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UNIVERSITY OF TORONTO

A THEORY OF THE SOUND FROM JETS AND OTHER FLOWS IN TERMS OF SIMPLE SOURCES

BY

H. S. RIBNER



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SUMMARY

An alternative to the quadrupole picture of the generation of flow noise is given in terms of simple sources. In this view the volume of a moving fluid element fluctuates inversely with the local inertial pressure, and this fluctuation radiates the sound. The effective acoustic source strength is $\sim D^2 \dot{p}^{(0)} / Dt^2$, where $\dot{p}^{(0)}$ is the pressure perturbation due to inertial effects and is determined as if the fluid were incompressible. The sources, although individually non-directional, jointly yield a directionality for the radiated sound from jets; it arises in part from convection of the sources as reflected in the character of their two-point covariance with retarded time. Further directionality arises from refraction of the sound field by the mean shear flow. These features are illustrated by examples.

For unbounded low speed flows the equivalence of the simple-source integral and Lighthill's quadrupole integral is examined by means of a momentum balance. For bounded flows (e. g., the flow about a rod producing Aeolian tones) the volume integral of simple sources still describes the primary radiated sound; on the other hand, the volume integral of quadrupoles must be supplemented by a surface integral of dipoles. Similarity considerations for low speed jets recover not only the famous U^6 law for total noise power (U = nozzle velocity) but also the newer laws describing the distribution of noise energy emission with distance x along the jet: these go as x^0 (constant) in the mixing region with a transition to x^7 in the fully developed jet.

The power of the formalism employing the source covariance with retarded time - indicated in the first paragraph - is further demonstrated by additional examples. Calculations for a simulated jet show how narrow frequency bands of the source spectrum appear greatly broadened by convection of the sources past the observer. Corresponding calculations for the radiated sound field automatically produce the correct Doppler-shifted frequencies without implicit introduction of the shift. A final example for simulated static and subsonically moving jets yields comparative directional intensity plots for supersonic nozzle flow speeds: for the moving jet the directional peak is swept back the expected amount.

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NOTATION

A	parameter defined in Eq. (3.2)
A_0	generalization of A, Eq. (5.20)
a	reciprocal scale factor in \mathcal{K} (see e.g., Eq. (3.1))
B	parameter defined in Eq. (3.2)
B_0	generalization of B, Eq. (5.20)
c	speed of sound
c_0	time-average local speed of sound
D	jet nozzle diameter
D/Dt	$\partial/\partial t + u_i \partial/\partial y_i$ (follows instantaneous fluid motion)
$\bar{D}/\bar{D}t$	$\partial/\partial t + U \partial/\partial y_1$ (follows mean fluid motion)
\underline{F}	body force strength/unit volume
F_i	components of \underline{F}
\mathcal{F}_c	Fourier cosine transform (cf. Eqs. (2.26) and (2.26'))
f	frequency
$G(\underline{x}, y, t)$	nonimpulsive part of Green's function, Eq. (4.7)
h	distance of line source from velocity discontinuity, Fig. 11
J	Jacobian, Eqs. (5.16) and (5.20)
$K(\theta, y)$	directional factor in Green's function, Eq. (4.11)
L	a scale of turbulence
l_i	direction cosines of normal to surface element (taken out of the fluid)
M	Mach number, U/c_0 , of source motion or jet flow according to context
M_0	Mach number of uniform stream

m	parameter defined in Eq. (3. 2); also strength per unit volume of sources of matter
m_0	generalization of m , Eq. (5. 20)
P	acoustic power
p	instantaneous pressure ($p = p_0 + p^{(0)} + p^{(1)}$)
p_0	time-average pressure
$p^{(0)}$	'incompressible' component of pressure (\sim near-field)
$p^{(1)}$	'compressible' component of pressure (\sim far-field)
q	acoustic source strength per unit volume
\mathcal{R}	two-point space-time covariance, Eq. (2. 20)
R	nondimensional form of \mathcal{R} ($R = \mathcal{R}(\xi, \tau) / \mathcal{R}(0, 0)$)
\underline{r}	vector from source to field point ($\underline{x} - \underline{y}$)
r	magnitude of \underline{r}
\hat{r}	transform of r , Eq. (5. 3)
s	entropy per unit mass
T_{ij}	instantaneous Reynolds stress plus viscous stress ($\rho u_i u_j + \tau_{ij}$) (differs slightly from Lighthill's definition, Ref. 1)
t	time
\hat{t}	retarded time in fluid at rest ($t - r/c_0$)
$\hat{\hat{t}}$	retarded time in moving stream, Eq. (5. 3)
U	time-average velocity (taken along $y, -$ axis)
U_0	jet nozzle velocity
\underline{u}'	unsteady part of fluid velocity
u'_i	components of \underline{u}'
\underline{u}	resultant fluid velocity

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u_i	components of \underline{u} ($u_i = U\delta_i + u'_i$)
u_x	component of \underline{u} in the \underline{x} direction ($u_x = u_i x_i = \underline{u} \cdot \underline{x}$)
V	effective volume of region of flow
V_0	volume of control surface in momentum balance
$W(\omega_0)$	weighting function defined in Eqs. (3. 10) and (3. 11)
\underline{x}	vector position of observer
x_i	position coordinates
$\hat{\underline{x}}$	transform of \underline{x} , Eq. (5. 5)
Y	length of line distribution of sources
\underline{y}	vector position of source
y_i	position coordinates
$\hat{\underline{y}}$	transform of \underline{y} , Eq. (5. 5)
\mathcal{L}	fluctuation parameter in \mathcal{R} (e. g., Eq. (3. 1))
β	frequency parameter, Eq. (3. 9)
β_0	Mach number parameter ($\beta_0 = \sqrt{1 - M_0^2}$)
Υ	angle defined in Sketch 1
$\delta(t-t')$	Dirac δ -function ($\int_{-\infty}^{\infty} f(t)\delta(t-t')dt = f(t')$)
δ_{ij}	Kronecker δ with values 1 or 0 according as i and j are equal or unequal
\textcircled{H}	Doppler shift coefficient, Eq. (3. 9, 3. 20)
\textcircled{H}_i	fluctuation parameter, Eq. (3. 20)
θ	angle between \underline{x} and x_r -axis
θ_0	transform of θ , Eq. (5. 20)
θ_e	transform of θ for peak intensity, Eq. (5. 21)
λ	wave length

(vii)

ξ	vector separation in two-point covariance
ξ_i	components of separation
$\hat{\xi}$	transform of ξ , Eq. (5.5)
ρ	instantaneous density ($\rho = \rho_0 + \rho^{(0)} + \rho^{(1)}$)
ρ_0	time-average density
$\rho^{(0)}$	'incompressible' component of density ($\rho^{(0)} \approx c_0^{-2} p^{(0)}$)
$\rho^{(1)}$	'compressible' component of density ($\rho^{(1)} \approx c_0^{-2} p^{(1)}$)
σ	parameter proportional to acoustic source strength per unit volume ($\sigma = D^2 p^{(0)} / Dt^2$); also an inverse length scale in one instance
τ	difference in emission times for two source points of separation ξ in fluid at rest; value for simultaneous reception by observer given by Eq. (2.22)
$\hat{\tau}$	generalization of τ for moving stream
τ'	arbitrary increment to the τ of Eq. (2.22)
τ_{ij}	viscous stress tensor: stress in j direction on area with normal in i direction
ω	angular frequency ($\omega = 2\pi f$)
ω_0	specified frequency in the source pattern
ω_c	typical frequency associated with convection of a space pattern ($\omega_c = aU$)
ω_f	typical frequency associated with fluctuation of a space pattern ($\omega_f = \mathcal{L}aU$)

Superscripts

- (o) computed as though flow were incompressible (except $\beta^{(o)}$)
(1) increment due to compressibility
~ associated with single-frequency source pattern

Subscripts

- o time-average or ambient value
 y, \hat{t} evaluated at point y at time \hat{t}

Other

- $\langle \rangle_{AV}$ time average

Note: other symbols of limited use are explained in the text.

I. INTRODUCTION

The mechanism of flow noise was first put on a firm theoretical basis by M. J. Lighthill in fundamental papers published in 1952 and 1954 (Ref. 1). He demonstrated that the sound field could be regarded as generated primarily by fluctuations of momentum flux in the flow. In this view the nine components of momentum flux $\rho u_i u_j$ ($i, j = 1, 2, 3$) in an element of fluid each radiate sound like an acoustic quadrupole.

The basic formalism of the theory is widely accepted as correct. Moreover a simplified dimensional analysis based on the equations was highly successful: it led to the U^8 law relating noise power to jet velocity which has been widely confirmed by experiment. On the other hand a more detailed development accounting for eddy convection was faulty in predicting powers higher than U^8 . Furthermore, attempts to explain the observed directionality in terms of superpositions of the four-leaf-clover pattern of a quadrupole have not been wholly satisfactory (Refs. 1, 2).

Many have found the concept of quadrupole noise generation difficult to visualize correctly. It is true that the fluid elements must distort with virtually no change in volume in a low-speed eddying flow. This gives, for example, the picture of an element being squeezed in at the "waist" and bulging out at the top and bottom. Such a deformation is essentially equivalent to an acoustic quadrupole. This, however, is not the basic Lighthill quadrupole of strength $\rho u_i u_j$. The former is determined by local velocity gradients, the latter by local velocity alone. (Lighthill transformed the basic quadrupole into another compounded of pressure and shear: this quadrupole does correspond in part to the simple deformation).

The conceptual complexities of the quadrupole mechanism together with some of the shortcomings of the theoretical development motivated the search for a simpler picture. This led to the discovery (Ref. 3) that a source-like pulsation of the moving fluid elements can be regarded as generating the sound. The pulsation or fluctuating compression is proportional to the local fluctuating pressure in the flow.* To a sufficient accuracy this pressure may be attributed solely to inertial effects: it may be determined as though the fluid were incompressible.

* The generation of flow noise in terms of simple sources corresponding to the pressure fluctuation rate was first implied in the work of Meecham and Ford (Ref. 4). It was shown explicitly in a development of Corcos and Broadwell (Ref. 5) and in independent work of the present author (Ref. 3). Reference 3 brought the density fluctuations into the picture and gave the physical interpretation.

A simple source at rest radiates sound with spherical symmetry. How then, can we explain the more-or-less heart-shaped emission pattern of a jet in terms of a pattern of sources and sinks? The related problem of directionality from an array of loudspeakers or antennas supplies a partial answer: we know that a proper phasing of sinusoidal source arrays can provide lobes in chosen directions. In effect, such a phasing is provided by the convection of the sound-emitting eddies in a jet. It was found (Ref. 3) that the convection can be introduced into the function describing the statistics of the fluctuating acoustic source strength. This avoids the Lighthill moving-axis technique as well as alleviating the increase over the U^8 law predicted by that technique.

The statistical convection approach yields increased emission in the downstream direction for subsonic speeds. This is hardly the heart-shaped emission pattern of a jet. To explain the downstream dimple in the heart we must consider the refraction of the emitted sound by the velocity gradients of the mean jet flow: the sound rays are turned away from the jet axis to produce (qualitatively at least) such a dimple.

These departures from the concepts of the Lighthill theory were developed with extreme brevity in Reference 3 cited above. The present paper is an attempt at a fairly comprehensive account of this new viewpoint and of the associated mathematical formalism. Moreover, the theory is developed further in several directions. An important refinement is the generalization of the acoustic source strength - $D^2 \rho^{(0)} / Dt^2$ herein so that the derivative D/Dt follows the instantaneous fluid motion rather than the mean motion as in Reference 3.

II. GOVERNING EQUATIONS AND PRIMARILY LOW-SPEED APPLICATIONS

The basic equations governing flow noise are derived in Section 2.1. In later sections convection of the acoustic sources (elements of the "eddies") by any mean flow is neglected although self-convection is allowed for. This effectively restricts the applications in the present chapter to low-speed flows although the limitation is stretched in the treatment of jets.

The main acoustic features - in the absence of surfaces - are exhibited by a fluid without viscosity and heat conduction and with uniform initial entropy. Such a fluid will be postulated in the main text (except Section 2.5) and slight approximations will be made to simplify the analysis. A treatment for a general fluid and of greater rigor is given in Appendices A and B.

(3)

2.1 Governing Equations

Lighthill Equation - Choose a frame of reference at rest in the quiescent fluid outside the disturbed flow. For the specified fluid the exact equations of a continuity and momentum may be combined to give (Ref. 1 and Appendix A herein)

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} \quad (2.1)$$

Here ρ is the density, p the pressure, and u_i is the i th component of the fluid velocity; the indices i, j are summed over 1, 2, 3 when repeated. Postulation of the small disturbance form of the equation of state

$$\rho - \rho_0 = c^{-2}(p - p_0) \quad (2.2)$$

yields approximately

Lighthill
Equation

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} \quad (2.3)$$

(The speed of sound c has been replaced by its time-average C_0 , and higher order terms arising from derivatives of c have been ignored).

Equation (2.3) is an approximate form of Lighthill's equation governing flow noise (cf. Appendix A. 1). Mathematically the expression is of the form of the acoustic wave equation for a spatial distribution of sound sources whose strength per unit volume is given by the right-hand side, $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$. It is equally valid, if the fluid is unbounded, to regard the sound field as generated by quadrupoles of strength $\rho u_i u_j$ (Ref. 1). The equivalence results from the fact that the source strength has the mathematical form of a double divergence. (The equivalence can be demonstrated by two applications of the divergence theorem to the solution of Eq. (2.3) in terms of simple sources (Ref. 6); the procedure is not quite straightforward: see Appendix B. 1.)

Simple Source Equation - Lighthill's effective acoustic source strength $\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j}$ involves in general nonnegligible gradients of the density ρ . These can be eliminated if we reformulate Eq. (2.1) to refer to a volume element moving with the fluid, in a frame moving with the element (cf. Appendix A. 2):

Moving
Frame

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \rho \left[\frac{\partial^2 u_i u_j}{\partial y_i \partial y_j} \right]_{u_i=0} \quad (2.4)$$

(4)

The expression on the right hand side designates the value of $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$ in the moving frame in terms of velocities u_i referred to the stationary frame; this is effected by setting $u_i = u_j = 0$ after the differentiation.

Conversion to a stationary frame changes the operator $\partial/\partial t$ to $D/Dt = \partial/\partial t + u_i \partial/\partial y_i$

$$\text{stationary frame} \quad \frac{D^2 \rho}{Dt^2} - \nabla^2 p = \rho \left[\frac{\partial^2 u_i u_j}{\partial y_i \partial y_j} \right]_{u_i = 0} \quad (2.5)$$

The space gradients, being instantaneous values, are unchanged.

If the fluid were incompressible (but not necessarily of uniform density) the density derivative $D\rho/Dt$ following the fluid motion would vanish. We shall, however, specialize further to a uniform density ρ_0 . For such a fluid

$$- \nabla^2 p^{(0)} = \rho_0 \left[\frac{\partial^2 u_i^{(0)} u_j^{(0)}}{\partial y_i \partial y_j} \right]_{u_i = 0} \quad (2.6)$$

where the superscript (0) designates values as modified by the postulated incompressibility.

Now the turbulent component of a jet flow behaves almost incompressibly up to even low supersonic Mach numbers of the mean flow if shock waves are avoided. This is because the turbulent velocities are an order of magnitude smaller than the mean flow speed. Therefore the velocity gradients and the density appearing on the right-hand side of Eq. (2.5) - in the acoustic source term - may be replaced by incompressible-flow values. That is, the right-hand sides of Eqs. (2.5) and (2.6) may be taken to be equal. (A consideration of the error entailed is given in Appendix A.4). It follows that

$$\frac{D^2 \rho}{Dt^2} - \nabla^2 p = - \nabla^2 p^{(0)} \quad (2.7)$$

Now write the pressure and density as *

*See note following Eq. (2.3) and compare the more accurate Eq. (2.2).

(5)

$$\left. \begin{aligned}
 p &= p_0 + p^{(0)} + p^{(1)} \\
 \rho &= \rho_0 + \rho^{(0)} + \rho^{(1)} \\
 \rho^{(0)} &= c_0^{-2} p^{(0)} \\
 \rho^{(1)} &= c_0^{-2} p^{(1)}
 \end{aligned} \right\} \quad (2.8)$$

where p_0, ρ_0 are the ambient values far from the disturbed region, $p^{(0)}$ satisfies the incompressible flow equation (2.6), and $p^{(1)}$ is the remaining increment of pressure which contains all of the compressibility effects. (The definition of a compression $\rho^{(0)}$ in terms of a pressure $p^{(0)}$ for an incompressible flow may seem odd; it is, however, the first step in an iteration procedure for evaluating a weakly compressible flow). Eq. (2.7) reduces to

$$\frac{D^2 p^{(1)}}{Dt^2} - \nabla^2 p^{(1)} = - \frac{D^2 p^{(0)}}{Dt^2} \quad (2.9)$$

or

$$\frac{1}{c_0^2} \frac{D^2 p^{(1)}}{Dt^2} - \nabla^2 p^{(1)} = - \frac{D^2 p^{(0)}}{Dt^2} = - \frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} \quad (2.10)$$

In the absence of a mean flow ($U = 0$) the convective derivative D^2/Dt^2 on the left-hand side of Eq. (2.10) accounts for the erratic convective-refractive effects of an eddying flow on sound waves passing through it; i. e., it accounts for scattering of the sound by turbulence. This scattering is probably small for jet flow since the wavelengths tend to be \gg the eddy sizes. In any event it can be treated by alternate methods (see e. g., Refs. 7 - 10). We can suppress this scattering by replacing D^2/Dt^2 on the left-hand side by $\partial^2/\partial t^2$ in the case $U = 0$ or by $\bar{D}^2/\bar{D}t^2$ in the more general case of a mean flow $U(y_2, y_3)$ along the y_1 -axis:

$$\boxed{\frac{1}{c_0^2} \frac{\bar{D}^2 p^{(1)}}{\bar{D}t^2} - \nabla^2 p^{(1)} = - \frac{D^2 p^{(0)}}{Dt^2} = - \frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2}} \quad (2.11)$$

where $\frac{\bar{D}}{\bar{D}t} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial y_1}$

The operator $\bar{D}/\bar{D}t$ allows for the effects of the mean flow when $U \neq 0$. These take the form of refraction and diffraction of the sound field together with modifications of the Doppler frequency shift.

Equation (2.10) in the form of the approximation Eq. (2.11) is the governing relation in terms of simple sources for the sound radiated by quasi-incompressible flows.* Mathematically the expression has the form of a modified wave equation for a spatial distribution of sound sources whose strength per unit volume is given by $-D^2\phi^{(0)}/Dt^2$ or its equal $-c_0^{-2} D^2\beta^{(0)}/Dt^2$. This single source term replaces the nine terms of Lighthill's expression $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$ or alternatively the nine quadrupoles $\rho u_i u_j$.

The first form of Eq. (2.11) remains valid for more general flows involving viscosity and even added body forces and sources of matter and heat. (see Appendix A.3). The viscous stresses and other disturbances affect the compressible and incompressible flows similarly: they add virtually identical terms to the respective right-hand sides of Eqs. (2.5) and (2.6) that cancel in the subtraction. (The second form of Eq. (2.11) requires an added entropy term in the case of heat addition).

Physical Interpretation and Discussion - The perturbation pressure within the turbulence and nearby - the acoustic near field - is dominated by $\beta^{(0)}$, the pressure calculated as though the flow were incompressible. The weak compressive part of the pressure, $\beta^{(1)}$, attenuates more slowly with distance than $\beta^{(0)}$ (Appendix B.2) so that $\beta^{(1)}$ ultimately dominates the acoustic far field. Eq. (2.9) states, in effect, that the far field noise (dominated by $\beta^{(1)}$) is driven by the essentially incompressible near field noise (dominated by $\beta^{(0)}$).

The effective strength of simple sources per unit volume - insofar as the far field is concerned - has been determined as $-c_0^{-2} D^2\beta^{(0)}/Dt^2$ or its equal $-D^2\phi^{(0)}/Dt^2$. The physical interpretation of this reverts to an intuitive notion that was abandoned for the more sophisticated picture in terms of quadrupoles: the moving fluid elements simply pulsate (very minutely) and it is this source-like pulsation or dilatation that generates the sound.

