \in [0,1]$ that

$$\begin{aligned}\langle\xi',\lambda\rangle_{d,\boldsymbol{a}}^{(m+\theta)a_1} x_1^{p(m+\theta)} &= \langle\xi',\lambda\rangle_{d,\boldsymbol{a}}^{m(1-\theta)a_1} x_1^{pm(1-\theta)} \langle\xi',\lambda\rangle_{d,\boldsymbol{a}}^{(m+1)\theta a_1} x_1^{p(m+1)\theta}\\ &\le (1-\theta)\langle\xi',\lambda\rangle_{d,\boldsymbol{a}}^{ma_1} x_1^{pm} + \theta\langle\xi',\lambda\rangle_{d,\boldsymbol{a}}^{(m+1)a_1} x_1^{p(m+1)}.\end{aligned}$$

Using the triangle inequality for the $L_p(\mathbb{R}_+,Z)$-norm yields the desired estimate.

(b) Let $q>0$. Then we have that $S_{d,\boldsymbol{a}}^{d,\infty} = S_{d,q\boldsymbol{a}}^{qd,\infty}$ and that $S_{d,\boldsymbol{a}}^{d,\infty}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z))$, since

$$\begin{aligned}\langle\xi',\lambda\rangle_{d,a} &= (1+|\xi_2|^{2/a_2}+\ldots+|\xi_l|^{2/a_l}+|\lambda|^{2/a_{l+1}})^{1/2}\\ &\eqsim (1+|\xi_2|^{2/qa_2}+\ldots+|\xi_l|^{2/qa_l}+|\lambda|^{2/qa_{l+1}})^{q/2} = \langle\xi',\lambda\rangle_{d,qa}^q.\end{aligned}$$

The same assertion also holds in the parameter-independent case.

(c) Let $m,m' \in \mathbb{N}$, $\gamma \in \mathbb{N}^2$ as well as $\alpha_k \in \mathbb{N}^{d_k}$ $(k = 1.\ldots,l)$ and $\alpha' = (\alpha_2,\ldots,\alpha_k) \in \mathbb{N}^{n-1}$. Then, it follows from the definition of the symbol-kernels that $\widetilde{k} \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} D_\lambda^\gamma \widetilde{k}$ is a continuous mapping

$$S_{d,\boldsymbol{a}}^{d,\infty}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z)) \longrightarrow S_{d,\boldsymbol{a}}^{d-(m-m')a_1-a_2|\alpha_2|-\ldots-a_l|\alpha_l|-|\gamma|a_{l+1},\infty}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_p}(\mathbb{R}_+;Z)).$$

The respective assertion also holds for the other symbol-kernel classes as well as for the Hörmander symbols.

(d) Let $d_1,d_2 \in \mathbb{R}$ and suppose that we have a continuous bilinear mapping $Z_1\times Z_2 \to Z$ for the Banach spaces Z_1,Z_2 and Z. Then, the bilinear mapping

$$S_{d,\boldsymbol{a}}^{d_1,\infty}(\mathbb{R}^n\times\Sigma,Z_1)\times S_{d,\boldsymbol{a}}^{d_2,\infty}(\mathbb{R}^n\times\Sigma;Z_2) \to S_{d,\boldsymbol{a}}^{d_1+d_2,\infty}(\mathbb{R}^n\times\Sigma;Z),\ (p_1,p_2)\mapsto p\cdot p$$

is continuous. The respective assertions also hold for the other classes of Hörmander symbols.

Remark 6.5.7. Consider the situation of Definition 6.5.4. Suppose that $\Sigma = \Sigma_\varphi$ is a sector with opening angle $\varphi > \pi/2$ and let $\widetilde{k} \in S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$. If we just take $\lambda = 1 + i\theta$ with $\theta \in \mathbb{R}$, then $(x_1, \xi', \theta) \mapsto k(x_1, \xi', 1 + i\theta)$ is an anisotropic Poisson symbol kernel in the sense of Definition 6.5.3.

Proposition 6.5.8. *Let Z be a Banach space and $d \in \mathbb{R}$. Let further $l \in \mathbb{Z}_{\geq 1}$, $\mathscr{d} = (\mathscr{d}_1, \ldots, \mathscr{d}_l) \in (\mathbb{Z}_{\geq 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \ldots, a_{l+1}) \in (0, \infty)^{l+1}$. Let $q \in \mathbb{N}$ and $\Sigma_q := \{z^q : z \in \Sigma\}$ for some open $\Sigma \subset \mathbb{C}$. Let $\widetilde{k} \in S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma_q; \mathscr{S}_{L_1}(\mathbb{R}_+, Z))$ an anisotropic symbol. Then, the transformation $\lambda = \mu^q$ leads to a symbol in $S^{qd,\infty}_{\mathscr{d},\boldsymbol{a}_q}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}_+, Z))$ where $\boldsymbol{a}_q := (qa_1, \ldots, qa_l, a_{l+1})$, i.e. we have that*

$$[(x_1, \xi', \mu) \mapsto \widetilde{k}(x_1, \xi', \mu^q)] \in S^{qd,\infty}_{\mathscr{d},\boldsymbol{a}_q}(\mathscr{S}_{L_1}).$$

Proof. First, we note that

$$\mu^q = (\mu_1 + i\mu_2)^q = \sum_{\widetilde{q}=0}^{q} \binom{q}{\widetilde{q}} \mu_1^{\widetilde{q}} (i\mu_2)^{q-\widetilde{q}}.$$

We show by induction on $|\gamma|$ with $\gamma \in \mathbb{N}^2$ that $\partial^{\alpha'}_{\xi'} \partial^{\gamma}_{\mu} \widetilde{k}(x_1, \xi', \mu^q)$ is a linear combination of terms of the form $\mu^{j-i} f(x_1, \xi', \mu^q)$ where $j, i \in \mathbb{N}^2$ such that $j - i \in \mathbb{N}^2$ and $\frac{|j|}{q-1} + |i| = |\gamma|$ as well as $f \in S^{d - \boldsymbol{a}'' \cdot_{\mathscr{d}'} \alpha' - (|j| a_{l+1})/(q-1)}_{\mathscr{d},\boldsymbol{a}}(\mathscr{S}_{L_1})$. Obviously, this is true for $\gamma = 0$. So let $\partial^{\alpha}_{\xi'} \partial^{\gamma}_{\mu} \widetilde{k}(x_1, \xi', \mu^q)$ be a sum of terms of the form $\mu^{j-i} f(x_1, \xi', q)$ where $j, i \in \mathbb{N}^2$ such that $j - i \in \mathbb{N}^2$ and $\frac{|j|}{q-1} + i = |\gamma|$ as well as $f \in S^{d - \boldsymbol{a}'' \cdot_{\mathscr{d}'} \alpha' - (|j| a_{l+1})/(q-1)}_{\mathscr{d},\boldsymbol{a}}(\mathscr{S}_{L_1})$. We consider the summands separately. Then, we have

$$\begin{aligned} &\partial_{\mu_1} [\mu^{j-i} f(x_1, \xi', \mu^q)] \\ &= [j_1 - i_1]_+ \mu^{j-i-e_1} f(x_1, \xi', \mu^q) + \widetilde{q} \mu^{j-i} \Big(\sum_{\widetilde{q}=1}^{q} \binom{q}{\widetilde{q}} \mu_1^{\widetilde{q}-1} (i\mu_2)^{q-\widetilde{q}} \Big) (\partial_{\mu_1} f)(x_1, \xi', \mu^q). \end{aligned}$$

A similar computations holds for $\partial_{\mu_2} [\mu^{j-i} f(x_1, \xi', \mu^q)]$. Hence, by Remark 6.5.6 (c) the induction is finished. Estimating such terms, we obtain

$$\begin{aligned} \| x_1 \mapsto x_1^m D^{m'}_{x_1} \mu^{j-i} f(x_1, \xi', \mu^q) \|_{L_1} &\leq |\mu^{j-i}| \langle \xi', \mu^q \rangle_{\mathscr{d},\boldsymbol{a}}^{d - \boldsymbol{a}'' \cdot_{\mathscr{d}'} \alpha' - (m-m') a_l - (|j| a_{l+1})/(q-1)} \\ &\eqsim |\mu^{j-i}| \langle \xi', \mu \rangle_{\mathscr{d},\boldsymbol{a}_q}^{q[d - \boldsymbol{a}'' \cdot_{\mathscr{d}'} \alpha' - (m-m') a_l] - (q|j| a_{l+1})/(q-1)} \\ &\leq \langle \xi', |\mu| \rangle_{\mathscr{d},\boldsymbol{a}_q}^{qd - |\alpha_1| a_1 - \ldots - |\alpha_{l-1}| a_{l-1} - (m-m') a_l - (|j|/(q-1) + |i|) a_{l+1}} \\ &= \langle \xi', |\mu| \rangle_{\mathscr{d},\boldsymbol{a}_q}^{qd - |\alpha_1| a_1 - \ldots - |\alpha_{l-1}| a_{l-1} - (m-m') a_l - |\gamma| a_{l+1}} \end{aligned}$$

This proves the assertion. □

Definition 6.5.9. (a) Given a Hörmander symbol with constant coefficients p or $p_\mu := p(\cdot,\mu)$ in the parameter-dependent case, we define the associated operator

$$Pf := \mathrm{OP}(p)f = \mathscr{F}^{-1}p\mathscr{F}f \quad (f \in \mathscr{S}'(\mathbb{R}^n, X)).$$

or

$$P_\mu f := \mathrm{OP}(p_\mu)f = \mathscr{F}^{-1}p_\mu\mathscr{F}f \quad (f \in \mathscr{S}'(\mathbb{R}^n, X)).$$

respectively.

(b) Given a Poisson symbol-kernel k or $k_\mu := k(\cdot,\cdot,\mu)$ in the parameter-dependent case, we define the associated operator

$$K := \mathrm{OPK}(\widetilde{k})g(x) := (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{ix'\xi'}\widetilde{k}(x_1,\xi')\mathscr{F}g(\xi')\,d\xi' \quad (x \in \mathbb{R}^n_+,\ g \in \mathscr{S}(\mathbb{R}^{n-1}, X)).$$

or

$$K_\mu := \mathrm{OPK}(\widetilde{k}_\mu)g(x) := (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{ix'\xi'}\widetilde{k}(x_1,\xi',\mu)\mathscr{F}g(\xi')\,d\xi' \quad (x \in \mathbb{R}^n_+,\ g \in \mathscr{S}(\mathbb{R}^{n-1}, X)),$$

respectively.

Definition 6.5.10. Let Z be a Banach space, $d \in \mathbb{R}$ and $1 \le p \le \infty$. Let further $l \in \mathbb{Z}_{\ge 1}$, $\mathscr{d} = (\mathscr{d}_1,\dots,\mathscr{d}_l) \in (\mathbb{Z}_{\ge 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1,\dots,a_l) \in (0,\infty)^l$.

(a) We denote by $S^d(\mathbb{R}^{n-1};\mathscr{S}_{L_\infty}(\mathbb{R};Z))$ the space of all smooth functions

$$p : \mathbb{R}\times\mathbb{R}^{n-1}\times\Sigma \longrightarrow Z,\ (\xi_1,\xi') \mapsto p(\xi_1,\xi')$$

satisfying

$$\|p\|_{S^d(\mathscr{S}_{L_\infty}(\mathbb{R};Z)),\alpha',m,m'} := \sup_{\xi\in\mathbb{R}^n}\langle\xi'\rangle^{-(d+m-m'-|\alpha'|)}\|\xi_1^m D_{\xi_1}^{m'} D_{\xi'}^{\alpha'} p(\xi_1,\xi')\| < \infty$$

for every $\alpha' \in \mathbb{N}^{n-1}$ and $m, m' \in \mathbb{N}$.

(b) We denote by $S^d_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1};\mathscr{S}_{L_\infty}(\mathbb{R};Z))$ the space of all smooth functions

$$p : \mathbb{R}\times\mathbb{R}^{n-1}\times\Sigma \longrightarrow Z,\ (\xi_1,\xi') \mapsto p(\xi_1,\xi')$$

satisfying

$$\|p\|_{S^d_{\mathscr{d},\boldsymbol{a}}(\mathscr{S}_{L_\infty}(\mathbb{R};Z)),\alpha',m,m} := \sup_{\xi\in\mathbb{R}^n}\langle\xi'\rangle_{\mathscr{d}',\boldsymbol{a}'}^{-(d+(m-m')a_1-|\alpha_2|a_2-\dots-|\alpha_l|a_l)}\|\xi_1^m D_{\xi_1}^{m'} D_{\xi'}^{\alpha'} p(\xi_1,\xi')\| < \infty$$

for every $\alpha \in \mathbb{N}^n$ and $m, m' \in \mathbb{N}$.

Definition 6.5.11. Let Z be a Banach space, $d \in \mathbb{R}$ and $1 \le p \le \infty$. Let further $l \in \mathbb{Z}_{\ge 1}$, $\mathscr{d} = (\mathscr{d}_1,\dots,\mathscr{d}_l) \in (\mathbb{Z}_{\ge 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1,\dots,a_{l+1}) \in (0,\infty)^{l+1}$.

(a) We denote by $S^{d,\infty}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_\infty}(\mathbb{R};Z))$ the space of all smooth functions

$$p:\mathbb{R}\times\mathbb{R}^{n-1}\times\Sigma\longrightarrow Z,\ (\xi_1,\xi',\mu)\mapsto p(\xi_1,\xi',\mu)$$

satisfying

$$||p||_{S^{d,\infty}(\mathscr{S}_{L_\infty}(\mathbb{R};Z)),\alpha',m,m',\gamma}:= \sup_{\xi\in\mathbb{R}^n,\mu\in\Sigma}\langle\xi',\mu\rangle^{-(d+m-m'-|\alpha'|-|\gamma|)}||\xi_1^m D_{\xi_1}^{m'} D_{\xi'}^{\alpha'} D_\mu^\gamma p(\xi_1,\xi',\mu)||<\infty$$

for every $\alpha'\in\mathbb{N}^{n-1}$, $m,m'\in\mathbb{N}$ and $\gamma\in\mathbb{N}^2$.

(b) We denote by $S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_\infty}(\mathbb{R};Z))$ the space of all smooth functions

$$p:\mathbb{R}\times\mathbb{R}^{n-1}\times\Sigma\longrightarrow Z,\ (\xi_1,\xi',\lambda)\mapsto p(\xi_1,\xi',\lambda)$$

satisfying

$$||p||_{S^{d,\infty}_{d,\boldsymbol{a}}(\mathscr{S}_{L_\infty}(\mathbb{R};Z)),\alpha',m,m',\gamma}:= \sup_{\xi\in\mathbb{R}^n,\mu\in\Sigma}\langle\xi',|\lambda|\rangle_{d',\boldsymbol{a}'}^{-(d+(m-m')a_1-|\alpha_2|a_2-\ldots-|\alpha_l|a_l-\gamma a_{l+1}|)}||\xi_1^m D_{\xi_1}^{m'} D_{\xi'}^{\alpha'} D_\lambda^\gamma p(\xi_1,\xi',\lambda)||<\infty$$

for every $\alpha\in\mathbb{N}^n$, $m,m'\in\mathbb{N}$ and $\gamma\in\mathbb{N}^2$.

Lemma 6.5.12. *We have the continuous embedding*

$$S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_\infty}(\mathbb{R};Z))\hookrightarrow S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n}\times\Sigma;Z).$$

The respective assertion holds within the isotropic or parameter-independent classes.

Proof. We only prove the result for the anisotropic and parameter-dependent case, as the other cases can be proven in the exact same way. For given $\alpha\in\mathbb{N}^n$ and $\gamma\in\mathbb{N}^2$ we obtain

$$\begin{aligned}
&\sup_{\xi,\lambda\in\mathbb{R}^n\times\Sigma}\langle\xi,|\lambda|\rangle_{d,\boldsymbol{a}}^{-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|-a_{l+1}\gamma)}\|\partial_\xi^\alpha\partial_\lambda^\gamma p(\xi,\lambda)\|_Z\\
\lesssim&\sup_{\xi,\lambda\in\mathbb{R}^n\times\Sigma}\Big[\langle\xi',|\lambda|\rangle_{d,\boldsymbol{a}}^{-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|-a_{l+1}\gamma)}\|\partial_\xi^\alpha\partial_\lambda^\gamma p(\xi,\lambda)\|_Z\\
&\quad+[-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|-a_{l+1}\gamma)]_+\|\xi_1^{\frac{1}{a_1}[-(d-a_1|\alpha_1|-\ldots-a_l|\alpha_l|-a_{l+1}\gamma)]_+}\partial_\xi^\alpha\partial_\lambda^\gamma p(\xi,\lambda)\|_Z\Big]\\
<&\infty
\end{aligned}$$

□

Note that Lemma 6.5.12 shows us that we can define an operator to a symbol in $S^{d,\infty}_{d,\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma;\mathscr{S}_{L_\infty}(\mathbb{R};Z))$ by the means of Definition 6.5.9.

Lemma 6.5.13. *Let X, Y be a Banach spaces and $d \in \mathbb{R}$. Let further $l \in \mathbb{Z}_{\geq 1}$, $\mathscr{d} = (\mathscr{d}_1, \dots, \mathscr{d}_l) \in (\mathbb{Z}_{\geq 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \dots, a_{l+1}) \in (0, \infty)^{l+1}$. There is a continuous linear mapping*

$$S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X,Y))) \longrightarrow S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_\infty}(\mathbb{R}; \mathscr{B}(X,Y))),\ \widetilde{k} \mapsto p,$$

which assigns to each $\widetilde{k}$ a p such that $r_+ \operatorname{OP}[p](\delta_0 \otimes \cdot) = \operatorname{OPK}(\widetilde{k})$. More explicitly, the mapping $\widetilde{k} \mapsto p$ can be defined by means of the diagram

$$\begin{array}{ccc} S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X,Y))) & \xrightarrow{\ E\ } & S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}; \mathscr{B}(X,Y))) \\ & \searrow^{\widetilde{k} \mapsto p} & \Big\downarrow \mathscr{F}_{x_1 \mapsto \xi_1} \\ & & S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_\infty}(\mathbb{R}; \mathscr{B}(X,Y))) \end{array}$$

where E denotes the Seeley extension as in [225] and the space $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}; \mathscr{B}(X,Y)))$ is defined analogously to $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X,Y)))$. The respective assertions also hold within the isotropic or parameter-independent classes.

