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THREE CONSECUTIVE APPROXIMATION COEFFICIENTS: ASYMPTOTIC FREQUENCIES IN SEMI-REGULAR CASES

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Abstract. Denote by p_n/q_n, $n = 1, 2, 3, \dots$, the sequence of continued fraction convergents of a real irrational number x. Define the sequence of approximation coefficients by $\theta_n(x) := q_n |q_n x - p_n|$, $n = 1, 2, 3, \dots$. In the case of regular continued fractions the six possible patterns of three consecutive approximation coefficients, such as $\theta_{n-1} < \theta_n < \theta_{n+1}$, occur for almost all x with only two different asymptotic frequencies. In this paper it is shown how these asymptotic frequencies can be determined for two other semi-regular cases. It appears that the *optimal continued fraction* has a similar distribution of only two asymptotic frequencies, albeit with different values. The six different values that are found in the case of the *nearest integer continued fraction* will show to be closely related to those of the optimal continued fraction.

1. Introduction. In this paper we are concerned with a question that arises in the study of approximating real irrational numbers using continued fractions. Generally, we define a *continued fraction* of a real number x as a finite or infinite fraction

$$a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cfrac{\varepsilon_3}{a_3 + \ddots}}} \tag{1}$$

In this expression we have $\varepsilon_n = \pm 1, n \geq 1$, $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}, n \geq 1$. In the following we will use the more convenient notation $[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \varepsilon_3 a_3, \dots]$ for a continued fraction. A finite or infinite continued fraction is called a *semi-regular continued fraction* (SCRF) when $a_n \geq 1, n \geq 1$; $\varepsilon_{n+1} + a_n \geq 1, n \geq 1$, and, in the infinite case, $\varepsilon_{n+1} + a_n \geq 2$ infinitely often; see for instance [P] or [K1]. In this paper we will only deal with infinite continued fractions.

The SRCFs have been studied extensively (e.g. [B, K1]), as have their *approximation coefficients*, defined by

$$\theta_n(x) := q_n^2 \left| x - \frac{p_n}{q_n} \right|, \quad n = 1, 2, 3, \dots,$$

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where x is a real irrational with continued fraction expansion (1) and $p_n/q_n,\ n = 1, 2, 3, \ldots,$ is the corresponding sequence of convergents, obtained by truncation of the infinite continued fraction (1). In the sequel we will omit the suffix '(x)' behind θ_n.

For the *regular continued fraction* (RCF) expansion, being the SRCF with $\varepsilon_n = +1$ for all $n \geq 1$, Legendre showed in 1798 ([L]) that if $p, q \in \mathbb{Z}$, $q > 0$, $\gcd(p, q) = 1$, and

$$\left| x - \frac{p}{q} \right| < \frac{1}{2} \frac{1}{q^2} ,$$

then there exists an $n \geq 1$ for which $p_n/q_n = p/q$, where p_n/q_n is the nth RCF-convergent of x. In 1895, Vahlen ([V]) showed that for all irrational x and all $n \geq 2$,

$$\min\{\theta_{n-1}, \theta_n\} < \frac{1}{2} ,$$

while Borel ([Bor]) showed in 1905 that

$$\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} < \frac{1}{\sqrt{5}} .$$

In the course of the 20th century several authors sharpened Borel's result:

$$\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} < \frac{1}{\sqrt{a_{n+1}^2 + 4}} ;$$

see e.g. [BM]. In fact, J. Tong ([T1]) showed in 1983 that one also has the *converse property*:

$$\max\{\theta_{n-1}, \theta_n, \theta_{n+1}\} > \frac{1}{\sqrt{a_{n+1}^2 + 4}} .$$

For the *optimal continued fraction* (OCF) expansion, which we will discuss in Section 3, one has even more impressive Diophantine properties such as:

$$\min\{\theta_{n-1}, \theta_n\} < \frac{1}{\sqrt{5}} .$$

Unfortunately, this is not the case for the *nearest integer continued fraction* (NICF) expansion, which we will discuss in Section 4. In 1995, J. Tong showed in [T2] that for all irrational x, for all $n \geq 2$ and for all $k \geq 1$ one has that:

$$\min\{\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}\} < \left(1 + \left((3 - \sqrt{5})/2 \right)^{2k+3} \right) / \sqrt{5} .$$

Note that

$$\lim_{k \to \infty} \left(1 + \left((3 - \sqrt{5})/2 \right)^{2k+3} \right) / \sqrt{5} = 1/\sqrt{5} .$$

In various papers the distribution for almost all x for the sequences $(\theta_n)_{n \geq 1}$ and $((\theta_{n-1}, \theta_n))_{n \geq 2}$ has been studied for the RCF, OCF, NICF, and several other continued fraction algorithms; see e.g. [BJW, BK1]. In this paper, we will focus on the asymptotic frequency of triplets $(\theta_{n-1}, \theta_n, \theta_{n+1})$.

In [JJ] the six patterns

$$\mathcal{A}: \theta_{n-1} < \theta_n < \theta_{n+1}\,, \qquad \mathcal{B}: \theta_{n-1} < \theta_{n+1} < \theta_n\,, \qquad \mathcal{C}: \theta_n < \theta_{n-1} < \theta_{n+1}\,,$$
$$\mathcal{D}: \theta_n < \theta_{n+1} < \theta_{n-1}\,, \qquad \mathcal{E}: \theta_{n+1} < \theta_{n-1} < \theta_n\,, \qquad \mathcal{F}: \theta_{n+1} < \theta_n < \theta_{n-1}$$

are studied in the case of the *regular continued fraction* (RCF), being the SRCF with $\varepsilon_n = 1$ for all $n \geq 1$. Defining the *asymptotic frequency* (AF) of $\mathcal{A}$ as

$$AF(\mathcal{A}) = \lim_{N\to\infty} \frac{1}{N} \#\{n \in \mathbb{N} \mid 2 \leq n \leq N, \theta_{n-1} < \theta_n < \theta_{n+1}\}$$

and the asymptotic frequencies of the other patterns likewise, the following result was derived:

THEOREM 1. *Define the constant* ρ *by*

$$\rho := \prod_{a=1}^{\infty} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{a^2}} \right) = 2.82698 \cdots .$$

Then for almost all x[1]

$$AF(\mathcal{A}) = AF(\mathcal{F}) = \frac{1}{2} - \frac{1}{\log 2} \cdot \log \frac{\sinh \pi}{\pi \rho} = 0.12109 \cdots$$

and

$$AF(\mathcal{B}) = AF(\mathcal{C}) = AF(\mathcal{D}) = AF(\mathcal{E}) = \frac{1}{2 \log 2} \cdot \log \frac{\sinh \pi}{\pi \rho} = 0.18945 \cdots .$$

In this paper we will investigate these asymptotic frequencies for two other SRCFs: the *optimal continued fraction* (OCF) and the *nearest integer continued fraction* (NICF). In general, we take the same approach as we used for the regular RCF, which involves the following definitions and notations. We write

$$t_n = [0; \varepsilon_{n+1}a_{n+1}, \varepsilon_{n+2}a_{n+2}, \dots],\ n \geq 1$$

and

$$v_n = \frac{q_{n-1}}{q_n} = [0; a_n, \varepsilon_n a_{n-1}, \dots, \varepsilon_2 a_1],\ n \geq 1\,,$$

which are the 'future' respectively the 'past' of x 'at time n'. We have the following relations, which can be found in, for instance, [JK, p. 303]:

$$\theta_{n-1} = \frac{v_n}{1 + t_n v_n},\ n \geq 1\,, \tag{2}$$

$$\theta_n = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n},\ n \geq 1\,, \tag{3}$$

and

$$\theta_{n+1} = \varepsilon_{n+2}\big(\varepsilon_{n+1}\theta_{n-1} + a_{n+1}\sqrt{1 - 4\varepsilon_{n+1}\theta_{n-1}\theta_n} - a_{n+1}^2\theta_n\big),\ n \geq 1\,. \tag{4}$$

We see that we can express both θ_{n-1}, θ_n and θ_{n+1} in terms of t_n and v_n.

[1] All *almost all* statements in this paper are with respect to Lebesgue meausure λ.

In [JJ] the approach for the RCF is based upon the (*one-sided*) *shift operator* (or *Gauss map*) $T : \Omega \to \Omega$, defined by

$$T(t) := \frac{1}{t} - \left\lfloor \frac{1}{t} \right\rfloor,$$

with $\Omega := [0, 1] \setminus \mathbb{Q}$, but especially on the *natural extension* of T, with the *two-sided shift operator* $\mathcal{T} : \bar{\Omega} \to \bar{\Omega}$, defined by

$$\mathcal{T}(t, v) := \left(T(t), \frac{1}{v + a_1} \right) = \left(\frac{1}{t} - a_1, \frac{1}{v + a_1} \right), \tag{5}$$

with $\bar{\Omega} := \Omega \times [0, 1]$.

In particular, we can show that

$$(t_n, v_n) = \mathcal{T}^n(t, 0), \quad n \geq 0.$$

Now the following theorem of Jager ([J]) is used:

THEOREM 2. *The two-dimensional sequence* (t_n, v_n), $n = 1, 2, \ldots,$ *is for almost all irrational* x *distributed over the unit square in the* (t, v)*-plane according to the density function* d *defined by*

$$d(t, v) := \frac{1}{\log 2} \cdot \frac{1}{(1 + tv)^2}.$$

For a proof of this theorem, see [DK], Lemma 5.3.11.

From Theorem 2 we derive that for every Borel measurable set $A \subseteq \bar{\Omega}$

$$AF(A) = \iint_A \frac{1}{\log 2} \cdot \frac{1}{(1 + tv)^2} \, \mathrm{d}t \, \mathrm{d}v.$$

For convenience we will put aside *normalizing factors* such as $\frac{1}{\log 2}$ in our calculations and will mostly use on the appropriate space the measure m, defined by

$$m(A) := \iint_A \frac{1}{(1 + tv)^2} \, \mathrm{d}t \, \mathrm{d}v, \tag{6}$$

processing the normalizing factor in the final stage of the calculations.

In [JJ] the (t, v)-plane is divided in vertical strips

$$R_a := \left(\frac{1}{a + 1}, \frac{1}{a} \right) \times [0, 1], \quad a = 1, 2, \ldots,$$

in each of which the measures of all areas corresponding to the six patterns are computed, by applying Theorem 2. The asymptotic frequencies are found by summing all measures over a. In the following sections we will show how to adopt this approach to the OCF and the NICF, where the situation is more complicated. For convenience we will identify a pattern $\mathcal{P} \in \{\mathcal{A}, \ldots, \mathcal{F}\}$, with the region corresponding to this pattern. Throughout this paper, we will use $g := \frac{1}{2}\sqrt{5} - \frac{1}{2} = 0.6180\cdots$ and $G := \frac{1}{2}\sqrt{5} + \frac{1}{2} = 1.6180\cdots$ as abbreviations of the two golden means. Note that $G = g + 1$ and that $g = \frac{1}{G}$.

2. The asymptotic frequencies in the case of the Optimal Continued Fraction. As remarked above, we obtain the convergents p_n/q_n, $n = 1, 2, 3, \ldots$, by truncating the infinite continued fraction (1) expansion of a real irrational number x, so as to obtain good approximations of x. The approximation coefficients θ_n, $n = 1, 2, 3, \ldots$, provide a way of measuring the quality of the approximants. In [B], Wieb Bosma introduced the optimal continued fraction as a continued fraction that is both *fastest* (i.e. having an expansion for which the growth rate of the denominators is *maximal*) and *closest* (i.e. having expansions for which $\sup\{\theta_k : \theta_k = q_k|q_k x - p_k|\}$ is *minimal*).

Optimal as this fraction may be as to its approximating qualities, in [B, BK1] it is shown that both the subset of $\mathbb{R}^2$, which we denote by Υ_O, and the two-sided shift operator $\mathcal{T}_O : \Upsilon_O \to \Upsilon_O$ of the ergodic system underlying the OCF are less accessible than those of the RCF:

$$\Upsilon_O = \left\{(t, v) \in (-1, 1) \times (-1, 1) : v \leq \min\left(\frac{2t+1}{t+1}, \frac{t+1}{t+2}\right) \text{ and } v \geq \max\left(0, \frac{2t-1}{1-t}\right)\right\},$$

see also Figure 1, and

$$\mathcal{T}_O(t, v) := \left(\left|\frac{1}{t}\right| - a(t, v), \frac{1}{a(t, v) + \operatorname{sign}(t)v}\right)$$

where

$$a(t, v) := \left\lfloor \left|\frac{1}{t}\right| + \frac{\left\lfloor\left|\frac{1}{t}\right|\right\rfloor + \operatorname{sign}(t)v}{2\left(\left\lfloor\left|\frac{1}{t}\right|\right\rfloor + \operatorname{sign}(t)v\right) + 1} \right\rfloor. \tag{7}$$

It is not hard to see that $\mathcal{T}_O$ works on Υ_O in way similar to $\mathcal{T}$ on $\bar{\Omega}$ in the RCF case:

$$\mathcal{T}_O(t, v) = \left(\frac{\varepsilon_1}{t} - a_1, \frac{1}{\varepsilon_1 v + a_1}\right), \quad \text{and} \quad (t_n, v_n) = \mathcal{T}_O^n(t, 0), \quad n \geq 0.$$

In [BK1] it is shown that $(\Upsilon_O, \mathscr{B}_{\Upsilon_O}, \bar{\mu}_{\Upsilon_O}, \mathcal{T}_O)$ forms an ergodic system, with $\mathscr{B}_{\Upsilon_O}$ the collection of Borel subsets of Υ_O and $\bar{\mu}_{\Upsilon_O}$ the measure with density function

$$d_O(t, v) := \frac{1}{\log G} \cdot \frac{1}{(1 + tv)^2}, \quad \text{for } (t, v) \in \Upsilon_O. \tag{8}$$

In particular, we have that (apart from sets with Lebesgue measure 0) $\mathcal{T}_O : \Upsilon_O \to \Upsilon_O$ is bijective and that $\bar{\mu}_{\Upsilon_O}$ is invariant under $\mathcal{T}_O$. In [BK1] the following version of Theorem 2 was obtained:

THEOREM 3. *The two-dimensional sequence* (t_n, v_n), $n = 1, 2, \ldots$, *is for almost all irrational* x *distributed over* Υ_O *according to the density function* d_O *in* (8).