The physical reasoning is this. The assumption of uniform density (incompressible flow) yields the field of perturbation pressures $\beta^{(0)}$; these pressures result solely from inertial effects but within the flow they closely approximate the actual pressures. The neglected

* Equations (2.7) and (2.10) are evidently equivalent (see also Appendix B.1), although (2.10) appears the more useful. Similar equations, but with D^2/Dt^2 approximated as $\partial^2/\partial t^2$ have been presented previously (Refs. 3 - 5; see footnote p. 2)

density perturbations are restored in first approximation by putting $\rho^{(0)} = c_0^{-2} p^{(0)}$ according to the isentropic law. One need proceed no further in this iteration procedure: in the actual slightly compressible flow $\rho - \rho_0$ is approximated by $\rho^{(0)}$ to the same order of accuracy as $p - p_0$ is approximated by $p^{(0)}$. A moving element of fluid experiences a rate of dilatation or expansion given by $-D\rho^{(0)}/Dt$ divided by the total density: this is attributable jointly to the inertial effects of the flow and to the fluid compressibility. The same dilatation would be produced in a uniform medium at rest by a rate of mass addition (matter source strength) $-D\rho^{(0)}/Dt$. This dilatation corresponds to an acoustical source strength $-D^2\rho^{(0)}/Dt^2$, the time derivative of the matter source strength; this is the quantity arrived at in our formal analysis.

We can sum up and illustrate as follows. Inertial effects in an eddying flow give rise to regions of high and low pressure (Fig. 1). These are also, respectively, regions of compression and rarefaction. The volume of a moving fluid element fluctuates with time (as the pattern of pressure changes) and with position (as it moves from a rarefied region to a compressed region). The latter aspect is illustrated and enlarged upon in Fig. 2. Both aspects are included in the relation

$$\frac{D\rho^{(0)}}{Dt} = \frac{\partial\rho^{(0)}}{\partial t} + u_i \frac{\partial\rho^{(0)}}{\partial y_i}$$

As a matter of generality it is noted that the acoustic source term allows for non-acoustic or steady aerodynamic compressibility effects. Thus in a steady flow of speed U along y_1 plus perturbations, the term reduces essentially to $-U^2 \partial^2 \rho^{(0)} / \partial y_1^2$ or $-M^2 \partial^2 p^{(0)} / \partial y_1^2$. The left-hand side of Eq. (2.10) reduces to just the Laplacian, giving

$$-\nabla^2 p^{(0)} = -M^2 \frac{\partial^2 p^{(0)}}{\partial y_1^2}$$

Solution of this equation for, say, the flow over a wavy wall would constitute the first step in iteration procedure to allow for $M \neq 0$. The same equation would result from the well known linearized equation of steady compressible flow (Ref. 11).

In conclusion, it is emphasized that the acoustic source term $-D^2\rho^{(0)}/Dt^2 = -c_0^{-2} D^2 p^{(0)}/Dt^2$ yields only the far-field pressure $p^{(0)}$. Knowledge of the near-field pressure $p^{(0)}$ within the flow and nearby is a prerequisite. In the present state of knowledge of turbulent flows neither $p^{(0)}$ nor the associated velocity field $u_i^{(0)}$ is well known: either could be taken as the primary variable. In case $u_i^{(0)}$ is taken as the primary (hypothetically known) variable it is necessary to revert to $\rho_0 u_i^{(0)} u_j^{(0)}$ (quadrupoles) or to $\partial^2 \rho_0 u_i^{(0)} u_j^{(0)} / \partial y_i \partial y_j$ ("divergence" sources) for calculation of $p^{(0)}$. (see Appendices B.2 and B.3 and Refs. 12, 2).

(8)

Magnitude of Self-Convection Source Terms - The acoustic source strength $-\partial^2 \rho^{(0)}/\partial t^2$ may be written in expanded form as

$$-\frac{\partial^2 \rho^{(0)}}{\partial t^2} = -\frac{\partial^2 \rho^{(0)}}{\partial t^2} - 2u_i \frac{\partial^2 \rho^{(0)}}{\partial y_i \partial t} - u_i u_j \frac{\partial^2 \rho^{(0)}}{\partial t^2} \quad (2.12)$$

In the absence of a mean flow u_i and u_j result solely from the unsteady flow. The last two terms of Eq. (2.12) represent dilatational effects (effective acoustic sources) due to convection of fluid from regions of rarefaction to regions of compression by this eddy flow.

In earlier presentations (Refs. 3 - 5) only the term $-\partial^2 \rho^{(0)}/\partial t^2$ was in effect employed: the eddy self-convection terms were neglected. Let us estimate the relative orders of magnitudes of the neglected terms which have now been restored. Take a typical Fourier component of the density field $\rho^{(0)}$ as

$$\rho^{(0)} = A \exp i(\omega t + ky) ; \quad \begin{aligned} \omega &= 2\pi f \\ k &= 2\pi/L \end{aligned}$$

Then the ratios of the added terms to $\partial^2 \rho^{(0)}/\partial t^2$ are approximated by

$$\left. \begin{aligned} \frac{|u \partial^2 \rho^{(0)}/\partial y \partial t|}{|\partial^2 \rho^{(0)}/\partial t^2|} &\sim \frac{ku}{\omega} = \frac{u}{fL} \\ \frac{|u^2 \partial^2 \rho^{(0)}/\partial y^2|}{|\partial^2 \rho^{(0)}/\partial t^2|} &\sim \left(\frac{ku}{\omega}\right)^2 = \left(\frac{u}{fL}\right)^2 \end{aligned} \right\} \quad (2.13)$$

where f is a typical frequency and L is the associated length scale in the turbulence.

The quantity u/fL is a sort of reciprocal Strouhal number (cf. Ref. 13); its value is estimated in Appendix C from considerations concerning the two-point space covariance of pressure or velocity, using data of Richards and Williams (Ref. 14). A typical value for the jet mixing region is

$$\frac{|u \partial \rho^{(0)}/\partial y|}{|\partial \rho^{(0)}/\partial t|} \sim \frac{u}{fL} = O(2/3) \quad (2.14)$$

Upon insertion of this value in Eqs. (2.13) it appears that the added terms are quite comparable in magnitude with $\partial^2 \rho^{(0)}/\partial t^2$. Thus their omission cannot be justified for jet flows.

This assessment refers to turbulent shear flows such as boundary layers and jets; the order-of-magnitude estimate does not apply to homogeneous turbulence.

2.2 Radiated Sound Pressure and Spectrum

Mean Square Pressure - It will be recalled that the perturbation pressure is divided herein into an "incompressible" part $p^{(i)}$ and a "compressible" part $p^{(c)}$. The part $p^{(i)}$ dominates within the basic aerodynamic flow but attenuates rapidly (as x^{-3}) with distance (Appendix B. 2). The part $p^{(c)}$ attenuates more slowly (as x^{-1}) with distance and so dominates at large distances from the flow. We call $p^{(c)}$ the radiated sound pressure.

For the low speed flows of this chapter the mean speed U may be neglected in the operator $\bar{\partial}/\bar{\partial}t \equiv \partial/\partial t + U\partial/\partial y_1$. The governing equation (2. 11) for the radiated sound pressure $p^{(c)}$ reduces to

$$\frac{1}{c^2} \frac{\partial^2 p^{(c)}}{\partial t^2} - \nabla^2 p^{(c)} = - \frac{1}{c^2} \frac{D^2 p^{(i)}}{Dt^2} \quad (2.15)$$

This is the acoustic wave equation for a spatial distribution of sources given by the right-hand side. The solution for the pressure $p^{(c)}$ at point \underline{x} and time t reads

$$p^{(c)}(\underline{x}, t) = - \frac{1}{4\pi c^2} \int \frac{\sigma(\underline{y}, \hat{t})}{|\underline{x} - \underline{y}|} d\underline{y} \quad ; \quad \begin{aligned} \sigma &\equiv D^2 p^{(i)} / Dt^2 \\ \hat{t} &= t - |\underline{x} - \underline{y}| / c. \end{aligned} \quad (2.16)$$

Strictly speaking, the integral is over all space, but in practice it may be limited to an effective volume V of the disturbed region - the aerodynamic flow. At distances large compared with the dimensions of V this reduces to

$$\begin{array}{l} \text{Far} \\ \text{Field} \end{array} \quad p^{(c)}(\underline{x}, t) = - \frac{1}{4\pi c^2 x} \int \sigma(\underline{y}, \hat{t}) d\underline{y} \quad (2.17)$$

The approximation $|\underline{x} - \underline{y}| \approx x$ is made in the denominator but not in the retarded time \hat{t} .

The corresponding mean square pressure at large distances may be written*

* The superscript ^(c) has been dropped for simplicity, since $p^{(c)}$ is the sole surviving pressure perturbation in the far-field.

(10)

$$\text{Far Field} \quad \langle p^2(x) \rangle_{AV} = \frac{1}{16\pi^2 c_0^2 x^2} \iint \langle \sigma(y', \hat{t}') \sigma(y'', \hat{t}'') \rangle_{AV} dy' dy'' \quad (2.18)$$

the average being over time. The adjective "far-field" applies if we make the additional restriction that $2\pi x \gg$ wave length of sound. In this case the plane-wave relations are valid, and $\langle p^2 \rangle_{AV} = \text{acoustic intensity} / \rho_0 c_0$.

It will be convenient to change variables. Let y be the midpoint between the two points y' and y'' , and let ξ be their separation:

$$\left. \begin{aligned} y &= \frac{1}{2} (y' + y'') \\ \xi &= y' - y'' \end{aligned} \right\} \quad (2.19)$$

The integration limits are again infinite in y and ξ . In practice the limit on y may be reduced to the effective flow volume V while retaining (for convenience in calculation) the infinite limits on ξ . The mean square pressure now reads \bar{p}

$$\langle p^2(x) \rangle_{AV} = \frac{1}{16\pi^2 c_0^2 x^2} \int_V dy \int_{-\infty}^{\infty} \mathcal{R}(\xi, \tau; y) d\xi \quad (2.20)$$

$$\mathcal{R} \equiv \langle \sigma(y + \xi/2, \hat{t}') \sigma(y - \xi/2, \hat{t}'') \rangle_{AV}$$

Here \mathcal{R} is the two-point covariance of the source strength* $\sigma \equiv D^2 p^{(s)} / Dt^2$; \mathcal{R} is a function of the space separation ξ of the two points, the time separation or delay τ and the midpoint y . The time relay τ is given by

$$\begin{aligned} \tau &= \tau(\xi, \theta) = \hat{t}' - \hat{t}'' & ; & \quad r' = |r'| = |x - y'| \\ & & & \quad r'' = |r''| = |x - y''| \\ &= (r'' - r') / c_0 & ; & \quad r' = |r'| = |x - y'| \end{aligned} \quad (2.21)$$

\bar{p} It is assumed in what follows that the flow field is a statistically stationary (not necessarily random) process; that is, long-term time averages do not vary with time.

* A factor of $-c_0^{-2}$ has been omitted for simplicity in indentifying σ with the source strength.

(11)

Since \underline{x} , \underline{r}' and \underline{r}'' are essentially parallel at inclination θ to the x_1 axis, the construction in Sketch 1 shows that *

$$\begin{aligned} \tau(\underline{x}, \theta) &\approx c_0^{-1} |\underline{y}' - \underline{y}''| \cos \gamma \\ &\approx c_0^{-1} \underline{\xi} \cos \gamma \end{aligned}$$

or

$\tau \approx \underline{\xi} \cdot \underline{x} / x c_0$

(2.22)

Sketch 1. Determination of Time Delay τ

Thus to a close approximation the time delay τ equals c_0^{-1} times the component of the space separation $\underline{\xi}$ in the direction of propagation.

Autocovariance and Spectrum. - The autocovariance $\langle p p' \rangle_{AV}$ of the far-field sound pressure at a point \underline{x} is the time average $\langle p(t) p(t + \tau) \rangle_{AV}$, where the time delay τ is held fixed. We proceed now to evaluate the autocovariance as a step toward determining the pressure spectrum.

The derivation of $\langle p^2 \rangle$, Eqs. (2.18) to (2.20), is generalized to yield $\langle p p' \rangle_{AV}$ by addition of the arbitrary time delay τ to the value of t'' . This yields the autocovariance in the form

$$\langle p p'(\underline{x}, \tau) \rangle_{AV} = \frac{1}{16\pi^2 c_0^2 x^2} \int_V d\underline{y} \int_{-\infty}^{\infty} R(\underline{\xi}, \tau + \tau'; \underline{y}) d\underline{\xi} \quad (2.23)$$

where τ remains the time for sound to travel the projected distance $\underline{\xi} \cdot \underline{x} / x$ (Sketch 1 and Eq. (2.22).)

The spectral density or spectrum function may be written as

* The points $O, \underline{y}', \underline{y}''$, and \underline{x} do not, in general, lie in the same plane.

(12)

$$\Phi(\omega) = \frac{d\langle p^2 \rangle_{AV}}{d\omega} \quad (2.24)$$

This form illustrates the property

$$\langle p^2 \rangle_{AV} = \int_0^{\infty} \Phi(\omega) d\omega = \int_0^{\infty} \frac{d\langle p^2 \rangle_{AV}}{d\omega} d\omega \quad (2.25)$$

Physically, $\Phi(\omega)d\omega$ is the contribution to the mean square pressure $\langle p^2 \rangle_{AV}$ from angular frequencies in a band $d\omega$ centered about ω where $\omega = 2\pi f$.

It is well-known that the spectral density and autocovariance are simply Fourier cosine transforms of each other (cf. e. g., Ref. 15) Thus in a shorthand notation

$$\begin{aligned} \frac{d\langle p^2 \rangle_{AV}}{d\omega} = \Phi(\omega) &= \frac{2}{\pi} \mathcal{F}_c \{ \langle p p'(\tau) \rangle_{AV}; \omega \} \\ \langle p p'(\tau) \rangle_{AV} &= \mathcal{F}_c \{ \Phi(\omega); \tau' \} \end{aligned} \quad (2.26)$$

or written out

$$\begin{aligned} \frac{d\langle p^2 \rangle_{AV}}{d\omega} = \Phi(\omega) &= \frac{2}{\pi} \int_0^{\infty} \langle p p'(\tau) \rangle_{AV} \cos \omega \tau' d\tau' \\ \langle p p'(\tau) \rangle_{AV} &= \int_0^{\infty} \Phi(\omega) \cos \omega \tau' d\omega \end{aligned} \quad (2.26')$$

The first of Eqs. (2.26) or (2.26') is the desired relation for the spectral density. A prerequisite is the evaluation of the autocovariance, Eq. (2.23). (In a variation of this approach $\Phi(\omega)$ is given by an integral of the form of Eq. (2.23) with \mathcal{R} therein replaced by its (complex) Fourier transform, the two-point cross-spectral density of the source strength with retarded time (Ref. 3). The present approach appears, however, to simplify the integrations involved).

Since $\langle p p'(0) \rangle_{AV}$ is just $\langle p^2 \rangle_{AV}$, the second of Eqs. (2.26') includes Eq. (2.25) as a special case.

The use of these equations is illustrated in Section III.

Correlation and Correlation Volume - For some applications it will be convenient to replace the covariance \mathcal{R} by the nondimensional 'correlation' R , given by*

* The function R is more convenient for the present purpose than the true correlation $\langle \sigma \sigma' \rangle_{AV} / [\langle \sigma^2 \rangle_{AV} \langle \sigma'^2 \rangle_{AV}]^{1/2}$; they differ only for nonhomogeneous turbulence, for which R may exceed unity.

(13)

$$R = \langle \sigma_y^2 \rangle_{AV} R = \left\langle \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_y \right\rangle_{AV} R \quad (2.27)$$

Eq. (2.20) becomes

$$\langle p^2(\underline{x}) \rangle_{AV} = \frac{1}{16\pi^2 c_0^4 x^2} \int_V \left\langle \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_y^2 \right\rangle_{AV} dy \int_{-\infty}^{\infty} R(\underline{\xi}, \tau; y) d\underline{\xi} \quad (2.28)$$

The inner integral has the dimensions of a length L cubed and may be called the "correlation volume": if the two-point correlation R were unity within this volume (i. e., $R = \langle \sigma_y^2 \rangle_{AV}$ therein) and zero outside, the noise radiation would be the same. Thus Eq. (2.28) may be written in the form

$$\langle p^2(\underline{x}) \rangle_{AV} = \frac{1}{16\pi^2 c_0^4 x^2} \int_V \left\langle \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_y \right\rangle_{AV} L^3 dy \quad (2.29)$$

where $\overline{L^3}$

$$L^3 = L^3(y, \underline{x}/x) \equiv \int_{-\infty}^{\infty} R(\underline{\xi}, \tau; y) d\underline{\xi} \quad (2.30)$$

Physically the correlation volume L^3 may be interpreted as the effective volume of a turbulent eddy considered as a coherently radiating entity. Note that the retarded time τ introduces a dependence of L^3 on direction \underline{x}/x (see Eq. (2.22)).

2.3 Are the Quadrupole and Simple Source Solutions Equivalent?

Momentum Balance. - Lighthill's quadrupole solution (Ref. 1) for the acoustic density perturbation at a large distance x from the generating flow ($2\pi x \gg \lambda$) is given by his Eq. (17) (Part I). The corresponding pressure perturbation is c_0^2 times this according to the isentropic law:

$$p(\underline{x}, t) = \frac{x_i x_j}{4\pi c_0^2 x^3} \int_V \left[\frac{\partial^2 \rho u_i u_j}{\partial t^2} \right]_{\hat{t}} dy \quad ; \quad \hat{t} = t - r/c_0 \quad (2.31)$$

\overline{L} The length L as here defined is a special scale of turbulence; it differs from the so-called integral scale

$$L_1 = \int_0^{\infty} R(\xi, 0, 0) d\xi$$

A generalization of L to allow for variation in scale for different directions (nonspherical correlation volume) is given in Appendix B. 4.

(The superscript ⁽⁰⁾ has been dropped from the left-hand side, for simplicity). Proudman (Ref. 16) has pointed out that the double summation $x_i x_j u_i u_j / x^2$ reduces to just u_x^2 , the square of the velocity component in the direction of x . Thus the quadrupole relation Eq. (2.31) and the simple-source relation Eq. (2.17) assume somewhat parallel forms:

$$\text{quadrupole: } \phi = \frac{1}{4\pi c_0^2 x} \int_V \left[\frac{\partial^2 (\rho u_x^2)}{\partial t^2} \right]_{\hat{t}} dy \quad (2.32)$$

$$\text{simple-source } \phi = - \frac{1}{4\pi c_0^2 x} \int_V \left[\frac{D^2 \phi^{(0)}}{Dt^2} \right]_{\hat{t}} dy \quad (2.33)$$

Now it can be argued that Eqs. (2.32 and (2.33) should be equivalent since both were derived with negligible approximation from the exact equations of continuity and momentum for a fluid. However, the derivations were indirect and the negligibility of the approximations may be a matter of controversy. Therefore a fairly direct proof of the equivalence of the two equations would appear to be desirable. There will be no loss in generality if the vector x is taken to lie along the y_1 -axis. Then the compatibility of Eqs. (2.32) and (2.33) would require that

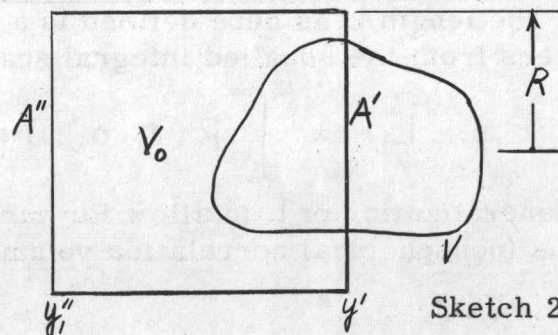
$$\int \left(\frac{D^2 \phi^{(0)}}{Dt^2} + \frac{\partial^2 \rho u_x^2}{\partial t^2} \right)_{\hat{t}} dy = 0 \quad (2.34)$$

No direct proof of this relation has been found. However, if we neglect the compressibility of the fluid in ρu_x^2 and the convective terms in $D^2 \phi^{(0)} / Dt^2$ (which on one interpretation result from compressibility in the expansion of $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$) there results

$$\int \frac{\partial^2}{\partial t^2} \left(\phi^{(0)} + \rho_0 u_x^{(02)} \right)_{\hat{t}} dy = 0 \quad (2.35)$$

This last expression can be proved by use of a momentum balance for incompressible flow: this proof is given below. Since the omitted convective terms in $D^2 \phi^{(0)} / Dt^2$ are comparable with the retained term $\partial^2 \phi^{(0)} / \partial t^2$ (see Sec. 2.1) proof of Eq. (2.35) indicates at least an order of magnitude agreement between Eqs. (2.32) and (2.33).