Proof. We only prove the result for the anisotropic and parameter-dependent case, as the other cases can be proven in the exact same way. The proof consists of three steps:

(i) We show that the Seeley extension is bounded from $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X,Y)))$ to $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}; \mathscr{B}(X,Y)))$.

(ii) We show that $\mathscr{F}_{x_1 \mapsto \xi_1}$ is bounded from $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_1}(\mathbb{R}; \mathscr{B}(X,Y)))$ to $S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_\infty}(\mathbb{R}; \mathscr{B}(X,Y)))$.

(iii) We show that $\operatorname{OP}[\mathscr{F}_{x_1 \mapsto \xi_1} E\widetilde{k}](\delta_0 \otimes \cdot) = \operatorname{OPK}[\widetilde{k}]$.

So let us prove the three steps one by one:

(i) For the Seeley extension we fix two sequences $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ such that

   (i) $b_k < 0$ for all $k \in \mathbb{N}$,

   (ii) $\sum_{k=1}^{\infty} |a_k||b_k|^l < \infty$ for all $l \in \mathbb{N}$,

   (iii) $\sum_{k=1}^{\infty} a_k b_k^l = 1$ for all $l \in \mathbb{N}$,

   (iv) $b_k \to -\infty$ as $k \to \infty$.

   It was proven in [225] that such sequences indeed exist. Moreover, we take a function $\phi \in C^\infty(\mathbb{R}_+)$ with $\phi(t) = 1$ for $0 \leq t \leq 1$ and $\phi(t) = 0$ for $t \geq 2$. Then, the Seeley extension for a function $f \in S^{d,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_p}(\mathbb{R}_+; Z))$ is defined by

$$(Ef)(t, \xi', \lambda) = \sum_{k=1}^{\infty} a_k \phi(b_k t) f(b_k t, \xi', \lambda) \quad (t < 0).$$

The assertion regarding the smoothness has already been proved by Seeley in [225]. Hence, we only have to show that the estimates of the symbol classed are preserved under the Seeley extension. But they indeed hold as

$$
\begin{aligned}
&\|x_1 \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} D_\lambda^\gamma E\widetilde{k}(x_1,\xi',\lambda)\|_{L_1(\mathbb{R}_-,\mathscr{B}(X,Y))}\\
&= \Big\| x_1 \mapsto x_1^m D_{x_1}^{m'} D_{\xi'}^{\alpha'} D_\lambda^\gamma \sum_{k=1}^\infty a_k \phi(b_k x_1)\widetilde{k}(b_k x_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_-,\mathscr{B}(X,Y))}\\
&= \Big\| x_1 \mapsto x_1^m D_{\xi'}^{\alpha'} D_\lambda^\gamma \sum_{k=1}^\infty a_k \sum_{q=0}^{m'} \binom{m'}{q} b_k^{m'} (D_{x_1}^q \phi)(b_k x_1)(D_{x_1}^{m'-q}\widetilde{k})(b_k x_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_-,\mathscr{B}(X,Y))}\\
&\le \sum_{k=1}^\infty a_k b_k^{m'} \sum_{q=0}^{m'} \binom{m'}{q} \Big\| x_1 \mapsto x_1^m D_{\xi'}^{\alpha'} D_\lambda^\gamma (D_{x_1}^q \phi)(b_k x_1)(D_{x_1}^{m'-q}\widetilde{k})(b_k x_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_-,\mathscr{B}(X,Y))}\\
&\le \sum_{k=1}^\infty a_k b_k^{m'} \sum_{q=0}^{m'} \binom{m'}{q} \Big\| x_1 \mapsto x_1^m D_{\xi'}^{\alpha'} D_\lambda^\gamma (D_{x_1}^q \phi)(b_k x_1)(D_{x_1}^{m'-q}\widetilde{k})(b_k x_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_-,\mathscr{B}(X,Y))}\\
&\le \sum_{k=1}^\infty a_k b_k^{m'-m-1/p} \sum_{q=0}^{m'} \binom{m'}{q} \Big\| y_1 \mapsto y_1^m D_{\xi'}^{\alpha'} D_\lambda^\gamma (D_{y_1}^q \phi)(y_1)(D_{y_1}^{m'-q}\widetilde{k})(y_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_+,\mathscr{B}(X,Y))}\\
&\le \sum_{k=1}^\infty a_k b_k^{m'-m-1/p} \sum_{q=0}^{m'} \binom{m'}{q} \|D_{y_1}^q \phi\|_{L^\infty(\mathbb{R}_+^n)} \Big\| y_1 \mapsto y_1^m D_{\xi'}^{\alpha'} D_\lambda^\gamma (D_{y_1}^{m'-q}\widetilde{k})(y_1,\xi',\lambda)\Big\|_{L_1(\mathbb{R}_+,\mathscr{B}(X,Y))}\\
&\le \sum_{k=1}^\infty a_k b_k^{m'-m-1/p} \sum_{q=0}^{m'} \binom{m'}{q} \|D_{y_1}^q \phi\|_{L^\infty(\mathbb{R}_+^n)} C_{\alpha',m,m'-q,\gamma} \langle \xi',|\lambda|\rangle_{d,\boldsymbol{a}}^{d-(m+m')a_1-q-|\alpha|_2 a_2-\ldots-|\alpha|_l a_l-\gamma a_{l+1}}\\
&\le C\langle \xi',|\lambda|\rangle_{d,\boldsymbol{a}}^{d-(m+m')a_1-q-|\alpha|_2 a_2-\ldots-|\alpha|_l a_l-\gamma a_{l+1}}
\end{aligned}
$$

(ii) This follows directly from the above computation together with the definition of the symbol classes and the fact that $\mathscr{F}_{x_1\mapsto\xi_1}$ maps $L_1(\mathbb{R},\mathscr{B}(X,Y))$ continuously into $L_\infty(\mathbb{R},\mathscr{B}(X,Y))$.

(iii) For all $g \in \mathscr{S}(\mathbb{R}^{n-1})$ and all $x \in \mathbb{R}_+^n$ we have that

$$
\begin{aligned}
\mathrm{OP}(\mathscr{F}_{x_1\mapsto\xi_1} E\widetilde{k})(\delta_0 \otimes g)(x) &= \int_{\mathbb{R}^n} e^{ix\xi}[\mathscr{F}_{x_1\mapsto\xi_1} E\widetilde{k}(\xi_1,\xi',\mu)]\mathscr{F}_{x\mapsto\xi}(\delta_0\otimes g)\,d\xi\\
&= \int_{\mathbb{R}^n} e^{ix\xi}[\mathscr{F}_{x_1\mapsto\xi_1} E\widetilde{k}(\xi_1,\xi',\mu)]1(\xi_1)\mathscr{F}_{x'\mapsto\xi'} g(\xi')\,d\xi\\
&= \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} E\widetilde{k}(x_1,\xi',\mu)\widehat{g}(\xi')\,d\xi\\
&= \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \widetilde{k}(x_1,\xi',\mu)\widehat{g}(\xi')\,d\xi\\
&= \mathrm{OPK}(\widetilde{k})g(x).
\end{aligned}
$$

This finishes the proof.

□

Remark 6.5.14. Note that in Lemma 6.5.13 we can also apply $r_+ \operatorname{OP}[p](\delta_0 \otimes \cdot)$ to elements of $\mathscr{S}'(\mathbb{R}^{n-1})$, cf. Section 6.3.4.

Lemma 6.5.15. *Let Z, Z_1, Z_2, Z_3 be Banach spaces and $d_1, d_2, d_3 \in \mathbb{R}$. Let further $l \in \mathbb{Z}_{\geq 1}$, $\mathscr{d} = (\mathscr{d}_1, \dots, \mathscr{d}_l) \in (\mathbb{Z}_{\geq 1})^l$ with $\mathscr{d}_1 = 1$ such that $|\mathscr{d}| = n$ and $\boldsymbol{a} = (a_1, \dots, a_{l+1}) \in (0,\infty)^{l+1}$. A continuous trilinear mapping $Z_1 \times Z_2 \times Z_3 \to Z$ induces by pointwise multiplication a continuous trilinear mapping*

$$\begin{array}{c} S^{d_1,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^n \times \Sigma; Z_1) \\ \times \\ S^{d_2,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_\infty}(\mathbb{R}; Z_2)) \\ \times \\ S^{d_3,\infty}_{\mathscr{d}',\boldsymbol{a}'}(\mathbb{R}^{n-1} \times \Sigma; Z_3) \end{array} \longrightarrow S^{d_1+d_2+d_3,\infty}_{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \Sigma; \mathscr{S}_{L_\infty}(\mathbb{R}; Z)),$$

where $(p_1, p_2, p_3) \mapsto p$ given by

$$p(\xi_1, \xi', \mu) = p_1(\xi, \mu) p_2(\xi, \mu) p_3(\xi', \mu).$$

It also holds that

$$\operatorname{OP}[p] = \operatorname{OP}[p_1] \circ \operatorname{OP}[p_2] \circ \operatorname{OP}[p_3]$$

Again, the respective assertions also hold within the isotropic or parameter-independent classes.

Proof. In order to keep notations shorter, we first show the assertion for constant p_3. Hence, we omit it in the notation and estimate the term

$$\|\xi_1^m D_{\xi_1}^{m'} D_{\xi'}^{\alpha'} D_\lambda^j p_1(\xi_1, \xi', \lambda) p_2(\xi_1, \xi', \lambda)\|_Z.$$

By the product rule and the triangle inequality, it suffices to estimate expressions of the form

$$\|D_{\xi_n}^{\widetilde{m}'} D_{\xi'}^{\widetilde{\alpha}'} D_\lambda^{\widetilde{j}} p_1(\xi_1, \xi', \lambda) \xi_n^m D_{\xi_n}^{\overline{m}'} D_{\xi'}^{\overline{\alpha}'} D_\lambda^{\overline{j}} p_2(\xi_1, \xi', \lambda)\|_Z,$$

where $|\overline{\alpha}'| + |\widetilde{\alpha}'| = |\alpha'|$, $|\widetilde{j}| + |\overline{j}| = |j|$ and $\overline{m}' + \widetilde{m}' = m'$. But for such an expression, we obtain

$$\begin{aligned} &\|D_{\xi_1}^{\widetilde{m}'} D_{\xi'}^{\widetilde{\alpha}'} D_\lambda^{\widetilde{j}} p_1(\xi_1, \xi', \lambda) \xi_1^m D_{\xi_1}^{\overline{m}'} D_{\xi'}^{\overline{\alpha}'} D_\lambda^{\overline{j}} p_2(\xi_1, \xi', \lambda)\|_Z \\ &\lesssim \langle \xi, |\lambda| \rangle_{\mathscr{d},\boldsymbol{a}}^{d_1 - \widetilde{m}' a_1 - |\alpha_2| a_2 - \dots - |\alpha_l| a_l - |\widetilde{j}| a_{l+1}} \|\xi_1^m D_{\xi_1}^{\overline{m}'} D_{\xi'}^{\overline{\alpha}'} D_\lambda^{\overline{j}} p_2(\xi_1, \xi', \lambda)\|_{Z_2} \\ &\lesssim \langle \xi', |\lambda| \rangle_{\mathscr{d},\boldsymbol{a}}^{d_1 - \widetilde{m}' a_1 - |\alpha_2| a_2 - \dots - |\alpha_l| a_l - |\widetilde{j}| a_{l+1}} \|\xi_1^m D_{\xi_1}^{\overline{m}'} D_{\xi'}^{\overline{\alpha}'} D_\lambda^{\overline{j}} p_2(\xi_1, \xi', \lambda)\|_{Z_2} \\ &\quad + [d_1 - \widetilde{m}' a_1 - |\alpha_2| a_2 - \dots - |\alpha_l| a_l - |\widetilde{j}| a_{l+1}]_+ \\ &\quad \cdot \quad \|\xi_1^{m + \frac{1}{a_1}[d_1 - \widetilde{m}' a_1 - |\alpha_2| a_2 - \dots - |\alpha_l| a_l - |\widetilde{j}| a_{l+1}]_+} D_{\xi_1}^{\overline{m}'} D_{\xi'}^{\overline{\alpha}'} D_\lambda^{\overline{j}} p_2(\xi_1, \xi', \lambda)\|_{Z_2} \\ &\lesssim \langle \xi', |\lambda| \rangle_{\mathscr{d},\boldsymbol{a}}^{d_1 + d_2 - (\widetilde{m}' + \overline{m}' - m) a_1 - (\widetilde{\alpha}_2 + \overline{\alpha}_2) a_2 - \dots - (\widetilde{\alpha}_l + \overline{\alpha}_l) a_l - (|\widetilde{j}| + |\overline{j}|) a_{l+1}} \end{aligned}$$

$$+ [d_1 - \widetilde{m}' a_1 - |\alpha_2| a_2 - \ldots - |\alpha_l| a_l - |\widetilde{j}| a_{l+1}]_+ \cdot \langle \xi', |\lambda| \rangle_{d,\boldsymbol{a}}^{d_1+d_2-(\widetilde{m}'+\overline{m}'-m)a_1-(\widetilde{\alpha}_2'+\overline{\alpha}_2)a_2-\ldots-(\widetilde{\alpha}_l+\overline{\alpha}_l)a_l-(|\widetilde{j}|+|\overline{j}|)a_{l+1}}$$
$$\lesssim \langle \xi', |\lambda| \rangle_{d,\boldsymbol{a}}^{d_1+d_2-(m'-m)a_1-\alpha_2 a_2-\ldots-\alpha_l a_l-|j|a_{l+1}}.$$

A similar computation shows the respective assertion for the case that p_1 is constant and p_3 is arbitrary. The formula for the operators is trivial. □

6.5.2. Solution Operators for Model Problems

In this subsection we consider the boundary value model problems

$$\begin{cases} (\varepsilon+\lambda)v + \mathscr{A}(D)v = 0, & \text{on } \mathbb{R}^n_+, \\ \mathscr{B}_j(D)v = g_j, & \text{on } \mathbb{R}^{n-1}, \quad j = 1,\ldots,m. \end{cases} \tag{6.51}$$

with arbitrary but fixed $\varepsilon > 0$ and

$$\begin{cases} \partial_t u + (1+\mathscr{A}(D))u = 0 & \text{on } \mathbb{R}^n_+ \times \mathbb{R}, \\ \mathscr{B}_j(D)u = g_j, & \text{on } \mathbb{R}^{n-1}\times\mathbb{R}, \quad j = 1,\ldots,n, \end{cases} \tag{6.52}$$

Here, $\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D)$ is a constant coefficient homogeneous $\mathscr{B}(X)$-valued differential boundary value system on $\mathbb{R}^n_+$ as considered in Section 6.3.5. In this subsection we restrict ourselves to $g_1,\ldots,g_m \in \mathscr{S}(\mathbb{R}^{n-1};X)$ so that we can later extend the solution by density to the desired spaces.

The following proposition and its corollary are the main results of this subsection. They (together with the mapping properties that will be obtained in Section 6.5.3) show that the Poisson operators introduced in Section 6.5.1 provide the right classes of operators for solving (6.51) and (6.52).

Proposition 6.5.16. *Let X be a Banach space and assume that $(\mathscr{A},\mathscr{B}_1,\ldots,\mathscr{B}_n)$ satisfies* (E) *and* (LS) *for some $\phi \in (0,\pi)$. Then there exist $\widetilde{k}_j \in S^{-\frac{m_j+1}{2m},\infty}_{(\frac{1}{2m},\frac{1}{2m},1)}(\mathbb{R}^{n-1}\times\Sigma_{\pi-\phi};\mathscr{S}_{L_1}(\mathbb{R}_+;\mathscr{B}(X)))$, $j = 1,\ldots,m$, such that, for each $\lambda \in \Sigma_{\pi-\phi}$,*

$$K_\lambda : \mathscr{S}(\mathbb{R}^{n-1};X)^m \longrightarrow \mathscr{S}(\mathbb{R}^n_+;X),\ (g_1,\ldots,g_m) \mapsto \sum_{j=1}^m \mathrm{OPK}(\widetilde{k}_{j,\lambda})g_j,$$

is a solution operator for the elliptic differential boundary value problem (6.51)*. Moreover, there is uniqueness of solutions in $\mathscr{S}(\mathbb{R}^n_+;X)$: if $u \in \mathscr{S}(\mathbb{R}^n_+;X)$ is a solution of* (6.51)*, then $u = K_\lambda(g_1,\ldots,g_m)$.*

Remark 6.5.17. Proposition 6.5.16 together with Proposition 6.5.8 shows that $\widetilde{k_j}$ belongs to $S^{-m_j-1,\infty}(\mathbb{R}^{n-1}\times\Sigma_{(\pi-\phi)/2m};S_{L_1}(\mathbb{R}_+,\mathscr{B}(X)))$ after the substitution $\lambda = \mu^{2m}$. To be more precise:

$$\widetilde{k_j}^{[2m]} := [(x_1,\xi',\mu) \mapsto \widetilde{k_j}(x_1,\xi',\mu^{2m})] \in S^{-m_j-1,\infty}(\mathbb{R}^{n-1}\times\Sigma_{(\pi-\phi)/2m};S_{L_1}(\mathbb{R}_+,\mathscr{B}(X))).$$

Corollary 6.5.18. *Let X be a Banach space and assume that $(\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n)$ satisfies* (E) *and* (LS) *for some $\phi \in (0, \frac{\pi}{2})$. Then there exist $\widetilde{k}_j \in S^{-\frac{m_j+1}{2m}}_{(\frac{1}{2m}, \frac{1}{2m}, 1)}(\mathbb{R}^{n-1} \times \mathbb{R}; \mathcal{S}_{L_1}(\mathbb{R}_+; \mathcal{B}(X)))$, $j = 1, \dots, m$, such that*

$$K : \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R}; X)^m \longrightarrow \mathcal{S}(\mathbb{R}^n_+ \times \mathbb{R}; X), (g_1, \dots, g_m) \mapsto \sum_{j=1}^{m} \mathrm{OPK}(\widetilde{k}_j) g_j,$$

is a solution operator for the parabolic differential boundary value problem (6.52). *Moreover, there is uniqueness of solutions in $\mathcal{S}(\mathbb{R}^n_+ \times \mathbb{R}; X)$: if $u \in \mathcal{S}(\mathbb{R}^n_+ \times \mathbb{R}; X)$ is a solution of* (6.52)*, then $u = K(g_1, \dots, g_m)$.*

Proof. Under Fourier transformation in time, (6.52) turns into

$$\begin{cases} (1+\tau)\mathcal{F}_{t\to\tau} u + \mathcal{A}(D)\mathcal{F}_{t\to\tau} u &= 0, \\ \mathcal{B}_j(D)\mathcal{F}_{t\to\tau} u &= \mathcal{F}_{t\to\tau} g_j, \quad j = 1, \dots, n. \end{cases}$$

The result thus follows from Proposition 6.5.16 through a substitution as in Remark 6.5.7. □

In order to prove Proposition 6.5.16, we use a certain solution formula (6.51). Following the considerations in [68] we can represent the solution in the Fourier image as

$$\widehat{u} = e^{i\rho A_0(b,\sigma)x_n} M(b,\sigma)\widehat{g}_\rho$$

where

- A_0 is some smooth function with values in $\mathcal{B}(X^{2m}, X^{2m})$ that one obtains from A_0 after some reduction to a first-order system,
- M is some smooth function with values in $\mathcal{B}(X^m, X^{2m})$ which maps the values of the boundary operator applied to the stable solution to the respective Dirichlet traces,
- ρ is a positive parameter that can be chosen in different ways and in dependence of ξ' and λ,
- $b = \xi'/\rho$, $\sigma = (1+\lambda)/\rho^{2m}$ and $\hat{g}_\rho = (\hat{g}_1/\rho^{m_1}, \dots, \hat{g}_m/\rho^{m_m})^T$.