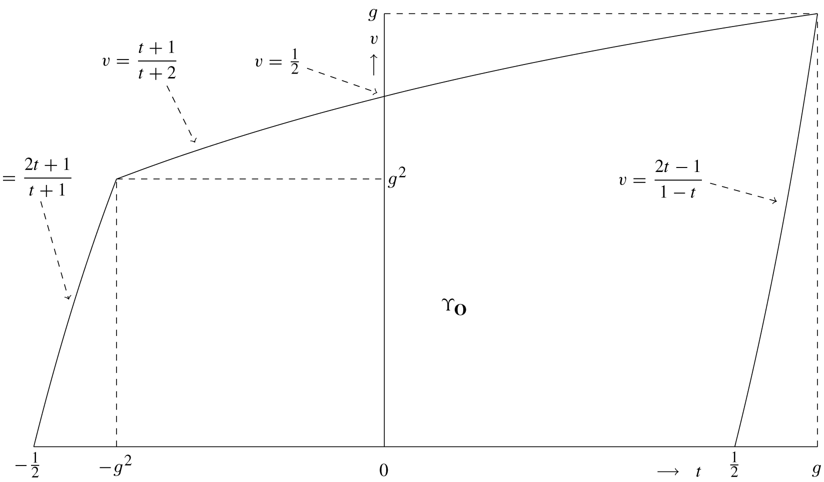

FIGURE 1. The domain of the OCF.

For more detailed information about the metric properties of the OCF, see for instance [BK1].

From Theorem 3 we derive that for every Borel measurable set $A \subseteq \Upsilon_O$

$$AF_O(A) = \iint_A \frac{1}{\log G} \cdot \frac{1}{(1+tv)^2}\, \mathrm{d}t\, \mathrm{d}v\,. \tag{9}$$

The most important obstacle to following the approach taken in the case of the RCF is that a 'stripwise' computation (with $1/a_{n+1} < t_n < 1/a_n$, $n = 1, 2, \ldots$) is not possible, due to the curved boundary of the domain of the natural extension of the OCF; see Figure 1. In view of (7), an obvious solution of this problem is regarding curved regions for every $a_{n+1} = 2, 3, \ldots$. For $(t, v) \in \Upsilon_O$, the sign of t is obvious. However, it is not easy to find the regions of Υ_O where the value of the digit $a(t, v)$ is fixed using (7). We first will show that, apart from sets of Lebesgue measure 0, for every $(t, v) \in \Upsilon_O$ a unique integer $a \geq 2$ exists for which

$$\mathcal{T}_O(t, v) = \left(\frac{\varepsilon}{t} - a, \frac{1}{a + \varepsilon v}\right) =: (T, V) \in \Upsilon_O\,,$$

where $\varepsilon = \text{sign}(t)$. We first consider the points (t, v) that are sent under $\mathcal{T}_O$ to the boundary of Υ_O. Let $(T, V) \in \partial(\Upsilon_O)$, then we have the following three cases:

(1) (T, V) satisfies $V = \frac{2T+1}{1+T}$. In this case we obviously have that

$$(\alpha, \beta) := \left(\frac{\varepsilon_1}{t} - a - \mathbf{1}, \frac{1}{a + \mathbf{1} + \varepsilon v}\right) \notin \Upsilon_O\,,$$

since $T - \mathbf{1} = \frac{\varepsilon_1}{t} - a - \mathbf{1} < -\frac{1}{2}$. Now consider the point (α, β), given by

$$(\alpha, \beta) := \left(\frac{\varepsilon_1}{t} - a + \mathbf{1}, \frac{1}{a - \mathbf{1} + \varepsilon v}\right)\,.$$

In this case we have

$$\beta = \frac{1}{a+\varepsilon v-1} = \frac{1}{\frac{1}{V}-1} = \frac{2\alpha-1}{1-\alpha};$$

i.e. (α,β) is on one of the other boundary curves of Υ_O. We conclude that $(t,v)\in\Upsilon_O$ was on the boundary of the regions where the digit is either equal to a or to $a-1$.

(2) (T,V) satisfies $V=\frac{2T-1}{1-T}$. In this case we obviously have that

$$(\alpha,\beta) := \left(\frac{\varepsilon_1}{t} - a+\mathbf{1}, \frac{1}{a-\mathbf{1}+\varepsilon v}\right) \notin \Upsilon_O\,,$$

since $T+\mathbf{1}=\frac{\varepsilon_1}{t}-a+\mathbf{1}>\frac{3}{2}>g$. Now consider the point (α,β), given by

$$(\alpha,\beta) := \left(\frac{\varepsilon_1}{t} - a-\mathbf{1}, \frac{1}{a+\mathbf{1}+\varepsilon v}\right).$$

In this case we have

$$\beta = \frac{1}{a+1+\varepsilon v} = \frac{1}{\frac{1}{V}+1} = \frac{2T-1}{T} = \frac{2\alpha+1}{1+\alpha};$$

i.e. (α,β) is on one of the other boundary curves of Υ_O. Again we conclude that $(t,v)\in\Upsilon_O$ was on the boundary of the regions where the digit is either equal to a or to $a+1$.

(3) (T,V) satisfies $V=\frac{1+T}{2+T}$. In this case we obviously have that

$$(\alpha,\beta) := \left(\frac{\varepsilon_1}{t} - a+\mathbf{1}, \frac{1}{a-\mathbf{1}+\varepsilon v}\right) \notin \Upsilon_O\,,$$

since $T+1\geq g$. Now consider the point (α,β), given by

$$(\alpha,\beta) := \left(\frac{\varepsilon_1}{t} - a-\mathbf{1}, \frac{1}{a+\mathbf{1}+\varepsilon v}\right).$$

In this case we have

$$\beta = \frac{1}{a+1+\varepsilon v} = \frac{1}{\frac{1}{V}+1} = \frac{2+T}{3+2T} = \frac{\alpha+3}{2\alpha+5} \notin \Upsilon_O\,.$$

In this case the digit a was unique.

Now let $(t,v)\in\Upsilon_O$ be such, that $\mathcal{T}_O(t,v)\in \mathrm{Int}(\Upsilon_O)$ (here $\mathrm{Int}(S)$ denotes the interior of the set S). Then from the above it follows that we must have that

$$(\alpha,\beta) := \left(\frac{\varepsilon_1}{t} - a \pm 1, \frac{1}{a\mp 1+\varepsilon v}\right) \notin \Upsilon_O\,,$$

so we must have that $a=a(t,v)$, i.e.

$$a = \left\lfloor \left|\frac{1}{t}\right| + \frac{\left\lfloor\left|\frac{1}{t}\right|\right\rfloor + \mathrm{sign}(t)v}{2\left(\left\lfloor\left|\frac{1}{t}\right|\right\rfloor + \mathrm{sign}(t)v\right)+1} \right\rfloor.$$

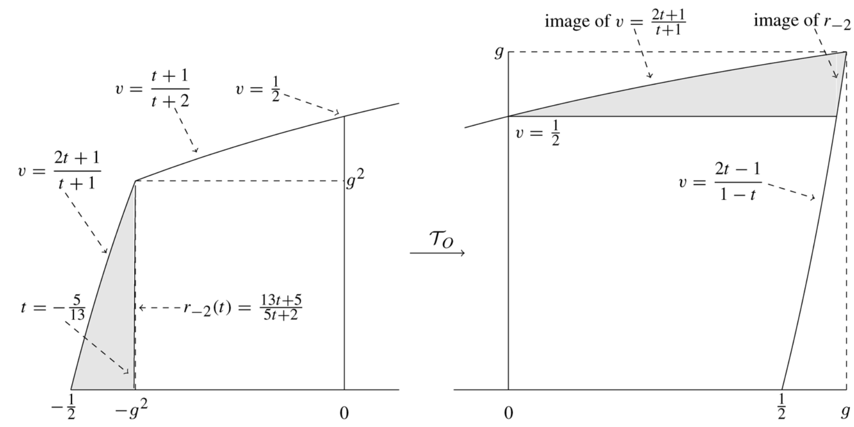

FIGURE 2. The map of the leftmost corner of Υ_O applying $\mathcal{T}_O$.

In the regular case, the value of a_{n+1} depends on t_n only, but in the optimal case it depends on both t_n and v_n. We want to know how to determine the curves between which a_{n+1} is constant, given t_n and v_n. For convenience, we will generally omit the indices n for t and v and $n+1$ for a in what follows. We start in the leftmost corner of Υ_O, where $a = 2$, $\varepsilon_{n+1} = -1$ and $\varepsilon_{n+2} = +1$. So

$$\mathcal{T}_O(t, v) = \left(\frac{-1}{t} - 2, \frac{1}{2-v}\right) \quad (a = 2, \varepsilon_{n+1} = -1, \varepsilon_{n+2} = +1)\,. \tag{10}$$

The left boundary is given by $\left(t, \frac{2t+1}{t+1}\right)$, for t between $-\frac{1}{2}$ and $-g^2$, which $\mathcal{T}_O$ maps to the curve $(T, V) = \left(\frac{-1}{t} - 2, \frac{1}{2-\frac{2t+1}{t+1}}\right) = \left(\frac{-1}{t} - 2, t+1\right)$, which we can write as $\left(T, \frac{T+1}{T+2}\right)$, for T between 0 and g. The horizontal line segment with v-coordinate 0 is mapped to the horizontal line segment with V-coordinate $\frac{1}{2}$. We now determine the right boundary, denoted by $r_{-2} = r_{-2}(t)$, such that r_{-2} is mapped to the upper right boundary of Υ_O. Applying (10), we want to be able to write $\left(\frac{-1}{t} - 2, \frac{1}{2-r_{-2}(t)}\right)$ as $\left(T, \frac{2T-1}{1-T}\right)$. A straightforward calculation yields $r_{-2}(t) = \frac{13t+5}{5t+2}$ (see Figure 2).

This procedure is easily copied to the rightmost side of Υ_O. This time we have $a = 2$, $\varepsilon_{n+1} = \varepsilon_{n+2} = +1$. Now

$$\mathcal{T}_O(t, v) = \left(\frac{1}{t} - 2, \frac{1}{2+v}\right) \quad (a = 2, \varepsilon_{n+1} = \varepsilon_{n+2} = 1)\,. \tag{11}$$

The right boundary is given by $\left(t, \frac{2t-1}{1-t}\right)$, for t between $\frac{1}{2}$ and g, which $\mathcal{T}_O$ maps to $\left(T, \frac{T+1}{T+2}\right)$, for T between $-g^2$ and 0. The upper boundary is part of $\left(t, \frac{t+1}{t+2}\right)$, its rightmost point being (g, g), which is mapped to $\left(T, \frac{2T+5}{5T+13}\right)$, with leftmost point on $T = -g^2$. We now determine

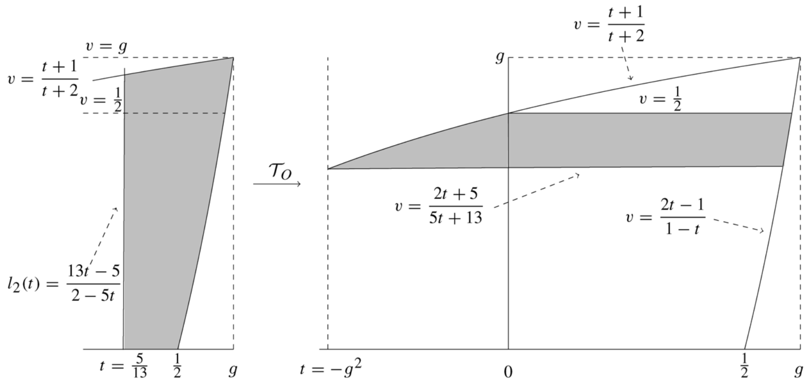

FIGURE 3. The map of the rightmost strip applying $\mathcal{T}_O$.