As a control surface take a cylinder of radius R concentric with the y_1 -axis and with the face A' (given by $y_1 = y_1'$) cutting through the region V of flow (Sketch 2). The conservation equations for momentum and mass in an incompressible flow read (where the superscript ⁽⁰⁾ has been dropped from u_i for simplicity),



Control Surface for Momentum Balance

(15)

$$\frac{\partial}{\partial t} \iiint_{V_0} \rho_0 u_i dV \pm \iint_{A, A''} (\rho + \rho^{(0)} + \rho_0 u_i^2) dA + \int_C \rho_0 u_i u_r dA = -X_i \quad (2.36)$$

$$\pm \iint_{A, A''} \rho_0 u_i dA + \iint_C \rho_0 u_r dA = 0 \quad \text{or constant} \quad (2.37)$$

where the + of \pm refers to A', the - to A'', C to the curved boundary, and X_i is the force on the fluid (e. g., negative of the jet thrust), specified to be constant.

It is assumed that the fluctuating terms fall off sufficiently fast with distance so that for $R \rightarrow \infty$, $y_i^* \rightarrow -\infty$ the fluctuating parts of the integrals over C and A'' approach zero. Then differentiation of Eq. (2.37) yields

$$\frac{\partial}{\partial t} \iint_{A'} \rho_0 u_i dA = 0 \quad (2.38)$$

This holds for any location $y_i = y_i'$ of A'. If we multiply by dy_i' and integrate with respect to y_i' the integral can range over the volume V_0 . The result is

$$\frac{\partial}{\partial t} \iiint_{V_0} \rho_0 u_i dV = 0 \quad (2.39)$$

Thus the first term in Eq. (2.36) vanishes because of conservation of mass.

If now the differentiation $\partial^2/\partial t^2$ is applied to Eq. (2.36) only the integral over A' will contribute: the other integrals, it has been noted, are zero or constant in time. The result is

$$\frac{\partial^2}{\partial t^2} \iint_{A'} (\rho^{(0)} + \rho_0 u_i^2) dA = 0 \quad (2.40)$$

A righthand term $-\partial^2 X_i / \partial t^2$ vanishes because we postulate zero or constant thrust: this restriction is implicit in Lighthill's work (i. e., unbounded flow, no immersed bodies such as a jet engine which may experience fluctuating surface forces).

Eq. (2.40) holds for any choice of the time; thus the retarded time \hat{t} may be selected. Furthermore, the differentiation may be carried under the integration sign since the limits are constant. Again we multiply by dy_i' - so that $dA dy_i' = dV$ - and allow the integration to range over V . The result is

$$\iiint_V \frac{\partial^2}{\partial t^2} [\rho^{(0)} + \rho_0 u_i^2]_{\hat{t}} dV = 0$$

Thus Eq. (2.35) is proved.

Discussion of Lighthill Source Term. - Appendix A. 2
 develops the following expansion of Lighthill's acoustic source term for
 flow noise:

$$\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} = \rho \left[\frac{\partial^2 u_i u_j}{\partial y_i \partial y_j} \right]_{u_i=0} - 2 u_i \frac{\partial^2 \rho}{\partial y_i \partial t} - u_i u_j \frac{\partial^2 \rho}{\partial y_i \partial y_j} \quad (2.41)$$

Only the first term on the right-hand side may in general be approximated closely in terms of quantities computed for an incompressible flow. The density gradients in the remaining terms represent initially unknown compressibility effects.

If these unknown density gradient terms are moved to the left-hand side of Eq. (2.1) to join the equally unknown $\partial^2 \rho / \partial t^2$, the result is Eq. (2.5). This is a key step in the derivation of the simple-source relation for flow noise, Eq. (2.11).

Lighthill, on the other hand, in effect circumvented the difficulty posed by Eq. (2.41) by avoiding direct use of the simple-source term $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$. He employed transformations like those of Appendix B to obtain an integral (Eq. 2.31) involving an integrand $\partial^2 \rho u_i u_j / \partial t^2$ corresponding to a quadrupole distribution. It is easy to show that the density derivatives in this are negligible when $U=0$.

For the case of a substantial mean flow, as in a jet of high subsonic speed, the density derivatives are not negligible: Lighthill's quadrupole integrand can no longer be approximated as $\partial^2 \rho_0 u_i^{(0)} u_j^{(0)} / \partial t^2$, the value for an incompressible flow. To show this we first restore the omitted directionality factor (Eq. B2.5c) to the integrand: it is

$$\frac{x_i x_j}{x^2} \frac{\partial^2 \rho u_i u_j}{\partial t^2}$$

Now rotate the y_1 -axis to coincide with the \underline{x} -direction which makes an angle θ with the stream direction: the integrand becomes

$$\frac{\partial^2 \rho u_1^2}{\partial t^2}$$

where $u_1 = U \cos \theta + u_1'$

The formal differentiation yields

$$\frac{\partial^2 (\rho u_1^2)}{\partial t^2} = \rho \frac{\partial^2 u_1^2}{\partial t^2} + 2 u_1 \frac{\partial u_1}{\partial t} \frac{\partial \rho}{\partial t} + u_1^2 \frac{\partial^2 \rho}{\partial t^2} \quad (2.42)$$

If we neglect u_1 in comparison with $U \cos \theta$ and approximate ρ by $\rho_0 + \rho^{(0)}$, u_1' by $u_1^{(0)}$, the quasi-incompressible values, there results

$$\frac{\partial^2(\rho u_1^2)}{\partial t^2} \approx \rho_0 \frac{\partial^2 u_1^{(0)2}}{\partial t^2} + 2U \cos \theta \frac{\partial u_1^{(0)}}{\partial t} \frac{\partial \rho^{(0)}}{\partial t} + U^2 \cos^2 \theta \frac{\partial^2 \rho^{(0)}}{\partial t^2} \quad (2.43)$$

Now $\rho^{(0)} \sim \rho_0 u_1^{(0)2}$ (neglecting the mean shear)

whence $\rho^{(0)} \sim c_c^{-2} \rho_c u_1^{(0)2}$

and $U^2 \cos^2 \theta \frac{\partial^2 \rho^{(0)}}{\partial t^2} \sim \rho_0 \frac{U^2}{c_c^2} (\cos^2 \theta) \frac{\partial^2 u_1^{(0)2}}{\partial t^2}$

Thus the third term on the right-hand side $\sim M^2 \cos^2 \theta$ times the first term. A similar argument suggests that the second term is smaller by a factor $2u_1^{(0)}/U \cos \theta$. It appears then that the incompressibly calculated first term must be supplemented by at least the third when $M^2 \cos^2 \theta$ is not $\ll 1$.

The above result neglects amplification of $\rho^{(0)}$ by the mean shear in a jet (see Sec. 4.2). The amplification factor in an idealized case is the effective nondimensionalized mean shear $(4/15)^{1/2} \partial(U/u)/\partial \sigma n$ where σ is an eddy scale length (Eq. 4.4'). This factor, which can exceed unity, may magnify the second and third terms.

2.4 Jet Acoustic Power: the U_0^8 , x^0 , and x^{-7} Laws

The most well known result of Lighthill's quadrupole theory was obtained by the use of considerations of flow similarity (or, from another point of view, dimensional analysis) in comparing the noise output of jets of different nozzle diameter D and velocity U_0 . By these means he deduced the famous $D^2 U_0^8$ law (Ref. 1). A more detailed application of similarity considerations - again in the context of the quadrupole theory - has yielded the noise power emitted by successive "slices" of an idealized cold jet as a function of axial distance x downstream of the nozzle (Ref. 17)*. Slices of jet in the mixing region are predicted to emit the same noise power (x^0 law), and in the fully developed jet the emission of successive slices is predicted to fall off extremely fast (x^{-7} law). The present source-sink theory of jet noise leads very simply to the same laws. This is demonstrated below.

* The deduction of the x^0 and x^{-7} similarity laws was first reported at the Ann. Mtg., Acous. Soc. Amer., Washington, May 7-10, 1958. At a later presentation (1st. Internat. Cong. Aero. Sci. (ICAS), Madrid, Sept. 8-13, 1958; see Ref. 18) E. J. Richards and M. J. Lighthill took exception to some of the theoretical foundations. Confirmation of these laws has in the meantime come from independent work of Lilley (Ref. 2) and of Powell (Ref. 19). Conversations with Richards and with Lighthill have indicated that they no longer maintain their objections.

(18)

In all of the cited work and in the present section convection of the sources by the mean jet flow is neglected insofar as it affects the sound power. This point is commented on at the end of the section.

We return to the mean square pressure $\langle p^2 \rangle_{AV}$ in the far field, Eqs. (2.29) and (2.30). The acoustic intensity I (energy flow in the \underline{x} direction per unit time per unit area normal to \underline{x}) is $\langle p^2 \rangle_{AV} / \rho_0 c_0$. The total power P is the integral over a sphere of radius x ; this introduces a factor $4\pi x^2$ and yields

$$P = \frac{1}{2\pi\rho_0 c_0^5} \int_V L^3 \left\langle \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_y^2 \right\rangle_{AV} dy_f \quad (2.44)$$

if as a simplifying approximation the time delay τ is ignored. (If τ is retained the correlation volume or effective source size L^3 is a function of direction \underline{x} , and an average value must be used. Further consequences of the neglect of τ - and of source convection - are discussed at the end of this section).

For the purpose of developing similarity laws write Eq. (2.44) in differential form, omitting the proportionality constant and writing the volume element dy as dV

$$dP \sim \frac{L^3}{\rho_0 c_0^5} \left\langle \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_y^2 \right\rangle_{AV} dV \quad (2.45)$$

Thus dP is the acoustic power emitted by a volume element dV . Now in an idealized model of a jet there exist two regions where the profiles of mean and turbulent velocities are invariant with x when expressed nondimensionally; at corresponding points of these regions (i. e., along certain rays) the turbulent and mean velocities maintain a fixed proportionality: $\langle u_i^2 \rangle_{AV} \sim U^2$. A similar proportionality applies to corresponding points of two complete jets. Also (Refs. 12, 2) $p^{(0)} \sim \rho_0 \bar{u}_i^2 \sim \rho_0 U^2$ (similar regions)* and according to Lighthill's ideas typical frequencies are proportional to U/L , whence

$$\partial/\partial t \sim U/L \quad (\text{similar regions})$$

Accordingly Eq. (2.45) yields

$$dP \sim \frac{\rho_0 U^8}{c_0^5 L} dV \quad (\text{similar regions}) \quad (2.46)$$

Eq. (2.46) is the basic relation for comparing similar regions and it is identical with Eq. (14) of Ref. 17(a) and Eq. (2) of Ref. 17(b). The equation may be applied as shown in Table I (the integral form is used in column 1).

* An amplification factor representing the nondimensionalized mean shear has been omitted (cf. last paragraph of Sec. 2.3) since it is invariant for similar regions.

TABLE I

Complete Jet	Annular Mixing Region	Fully Developed Region
Assumed similarity:	$0 < x \leq 4D$ Assumed similarity:	$8D < x < \infty$ Assumed similarity:
$\int dV \sim D^3$	$dV \sim x D dx$ (annulus)	$dV \sim x^2 dx$ (circle)
$L_{AV} \sim D$	$L \sim x$	$L \sim x$
$U \sim U_0$	$U \sim U_0$	$U \sim U_0 D/x$
$P \sim \frac{\rho_0 U_0^8 D^2}{c_0^5}$	$dP \sim \frac{\rho_0 U_0^8 D}{c_0^5} dx$	$dP \sim \frac{\rho_0 U_0^8 D}{c_0^5} \left(\frac{x}{D}\right)^{-7} dx$
	$\frac{dP}{dx} \sim \frac{\rho_0 U_0^8 D}{c_0^5} = (\text{constant})$	$\frac{dP}{dx} \sim \frac{\rho_0 U_0^8 D}{c_0^5} \left(\frac{x}{D}\right)^{-7}$
$(U_0^8 \text{ Law})$	$(x^0 \text{ Law})$	$(x^{-7} \text{ Law})$

(19)

These results are, in part, exhibited in Fig. 3. Slices of jet within four diameters of the nozzle are predicted to emit the same noise power ($dP/dx = \text{constant}$: x^0 law); and beyond eight diameters the emission decreases like x^{-7} . The area under the curve represents the noise power emitted by the entire jet, and this is proportional to U_0^8 .

The nonuniform turbulent properties across each slice of the jet are bypassed (but not violated) in the foregoing derivation of the U_0^8 , χ^0 and χ^7 laws. A more detailed derivation employing functional expressions for the profiles of the turbulent properties is given in Ref. 17(a); the starting point is likewise Eq. (2.45)

The present derivation shares with Lighthill's original deduction of the U_0^8 law the neglect of convective and refractive effects of the mean flow including (cf. remark following Eq. (2.44)) suppression of the time retardation in the integrals. It is shown later in the present paper how these effects yield the directionality of the jet noise (in part via a dependence of L^3 on direction). The computations are, however, too idealized to determine whether the convective effect notably enhances the emitted power. The excellent agreement of the U_0^8 law with experiment (Ref. 20) suggests the enhancement effect is either constant with speed (Ref. 21) or small (Ref. 22). Theoretical arguments are given in the cited references.

2.5 Effects of Bounding Surfaces in the Flow

The primary sound field radiated by an aerodynamic flow can be represented in terms of simple sources alone regardless of the presence or absence of bounding surfaces. The expression, Eq. (2.17), for the primary sound field in terms of simple sources is unaffected by such surfaces: this is proved in Appendix B3*.

On the other hand, the sound generated can be represented in terms of quadrupoles alone only if the fluid is unbounded. If bounding surfaces are present surface distributions of sources and dipoles must be added; the appropriate expressions have been derived by Curle (Ref. 6).

The surface source-dipole distribution on the quadrupole theory by itself yields the dominant far-field radiation for low speed flows, but a misleading nonzero result near the surfaces.[†] Of course, the acoustic energy flux must approach zero close to a fixed surface: the energy flux is the product of the normal component of perturbation velocity near the surface - which vanishes - and the perturbation pressure.

* Bounding surfaces or obstacles serve, however, to reflect and diffract (scatter) the incident sound derivable from the simple source distribution. Added terms to describe this scattering are included in the derivation of Appendix B3.

[†] However, when taken together with the quadrupole distribution the resultant radiation is physically correct and it presumably exhibits a zero value at the surface.

The sound actually originates in a more or less extended region bounded by the surface, but not from the surface itself. The proof lies jointly in two facts: first, the (primary) sound field is given by an integral of simple sources over the extended region; second, the simple sources may be interpreted physically as direct sources of sound (Sec. 2. 1).

When the influence of the surface is dominant (i. e., Aeolian tones, boundary layer noise) the main virtue of the simple-source formulation may perhaps be limited to this physical interpretation. For calculation purposes for such flows the Curle surface dipole terms by themselves give a good approximation to the resultant sound field at a distance, provided the flow is of low speed. The surface terms, moreover, have a neat interpretation in terms of surface stresses and are mathematically simple.

From the comparison it is observed that Lamb's expressions for the acoustic radiation from a fluctuating force or surface stress (Refs. 23, 24) - although derived for a medium at rest - have a wider applicability. They give the correct far field whether the surface stresses arise from motion of a surface in a fluid otherwise at rest, or from unsteady motion of a fluid opposed by a stationary surface. In the latter case, however, they give the false result of nonzero radiation at the surface, and must on this account be supplemented by a quadrupole integral to provide the near field.

III. MOVING SOURCES IN A STATIONARY FLUID: 'CONVECTIVE' EFFECTS ON SOUND DIRECTIONALITY AND SPECTRA

3. 1 Relationship to Jet Noise

The sound sources in a jet - the turbulent eddies - are convected along by the mean flow. The effects on the directionality of the radiated sound are two-fold. A convective effect arises from the motion of the sound sources with respect to the quiescent fluid outside the jet. A refractive effect is due to gradient of the mean velocity within the jet.

These effects are not linearly superposable. However, it will be illuminating to look at the convective aspect separately in an idealized situation: we consider the jet turbulence to be replaced by a pattern of acoustic sources of strength σ moving through fluid at rest.* The mean jet flow that in a real jet transports the pattern does not then figure in the governing equations and the refractive aspect is suppressed.

*A factor $-c_0^{-2}$ is omitted for simplicity in referring to σ as the "source strength" here and later on.

The examples refer to a random distribution of acoustic sources but not necessarily to a possible turbulent flow; that is, the chosen covariance of the source strength σ is not necessarily compatible with $D^2 p^{(0)}/Dt^2$ in a realizable fluid motion*. The form of the covariance of σ has been chosen largely for mathematical simplicity. As a special feature functions with regions of negative covariance have been avoided because they complicate the otherwise simple picture of 'convective' effects on sound directionality and spectra. The degree to which this picture can be applied to jet noise can only be speculated on in the absence of experimental values of the σ covariance, and in view of the remarks of the first two paragraphs.

3.2 Convected Volume Pattern of Sources: Example

Consider a random pattern of acoustic sources of strength $\sigma(y, t)$, homogeneous and isotropic within a volume V , but vanishing outside. The pattern is continuously created at the left face, moves continuously through V with the uniform speed U , and is destroyed at the right face (see inset, Fig. 4). The picture is rather like that of the moving pattern of clouds seen through an airplane window, a two-dimensional analog of the volume V .

The statistics of the source pattern govern the sound radiation according to

$$\langle p^2 \rangle_{AV} = \frac{1}{16\pi^2 c_0^2 x^2} \int_V dy \int_{-\infty}^{\infty} \mathcal{R}(\xi, \tau; y) d\xi \quad (2.20)$$

in terms of the parameter \mathcal{R} . Here \mathcal{R} is the two-point space-time covariance of the source strength

$$\mathcal{R} \equiv \langle \sigma \sigma' \rangle_{AV} \equiv \langle \sigma(y + \frac{1}{2}\xi, t + \tau) \sigma(y - \frac{1}{2}\xi, t) \rangle_{AV}$$

the average being over the time t . A hypothetical form for \mathcal{R} that allows both for convection and fluctuation of the pattern is

* It is known that the covariance of the σ in the first example is incompatible with a real flow when σ is identified with $\partial^2 p^{(0)}/\partial t^2$. Such an identification was implied as an oversimplification in Ref. 3.

$$R = \underbrace{\langle \sigma^2 \rangle_{AV}}_{\text{convection}} \exp \left[\underbrace{-a^2(\xi_1 - U\tau)^2 - a^2(\xi_2^2 + \xi_3^2) - \mathcal{L}^2 a^2 U^2 \tau^2}_{\text{fluctuation}} \right] \quad (3.1)$$

With this choice of the covariance R , and with the time delay inserted according to $\tau = \underline{\xi} \cdot \underline{x} / x c_0$, integration of Eq. (2.20) yields an explicit result for $\langle p^2(x) \rangle_{AV}$. It is $\overline{\mathcal{F}}$

$$\langle p^2(x) \rangle_{AV} = \frac{\langle \sigma^2 \rangle_{AV}}{16\pi^2 c_0^4 a^3 x^2} \begin{cases} (1+m^2)(Am^2 - Bm + 1)^{-\frac{1}{2}} (m^2 + Bm + A)^{-\frac{1}{2}} \\ \xrightarrow{\theta \rightarrow 0} (1 + \mathcal{L}^2 M^2 + M^2 - 2M)^{-\frac{1}{2}} \\ \xrightarrow{\theta \rightarrow \pi/2} (1 + \mathcal{L}^2 M^2 + M^2 + 2M)^{-\frac{1}{2}} \end{cases} \quad (3.2)$$

where

$$\begin{aligned} m &= \left[(1-A)/B \right] - \left[1 + (1-A)^2/B^2 \right]^{\frac{1}{2}} \\ A &= 1 + \mathcal{L}^2 M^2 + M^2 - 2M \cos \theta \\ B &= 2M \sin \theta \\ M &= U/c_0 \\ \theta &= \text{angle between } \underline{x} \text{ and } \underline{U}, \text{ the latter} \\ &\text{being taken in the } x_1 \text{ direction; thus} \\ &\cos \theta = x_1/x. \end{aligned}$$

Equation (3.2) provides the mean square sound pressure at a radial distance x and an angle θ from the direction of source-motion \underline{U} . A polar plot of $\langle p^2 \rangle_{AV}/x^2$ versus θ is shown in Fig. 4. The different curves correspond to different Mach numbers $M = U/c_0$ formed from the speed U of source motion or "convection". For supersonic convection speeds the sound pressure peaks in a direction normal to the Mach cone ($\theta_p = \cos^{-1} 1/M$). The peak is directly downstream ($\theta_p = 0$) at the sonic speed and for lower speeds becomes progressively less pronounced in the same direction.