Another operator that we will use later is the spectral projection $\mathcal{P}_-$ of the matrix A_0 to the part of the spectrum that lies above the real line. This spectral projection hast the property that $\mathcal{P}_-(b,\sigma)M(b\sigma) = M(b,\sigma)$.
For our purposes, we will rewrite the above representation in the following way: For $j = 1, \dots, m$ we write

$$M_{\rho,j}(b,\sigma)\hat{g}_j := M(b,\sigma)\frac{\hat{g}_j \otimes e_j}{\rho^{m_j}}$$

so that we obtain

$$\widehat{u} = e^{i\rho A_0(b,\sigma)x_n} M(b,\sigma)\widehat{g}_\rho = \sum_{j=1}^{m} e^{i\rho A_0(b,\sigma)x_n} M_{\rho,j}(b,\sigma)\hat{g}_j. \tag{6.53}$$

The functions $(\xi',\lambda)\mapsto e^{i\rho A_0(b,\sigma)x_n}M_{\rho,j}(b,\sigma)$ (note that ρ, b and σ depend on (ξ,λ) where we oppress the dependence in the notation for the sake of readability) are exactly the Poisson symbol-kernels $\widetilde{k}_j$ in Proposition 6.5.16. In the following, we will show that they satisfy the symbol-kernel estimates in order to prove Proposition 6.5.16.

Lemma 6.5.19. *Let $N\in\mathbb{N}$ and let $\Sigma_1,\dots,\Sigma_N\subset\mathbb{C}$ be some sectors (or lines) in the complex plane. Let further $m\colon \prod_{i=1}^N \Sigma_i\setminus\{0\}\to C$ be differentiable and homogeneous in the sense that there are numbers $\alpha_1,\dots,\alpha_N,\alpha\in\mathbb{R}$ such that*

$$m(r^{\alpha_1}x_1,\dots,r^{\alpha_N}x_N) = r^{\alpha}m(x_1,\dots,x_N) \quad (r>0, x_i\in\Sigma_i, i=1,\dots,N).$$

Then, we have

$$(\partial_j m)(r^{\alpha_1}x_1,\dots,r^{\alpha_N}x_N) = r^{\alpha-\alpha_j}\partial_j m(x_1,\dots,x_N) \quad (r\geq 0, x_i\in\Sigma_i, i,j=1,\dots,N).$$

Proof. Let $r>0$ and $x_i\in\Sigma_i$ for $i=1,\dots,N$. Define $x=(x_1,\dots,x_N)$ and $x_{(r)} := (r^{\alpha_1}x_1,\dots,r^{\alpha_N}x_n)$. Let further e_j be the j-th unit normal vector. Then we have

$$(\partial_j m)(x_{(r)}) = \lim_{h\to 0}\frac{m(x_{(r)}+he_j)}{h} = \lim_{h\to 0} r^{\alpha}\frac{m(x+\frac{h}{r_j^{\alpha_j}}e_j)}{h} = \lim_{\widetilde{h}\to 0} r^{\alpha-\alpha_j}\frac{m(x+\widetilde{h}e_j)}{\widetilde{h}} = r^{\alpha-\alpha_j}(\partial_j m)(x).$$

□

Proposition 6.5.20. *Let $a_1,a_2>0$ such that $\frac{1}{a_1},\frac{1}{a_2}\in\mathbb{N}$. Then the function $(\xi,\lambda)\mapsto(\frac{\xi}{\langle\xi,\lambda\rangle_{\boldsymbol{a}}^{a_1}},\frac{\varepsilon+\lambda}{\langle\xi,\lambda\rangle_{\boldsymbol{a}}^{a_2}})$ is a symbol in $S^{0,\infty}_{\boldsymbol{a}}(\mathbb{R}^n\times\Sigma,\mathbb{C}^{n+1})$.*

Proof. The function

$$m\colon \mathbb{R}\times\mathbb{R}^n\times\Sigma\setminus\{0\}\colon (x,\xi,\lambda)\mapsto\frac{\varepsilon x+\lambda}{(x^{2/a_2}+|\xi|^{2/a_1}+|\lambda|^{2/a_2})^{a_2/2}}$$

is homogeneous in the sense that

$$m(r^{a_2}x, r^{a_1}\xi, r^{a_2}\lambda) = m(x,\xi,\lambda).$$

Moreover, since $\frac{1}{a_1},\frac{1}{a_2}\in\mathbb{N}$ we also have that $m\in C^\infty(\mathbb{R}\times\mathbb{R}^n\times\Sigma\setminus\{0\},\mathbb{C})$. In particular, for all $\alpha\in\mathbb{N}^n$ and $k\in\mathbb{N}^2$ we have that $\partial_\xi^\alpha\partial_\lambda^k m$ is bounded on the set

$$S_{\boldsymbol{a}} := \{(x,\xi,\lambda)\in\mathbb{R}\times\mathbb{R}^n\times\Sigma\setminus\{0\} : x^{2/a_2}+|\xi|^{2/a_1}+|\lambda|^{2/a_1}=1\}$$

and satisfies

$$(\partial_\xi^\alpha\partial_\lambda^k m)(r^{a_2}x, r^{a_1}\xi, r^{a_2}\lambda) = r^{-a_1|\alpha|-a_2|k|}(\partial_\xi^\alpha\partial_\lambda^k m)(x,\xi,\lambda).$$

Thus, we have the estimate

$$\begin{aligned}
&\sup_{(\xi,\lambda)\in\mathbb{R}^n\times\Sigma} \langle\xi,\lambda\rangle_{\boldsymbol{a}}^{a_1|\alpha|+a_2|k|}|\partial_\xi^\alpha\partial_\lambda^k m(1,\xi',\lambda)| \\
&\le \sup_{(x,\xi,\lambda)\in\mathbb{R}\times\mathbb{R}^n\times\Sigma\setminus\{0\}} (x^{2/a_2}+|\xi'|^{2/\alpha_1}+|\lambda|^{2/\alpha_2})^{\frac{a_1|\alpha|}{2}+\frac{a_2|k|}{2}}|\partial_\xi^\alpha\partial_\lambda^k m(x,\xi,\lambda)| \\
&\le \|\partial_\xi^\alpha\partial_\lambda^k m\|_{L_\infty(S_{\boldsymbol{a}})}
\end{aligned}$$

so that we obtain that $(\xi,\lambda)\mapsto\frac{\varepsilon+\lambda}{\langle\xi,\lambda\rangle_{\boldsymbol{a}}^{a_2}}$ is a symbol in $S^{0,\infty}_{\boldsymbol{a}}(\mathbb{R}^n\times\Sigma,\mathbb{C})$. A simmilar approach also shows the desired estimates for the other components. □

For the rest of this section, in (6.53) we fix

$$\rho(\xi',\lambda):=\langle\xi',\lambda\rangle_{\boldsymbol{a}}^{a_1}$$

$$b(\xi',\lambda):=\frac{\xi'}{\langle\xi',\lambda\rangle_{\boldsymbol{a}}^{a_1}}\quad\text{and}\quad\sigma(\xi',\lambda):=\frac{\varepsilon+\lambda}{\langle\xi',\lambda\rangle_{\boldsymbol{a}}^{2ma_1}}.$$

In particular, if we choose $\boldsymbol{a}=(a_1,a_2)=(\frac{1}{2m},1)$ then we obtain

$$b(\xi',\lambda):=\frac{\xi'}{\langle\xi',\lambda\rangle_{\boldsymbol{a}}^{a_1}}\quad\text{and}\quad\sigma(\xi',\lambda):=\frac{\varepsilon+\lambda}{\langle\xi',\lambda\rangle_{\boldsymbol{a}}^{a_2}}$$

so that (b,σ) coincides with the function in Proposition 6.5.20.

Proposition 6.5.21. *Let again $a_1,a_2>0$ such that $\frac{1}{a_1},\frac{1}{a_2}\in\mathbb{N}$ and let A be smooth with values in some Banach space Z. We further assume that A and all its derivatives are bounded on the range of (b,σ). Then, we have that*

$$A\circ(b,\sigma)\in S^{0,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1}\times\Sigma,Z).$$

Proof. We show by induction on $|\alpha'|+|\gamma|$ that $D_{\xi'}^{\alpha'}D_\lambda^\gamma(A\circ(b,\sigma))$ is a linear combination of terms of the form $(D_{\xi'}^{\tilde\alpha'}D_\lambda^{\tilde\gamma}A)\circ(b,\sigma)\cdot f$ with $f\in S_{\boldsymbol{a}}^{-a_1|\alpha'|-a_2|\gamma|,\infty}(\mathbb{R}^{n-1}\times\Sigma)$, $\tilde\alpha'\in N_0^{n-1}$ and $\tilde\gamma\in\mathbb{N}^2$. Obviously, this is true for $|\alpha'|+|\gamma|=0$. So let $j\in\{1,\dots,n-1\}$. By induction hypothesis, we have that $D_{\xi'}^{\alpha'}D_\lambda^kA\circ(b,\sigma)$ is a linear combination of terms of the form $(D_{\xi'}^{\tilde\alpha'}D_\lambda^{\tilde\gamma}A)\circ(b,\sigma)\cdot f$ with $f\in S_{\boldsymbol{a}}^{-a_1|\alpha'|-a_2|\gamma|,\infty}(\mathbb{R}^{n-1}\times\Sigma)$, $\tilde\alpha'\in N_0^{n-1}$ and $\tilde\gamma\in\mathbb{N}^2$. Hence, for $D_jD_{\xi'}^{\alpha'}D_\lambda^\gamma A\circ(b,\sigma)$ it suffices to treat the summands separately, i.e. we consider $D_j((D_{\xi'}^{\tilde\alpha'}D_\lambda^{\tilde\gamma}A)\circ(b,\sigma)\cdot f)$. By the product rule, we have

$$\begin{aligned}
&D_j((D_{\xi'}^{\tilde\alpha'}D_\lambda^{\tilde\gamma}A)\circ(b,\sigma)\cdot f) \\
&=(D_{\xi'}^{\tilde\alpha'}D_\lambda^{\tilde\gamma}A)\circ(b,\sigma))(D_jf)+\Big(\sum_{l=1}^{n-1}(D_j(\tfrac{\xi_l'}{\rho})\cdot f\cdot[(\partial_l\partial_{\xi'}^{\tilde\alpha'}\partial_\lambda^{\tilde\gamma}A)\circ(b,\sigma)]\Big)
\end{aligned}$$

By the induction hypothesis and Remark 6.5.6 (c) and (d) we have that

$$(D_jf),(D_j\frac{\xi_1'}{\rho})f,\dots,(D_j\frac{\xi_{n-1}'}{\rho})f\in S_{\boldsymbol{a}}^{-a_1(|\alpha'|+1)-a_2|\gamma|,\infty}(\mathbb{R}^{n-1}\times\Sigma).$$

The same computation for ∂_{λ_1} and ∂_{λ_2} instead of ∂_j also shows the desired behavior and hence, the induction is finished. Finally, the assertion follows now from Proposition 6.5.20 and Remark 6.5.6 (c) and (d). □

Lemma 6.5.22. *Let $n_1, n_2 \in \mathbb{R}$ and $a = (a_1, a_2) = (\frac{1}{2m}, 1)$. Let further $f_0 \in S^{n_1,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X^{2m}, X^{2m}))$ and $g_0 \in S^{n_2,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X, X^{2m}))$. Then, for all $\alpha' \in \mathbb{N}^{n-1}$ and $\gamma \in \mathbb{N}^2$ we have that*

$$\partial^{\alpha'}_{\xi'} \partial^{\gamma}_{\lambda} f_0 \exp(i\rho A_0(b,\sigma)x_1)\mathscr{P}_-(b,\sigma) g_0$$

is a linear combination of terms of the form

$$f \exp(i\rho A_0(b,\sigma)x_1)\mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2}$$

where $f \in S^{n_1-a_1|\tilde{\alpha}'|-a_2|\tilde{\gamma}|+(a_1-a_2)p_2,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X^{2m}, X^{2m}))$, $g \in S^{n_2-a_1|\overline{\alpha}'|-a_2|\overline{\gamma}|,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X, X^{2m}))$, $|\tilde{\gamma}| + |\overline{\gamma}| + p_2 = |\gamma|$ and $|\tilde{\alpha}'| + |\overline{\alpha}'| + p_1 = |\alpha'|$.

Proof. We show the assertion by induction on $|\alpha'| + |\gamma|$. Obviously, for $|\alpha'| + |\gamma| = 0$ the assertion holds true. So let $\alpha' \in \mathbb{N}^{n-1}$ and $\gamma \in \mathbb{N}^2$. Let further

$$\partial^{\alpha'}_{\xi'} \partial^{\gamma}_{\lambda} f_0 \exp(i\rho A_0(b,\sigma)x_1)\mathscr{P}_-(b,\sigma) g_0$$

be a linear combination of terms of the form

$$f \exp(i\rho A_0(b,\sigma)x_1)\mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2}$$

where $f \in S^{n_1-a_1|\tilde{\alpha}'|-a_2|\tilde{\gamma}|+(a_1-a_2)p_2,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X^{2m}, X^{2m}))$, $g \in S^{n_2-a_1|\overline{\alpha}'|-a_2|\overline{\gamma}|,\infty}_{\boldsymbol{a}}(\mathbb{R}^{n-1} \times \Sigma, \mathscr{B}(X, X^{2m}))$, $|\tilde{\gamma}| + |\overline{\gamma}| + p_2 = |\gamma|$ and $|\tilde{\alpha}'| + |\overline{\alpha}'| + p_1 = |\alpha'|$. We treat the summands separately. Then, for $j = 1, \dots, n-1$ we have

$$\begin{aligned}
\partial_j [f e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^p] &= \partial_j [f \mathscr{P}_-(b,\sigma) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2}] \\
&= [\partial_j f] \mathscr{P}_-(b,\sigma) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2} \\
&\quad + f [\partial_j \mathscr{P}_-(b,\sigma)] e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2} \\
&\quad + f e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) [\partial_j i\rho A_0(b,\sigma)] e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2+1} \\
&\quad + f \mathscr{P}_-(b,\sigma) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) \partial_j [\mathscr{P}_-(b,\sigma)] g x_1^{p_1+p_2} \\
&\quad + f \mathscr{P}_-(b,\sigma) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) [\partial_j g] x_1^{p_1+p_2}.
\end{aligned}$$

Here, we used that the spectral projection $\mathscr{P}_-(b,\sigma)$ commutes with $e^{i\rho A_0(b,\sigma)x_1}$. Using Remark 6.5.6 (c) and (d) and Proposition 6.5.21 we obtain that in each summand, we have that either $|\tilde{\alpha}'|$, $|\overline{\alpha}'|$ or p_1 increases by 1. The same computation as above also yields the desired estimate for ∂_{λ_1} and ∂_{λ_2}, where either $|\tilde{\gamma}|$. $|\overline{\gamma}|$ or p_2 increases by 1. Hence, we obtain the assertion. □

Proposition 6.5.23. *Let again $a = (a_1, a_2) = (\frac{1}{2m}, 1)$. Then, we have the estimate*

$$\| x_1^r D^k_{x_1} D^{\alpha'}_{\xi'} D^{\gamma}_{\lambda} e^{i\rho A_0(b,\sigma)x_1} M_{\rho,j}(b,\sigma) \|_{\mathscr{B}(X, X^{2m})} \leq C \rho^{k - m_j - r - |\alpha'| - \frac{a_2}{a_1}|\gamma|} e^{-\frac{c}{2}\rho x_1}.$$

for all $r, k \in \mathbb{N}_0$, $\alpha' \in \mathbb{N}_0^{n-1}$ and $\gamma \in \mathbb{N}_0^2$.