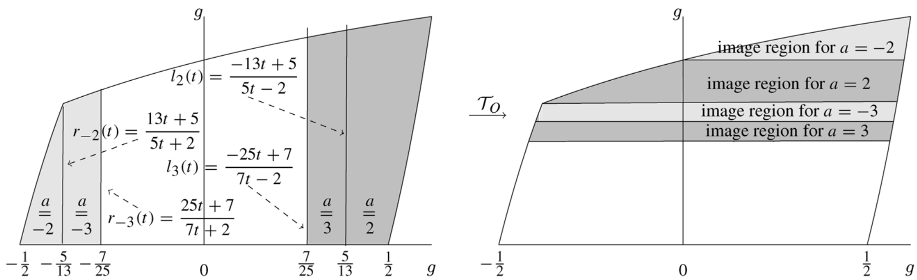

FIGURE 4. The alternating character of the map $\mathcal{T}_O$.

the left boundary, denoted by $l_2 = l_2(t)$, such that l_2 is mapped to $\left(T, \frac{2T-1}{1-T}\right)$. Applying (11), we want to be able to write $\left(\frac{1}{t} - 2, \frac{1}{2+l_2(t)}\right)$ as $\left(T, \frac{2T-1}{1-T}\right)$. We find $l_2(t) = \frac{13t-5}{2-5t}$; see Figure 3.

Proceeding similarly, we establish formulas for all combinations of $a = 3, 4, \ldots, \varepsilon_{n+1}$ and ε_{n+2}. We remark that the boundary between two regions with equal a and ε_{n+1} are separated by the line $t = \frac{\varepsilon_{n+1}}{a}$, where

$$r_{-a}(t) = \frac{(2a^2 + 2a + 1)t + 2a + 1}{(2a + 1)t + 2}$$

and

$$l_a(t) = \frac{-(2a^2 + 2a + 1)t + 2a + 1}{(2a + 1)t - 2}.$$

We conclude that $\mathcal{T}_O$ maps vertical regions from the left and the right side of Υ_O alternately to horizontal regions from the top of Υ_O downwards; see Figure 4.

In Figure 4 we have not yet processed the value of ε_{n+2}, which is indispensable for determining the six patterns. In Figure 5, confining ourselves to the leftmost part of Υ_O, we

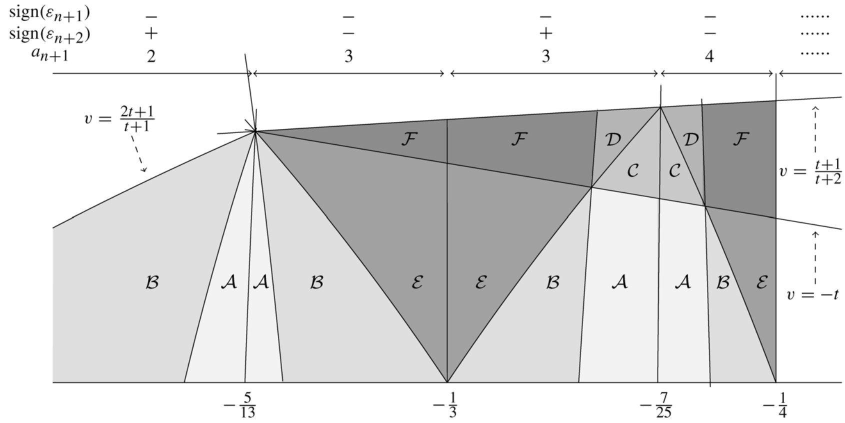

FIGURE 5. The six patterns for $\varepsilon_{n+1} = -1$.

show how the six patterns are spread out over Υ_O, for $a = 2, 3, \ldots$. We have filled the regions with different shades of grey, such that pattern $\mathcal{A}$ has the lightest shade and $\mathcal{F}$ has the darkest. Now that we have established a way of dividing Υ_O in regions where a, ε_{n+1} and ε_{n+2} are constant, we will show how we compute the measure of all regions. From the relations (2), (3) and (4) we derive for each of the four possible ordered sign tuples $(\varepsilon_{n+1}, \varepsilon_{n+2})$ the three curves that establish the six patterns. In each strip, that is, for every $a \geq 2$, we will now draw the curves that divide the strip in regions that correspond with the patterns $\mathcal{A}$ through $\mathcal{F}$, for which we will use the table on the next page. Recall that for convenience we use $t := t_n$, $v := v_n$ and $a := a_{n+1}$. Finally, in Figure 6 we have a generic situation for the patterns: we know the values of ε_{n+1} and ε_{n+2} (which in Figure 6 is -1 for both of them) and all patterns actually occur, which is not the case in the leftmost and the rightmost regions. In Figure 6 we have indicated the formulas belonging to the curves drawn and some noteworthy values of t. For convenience, we have omitted the coordinates of most intersection points, which are a bit lenghty in some cases. For instance, the t-coordinate of the intersection of $v = \frac{(2a^2-2a+1)t+2a-1}{(2a-1)t+2}$ (which is actually r_{a-1}) and $v = -t$ is

$$\frac{\sqrt{4a^4 - 8a^3 + 4a + 5} - (2a^2 - 2a + 3)}{4a - 2}. \tag{12}$$

The calculation of the measure of areas such as $\mathcal{C}_a$ involves computing the sum of two double integrals, the limits of which are expressions such as (12). Computing the measures of all pattern regions for all four cases would obviously be very tedious and demanding, and therefore it is convenient that several areas prove to have the same measure. As in [JJ], we use a composed operator, which in the case of $\varepsilon_{n+1} = -1$ is

$$\mathcal{S}_O^- := \mathcal{R}^- \mathcal{T}_O\,,$$

$(\varepsilon_{n+1}, \varepsilon_{n+2})$	$(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\boldsymbol{\theta_{n-1} = \theta_n}$	$v = -t$	$v = -t$	$v = t$	$v = t$
$\boldsymbol{\theta_{n-1} = \theta_{n+1}}$	$v = a + \frac{1}{t}$	$v = \frac{a^2t+a}{at+2}$	$v = \frac{a^2t-a}{-at+2}$	$v = -a + \frac{1}{t}$
$\boldsymbol{\theta_n = \theta_{n+1}}$	$v = \frac{(a^2-1)t+a}{at+1}$	$v = \frac{(a^2+1)t+a}{at+1}$	$v = \frac{(-a^2+1)t+a}{at-1}$	$v = \frac{(a^2+1)t-a}{-at+1}$
$\boldsymbol{\theta_{n-1} = \theta_n = \theta_{n+1}}$	$t = \frac{-a+\sqrt{a^2-4}}{2}$	$t = \frac{-a^2-2+\sqrt{a^4+4}}{2a}$	$t = \frac{2-a^2+\sqrt{a^4+4}}{2a}$	$t = \frac{-a+\sqrt{a^2+4}}{2}$

TABLE 1. The curves and their intersection per sign tuple.

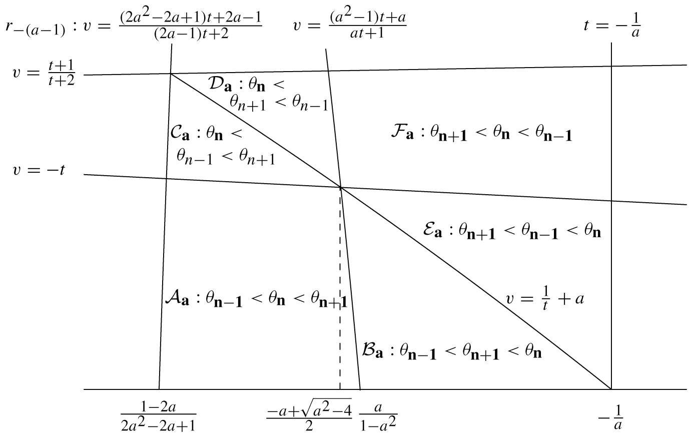

FIGURE 6. The six regions for $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$.

$\mathcal{R}^-$ being the reflection

$$\mathcal{R}^-(t, v) = (-v, -t)\,.$$

This operator $\mathcal{S}_O^-$ is an involution that is measure-preserving with respect to the measure m in (6). We will show how $\mathcal{S}_O^-$ works on the regions shown in Figure 6, where $\varepsilon_{n+2} = -1$ holds as well. We have (leaving the computations to the reader)

$$\begin{cases} \mathcal{S}_O^-\{(t,v) : v = \frac{1}{t} + a)\} = \{(t,v) : v = \frac{1}{t} + a)\}\,; \\ \mathcal{S}_O^- r_{-(a-1)} = \{(t,v) : v = \frac{t+1}{t+2}\}\,; \\ \mathcal{S}_O^-\{(t,v) : v = 0\} = \{(t,v) : t = -\frac{1}{a}\}\,; \\ \mathcal{S}_O^-\{(t,v) : v = -t\} = \{(t,v) : t = \frac{(a^2-1)t+a}{at+1}\}\,. \end{cases} \tag{13}$$

For $\mathcal{X} \in \{\mathcal{A}, \ldots, \mathcal{F}\}$, we set $\mathcal{X}_a^{\varepsilon_1/\varepsilon_2} = \{(x, y) \in \mathcal{X} \mid a_1(x) = a, \varepsilon_1(x) = \varepsilon_1, \varepsilon_2(x) = \varepsilon_2\}$. Now, using (13), we easily derive the following (while $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$):

$$\begin{cases} m(\mathcal{A}_a^{-/-}) = m(\mathcal{F}_a^{-/-})\,; \\ m(\mathcal{B}_a^{-/-}) = m(\mathcal{E}_a^{-/-})\,; \\ m(\mathcal{C}_a^{-/-}) = m(\mathcal{D}_a^{-/-})\,; \\ m(\mathcal{A}_a^{-/-} \cup \mathcal{B}_a^{-/-} \cup \mathcal{C}_a^{-/-}) = m(\mathcal{D}_a^{-/-} \cup \mathcal{E}_a^{-/-} \cup \mathcal{F}_a^{-/-})\,. \end{cases} \tag{14}$$

This is exactly what was found in the case of the RCF. We note, however, that at this place we are only dealing with the situation $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$, while in the case of the RCF one always has $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$. Still, we can confine ourselves to computing three relatively easy measures, say $m(\mathcal{C}_a^{-/-})$, $m(\mathcal{E}_a^{-/-})$ and $m(\mathcal{D}_a^{-/-} \cup \mathcal{E}_a^{-/-} \cup \mathcal{F}_a^{-/-})$.

We have (in the case $\varepsilon_{n+1} = \varepsilon_{n+2} = -1,\ a \geq 4$)

$$m(\mathcal{C}_a^{-/-}) = \int_{\frac{\sqrt{4a^4-8a^3+4a+5}-(2a^2-2a+3)}{4a-2}}^{\frac{-a+\sqrt{a^2-2a+2}}{a-1}} \int_{-t}^{\frac{(2a^2-2a+1)t+2a-1}{(2a-1)t+2}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} + \int_{\frac{-a+\sqrt{a^2-2a+2}}{a-1}}^{\frac{-a+\sqrt{a^2-4}}{2}} \int_{-t}^{\frac{1}{t}+a} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2},$$

which is

$$\frac{1}{2}\Bigg(\log \frac{\sqrt{4a^4-8a^3+4a+5}+2a^2-2a-1}{2} + \log \frac{a-\sqrt{a^2-4}}{2} + \log\left(\sqrt{a^2-2a+2}-a+1\right)\Bigg)$$

and can be written as

$$\frac{1}{2} \log \frac{\sqrt{(2a^2-2a-1)^2+4}+2a^2-2a-1}{(a+\sqrt{a^2-4})\left(\sqrt{(a-1)^2+1}+a-1\right)}\,.$$

Then,

$$m(\mathcal{E}_a^{-/-}) = \int_{\frac{-a+\sqrt{a^2-4}}{2}}^{\frac{-1}{a}} \int_{\frac{1}{t}+a}^{-t} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} = \frac{1}{2}\log\frac{a-\sqrt{a^2-4}}{2} + \frac{1}{2}\log\frac{a^2-1}{a}\,.$$

Finally,

$$\begin{aligned} m(\mathcal{D}_a^{-/-} \cup \mathcal{E}_a^{-/-} \cup \mathcal{F}_a^{-/-}) &= \int_{\frac{-a+\sqrt{a^2-2a+2}}{a-1}}^{\frac{-1}{a}} \int_{\frac{1}{t}+a}^{\frac{t+1}{t+2}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} \\ &= \frac{1}{2}\log\frac{2a^2-2a+1}{a} + \frac{1}{2}\log\left(\sqrt{(a-1)^2+1}-(a-1)\right). \end{aligned}$$

Applying (14), we find for $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$ and $a \geq 4$:

$$(15)\qquad \begin{cases} m(\mathcal{A}_a^{-/-}) = m(\mathcal{F}_a^{-/-}) = \frac{1}{2}\log \frac{(2a^2-2a+1)\left(a\sqrt{a^2-4}+a^2-2\right)}{(a^2-1)\left(\sqrt{(2a^2-2a-1)^2+4}+(2a^2-2a-1)\right)}\,; \\ m(\mathcal{B}_a^{-/-}) = m(\mathcal{E}_a^{-/-}) = \frac{1}{2}\log \frac{a-\sqrt{a^2-4}}{2} + \frac{1}{2}\log \frac{a^2-1}{a}\,; \\ m(\mathcal{C}_a^{-/-}) = m(\mathcal{D}_a^{-/-}) = \frac{1}{2}\log \frac{\sqrt{(2a^2-2a-1)^2+4}+2a^2-2a-1}{(a+\sqrt{a^2-4})\left(\sqrt{(a-1)^2+1}+a-1\right)}\,. \end{cases}$$

We remark that although for $a = 3$ patterns $\mathcal{C}_a^{-/-}$ and $\mathcal{D}_a^{-/-}$ do not occur, the formula in (15) still holds, for it gives $m(\mathcal{C}_a^{-/-}) = m(\mathcal{D}_a^{-/-}) = 0$.