$\overline{\mathcal{F}}$ Because of the symmetry about the x_1 axis, it suffices to limit \underline{x} to the x_1, x_2 plane. With $x_3 = 0$, $\tau = (\xi_1 \cos \theta + \xi_2 \sin \theta) / c_0$. Insertion of this value into Eq. (2.20) yields cross-product terms $\xi_1 \xi_2$; these are eliminated by a suitable transformation of coordinates - essentially a rotation $\overline{\mathcal{F}}$ simplify the integration. The detailed procedure is exhibited in the more general case of Chap. V, and is obtained by setting $M_0 = 0$ therein.

The source-pattern fluctuation parameter α was taken as 0.1 in Fig. 4. This implies (cf. Eq. (3.1)) that the effective length of a coherent source patch or 'eddy' ($\sim 1/a$) is one-tenth its decay length ($\sim 1/\alpha a$). In other words for $\alpha = 0.1$ an 'eddy' travels about ten times its length before the pattern fluctuation has altered it very greatly. This appears to be about the right order of magnitude for the eddies in a boundary layer (Ref. 25). Recent data suggests a better value for the mixing region of a jet is $\alpha \sim .2$ or $.25$ (Ref. 14; see Appendix C herein).

If the fluctuation parameter α were taken zero ("frozen" convected source pattern) the directionality peaks normal to the Mach cone in Fig. 4 would become infinite. This is exhibited in Fig. 5 for the special case of sonic speed of source motion, $M=1$. The downstream lobe with peak at $\theta_{\text{peak}} = 0$ grows to infinity as α is decreased progressively from 1 to 0. It is clear that pattern fluctuation is a moderating influence, reducing and rounding off the otherwise infinite peaks.

Lighthill has already predicted intensity peaks normal to the Mach cone in terms of his Mach-number factors (Ref. 1). However, those factors fail to allow for pattern fluctuation and yield only infinities at the peaks. It is implicit in his derivation that each correlation volume (i. e., "eddy volume") has existed since minus infinity in time without decay due to fluctuation: the finite lifetime of the eddies is not taken into account.

In summary, the source mean motion or "convection" in conjunction with the time retardation $\tau(\xi, \theta)$, as expressed in the source-pattern space-time covariance \mathcal{R} , account for the strong directionality of the sound radiation in the example. The directionality is softened by pattern fluctuation, specified by $\alpha \neq 0$. The ability to allow for pattern fluctuation provides a large reduction of the convective enhancement of power.

The primary purpose of this example was to show how the motion of a random pattern of acoustic sources through a stationary fluid can give rise to pronounced directionality of the radiated sound. A very simple choice for the pattern covariance \mathcal{R} was made (Eq. 3.1) to ease both the mathematics and the physical argument (which follows later). The example constitutes an idealization of a real flow wherein the mean motion of the fluid is suppressed. Thus no account is taken of refraction by the gradients of the mean velocity (cf. Chap. IV).

3.3 Effects of Pattern Convection on Directionality

Mathematical Interpretation of Peak at $M \cos \theta = 1$

The function \mathcal{R} is essentially the correlation* of acoustic source strength σ at two points separated in space and time. The space separation is ξ and the time separation is τ . The correlation must be unity at the origin ($\xi = \tau = 0$) and must approach zero at large separations (but not necessarily monotonically). A hypothetical case is sketched in Fig. 6 as a plot of contours of constant \mathcal{R} in the ξ, τ plane (for convenience τ is replaced by $U\tau$).

The choice of \mathcal{R}_0 here is that of a Gaussian function in both ξ and $U\tau$ (Eq. 3.1). Thus the contour plot of a stationary pattern represents a hill or ridge; the long axis is along $U\tau$ by the choice $\mathcal{A} \approx .2$. This implies the 'eddy' length is about 0.2 the decay length. Convection of the source pattern with uniform velocity U is introduced by the change $\mathcal{R}_0(\xi, \tau) \longrightarrow \mathcal{R}_0(\xi, -U\tau, \tau)$. The contour plot (Fig. 6) shows the long axis of the ridge has been sheared over to a 45° inclination with the $U\tau$ axis.

We shall employ the convected form of \mathcal{R} as the integrand in a one-dimensional version of Eq. (3.1) for the mean square radiated sound pressure. The path of integration follows a radial time-delay line $U\tau = \xi, M \cos \theta$ whose inclination depends on M and θ ; several possibilities are shown dotted for $M = 1$. It is evident that the value of the integral depends likewise on M and θ . The particular choice $M \cos \theta = 1$ (or $U\tau = \xi$), by traversing the long axis of the ridge, maximizes the integral. \bar{F} In other words, the mean square radiated sound pressure is a maximum in the direction normal to the Mach cone, $\theta = \cos^{-1} 1/M$.

* More precisely, we distinguish between the dimensional covariance $\mathcal{R} = \langle \sigma \sigma' \rangle_{AV}$ and the nondimensional correlation $\langle \sigma \sigma' \rangle_{AV} / [\langle \sigma^2 \rangle_{AV} \langle \sigma'^2 \rangle_{AV}]^{1/2}$

\bar{F} A similar observation has been made by Lighthill (Ref. 1) with regard to the integrand of the integral for the density perturbation (not its mean square) in the far field. In that case the ridge line was taken as infinitely long, corresponding to an infinite eddy lifetime. (The integral for $M \cos \theta = 1$ was therefore infinite, which explained the infinite pressure peak at that value of θ).

Thus we have an explanation of a kind for the directional properties exhibited in Fig. 4 for the sound radiated by a convected random pattern of acoustic sources. The one-dimensional example shows qualitatively how the pattern convection (at Mach number M) and the time delay respectively determine the yaw of the ridge line (range of large values) of the integrand and the path of integration. The dependence of the path or time-delay on the direction θ of the observer gives rise to a pronounced dependence of the integral - the mean square sound pressure - on θ .

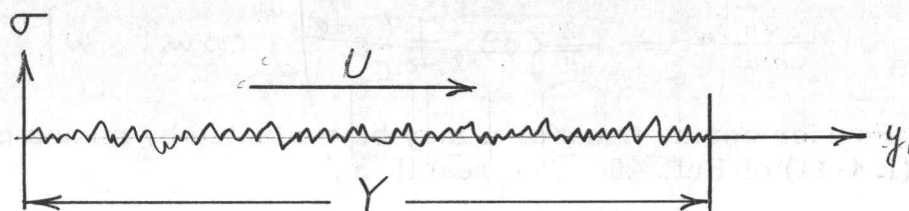
Physical Interpretation of Peak at $M \cos \theta = 1$. - In Figure 7 an intermittent sound source has moved from left to right across the volume V with supersonic speed, emitting pulses at the points marked x . The two sketches portray the sound field at an early time t_1 , and at a later time t_2 . In each case the sound waves toward the left, having been emitted earlier, have grown larger. Note how the sound waves coalesce to form an envelope - an annular segment of a supersonic Mach cone. The sound intensity is maximum in the directions normal to the Mach cone ($\theta = \cos^{-1} 1/M$). (Since a transient event has been considered an observer at $\theta = \cos^{-1} 1/M$ will experience only a single sound pulse as the envelope moves past him).

The wave pattern grows even after the sound source has died, enveloping "upstream" points P as well as "downstream" points P' . Thus there is no zone of silence as in steady supersonic flow. The difference lies in our use of a stationary frame of reference that does not follow the moving source. Further, the finite lifetime of the source accounts for the truncation of the Mach cone.

The situation of Fig. 7 is an idealization of the convection of a continuous random fluctuating pattern of sources through the volume V . The continuous pattern will provide a succession of wave trains, not just the single growing pattern portrayed. Points like Q, P, P' will each receive a continuous fluctuating sound pressure instead of one or several pulses. Further, the strength fluctuation in time - but not the randomness in space - will tend to smear out or impair the sharpness of the envelope at $\theta_p = \cos^{-1} 1/M$. This will reduce an otherwise infinite intensity peak at θ_p which would occur for a nonfluctuating pattern. It is again noted that Lighthill's Mach number factors imply the latter situation (cf. last footnote).

3.4 'Convected' Single-Frequency Sources: Example

Near Field Spectrum: Convection Broadening. - Consider just a single Fourier component (in the time domain) of a convected random pattern of acoustic sources: that is, consider a pattern that is random in space but sinusoidal in time. Furthermore, we specify a one-dimensional pattern distributed along a line segment rather than within a volume: this simplifies the mathematics without, it is thought, sacrificing the essential physical features.



Sketch 3. Line Distribution of Acoustic Sources

The pattern is taken to be a stationary (homogeneous) random function in space and time. The assumed two-point source strength covariance $\langle \sigma \sigma' \rangle_{AV}$ is

$$\tilde{R}(\tau, \xi_1) = \langle \sigma^2 \rangle_{AV} e^{-a^2(\xi_1 - U\tau)^2} \cos \omega_0 \tau \quad ; \quad \sigma = \frac{D^2 b^{(0)}}{Dt^2} \quad (3.3)$$

where σ is the source strength at (y_1, t) and σ' is the source strength at $(y_1 + \xi_1, t + \tau)$. The randomness in space is described by the exponential factor, with a^{-1} serving as the length scale or average "eddy" length. The strength of the pattern oscillates through positive and negative values by means of the sinusoidal factor $\cos \omega_0 \tau$: the entire pattern oscillates in phase.

Equation (3.3) is referred to a stationary frame of reference with respect to which the pattern is convected with velocity U (toward the right in the sketch). In the corresponding equation for an observer moving with the pattern the argument $(\xi_1 - U\tau)$ in the exponential function would be replaced by ξ_1 alone.

It is evident that the moving observer hears just the single source frequency ω_0 : for him the frequency spectrum is just a single line*. The question now arises: what sort of spectrum does a stationary observer record? To answer this we employ the formalism

* The angular frequency $\omega = 2\pi \times$ frequency is for brevity referred to as the "frequency" in this section.

(28)

Spectrum function \sim Fourier transform of time covariance ($\xi_1 = 0$), or more specifically

$$\frac{d\langle \tilde{\sigma}^2 \rangle_{AV}}{d\omega} = \frac{2}{\pi} \mathcal{J}_c \left\{ \tilde{R}(\tau, 0); \omega \right\} \quad (3.4)$$

In the present case this is

$$\frac{d\langle \tilde{\sigma}^2 \rangle_{AV}}{d\omega} = \frac{2}{\pi} \langle \sigma^2 \rangle_{AV} \mathcal{J}_c \left\{ e^{-a^2 U^2 \tau^2} \cos \omega_0 \tau; \omega \right\} \quad (3.5)$$

This Fourier cosine transform may be evaluated by means of Eqs. (1.1-3) and (1.4-11) of Ref. 26. The result is

$$\frac{d\langle \tilde{\sigma}^2 \rangle_{AV}}{d\omega} = \frac{\langle \sigma^2 \rangle_{AV}}{2\omega_c \sqrt{\pi}} \left\{ \exp \left[-\frac{(\omega_0 + \omega)^2}{4\omega_c^2} \right] + \exp \left[-\frac{(\omega - \omega_0)^2}{4\omega_c^2} \right] \right\} \quad (3.6)$$

where $\omega_c = \frac{U}{a^1} \sim \frac{\text{speed}}{\text{eddy size}}$

The spectrum Eq. (3.6) seen by a stationary observer is plotted in Fig. 8 for several ratios of ω_0/ω_c . When $\omega_0 = 20\omega_c$ the apparent frequency ω_c produced by convection of eddies of size a^1 at speed U past the observer is relatively small compared with the pattern oscillation frequency ω_0 . The corresponding spectrum function in Fig. 8 is very nearly the δ -function or single line seen by a moving observer. By virtue of the added convective frequencies the line has been broadened to a band width $\sim 2\omega_c$, but it is still narrow.

When $\omega = 2\omega_c$ the frequency produced by pattern oscillation is comparable with the apparent frequencies produced by pattern convection. The corresponding curve of Fig. 8 shows how the apparent convective frequencies, by addition and subtraction from ω_0 , have broadened the spectral line at $\omega/\omega_0 = 1$ so that it is now a peaked broad-band spectrum.

When $\omega_0 = 0.2\omega_c$ the apparent frequencies produced by pattern convection completely dominate over the oscillation frequency ω_0 . The curve in Fig. 8 is that of a broad band spectrum with no discernible peak at the pattern oscillation frequency ω_0 . In this case the true time fluctuations of the pattern are so slow as to be unimportant: the frequencies seen by the observer are essentially produced by the convection past him of a random space pattern.

The last-described case, that of the convection of a "quasi-frozen" pattern of sources, appears to approximate best the turbulence in a boundary layer (Ref. 25) or in a jet (Ref. 14; see Appendix C herein).

Far Field Spectrum: Doppler Shift

In Section 2.2 an expression is worked out for the auto-covariance of the sound pressure in the far field. We modify this three-dimensional result to apply to the one-dimensional case of this section wherein the acoustic sources are distributed along a line of length Y . The modified form of Eq. (2.23) reads

$$\begin{aligned} \langle \tilde{p} \tilde{p}' \rangle_{\text{av}}(\tau, \underline{x}) &= \frac{Y^2 \bar{a}^2}{16\pi^2 c_0^2 x^2} \int_0^Y dy_1 \int_{-\infty}^{\infty} \tilde{\mathcal{R}}(\xi_1, \tau + \tau') d\xi_1 \\ &= \frac{Y^3 \bar{a}^2}{16\pi^2 c_0^2 x^2} \int_{-\infty}^{\infty} \tilde{\mathcal{R}}(\xi_1, \tau + \tau') d\xi_1 \end{aligned} \quad (3.7)$$

with
$$\tau = \underline{\xi} \cdot \underline{x} / c_0 x = \frac{U}{c_0} \xi_1 \cos \theta$$

The factor $Y^2 \bar{a}^2$ in the first integral serves in effect to convert source strength/unit volume into source strength/unit length. The particular combination of physical length Y and 'eddy scale' \bar{a} is one that arises naturally in the corresponding integral in three dimensions for sources in a volume Y^3 .

The desired spectrum function is the Fourier cosine transform of the auto-correlation:

$$\frac{d\langle \tilde{p}^2 \rangle_{\text{av}}}{d\omega} = \frac{2}{\pi} \mathcal{J}_c \left\{ \langle \tilde{p} \tilde{p}' \rangle_{\text{av}}(\tau, \underline{x}) ; \omega \right\} \quad (3.8)$$

Eqs. (3.7) and (3.8) are evaluated in Appendix D for the case that $\tilde{\mathcal{R}}$ has the form of Eq. (3.3). The resulting spectrum function is (Eq. (D7)):

$$\left. \frac{d\langle \tilde{p}^2 \rangle_{\text{av}}}{d\omega} = \frac{\langle \sigma^2 \rangle Y^3}{192 \pi^{3/2} \bar{a}^3 c_0^2 x^2 \Theta} e^{-\frac{\beta^2}{4\bar{a}^2 \Theta^2}} \delta\left(\omega - \frac{\omega_0}{|\Theta|}\right) \right\} \quad (3.9)$$

where
$$\beta = \frac{\omega_0 \cos \theta}{c_0}, \quad \Theta = 1 - M \cos \theta$$

Eq. (3.9) shows that the spectrum is a single line at angular frequency $\omega_c / \Theta \equiv \omega_0 / |1 - M \cos \theta|$. Thus a frequency ω_0 in the sound source pattern moving at Mach M (moving observer) produces sound

(30)

of frequency $\omega_0 / |1 - M \cos \theta|$, the Doppler-shifted frequency, in the far field (stationary observer). On physical grounds this result could have been written down without analysis. The fact that the Doppler shift results automatically from the mathematics is a testimony to the power of the correlation formalism when 'convection' and time-delay are included.

3.5 'Convected' Randomly Fluctuating Sources: Example

Choice of the Source Strength Covariance. - The single-frequency line distribution of acoustic sources of Sec. 3.4 was by hypothesis just a single Fourier component of a pattern randomly fluctuating in time. The two-point space-time covariance of this more general pattern in a stationary reference frame may be written

$$\langle \sigma \sigma' \rangle_{AV} \equiv \mathcal{R} = \int_0^\infty W(\omega_0) \tilde{\mathcal{R}} d\omega_0 \quad ; \quad \sigma = \frac{D^2 p^{(0)}}{Dt^2} \quad (3.10)$$

where $\tilde{\mathcal{R}}$ is the single-frequency covariance Eq. (3.3) and $W(\omega_0)$ is some weighting function. Suppose we choose $W(\omega_0)$ so that

$$\langle \sigma \sigma' \rangle_{AV} = \int_0^\infty \underbrace{\frac{1}{12\sqrt{\pi}} \omega_f \left(\frac{\omega_0}{\omega_f}\right)^4 e^{-\frac{\omega_0^2}{4\omega_f^2}}}_{W(\omega_0)} \cdot \underbrace{\langle \sigma^2 \rangle e^{-\alpha^2(\xi_1 - U\tau)^2} \cos \omega_0 \tau}_{\tilde{\mathcal{R}}} d\omega_0 \quad (3.11)$$

where $\omega_f = \mathcal{L}aU$

Then $\mathcal{R} \equiv \langle \sigma \sigma' \rangle_{AV}$ has the value

$$\text{stationary frame } \langle \sigma \sigma' \rangle_{AV} = e^{-\alpha^2(\xi_1 - U\tau)^2} \frac{\langle \sigma^2 \rangle_{AV}}{12\omega_f^4} \frac{\partial^4}{\partial \tau^4} e^{-\omega_f^2 \tau^2} \quad (3.12)$$

(A reverse procedure was actually used, with Eq. (3.12) specified at the outset; then Eq. (3.11) was obtained as an inverse cosine transform with use of Eq. (1.1-5) of Ref. 26).

Motivation for Choice of Source Covariance. - The form of the source covariance, Eq. (3.12), for this example was originally chosen on the basis of σ being defined as $\bar{D}^2 p^{(0)} / \bar{D}t^2$ (Ref. 3). This definition has now been superseded so that the motivation is no longer very relevant. The reasoning on the basis of $\sigma = D^2 p^{(0)} / Dt^2$ is, however, set forth below as a matter of interest. In the later applications, the new definition $\sigma = D^2 p^{(0)} / Dt^2$ is to be understood.

(The factor $\partial^4 / \partial t^4 e^{-\omega_f^2 \tau^2}$ in Eq. (3.12) provides a region of negative covariance. Such a negative region, while satisfactory in the present one-dimensional example, can lead to absurd results (i. e., directions of negative $\langle p \rangle_{AV}$) in the three-dimensional case.

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Since the two-point covariance arises from the process of squaring an integral, any function whose integral over covariance space (with or without retarded time) is negative is not an admissible covariance).

The significance of Eq. (3.12) is best displayed by shifting to a moving reference frame: $\xi_1 - U\tau$ is replaced by ξ_1 to give

$$\langle \sigma\sigma' \rangle_{AV} = e^{-a^2\xi_1^2} \frac{\langle \sigma^2 \rangle_{AV}}{12\omega_f^4} \frac{\partial^4}{\partial \tau^4} e^{-\omega_f^2 \tau^2} \quad (3.13)$$

Now $\bar{\partial}/\bar{\partial}t \longrightarrow \partial/\partial t$ in the moving frame, whence

$$\langle \sigma\sigma' \rangle_{AV} = \left\langle \frac{\partial^2 p^{(0)}}{\partial t^2} \frac{\partial^2 p^{(0)'}}{\partial t'^2} \right\rangle_{AV}$$

in that frame. If we postulate that the pressure pattern be a stationary random function in time (i. e., time averages that are nonzero approach a fixed limit for infinite averaging time) it can be shown that

$$\left\langle \frac{\partial^2 p^{(0)}}{\partial t^2} \frac{\partial^2 p^{(0)'}}{\partial t'^2} \right\rangle_{AV} = \frac{\partial^4}{\partial \tau^4} \langle p^{(0)} p^{(0)'} \rangle_{AV} \quad (3.14)$$

Thus Eq. (3.13) is compatible with

$$\langle p^{(0)} p^{(0)'} \rangle_{AV} = e^{-a^2\xi_1^2} \frac{\langle \sigma^2 \rangle_{AV}}{12\omega_f^4} e^{-\omega_f^2 \tau^2} \quad (3.15)$$

The left-hand side reduces to $\langle p^{(0)2} \rangle_{AV}$ on setting $\xi_1 = \tau = 0$. Therefore

$$\langle p^{(0)2} \rangle_{AV} = \frac{\langle \sigma^2 \rangle_{AV}}{12\omega_f^2} \equiv \frac{\langle (\frac{\partial^2 p^{(0)}}{\partial t^2})^2 \rangle_{AV}}{12\omega_f^4} \quad (3.16)$$

and Eq. (3.15) may be written in the alternative form

$$\langle p^{(0)} p^{(0)'} \rangle_{AV} = \langle p^{(0)2} \rangle_{AV} e^{-a^2\xi_1^2 - \omega_f^2 \tau^2} \quad (3.15')$$

Sound Source Strength Spectrum (Moving Frame). - The frequency spectrum of the pattern of sound source strength is the Fourier cosine transform of the covariance in the time domain ($\xi_1 = 0, \tau \neq 0$)

$$\frac{d\langle \sigma^2 \rangle_{AV}}{d\omega} = \frac{2}{\pi} \int_c \{ \mathcal{R}(\tau, 0); \omega \} \quad (3.17)$$

The frame of reference for the spectrum is the same as that for \mathcal{R} : in this case we choose a frame moving with the velocity U of the pattern. With use of Eq. (3.13) for \mathcal{R} the result is

(32)

$$\frac{d\langle \sigma^2 \rangle_{AV}}{d\omega} = \frac{\langle \sigma^2 \rangle_{AV}}{12\sqrt{\pi}} \frac{\omega^4}{\omega_f^4} e^{-\frac{\omega^2}{4\omega_f^2}} \quad (3.18)$$

This source strength spectrum is shown in Fig. 9 for later comparison with the radiated sound field. The value of \mathcal{L} is taken as 0.1.