Proof. By Lemma 6.5.22, we have that $D_{\xi'}^{\alpha'} D_{\lambda}^{\gamma} e^{i\rho A_0(b,\sigma)x_1} M_{\rho,j}(b,\sigma)$ is a linear combination of terms of the form

$$f e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g x_1^{p_1+p_2}$$

where $f \in S_{\boldsymbol{a}}^{-a_1|\tilde{\alpha}'|-a_2|\tilde{\gamma}|+p_2(a_1-a_2),\infty}(\mathbb{R}^{n-1}\times\Sigma, \mathscr{B}(X^{2m}, X^{2m}))$, $g \in S_{\boldsymbol{a}}^{-a_1 m_j - a_1|\overline{\alpha}'|-a_2|\overline{\gamma}|,\infty}(\mathbb{R}^{n-1}\times\Sigma, \mathscr{B}(X, X^{2m^i}))$, $|\tilde{\gamma}| + |\overline{\gamma}| + p_2 = |\gamma|$ and $|\tilde{\alpha}'| + |\overline{\alpha}'| + p_1 = |\alpha'|$. But for such a term, we have that

$$\begin{aligned}
&\| x_1^r D_{x_1}^k f(\xi',\mu) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) g(\xi',\mu) x_1^{p_1+p_2} \|_{\mathscr{B}(X,X^{2m})} \\
&\leq C x_1^r \sum_{l=0}^{k} \| f(\xi',\mu) e^{i\rho A_0(b,\sigma)x_1} \mathscr{P}_-(b,\sigma) [i\rho A_0(b,\sigma)]^{k-l} g(\xi',\mu) x_1^{[p_1+p_2-l]_+} \|_{\mathscr{B}(X,X^{2m})} \\
&\leq C x_1^r \sum_{l=0}^{k} \rho^{k-l-m_j-|\tilde{\alpha}'|-|\overline{\alpha}'|-\frac{a_2}{a_1}(|\tilde{\gamma}|+|\overline{\gamma}|)+p_2-\frac{a_2}{a_1}p_2} e^{-c\rho x_1} x_1^{[p_1+p_2-l]_+} \\
&\leq C x_1^r \sum_{l=0}^{k} \rho^{k-l-m_j-[p_1+p_2-l]_+-r-|\tilde{\alpha}'|-|\overline{\alpha}'|-\frac{a_2}{a_1}(|\tilde{\gamma}|+|\overline{\gamma}|)+p_2-\frac{a_2}{a_1}p_2} e^{-\frac{c}{2}\rho x_1} x_1^{[p_1+p_2-l]_+-[p_1+p_2-l]_+-r} \\
&\leq C \sum_{l=0}^{k} \rho^{k-m_j-r-(|\tilde{\alpha}'|+|\overline{\alpha}'|+p_1)-\frac{a_2}{a_1}(\tilde{k}+\overline{k}+p_2)} e^{-\frac{c}{2}\rho x_1} \\
&\leq C \rho^{k-m_j-r-|\alpha'|-\frac{a_2}{a_1}|\gamma|} e^{-\frac{c}{2}\rho x_1}
\end{aligned}$$

This is the desired estimate. □

Corollary 6.5.24. *We have that* $e^{i\rho A_0(b,\sigma)x_1} M_{\rho,j}(b,\sigma) \in S_{(1/2m,1)}^{-(1+m_j)/2m,\infty}(\mathbb{R}^{n-1}\times\Sigma; S_{L_1}(\mathbb{R}_+, \mathscr{B}(X, X^{2m})))$.

Proof. This is obtained by computing the L_1-norms in Proposition 6.5.23. □

Putting together the above gives Proposition 6.5.16:

Proof of Proposition 6.5.16. A combination of the solution formula (6.53) and Corollary 6.5.24 gives the desired result, where the uniqueness statement is clear from the construction of the solution formula. □

6.5.3. Mapping Properties

Recall the notation from Section 6.4.2.

Theorem 6.5.25. *Let X be a Banach space, $d_1 = 1$, $\boldsymbol{p} \in (1,\infty)^l$, $r \in [1,\infty]$, $\gamma \in (-1,\infty)$, $\boldsymbol{w}'' \in \prod_{j=2}^{l} A_\infty(\mathbb{R}^{d_j})$, $s \in \mathbb{R}$ and $\boldsymbol{a} \in (0,\infty)^l$. Then $(\tilde{k}, g) \mapsto \mathrm{OPK}(\tilde{k})g$ defines continuous bilinear operators*

$$S_{d,\boldsymbol{a}}^{d-a_1}(\mathbb{R}^{n-1}; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X))) \times F_{\boldsymbol{p}'',p_1}^{s+d-a_1\frac{1+\gamma}{p_1},\boldsymbol{a}'',d''}(\mathbb{R}^{n-1}, \boldsymbol{w}''; X) \longrightarrow F_{\boldsymbol{p},r}^{s,\boldsymbol{a},d}(\mathbb{R}^n_+, (w_\gamma, \boldsymbol{w}''); X)$$

and

$$S_{d,\boldsymbol{a}}^{d-a_1}(\mathbb{R}^{n-1}; \mathscr{S}_{L_1}(\mathbb{R}_+; \mathscr{B}(X))) \times B_{\boldsymbol{p}'',r}^{s+d-a_1\frac{1+\gamma}{p_1},\boldsymbol{a}'',d''}(\mathbb{R}^{n-1}, \boldsymbol{w}''; X) \longrightarrow B_{\boldsymbol{p},r}^{s,\boldsymbol{a},d}(\mathbb{R}^n_+, (w_\gamma, \boldsymbol{w}''); X).$$

Corollary 6.5.26. *Let X be a UMD Banach space, $q,p,r \in (1,\infty)$, $v \in A_q(\mathbb{R})$, $\gamma \in (-1,\infty)$, $s \in \mathbb{R}$ and $\rho \in (0,\infty)$. Let $\mathpzc{d} = (1,n-1,1)$ and $\boldsymbol{a} = (\frac{1}{\rho},\frac{1}{\rho},1)$. Then $(\widetilde{k},g) \mapsto \mathrm{OPK}(\widetilde{k})g$ defines a continuous bilinear operator*

$$S^{d-\frac{1}{\rho}}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^n;\mathscr{S}_{L_1}(\mathbb{R}_+;\mathscr{B}(X)))\times F^{\frac{1}{\rho}(s-\frac{1+\gamma}{p})+d,(\frac{1}{\rho},1)}_{(p,q),p}(\mathbb{R}^{n-1}\times\mathbb{R},(1,v);X)$$
$$\longrightarrow W^1_q(\mathbb{R},v;F^s_{p,r}(\mathbb{R}^n_+,w_\gamma;X))\cap L_q(\mathbb{R},v;F^{s+\rho}_{p,r}(\mathbb{R}^n_+,w_\gamma;X)).$$

Proof. Let s_+, γ_+, σ and η be as in Lemma 6.4.3. Then note that we have the embedding (6.44) while

$$\left(\sigma+\frac{\eta}{\rho}\right)+d-\frac{1}{\rho}\frac{1+(\gamma_++\eta p)}{p}=\frac{1}{\rho}(s-\frac{1+\gamma}{p})+d.$$

Observing that

$$F^{\sigma+\frac{\eta}{\rho},(\frac{1}{\rho},1)}_{(p,q),1}(\mathbb{R}^n_+\times\mathbb{R},(w_{\gamma_++\eta p},v);X)=F^{\sigma+\frac{\eta}{\rho},(\frac{1}{\rho},\frac{1}{\rho},1)}_{(p,p,q),1}(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R},(w_{\gamma_++\eta p},1,v);X),$$

the result thus follows from Theorem 6.5.25. □

Remark 6.5.27. In case $s=0$, $\rho=k\in\mathbb{Z}_{\geq 1}$ and $r=1$ in Corollary 6.5.26, the elementary embedding

$$F^0_{p,1}(\mathbb{R}^n_+,w_\gamma;X)\hookrightarrow L_p(\mathbb{R}^n_+,w_\gamma;X),\qquad F^k_{p,1}(\mathbb{R}^n_+,w_\gamma;X)\hookrightarrow W^k_p(\mathbb{R}^n_+,w_\gamma;X)$$

yield that $(\widetilde{k},g)\mapsto\mathrm{OPK}(\widetilde{k})g$ defines a continuous bilinear operator

$$S^{d-\frac{1}{k}}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^n;\mathscr{S}_{L_1}(\mathbb{R}_+;\mathscr{B}(X)))\times F^{-\frac{1+\gamma}{kp}+d,(\frac{1}{k},1)}_{(p,q),p}(\mathbb{R}^{n-1}\times\mathbb{R},(1,v);X)$$
$$\longrightarrow W^1_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;X))\cap L_q(\mathbb{R},v;W^k_p(\mathbb{R}^n_+,w_\gamma;X)).$$

However, this could also directly be derived from Theorem 6.5.25 using the elementary embedding ([**?** , Lemma 7.2])

$$F^{1,(\frac{1}{k},\frac{1}{k},1)}_{(p,p,q),1}(\mathbb{R}_+\times\mathbb{R}^{n-1}\times\mathbb{R},(w_\gamma,1,v);X)=F^{1,(\frac{1}{k},1)}_{(p,q),1}(\mathbb{R}^n_+\times\mathbb{R},(w_\gamma,v);X)$$
$$\hookrightarrow W^1_q(\mathbb{R},v;L_p(\mathbb{R}^n_+,w_\gamma;X))\cap L_q(\mathbb{R},v;W^k_p(\mathbb{R}^n_+,w_\gamma;X)).$$

Theorem 6.5.28. *Let X be a reflexive Banach space, $d\in\mathbb{R}$ and $p,q\in(1,\infty)$. Let further $(\mathscr{A},\gamma)\in\{B,F\}\times(-1,\infty)\cup\{\mathscr{B},\mathscr{F}\}\times(-\infty,p-1)$, $s\in\mathbb{R}$ and $s_0\in(\frac{1+\gamma}{p}-1,\infty)$. Then $(\widetilde{k},g)\mapsto\mathrm{OPK}(\widetilde{k})g$ defines a continuous bilinear operator*

$$S^{d-1,\infty}(\mathbb{R}^{n-1};\mathscr{S}_{L_1}(\mathbb{R}_+;\mathscr{B}(X)))\times\partial\mathscr{A}^{s,|\mu|}_{p,q,\gamma}(\mathbb{R}^n_+,w_\gamma;X)\longrightarrow\mathscr{A}^{s-d,|\mu|,s_0}_{p,q}(\mathbb{R}^n_+,w_\gamma;X)$$

Lemma 6.5.29. *Let X be a Banach space, $i\in\{1,\dots,l\}$, $\boldsymbol{a}\in(0,\infty)^l$, $\boldsymbol{p}\in[1,\infty)^l$, $q\in[1,\infty]$, $\gamma\in(-\mathpzc{d}_1,\infty)$ and $s\in(-\infty,a_1\left[\frac{\mathpzc{d}_1+\gamma}{p_1}-\mathpzc{d}_1\right])$. Let $\boldsymbol{w}\in\prod_{j=1}^l A_\infty(\mathbb{R}^{\mathpzc{d}_j})$ be such that $w_1(x_1)=|x_1|^\gamma$. The linear operator*

$$T:\mathscr{S}'(\mathbb{R}^{n-\mathpzc{d}_1})\longrightarrow\mathscr{S}'(\mathbb{R}^n),\ f\mapsto\delta_0\otimes f.$$

restricts to bounded linear operators from $B^{s+a_1\left(d_1-\frac{d_1+\gamma}{p_1}\right),\boldsymbol{a}'}_{\boldsymbol{p}',q,\mathpzc{d}'}(\mathbb{R}^{n-d_1},\boldsymbol{w}';X)$ *to* $B^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X)$ *and from* $F^{s+a_1\left(d_1-\frac{d_1+\gamma}{p_1}\right),\boldsymbol{a}'}_{\boldsymbol{p}',p_1,\mathpzc{d}'}(\mathbb{R}^{n-d_1},\boldsymbol{w}';X)$ *to* $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w};X)$.

Proof. The Besov case is contained in [160, Lemma 4.14]. So let us consider the Triebel-Lizorkin case. Using the Sobolev embedding from Proposition 6.4.1, we may without loss of generality assume that $p_1 = q$, so that

$$L_{\boldsymbol{p},\mathpzc{d}}(\mathbb{R}^n,\boldsymbol{w})[\ell_q(\mathbb{N})] = L_{\boldsymbol{p}',\mathpzc{d}'}(\mathbb{R}^{n-d_1},\boldsymbol{w}')[\ell_q(\mathbb{N})[L_{p_1}(\mathbb{R}^{d_1},|\cdot|^{\gamma})]].$$

Now the desired estimate can be obtained as in the proof of [160, Lemma 4.14(i)]. □

Lemma 6.5.30. *Let X be a Banach space, $\boldsymbol{a}\in(0,\infty)$, $s\in\mathbb{R}$, $\boldsymbol{p}\in[1,\infty)$, $q\in[1,\infty]$ and $d\in\mathbb{R}$. Then $(f,p)\mapsto \mathrm{OP}(p)f$ defines continuous bilinear mappings*

$$S^{d}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^n;\mathscr{B}(X))\times F^{s,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)\longrightarrow F^{s-d,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)$$

and

$$S^{d}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^n;\mathscr{B}(X))\times B^{s,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)\longrightarrow B^{s-d,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X).$$

Proof. This follows directly from the fact that $F^{s,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)$ and $B^{s,\boldsymbol{a},\mathpzc{d}}_{\boldsymbol{p},q}(\mathbb{R}^n,\boldsymbol{w};X)$ are $(\mathpzc{d},\boldsymbol{a})$-admissible Banach spaces of tempered distributions with (6.27). □

Proof of Theorem 6.5.25. Let $\widetilde{k}\in S^{d-a_1}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^{n-1};\mathscr{S}_{L_1}(\mathbb{R}_+;\mathscr{B}(X)))$. Let $p\in S^{d-a_1}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^{n-1};\mathscr{S}_{L_\infty}(\mathbb{R};\mathscr{B}(X)))$ be as in Lemma 6.5.13 for this given $\widetilde{k}$; so $\mathrm{OPK}(\widetilde{k}) = r_+\,\mathrm{OP}[p](\delta_0\otimes\cdot)$. Then, for every $\sigma\in\mathbb{R}$,

$$\mathrm{OPK}(\widetilde{k}) = r_+\,\mathrm{OP}[p]\mathscr{J}^{\mathpzc{d}'',\boldsymbol{a}''}_{-\sigma}(\delta_0\otimes\cdot)\mathscr{J}^{\mathpzc{d}'',\boldsymbol{a}''}_{\sigma} = r_+\,\mathrm{OP}[p_\sigma](\delta_0\otimes\cdot)\mathscr{J}^{\mathpzc{d}'',\boldsymbol{a}''}_{\sigma}, \tag{6.54}$$

where $p_\sigma(\xi) := p(\xi)J^{\mathpzc{d}'',\boldsymbol{a}''}_{-\sigma}(\xi'')$. By Lemmas 6.5.12 and 6.5.15, $p\mapsto p_\sigma$ defines a continuous linear mapping

$$S^{d-a_1}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^{n-1};\mathscr{S}_{L_\infty}(\mathbb{R};\mathscr{B}(X)))\stackrel{p\mapsto p_\sigma}{\longrightarrow} S^{d-a_1-\sigma}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^{n-1};\mathscr{S}_{L_\infty}(\mathbb{R};\mathscr{B}(X)))\hookrightarrow S^{d-a_1-\sigma}_{\mathpzc{d},\boldsymbol{a}}(\mathbb{R}^{n};\mathscr{B}(X)). \tag{6.55}$$

Choosing $\sigma\in\mathbb{R}$ such that $s-\sigma < a_1(\frac{1+\gamma}{p_1}-1)$, a combination of (6.54), (6.55), Lemma 6.5.29, Lemma 6.5.30 and the lifting property of weighted mixed-norm anisotropic B- and F-

spaces (see (6.27)) gives the desired result. Indeed, we obtain the commutative diagram:

$$\begin{array}{ccc}
F^{s+d-a_1\frac{1+\gamma}{p_1},\boldsymbol{a}'',\boldsymbol{d}''}_{\boldsymbol{p}'',p_1}(\mathbb{R}^{n-1},\boldsymbol{w}'';X) & \xrightarrow{\mathscr{I}_\sigma^{\boldsymbol{d}'',\boldsymbol{a}''}} & F^{s+d-\sigma-a_1\frac{1+\gamma}{p_1},\boldsymbol{a}'',\boldsymbol{d}''}_{\boldsymbol{p}'',p_1}(\mathbb{R}^{n-1},\boldsymbol{w}'';X) \\
 & & \Big\downarrow{\scriptstyle(\delta_0\otimes\cdot)} \\
\Big\downarrow{\scriptstyle \mathrm{OPK}(\widetilde{k})} & & F^{s+d-\sigma,\boldsymbol{a},\boldsymbol{d}}_{\boldsymbol{p},r}(\mathbb{R}^n,(w_\gamma,\boldsymbol{w}'');X) \\
 & & \Big\downarrow{\scriptstyle \mathrm{OP}[p_\sigma]} \\
F^{s,\boldsymbol{a},\boldsymbol{d}}_{\boldsymbol{p},r}(\mathbb{R}^n_+,(w_\gamma,\boldsymbol{w}'');X) & \xleftarrow[r_+]{} & F^{s,\boldsymbol{a},\boldsymbol{d}}_{\boldsymbol{p},r}(\mathbb{R}^n,(w_\gamma,\boldsymbol{w}'');X)
\end{array}$$

□

Proof of Theorem 6.5.28. We take p as defined in Lemma 6.5.13 so that we have the identity

$$r_+\,\mathrm{OP}(p)(\delta_0\otimes g)=\mathrm{OPK}(\widetilde{k})g.$$

Now, for $\sigma\in\mathbb{R}$ we define

$$p^\sigma:=\langle\xi,\mu\rangle^{s-s_0-d}\,p\,(\langle\xi,\mu\rangle^{-s+s_0+\sigma+1}\langle\xi',\mu\rangle^{-\sigma})$$

so that we obtain

$$\mathrm{OPK}(\widetilde{k})=r_+\,\mathrm{OP}(p)(\delta_0\otimes\,\cdot)=r_+\Xi_\mu^{d+s_0-s}\,\mathrm{OP}(p^\sigma)\Xi_\mu^{s-\sigma+d-1-s_0}[\delta_0\otimes\,\cdot\,]\Xi_\mu^\sigma.$$