In the case that $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$, the approach is completely analogous, including the use of

$$\mathcal{S}_O^+ := \mathcal{R}^+\mathcal{T}_O\,,$$

$\mathcal{R}^+$ being the reflection

$$\mathcal{R}^+(t, v) = (v, t)\,,$$

instead of $\mathcal{S}_O^-$. In this case we find, for $a \geq 3$,

$$(16)\qquad \begin{cases} m(\mathcal{A}_a^{+/+}) = m(\mathcal{F}_a^{+/+}) = \frac{1}{2}\log \frac{\left(a^2+2+a\sqrt{a^2+4}\right)(2a^2+2a+1)}{(a^2+1)\left(2a^2+2a+3+\sqrt{(2a^2+2a+3)^2-4}\right)}; \\ m(\mathcal{B}_a^{+/+}) = m(\mathcal{E}_a^{+/+}) = \frac{1}{2}\log \frac{\sqrt{a^2+4}-a}{2} + \frac{1}{2}\log \frac{a^2+1}{a}; \\ m(\mathcal{C}_a^{+/+}) = m(\mathcal{D}_a^{+/+}) = \frac{1}{2}\log \frac{2a^2+2a+3+\sqrt{(2a^2+2a+3)^2-4}}{\left(\sqrt{a^2+4}+a\right)\left(\sqrt{(a+1)^2+1}+(a+1)\right)}. \end{cases}$$

In the cases where $\varepsilon_{n+1} \cdot \varepsilon_{n+2} = -1$, we get hold of the six patterns with a mixture of $\mathcal{S}_O^-$ and $\mathcal{S}_O^+$:

$$(17)\qquad \begin{cases} \mathcal{S}_O^+(\mathcal{A}_a^{-/+}) = \mathcal{F}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{F}_a^{+/-}) = \mathcal{A}_a^{-/+}\,; \\ \mathcal{S}_O^+(\mathcal{B}_a^{-/+}) = \mathcal{E}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{E}_a^{+/-}) = \mathcal{B}_a^{-/+}\,; \\ \mathcal{S}_O^+(\mathcal{C}_a^{-/+}) = \mathcal{D}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{D}_a^{+/-}) = \mathcal{C}_a^{-/+}\,; \\ \mathcal{S}_O^+(\mathcal{D}_a^{-/+}) = \mathcal{C}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{C}_a^{+/-}) = \mathcal{D}_a^{-/+}\,; \\ \mathcal{S}_O^+(\mathcal{E}_a^{-/+}) = \mathcal{B}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{B}_a^{+/-}) = \mathcal{E}_a^{-/+}\,; \\ \mathcal{S}_O^+(\mathcal{F}_a^{-/+}) = \mathcal{A}_a^{+/-}\,; & \mathcal{S}_O^-(\mathcal{A}_a^{+/-}) = \mathcal{F}_a^{-/+}\,, \end{cases}$$

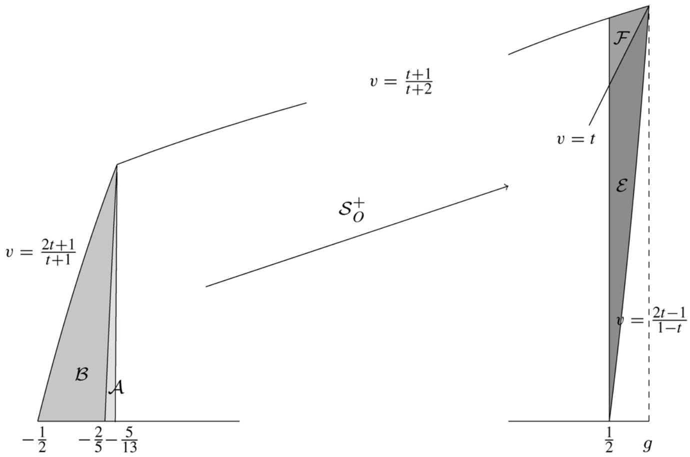

FIGURE 7. The four deviant regions, where $a = 2$ and $\varepsilon_{n+1} \cdot \varepsilon_{n+2} = -1$.

and we find, for $a \geq 3$,

$$
\begin{cases}
m(\mathcal{A}_a^{-/+}) = m(\mathcal{F}_a^{+/-}) = \frac{1}{2} \log \dfrac{\left(\sqrt{a^4+4}-a^2\right)(a^2+1)}{(2a^2+2a+1)\left(\sqrt{(2a^2+2a-1)^2+4}-(2a^2+2a-1)\right)}; \\[2ex]
m(\mathcal{B}_a^{-/+}) = m(\mathcal{E}_a^{+/-}) = \frac{1}{2} \log \dfrac{a^2+2+\sqrt{a^4+4}}{2(a^2+1)}; \\[2ex]
m(\mathcal{C}_a^{-/+}) = m(\mathcal{D}_a^{+/-}) = \frac{1}{2} \log \dfrac{\left(a^2-2+\sqrt{a^4+4}\right)\left(a^2+a+1+(a+1)\sqrt{a^2+1}\right)}{a^2\left(\sqrt{(2a^2+2a-1)^2+4}+(2a^2+2a-1)\right)}; \\[2ex]
m(\mathcal{D}_a^{-/+}) = m(\mathcal{C}_a^{+/-}) = \frac{1}{2} \log \dfrac{\left(a^2+2+\sqrt{a^4+4}\right)\left(a^2-a+1+(a-1)\sqrt{a^2+1}\right)}{a^2\left(2a^2-2a+3+\sqrt{(2a^2-2a+3)^2-4}\right)}; \\[2ex]
m(\mathcal{E}_a^{-/+}) = m(\mathcal{B}_a^{+/-}) = \frac{1}{2} \log \dfrac{a^2-2+\sqrt{a^4+4}}{2(a^2-1)}; \\[2ex]
m(\mathcal{F}_a^{-/+}) = m(\mathcal{A}_a^{+/-}) = \frac{1}{2} \log \dfrac{\left(\sqrt{a^4+4}-a^2\right)(a^2-1)}{(2a^2-2a+1)\left(2a^2-2a+3-\sqrt{(2a^2-2a+3)^2-4}\right)}.
\end{cases}
\tag{18}
$$

We are almost able to give the total sum measure of all the six patterns. To actually do so, we have yet to compute the measures of the regions in the leftmost and the rightmost part of Υ_O, where $a = 2$. In the case $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$, we can apply the formulas of (16). In the case $\varepsilon_{n+1} \cdot \varepsilon_{n+2} = -1$, the patterns $\mathcal{C}$ and $\mathcal{D}$ do not occur. In fact, on the left side we only have $\mathcal{A}$

and $\mathcal{B}$ and on the right side we only have $\mathcal{E}$ and $\mathcal{F}$, which can be mutually mapped onto each other as in (17); see Figure 7[2]. To compute their measures, we apply the formulas in (18).

Now we can compute the total sum measures of all regions:

PATTERNS $\mathcal{A}$ AND $\mathcal{F}$:

$$m(\mathcal{A}) = m(\mathcal{F}) = m(\mathcal{A}_2^{-/+}) + m(\mathcal{A}_2^{+/+}) + \sum_{a=3}^{\infty} m(\mathcal{A}_a^{-/-}) + m(\mathcal{A}_a^{-/+}) + m(\mathcal{A}_a^{+/-}) + m(\mathcal{A}_a^{+/+})$$

$$= \tfrac{1}{2}\log\frac{3+\sqrt{5}}{2} + \tfrac{1}{2}\log\frac{5}{13} + \tfrac{1}{2}\log\frac{13(6+4\sqrt{2})}{5(15+\sqrt{221})}$$
$$+ \sum_{a=3}^{\infty}\left(\frac{1}{2}\log\frac{\big(2a^2+2a-1+\sqrt{4a^4+8a^3-4a+5}\big)\big(2a^2-2a+3+\sqrt{4a^4-8a^3+16a^2-12a+5}\big)}{2\big(2a^2-2a-1+\sqrt{4a^4-8a^3+4a+5}\big)\big(2a^2+2a+3+\sqrt{4a^4+8a^3+16a^2+12a+5}\big)}\right.$$
$$\left. + \frac{1}{2}\log\frac{\big(a\sqrt{a^2-4}+a^2-2\big)\big(a\sqrt{a^2+4}+a^2+2\big)}{\big(a^4+2+a^2\sqrt{a^4+4}\big)}\right).$$

Applying the principle of telescoping series, we can reduce this to

$$\frac{1}{2}\log\big(3+2\sqrt{2}\big) + \frac{1}{2}\log\big(\sqrt{5}-2\big) + \sum_{a=3}^{\infty}\frac{1}{2}\log\frac{\big(a\sqrt{a^2-4}+a^2-2\big)\big(a\sqrt{a^2+4}+a^2+2\big)}{2\big(a^4+2+a^2\sqrt{a^4+4}\big)},$$

and finally to

$$\log\big(\sqrt{2}+1\big) - \frac{1}{2}\log G^3 + \sum_{a=3}^{\infty}\log\frac{\big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)}{2\big(a^2+\sqrt{a^4+4}\big)},$$

which can be simplified further to

$$\frac{3}{2}\log G + \sum_{a=2}^{\infty}\log\frac{\big(a+\sqrt{a^2-4}\,\big)\big(a+\sqrt{a^2+4}\,\big)}{2\big(a^2+\sqrt{a^4+4}\big)}.$$

In order to facilitate numerical computations, we write

$$\frac{3}{2}\log G + \sum_{a=2}^{\infty}\log\frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}.$$

PATTERNS $\mathcal{B}$ AND $\mathcal{E}$:

$$m(\mathcal{B}) = m(\mathcal{E}) = m(\mathcal{B}_2^{-/+}) + m(\mathcal{B}_2^{+/+}) + \sum_{a=3}^{\infty} m(\mathcal{B}_a^{-/-}) + m(\mathcal{B}_a^{-/+}) + m(\mathcal{B}_a^{+/-}) + m(\mathcal{B}_a^{+/+})$$

$$= \frac{1}{2}\log\left(\frac{3+\sqrt{5}}{5}\right) + \frac{1}{2}\log\left(\sqrt{2}-1\right) + \frac{1}{2}\log\frac{5}{2}$$

[2]For visual purposes, we used different scaling for the left part and the right of Υ_O, as a result of which not everything seems to fit.

$$+\sum_{a=3}^{\infty}\frac{1}{2}\log\frac{2\big(a^2+\sqrt{a^4+4}\big)}{\big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)},$$

shortly

$$\frac{1}{2}\log G^2+\frac{1}{2}\log\big(\sqrt{2}-1\big)$$
$$+\sum_{a=3}^{\infty}\frac{1}{2}\log\frac{2\big(a^2+\sqrt{a^4+4}\big)}{\big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)},$$

which can be simplified further to

$$-\frac{1}{2}\log G+\sum_{a=2}^{\infty}\frac{1}{2}\log\frac{2\big(a^2+\sqrt{a^4+4}\big)}{\big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)}\quad\text{or}$$
$$-\frac{1}{2}\log G+\sum_{a=2}^{\infty}\frac{1}{2}\log\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}.$$

PATTERNS $\mathcal{C}$ AND $\mathcal{D}$:

$$m(\mathcal{C})=m(\mathcal{D})=m(\mathcal{C}_2^{+/+})+\sum_{a=3}^{\infty}m(\mathcal{C}_a^{-/-})+m(\mathcal{C}_a^{-/+})+m(\mathcal{C}_a^{+/-})+m(\mathcal{C}_a^{+/+})$$

$$=\sum_{a=3}^{\infty}\left(\frac{1}{2}\log\frac{\left(2a^2-2a-1+\sqrt{4a^4-8a^3+4a+5}\right)\left(2a^2+2a+3+\sqrt{4a^4+8a^3+16a^2+12a+5}\right)}{\left(2a^2+2a-1+\sqrt{4a^4+8a^3-4a+5}\right)\left(2a^2-2a+3+\sqrt{4a^4-8a^3+16a^2-12a+5}\right)}\right.$$
$$\left.+\frac{1}{2}\log\frac{2\left(a^2+\sqrt{a^4+4}\right)\left(2a^2+1+2a\sqrt{a^2+1}\right)}{\left(a+\sqrt{a^2-4}\right)\left(a+\sqrt{a^2+4}\right)\left(a+1+\sqrt{a^2+2a+2}\right)\left(a-1+\sqrt{a^2-2a+2}\right)}\right)$$
$$+\frac{1}{2}\log\frac{15+\sqrt{221}}{2}+\frac{1}{2}\log\left(\sqrt{10}-3\right)+\frac{1}{2}\log\left(\sqrt{2}-1\right).$$

Again applying the principle of telescoping series, we can reduce this to

$$\frac{1}{2}\log G^2+\frac{1}{2}\log\big(\sqrt{2}-1\big)+\sum_{a=3}^{\infty}\frac{1}{2}\log\frac{2\big(a^2+\sqrt{a^4+4}\big)}{\big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)},$$

which can be simplified further to

$$-\frac{1}{2}\log G+\sum_{a=2}^{\infty}\frac{1}{2}\log\frac{2\big(a^2+\sqrt{a^4+4}\big)}{big(a+\sqrt{a^2-4}\big)\big(a+\sqrt{a^2+4}\big)}\quad\text{or}$$
$$-\frac{1}{2}\log G+\sum_{a=2}^{\infty}\frac{1}{2}\log\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)},$$

being the same as for patterns $\mathcal{B}$ and $\mathcal{E}$.