Far-Field Pressure Spectrum (Stationary Frame). - In Section 3.4 we saw that a single-frequency line pattern of acoustic sources with covariance $\tilde{\mathcal{R}}$ (Eq. 3.3) radiates a far-field pressure spectrum $\frac{d\langle \phi^2 \rangle_{AV}}{d\omega}$ (Eq. 3.9). It follows that the superposition

$$\mathcal{R} = \int_0^\infty W(\omega_0) \tilde{\mathcal{R}} d\omega_0 \quad (3.10)$$

which represents the chosen random source pattern will yield a far-field pressure spectrum

$$\frac{d\langle \phi^2 \rangle_{AV}}{d\omega} = \int_0^\infty W(\omega_0) \frac{d\langle \tilde{\phi}^2 \rangle_{AV}}{d\omega} d\omega_0 \quad (3.19)$$

The values of $W(\omega_0)$ and $d\langle \tilde{\phi}^2 \rangle_{AV}/d\omega$ are obtained from Eqs. (3.11) and (3.9), respectively. The integration yields

$$\frac{d\langle \phi^2 \rangle}{d\omega} = \frac{\langle \sigma^2 \rangle \Upsilon^3 \Theta^4}{192\pi^2 a^3 c^2 x^2} \frac{\omega^4}{\omega_f^4} e^{-\frac{\omega^2}{\omega_f^2} \frac{\Theta^2 + \Theta_1^2}{4}} \quad (3.20)$$

where

$$\left. \begin{aligned} \Theta &= 1 - M \cos \theta \\ \Theta_1 &= \mathcal{L} M \cos \theta \\ \omega_f &= \mathcal{L} a U \end{aligned} \right\}$$

The radiation spectrum Eq. (3.20) is compared with the acoustic source spectrum Eq. (3.18) in Fig. 9. A particular case is chosen: pattern convection speed $U = .5$ speed of sound c_0 , axial direction ($\theta = 0$), and $\mathcal{L} = 0.1$ ('eddy' length ≈ 0.1 decay length). The abscissa scale is $\omega/2\omega_f$ where $\omega_f = \mathcal{L} a U$ is a characteristic frequency associated with fluctuation of the pattern. The ordinate scale is arbitrary.

The radiation spectrum displays the same functional shape as the source spectrum, but stretched toward higher frequencies. In the graph this stretch (with particular reference to the displacement of the peaks) is labelled as the "Doppler shift".

Comparison of the arguments of the exponential

$$-\frac{\omega^2}{4\omega_f^2} \quad \text{vs.} \quad -\frac{\omega^2}{4\omega_f^2} (\Theta^2 + \Theta_1^2)$$

shows the label is only approximate: the actual scale factor $[\Theta^2 + \Theta_1^2]^{-1/2}$ is slightly less than the Doppler factor Θ^{-1} . The reason lies (see below) in a progressively decreasing efficiency of radiation with increasing frequency. Functionally, this modifies the spectrum very much like a Doppler shift via the argument

$$-\frac{\omega^2 \Theta_1^2}{4\omega_f^2} = -\frac{\omega^2 \cos^2 \theta}{4a^2 c_0^2}$$

Since $\sqrt{\pi}/a$ is a scale L of the source pattern,

$$\exp -\frac{\omega^2 \cos^2 \theta}{4a^2 c_0^2} = \exp -\pi \frac{L^2}{\lambda^2} \cos^2 \theta$$

Thus the radiation attenuates as the wavelength λ becomes smaller in comparison with the 'eddy size' L .

The physical explanation of the inefficiency of high-frequency sound radiation can be discerned from an alternative derivation of Eq. (3.20) by a more direct process (integration of $\mathcal{R} d\xi_1$ over ξ_1 followed by the Fourier transform operation). It is found that when a coherent sound source ('eddy') is not negligibly small compared with a wavelength of the emitted sound there may be phase cancellation of the sound emitted from different parts of the source. The phenomenon bears some similarity to the cancellation effects underlying diffraction of waves through an aperture (Ref. 27). In the case plotted in Fig. 9 the choice $\alpha = 0.1$ implies that the eddy size \ll than the characteristic wave length so that the effect on the apparent Doppler shift is small.

'Turbulence' Pressure Spectrum (Both Frames). - An ideal microphone* within the 'turbulence' of our one-dimensional example would record essentially $p^{(o)}$. On the other hand, the acoustic source strength is proportional to the second time derivative following the fluid motion $D^2 p^{(o)}/Dt^2$. The frequency spectrum of the latter is exhibited in Fig. 9, and we seek now to obtain the frequency spectrum of $p^{(o)}$ itself.

* Such a pressure transducer presents a development problem since an ordinary microphone will interfere with the flow; e. g., a microphone facing upstream will read stagnation pressure. A successful static pressure microphone with good frequency response is claimed in Ref. 28.

The difficulties force us to approximate

$$\sigma = D^2 \beta^{(0)} / Dt^2 = (\partial/\partial t + u_i \partial/\partial y_i)^2 \beta^{(0)} \quad \text{by} \quad \bar{D}^2 \beta^{(0)} / Dt^2 = (\partial/\partial t + U \partial/\partial y_1)^2 \beta^{(0)}$$

that is, we neglect the eddy self-convection, even though the terms are not negligible. To this approximation the space-time covariance of $\beta^{(0)}$ in our example is given by Eq. (3.15) when the reference frame moves with the source-pattern speed U . Transformation to a stationary frame is effected by replacement of ξ_1 therein by $\xi_1 - U\tau$. Finally the covariance in the time domain alone (autocovariance) is obtained by setting $\xi_1 = 0$ in both cases. This yields

$$\begin{array}{l} \text{moving} \\ \text{frame} \end{array} \quad \langle \beta^{(0)} \beta^{(0')} \rangle_{AV}(\tau) = \langle \beta^{(0)^2} \rangle_{AV} e^{-\omega_f^2 \tau^2} \quad (3.21)$$

$$\begin{array}{l} \text{stationary} \\ \text{frame} \end{array} \quad \langle \beta^{(0)} \beta^{(0')} \rangle_{AV}(\tau) = \langle \beta^{(0)^2} \rangle_{AV} e^{-(\omega_f^2 + \omega_c^2) \tau^2} \quad (3.22)$$

$$\text{where} \quad \omega_f = \mathcal{L} a U, \quad \omega_c = a U$$

The frequency spectrum is given by the Fourier cosine transform

$$\frac{d \langle \beta^{(0)^2} \rangle_{AV}}{d\omega} = \frac{2}{\pi} \int_c \left\{ \langle \beta^{(0)} \beta^{(0')} \rangle_{AV}(\tau); \omega \right\} \quad (3.23)$$

The results for the two frames may be written

$$\begin{array}{l} \text{moving} \\ \text{frame} \end{array} \quad \frac{d \langle \beta^{(0)^2} \rangle_{AV}}{d\omega} = \frac{\langle \beta^{(0)^2} \rangle_{AV}}{\omega_f \sqrt{\pi}} e^{-\frac{\omega^2}{4\omega_f^2}} \quad (3.24)$$

$$\begin{array}{l} \text{stationary} \\ \text{frame} \end{array} \quad \frac{d \langle \beta^{(0)^2} \rangle_{AV}}{d\omega} = \frac{\langle \beta^{(0)^2} \rangle_{AV}}{\sqrt{(\omega_f^2 + \omega_c^2) \pi}} e^{-\frac{\omega^2}{4(\omega_f^2 + \omega_c^2)}} \quad (3.25)$$

Fluctuation versus convection. - It is seen that the 'turbulence' pressure spectrum in our one-dimensional example has the same form in either frame: only the scale-factor changes. This is a fortuitous consequence of the form of covariance chosen and our approximations, since the two spectra are governed by different phenomena: the moving-frame spectrum is dominated by the randomness of the pattern in time (fluctuation) whereas the stationary-frame spectrum may be dominated by the randomness of the pattern in space (the 'eddy' space structure). In fact, Eq. (3.25) shows that for $\omega_f^2 \ll \omega_c^2$ ($\mathcal{L}^2 \ll 1$), fluctuation of the pattern (typical frequency ω_f) contributes negligibly to the frequency recorded by a stationary observer: he sees effectively a frozen convected pattern. The case $\mathcal{L}^2 \ll 1$ is tentatively thought to be characteristic of a boundary layer (Ref. 25) or of a jet (Ref. 14).

The two frequency spectra are compared in Fig. 10 for the case $\mathcal{L}^2 = .01$ ($\omega_f^2/\omega_c^2 = .01$). It is seen that the moving-frame spectrum is concentrated at the low-frequency end, whereas the convective effects stretch the stationary-frame spectrum one-hundred-fold.

Comparison with far-field spectrum. - A stationary ideal microphone* in or near the 'turbulence' would record the stationary frame spectrum of Fig. 10. A second microphone at a large distance would record the radiation spectrum of Fig. 9. These two spectra may be compared as constituting the (very) near-field and the far-field of the same 'turbulence' noise field. The two spectra behave quite differently as the frequency approaches zero: the near-field spectrum level approaches a constant value whereas the far-field level approaches zero. In this respect the situation resembles that of jet noise although the example is over idealized: in particular, the assumption of a homogeneous source pattern along a line is far from reality.

IV. MOVING SOURCES IN A JET FLOW: REFRACTIVE EFFECTS ON SOUND DIRECTIONALITY

4.1 Introduction and Governing Equation

The applications in Section II do not allow for any mean motion of the fluid as in a jet. Section III improves the situation by allowing the turbulence pattern to move with velocity U through fluid at rest. In this way certain convective effects on jet noise directionality and on Doppler shifts of frequency are allowed for[†]. The mean motion that transports the turbulence pattern is, however, still neglected. In the present chapter the mean motion is brought into the picture, with particular attention to the mean shear: the shear enhances the noise generation and it provides a refractive effect on the directionality.

The mean motion appears explicitly in the operator $\bar{D}/\bar{D}t = \partial/\partial t + U \partial/\partial y_1$ of the fundamental equation Eq. (2.11):

* see footnote page 33.

[†] This allowance is only approximate. The "convective" effects calculated herein in this manner (by motion of acoustic sources through fluid at rest) are oversimplified in the use of a time delay τ in the integrand that does not allow for convection of the sound waves by the mean flow. Inclusion of such an allowance is expected to modify the predicted directionality and Doppler shift in the vicinity of $\theta \rightarrow 0$. A further discussion is given in Sec. 4.4.

$$\frac{1}{c_0^2} \frac{D^2 p'''}{Dt^2} - \nabla^2 p'' = - \frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} \quad (2.11)$$

In what follows (except Sec. 4.2) we shall deal with solutions of this equation. (Insertion of $U = 0$ will recover the simplified equation

$$\frac{1}{c_0^2} \frac{\partial^2 p'''}{\partial t^2} - \nabla^2 p'' = - \frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} \quad (2.15)$$

employed in earlier chapters.)

4.2 Amplifying Effect of Mean Shear

Eq. (2.11) serves for the computation of p''' in terms of the acoustic sources of strength $-c_0^{-2} D^2 p^{(0)}/Dt^2$. The far-field sound pressure p''' is essentially driven by the pressure $p^{(0)}$ that dominates within and near the turbulent field. It will be shown how the shear in the mean flow amplifies the amplitude of $p^{(0)}$ and thereby the amplitude of p''' .

The expanded form of Eq. (2.6) governing $p^{(0)}$ reads (Appendix A.2, Eq. (A16)):

$$-\nabla^2 p^{(0)} = \rho_0 \left(\frac{\partial u_i}{\partial y_i} \frac{\partial u_i}{\partial y_i} + \frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} \right) \quad (4.1)$$

where superscripts (0) are to be understood as applying to u_i , u_j (omitted for simplicity). Allowance for a mean flow* $U(y_2, y_3)$ along y_1 is made by writing

$$\begin{aligned} u_i &= U \delta_{i1} + u'_i & \delta_{i1} &= 1 & i &= 1 \\ u_j &= U \delta_{ij} + u'_j & &= 0 & i &\neq 1 \end{aligned} \quad (4.2)$$

where u'_i, u'_j is the unsteady part of the velocity. If the cross-stream gradients of $U \gg$ any gradients of u'_i the first term on the right-hand side of Eq. (4.1) dominates, giving approximately

$$-\nabla^2 p^{(0)} = 2\rho_0 \frac{\partial U}{\partial y_i} \frac{\partial u_j}{\partial y_i} \quad ; \quad j = 2, 3 \quad (4.3)$$

This may be written

$$-\nabla^2 p^{(0)} = 2\rho_0 \frac{\partial U}{\partial n} \frac{\partial v}{\partial y_i} \quad (4.3')$$

where n has the direction of the cross-stream gradient of U and v is the unsteady component of velocity in the direction of n . This equation is due to Kraichman (Ref. 12): he considered it to be a good approximation in a turbulent shear flow.[†]

* In a jet flow there is also a weak dependence of U on y_1 : this is neglected for simplicity, as well as very weak mean flow components in the y_2 and y_3 directions.

† Mollø-Christensen (Ref. 30) and O.M. Phillips (lecture) recognized that for turbulent shear flow the term $2\rho_0 \frac{\partial U}{\partial n} \frac{\partial v}{\partial y_i}$ is the dominant part of $\partial^2 T_{ij} / \partial y_i \partial y_j$, considered as the source term of Lighthill's equation.

(37)

Kraichman solved Eq. (4.3') for an idealized model consisting of a certain homogeneous turbulence superposed on a uniform shear flow. He obtained

$$\langle p^{(0)2} \rangle_{AV} = \frac{4}{15} \rho_0^2 \left(\frac{\partial U}{\partial n} \right)^2 \langle u_1^2 \rangle_{AV} \sigma^{-2} \quad (4.4)$$

$$= \frac{4}{15} \rho_0^2 \left[\frac{\partial (U/\sqrt{\langle u_1^2 \rangle_{AV}})}{\partial (\sigma n)^2} \right]^2 \langle u_1^2 \rangle_{AV}^2 \quad (4.4')$$

where σ is an inverse length scale of the turbulence (Ref. 12; see also Ref. 2) The corresponding result in the absence of a shear flow is

$$\langle p^{(0)2} \rangle_{AV} = c^2 \cdot \rho_0^2 \langle u_1^2 \rangle_{AV}^2 \quad (4.5)$$

where Batchelor (Ref. 29) obtained $c^2 = 0.34$ and Kraichman (Ref. 12) obtained $c^2 = 1.00$. Thus with this flow model amplification will result when the effective nondimensional mean shear $(4/15)^{1/2} \partial(U/\sqrt{\langle u_1^2 \rangle_{AV}})/\partial(\sigma n)$ exceeds c . The criterion alters with other flow models (Ref. 12).

4.3 Green's Function Describing Refraction and Diffraction from Point Source

Formulation of the Problem. - The governing equation for aerodynamic sound sources in a mean flow, Eq. (2.11), may be solved by a familiar technique: the right-hand-side is replaced by a δ -function to give

$$\frac{1}{c_0^2} \frac{\partial^2 p^{(0)}}{\partial t^2} - \nabla^2 p^{(0)} = \delta(\underline{x} - \underline{y}) \delta(t - t') \quad (4.6)$$

and a solution of this subsidiary equation is sought (\underline{y} is replaced by \underline{x} in the operators $\partial/\partial t$ and ∇^2). This solution, the Green's function, represents the pressure $p^{(0)}$ at a point \underline{x} and time t due to a unit point source at \underline{y} emitting impulsively at time t' into unbounded space. Suppose the Green's function is written

$$p^{(0)} = \frac{1}{4\pi} G(\underline{x}, \underline{y}, t) \delta(t - t') \quad (4.7)$$

where $t' = t'(\underline{x}, \underline{y}, t) =$ known function.

Then the desired solution of Eq. (4.6) would read

$$p^{(0)} = \frac{1}{4\pi} \iint_V \left[\frac{1}{c_0^2} \frac{\partial^2 p^{(0)}}{\partial t^2} G(\underline{x}, \underline{y}, t) \delta(t - t') \right] dy dt \quad (4.8)$$

$$= \frac{1}{4\pi} \int_V \left[\frac{1}{c_0^2} \frac{\partial^2 p^{(0)}}{\partial t^2} G(\underline{x}, \underline{y}, t') \right]_{t=t'} dy \quad (4.9)$$

where c_0 may be $c_0(y)$ as in a hot jet or jet of foreign fluid.

Now what can we say about the Green's function G short of actually solving for it? Outside the jet where U has fallen to essentially zero Eq. (4.6) reduces to the inhomogeneous wave equation. By Kirchoff's formula the sound pressure radiated outside some control surface S' enclosing the jet may be determined by (Ref. 31)

$$p(\underline{x}, t) = \frac{1}{4\pi} \int_{S'} \left[\frac{1}{r} \frac{\partial p}{\partial n} + \frac{\partial r}{\partial n} \frac{p}{r^2} + \frac{1}{c_0 r} \frac{\partial r}{\partial n} \frac{\partial p}{\partial t} \right]_{t-\frac{r}{c_0}} ds' \quad (4.10)$$

that is, as the emission (into air at rest) from a certain virtual distribution of sources and dipoles on S' . To an observer at a sufficiently large distance x from the centroid the surface S' seems essentially like a point at $x = 0$. The pressure p radiated by the surface distribution falls off like x^{-1} and depends also on the direction of \underline{x} . Thus we may write

$$G \approx x^{-1} K(\theta, y) \quad \left\{ \begin{array}{l} x \gg \text{radius of } S' \\ 2\pi x \gg \text{wave length} \end{array} \right\} \quad (4.11)$$

where θ is the angle of \underline{x} with the jet axis, with \underline{x} restricted to the x, x_2 plane. (This restriction imposes no loss in generality in view of the axial symmetry of the jet).

Qualitative Effects on Directionality. - The factor $K(\theta, y)$ of the Green's function describes the directional distribution of sound pressure in the far-field radiated from a point acoustic source at point y in a specified jet flow: the refractive and diffractive effects of the flow are embodied in this directionality function. The analytical determination of $K(\theta, y)$ is a formidable task (see Refs. 32 and 33 for similar problems) and only idealized cases have been treated (Refs. 34, 35).

Thus consider a two-dimensional oscillating acoustic line source lying cross-stream in an infinite plane jet (jet width \gg wave length). Such a source can be built up from plane waves disposed radially like the spokes of a wheel (Refs. 36, 37). Previous papers (Refs. 38, 39) show that these waves are refracted outward and forward (upstream), leaving a wedge-like zone of silence opening downstream. In a real jet the diffraction due to the great reduction in jet size/wave length must greatly weaken the refraction. The wedge (cone in the axisymmetric case) is thus perhaps reduced to just a deep inward dimple in the directionality curve as $\theta \rightarrow 0$.

Dr. P. Gottlieb (Ref. 34) has obtained an analytical solution for the closely related problem of an acoustic line source in a semi-infinite plane flow: the source lies cross-stream and parallel to the interface*. Figure 11 shows the directional distribution of the refracted sound pressure at large distances from this source. The curve segments labelled $\lambda = \infty$, $\lambda = 2\pi h$, $\lambda = \pi h$ lie in the wedge of silence; they represent diffractive "leakage" of sound into this zone at small distances h from the interface. It is expected (in the absence of further curves on Gottlieb's figure) that the penetration must approach zero as $\lambda \rightarrow 0$.

For a point source centered in a round jet the qualitative directionality pattern may resemble a figure of revolution developed from Fig. 11. For off-center source positions the pattern will be distorted. A common feature will, however, be the inward dimple, or reduced intensity, as $\theta \rightarrow 0$. Such a dimple in the downstream direction is characteristic of jet noise.