By Lemma 6.5.15 and Lemma 6.5.12 we obtain that

$$S^{d-1,\infty}(\mathbb{R}^{n-1};\mathscr{S}_{L_1}(\mathbb{R}_+,\mathscr{B}(X)))\to S^{0,\infty}(\mathbb{R}^n\times\Sigma),\;p\mapsto p^\sigma$$

is continuous. We even obtain that $(p^\sigma(\cdot,\mu))_{\mu\in\Sigma}$ defines a bounded family in $S^0(\mathbb{R}^n)$. Taking $\sigma>s-\frac{1+\gamma}{p}$ in combination with Corollary 6.3.10 yields the desired result as can be seen in the following commutative diagram

$$
\begin{array}{ccc}
\partial\mathscr{A}^{s,|\mu|}_{p,q,\gamma}(\mathbb{R}^n_+, w_\gamma; X) & \xrightarrow{\;\Xi^{\sigma}_{\mu}\;} & \partial\mathscr{A}^{s-\sigma,|\mu|}_{p,q,\gamma}(\mathbb{R}^n_+, w_\gamma; X)) \\
\Big\downarrow {\scriptstyle \mathrm{OPK}(\widetilde{k})} & & \Big\downarrow {\scriptstyle (\delta_0\otimes\cdot)} \\
 & & \mathscr{A}^{s-1-\sigma,|\mu|,s_0}_{p,q,\gamma}(\mathbb{R}^n_+, w_\gamma; X)) \\
 & & \Big\downarrow {\scriptstyle \Xi^{s-\sigma-1-s_0}_{\mu}} \\
 & & \mathscr{A}^{s_0,|\mu|,s_0}_{p,q,\gamma}(\mathbb{R}^n_+, w_\gamma; X)) \\
 & & \Big\downarrow {\scriptstyle \mathrm{OP}[p_\sigma]} \\
\mathscr{A}^{s-d,|\mu|,s_0}_{p,q}(\mathbb{R}^n_+, w_\gamma; X)) & \xleftarrow[\;r_+\Xi^{s_0+d-s}_{\mu}\;]{} & \mathscr{A}^{s_0,|\mu|,s_0}_{p,q,\gamma}(\mathbb{R}^n_+, w_\gamma; X))
\end{array}
$$

□

6.6. PARABOLIC PROBLEMS

In this section we consider the linear vector-valued parabolic initial-boundary value problem (6.2). As the main result of the paper, we solve the $L_{q,\mu}$-$H^s_{p,\gamma}$-maximal regularity problem and the $L_{q,\mu}$-$F^s_{p,r,\gamma}$-maximal regularity problem for (6.2) in Theorem 6.6.2. This simultaneously generalizes [160, Theorem 3.4] and [162, Theorem 4.2].

Before can state Theorem 6.6.2, we first need to introduce some notation.

6.6.1. Some notation and assumptions

Let $\mathscr{O}$ be either $\mathbb{R}^n_+$ or a bounded C^∞-domain in $\mathbb{R}^n$ and $J = (0, T)$ with $T \in (0,\infty)$. Let X be a Banach space and let $\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D)$ be a $\mathscr{B}(X)$-valued BC^∞-differential boundary value system on $\mathscr{O} \times J$ as considered in Section 6.3.5 where the coefficients satisfy certain smoothness conditions which we are going to introduce later. Put $m_* := \max\{m_1, \ldots, m_m\}$.

Let $q \in (1,\infty)$ and $\mu \in (-1, q-1)$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as either

(a) $\mathbb{E} = H^s_{p,\gamma}(\mathscr{O}; X)$ and $\mathbb{E}^{2m} = H^{s+2m}_{p,\gamma}(\mathscr{O}; X)$ with $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $s \in (\frac{1+\gamma}{p} + m_* - 2m, \frac{1+\gamma}{p})$ *(the Bessel potential case)*; or

(b) $\mathbb{E} = F^s_{p,r,\gamma}(\mathscr{O}; X)$ and $\mathbb{E}^{2m} = F^{s+2m}_{p,r,\gamma}(\mathscr{O}; X)$ with $p, r \in (1,\infty)$, $\gamma \in (-1,\infty)$ and $s \in (\frac{1+\gamma}{p} + m_* - 2m, \frac{1+\gamma}{p})$ *(the Triebel-Lizorkin case),*

and set

$$\kappa_{j,\mathbb{E}} = \kappa_{j,p,\gamma,s} := \frac{s+2m-m_j}{2m} - \frac{1+\gamma}{2mp} \in (0,1), \qquad j = 1, \ldots, m.$$

In the $L_{q,\mu}$-$\mathbb{E}$-maximal regularity approach in Theorem 6.6.2 we look for solutions

$$u \in W^1_q(J, v_\mu; \mathbb{E}) \cap L_q(J, v_\mu; \mathbb{E}^{2m})$$

of the problem

$$\begin{cases} \partial_t u + \mathscr{A}(D)u &= f, \quad \text{on } \mathscr{O} \times J, \\ \mathscr{B}_j(D)u &= g_j, \quad \text{on } \partial\mathscr{O} \times J, \quad j = 1, \ldots, m, \\ u(0) &= u_0, \quad \text{on } \mathscr{O}. \end{cases} \tag{6.56}$$

and find characterize the data $g = (g_1, \ldots, g_m)$ and u_0 for which this actually can be solved.

Let us now introduce some notation for the function spaces appearing in this problem. For an open interval $I \subset \mathbb{R}$ and $v \in A_q(\mathbb{R})$, we put

$$\begin{aligned} \mathbb{D}_{q,v}(I;\mathbb{E}) &:= L_q(I, v; \mathbb{E}), \\ \mathbb{M}_{q,v}(I;\mathbb{E}) &:= W^1_q(I, v; \mathbb{E}) \cap L_q(I, v; \mathbb{E}^{2m}), \\ \mathbb{B}_{q,v,j}(I;\mathbb{E}) &:= F^{\kappa_{j,\mathbb{E}}}_{q,p}(I, v; L_p(\partial\mathscr{O}; X)) \cap L_q(I, v; F^{2\kappa_{j,\mathbb{E}}}_{p,p}(\partial\mathscr{O}; X)), \qquad j = 1, \ldots, m, \\ \mathbb{B}_{q,v}(I;\mathbb{E}) &:= \bigoplus_{j=1}^m \mathbb{B}_{q,v,j}(I;\mathbb{E}). \end{aligned}$$

For the power weight $v = v_\mu$, with $\mu \in (-1, q-1)$, we simply replace v by μ in the subscripts: $\mathbb{D}_{q,\mu}(I;\mathbb{E}) := \mathbb{D}_{q,v_\mu}(I;\mathbb{E})$, $\mathbb{M}_{q,\mu}(I;\mathbb{E}) := \mathbb{M}_{q,v_\mu}(I;\mathbb{E})$, $\mathbb{B}_{q,\mu,j}(I;\mathbb{E}) = \mathbb{B}_{q,v_\mu,j}(I;\mathbb{E})$ and $\mathbb{B}_{q,\mu}(I;\mathbb{E}) = \mathbb{B}_{q,v_\mu}(I;\mathbb{E})$. In this case we furthermore define

$$\mathbb{I}_{q,\mu}(I;\mathbb{E}) := B^{s+2m(1-\frac{1+\mu}{q})}_{p,q,\gamma}(\mathscr{O}; X).$$

In Theorem 6.6.2 we will in particularly see that

$$\mathbb{M}_{q,\mu}(J;\mathbb{E}) \longrightarrow \mathbb{B}_{q,\mu}(J;\mathbb{E}) \oplus \mathbb{I}_{q,\mu}(J;\mathbb{E}),\ u \mapsto (\mathscr{B}(D)u, u_0),$$

which basically just is a trace theory part of the problem. In view of the commutativity of taking traces, $\mathrm{tr}_{\partial\mathscr{O}} \circ \mathrm{tr}_{t=0} = \mathrm{tr}_{t=0} = \mathrm{tr}_{\partial\mathscr{O}}$, when well-defined, we also have to impose a compatibility condition on g and u_0 in (6.56). In order to formulate this precisely, let us define

$$\mathscr{B}^{t=0}_j(D) := \sum_{|\beta| \le m_j} b_{j,\beta}(0,\cdot)\mathrm{tr}_{\partial\mathscr{O}} D^\beta, \qquad j = 1, \ldots, m,$$

and

$$\mathbb{IB}_{q,\mu}(J;\mathbb{E}) := \left\{ (g, u_0) \in \mathbb{B}_{q,\mu}(J;\mathbb{E}) \oplus \mathbb{I}_{q,\mu}(J;\mathbb{E}) : \mathrm{tr}_{t=0} g_j - \mathscr{B}^{t=0}_j(D)u_0 = 0 \text{ when } \kappa_{j,\mathbb{E}} > \frac{1+\mu}{q} \right\}.$$

Remark 6.6.1. Regarding the compatibility condition

$$\mathrm{tr}_{t=0} g_j - \mathscr{B}^{t=0}_j(D)u_0 = 0 \text{ when } \kappa_{j,\mathbb{E}} > \frac{1+\mu}{q}$$

in the definition of $\mathbb{IB}_{q,\mu}(J;\mathbb{E})$, let us remark the following. Suppose $\kappa_{j,\mathbb{E}} > \frac{1+\mu}{q}$. Then $(g_j, u_0) \mapsto \mathrm{tr}_{t=0} g_j - \mathscr{B}_j^{t=0}(D)u_0$ is a well-defined bounded linear operator $\mathbb{B}_{q,\mu}(J;\mathbb{E}) \oplus \mathbb{I}_{q,\mu}(J;\mathbb{E}) \to L_p(\partial\mathscr{O};X)$ as

$$\mathbb{B}_{q,\mu,j}(I;\mathbb{E}) \hookrightarrow F^{\kappa_{j,\mathbb{E}}}_{q,p}(I, v_\mu; L_p(\partial\mathscr{O};X))$$

and

$$D^\beta : \mathbb{I}_{q,\mu}(J;\mathbb{E}) \longrightarrow B^{s+2m(1-\frac{1+\mu}{q})-m_j}_{p,q,\gamma}(\mathscr{O};X), \qquad |\beta| \leq m_j,$$

with

$$s+2m(1-\frac{1+\mu}{q})-m_j = 2m\left(\kappa_{j,\mathbb{E}} - \frac{1+\mu}{q} + \frac{1+\gamma}{2mp}\right) > \frac{1+\gamma}{p}.$$

6.6.2. Statement of the Main Result

Theorem 6.6.2. *Let the notations be as in Subsection 6.6.1 with $v = v_\mu$, $\mu \in (-1, q-1)$. Suppose that X is a UMD space, that $\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D)$ satisfies the conditions* $(\mathrm{E})_\phi$, $(\mathrm{LS})_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$, and that $\kappa_{j,\mathbb{E}} \neq \frac{1+\mu}{q}$ for all $j \in \{1,\ldots,n\}$. Then the problem* (6.2) *enjoys the property of maximal $L_{q,\mu}$-$\mathbb{E}$-regularity with $\mathbb{IB}_{q,\mu}(J;\mathbb{E})$ as the optimal space of initial-boundary data, i.e.*

$$\mathbb{M}_{q,\mu}(J;\mathbb{E}) \longrightarrow \mathbb{D}_{q,\mu}(J;\mathbb{E}) \oplus \mathbb{IB}_{q,\mu}(J;\mathbb{E}),\ u \mapsto (\partial_t u + \mathscr{A}(D)u, \mathscr{B}(D)u, u_0)$$

defines an isomorphism of Banach spaces. In particular, the problem (6.56) *admits a unique solution $u \in \mathbb{M}_{q,\mu}(J;\mathbb{E})$ if and only if $(f, g, u_0) \in \mathbb{D}_{q,\mu}(J;\mathbb{E}) \oplus \mathbb{IB}_{q,\mu}(J;\mathbb{E})$.*

Remark 6.6.3. In the $L_{q,\mu}$-$L_{p,\gamma}$-case the proof simplifies a bit on the function space theoretic side of the problem, yielding a simpler proof than the previous ones ([62] ($\mu = 0$, $\gamma = 0$), [181] ($q = p$, $\mu \in [0, p-1)$, $\gamma = 0$) and [160]).

Analogously to [162, Section 4.3], we obtain the following smoothing result as a corollary to Theorem 6.6.2. It basically says that, in the case of smooth coefficients, there is C^∞-regularity in the spatial variable with some quantitative blow-up near the boundary for the solution u when $f = 0$ and $u_0 = 0$ (see the discussion after [162, Corollary 1.3]).

Corollary 6.6.4. *Let the notations and assumptions be as in Theorem 6.6.2. Then*

$$\begin{aligned}
&\{u \in \mathbb{M}_{q,\mu}(J;\mathbb{E}) : \partial_t u + \mathscr{A}(D)u = 0, u_0 = 0\} \\
&\quad \hookrightarrow \bigcap_{\nu > -1} \left[W^1_{q,\mu}(J; F^{s+\frac{\nu-\gamma}{p}}_{p,1,\nu}(\mathscr{O};X)) \cap L_{q,\mu}(J; F^{s+\frac{\nu-\gamma}{p}+2m}_{p,1,\nu}(\mathscr{O};X)) \right] \\
&\quad \hookrightarrow \bigcap_{k \in \mathbb{N}} \left[W^1_{q,\mu}(J; W^k_p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+(k-s)p}; X)) \cap L_\mu(J; W^{k+2m}_p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+(k-s)p}; X)) \right].
\end{aligned}$$

6.6.3. The Proof of Theorem 6.6.2

For the proof of Theorem 6.6.2 we will first look at model problems on $\mathscr{O} = \mathbb{R}^n_+$, from which the general case can be derived by means of a localization procedure.

Proposition 6.6.5. *Let X be a UMD Banach space and assume that $(\mathscr{A}, \mathscr{B}_1, \dots, \mathscr{B}_m)$ is homogeneous with constant-coefficients on $\mathscr{O} = \mathbb{R}^n_+$ and satisfies* (E)$_\phi$ *and* (LS)$_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$. Let $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as in either* (a) *or* (b) *(with $\mathscr{O} = \mathbb{R}^n_+$). Then $u \mapsto (\partial_t u + (1+\mathscr{A}(D))u, \mathscr{B}(D)u)$ defines an isomorphism of Banach spaces*

$$\mathbb{M}_{q,v}(\mathbb{R};\mathbb{E}) \longrightarrow \mathbb{D}_{q,v}(\mathbb{R};\mathbb{E}) \oplus \mathbb{B}_{q,v}(\mathbb{R};\mathbb{E}),$$

where $\mathbb{M}_{q,v}(\mathbb{R};\mathbb{E})$, $\mathbb{D}_{q,v}(\mathbb{R};\mathbb{E})$, $\mathbb{B}_{q,v}(\mathbb{R};\mathbb{E})$ are as in Subsection 6.6.1.

Proposition 6.6.6. *Let X be a UMD Banach space and assume that $(\mathscr{A}, \mathscr{B}_1, \dots, \mathscr{B}_m)$ is homogeneous with constant-coefficients on $\mathscr{O} = \mathbb{R}^n_+$ and satisfies* (E)$_\phi$ *and* (LS)$_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$. Let $J = (0,T)$ with $T \in (0,\infty]$. Let $q \in (1,\infty)$ and $\mu \in (-1, q-1)$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as in either* (a) *or* (b) *(with $\mathscr{O} = \mathbb{R}^n_+$). Then $u \mapsto (\partial_t u + (1+\mathscr{A}(D))u, \mathscr{B}(D)u, u(0))$ defines an isomorphism of Banach spaces*

$$\mathbb{M}_{q,\mu}(J;\mathbb{E}) \longrightarrow \mathbb{D}_{q,\mu}(\mathbb{R};\mathbb{E}) \oplus \mathbb{IB}_{q,\mu}(\mathbb{R};\mathbb{E}),$$

where $\mathbb{M}_{q,\mu}(\mathbb{R};\mathbb{E})$, $\mathbb{D}_{q,\mu}(\mathbb{R};\mathbb{E})$, $\mathbb{IB}_{q,\mu}(\mathbb{R};\mathbb{E})$ are as in the beginning of this section.