All we have to do to find the asymptotic frequencies for the six patterns $\mathcal{A}$ through $\mathcal{F}$ is dividing these expressions by the normalising constant $\frac{1}{\log G}$ from (9) and rendering these numerically[3]. We have now proved the following theorem:

THEOREM 4. *For the optimal continued fraction, the asymptotic frequencies of the six patterns of three consecutive approximation constants are given by*

$$\begin{aligned} AF_O(\mathcal{A}) = AF_O(\mathcal{F}) &= \frac{3}{2} + \frac{1}{\log G}\left(\sum_{a=2}^{\infty} \log \frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{1}{a^4}}\right)}\right) \\ &\approx 0.160377\ldots ; \\ AF_O(\mathcal{B}) = AF_O(\mathcal{C}) &= AF_O(\mathcal{D}) = AF_O(\mathcal{E}) \\ &= -\frac{1}{2} + \frac{1}{2\log G}\left(\sum_{a=2}^{\infty} \log \frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right) \\ &\approx 0.169811\ldots . \end{aligned}$$

We conclude that, similar to the case of the RCF, the OCF has only two different values for the asymptotic frequencies of the six patterns, similarly divided over these patterns, but mutually differing considerably less than in the case of the RCF (where these values are 0.12109... and 0.18945...).

3. The asymptotic frequencies in the case of the Nearest Integer Continued Fraction. Like the OCF, the NICF is an example of a continued fraction with better approximation properties than those of the regular one. Although the NICF is merely fastest (and not closest), it is a continued fraction that is much studied; see for instance [WB] and [W]. As with the OCF, the convergents of the NICF and the OCF form a subsequence of the sequence of the RCF-convergents. It can be obtained from the RCF by a singularization process concerning all partial quotients with value 1 ([K1]), yielding a continued fraction $[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ that – in the case of the NICF – satisfies

$$\varepsilon_n \in \{-1, 1\}, n \geq 1; \quad a_0 \in \mathbb{Z}; \quad a_n \geq 2, n \geq 1; \quad \varepsilon_{n+1} + a_n \geq 2, \;\; n \geq 1 .$$

The NICF of an $x \in \mathbb{R} \setminus \mathbb{Q}$ can also be obtained directly, by an algorithm explaining the name of this continued fraction. We remark that for $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ the expression $\lfloor \frac{1}{\alpha} + \frac{1}{2} \rfloor$ is the rounding of $\frac{1}{\alpha} + \frac{1}{2}$ to its *nearest integer*, the absolute value of which is at least 2. It is also this expression in the NICF operator $\tau : (-\frac{1}{2}, \frac{1}{2}) \to (-\frac{1}{2}, \frac{1}{2})$ that yields the partial quotients a_n of the NICF, where

$$\tau(t) := \frac{\varepsilon}{t} - \left\lfloor \frac{\varepsilon}{t} + \frac{1}{2} \right\rfloor, \; t \neq 0; \quad \tau(0) := 0,$$

with ε being the sign of t.

[3] For obtaining numerical values we used Mathematica from WolframAlpha.

The values of ε_n and a_n, $n \geq 1$, are determined by repeated application of this operator:

$$\varepsilon_n = \operatorname{sgn}\big(\tau^{n-1}(t)\big) \quad \text{and} \quad a_n = \left\lfloor \frac{\varepsilon_n}{\tau^{n-1}(t)} + \frac{1}{2} \right\rfloor,$$

provided $\tau^{n-1}(t) \neq 0$ – which is always true in the case of $t \in \mathbb{R} \setminus \mathbb{Q}$.

Now put $\Omega_N := [-\frac{1}{2}, \frac{1}{2}] \setminus \mathbb{Q}$ and let $[0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots]$ be the NICF expansion of $t \in \Omega_N$.

We define

$$\rho(t) := \begin{cases} \dfrac{1}{\log G} \cdot \dfrac{1}{G+t}, & 0 \leq t \leq \frac{1}{2} \\[2ex] \dfrac{1}{\log G} \cdot \dfrac{1}{G+t+1}, & -\frac{1}{2} \leq t < 0 \end{cases}.$$

Let μ be the measure with density function ρ. Then (Ω_N, μ, τ) forms an ergodic system, as was proved by G. J. Rieger ([Ri]) and A. M. Rockett ([Ro]).

For our investigation we use the natural extension of τ, which is the same as the one of the OCF, but defined on a different domain:

$$\mathcal{T}_N(t, v) := \left(\tau(t), \frac{1}{\varepsilon_1 v + a_1}\right) = \left(\frac{\varepsilon_1}{t} - a_1, \frac{1}{\varepsilon_1 v + a_1}\right),$$

where

$$(t, v) \in \Upsilon_N := [-\tfrac{1}{2}, 0] \setminus \mathbb{Q} \times [0, g^2] \cup [0, \tfrac{1}{2}] \setminus \mathbb{Q} \times [0, g];$$

see Figure 8. This natural extension domain was first obtained by H. Nakada ([N]), who showed that $(\Upsilon_N, \mathscr{B}_{\Upsilon_N}, \bar{\mu}_{\Upsilon_N}, \mathcal{T}_N)$ forms an ergodic system, where the $\mathcal{T}_N$-invariant probability measure $\bar{\mu}_{\Upsilon_N}$ has density function

$$d_N(t, v) := \frac{1}{\log G} \cdot \frac{1}{(1+tv)^2} 1_{\Upsilon_N}(t, v).$$

Note that projecting this measure on the first coordinate axis yields a τ-invariant probability measure with Rieger's density function ρ; see also [K1]. As in the case of the OCF, we have to deal with ε_{n+1} and ε_{n+2}, yielding four different cases. At first, it seems convenient that we can take the approach from the RCF by regarding vertical strips in the (t, v)-plane. These strips $R_a^{\varepsilon_{n+1}/\varepsilon_{n+2}}$, defined below, are determined by the values of ε_{n+1} and ε_{n+2} and a_{n+1}, about which we remark that

$$\mathcal{T}_N^n(t, 0) \in R_a^{\varepsilon_{n+1}/\varepsilon_{n+2}} \Leftrightarrow a_{n+1} = a, \ n \geq 0.$$

We define

$$R_a^{-/-} := \left(-\frac{2}{2a-1}, -\frac{2}{2a}\right) \times [0, g^2], \quad a = 3, 4, 5, \dots$$

$$R_a^{-/+} := \left(-\frac{2}{2a}, -\frac{2}{2a+1}\right) \times [0, g^2], \quad a = 2, 3, 4, \dots$$

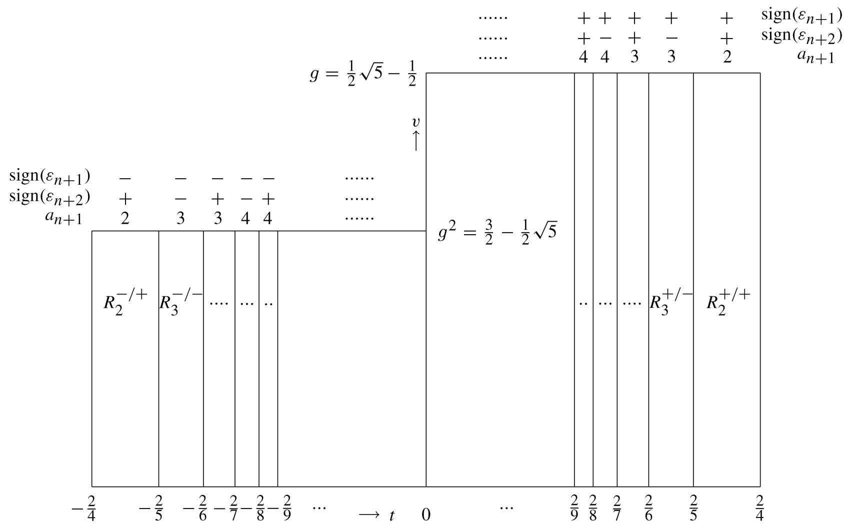

FIGURE 8. The strips $R_a^{*/*}$.

$$R_a^{+/-} := \left(\frac{2}{2a},\frac{2}{2a-1}\right) \times [0,g], \quad a = 3,4,5,\dots$$

$$R_a^{+/+} := \left(\frac{2}{2a+1},\frac{2}{2a}\right) \times [0,g], \quad a = 2,3,4,\dots$$

In Figure 8 we have drawn these strips in the (t, v)-square. Note that $g \approx 0.618$ and $g^2 \approx 0.382$. Also, $\varepsilon_{n+1} = -1$ implies $t < 0$ and $a_n \geq 3$, from which follows $v < g^2$. Secondly, $\varepsilon_{n+2} = -1$ implies $a = a_{n+1} \geq 3$, and therefore $|t| < \frac{2}{5}$.

An important difference with the regular case is that the measure of each region is not given by one formula for every a for which the patterns exist. As we can see in Figure 9, in both $R_2^{-/+}$ and $R_3^{-/-}$ not all patterns are present. In Figures 10 and 11, we have filled all regions according to the same pattern with the same shade of grey, where – similar to the case of the OCF – darker shades correspond with ascending alphabetical order from $\mathcal{A}$ to $\mathcal{F}$.

In the case of the patterns $\mathcal{C}$ and $\mathcal{D}$ it is only from $a = 6$ on that the same formula holds for every value of a. This absence of one formula for every a is connected with the inutility of measure-preserving maps within the strips, from which the clear distribution of asymptotic frequencies was derived, in case of the RCF in [JJ] and in the case of the OCF in the previous sections. Actually, if both regions would have $0 \leq v \leq \frac{1}{2}$ instead of $0 \leq v \leq g^2$ (in the case $\varepsilon_{n+1} = -1$) and $0 \leq v \leq g$ (in the case $\varepsilon_{n+1} = 1$), we could have applied the same maps for $R_a^{-/-}$ and $R_a^{+/+}$ as in the regular and the optimal case. But even then we would still have to deal with the less 'convenient' two other strips. There is not much we can do but calculate the measure for each region in the most exterior strips and then find general formulas for the regions in all other strips.

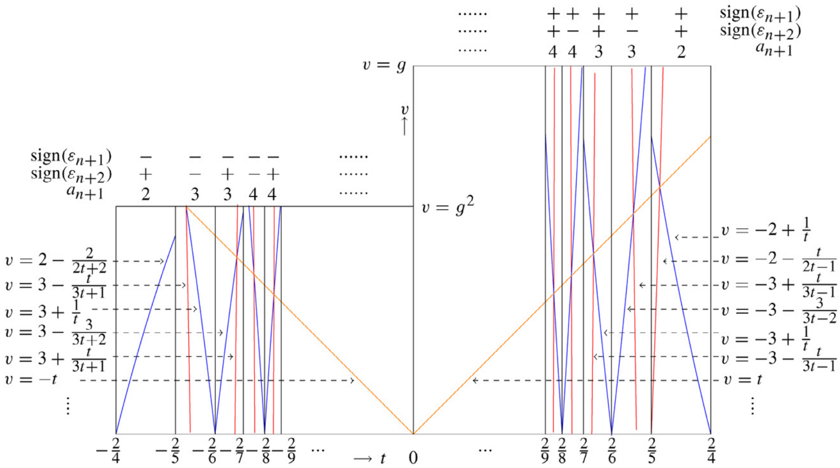

FIGURE 9. The curves dividing the regions of the patterns $\mathcal{A}$ through $\mathcal{F}$.

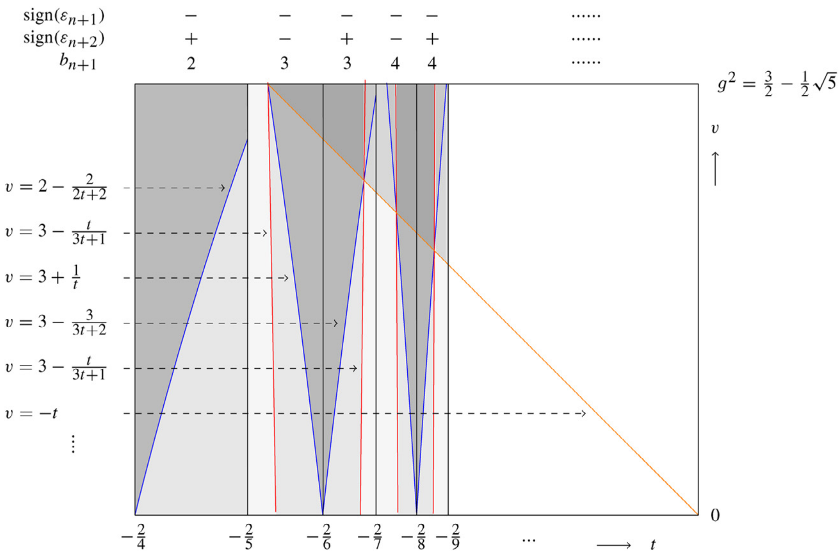

FIGURE 10. The patterns $\mathcal{A}$ through $\mathcal{F}$ for $\varepsilon_{n+1} < 0$.