4.4 Mean Square Pressure Integral and Acoustic Time Delay Therein

Use of the approximate Green's function for large x , Eq. (4.11) simplifies Eq. (4.9) to

$$p^{(1)}(\underline{x}, t) = \frac{1}{4\pi x} \int \left[\frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} K(\theta, y) \right]_{t'(x, y, t)} dy \quad ; \quad (4.12)$$

this is the generalization of Eq. (2.17) to allow for a mean flow. The corresponding generalization of Eq. (2.20) for the mean square pressure yields

$$\langle p^{(1)2}(\underline{x}) \rangle_{AV} = \frac{1}{16\pi^2 x^2} \int_V c_0^{-4} dy \int_{-\infty}^{\infty} K(\theta, y + \xi/2) K(\theta, y - \xi/2) \mathcal{R}(\xi, \tau; y) d\xi \quad (4.13)$$

If the geometric average of K at $\pm \xi/2$ may be approximated by the value of K at the midpoint $\xi = 0$, then

$$\langle p^{(1)2}(\underline{x}) \rangle_{AV} = \frac{1}{16\pi^2 x^2} \int_V c_0^{-4} K^2(\theta, y) dy \int_{-\infty}^{\infty} \mathcal{R}(\xi, \tau; y) d\xi \quad (4.14)$$

where $\tau = \tau(x, y, \xi)$

The functional dependence of the time delay τ requires further discussion. Formally τ is determined as the difference in the times for sound to travel from points y' and y'' to the observer at \underline{x} .

* Similar analyses for a line source in a plane jet and a point source in a cylindrical jet have very recently been published by Moretti and Slutsky (Ref. 35). They were apparently unaware of Gottlieb's work.

(40)

$$\tau = \hat{t}'(\underline{x}, y', t) - \hat{t}''(\underline{x}, y'', t)$$

where

$$y' = y + \xi/2$$

$$y'' = y - \xi/2$$

(4. 15)

In the absence of a mean flow this reduces to Eq. (2. 22)

$$\tau \approx \xi \cdot \underline{x} / \chi c_0 \quad (2. 22)$$

But the presence of a mean flow gives rise to a more complicated functional dependence described by $\hat{t}' - \hat{t}''$. The \hat{t} function cannot be determined from a simple geometrical consideration as in Sketch 1: it arises instead as part of the solution of Eq. (4. 6) for the Green's function.

As a matter of interest a crude approximation to the effects of the jet flow on \hat{t} and τ are worked out in Appendix E.

4. 5 Pressure Autocovariance and Spectral Density

A simple extension of the expression, Eq. (4. 14), for the mean square pressure is effected by inserting a time delay τ' :

$$\langle p(\underline{x}, t) p(\underline{x}, t + \tau') \rangle_{AV} \approx \frac{1}{16\pi^2 \chi^2} \int_V c_0^{-4} K^2(\theta, y) dy \int_{-\infty}^{\infty} \mathcal{R}(\xi, \tau + \tau'; y) d\xi \quad (4. 16)$$

If we abbreviate the left-hand-side as $\langle p p' \rangle_{AV}(\tau'; \underline{x})$ the Fourier cosine transform reads

$$\Phi(\omega; \underline{x}) = \frac{2}{\pi} \mathcal{F}_C \{ \langle p p' \rangle_{AV}(\tau', \underline{x}); \omega \} \quad (4. 17)$$

Eq. (4. 16) gives the autocovariance of the sound pressure in the far field and Eq. (4. 17) gives the corresponding spectral density, (Note that

$\Phi(\omega; \underline{x})$ may be written as $d\langle p \rangle_{AV} / d\omega$ whence $\int_0^{\infty} \Phi(\omega; \underline{x}) d\omega = \langle p^2 \rangle_{AV}$).

V. MOVING SOURCES IN A UNIFORM STREAM SIMULATION OF A MOVING JET

5.1 Fundamental Solution for Subsonic Stream

The sound field of a jet on an aircraft in flight as recorded by an observer in a companion aircraft may be of interest. The technique of employing, in effect, moving sources in fluid at rest (Chap. III) may be generalized to allow for uniform motion of the medium.

The governing equation is Eq. (2.11) in the form

$$\frac{1}{c_0^2} \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial y_1} \right)^2 p^{(n)} - \nabla^2 p^{(n)} = - \frac{\sigma}{c_0^2} ; \quad \sigma \equiv \frac{D^2 p^{(0)}}{Dt^2} \quad (5.1)$$

When the stream speed U_0 is limited to subsonic values this has the solution (see, e. g., Ref. 40, Eqs. (3.5.0) and (3.5.2)):

$$p^{(n)} = - \frac{1}{4\pi c_0^2} \int \frac{\sigma(y, \hat{t})}{\hat{r}} dy \quad (5.2)$$

where

$$\left. \begin{aligned} \hat{t} &= t + \frac{M_0(x_1 - y_1)}{c_0 \beta_0^2} - \frac{\hat{r}}{c_0 \beta_0^2} \\ \hat{r} &= \sqrt{(x_1 - y_1)^2 + \beta_0^2 [(x_2 - y_2)^2 + (x_3 - y_3)^2]} \\ \beta_0^2 &= 1 - U_0^2/c_0^2 = 1 - M_0^2 \end{aligned} \right\} \quad (5.3)$$

The mean square pressure in the far-field, where $|\underline{x}| \gg |\underline{y}|$, may be written (dropping the superscript $^{(n)}$)

$$\langle p^2(\underline{x}) \rangle = \frac{1}{16\pi^2 c_0^4 r^2} \iint \langle \sigma(y', \hat{t}') \sigma(y'', \hat{t}'') \rangle_{AV} dy' dy'' \quad (5.4)$$

where $\underline{\hat{x}}$, $\underline{\hat{y}}$, and $\underline{\hat{\xi}}$ (to be introduced later) are "reduced coordinates" given by

$$\left. \begin{aligned} \underline{\hat{x}} &= \hat{x}_1, \hat{x}_2, \hat{x}_3 = x_1, \beta_0 x_2, \beta_0 x_3 \\ \underline{\hat{y}} &= \hat{y}_1, \hat{y}_2, \hat{y}_3 = y_1, \beta_0 y_2, \beta_0 y_3 \\ \underline{\hat{\xi}} &= \hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3 = \xi_1, \beta_0 \xi_2, \beta_0 \xi_3 \end{aligned} \right\} \quad (5.5)$$

(42)

$$\text{Put } y = (y' + y'')/2, \quad \xi = y' - y'', \quad \hat{\tau} = \hat{\tau}' - \hat{\tau}'' \quad (5.6)$$

as in Sec. 2.2. Then

$$\langle \rho^2(x) \rangle = \frac{1}{16\pi^2 c^4 \lambda^2} \int_V dy \int_{-\infty}^{\infty} R(\xi, \hat{\tau}; y) d\xi \quad (5.7)$$

where V is the effective volume of eddying flow,

$$R(\xi, \hat{\tau}; y) \equiv \langle \sigma(y + \xi/2, \hat{\tau}'' + \hat{\tau}) \sigma(y - \xi/2, \hat{\tau}'') \rangle_{AV} \quad (5.8)$$

and the average is over $\hat{\tau}''$. By procedures similar to those of Appendix E it can be established that for $|\xi| \gg |\hat{y}|$,

$$\beta_0^2 c_0 \hat{\tau} = -M_0 \xi_1 + \xi_2 \lambda / \lambda \quad (5.9)$$

approximately.

5.2 Example: Fluctuating Sources Moving with Speed U in Stream of Speed U_0

Take for the covariance R the same form assumed in Section 3.2:

$$R = \langle \sigma^2 \rangle_{AV} \exp -a^2 \left[\underbrace{(\xi_1 - U\hat{\tau})^2}_{\text{convection}} + \xi_2^2 + \xi_3^2 + \underbrace{\mathcal{L}^2 U^2 \hat{\tau}^2}_{\text{fluctuation}} \right] \quad (5.10)$$

Then to evaluate $\langle \rho^2 \rangle$, Eq. (5.7), we require

$$I = \iiint_{-\infty}^{\infty} R(\xi, \hat{\tau}) d\xi_1 d\xi_2 d\xi_3 \quad (5.11)$$

By virtue of the symmetry about the x_1 -axis it will suffice to limit \underline{x} to the x_1, x_2 plane. With $\xi_3 = 0$, Eq. (5.9) reduces to

$$\beta_0^2 c_0 \hat{\tau} = \xi_1 (\cos \theta_0 - M_0) + \xi_2 \beta_0 \sin \theta_0 \quad (5.12)$$

where $\tan \theta_0 = \beta_0 \tan \theta = \beta_0 x_2 / x_1$ (5.13)

The insertion of $\hat{\tau}$ into Eq. (5.10) will be simplified if we transform coordinates so that

$$\left\{ \begin{array}{l} \tilde{\xi}_1 = \xi_1 (\cos \theta_0 - M_0) + \xi_2 \beta_0 \sin \theta_0 \\ \tilde{\xi}_2 = -\xi_1 \beta_0 \sin \theta_0 + \xi_2 (\cos \theta_0 - M_0) \\ \tilde{\xi}_3 = \xi_3 \end{array} \right\} \left\{ \begin{array}{l} J\xi_1 = \tilde{\xi}_1 (\cos \theta_0 - M_0) - \tilde{\xi}_2 \beta_0 \sin \theta_0 \\ J\xi_2 = \tilde{\xi}_1 \beta_0 \sin \theta_0 + \tilde{\xi}_2 (\cos \theta_0 - M_0) \\ \tilde{\xi}_1^2 + \tilde{\xi}_2^2 = J(\xi_1^2 + \xi_2^2) \end{array} \right\} \quad (5.14)$$

where the Jacobian J is

$$\begin{aligned} J &= (\cos \theta_0 - M_0)^2 + \beta_0^2 \sin^2 \theta_0 \\ &= (1 - M_0 \cos \theta_0)^2 \end{aligned} \quad (5.15)$$

(43)

Then

$$I = \iiint_{-\infty}^{\infty} J^{-1} \mathcal{R}(\underline{x}, \hat{\tau}) d\tilde{\xi}_1 d\tilde{\xi}_2 d\tilde{\xi}_3 \quad (5.16)$$

wherein, after reduction,

$$\mathcal{R} = \langle \sigma^2 \rangle_{AV} \exp -J^{-1} a^2 [A_0 \tilde{\xi}_1^2 + \tilde{\xi}_2^2 + J \tilde{\xi}_3^2 + B_0 \tilde{\xi}_1 \tilde{\xi}_2] \quad (5.16')$$

with A_0, B_0 functions given in Eq. (5.20).

The cross-product term may be eliminated by means of the transformation

$$\left. \begin{aligned} \tilde{\xi}_1 &= m_0 r_1 + r_2 \\ \tilde{\xi}_2 &= -r_1 + m_0 r_2 \\ B_0 m_0 &= 1 - A_0 - \sqrt{B_0^2 + (1 - A_0)^2} \end{aligned} \right\} \quad (5.17)$$

where

with the result

(5.18)

$$\mathcal{R} d\tilde{\xi}_1 d\tilde{\xi}_2 d\tilde{\xi}_3 = \left\{ \langle \sigma^2 \rangle_{AV} \exp -J^{-1} a^2 [R_1 r_1^2 + R_2 r_2^2 + \tilde{\xi}_3^2] \right\} (1+m^2) dr_1 dr_2 d\tilde{\xi}_3$$

where R_1, R_2 are constant for the integration (given in Eq. (5.20)).

The variables are now separated and the integration of Eq. (5.16) can be carried out readily. With use of Eq. (5.7) the final result for $\langle \phi^2 \rangle_{AV}$ is

$$\langle \phi^2(x) \rangle_{AV} = \frac{\langle \sigma^2 \rangle_{AV} V}{16 \pi^2 c^4 a^3 \lambda^2} \frac{m_0^2 + 1}{\sqrt{R_1 R_2}} \quad (5.19)$$

(44)

where

$$\hat{x}^2 = x^2 (1 - M_0^2 \sin^2 \theta) = x^2 (1 - M_0^2 x_2^2 / x^2)$$

$$R_1 = A_0 m_0^2 - B_0 m_0 + 1$$

$$R_2 = m_0^2 + B_0 m_0 + A_0$$

$$A_0 = 1 + (1 + \mathcal{L}^2) J M^2 \beta_0^{-4} - 2 M \beta_0^{-2} (\cos \theta_0 - M_0)$$

$$B_0 = 2 M \beta_0^{-1} \rho \omega \theta_0$$

$$J = (1 - M_0 \cos \theta_0)^2$$

$$m_0 = \frac{1 - A_0}{B_0} - \sqrt{1 + \frac{(1 - A_0)^2}{B_0^2}}$$

$$\theta_0 = \arctan (\beta_0 \tan \theta)$$

$$\beta_0 = \sqrt{1 - M_0^2}$$

$$M = \text{source speed } U / \text{speed of sound}$$

$$M_0 = \text{flow speed } U_0 / \text{speed of sound}$$

(5.20)

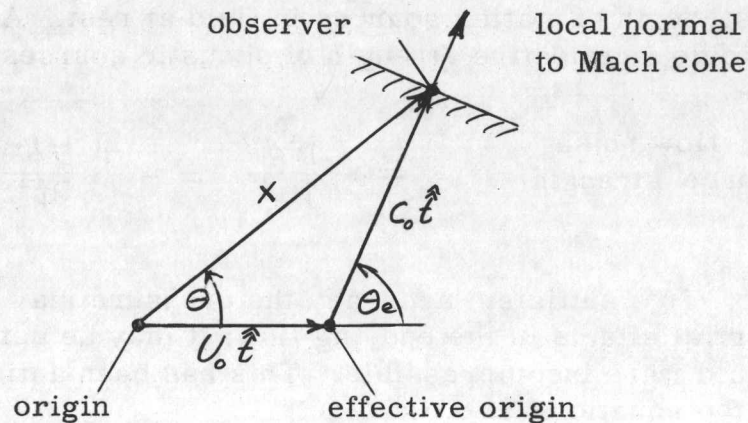
Eqs. (5.19) and (5.20) give the mean square pressure radiated by acoustic sources characterized by the covariance Eq. (5.10) moving with speed U through a certain volume V in a uniform stream of speed U_0 . The source speed U simulates the effects of eddy convection by a jet. In a numerical example we compare the case $M_0 = 0$ (\sim jet noise in fluid at rest) with the case $M_0 = 0.8$ (\sim jet noise in a stream with $M_0 = 0.8$) for a constant difference between source speed (\sim jet flow speed) and stream speed: $M - M_0 = 2.0$. The results are exhibited in Fig. 12 as curves of $\langle p^2 \rangle$ versus θ for a fixed radial distance x .

The interesting feature is the sweepback of the peak of the lobe from $\theta_e = 60^\circ$ at $M_0 = 0$ to $\theta \approx 33.5^\circ$ * at $M_0 = 0.8$. The peak at θ_e is normal to the Mach cone and satisfies $(M - M_0) \cos \theta_e = 1$. The angle θ of the $M = 0.8$ peak appears to be related† to the angle θ_e of the $M = 0$ peak by the construction shown in Sketch 4.

* The graph is faired improperly so as to show the peak at $\theta \approx 31^\circ$.

† This relation has been checked by careful numerical computation for one case at $\mathcal{L} = 0$ (no source fluctuation). It may be that when \mathcal{L} is not $\ll 1$ the two peaks are shifted slightly.

(45)



Sketch 4. Sweepback of the Noise Peak

The sweepback results from the convection of the wave pattern. Calculation gives

$$\sin \theta_e = \frac{(1 - M_0^2) \sin \theta}{\sqrt{1 - M_0^2 \sin^2 \theta} - M_0 \cos \theta} \quad (5.21)$$

VI. ASSESSMENT AND RESUMÉ OF MAJOR POINTS

In this chapter a number of the more important matters relevant to or dealt with in the theory are summed up from a critical point of view. There is no attempt at completeness: this is left to the Summary.

6.1 Pulsating Fluid Elements as Sound Sources

The quadrupole picture of the mechanism for generation of flow noise is correct but not unique. A much simpler picture is afforded in terms of simple sources. The volume of a moving fluid element fluctuates inversely with the local pressure, and this fluctuation radiates the sound.

The volume, density, and pressure fluctuations are related by

$$\rho_0 \frac{D \text{ vol} / \text{vol.}}{Dt} = - \frac{D \rho^{(0)}}{Dt} = - \frac{1}{c_0^2} \frac{D p^{(0)}}{Dt} \quad (6.1)$$

where the derivative follows the fluid motion. Any of these is equivalent

to a virtual strength of matter sources in fluid at rest. A second time derivative yields an effective strength of acoustic sources per unit volume:

$$\begin{array}{l} \text{flow noise} \\ \text{source strength} \end{array} = - \frac{D^2 \rho^{(0)}}{Dt^2} = - \frac{1}{c_0^2} \frac{D^2 \dot{p}^{(0)}}{Dt^2} \quad (6.2)$$

To a sufficient accuracy the pressure may be attributed solely to inertial effects in the eddying flow; it may be determined as though the fluid were incompressible. This has been anticipated in the use of $\dot{p}^{(0)}$ in the equations.

For bounded flows (e.g., the flow about a rod producing Aeolian tones) a volume integral of the simple sources of Eq. (5.2) still describes the primary radiated sound. On the other hand, the volume integral of quadrupoles must be supplemented by a surface integral of dipoles (Refs. 1, 6).

The simple-source formulation shows that the sound originates in a more or less extended region bounded by the surface, but not from the surface itself. The dipole integral on the other hand implies a finite sound emission from the surface; addition of the quadrupole integral is required to bring the predicted surface sound emission to zero. This fact has more conceptual than practical significance since the dipole integral will dominate at large distances from low speed flows.

6.2 Comparisons with Quadrupole Theory.

For low speed flow the sound pressure at a large distance \underline{x} from the region V occupied by the flow may be expressed in the alternate forms

$$p \sim \frac{1}{c_0^2 x} \int_V \frac{\partial^2 (\rho_0 u_x^2)}{\partial t^2} dV \quad \text{quadrupole theory} \quad (6.3)$$

$$p \sim \frac{1}{c_0^2 x} \int_V \frac{D^2 \dot{p}^{(0)}}{Dt^2} dV \quad \text{simple source theory} \quad (6.4)$$

where u_x is the component perturbation velocity along \underline{x} . Which should be used in particular applications is, thus, just a matter of convenience or preference.

Either of the above expressions leads, on neglect of convective effects, to

$$dP \sim \frac{\rho_0 \omega^4 u^4}{c_0^5} L^3 dV \sim \frac{\rho_0 U^8}{c_0^5 L} dV \quad (6.5)$$

as the relative acoustic power emitted by a turbulent volume element dV for similar flows. The famous U_0^8 law for the total noise power of a jet follows at once. Simple similarity considerations lead to laws describing the power emitted by successive slices of a jet as a function of distance x (not to be confused with the field point \underline{x} referred to earlier) from the nozzle: these go as x^0 (constant) in the mixing region with a transition to x^{-7} in the fully developed jet.

6.3 Amplifying Effect of Mean-Flow Shear

The effect of the strongly sheared mixing region in a jet in intensifying the generation of flow noise was first pointed out by Lighthill (Ref. 1). He reexpressed his quadrupole integrand as essentially the product of the shear and the time derivative of the pressure. Thus the presence of a mean shear \gg the fluctuating shear provides an amplification.

In the simple source theory the amplification is indirect. The near field pressure $p^{(0)}$ is amplified, which augments the simple-source strength $\sim D^2 p^{(0)} / Dt^2$. The far-field pressure is amplified in the same proportion. The amplification of $p^{(0)}$ is exhibited in the term $2 \rho_0 (\partial U / \partial n) (\partial v / \partial y_i)$: this term dominates the pressure source strength $\partial^2 \rho_0 u_i^{(0)} u_j^{(0)} / \partial y_i \partial y_j$ when the mean shear $\partial U / \partial n \gg$ the fluctuating shear.

6.4 Convective and Refractive Effects of the Mean Flow

The mean jet flow convects the acoustic sources and refracts or distorts their individual sound fields. We have approximated the convective aspect by calculating the sound emission from sources effectively moving through fluid at rest. The formalism actually employs sources at rest; the effective motion is imparted by replacing ξ_i by $\xi_i - U\tau$ in the source-strength covariance. Allowance is also made for fluctuation of the source strength with time (through a term in the covariance) and for the time delays of emission from different points.