Lemma 6.6.7. *Let X be a UMD Banach space and assume that $(\mathscr{A}, \mathscr{B}_1, \dots, \mathscr{B}_m)$ is homogeneous with constant-coefficients on $\mathscr{O} = \mathbb{R}^n_+$ and satisfies* (E)$_\phi$ *and* (LS)$_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$. Let $q \in (1,\infty)$ and $v \in A_q(\mathbb{R})$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as in either* (a) *or* (b) *(with $\mathscr{O} = \mathbb{R}^n_+$). Then*

$$\mathscr{B}(D) : \mathbb{M}_{q,v}(\mathbb{R};\mathbb{E}) \longrightarrow \mathbb{B}_{q,v}(\mathbb{R};\mathbb{E}),\ u \mapsto (\mathscr{B}_1(D)u, \dots, \mathscr{B}_m u), \tag{6.57}$$

is a well-defined bounded linear operator and the differential parabolic boundary value problem

$$\begin{cases} \partial_t u + (1+\mathscr{A}(D))u &= 0, \\ \mathscr{B}_j(D)u &= g_j, \quad j = 1, \dots, n, \end{cases} \tag{6.58}$$

admits a bounded linear solution operator

$$\mathscr{S} : \mathbb{B}_{q,v}(\mathbb{R};\mathbb{E}) \longrightarrow \mathbb{M}_{q,v}(\mathbb{R};\mathbb{E}),\ (g_1, \dots, g_m) \mapsto u,$$

where $\mathbb{M}_{q,v}(\mathbb{R};\mathbb{E})$, $\mathbb{B}_{q,v}(\mathbb{R};\mathbb{E})$ are as in the beginning of this section. Moreover, there is uniqueness of solutions in $\mathbb{M}_{q,v}(\mathbb{R};\mathbb{E})$: if $u \in \mathbb{M}_{q,v}(\mathbb{R};\mathbb{E})$ and $g = (g_1, \dots, g_m) \in B_{q,v}(\mathbb{R};\mathbb{E})$ satisfy (6.58)*, then $u = \mathscr{S} g$.*

Proof. That (6.57) is a well-defined bounded linear operator follows from Corollary 6.4.8, where we use the elementary embedding $\mathbb{E} \hookrightarrow F^s_{p,\infty,\gamma}(\mathbb{R}^n_+;X)$ and $\mathbb{E}^{2m} \hookrightarrow F^{s+2m}_{p,\infty,\gamma}(\mathbb{R}^n_+;X)$ in case (a). So we just need to establish the existence of a bounded linear solution operator

$\mathscr{S} : (\mathscr{S}(\mathbb{R}^n;X)^m, ||\cdot||_{\mathbb{B}}) \longrightarrow \mathbb{M}$. But the existence of such a solution operator follows from a combination of Corollaries 6.5.18 and 6.5.26, where we use the elementary embedding $F^s_{p,1,\gamma}(\mathbb{R}^n_+;X) \hookrightarrow \mathbb{E}$ and $F^{s+2m}_{p,1,\gamma}(\mathbb{R}^n_+;X) \hookrightarrow \mathbb{E}^{2m}$ in case (a).

Finally, let us prove the uniqueness of solutions. For this it suffices to show that $\mathscr{S}(\mathbb{R}^n_+ \times \mathbb{R};X)$ is dense in $\mathbb{M}$. Indeed, using this density, the uniqueness statement follows from a combination of (6.57), the uniqueness statement in Corollary 6.5.18 and the continuity of our solution operator $\mathscr{S} : \mathbb{B} \longrightarrow \mathbb{M}$.

For this density, note that $W^1_q(\mathbb{R},v;\mathbb{E}^{2m})$ is dense in $\mathbb{M}$ by a standard convolution argument (in the time variable). So

$$\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}^n_+;X) \overset{d}{\subset} \mathscr{S}(\mathbb{R}) \otimes \mathbb{E}^{2m} \overset{d}{\subset} \mathscr{S}(\mathbb{R};\mathbb{E}^{2m}) \overset{d}{\hookrightarrow} W^1_q(\mathbb{R},v;\mathbb{E}^{2m}) \overset{d}{\hookrightarrow} \mathbb{M},$$

yielding the required density. □

Lemma 6.6.8. *Let X be a Banach space, $p,r \in [1,\infty)$, $w \in A_\infty(\mathbb{R}^n)$ and $s \in \mathbb{R}$. Suppose $\mathscr{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ with $a_\alpha \in \mathscr{B}(X)$ is parameter-elliptic with angle of ellipticity $\phi_{\mathscr{A}}$ and let $\phi > \phi_{\mathscr{A}}$. Then, for all $s = (s_1,s_2,s_3) \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}^n$, we have that*

$$\kappa_\alpha := \mathscr{R}\left\{\langle\xi\rangle^{|\alpha|} D^{s,\alpha}_\xi (s_1 + s_2\lambda + s_3|\xi|^{2m})(1+\lambda+\mathscr{A}(\xi))^{-1} : \lambda \in \Sigma_{\pi-\phi}, \xi \in \mathbb{R}^n\right\} < \infty \quad in \quad \mathscr{B}(X). \tag{6.59}$$

Proof. In order to establish (6.59), we define

$$f : \mathbb{R}\times\mathbb{R}^n \times \Sigma_{\pi-\phi'} \to \mathscr{B}(X),\ (x,\xi,\lambda) \mapsto (s_1 x^{2m} + s_2\lambda + s_3|\xi|^{2m})(x^{2m} + \lambda + A(\xi))^{-1},$$

where $\phi_{\mathscr{A}} < \phi' < \phi$ as well as $f_\alpha(x,\xi,\lambda) := (x^2+|\xi|^2)^{|\alpha|/2}\partial^\alpha_\xi f(x,\xi,\lambda)$

$$f_\alpha(x,\xi,\lambda) := (x^2+|\xi|^2)^{|\alpha|/2}\partial^\alpha_\xi f(x,\xi,\lambda) \quad g_\alpha(x,\xi,\lambda) := (x^2+|\xi|^2+\lambda^{1/m})^{|\alpha|/2}\partial^\alpha_\xi f(x,\xi,\lambda)$$

for $\alpha \in \mathbb{N}^n$. By geometric considerations, we obtain that

$$x^2+|\xi|^2 \le \frac{\left|x^2+|\xi|^2+\lambda^{1/m}\right|}{\cos\left(\frac{\pi}{2} - \max\{\frac{\pi}{2},\phi'\}\right)}$$

for all $(x,\xi,\lambda) \in \mathbb{R}\times\mathbb{R}^n\times\Sigma_{\pi-\phi'}$. Hence, Kahane's contraction principle yields

$$\begin{aligned}\kappa_\alpha &= \mathscr{R}\left\{f_\alpha(1,\xi,\lambda) : \lambda \in \Sigma_{\pi-\phi}, \xi\in\mathbb{R}^n\right\}\\ &\lesssim \mathscr{R}\left\{f_\alpha(1,\xi,\lambda) : \lambda \in \Sigma_{\pi-\phi}, \xi\in\mathbb{R}^n \text{ such that } |\lambda| \le |\xi|^{2m} \text{ or } |\lambda|\le 1\right\}\\ &\quad+\mathscr{R}\left\{g_\alpha(1,\xi,\lambda) : \lambda \in \Sigma_{\pi-\phi}, \xi\in\mathbb{R}^n \text{ such that } |\lambda| \ge |\xi|^{2m} \text{ and } |\lambda|\ge 1\right\}.\end{aligned}$$

Obviously, we have that $f(cx,c\xi,c^{2m}\lambda) = f(x,\xi,\lambda)$ for all $c>0$. Lemma 6.5.19 shows that the same holds for f_α and g_α. Hence, by choosing $c = (1+|\xi|^2+|\lambda|^{1/m})^{1/2}$ and defining

$$D_1 := \mathrm{cl}\left\{\left(\frac{1}{c},\frac{\xi}{c},\frac{\lambda}{c^{2m}}\right) : \lambda\in\Sigma_{\pi-\phi}, \xi\in\mathbb{R}^n \text{ such that } |\lambda|\le|\xi|^{2m} \text{ or } |\lambda|\le 1\right\},$$

$$D_2 := \mathrm{cl}\left\{\left(\frac{1}{c}, \frac{\xi}{c}, \frac{\lambda}{c^{2m}}\right) : \lambda \in \Sigma_{\pi-\phi}, \xi \in \mathbb{R}^n \text{ such that } |\lambda| \geq |\xi|^{2m} \text{ and } |\lambda| \geq 1\right\},$$

we obtain

$$\kappa_\alpha \lesssim \mathscr{R}(f(D_1)) + \mathscr{R}(f(D_2)).$$

But since f_α is holomorphic on

$$\mathbb{R} \times \mathbb{R}^n \times \Sigma_{\pi-\phi'} \setminus \{(0,0,c_3) : c_3 \in \Sigma_{\pi-\phi'}\} \supset D_1$$

and since g_α is holomorphic on

$$\mathbb{R} \times \mathbb{R}^n \times \Sigma_{\pi-\phi'} \setminus \{(c_1, c_2, 0) : c_1 \in \mathbb{R}, c_2 \in \mathbb{R}^n\} \supset D_2$$

we obtain that [60, Proposition 3.10] implies

$$\kappa_\alpha \lesssim \mathscr{R}\{f_\alpha(D_1)\} + \mathscr{R}\{g_\alpha(D_2)\} < \infty.$$

by the compactness of D_1 and D_2. □

Lemma 6.6.9. *Let X be a Banach space, $p, r \in [1,\infty)$, $w \in A_\infty(\mathbb{R}^n)$ and $s \in \mathbb{R}$. Suppose $\mathscr{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ with $a_\alpha \in \mathscr{B}(X)$ is parameter-elliptic with angle of ellipticity $\phi_{\mathscr{A}}$. Let A be the realization of $\mathscr{A}(D)$ in $F^s_{p,r}(\mathbb{R}^n, w; X)$ with domain $D(A) = F^{s+2m}_{p,r}(\mathbb{R}^n, w; X)$. Then $0 \in \rho(1 + A)$ and $1 + A$ is R-sectorial with angle $\omega_R(1 + A) \leq \phi_{\mathscr{A}}$.*

Proof. By Lemma 6.6.8 and Lemma 6.3.4, we have

$$\mathscr{R}\{(s_1 + s_2\lambda + s_3|\xi|^{2m})(1 + \lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\} < \infty. \tag{6.60}$$

For $(s_1, s_2, s_3) = (0,1,0)$ this shows the $\mathscr{R}$-sectoriality of $1 + A$. Hence, it only remains to show that $D(A) = F^{s+2m}_{p,r}(\mathbb{R}^n, w; X)$. But Kahane's contraction principle together with $(s_1, s_2, s_3) = (\sqrt{2}, 0, \sqrt{2})$ in (6.59) shows that

$$\kappa_\alpha := \mathscr{R}\left\{\langle\xi\rangle^{|\alpha|} D^\alpha_\xi (1 + \lambda + \mathscr{A}(\xi))^{-1} : \lambda \in \Sigma_{\pi-\phi}, \xi \in \mathbb{R}^n\right\} < \infty \quad \text{in} \quad \mathscr{B}(X), \quad \alpha \in \mathbb{N}^n,\ |\alpha| \leq 2m.$$

Using Lemma 6.3.4 together with (6.27), we obtain that $(1 + \lambda + A)^{-1}$ maps $F^s_{p,r}(\mathbb{R}^n, w; X)$ into $F^{s+2m}_{p,r}(\mathbb{R}^n, w; X)$. This shows that $D(A) = F^{s+2m}_{p,r}(\mathbb{R}^n, w; X)$. □

Proof of Proposition 6.6.5. We first show that the differential parabolic boundary value problem

$$\begin{aligned} \partial_t u + (1 + \mathscr{A}(D))u &= f, \\ \mathscr{B}_j(D)u &= g_j, \quad j = 1, \ldots, n, \end{aligned} \tag{6.61}$$

admits a bounded linear solution operator

$$\mathscr{T} : L_q(\mathbb{R}, v; \mathbb{E}) \oplus \mathbb{B} \longrightarrow \mathbb{M}, (f, g_1, \ldots, g_m) \mapsto u$$

To this end, for $k \in \{0, 2m\}$ let

$$\overline{\mathbb{E}}^k := \begin{cases} H_p^{s+k}(\mathbb{R}^n, w_\gamma; X), & \text{in case (a)}, \\ F_{p,r}^{s+k}(\mathbb{R}^n, w_\gamma; X), & \text{in case (b)}, \end{cases}$$

and put $\overline{\mathbb{M}} := W_q^1(\mathbb{R}, v; \overline{\mathbb{E}}) \cap L_q(\mathbb{R}, v; \overline{\mathbb{E}}^{2m})$. The realization of $1 + \mathscr{A}(D)$ in $\overline{\mathbb{E}}$ with domain $\overline{\mathbb{E}}^{2m}$ has 0 in its resolvent and is R-sectorial with angle $< \frac{\pi}{2}$, which in case (b) is contained in Lemma 6.6.9 and which in case (a) can be derived as in [60, Corollary 5.6] using the operator-valued Mikhlin theorem for $H_p^s(\mathbb{R}^n, w_\gamma; X)$ (see Proposition 6.3.6). As a consequence (see Section 6.3.2), the parabolic problem

$$\partial_t \overline{u} + (1 + \mathscr{A}(D))\overline{u} = \overline{f} \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R}$$

admits a bounded linear solution operator

$$\mathscr{R} : L_q(\mathbb{R}, v; \overline{\mathbb{E}}) \longrightarrow \overline{\mathbb{M}}, \; \overline{f} \mapsto \overline{u}.$$

Choosing an extension operator

$$\mathscr{E} \in \mathscr{B}\left(L_q(\mathbb{R}, v; \mathbb{E}), L_q(\mathbb{R}, v; \overline{\mathbb{E}})\right),$$

recalling (6.57), denoting by $r_+ \in \mathscr{B}(\overline{\mathbb{M}}, \mathbb{M})$ the operator of restriction from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n_+ \times \mathbb{R}$ and denoting by $\mathscr{S}$ the solution operator from Lemma 6.6.7, we find that

$$\mathscr{T}(f, g_1, \ldots, g_m) := r_+ \mathscr{R}\mathscr{E} f - \mathscr{S}\mathscr{B}(D) r_+ \mathscr{R}\mathscr{E} f + \mathscr{S}(g_1, \ldots, g_m)$$

defines a solution operator as desired.

Finally, the uniqueness follows from the uniqueness obtained in Lemma 6.6.7. □

Lemma 6.6.10. *Let X be a UMD Banach space and assume that $(\mathscr{A}, \mathscr{B}_1, \ldots, \mathscr{B}_m)$ is homogeneous with constant-coefficients on $\mathscr{O} = \mathbb{R}^n_+$ and satisfies* (E)$_\phi$ *and* (LS)$_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$. Let $q \in (1, \infty)$ and $v \in A_q(\mathbb{R})$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as in either* (a) *or* (b) *(with $\mathscr{O} = \mathbb{R}^n_+$). Let A_B be the realization of $\mathscr{A}(D)$ in $\mathbb{E}$ with domain $D(A_B) = \{u \in \mathbb{E}^{2m} : \mathscr{B}(D)u = 0\}$. Then there is an equivalence of norms in $D(A_B) = \{u \in \mathbb{E}^{2m} : \mathscr{B}(D)u = 0\}$, $-(1 + A_B)$ is the generator of an exponentially stable analytic semigroup on $\mathbb{E}$ and $1 + A_B$ enjoys the property of $L_q(\mathbb{R}_+, v_\mu)$-maximal regularity.*

Proof. As a consequence of Proposition 6.6.5, $1 + A_B$ satisfies the conditions of Lemma 6.3.1 with $|||\cdot||| = ||\cdot||_{\mathbb{E}^{2m}}$. Therefore, there is an equivalence of norms in $D(1 + A_B) = D(A_B) = \{u \in \mathbb{E}^{2m} : \mathscr{B}(D)u = 0\}$ and $1 + A_B$ is a closed linear operator on $\mathbb{E}$ enjoying the property of $L_q(\mathbb{R}, v)$-maximal regularity. Moreover, it follows from Lemma 6.6.9 together with Proposition 6.5.16 that $\mathbb{C}_+ \subset \varrho(1 + A_B)$ and that $\lambda \mapsto (\lambda + 1 + A_B)^{-1}$ is bounded. Thus, $1 + A_B$ satisfies the conditions of Lemma 6.3.2 and the desired result follows. □

Lemma 6.6.11. *Let the notations be as in Subsection 6.6.1 with $v = v_\mu$, $\mu \in (-1, q-1)$, and suppose that X is a UMD space. Then $tr_{t=0} : u \mapsto u(0)$ is a retraction*

$$tr_{t=0} : \mathbb{M}_{q,\mu}(J; \mathbb{E}) \longrightarrow \mathbb{I}_{q,\mu}(J; \mathbb{E}).$$

Proof. This can be derived from [186, Theorem 1.1]/[198, Theorem 3.4.8], see [160, Section 6.1] and [162, Lemma 4.8]. □

Proof of Proposition 6.6.6. That $u \mapsto (u' + (1+\mathscr{A}(D))u, \mathscr{B}(D)u, u(0))$ is a bounded operator

$$\mathbb{M}_{q,\mu}(\mathbb{R}_+;\mathbb{E}) \longrightarrow \mathbb{D}_{q,\mu}(\mathbb{R}_+;\mathbb{E}) \oplus \mathbb{B}_{q,\mu}(\mathbb{R}_+;\mathbb{E}) \oplus \mathbb{I}_{q,\mu}(\mathbb{E})$$

follows from a combination of Proposition 6.6.5 (choosing an extension operator $\mathbb{M}_{q,\mu}(\mathbb{R}_+;\mathbb{E}) \to \mathbb{M}_{q,\mu}(\mathbb{R};\mathbb{E})$) and Lemma 6.6.11. That it maps to $\mathbb{D}_{q,\mu}(\mathbb{R}_+;\mathbb{E}) \oplus \mathbb{IB}_{q,\mu}(\mathbb{R}_+;\mathbb{E})$ can be seen as follows: we only need to show that

$$\mathrm{tr}_{t=0}\mathscr{B}_j(D)u = \mathscr{B}_j(D)\mathrm{tr}_{t=0}u, \qquad u \in \mathbb{M}_{q,\mu}(\mathbb{R}_+;\mathbb{E}), \tag{6.62}$$

when $\kappa_{j,\mathbb{E}} > \frac{1+\mu}{q}$ (also see Remark 6.6.1), which simply follows from

$$W^1_{q,\mu}(\mathbb{R}_+;\mathbb{E}^{2m}) \overset{d}{\hookrightarrow} \mathbb{M}_{q,\mu}(\mathbb{R}_+;\mathbb{E}).$$

Here this density follows from a standard convolution argument (in the time variable) in combination with an extension/restriction argument.