4. The measures of the six patterns of the NICF. For the frequencies of the patterns $\mathcal{A}$ through $\mathcal{F}$ that we are investigating, we define

$$\mathcal{X} := \bigcup_{a=2}^{\infty} \mathcal{X}_a, \ \mathcal{X} = \mathcal{A}, \dots, \mathcal{F}.$$

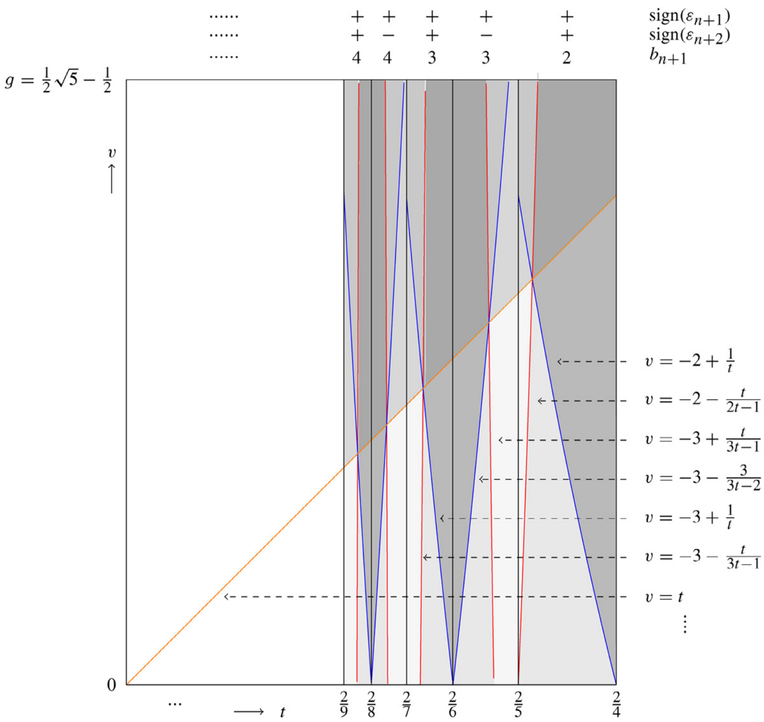

FIGURE 11. The patterns $\mathcal{A}$ through $\mathcal{F}$ for $\varepsilon_{n+1} > 0$.

A major complication in trying to determine a clear expression for the sum of the measures of all regions belonging to each pattern, is that quite a lot of different smaller expressions are involved, among which three different square root arguments, as we have seen in Table 1. In this section we wil evaluate these measures for each pattern separately. The calculations involve laborious double integrals that we will mostly omit. In establishing the formulas below, however, it is interesting to see how some terrifying expressions can be reduced using basic calculus. We come across expressions such as (in case $\mathcal{A}$)

$$\frac{1}{2}\log\Big(a^6 + a^4 + 4a^2 + 4 - (a^4 + a^2 + 2)\sqrt{a^4+4}\Big) - \frac{1}{2}\log\Big(-a^4 - 4 + (a^2+2)\sqrt{a^4+4}\Big),$$

which at first seem hard to handle. Applying long division and some other basic techniques, though, we can reduce this to

$$\frac{1}{2}\log\frac{-a^2 + \sqrt{a^4+4}}{2}.$$

It is in fact possible to evaluate the sum measures for four patterns as more or less well-arranged expressions of a definite form; for the patterns $\mathcal{C}$ and $\mathcal{D}$ a bit too many terms are involved. To find the asymptotic frequency of each pattern, we merely have to divide these measures by the normalizing constant, that is by

$$m(\Upsilon_N) = \int_{-\frac{1}{2}}^{0} \int_{0}^{g^2} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} + \int_{0}^{\frac{1}{2}} \int_{0}^{g} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} = \log G\,.$$

Unfortunately but not unexpectedly, the attractive conciseness of the regular case will not be reached, and we shall see that the asymptotic frequencies are different in all six cases.

4.1. Pattern $\mathcal{A}$. There are two regions of pattern $\mathcal{A}$, in the strips $R_2^{+/+}$ and $R_3^{-/-}$, with deviant measures, that is, not computable with the formulas for all other strips stated below. These measures are $\log(\sqrt{2}+1) + \frac{1}{2}\log\frac{5}{29}$ and $\frac{1}{2}\log(7\sqrt{5}-15) + \log\frac{5}{4}$, respectively. The measures of all other $\mathcal{A}$-regions can be expressed as functions of a_{n+1} or shortly a, for $a \geq 3$, as in the previous section. These are

$$\begin{array}{lll}
R_a^{-/-}, a \geq 4: & \frac{1}{2}\log\frac{(2a-1)^2}{((2a-1)^2-4)(a^2-1)} & -\log\frac{a-\sqrt{a^2-4}}{2}\,; \\
R_a^{-/+}, a \geq 3: & \frac{1}{2}\log\frac{((2a+1)^2-4)(a^2+1)}{(2a+1)^2} & +\frac{1}{2}\log\frac{-a^2+\sqrt{a^4+4}}{2}\,; \\
R_a^{+/-}, a \geq 3: & \frac{1}{2}\log\frac{((2a-1)^2+4)(a^2-1)}{(2a-1)^2} & +\frac{1}{2}\log\frac{-a^2+\sqrt{a^4+4}}{2}\,; \\
R_a^{+/+}, a \geq 3: & \underbrace{\frac{1}{2}\log\frac{(2a+1)^2}{((2a+1)^2+4)(a^2+1)}}_{\text{part I}} & \underbrace{-\log\frac{a-\sqrt{a^2+4}}{2}\,.}_{\text{part II}}
\end{array}$$

A close inspection of the factors in the arguments of the logarithms in part I unfolds that all terms in part I are mutually canceled, except for the one of $R_3^{+/-}$, the canceling term of which would be found partly in $R_2^{+/+}$ and partly in $R_3^{-/-}$. This yields a measure of $\frac{1}{2}\log\frac{29\cdot 8}{25}$. If we now sum all terms of both parts from $a = 3$, we mistakenly add a 'virtual' $-\frac{1}{2}\log\frac{3^2-2-3\sqrt{3^2-4}}{2}$ from $R_3^{-/-}$ that we have to cancel by adding $\frac{1}{2}\log\frac{7-3\sqrt{5}}{2}$. Finally, summing up to $a = n$, we should add the value $\frac{1}{2}\log(n^2+4n-3) - \frac{1}{2}\log(n^2+4n+5)$ from $R_n^{-/+}$ and $R_n^{+/+}$ that has by then not yet been canceled by the corresponding terms in $R_{n+1}^{-/-}$ and $R_{n+1}^{+/-}$. However, $\lim_{n\to\infty}\frac{1}{2}\log(n^2+4n-3) - \frac{1}{2}\log(n^2+4n+5) = 0$, so for the infinite summation this makes no difference. So far, having summed the aforementioned constants, we have found the sum measure of the regions $\mathcal{A}$ to be:

$$\frac{1}{2}\log\frac{235\sqrt{5}-525}{2} + \log(\sqrt{2}+1) + \sum_{a=3}^{\infty}(\text{expressions in part II})\,. \tag{19}$$

The final step in getting hold of the sum measure is reducing the sum of the four expressions in part II, which turns out to be $\log\frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}$. This means that we can write (19) as

$$\frac{1}{2}\log\frac{235\sqrt{5}-525}{2}+\log\left((\sqrt{2}+1)\prod_{a=3}^{\infty}\frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}\right). \tag{20}$$

Unfortunately, we cannot but give a numerical approximation for this expression, which is 0.08122410.... Dividing this by the normalizing constant, $\log G$, we find

$$AF_N(\mathcal{A}) = 0.168790\ldots.$$

4.2. Pattern $\mathcal{B}$. There is one region of pattern $\mathcal{B}$, in the strip $R_2^{-/+}$, with a deviant (see 4.1) measure, which is $\frac{1}{2}\log\frac{26}{25}$. The measures of all other $\mathcal{B}$-regions can be expressed as functions of a. These are

$$\begin{array}{lll}
R_a^{-/-}, a\geq 3: & \frac{1}{2}\log\left(a-\frac{1}{a}\right) & +\frac{1}{2}\log\frac{a-\sqrt{a^2-4}}{2};\\
R_a^{-/+}, a\geq 3: & -\frac{1}{2}\log\left(a+\frac{1}{a}\right) & +\frac{1}{2}\log\frac{a^2+2+\sqrt{a^4+4}}{2a};\\
R_a^{+/-}, a\geq 3: & -\frac{1}{2}\log\left(a-\frac{1}{a}\right) & +\frac{1}{2}\log\frac{a^2-2+\sqrt{a^4+4}}{2a};\\
R_a^{+/+}, a\geq 2: & \underbrace{\frac{1}{2}\log\left(a+\frac{1}{a}\right)}_{\text{part I}} & \underbrace{+\frac{1}{2}\log\frac{-a+\sqrt{a^2+4}}{2}.}_{\text{part II}}
\end{array}$$

It appears that in part I all terms are mutually canceled, except for $R_2^{+/+}$, where we also have to take the corresponding value in part II into account, yielding the sum value $\frac{1}{2}\log 5-\frac{1}{2}\log 2+\frac{1}{2}\log(\sqrt{2}-1)=\frac{1}{2}\log\frac{5\sqrt{2}-5}{2}$. Adding this number to the aforementioned $\frac{1}{2}\log\frac{26}{25}$, we find the sum measure of the regions $\mathcal{B}$ to be:

$$\frac{1}{2}\log\frac{13\sqrt{2}-13}{5}+\sum_{a=3}^{\infty}(\text{expressions in part II}). \tag{21}$$

We can write the sum of the four expressions in part II as $\frac{1}{2}\log\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}$. This means that we can write (21) as

$$\frac{1}{2}\log\left(\frac{13\sqrt{2}-13}{5}\prod_{a=3}^{\infty}\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right). \tag{22}$$

A numerical approximation for this expression is 0.07825923.... Dividing this by the normalizing constant, we find

$$AF_N(\mathcal{B}) = 0.162629\ldots.$$

4.3. Pattern $\mathcal{C}$. There is one region of pattern $\mathcal{C}$, in the strip $R_3^{-/+}$, with deviant (see 4.1) measure, which is $\frac{1}{2}\log(7+\sqrt{85})+\frac{1}{2}\log\frac{5}{81}$. The measures of all other $\mathcal{C}$-regions can be expressed as generic functions only from $a \geq 6$. These are

$$R_a^{-/-},\ a \geq 4:$$
$$\frac{1}{2}\log\frac{(2a-1)^2-4}{(2a+\sqrt{5}-4)^2}+\frac{\log(a+\sqrt{5}-3)+\log\frac{a-\sqrt{a^2-4}}{2}}{2};$$
$$R_a^{-/+},\ a \geq 4:$$
$$\frac{1}{2}\log\frac{(2a+\sqrt{5}-2)^2}{(2a+1)^2-4}-\frac{\log\left(a^2+(\sqrt{5}-3)a+7-3\sqrt{5}\right)-\log\frac{a^2-2+\sqrt{a^4+4}}{2}}{2};$$
$$R_a^{+/-},\ a \geq 6:$$
$$-\frac{1}{2}\log((2a-1)^2+4)+\frac{\log\left(4-\frac{4}{a}+\frac{2}{a^2}\right)+\log\frac{a^2+2+\sqrt{a^4+4}}{2}}{2};$$
$$R_a^{+/+},\ a \geq 2:$$
$$\underbrace{\frac{1}{2}\log((2a+1)^2+4)}_{\text{part I}}\underbrace{-\frac{\log(4a+4)-\log\frac{-a+\sqrt{a^2+4}}{2}}{2}.}_{\text{part II}}$$

Unfortunately, we also have to deal with

$$R_a^{+/-},\ a \in \{3,4,5\}:\quad \underbrace{-\frac{1}{2}\log((2a-1)^2+4)}_{\text{part I}}\underbrace{+\frac{\log\frac{(2a+\sqrt{5}-2)^2}{a^2+(\sqrt{5}-1)a+3-\sqrt{5}}+\log\frac{a^2+2+\sqrt{a^4+4}}{2}}{2}.}_{\text{part II}}$$

Both parts make the computations for pattern $\mathcal{C}$ more troublesome than for $\mathcal{A}$ and $\mathcal{B}$. Luckily, in part I all terms, including those for $a \in \{3,4,5\}$, are mutually canceled, except for $\frac{1}{2}\log\frac{45}{21+8\sqrt{5}}$ in $R_4^{-/-}$ when we sum the terms of both parts from $a=4$. This summation seems the most convenient in the sense of restricting the number of loose constants to be summed separately. Still, there are too many of those to write them sensibly as one logarithm. They are:

$$\text{from } R_a^{+/-}: \quad -\frac{1}{2}\log(2\sqrt{5}+9),\ -\frac{1}{2}\log(3\sqrt{5}+15),$$
$$-\frac{1}{2}\log(4\sqrt{5}+23),\ \log(4+\sqrt{5}),\ \log(6+\sqrt{5}),$$
$$\log(8+\sqrt{5}),\ \frac{1}{2}\log\frac{11+\sqrt{85}}{2},\ \frac{1}{2}\log\frac{16}{50},\ \frac{1}{2}\log\frac{25}{82};$$
$$\text{from } R_a^{+/+}: \quad -\frac{1}{2}\log 12,\ \frac{1}{2}\log(\sqrt{2}-1),\ -\frac{1}{2}\log 16,\ \frac{1}{2}\log\frac{\sqrt{13}-3}{2}.$$