Computations using this formalism yield sound directionality curves; they show generally enhanced downstream emission due to the source motion. However, the infinite enhancement predicted by Lighthill at $M = 1$ is reduced to a bounded value by the fluctuation term. (It is, in fact, implicit in Lighthill's formulation via moving axes that the source pattern is convected without fluctuation).

A refinement would replace the nominal emission time $t - r/c_0$ in the integrand by a value approximately corrected to allow for convection of the sound waves. The effect of this correction may be important for directions approaching the jet axis.

A further refinement would use the accurately computed emission time and would correct for refraction by the shear flow: the spherically symmetric radiation term in the integrand would be multiplied by a suitable directionality factor. This modified source term (Green's function) and its emission time present a formidable computation problem and have not been worked out. Calculations of the Green's function for idealized models of jet flow have been made by Gottlieb (Ref. 34) and by Moretti and Slutsky (Ref. 35).

6.5 Role of the Covariance

The central element of the formalism is the covariance of the acoustic source strength at two points with fixed separation in space and time - the time average of the products of the two source strengths. If the functional form of this covariance were known formal integrations would yield the mean square noise pressure and other properties. However, experimental determinations have not been carried out and theoretical guides are limited. Resort has therefore been had herein to purely speculative assumptions for the form of the covariance.

The computed results are in certain respects sensitive to the assumed functional form. It is quite possible to obtain the impossible result of negative $\langle p^2 \rangle_{AV}$ for certain directions by a choice of covariance function that has extensive negative regions. This must be construed as a fault of the assumed covariance rather than of the method. The covariance formalism applies equally as well with Lighthill's quadrupoles as it does with simple sources, so the difficulties cannot be associated with the kind of elementary source. In the classical treatment (Ref. 1), in fact, the difficulties were suppressed by postulating a covariance equivalent to perfect correlation and zero time delay within a cubical box and zero correlation outside: there were by hypothesis no negative regions.

6.6 Convection vs. Fluctuation in the Near Field

An observer moving with a non-fluctuating or "frozen" random pattern of convected acoustic sources would record zero frequency. On the other hand the spatial variations of the same pattern moving past a stationary observer will appear to him as variations with time: he will record a broad band of frequencies. The covariance formalism herein provides the mathematical apparatus for making the comparison. "Frozen" patterns and those with arbitrary amounts of fluctuation may be treated.

The corresponding comparison for a single-frequency convected pattern (single-Fourier-component of a random pattern) is illuminating. The moving observer sees, of course, the single frequency - a line. The stationary observer sees, by calculation with the covariance technique, a line broadened by addition and subtraction of the apparent frequencies due to convection (a band). If the true frequency is relatively low compared with the apparent convective frequencies ("semi-frozen" pattern) the line is submerged in a broad-band convective spectrum. (see Fig. 8).

The foregoing remarks are concerned with the spectrum of the sound source strength $-c_0^2 D^2 \bar{p}^{(s)} / Dt^2$. Similar comparisons apply to the pressure $\bar{p}^{(s)}$ within and near the turbulent flow.

6.7 Doppler Shift

It has been noted earlier that the use of moving axes to account for source-convection effects (Ref. 1) effectively suppresses fluctuation of the source pattern. The covariance formalism herein, on the other hand, allows for arbitrary amounts of fluctuation: both the convection and the fluctuation are accounted for in the form of the covariance function. This may be either theoretical or experimental.

Either approach should, among other things, correctly predict the Doppler shift of the radiated sound spectrum. This prediction is implicit in the moving-axis technique. But with the covariance formalism a calculation is required. In the example of a convected single-frequency source pattern the correct* Doppler-shifted radiation pattern emerges automatically. The power of the covariance approach - already demonstrated in its prediction of directionality and its ability to discriminate between convective and fluctuative effects on spectra - is confirmed again.

6.8 Moving Jets

The formalism is readily extended to treat the sound field of a moving jet as recorded by a similarly moving observer. In a first approximation the jet flow that transports the eddy pattern - and its refractive effects on the sound field - was neglected. The eddy elements (acoustic sources) were taken to move at essentially the jet velocity through a parallel uniform stream of different velocity. An example has been worked out wherein the excess of the source velocity U over the stream velocity U_0 is supersonic: $M - M_0 = 2$. The cases $M_0 = 0$ and $M_0 = 0.8$ simulate static and moving jets respectively. The sound field for $M_0 = 0$ exhibits an intensity peak at 60° from the "jet" axis, i. e., normal to the Mach cone for Mach number 2. On the other hand, motion of the jet at $M_0 = 0.8$ sweeps the intensity peak downstream closer to the axis ($\theta \approx 33.5^\circ$). The effect is simply explained by a vectorial composition of wave convection and propagation.

*On the basis of the idealized model of sound sources moving through fluid at rest: this model is implicit in both calculations. Allowance for convection of the sound waves by the mean flow can in principle be made in the covariance integral: see Chapter IV.

APPENDIX A

GOVERNING EQUATIONS FOR SOUND PRODUCED BY UNSTEADY FLOWS
PLUS OTHER DISTURBANCES

In the main text generality was sacrificed in the interest of simplicity. The situation is reversed in this appendix. We postulate a general fluid possessing viscosity and heat conduction. There may be body forces F_i per unit volume and sources of heat and matter. The effects of heat sources, conduction, and dissipation are expressed jointly in terms of the entropy s for the case that the fluid obeys an equation of state.[†]

A1. Generalized Form of Lighthill Equation

In what follows, unless stated otherwise, the reference frame will be fixed in quiescent fluid outside the region of disturbed flow. Under the postulated assumptions the governing equations are

continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial y_i} = m \quad (\text{A1})$$

momentum:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial y_j} (\rho u_i u_j + \tau_{ij}) + \frac{\partial p}{\partial y_i} = F_i + m u_i \quad (\text{A2})$$

state:

(where applicable)[†]

$$d\rho = c^{-2} dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds \quad (\text{A3})$$

The momentum equation is in Reynolds form: it differs from the usual form by the addition of u_i times the continuity equation. The dependence of the viscous stress tensor τ_{ij} on the rates of strain in the fluid is left unspecified for generality. The equation of state is in differential form, which suffices for the present purpose; the reciprocal of the speed of sound squared, c^{-2} , is written in place of its equal $(\partial \rho / \partial p)_s$. For a perfect gas $(\partial \rho / \partial s)_p = -\rho / c_p$.

The elimination of ρu_i between the first two equations results in

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \tau_{ij}}{\partial y_i \partial y_j} - \frac{\partial F_i}{\partial y_i} + \frac{\partial m}{\partial t} - \frac{\partial (m u_i)}{\partial y_i} \quad (\text{A4})$$

[†] An equation of state is here taken to mean a fixed functional relation connecting density, pressure and entropy, free of time-dependent relaxation effects. Such a relation does not hold for ultrasonic frequencies.

where $T_{ij} \equiv \rho u_i u_j + \tau_{ij}$ (A5)

\equiv instantaneous Reynolds stress plus shear stress

With use of the symbol $\delta_{ij} \left\{ \begin{array}{l} = 0 \quad i \neq j \\ = 1 \quad i = j \end{array} \right\}$ this can be written in the alternate forms:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 [T_{ij} + (\rho - c_0^2 \rho) \delta_{ij}]}{\partial y_i \partial y_j} - \frac{\partial F_i}{\partial y_i} + \frac{\partial m}{\partial t} - \frac{\partial (m u_i)}{\partial y_i} \quad (A6)$$

$$\frac{1}{c_0^2} \frac{\partial^2 \rho}{\partial t^2} - \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} + \frac{1}{c_0^2} \frac{\partial^2 (\rho - c_0^2 \rho)}{\partial t^2} - \frac{\partial F_i}{\partial y_i} + \frac{\partial m}{\partial t} - \frac{\partial (m u_i)}{\partial y_i} \quad (A7)$$

Eq. (A6) is Lighthill's equation governing flow noise plus additional terms accounting for body forces F_i and mass addition m . Equation (A7) is an equivalent equation with pressure p replacing density ρ as the dependent variable. Neither equation involves the restrictive assumption that the fluid obeys an equation of state; thus they have maximum generality.

Equation (A6) may be written as

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \text{R.H.S.} \quad (A8)$$

where R. H. S. signifies "right-hand-side". This is of the form of the inhomogeneous wave equation, with R. H. S. interpretable as a strength distribution of simple acoustic sources per unit volume. Lighthill examined the terms in R. H. S. and showed that for unbounded flows* (a) a simple-source strength $-\partial F_i / \partial y_i$ / unit volume is equivalent to a dipole strength $-F_i$ / unit volume; (b) a simple source strength $\partial^2 T_{ij} / \partial y_i \partial y_j$ / unit volume is equivalent to a quadrupole strength T_{ij} / unit volume. The equivalence can be demonstrated by the methods of Appendix B.

For most cases of interest the fluid obeys an equation of state. For these cases it will be convenient to introduce the entropy S into the formulation. Thus insertion of Eq. (A3) into either Eq. (A4) or Eq. (A7) yields approximately †

* For bounded flows the volume distribution of simple sources $\partial^2 T_{ij} / \partial y_i \partial y_j$ is equivalent to the sum of a volume distribution of quadrupoles T_{ij} plus a surface distribution of dipoles. The present author believes that Doak (Ref. 40) errs in failing to include this surface distribution in his expression for boundary layer noise.

† The local values of c (speed of sound) and $(\partial \rho / \partial s)_p$ have been replaced by their mean values c_0 and $\langle (\partial \rho / \partial s)_p \rangle_{AV}$, respectively, and higher order terms arising from their derivatives have been neglected.

$$\frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \underbrace{\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}}_{\substack{\text{flow} \\ \text{noise}}} - \frac{1}{c_0^2} \left\langle \left(\frac{\partial p}{\partial s} \right)_p \right\rangle_{AV} \frac{\partial^2 s}{\partial t^2} - \frac{\partial f_i}{\partial y_i} + \frac{\partial m}{\partial t} - \frac{\partial(m u_i)}{\partial y_i} \quad (\text{A9})$$

heat
body
mass
addition
forces
addition

where

$$T_{ij} = \rho u_i u_j + \tau_{ij} \quad (\text{A5})$$

The labels appended to Eq. (A9) show the interpretation of the terms of the right-hand-side as sources of sound. The term in T_{ij} lumps together the effects of momentum flux $\rho u_i u_j$ and shear stress τ_{ij} . The term in $\partial^2 s / \partial t^2$ lumps together direct heat addition, viscous dissipation, and heat conduction; for low Mach numbers and the audible frequency range the latter two will be unimportant. The terms $\partial m / \partial t - \partial(m u_i) / \partial y_i$ appear to account for the mass addition. However, as will be seen later (Eq. A18), additional terms in m are implicit in the density gradients of the T_{ij} term (these gradients also include secondary effects of heat addition). With inclusion of the terms implicit in the T_{ij} term the acoustic source strength associated with mass addition becomes Dm/Dt (Eq. A20).

A2. Expansion of Lighthill Source Term

In this section Lighthill's source term $\partial T_{ij} / \partial y_i \partial y_j$ will be expanded to exhibit non-negligible density gradients. Since $T_{ij} = \rho u_i u_j + \tau_{ij}$ is dominated by $\rho u_i u_j$ except near surfaces, we shall deal first with $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$. The first step in the expansion is

$$\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} = \frac{\partial}{\partial y_i} \left[u_i \frac{\partial \rho u_j}{\partial y_j} + \left(\frac{\partial u_i}{\partial y_j} \right) \rho u_j \right] \quad (\text{A10})$$

Eliminate $\partial \rho u_j / \partial y_j$ by means of the continuity equation (A1), carry out the differentiation, then again use (A1) to eliminate $\partial u_i / \partial y_i$; this yields

$$\begin{aligned} \frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} &= -u_i \frac{\partial^2 \rho}{\partial y_i \partial t} + \frac{\partial \rho}{\partial t} \left[\left(\frac{\partial \rho}{\partial y_i} \right) \frac{u_i}{\rho} + \left(\frac{\partial \rho}{\partial t} \right) \frac{1}{\rho} - \frac{m}{\rho} \right] + \frac{\partial(\rho u_j)}{\partial y_i} \frac{\partial u_i}{\partial y_j} \\ &\quad - \rho u_j \frac{\partial}{\partial y_j} \left[\left(\frac{\partial \rho}{\partial y_i} \right) \frac{u_i}{\rho} + \left(\frac{\partial \rho}{\partial t} \right) \frac{1}{\rho} - \frac{m}{\rho} \right] \quad (\text{A11}) \end{aligned}$$

$$\begin{aligned} &= -u_i \frac{\partial^2 \rho}{\partial y_i \partial t} + \frac{\partial \rho}{\partial t} \left(\frac{\partial \rho}{\partial y_i} \right) \frac{u_i}{\rho} + \left(\frac{\partial \rho}{\partial t} \right)^2 \frac{1}{\rho} - \left(\frac{\partial \rho}{\partial t} \right) \frac{m}{\rho} + \frac{\partial(\rho u_j)}{\partial y_i} \frac{\partial u_i}{\partial y_j} + \frac{\partial(m u_i)}{\partial y_i} \\ &\quad - \rho u_j \left[\frac{u_i}{\rho} \frac{\partial^2 \rho}{\partial y_i \partial y_j} + \frac{1}{\rho} \frac{\partial u_i}{\partial y_j} \frac{\partial \rho}{\partial y_i} + \frac{u_i}{\rho} \frac{\partial^2 \rho}{\partial y_j \partial y_j} - \frac{1}{\rho} \frac{\partial \rho}{\partial y_j} \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \frac{\partial^2 \rho}{\partial y_j \partial t} - \frac{\partial}{\partial y_j} \left(\frac{m}{\rho} \right) \right] \quad (\text{A12}) \end{aligned}$$

In our efforts to simplify this equation m will be considered sufficiently small so that m^2 and products $mu_i \partial \rho / \partial y_i$ and $m \partial \rho / \partial t$ can be neglected. To this accuracy we can use Eq. (A1) to establish the useful identity

$$\rho \frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial y_i} - \frac{m}{\rho} \right) \left(\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial y_j} - \frac{m}{\rho} \right) \quad (\text{A13})$$

$$\approx \frac{1}{\rho} \left[\left(\frac{\partial \rho}{\partial t} \right)^2 + 2 u_i \frac{\partial \rho}{\partial t} \frac{\partial \rho}{\partial y_i} + u_i u_j \frac{\partial \rho}{\partial y_i} \frac{\partial \rho}{\partial y_j} \right] \quad (\text{A14})$$

The left-hand-side thus replaces the four terms marked • in Eq. (A12). That equation may now be written to a consistent order of accuracy as

$$\begin{aligned} \frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} &= \rho \left(\frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} + \frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} \right) - 2 u_i \frac{\partial^2 \rho}{\partial y_i \partial t} - u_i u_j \frac{\partial^2 \rho}{\partial y_i \partial y_j} \\ &+ u_i \frac{\partial m}{\partial y_i} + \frac{\partial m u_i}{\partial y_i} \end{aligned} \quad (\text{A15})$$

after slight simplification.

Suppose $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$ were evaluated in a frame of reference following the motion of a fluid element, but with u_i still referred to the stationary frame. The space derivatives $\partial u_j / \partial y_i$ etc., being instantaneous, will be unaffected by the motion. But the terms multiplied by u_i vanish, since u_i must be replaced there by the zero velocity of the element in the moving frame. If in addition there were no matter sources ($m = 0$) $\partial^2 \rho u_i u_j / \partial y_i \partial y_j$ would reduce to

$$\begin{array}{l} \text{moving} \\ \text{frame,} \\ m = 0 \end{array} \left[\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} \right]_{u_i = m = 0} = \rho \left(\frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} + \frac{\partial u_i}{\partial y_i} \frac{\partial u_j}{\partial y_j} \right) \quad (\text{A16})$$

Accordingly Eq. (A15) may be written in the useful form

$$\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} = \left[\frac{\partial^2 \rho u_i u_j}{\partial y_i \partial y_j} \right]_{u_i = m = 0} - 2 u_i \frac{\partial^2 \rho}{\partial y_i \partial t} - u_i u_j \frac{\partial^2 \rho}{\partial y_i \partial y_j} + u_i \frac{\partial m}{\partial y_i} + \frac{\partial m u_i}{\partial y_i} \quad (\text{A17})$$

Since $T_{ij} = \rho u_i u_j + \tau_{ij}$, it remains to deal with the shear stress tensor τ_{ij} . The velocity gradients in τ_{ij} will be unaltered in the moving frame, and the viscosity gradients can be neglected in either frame. Therefore we can take τ_{ij} as unaltered in the moving frame. Thus finally

$$\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} = \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{u_i=m=0} - 2 u_i \frac{\partial^2 \rho}{\partial y_i \partial t} - u_i u_j \frac{\partial^2 \rho}{\partial y_i \partial y_j} + u_i \frac{\partial m}{\partial y_i} + \frac{\partial m u_i}{\partial y_i} \quad (\text{A18})$$

where the subscript $u_i = 0$ (to be applied after the differentiation) designates the value following the motion of a fluid element.

Equation (A18) is the required expansion of Lighthill's source term. The relative magnitude of the density derivatives will be considered later.

A3. Generalized Form of Simple Source Equation

Case (a)-Body Forces and Matter Sources Accounted for in the Incompressible Flow. - If the expanded form Eq. (A18) of Lighthill's source term is inserted in Eq. (A4) there results

$$\frac{\partial^2 \rho}{\partial t^2} + 2 u_i \frac{\partial^2 \rho}{\partial y_i \partial t} + u_i u_j \frac{\partial^2 \rho}{\partial y_i \partial y_j} - \nabla^2 p = \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{u_i=m=0} - \frac{\partial f_i}{\partial y_i} + \frac{\partial m}{\partial t} + u_i \frac{\partial m}{\partial y_i} \quad (\text{A19})$$

This may be written as

$$\frac{D^2 \rho}{Dt^2} - \nabla^2 p = \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{u_i=m=0} - \frac{\partial f_i}{\partial y_i} + \frac{Dm}{Dt} \quad (\text{A20})$$

where $D/Dt = \partial/\partial t + u_i \partial/\partial y_i$ and $[\partial^2 \rho u_i u_j / \partial y_i \partial y_j]_{u_i=0}$ both designate operations following the fluid motion.

If the fluid were incompressible but not necessarily of uniform density the density derivative following the fluid motion would vanish. (It is for this reason that the convective parts of $D^2 \rho / Dt^2$ were put into evidence in the expansion of the Lighthill source term). This yields the equation

$$-\nabla^2 p^{(0)} = \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{u_i=m=0} - \frac{\partial f_i}{\partial y_i} + \left(\frac{Dm}{Dt} \right)^{(0)} \quad (\text{A21})$$

where the superscript (0) designates values as modified by the postulation of incompressible flow.

It is evident from Eqs. (A20) and (A21) that the body forces and matter sources m are accounted for in the incompressible flow as well as in the compressible flow. This leads to results of great formal simplicity, which are developed below. An alternate treatment in which F_i and m are omitted from the incompressible flow is given in the later case (b).

Subtraction of the second equation from the first yields

$$\frac{D^2 p}{Dt^2} - \nabla^2 p = -\nabla^2 p^{(0)} - \left[\frac{\partial^2 (T_{ij} - T_{ij}^{(0)})}{\partial y_i \partial y_j} \right]_{u_i = m = 0} \quad (\text{A22})$$

The difference between T_{ij} and $T_{ij}^{(0)}$ arises from the compressibility of the fluid; the difference is small for the postulated low speeds of the unsteady components of the flow. In what follows, we shall neglect the affected terms completely, the justification being given in Appendix B2. The result is

$$\frac{D^2 p}{Dt^2} - \nabla^2 p = -\nabla^2 p^{(0)} \quad (\text{A23})$$

By virtue of the equation of state this is approximately

$$\frac{1}{c_0^2} \frac{D^2 p}{Dt^2} - \nabla^2 p = -\nabla^2 p^{(0)} - \frac{1}{c^2} \left\langle \left(\frac{\partial p}{\partial s} \right)_p \right\rangle_{AV} \frac{D^2 s}{Dt^2} \quad (\text{A24})$$

(With heat conduction and viscous dissipation neglected and with no external heat addition this reduces to

$$\frac{1}{c_0^2} \frac{D^2 p}{Dt^2} - \nabla^2 p = -\nabla^2 p^{(0)} \quad (\text{A25})$$

Eq. (A24) is useful as it stands. However, it simplifies if we put

$$p = p_0 + p^{(0)} + p^{(1)} \quad (\text{A26})$$

where p_0 is the uniform, constant ambient pressure, $p^{(0)}$ is the pressure calculated as if the fluid were incompressible (i. e., via Eq. (A21)) and $p^{(1)}$ is the remaining increment in pressure, which must be attributed to the compressibility. This is inserted into Eq. (A24) to yield

$$\frac{1}{c_0^2} \frac{D^2 p^{(1)}}{Dt^2} - \nabla^2 p^{(1)} = -\frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} - \frac{1}{c^2} \left\langle \left(\frac{\partial p}{\partial s} \right)_p \right\rangle_{AV} \frac{D^2 s}{Dt^2} \quad (\text{A27})$$

since terms in $-\nabla^2 p^{(0)}$ cancel.