Let A_B be as in Lemma 6.6.10. Then there is an equivalence of norms in $D(A_B) = \{u \in \mathbb{E}^{2m} : \mathscr{B}(D)u = 0\}$, $-(1+A_B)$ is the generator of an exponentially stable analytic semigroup on $\mathbb{E}$ and $1+A_B$ enjoys the property of $L_q(\mathbb{R}_+, v_\mu)$-maximal regularity. Now the desired result can be derived from Proposition 6.6.5 as in Theorem 5.7.16. □

Proof of Theorem 6.6.2. This can be derived from the model problem case considered in Proposition 6.6.6 by a standard localization procedure, see [176, Sections 2.3 & 2.4] and [159, Appendix B]. □

6.7. ELLIPTIC PROBLEMS

6.7.1. Some notation and assumptions

Let $\mathscr{O}$ be either $\mathbb{R}^n_+$ or a bounded C^∞-domain in $\mathbb{R}^n$. Let further X be a reflexive Banach space and let $\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D)$ be a $\mathscr{B}(X)$-valued differential boundary value system on $\mathscr{O}$ as considered in Section 6.3.5, where the coefficients satisfy certain smoothness conditions which we are going to introduce later. Put $m_* := \max\{m_1, \ldots, m_m\}$. Let $p, q \in (1,\infty)$. For $s \in \mathbb{R}$ let $\mathbb{F}^s$ and $\partial\mathbb{F}^s$ be given as either

(A) $\mathbb{F}^s := H^s_{p,\gamma}(\mathscr{O}, X)$ and $\partial\mathbb{F}^s := \partial H^s_{p,\gamma}(\partial\mathscr{O}, X)$ where $\gamma \in (-1, p-1)$ and X is a UMD space.

(B) $\mathbb{F}^s := \mathscr{A}^s_{p,q,\gamma}(\mathscr{O}, X)$ and $\partial\mathbb{F}^s := \partial\mathscr{A}^s_{p,q,\gamma}(\partial\mathscr{O}, X)$ where $(\gamma, \mathscr{A}) \in (-1,\infty) \times \{B, F\} \cup (-\infty, p-1) \times \{\mathscr{B}, \mathscr{F}\}$.

6.7.2. Parameter-dependent Estimates

Theorem 6.7.1. *Let the notations be as in Subsection 6.7.1. Let further* $s_0 \in (\frac{1+\gamma}{p} - 1, \frac{1+\gamma}{p})$ *and* $s_1 \in [s_0, \infty)$. *Suppose that also* $(\mathrm{E})_\phi$ *and* $(\mathrm{LS})_\phi$ *are satisfied for some* $\phi \in (0, \pi)$. *Then, there is a* $\lambda_\phi > 0$ *such that for all* $\lambda \in \Sigma_{\pi-\phi}$, $t \in [s_0, s_1]$ *and all*

$$(f, g_1, \ldots, g_m) \in \mathbb{F}^t \oplus \bigoplus_{j=1}^{m} \partial\mathbb{F}^{t+2m-m_j}$$

there exists a unique solution $u \in \mathbb{F}^{t+2m,\mu}$ *of the problem*

$$\begin{cases} (\lambda + \lambda_\phi + \mathscr{A}(\cdot, D))u &= f \quad \text{on } \mathscr{O}, \\ \mathscr{B}_j(D)u &= g_j \quad \text{on } \partial\mathscr{O}, \qquad j = 1, \ldots, m. \end{cases} \tag{6.63}$$

Moreover, for this solution there are the parameter-dependent estimates (independent of t)

$$||u||_{\mathbb{F}^{t+2m}} + |\lambda|^{\frac{t+2m-s_0}{2m}} ||u||_{\mathbb{F}^{s_0}} \eqsim ||f||_{\mathbb{F}^t} + |\lambda|^{\frac{t-s_0}{2m}} ||f||_{\mathbb{F}^{s_0}} + \sum_{j=1}^{m} \left(||g_j||_{\partial\mathbb{F}^{t+2m-m_j}} + |\lambda|^{\frac{t+2m-m_j-\frac{1+\gamma}{p}}{2m}} ||g_j||_{L_p(\partial\mathscr{O};X)} \right).$$

In the proo of the above theorem we will use the following Lemmas.

Lemma 6.7.2. *Let X be a reflexive Banach space,* $p, q \in (1, \infty)$, $(w, \mathscr{A}) \in A_\infty(\mathbb{R}^n) \times \{B, F\} \cup [A_\infty]'_p(\mathbb{R}^n) \times \{\mathscr{B}, \mathscr{F}\}$ *and* $s_0, t \in \mathbb{R}$ *with* $t \geq s_0$. *Let* $\mathscr{A}(D)$ *be a differential operator of order* $2m$ *with constant* $\mathscr{B}(X)$*-valued coefficients satisfying* $(E)_\phi$ *for some* $\phi \in (0, \pi]$. *Given* $f \in \mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n, w; X)$ *let* $u := (\lambda + \mathscr{A}(D))^{-1} f$. *Then, for all* $\lambda_0 > 0$ *we have the estimate*

$$\|u\|_{\mathscr{A}^{t+2m}_{p,q}(\mathbb{R}^n,w;X)} + |\lambda|^{\frac{2m+t-s_0}{2m}} \|u\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)} \eqsim_{\lambda_0} \|f\|_{\mathscr{A}^{t}_{p,q}(\mathbb{R}^n,w;X)} + |\lambda|^{\frac{t-s_0}{2m}} \|f\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)}$$

for every $\lambda \in (\lambda_0 + \Sigma_\phi$.

Proof. We substitute $\lambda = \mu^{2m}$ so that $(\xi, \mu) \mapsto (\mu^{2m} + \mathscr{A}(\xi))^{-1}$ is a parameter-dependent Hörmander symbol of order $-2m$ and regularity ∞. Hence, if we define

$$p_{2m}(\xi, \mu) := \langle \xi, \mu \rangle^{2m} (\mu^{2m} + \mathscr{A}(\xi))^{-1} = \langle \xi, \mu \rangle^{t+2m-s_0} (\mu^{2m} + \mathscr{A}(\xi))^{-1} \langle \xi, \mu \rangle^{s_0 - t},$$

then $(p(\cdot, \mu))_{\mu \in \Sigma_{\phi/2m}}$ and $(p(\cdot, \mu)^{-1})_{\mu \in \Sigma_{\phi/2m}}$ are bounded families in the parameter-independent Hörmander symbols $S^0(\mathbb{R}^n, \mathscr{B}(X))$ of order 0. In particular, by (6.26) together with a duality argument for the dual scales, we have that

$$\begin{aligned} \|u\|_{\mathscr{A}^{t+2m,|\mu|,s_0}_{p,q}(\mathbb{R}^n,w,X)} &= \|(\mu^{2m} + \mathscr{A}(D))^{-1} f\|_{\mathscr{A}^{t+2m,|\mu|,s_0}_{p,q}(\mathbb{R}^n,w;X)} \\ &= \|\Xi_\mu^{s_0-t-2m} \operatorname{op}(p_{2m}(\cdot\mu) \Xi_\mu^{t-s_0} f\|_{\mathscr{A}^{t+2m,|\mu|,s_0}_{p,q}(\mathbb{R}^n,w;X)} \\ &= \|\operatorname{op}(p_{2m}(\cdot\mu) \Xi_\mu^{t-s_0} f\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)} \end{aligned}$$

$$\eqsim \|\Xi_\mu^{t-s_0} f\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)}$$
$$= \|f\|_{\mathscr{A}^{t,|\mu|,s_0}_{p,q}(\mathbb{R}^n,w;X)}.$$

Using the equivalence (6.38) we obtain

$$\|u\|_{\mathscr{A}^{t+2m}_{p,q}(\mathbb{R}^n,w;X)} + \langle\mu\rangle^{t-2m-s_0}\|u\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)} \eqsim \|f\|_{\mathscr{A}^{t}_{p,q}(\mathbb{R}^n,w;X)} + \langle\mu\rangle^{t-s_0}\|f\|_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w;X)}.$$

Replacing μ^{2m} by λ again yields the assertion. □

Lemma 6.7.3. *Let X be a UMD Banach space, $p, q \in (1,\infty)$, $w \in A_p(\mathbb{R}^n)$, $s_0, t \in \mathbb{R}$ with $t \geq s_0$. Suppose that $\mathscr{A}(D)$ is a homogeneous differential operator of order $2m$ with constant coefficients in $\mathscr{B}(X)$ satisfying $(E)_\phi$ for some $\phi \in (0,\pi]$. Given $f \in H^{s_0}_p(\mathbb{R}^n,w;X)$ let $u := (\lambda + \mathscr{A}(D))^{-1} f$. Then, for all $\lambda_0 > 0$ we have the estimate*

$$\|u\|_{H^{t+2m}_p(\mathbb{R}^n,w;X)} + |\lambda|^{\frac{2m+t-s_0}{2m}}\|u\|_{H^{s_0}_p(\mathbb{R}^n,w;X)} \eqsim_{\lambda_0} \|f\|_{H^{t}_p(\mathbb{R}^n,w;X)} + |\lambda|^{\frac{t-s_0}{2m}}\|f\|_{H^{s_0}_p(\mathbb{R}^n,w;X)}$$

for every $\lambda \in (\lambda_0 + \Sigma_\phi$.

Proof. The proof is almost the same as the one of Lemma 6.7.2. But instead of (6.26) we use Proposition 6.3.6 together with Lemma 6.6.8. □

Proof of Theorem 6.7.1. First, we consider case (B). By localization, we only have to treat the case of a homogeneous system with constant $\mathscr{B}(X)$-valued coefficients on $\mathscr{O} = \mathbb{R}^n_+$. Taking Rychkov's extension operator $\mathscr{E}$ (see Theorem 6.3.8), we can represent the solution as

$$u = r_+(\lambda - \mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E} f + \sum_{j=1}^m \operatorname{OPK}(\widetilde{k}_{j,\lambda})(g_j - \operatorname{tr}_{\mathbb{R}^{n-1}}\mathscr{B}_j(D)(\lambda+\mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E} f).$$

Here, $(\lambda - \mathscr{A}(D))^{-1}_{\mathbb{R}^n}$ denotes the resolvent in the whole space as in Lemma 6.7.2 and $\widetilde{k}_{j,\lambda}$ are the Poisson symbol kernels as in Proposition 6.5.16. For the estimate, we treat the summands separately. We write

$$\begin{aligned} u_1 &:= r_+(\lambda - \mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E} f,\\ u_{2,j} &:= \operatorname{OPK}(\widetilde{k}_{j,\lambda})\operatorname{tr}_{\mathbb{R}^{n-1}}\mathscr{B}_j(D)(\lambda+\mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E} f,\\ u_{3,j} &:= \operatorname{OPK}(\widetilde{k}_{j,\lambda})g_j. \end{aligned}$$

First, by Theorem 6.3.8 and Lemma 6.7.2 we have that

$$\begin{aligned} ||u_1||_{\mathscr{A}^{t+2m}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t+2m-s_0}{2m}}||u_1||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} &\eqsim ||\mathscr{E} f||_{\mathscr{A}^{t}_{p,q}(\mathbb{R}^n,w_\gamma;X)} + |\lambda|^{\frac{t-s_0}{2m}}||\mathscr{E} f||_{\mathscr{A}^{s_0}_{p,q}(\mathbb{R}^n,w_\gamma;X)}\\ &\eqsim ||f||_{\mathscr{A}^{t}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t-s_0}{2m}}||f||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)}. \end{aligned} \tag{6.64}$$

For u_2 we substitute $\lambda = \mu^{2m}$ again. Then, Theorem 6.5.28, Proposition, Lemma 6.7.2 and Theorem 6.3.8 yield

$$\begin{aligned}
\|u_{2,j}\|_{\mathscr{A}^{t+2m,|\mu|,s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} &\leq \|\mathscr{B}_j(D)(\lambda+\mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E}f\|_{\partial\mathscr{A}^{t+2m-m_j,|\mu|}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \\
&\lesssim \|(\lambda+\mathscr{A}(D))^{-1}_{\mathbb{R}^n}\mathscr{E}f\|_{\mathscr{A}^{t+2m,|\mu|,s_0+m_j}_{p,q}(\mathbb{R}^n,w_\gamma;X)} \\
&\lesssim \|f\|_{\mathscr{A}^{t,|\mu|,s_0+m_j}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \\
&\eqsim \|f\|_{\mathscr{A}^{t}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + \langle\mu\rangle^{t-s_0-m_j}\|f\|_{\mathscr{A}^{s_0+m_j}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \\
&\lesssim \|f\|_{\mathscr{A}^{t}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + \langle\mu\rangle^{t-s_0}\|f\|_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)}.
\end{aligned}$$

Substituting $\lambda = \mu^{2m}$ again yields

$$||u_{2,j}||_{\mathscr{A}^{t+2m}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t+2m-s_0}{2m}}||u_{2,j}||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \lesssim ||f||_{\mathscr{A}^{t}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t-s_0}{2m}}||f||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)}.$$

Finally, it follows from Theorem 6.5.28 that

$$\|u_{3,j}\|_{\mathscr{A}^{t+2m,|\mu|,s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \lesssim \|g_j\|_{\partial\mathscr{A}^{t+2m-m_j,|\mu|}_{p,q,\gamma}(\mathbb{R}^n_+;X)}$$

so that (6.40) together with the substitution $\lambda = \mu^{2m}$ yields

$$\begin{aligned}
&||u_{3,j}||_{\mathscr{A}^{t+2m}_{p,q,\gamma}(\mathscr{O};X)} + |\lambda|^{\frac{t+2m-s_0}{2m}}||u_{3,j}||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathscr{O};X)} \\
&\qquad\lesssim ||g_j||_{\partial\mathscr{A}^{t+2m-m_j}_{p,q,\gamma}(\partial\mathscr{O};X)} + |\lambda|^{\frac{t+2m-m_j-\frac{1+\gamma}{p}}{2m}}||g_j||_{L_p(\partial\mathscr{O};X)}.
\end{aligned}$$

Summing up $u = u_1\sum_{j=1}^m u_{2,j} + u_{3,j}$ yields

$$\begin{aligned}
&||u||_{\mathscr{A}^{t+2m}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t+2m-s_0}{2m}}||u||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \\
&\qquad\lesssim ||f||_{\mathscr{A}^{t}_{p,q,\gamma}(\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t-s_0}{2m}}||f||_{\mathscr{A}^{s_0}_{p,q,\gamma}(\mathbb{R}^n_+;X)} \\
&\qquad\quad + \sum_{j=1}^m\left(||g_j||_{\partial\mathscr{A}^{t+2m-m_j}_{p,q,\gamma}(\partial\mathbb{R}^n_+;X)} + |\lambda|^{\frac{t+2m-m_j-\frac{1+\gamma}{p}}{2m}}||g_j||_{L_p(\partial\mathbb{R}^n_+;X)}\right).
\end{aligned}$$

The inverse estimate follows from Proposition 6.3.9 together with the estimate

$$\|(\lambda+\mathscr{A}(D))u\|_{\mathscr{A}^{s,\mu,s_0}_{p,q,\gamma}(\mathbb{R}^n_+,X)} \leq \|u\|_{\mathscr{A}^{s+2m,\mu,s_0}_{p,q,\gamma}(\mathbb{R}^n_+,X)}.$$

Case (A) can be carried out in almost the exact same way. One just has to use the extension operator from Proposition 6.3.7 instead of Rychkov's extenstion operator, use Lemma 6.7.3 instead of Lemma 6.7.2 and use the elementary embedding $F^s_{p,1,\gamma}(\mathbb{R}^n,X) \hookrightarrow H^s_{p,\gamma}(\mathbb{R}^n,E)$ for the Poisson operators estimates. □

6.7.3. Operator Theoretic Results

The L_q-maximal regularity established in Theorem 6.6.2 for the special case of homogeneous initial-boundary data gives L_q-maximal regularity and thus R-sectoriality for the realizations of the corresponding elliptic differential operators:

Corollary 6.7.4. *Let $\mathscr{O}$ be either $\mathbb{R}^n_+$ or a bounded C^∞-domain in $\mathbb{R}^n$. Let X be a UMD Banach space and let $(\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D))$ be a $\mathscr{B}(X)$-valued differential boundary value system on $\mathscr{O}$ as considered in Section 6.3.5 and put $m_* := \max\{m_1, \ldots, m_m\}$. Let $\mathbb{E}$ and $\mathbb{E}^{2m}$ be given as in either* (a) *or* (b) *as in Section 6.6.1. Let $(\mathscr{A}(D), \mathscr{B}_1(D), \ldots, \mathscr{B}_m(D))$ be a $\mathscr{B}(X)$-valued differential boundary value system of order $2m$ on $\mathscr{O}$ that satisfies* $(\mathrm{E})_\phi$ *and* $(\mathrm{LS})_\phi$ *for some $\phi \in (0, \frac{\pi}{2})$. Moreover, we assume that the coefficients satisfy the conditions* $(\mathrm{SDP})_s$, $(\mathrm{SDL})_s$, $(\mathrm{SBP})_s$ *and* $(\mathrm{SBL})_s$ *from Section 6.7.1.*[5] *Let A_B be the realization of $\mathscr{A}(D)$ in $\mathbb{E}$ with domain $D(A_B) = \{u \in \mathbb{E}^{2m} : \mathscr{B}(D)u = 0\}$. For every $\theta \in (\phi, \frac{\pi}{2})$ there exists $\mu_\theta > 0$ such that $\mu_\theta + A$ is R-sectorial with angle $\omega_R(\mu_\theta + A_B) \leq \theta$.*

Proof. This is a direct consequence of Theorem 6.6.2 and Proposition 6.3.3. Indeed, if we write $\mathscr{A}(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ and $\mathscr{B}_j(x,D) = \sum_{|\beta|\leq m_j} b_{j,\beta}(x)D^\beta$, then it follows from our assumption that $\widetilde{a}_\alpha := a_\alpha \otimes 1_J$ and $\widetilde{b}_{j,\beta} := b_{j,\beta} \otimes 1_J$ satisfy the conditions (SDP), (SDL), (SBP) and (SBL) from Section 6.6.1. □

The following result is an immediate corollary to Theorem 6.7.1.