Summing all constants aforementioned yields a value of $-1.90648232\ldots$. Summing all terms of both parts up to $a = n$, we should add the value

$$-\tfrac{1}{2}\log(n^2+4n-3)+\log(2n+\sqrt{5}-2)+\tfrac{1}{2}\log(n^2+4n+5)$$

from $R_n^{-/+}$ and $R_n^{+/+}$ that has by then not yet been canceled by the corresponding terms in $R_{n+1}^{-/-}$ and $R_{n+1}^{+/-}$. Note that $\lim_{n\to\infty}\frac{1}{2}\log(n^2+4n-3)-\frac{1}{2}\log(n^2+4n+5)=0$ (as used in the case of pattern $\mathcal{A}$), so we have to compute the sum of the terms of part II from $a = 4$ up to n and $\log(2n+\sqrt{5}-2)$ as $n\to\infty$. It seems like we cannot write this as anything shorter than

$$\lim_{n\to\infty}\left(\sum_{a=4}^{n}\tfrac{1}{2}\log\frac{\left(1+\frac{\sqrt{5}-3}{a}\right)\left(1+\sqrt{1+\frac{4}{a^4}}\right)\left(2-\frac{2}{a}+\frac{1}{a^2}\right)}{\left(1+\frac{\sqrt{5}-3}{a}+\frac{7-3\sqrt{5}}{a^2}\right)\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)\left(1+\frac{1}{a}\right)} + \log(2n+\sqrt{5}-2)\right), \tag{23}$$

which is $1.98689899\ldots$. So we can approximate the sum measure of the $\mathcal{C}$-regions by $1.98689899\ldots - 1.90648232\ldots \approx 0.08041667$. Dividing this by $\log G$, we find

$$AF_N(\mathcal{C}) = 0.167112\ldots.$$

4.4. Pattern $\mathcal{D}$. Pattern $\mathcal{D}$ demands even more bothersome computations than pattern $\mathcal{C}$. There is also one region of pattern $\mathcal{D}$, in the strip $R_3^{-/+}$, with deviant (see 4.1) measure

$$\frac{1}{2}\log\frac{(3-\sqrt{5})(21+8\sqrt{5})(11+\sqrt{85})}{600},$$

which is $0.00057163\ldots$. The measures of all other $\mathcal{D}$-regions can be expressed as generic expressions only from $a \geq 6$. These are

$R_a^{-/-}$, $a \geq 4$:

$$\frac{1}{2}\log\left(a^2+(\sqrt{5}-3)a+\tfrac{5}{2}-\tfrac{3}{2}\sqrt{5}\right)-\frac{\log(a+\sqrt{5}-3)-\log\frac{a-\sqrt{a^2-4}}{2}}{2};$$

$R_a^{-/+}$, $a \geq 4$:

$$-\frac{1}{2}\log\left(a^2+(\sqrt{5}-3)a+\tfrac{9}{2}-\tfrac{3}{2}\sqrt{5}\right)+\frac{\log\left(a^2+(\sqrt{5}-3)a+7-3\sqrt{5}\right)-\log\frac{a^2+2-\sqrt{a^4+4}}{2}}{2};$$

$R_a^{+/-}$, $a \geq 6$:

$$\frac{1}{2}\log\frac{(2a+\sqrt{5}-2)^2}{a^2+(\sqrt{5}-1)a+\frac{1}{2}-\frac{1}{2}\sqrt{5}}-\frac{\log\left(4-\frac{4}{a}+\frac{2}{a^2}\right)+\log\frac{-a^2+2+\sqrt{a^4+4}}{2}}{2};$$

$R_a^{+/+}$, $a \geq 2$:

$$\underbrace{\frac{1}{2}\log\frac{a^2+(\sqrt{5}-1)a+\frac{5}{2}-\frac{1}{2}\sqrt{5}}{(2a+\sqrt{5})^2}}_{\text{part I}}+\underbrace{\frac{\log(4a+4)+\log\frac{-a+\sqrt{a^2+4}}{2}}{2}}_{\text{part II}}.$$

As in the case of pattern $\mathcal{C}$, there are some deviant expressions namely

$$R_a^{+/-},\ a\in\{3,4,5\}:\ \frac{1}{2}\log\frac{a^2+(\sqrt{5}-1)a+3-\sqrt{5}}{a^2+(\sqrt{5}-1)a+\frac{1}{2}-\frac{1}{2}\sqrt{5}}-\frac{1}{2}\log\frac{-a^2+2+\sqrt{a^4+4}}{2}.$$

Again, both parts make the computations more troublesome than for $\mathcal{A}$ and $\mathcal{B}$. Luckily, in part I almost all terms, including those for $a\in\{3,4,5\}$, are mutually canceled when we sum the terms of both parts from $a=4$. This summation seems the most convenient, similar to the case of pattern $\mathcal{A}$. The number of loose constants to be summed separately are:

$$\begin{aligned}\text{from } R_a^{+/-}:\quad & \frac{1}{2}\log(2\sqrt{5}+9),\ \frac{1}{2}\log(3\sqrt{5}+15),\ \frac{1}{2}\log(4\sqrt{5}+23),\\ & -\frac{1}{2}\log\left(-\frac{7}{2}+\frac{1}{2}\sqrt{85}\right),\ \frac{1}{2}\log\frac{50}{16},\ \frac{1}{2}\log\frac{82}{25};\\ \text{from } R_a^{+/+}:\quad & -\log(4+\sqrt{5}),\ -\log(6+\sqrt{5}),\ -\log(8+\sqrt{5}),\ \frac{1}{2}\log 12,\\ & \frac{1}{2}\log 16,\ \frac{1}{2}\log(\sqrt{2}-1),\ \frac{1}{2}\log\left(\frac{9}{2}+\frac{3}{2}\sqrt{5}\right),\ \frac{1}{2}\log\left(\frac{1}{2}\sqrt{13}-\frac{3}{2}\right).\end{aligned}$$

Summing all constants aforementioned yields a value of $2.03969322\ldots$. Summing all terms of both parts up to $a=n$, we should add the value

$$-\frac{1}{2}\log\left(n^2+(\sqrt{5}-1)n+\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)-\log(2n+\sqrt{5})+\frac{1}{2}\log\left(n^2+(\sqrt{5}-1)n+\frac{5}{2}-\frac{1}{2}\sqrt{5}\right)$$

from $R_n^{-/+}$ and $R_n^{+/+}$ that has by then not yet been canceled by the corresponding terms in $R_{n+1}^{-/-}$, $R_{n+1}^{-/+}$ and $R_{n+1}^{+/-}$. Note that $\lim_{n\to\infty}-\frac{1}{2}\log(n^2+(\sqrt{5}-1)n+\frac{1}{2}-\frac{1}{2}\sqrt{5})+\frac{1}{2}\log(n^2+(\sqrt{5}-1)n+\frac{5}{2}-\frac{1}{2}\sqrt{5})=0$, so we have to compute the sum of the terms of part II from $a=4$ up to n and $-\log(2n+\sqrt{5})$ as $n\to\infty$. It seems like we cannot write this as anything shorter than

$$\lim_{n\to\infty}\left(\sum_{a=4}^{n}\frac{1}{2}\log\frac{\left(1+\frac{\sqrt{5}-3}{a}+\frac{7-3\sqrt{5}}{a^2}\right)\left(1+\sqrt{1+\frac{4}{a^4}}\right)\left(4+\frac{4}{a}\right)}{\left(1+\frac{\sqrt{5}-3}{a}\right)\left(2-\frac{2}{a}+\frac{1}{a^2}\right)\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}-\log\left(2n+\sqrt{5}\right)\right),\tag{24}$$

which is $-1.95667971\ldots$. We approximate the sum measure of the $\mathcal{D}$-regions by $2.03969322\ldots-1.95667971=0.08301351\ldots$. Dividing this by $\log G$, we find

$$AF_N(\mathcal{D})=0.172509\ldots.$$

4.5. Pattern $\mathcal{E}$. There is one region of pattern $\mathcal{E}$, in the strip $R_2^{-/+}$, with a deviant (see 4.1) measure, which is $\log(3+\sqrt{5}) - \frac{1}{2}\log 26$. The measures of all other $\mathcal{E}$-regions can be expressed as functions of a. These are

$$\begin{array}{lll}
R_a^{-/-}, a \geq 3: & \frac{1}{2}\log(a^2-1) & +\frac{1}{2}\log\frac{a-\sqrt{a^2-4}}{2a}; \\
R_a^{-/+}, a \geq 3: & -\frac{1}{2}\log(a^2-1) & +\frac{1}{2}\log\frac{a^2-2+\sqrt{a^4+4}}{2}; \\
R_a^{+/-}, a \geq 3: & -\frac{1}{2}\log(a^2+1) & +\frac{1}{2}\log\frac{a^2+2+\sqrt{a^4+4}}{2}; \\
R_a^{+/+}, a \geq 2: & \underbrace{\frac{1}{2}\log(a^2+1)}_{\text{part I}} & \underbrace{+\frac{1}{2}\log\frac{-a+\sqrt{a^2+4}}{2a}.}_{\text{part II}}
\end{array}$$

It appears that in part I all terms are mutually canceled, except for $R_2^{+/+}$, where we also have to take the corresponding value in part II into account, with the sum value $\frac{1}{2}\log 5 + \frac{1}{2}\log(-\frac{1}{2}+\frac{1}{2}\sqrt{2})$. Adding this number to the aforementioned two constants, we find the sum measure of the regions $\mathcal{E}$ to be:

$$\frac{1}{2}\log\frac{35+15\sqrt{5}}{26(\sqrt{2}+1)} + \sum_{a=3}^{\infty}(\text{expressions in part II}). \tag{25}$$

Attentive inspection of the four expressions in part II reveals that they have the same sum value as part II of pattern $\mathcal{B}$. This means that we can write (25) as

$$\frac{1}{2}\log\left(\frac{35+15\sqrt{5}}{26(\sqrt{2}+1)}\prod_{a=3}^{\infty}\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right). \tag{26}$$

A numerical approximation for this expression, which is $0.08517144\ldots$. Dividing this by $\log G$, we find

$$AF_N(\mathcal{E}) = 0.176993\ldots.$$

4.6. Pattern $\mathcal{F}$. There is one region of pattern $\mathcal{F}$, in the strip $R_3^{-/-}$, with deviant (see 4.1) measures, which is $\frac{1}{2}\log(9\sqrt{5}+20) - \frac{1}{2}\log 40$.
The measures of all other $\mathcal{F}$-regions can be expressed as functions of a. These are

$$\begin{array}{ll}
R_a^{-/-}, a \geq 4: & \frac{1}{2}\log\frac{(2a+\sqrt{5}-3)^2}{(2a^2-2)\left(2a^2+2(\sqrt{5}-3)a+5-3\sqrt{5}\right)} - \log\frac{a-\sqrt{a^2-4}}{2}; \\
R_a^{-/+}, a \geq 3: & \frac{1}{2}\log\frac{(2a^2-2)\left(2a^2+2(\sqrt{5}-3)a+9-3\sqrt{5}\right)}{(2a+\sqrt{5}-3)^2} + \frac{1}{2}\log\frac{-a^2+\sqrt{a^4+4}}{2}; \\
R_a^{+/-}, a \geq 3: & \frac{1}{2}\log\frac{(2a^2-2)\left(2a^2+2(\sqrt{5}-1)a+1-\sqrt{5}\right)}{(2a+\sqrt{5}-1)^2} + \frac{1}{2}\log\frac{-a^2+\sqrt{a^4+4}}{2};
\end{array}$$

$$R_a^{+/+}, a \geq 2: \underbrace{\frac{1}{2}\log\frac{(2a+\sqrt{5}-1)^2}{(2a^2-2)\big(2a^2+2(\sqrt{5}-1)a+5-\sqrt{5}\big)}}_{\text{part I}} \underbrace{-\log\frac{a-\sqrt{a^2+4}}{2}}_{\text{part II}}.$$

All terms in part I are mutually canceled. If we sum all terms of both parts from $a = 3$, we have to add a corrective $\frac{1}{2}\log(\frac{7}{2}-\frac{3}{2}\sqrt{5})$ in $R_3^{-/-}$, $\frac{1}{2}\log 8$ in $R_3^{-/+}$ and of $-\frac{1}{2}\log(3-2\sqrt{2})-\frac{1}{2}\log 5$ in $R_2^{+/+}$. Finally, summing up to $a = n$, we should add the value $\frac{1}{2}(\log(2n^2+2(\sqrt{5}-1)n+1-\sqrt{5})-\log(2n^2+2(\sqrt{5}-1)n+5-\sqrt{5})$ from $R_n^{+/-}$ and $R_n^{+/+}$ that has by then not yet been canceled by the corresponding terms in $R_{n+1}^{-/-}$ and $R_{n+1}^{-/+}$. However, as $\lim_{n\to\infty}$, this difference approaches 0, so for the infinite summation this makes no difference. So far, having summed the aforementioned constants, we have found the sum measure of the regions $\mathcal{F}$ to be:

$$\frac{1}{2}\log\frac{\big(5+3\sqrt{5}\big)\big(3+2\sqrt{2}\big)}{50}+\sum_{a=3}^{\infty}(\text{expressions in part II}). \tag{27}$$

It appears that part II is completely the same as part II of pattern $\mathcal{A}$, and we can write (27) as

$$\frac{1}{2}\log\frac{5+3\sqrt{5}}{50}+\log\left((\sqrt{2}+1)\prod_{a=3}^{\infty}\frac{\Big(1+\sqrt{1-\frac{4}{a^2}}\Big)\Big(1+\sqrt{1+\frac{4}{a^2}}\Big)}{2\Big(1+\sqrt{1+\frac{4}{a^4}}\Big)}\right). \tag{28}$$

A numerical approximation for this expression is $0.07312636\ldots$. Dividing this by $\log G$, we find

$$AF_N(\mathcal{F}) = 0.151962\ldots.$$

Summarizing, we have obtained the following result.