Corollary 6.7.5. *Consider the situation of Theorem 6.7.1 with $s = s_0 = s_1$. Let A_B be the realization of $\mathscr{A}(D)$ in $\mathbb{F}^s$ with domain $D(A_B) = \{u \in \mathbb{F}^{s+2m} : \mathscr{B}(D)u = 0\}$. For every $\theta \in (\phi, \pi)$ there exists $\mu_\theta > 0$ such that $\mu_\theta + A_B$ is sectorial with angle $\phi_{\mu_\theta + A} \leq \theta$.*

Remark 6.7.6. From the R-sectoriality and sectoriality in Corollary 6.7.4 and Corollary 6.7.5, respectively, one can derive boundedness of the H^∞-functional calculus using interpolation techniques from [76, 133]: [133, Corollary 7.8] respectively [76, Theorem 3.1] gives a bounded H^∞-calculus of the part of A_B in the Rademacher interpolation space $\langle \mathbb{E}, D(A)\rangle_\theta$ respectively in the real interpolation space $(\mathbb{E}, D(A))_{\theta,q}$. In this way one could improve the R-sectoriality to a bounded H^∞-functional calculus in Corollary 6.7.4 and the sectoriality to a bounded H^∞-functional calculus in the B- and $\mathscr{B}$-cases Corollary 6.7.5. However, the required knowledge of interpolation with boundary conditions does not seem to be available in the literature at the moment.

Remark 6.7.7. The scales of weighted $\mathscr{B}$- and $\mathscr{F}$-spaces, the dual scales to the scales of weighted B- and F-spaces, naturally appear in duality theory. In [163] they were used to describe the adjoint operators for realizations of second order elliptic operators subject to the Dirichlet boundary condition in weighted B- and F-spaces (see [163, Remark 9.13]), which was an important ingredient in the application to the heat equation with multiplicative noise of Dirichlet type at the boundary in weighted L_p-spaces

[5] Here, we identify Case (a) and Case (b) from Section 6.6.1 with Case (A) and the Triebel-Lizorkin version of Case (B) from Section 6.7.1, respectively.

in [167] through the so-called Dirichlet map (see [163, Theorem 1.2]). The incorporation of the scales of weighted $\mathscr{B}$- and $\mathscr{F}$-spaces in Theorem 6.7.1 and Corollary 6.7.5 would allow us similarly to describe the adjoint of the operator A_B from Corollary 6.7.4, which could then be used to extend [167] to more general parabolic boundary value problems with multiplicative noise at the boundary.

6.A. A WEIGHTED VERSION OF A THEOREM DUE TO CLÉMENT AND PRÜSS

The following theorem is a weighted version of a result from [51] (see [126, Theorem 5.3.15]). For its statement we need some notation that we first introduce.

Let X be a Banach space. We write $\widehat{C_c^\infty}(\mathbb{R}^n;X) := \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^n;X)$ and $\widehat{L^1}(\mathbb{R}^n;X) := \mathscr{F}^{-1}L^1(\mathbb{R}^n;X)$. Then

$$L_{1,\mathrm{loc}}(\mathbb{R}^n;\mathscr{B}(X)) \times \widehat{C_c^\infty}(\mathbb{R}^n;X) \longrightarrow \widehat{L^1}(\mathbb{R}^n;X),\ (m,f) \mapsto \mathscr{F}^{-1}[m\hat{f}] =: T_m f.$$

For $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$ we define $\mathscr{M}L_p(\mathbb{R}^n,w;X)$ as the space of all $m \in L_{1,\mathrm{loc}}(\mathbb{R}^n;\mathscr{B}(X))$ for which T_m extends to a bounded linear operator on $L_p(\mathbb{R}^n,w;X)$, equipped with the norm

$$||m||_{\mathscr{M}L_p(\mathbb{R}^n,w;X)} := ||T_m||_{\mathscr{B}(L_p(\mathbb{R}^n,w;X))}.$$

Theorem 6.A.1. *Let X be a Banach space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$. For all $m \in \mathscr{M}L_p(\mathbb{R}^n,w;X)$ it holds that*

$$\{m(\xi) : \xi \text{ is a Lebesgue point of } m\}$$

is R-bounded with

$$||m||_{L_\infty(\mathbb{R}^n;\mathscr{B}(X))} \leq \mathscr{R}_p\left(\{m(\xi) : \xi \text{ is a Lebesgue point of } m\}\right) \lesssim_{p,w} ||m||_{\mathscr{M}L_p(\mathbb{R}^n,w;X)}.$$

Proof. This can be shown as in [126, Theorem 5.3.15]. Let us comment on some modifications that have to be made for the second estimate. Modifying the Hölder argument given there according to (6.13), the implicit constant $C_{p,w}$ of interest can be estimated by

$$C_{p,w} \leq \liminf_{\varepsilon\to 0} \varepsilon^d ||\phi(\varepsilon\,\cdot)||_{L_p(\mathbb{R}^n,w)} ||\psi(\varepsilon\,\cdot)||_{L_{p'}(\mathbb{R}^n,w'_p)},$$

where $\phi,\psi \in \mathscr{S}(\mathbb{R}^n)$ are such that $\hat{\phi}$, $\check{\psi}$ are compactly supported with the property that $\int \hat{\phi}\check{\psi}\,d\xi = 1$. By a change of variable,

$$\varepsilon^d ||\phi(\varepsilon\,\cdot)||_{L_p(\mathbb{R}^n,w)} ||\psi(\varepsilon\,\cdot)||_{L_{p'}(\mathbb{R}^n,w'_p)} = ||\phi||_{L_p(\mathbb{R}^n,w(\varepsilon\,\cdot))} ||\psi||_{L_{p'}(\mathbb{R}^n,w'_p(\varepsilon\,\cdot))}.$$

Since $\mathscr{S}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n,w)$ with norm estimate only depending on n, p and $[w]_{A_p}$ (as a consequence of [182, Lemma 4.5]) and since the A_p-characteristic is invariant under scaling, the desired result follows. □

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SUMMARY

The subject of this thesis is the maximal regularity problem for parabolic boundary value problems with inhomogeneous boundary conditions in the setting of weighted function spaces and related function space theoretic problems. This in particularly includes weighted L_q-L_p-maximal regularity but also weighted L_q-maximal regularity in weighted Triebel-Lizorkin spaces. The weights we consider are power weights in time and in space, and yield flexibility in the optimal regularity of the initial-boundary data and allow to avoid compatibility conditions at the boundary.

The first part of this thesis, Part I, consists of Chapters 2 and 3 and is completely devoted to harmonic analysis and function spaces.

In Chapter 2 we introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [119], which includes weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. The main result is an intersection representation, which in the special case of the classical Triebel-Lizorkin spaces yields an improvement of the well-known Fubini property. The motivation comes from the L_q-maximal regularity approach to parabolic boundary value problems with inhomogeneous boundary conditions in Part II, where weighted anisotropic mixed-norm Triebel-Lizorkin spaces occur as the optimal space of boundary data.

In Chapter 3 we study weighted Bessel potential spaces of tempered distributions taking values in UMD Banach spaces. The main result is a randomized difference norm characterization for such function spaces $H^s_p(\mathbb{R}^d, w; X)$ with $s > 0$, extending a classical square function difference norm characterization from the unweighted scalar-valued case due to Strichartz [230]. The main ingredients are R-boundedness results for Fourier multiplier operators, which are of independent interest. As an application of the randomized difference norm description we characterize the pointwise multiplier property of $1_{\mathbb{R}^d_+}$ on $H^s_p(\mathbb{R}^d, w; X)$.

In Chapter 4 we prove results on the complex interpolation of weighted Sobolev spaces of distributions taking values in UMD Banach spaces spaces on the half line with Dirichlet boundary condition. The weights that we consider are A_p-power weights, where p is the integrability parameter under consideration. The proof is based on the pointwise multiplier property of $1_{\mathbb{R}^d_+}$ on the corresponding weighted Bessel potential spaces $H^s_p(\mathbb{R}^d, w; X)$, of which we provide a new and simpler proof as well. We apply the results to characterize the fractional domain spaces of the first derivative operator on the half line.

The second part of this thesis, Part II, consists of Chapters 5 and 6 and is devoted to the study of elliptic and parabolic boundary value problems in weighted function spaces of Sobolev, Bessel potential, Besov and Triebel-Lizorkin type.

In Chapter 5 we study the Laplace operator subject to Dirichlet boundary conditions on a smooth domain in a weighted L_p-setting with power weights that fall outside the classical class of Muckenhoupt A_p-weights. We prove boundedness of the H^∞-calculus. Furthermore, we characterize the domain of the operator and derive several consequences on elliptic and parabolic regularity. In particular, we obtain a new maximal regularity result for the heat equation with rough inhomogeneous boundary data.

In Chapter 6 we consider infinite-dimensional systems of elliptic and parabolic boundary value problems with inhomogeneous boundary conditions under assumptions of Lopatisnkii-Shapiro type as considered by Denk, Hieber & Prüss [59, 61]. The main result provides a solution to the problem of weighted L_q-maximal regularity in weighted L_p-based UMD Banach space-valued Bessel potential and Triebel-Lizorkin spaces for the parabolic equations. Here the spatial weights, which are the same power weights as in Chapter 5, are restricted to the Muckenhoupt A_p-class in the Bessel potential case and to the Muckenhoupt A_∞-class in the Triebel-Lizorkin case. The use of scales of weighted Triebel-Lizorkin spaces enables us to treat rough inhomogeneous boundary data and also provides a quantitative smoothing effect for the solution on the interior of the domain. For the elliptic equations we furthermore obtain parameter-dependent estimates. The main technical ingredient is an analysis of parameter-dependent and anisotropic Poisson operators.

SAMENVATTING

Het onderwerp van dit proefschrift is het maximale regulariteitsprobleem voor randwaardeproblemen met inhomogene randvoorwaarden binnen het kader van gewogen functieruimten en gerelateerde functieruimtetheoretische problemen. Dit bevat in het bijzonder gewogen L_q-L_p-maximale regulariteit maar ook gewogen L_q-maximale regulariteit in gewogen Triebel-Lizorkin ruimten. De gewichten die we beschouwen zijn machtsgewichten in tijd en ruimte, en leveren flexibiliteit in de optimale regulariteit van de begin-rand data en laten het toe compatabiliteitsvoorwaarden op de rand te vermijden.

Het eerste deel van dit proefschrift, Part I, bestaat uit Chapters 2 en 3 en is volledig gewijd aan harmonische analyse en functieruimten.

In Chapter 2 introduceren we een nieuwe klasse van anisotrope vectorwaardige functieruimten in een axiomatisch raamwerk à la Hedberg&Netrusov [119], welk gewogen anisotrope gemixte-norm Besov and Triebel-Lizorkin ruimten bevat. Het hoofdresultaat is een doorsnederepresentatie, welke in het speciale geval van de klassike Triebel-Lizorkin ruimten een verbetering van bekende Fubini-eigenschap oplevert. De motivatie komt van de L_q-maximale regulariteitsbenadering van parabolische randwaardeproblemen met inhomogene randvoorwaarden in Part II, waar gewogen anisotrope gemixte-norm Triebel-Lizorkin ruimten opduiken als de optimal ruimte van randdata.

In Chapter 3 bestuderen we gewogen Bessel potentiaal ruimten van getempereerde distributies die waarden aannemen in OMV Banach ruimten. Het hoofdresultaat is een gerandomiseerde differentienorm karakterisatie voor zulke functieruimten $H^s_p(\mathbb{R}^d,w;X)$ met $s>0$, welk een uitbreiding is van een klassieke kwadraatfunctie differentienorm karakterisatie uit het ongewogen scalarwaardige geval van Strichartz [230]. De hoofdingrediënten zijn R-begrensdheidsresultaten voor Fourier vermenigvuldigingsoperatoren, welk van onafhankelijke interesse zijn. Als een toepassing gerandomiseerde differentienorm beschrijving karakteriseren we de puntsgewijze vermenigvuldigingseigenschap van $1_{\mathbb{R}^d_+}$ op $H^s_p(\mathbb{R}^d,w;X)$.

In Chapter 4 bewijzen we resultaten op het gebied van complexe interpolatie van gewogen Sobolev ruimten van distributies die waarden aannemen in OMV Banach ruimten op de reële halve met Dirichlet randvoorwaarden. De gewichten die we beschouwen zijn A_p-machtsgewichten, waar p de integreerbaarheidsparameter onder beschouwing is. Het bewijs is gebaseerd op de puntsgewijze vermenigvuldigingseigenschap van $1_{\mathbb{R}^d_+}$ op de bijbehorende gewogen Bessel potentiaal ruimten $H^s_p(\mathbb{R}^d,w;X)$, waarvan we tevens een nieuw en simpeler bewijs geven. We passen de resultaten toe om fractionele domeinruimten van de eerste afgeleide-operator op op de reële halve te karakteriseren.

Het tweede deel van dit proefschrift, Part II, bestaat uit Chapters 5 en 6 en is gewijd aan de studie van elliptische en parabolische randwaardeproblemen in gewogen functieruimten van Sobolev, Bessel potential, Besov en Triebel-Lizorkin type.

In Chapter 5 bestuderen we de Laplace operator onderworpen aan Dirichlet randvoorwaarden op een glad domein binnen een gewogen L_p-kader met machtsgewichten die buiten de klassike klasse van Muckenhoupt A_p-gewichten vallen. We bewijzen de begrensdheid van de H^∞-rekening. Verder karakterizeren we het domein van de operator en leiden verscheidene gevolgen af met betrekking tot elliptische en parabolische regulariteit. In het bijzonder verkrijgen we een nieuw maximaliteitsresultaat voor de warmtevergelijking met ruige inhomogene randdata.

In Chapter 6 beschouwen we oneindig-dimensionale systemen van elliptische en parabolische randwaardeproblemen met inhomogene randvoorwaarden onder aannames van Lopatisnkii-Shapiro type als beschouwd door Denk, Hieber & Prüss [59, 61]. Het hoofdresultaat verstrekt een oplossing van het probleem van gewogen L_q-maximale regulariteit in, op gewogen L_p-gebaseerde, OMV Banach ruimtewaardige Bessel potentiaal and Triebel-Lizorkin ruimten voor de parabolische vergelijkingen. Hier de ruimtelijke gewichten, welke dezelfde machtsgewichten zijn als in Chapter 5, zijn beperkt tot de Muckenhoupt A_p-klasse in het Bessel potentiaalgeval en tot de Muckenhoupt A_∞-klasse in het Triebel-Lizorkingeval. Het gebruik van schalen van gewogen Triebel-Lizorkin ruimten maakt het mogelijk ruige inhomogene randdata te behandelen en geeft ook een kwantitatief gladmaakeffect voor de oplossing op het inwendige van het domein. Voor de elliptische vergelijkingen verkrijgen we bovendien parameterafhankelijke afschattingen. Het voornaamste technische ingredient is een analyse van parameterafhankelijke en anisotrope Poisson operatoren.

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Delft, April 2019

Nick Lindemulder

CURRICULUM VITÆ

Nick LINDEMULDER

Nick Lindemulder was born on August 1, 1989, in Emmen, the Netherlands. He completed his secondary education in 2008 at Hondsrug College in Emmen. In the same year he started his studies in Econometrics at University of Groningen. He obtained his propaedeutic degree in 2009. From 2009 to 2012, he studied for his BSc degree in Mathematics at Utrecht University and wrote a small thesis ’Hamiltonians and Spectra’ under the supervision of dr. H. Hanßmann. He obtained his BSc degree cum laude in 2012. From 2012 to 2014, he studied for his MSc degree in Mathematical Sciences with specialization in Pure Analysis at Utrecht University and wrote a thesis ’Parabolic Initial-Boundary Value Problems with Inhomogeneous Data: A Maximal Weighted L^q-L^p-Regularity Approach’ under the supervision of prof. dr. ir. M.C. Veraar from Delft University of Technology and under the cosupervision of dr. H. Hanßmann and prof. dr. S.M. Verduyn Lunel. He obtained his MSc degree cum laude in 2014. In 2014 he started his PhD research under the supervision of prof. dr. ir. M.C. Veraar and under the cosupervision of prof. dr. J.M.A.M. van Neerven at Delft University of Technology. Part of this research was carried out during a four month visit to prof. dr. R. Denk at University of Konstanz, Germany.

List of Publications

Published/accepted papers

(P1) N. Lindemulder. Banach space-valued extensions of linear operators on L^∞. *Ordered structures and applications*, Trends Math., pages 281-306. Birkhauser/Springer, Cham, 2016.

(P2) N. Lindemulder. Difference norms for vector-valued Bessel potential spaces with an application to pointwise multipliers. *Journal of Functional Analysis*, 272(4):1435-1476, 2017. This paper corresponds to *Chapter 3* of this thesis.

(P3) N. Lindemulder, M. Meyries, and M.C. Veraar. Complex interpolation with dirichlet boundary conditions on the half line. *Mathematische Nachrichten*, 291(16):2435-2456. This paper corresponds to *Chapter 4* of this thesis.

(P4) N. Lindemulder, M.C. Veraar, and I. Yaroslavtsev. The UMD property for Musielak–Orlicz spaces. ArXiv e-prints (to appear in *Positivity and Noncommutative Analysis. Festschrift in honour of Ben de Pagter on the occasion of his 65th birthday*, Trends in Mathematics. Birkhauser, 2019), arXiv:1810.13362, October 2018.

(P5) N. Lindemulder. Maximal Regularity with Weights for Parabolic Problems with Inhomogeneous Boundary Data. ArXiv e-prints (to appear in *Journal of Evolution Equations*), arXiv:1702.02803, February 2017.

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(P8) N. Lindemulder. Second Order Operators Subject to Dirichlet Boundary Conditions in Weighted Triebel-Lizorkin Spaces: Parabolic Problems. ArXiv e-prints, arXiv:1812.05462, December 2018.

(P9) N. Lindemulder. An Intersection Representation for a Class of Anisotropic Vector-valued Function Spaces. ArXiv e-prints, arXiv:1903.02980, March 2019. This paper corresponds to *Chapter 2* of this thesis.

IN PREPARATION

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(P11) N. Lindemulder. Second Order Operators Subject to Dirichlet Boundary Conditions in Weighted Besov and Triebel-Lizorkin Spaces: Elliptic Problems, 2018.

(P12) N. Lindemulder and Veraar M.C. The Heat Equation with Multiplicative Noise of Dirichlet Type at the Boundary, 2019.