THEOREM 5. *For the nearest integer continued fraction, the asymptotic frequencies of the six patterns of three consecutive approximation constants are given by*

$$AF_N(\mathcal{A}) = 0.168790\ldots \quad AF_N(\mathcal{B}) = 0.162629\ldots \quad AF_N(\mathcal{C}) = 0.167112\ldots,$$
$$AF_N(\mathcal{D}) = 0.172509\ldots \quad AF_N(\mathcal{E}) = 0.176993\ldots \quad AF_N(\mathcal{F}) = 0.151962\ldots.$$

At first sight, these numbers seem quite unremarkable; they are all different, and that's it. But something interesting happens when we add the expressions for patterns $\mathcal{A}$ and $\mathcal{F}$, for patterns $\mathcal{B}$ and $\mathcal{E}$, and for patterns $\mathcal{C}$ and $\mathcal{D}$. First, adding (20) and (28), belonging to patterns $\mathcal{A}$ and $\mathcal{F}$, we get

$$\frac{1}{2}\log\frac{235\sqrt{5}-525}{2}+\log\left((\sqrt{2}+1)\prod_{a=3}^{\infty}\frac{\Big(1+\sqrt{1-\frac{4}{a^2}}\Big)\Big(1+\sqrt{1+\frac{4}{a^2}}\Big)}{2\Big(1+\sqrt{1+\frac{4}{a^4}}\Big)}\right)$$

$$+\frac{1}{2}\log\frac{5+3\sqrt{5}}{50}+\log\left((\sqrt{2}+1)\prod_{a=3}^{\infty}\frac{\Big(1+\sqrt{1-\frac{4}{a^2}}\Big)\Big(1+\sqrt{1+\frac{4}{a^2}}\Big)}{2\Big(1+\sqrt{1+\frac{4}{a^4}}\Big)}\right),$$

which equals

$$\log\left(g^3\left((\sqrt{2}+1)\prod_{a=3}^{\infty}\frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}\right)^2\right).$$

A close inspection of this last expression reveals that it equals

$$3\log G + 2\cdot\sum_{a=2}^{\infty}\log\frac{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}, \tag{29}$$

which is exactly the same as the sum of the expressions for patterns $\mathcal{A}$ and $\mathcal{F}$ in the OCF case.

Adding (22) and (26), belonging to patterns $\mathcal{B}$ and $\mathcal{E}$, yields

$$\frac{1}{2}\log\left(\frac{13\sqrt{2}-13}{5}\prod_{a=3}^{\infty}\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right)$$

$$+\frac{1}{2}\log\left(\frac{35+15\sqrt{5}}{26(\sqrt{2}+1)}\prod_{a=3}^{\infty}\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right),$$

which equals

$$\log\left(G^2(\sqrt{2}-1)\prod_{a=3}^{\infty}\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}\right). \tag{30}$$

Completely similarly to the case of patterns $\mathcal{A}$ and $\mathcal{F}$, we can rewrite (30) so as to find

$$-\log G + \sum_{a=2}^{\infty}\log\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}, \tag{31}$$

which is exactly the same as the sum of the expressions for patterns $\mathcal{B}$ and $\mathcal{E}$ in the OCF case. Since the sum of all expressions for patterns $\mathcal{A}$ through $\mathcal{F}$ equals $\log G$, the sum of the patterns $\mathcal{C}$ and $\mathcal{D}$ equals $\log G-$ (29) $-$ (31), equalling

$$-\log G + \sum_{a=2}^{\infty}\log\frac{2\left(1+\sqrt{1+\frac{4}{a^4}}\right)}{\left(1+\sqrt{1-\frac{4}{a^2}}\right)\left(1+\sqrt{1+\frac{4}{a^2}}\right)}, \tag{32}$$

which is exactly the same as the sum of the expressions for patterns $\mathcal{C}$ and $\mathcal{D}$ in the OCF case.

5. Discussion-How to connect the results with our understanding of the continued fractions involved. In the previous sections we showed that in the case of both the OCF and the NICF the distribution of asymptotic frequencies (as defined above) is more even than in the case of the RCF. Moreover, we showed that the two different values occurring in the case of the OCF were distributed over the six patterns in a way completely similar to the RCF case. Finally, the six different values in the case of the NICF proved to be remarkably connected with the values belonging to the OCF.

Although some adjustments were needed and the calculations were more troublesome, the approach we took in computing the asymptotic frequencies in the RCF case proved to be applicable to the other two continued fractions. In fact, it is applicable to all continued fractions of which we know the natural extension, such as the Nakada α-expansions (for $\sqrt{2}-1 \le \alpha \le 1$), all Rosen fractions, the odd and the even continued fraction expansion et cetera.

The fact that in the case of the RCF we have

$$\text{AF}(\mathcal{A}) = \text{AF}(\mathcal{F})\,, \quad \text{AF}(\mathcal{B}) = \text{AF}(\mathcal{E}), \quad \text{and } \text{AF}(\mathcal{C}) = \text{AF}(\mathcal{D})\,, \tag{33}$$

follows from the properties of the natural extension of the RCF, see also [N], with natural extension map $\mathcal{T} : \Omega = [0,1) \times [0,1] \to \Omega$ given by

$$\mathcal{T}(x,y) = \left(T(x) = \tfrac{1}{x} - a, \frac{1}{a+y} \right), \quad \text{for}(x,y) \in \Omega\,, \text{ with } \tfrac{1}{a+1} < x \le \tfrac{1}{a}\,,\ 0 \le y \le 1\,,$$

and $\mathcal{T}(0,y) = (0,y)$, for $0 \le y \le 1$; see also (5) on page 288.

As a natural extension we have that $\mathcal{T} : \Omega \to \Omega$ is bijective almost surely. So apart from a set of Lebesgue measure zero, $\mathcal{T}^{-1} : \Omega \to \Omega$ exists, and is also a bijection (almost surely). Note that

$$\mathcal{T}^{-1}(x,y) = \left(\frac{1}{a+x}, \tfrac{1}{y} - a \right), \quad \text{for } (x,y) \in \Omega\,, \text{ with } 0 \le x \le 1,\ \tfrac{1}{a+1} < y \le \tfrac{1}{a},$$

so essentially we again find the natural extension of the Gauss map T. We can understand this as follows: if $(x,y) \in \Omega$, where $x = [0; a_1, a_2, \dots]$ and $y = [0; a_0, a_{-1}, a_{-2}, \dots]$ are the RCF-expansions of x resp. y, then - apart from a set of measure zero - we can write (x,y) symbolically also as a bi-infinite sequence

$$[\dots, a_{-3}, a_{-2}, a_{-1}, a_0; a_1, a_2, a_3, \dots]\,,$$

where for each $m \in \mathbb{Z}$ the expression $[0; a_m, a_{m+1}, \dots]$ is the RCF-expansion of some point $x_m \in [0,1]$.

Now $\mathcal{T}$ acts on $[\dots, a_{-3}, a_{-2}, a_{-1}, a_0; a_1, a_2, a_3, \dots]$ as the left-shift τ, i.e. we can write $\mathcal{T}(x,y)$ as

$$\tau\left([\dots, a_{-3}, a_{-2}, a_{-1}, a_0; a_1, a_2, a_3, \dots]\right) = [\dots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1; a_2, a_3, \dots]\,.$$

Note that for each $m \in \mathbb{Z}$ the expression $[0; a_m, a_{m-1}, \dots]$ is the RCF-expansion of some point $y_m \in [0,1]$. Since in $\mathcal{T}^{-1}$ time runs "backwards," we can view $\mathcal{T}^{-1}$ also as the left-shift on

$$[\dots, a_3, a_2, a_1; a_0, a_{-1}, a_{-2}, a_{-3}, \dots].$$

Since $\mathcal{T}$ and $\mathcal{T}^{-1}$ are essentially the same algorithm (but with "time n running forward resp. backward") we find the equalities (33).

Due to the way the OCF and also Minkowski's Diagonal continued fraction (DCF) expansion can be described as S-expansions, it can be shown that these continued fraction algorithms have the same property: the second coordinate map of $\mathcal{T}^{-1}$ of the corresponding natural extension map $\mathcal{T}$ behaves essentially the same as the first coordinate map of $\mathcal{T}$; see [K1, BK1, BK2, K2]. This is essentially due to symmetry in the singularizations yielding these continued fraction expansions. Consequently, (33) also holds for these two continued fraction algorithms.

In fact, one can show[4] that for the DCF one has that

$$\mathrm{AF}(\mathcal{A}) = \mathrm{AF}(\mathcal{F}) = 0.168017\dots, \quad \mathrm{AF}(\mathcal{B}) = \mathrm{AF}(\mathcal{C}) = \mathrm{AF}(\mathcal{D}) = \mathrm{AF}(\mathcal{E}) = 0.165991\dots.$$

In the previous section we found that

$$\mathrm{AF}_N(\mathcal{A}) \neq \mathrm{AF}_N(\mathcal{F}), \quad \mathrm{AF}_N(\mathcal{B}) \neq \mathrm{AF}_N(\mathcal{E}), \quad \mathrm{AF}_N(\mathcal{C}) \neq \mathrm{AF}_N(\mathcal{D}), \tag{34}$$

and also that

$$\frac{\mathrm{AF}_N(\mathcal{A}) + \mathrm{AF}_N(\mathcal{F})}{2} = \mathrm{AF}_O(\mathcal{A}), \tag{35}$$

and similarly for the other two means.

Although we do not have an explanation for these phenomena, we think that they are due to the fact that the NICF and *Hurwitz' singular continued fraction* (see [P], §44) are closely related; again see [K1], Section (2.11). In the latter case, the continued fraction map $T_H : [g-1, g) \to [g-1, g)$ defined by

$$T_H(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - g \right\rfloor, \quad \text{for } x \in [g-1, g) \setminus \{0\},$$

and $T_H(0) = 0$, is used. Nakada showed in [N] that the natural extension Ω_H of this continued fraction algorithm can be obtained by reflecting the natural extension of the NICF in the line $Y = X$ for those points in Ω_N for which $x \geq 0$, and by reflecting (x, y) in $Y = -X$ if $x < 0$; so the natural extension region of T_H is given by

$$\Omega_H = [g-1, g) \times [0, \tfrac{1}{2}].$$

One could say that Hurwitz' singular continued fraction 'suffers reflectedly' from what we noted at the end of Section 3 about the NICF: "Actually, if both regions would have $0 \leq v \leq \frac{1}{2}$ instead of $0 \leq v \leq g^2$ (in the case $\varepsilon_{n+1} = -1$) and $0 \leq v \leq g$ (in the case $\varepsilon_{n+1} = 1$), we could have applied the same maps for $R_a^{-/-}$ and $R_a^{+/+}$ as in the regular and the optimal case."

[4]These calculations are quite similar to those for the OCF in Section 2.

It is not hard to show that the second coordinate-map of $\mathcal{T}_N$ is essentially the map T_H, **not** the map T_N, which explains (34). "Running back the time" now **only** yields that

$$\mathrm{AF}_N(\mathcal{A}) = \mathrm{AF}_H(\mathcal{F}), \quad \mathrm{AF}_N(\mathcal{B}) = \mathrm{AF}_H(\mathcal{E}), \quad \mathrm{AF}_N(\mathcal{C}) = \mathrm{AF}_H(\mathcal{D})$$

and that

$$\mathrm{AF}_N(\mathcal{F}) = \mathrm{AF}_H(\mathcal{A}), \quad \mathrm{AF}_N(\mathcal{E}) = \mathrm{AF}_H(\mathcal{B}), \quad \mathrm{AF}_N(\mathcal{D}) = \mathrm{AF}_H(\mathcal{C}),$$

something which is immediately clear if we view these two continued fraction algorithms as S-expansions; see the examples in [K1], Section (2.11). However, we think that an explanation of (35) should follow from understanding how the singularization areas of the OCF, NICF and Hurwitz' singular continued fraction are related.

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