As in the main text the (presumably weak) scattering of the emitted sound field by the unsteady flow is suppressed by replacing the convective derivative D/Dt following the instantaneous flow by the convective derivative $\bar{D}/\bar{D}t$ following the mean flow on the left-hand side. This replacement yields

$$\left. \frac{1}{c_0^2} \frac{\bar{D}^2 p^{(1)}}{\bar{D}t^2} - \nabla^2 p^{(1)} = -\frac{1}{c_0^2} \frac{D^2 p^{(0)}}{Dt^2} - \frac{1}{c_0^2} \left\langle \left(\frac{\partial p}{\partial s} \right)_p \right\rangle_{AV} \frac{D^2 s}{Dt^2} \right\} \quad (\text{A28})$$

where

$$\frac{\bar{D}}{\bar{D}t} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial y_1}$$

for the case of a mean flow $U(y_2, y_3)$ along the y_1 axis.

The equations may be simplified if the viscosity and heat conduction are neglected (they are small anyhow for low Mach numbers of the unsteady motion). Then the compressible part of the pressure, which can be recognized as the radiated sound field, is isentropic. All of the entropy perturbation results from heat addition and may be combined with the pressure perturbation to define a density perturbation

$$d\rho^{(0)} = c_0^{-2} dp^{(0)} + \left\langle \left(\frac{\partial \rho}{\partial s} \right)_p \right\rangle_{Av} ds \quad (\text{A29})$$

Equation (A28) then reduces to

$$\frac{1}{c_0^2} \frac{D^2 \rho^{(0)}}{Dt^2} - \nabla^2 \rho^{(0)} = - \frac{D^2 \rho^{(0)}}{Dt^2} \quad (\text{A30})$$

Equation (A30) is the equation governing flow noise in its simplest form. Its physical interpretation is this. The radiative sound field $\rho^{(0)}$ obeys an inhomogeneous modified wave equation corresponding to a spatial distribution of simple sources of strength $-D^2 \rho^{(0)}/Dt^2$ per unit volume. The quantity $\rho^{(0)}$ is the "zero-order" density perturbation in the fluid: it is related to the pressure $p^{(0)}$ (calculated as though the fluid were incompressible) through the equation of state. Sources of heat, sources of matter, body forces, and hydrodynamic effects are all included implicitly in $\rho^{(0)}$: the heat sources via the entropy term; and the matter sources, body forces and hydrodynamic effects through their influence on $p^{(0)}$. In summary, Eq. (A30) states that the radiation of sound from the interior of a fluid originates solely from local first-order density perturbations; it is immaterial how these perturbations were produced, whether from dynamical flow effects, body forces, matter sources or heat sources.

Case (b): Body Forces and Matter Sources Excluded from the Incompressible Flow. - The primary virtue of Case (a) is its conceptual and formal simplicity. It does not obviate the need for explicit introduction of the body forces and matter sources in the prior computation of $p^{(0)}$. In practice it is far easier to calculate the direct acoustic (i. e., compressibility) effects of body forces and matter sources than to

* A density perturbation rate $D\rho^{(0)}/Dt$ would produce a dilatation rate $\text{div } \underline{u}$ in a fluid free of matter sources m according to $D\rho^{(0)}/Dt + \rho_0 \text{div } \underline{u} = 0$ approximately. Thus the sound radiation from density perturbations in the actual fluid (governed by $-D^2 \rho^{(0)}/Dt^2$) is as though the sound were produced by equivalent dilatations (via the term $D/Dt(\rho_0 \text{div } \underline{u})$) in a source-free fluid. (We distinguish here between $\rho^{(0)}$ associated with sound generation and $\rho^{(0)}$ associated with sound propagation).

determine these effects indirectly via an incompressible flow. The direct approach constitutes the present Case (b).

In Eqs. (A20) and (A21) the effects of body forces F_i and matter sources m are seen to be linearly additive to the effects of flow u_i . Therefore we delete these terms and consider them later separately. This gives

$$\frac{D^2 \rho}{Dt^2} - \nabla^2 p = \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right] u_i = m = 0 \quad (\text{A31})$$

$$- \nabla^2 p = \left[\frac{\partial^2 T_{ij}^{(0)}}{\partial y_i \partial y_j} \right] u_i = m = 0 \quad (\text{A32})$$

where D/Dt has again been approximated as $\bar{D}/\bar{D}t$ on the left-hand side. Once more put $p = p_0 + p^{(0)} + p^{(1)}$, and follow procedures parallel to those of Case (a). The final result is

$$\frac{1}{c_0^2} \frac{D^2 p^{(1)}}{Dt^2} - \nabla^2 p^{(1)} = - \frac{D^2 p^{(0)}}{Dt^2} \quad (\text{A33})$$

The value of $p^{(0)}$ may include the effects of heat addition via the entropy relation Eq. (A29). The pressures p , $p^{(0)}$ and $p^{(1)}$, however apply to a flow with $F_i = m = 0$. The effects of F_i and m can now be restored by generalizing p to

$$p = p_0 + p^{(0)} + p^{(1)} + p^{(1)'} \quad (\text{A34})$$

where

$$p^{(1)'} = - \frac{1}{4\pi} \int \left(\frac{1}{r} \frac{\partial F_i}{\partial y_i} \right)_{\bar{t}} dy + \frac{1}{4\pi} \int \left(\frac{Dm}{Dt} \right)_{\bar{t}} dy \quad (\text{A35})$$

$$= - \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int \left(\frac{F_i}{r} \right)_{\bar{t}} dy + \frac{1}{4\pi} \int \left(\frac{Dm}{Dt} \right)_{\bar{t}} dy \quad (\text{A36})$$

(cf. Appendix B)

A4 Examination of Neglected Term in Simple Source Equation

In the derivation of Eqs. (27) and (2.10) of the main text and Eqs. (A24) and (A28) of this Appendix a term representing distributed acoustic sources was neglected. With the inclusion of viscosity this term may be written as

$$\left[\frac{\partial^2 T_{ij}^{(1)}}{\partial y_i \partial y_j} \right] u_i = m = 0 \quad (\text{A37})$$

(58)

where

$$\begin{aligned} T_{ij}^{(1)} &\equiv T_{ij} - T_{ij}^{(0)} \\ &\equiv (\rho u_i u_j + \tau_{ij}) - (\rho_0 u_i^{(0)} u_j^{(0)} + \tau_{ij}^{(0)}) \end{aligned} \quad (\text{A38})$$

The use of a reference frame following the local instantaneous fluid motion, designated by $u_i = 0$ above, is a source of difficulty. In what follows we shall assume that

$$\left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{U = m = 0} \quad (\text{A39})$$

where the reference frame follows the mean motion U , is of comparable order of magnitude. (It should certainly be as large or larger).

The neglected term may, perhaps, contribute significantly to $p^{(1)}$ in the near-field, since $\partial^2/\partial y_i \partial y_j \gg c_0^{-2} \partial^2/\partial t^2$ there. However, it is the far-field behaviour of $p^{(1)}$ that is of major interest. Lighthill's analysis (Ref. 1) shows that a volume distribution of simple sources of strength

$$\left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{m=0} \quad (\text{A40})$$

yields a pressure disturbance

$$\Delta p^{(1)}(\underline{x}, t) = \frac{1}{4\pi c_0^2} \int_V \frac{(x_i - y_i)(x_j - y_j)}{r^3} \left[\frac{\partial^2 T_{ij}}{\partial t^2} \right]_{m=0} dy \quad (\text{A41})$$

at distances $r = |\underline{x} - \underline{y}|$ that are at least a few acoustic wave lengths from all parts of V .

This formalism is no longer valid when the source strength is

$$\left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{U = m = 0} \quad (\text{A39})$$

i. e., when $\partial^2/\partial y_i \partial y_j$ is applied following the mean motion. However, it seems reasonable to infer from Eq. (A41) that the effect of this source term on pressure in the far field goes more or less as

$$\left[\frac{\partial^2 T_{ij}}{\partial t^2} \right]_{U=m=0} \quad (\text{A42})$$

that is

$$\frac{|\Delta\phi^{(n)}| \text{ due to } \left[\frac{\partial^2 T_{ij}^{(n)}}{\partial y_i \partial y_j} \right]_{U=m=0}}{|\Delta\phi^{(n)}| \text{ due to } \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right]_{U=m=0}} \sim \frac{\left| \frac{\partial^2 T_{ij}^{(n)}}{\partial t^2} \right|_{U=m=0}}{\left| \frac{\partial^2 T_{ij}}{\partial t^2} \right|_{U=m=0}} \quad (\text{A43})$$

The shear stresses τ_{ij} in T_{ij} are unimportant compared with $\rho u_i u_j$ except near surfaces; therefore they will be neglected. Accordingly $T_{ij}^{(n)}$ reduces in the moving frame to

$$\left[T_{ij}^{(n)} \right]_{U=m=0} = \rho u_i' u_j' - \rho_0 u_i^{(n)'} u_j^{(n)'} \quad (\text{A44})$$

where u_i', u_j' are perturbations from the mean velocity U . This may be expanded as

$$(\rho_0 + \rho^{(o)} + \rho^{(n)})(u_i^{(o)'} + u_i^{(n)'}) (u_j^{(o)'} + u_j^{(n)'}) - \rho_0 u_i^{(o)'} u_j^{(o)'} \quad (\text{A45})$$

A typical component term is

$$\rho_0 u_i^{(o)'} u_j^{(n)'} \quad (\text{A46})$$

Thus

$$\left[T_{ij}^{(n)} \right]_{U=m=0} = O(\rho_0 u^{(o)'} u^{(n)}) \quad (\text{A47})$$

$$\left[\frac{\partial^2 T_{ij}^{(n)}}{\partial t^2} \right]_{U=m=0} = O(\omega^2 \rho_0 u^{(o)'} u^{(n)}) \quad (\text{A48})$$

Compare with

$$\left[T_{ij} \right]_{U=m=0} = O(\rho_0 [u^{(o)'}]^2) \quad (\text{A49})$$

$$\left[\frac{\partial^2 T_{ij}}{\partial t^2} \right]_{U=m=0} = O(\omega^2 \rho_0 [u^{(o)'}]^2) \quad (\text{A50})$$

Under the low-speed condition $u^{(o)'} \ll \text{speed of sound}$, it follows that $|u^{(n)}| / |u^{(o)'}| \ll 1$; accordingly

$$\frac{\text{effect of neglected term}}{\text{effect of retained term}} \sim \frac{\left| \frac{\partial^2 T_{ij}^{(n)}}{\partial t^2} \right|_{U=m=0}}{\left| \frac{\partial^2 T_{ij}}{\partial t^2} \right|_{U=m=0}} \ll 1 \quad (\text{A51})$$

APPENDIX B

SOLUTIONS OF THE FLOW NOISE EQUATIONS WITH MEAN FLOW NEGLECTED

The equations with which we have to deal are of the form

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = q(y, t) \quad (\text{B1})$$

This is the inhomogeneous wave equation: the right-hand-side represents sources of sound of strength q per unit volume. The solution for p at point \underline{x} and time t in an unbounded fluid is

$$p(\underline{x}, t) = \frac{1}{4\pi} \int_{\infty} \frac{1}{r} q(y, \hat{t}) dy + \text{constant} \quad (\text{B2})$$

In the integral r is the distance between the point \underline{x} and the volume element dy and the source strength q is evaluated at the earlier time $\hat{t} = t - r/c_0$.

When the fluid is bounded either internally or externally by surfaces S the solution takes the form (Ref. 6, Eqs. (2.4) and (2.13))

$$p(\underline{x}, t) = \frac{1}{4\pi} \int_V \frac{1}{r} q(y, \hat{t}) dy + \frac{1}{4\pi} \int_S \frac{\partial}{\partial x_i} \left(\frac{p}{r} \right) \hat{t} l_i dS + \frac{1}{4\pi} \int_S \left(\frac{\partial p}{\partial y_j} \right) \hat{t} \frac{l_j dS}{r} \quad (\text{B3})$$

If q has the form of a divergence or a Laplacian certain transformations of the integral can be made. These are dealt with in later sections.

B1. Equality of $\frac{1}{4\pi} \int \frac{1}{r} \nabla^2 p^{(0)} dy$ and $p^{(0)} - \frac{1}{4\pi c_0^2} \int \frac{1}{r} \frac{\partial^2 p^{(0)}}{\partial t^2} dy$

It may be of interest to demonstrate that the respective integral solutions of the equivalent equations (2.7) and (2.10) are likewise equivalent. The proof is given below.

It has been argued that replacement of $D^2 e^{(0)}/Dt^2$ by $\partial^2 e^{(0)}/\partial t^2$ (for the case no mean flow, $U = 0$) merely suppresses turbulent scattering and is permissible. With this change Eqs. (2.7) and (2.10) read, since

$$p = p_0 + p^{(0)} + p^{(1)},$$

$$\left. \begin{aligned} \frac{D^2 p^{(0)}}{Dt^2} - \frac{\partial^2 p^{(0)}}{\partial t^2} - \nabla^2 p &= -\nabla^2 p^{(0)} \\ \frac{\partial^2 p^{(0)}}{\partial t^2} - \nabla^2 p^{(0)} &= -\frac{D^2 p^{(0)}}{Dt^2} \end{aligned} \right\} \quad (\text{B4})$$

or

$$\left. \begin{aligned} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p &= -\nabla^2 p^{(0)} - \left(\frac{D^2}{Dt^2} - \frac{\partial^2}{\partial t^2} \right) p^{(0)} \\ \frac{\partial^2 p^{(0)}}{\partial t^2} - \nabla^2 p^{(0)} &= -\frac{\partial^2 p^{(0)}}{\partial t^2} - \left(\frac{D^2}{Dt^2} - \frac{\partial^2}{\partial t^2} \right) p^{(0)} \end{aligned} \right\} \quad (\text{B5})$$

With use of $dp \approx c_0^{-2} dp$ and some abbreviations these are

$$\left. \begin{aligned} \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p &= -\nabla^2 p^{(0)} - \frac{\sigma''}{c_0^2} \\ \frac{1}{c_0^2} \frac{\partial^2 p^{(1)}}{\partial t^2} - \nabla^2 p^{(1)} &= -\frac{\sigma'}{c_0^2} - \frac{\sigma''}{c_0^2} \end{aligned} \right\} \quad (B6)$$

where

$$\begin{aligned} \sigma' &= \frac{\partial^2 p^{(0)}}{\partial t^2} \\ \sigma'' &= \left(\frac{D^2}{Dt^2} - \frac{\partial^2}{\partial t^2} \right) p^{(0)} \\ p &= p_0 + p^{(0)} + p^{(1)} \end{aligned}$$

and $p^{(0)}$ is a solution of $-\nabla^2 p^{(0)} = [\partial^2 T_{ij}^{(0)} / \partial y_i \partial y_j]_{u_i=0}$. By comparison with Eqs. (B1) and (B2), their respective solutions are

$$p = p_0 - \frac{1}{4\pi} \int (\nabla^2 p^{(0)})_{\hat{t}} \frac{1}{r} dy - \frac{1}{4\pi c_0^2} \int \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy \quad (B7)$$

$$p^{(1)} = -\frac{1}{4\pi c_0^2} \int \left(\frac{\partial^2 p^{(0)}}{\partial t^2} \right)_{\hat{t}} \frac{1}{r} dy - \frac{1}{4\pi c_0^2} \int \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy \quad (B8)$$

$$\text{or } p = p_0 + p^{(0)} - \frac{1}{4\pi c_0^2} \int \left(\frac{\partial^2 p^{(0)}}{\partial t^2} \right)_{\hat{t}} \frac{1}{r} dy - \frac{1}{4\pi c_0^2} \int \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy \quad (B9)$$

(By virtue of the definition of σ'' above, Eq. (B8) can be recognized as Eq. (2.16) of the main text). It will be of interest to demonstrate the equivalence of Eqs. (B7) and (B9); it will afford an alternative proof of the equivalences of Eqs. (2.7) and (2.10).

Noting that $\hat{t} = t - r/c_0$, $r = |\underline{x} - \underline{y}|$, we define the operators

$$\left. \begin{aligned} \left(\frac{\partial}{\partial y_i} \right)_t &= \left(\frac{\partial}{\partial y_i} \right)_{\hat{t}} + \left(\frac{\partial \hat{t}}{\partial y_i} \right)_t \frac{\partial}{\partial \hat{t}} \\ \left(\frac{\partial}{\partial x_i} \right)_t &= \left(\frac{\partial}{\partial x_i} \right)_{\hat{t}} + \left(\frac{\partial \hat{t}}{\partial x_i} \right)_t \frac{\partial}{\partial \hat{t}} \\ &= \left(\frac{\partial}{\partial x_i} \right)_{\hat{t}} - \left(\frac{\partial \hat{t}}{\partial y_i} \right)_t \frac{\partial}{\partial \hat{t}} \quad \text{since } \frac{\partial r}{\partial x_i} = -\frac{\partial r}{\partial y_i} \end{aligned} \right\} \quad (B10)$$

$$\text{Thus } \frac{1}{r} \left(\frac{\partial}{\partial y_i} \right)_{\hat{t}} = \left(\frac{\partial}{\partial x_i} \frac{1}{r} \right)_{\hat{t}} + \left(\frac{\partial y_i}{\partial y_i} \frac{1}{r} \right)_{\hat{t}}$$

Apply these to $\int \frac{1}{r} (\nabla^2 p^{(0)})_{\hat{t}} dy = \int \frac{1}{r} \left(\frac{\partial^2 p^{(0)}}{\partial y_i^2} \right)_{\hat{t}} dy$, taking for the present

a finite integration volume V ; later we shall require $V \rightarrow \infty$. There results

$$\int_V \left(\frac{\partial^2 p^{(0)}}{\partial y_i^2} \right)_{\hat{t}} \frac{1}{r} dy = \int_V \frac{\partial}{\partial x_i} \left[\left(\frac{\partial p^{(0)}}{\partial y_i} \right) \frac{1}{r} \right]_{\hat{t}} dy + \int_V \frac{\partial}{\partial y_i} \left[\left(\frac{\partial p^{(0)}}{\partial y_i} \right) \frac{1}{r} \right]_{\hat{t}} dy \quad (\text{B11})$$

$$= \int_V \frac{\partial}{\partial x_i} \left[\left(\frac{\partial p^{(0)}}{\partial y_i} \right) \frac{1}{r} \right]_{\hat{t}} dy + \int_S \left(\frac{\partial p^{(0)}}{\partial y_i} \right)_{\hat{t}} \frac{l_i dS(y)}{r} \quad (\text{B12})$$

by use of the divergence theorem on the second integral. Here S is the surface bounding V and l_i is the direction cosine of the normal to dS drawn out of the fluid.

A further reduction of the first integral is obtained by use of the relation (see Eq. (B10))

$$\left(\frac{\partial p^{(0)}}{\partial y_i} \right)_{\hat{t}} \frac{1}{r} = \left(\frac{\partial}{\partial x_i} \frac{p^{(0)}}{r} \right)_{\hat{t}} + \left(\frac{\partial p^{(0)}}{\partial y_i} \frac{p^{(0)}}{r} \right)_{\hat{t}} \quad (\text{B13})$$

Application of the divergence theorem then gives

$$\int_V \left(\frac{\partial^2 p^{(0)}}{\partial y_i^2} \right) \frac{1}{r} dy = \int_V \frac{\partial^2}{\partial x_i^2} \left(\frac{p^{(0)}}{r} \right) dy + \int_S \frac{\partial}{\partial x_i} \left(\frac{p^{(0)}}{r} \right) l_i dS + \int_S \left(\frac{\partial p^{(0)}}{\partial y_i} \right) \frac{l_i dS}{r} \quad (\text{B14})$$

the integrands being evaluated at time \hat{t} .

By Eq. (B7), $p - p_0$ is just $-1/4\pi$ times the right-hand side evaluated with V taken as infinite. We shall limit attention to the case where the unsteady flow that generates the sound is of limited spatial extent. Then it can be established that the integrands of the surface integrals attenuate like r^{-4} or faster for large r , whence the surface integrals vanish in the limit $r \rightarrow \infty$. Equation (B7) becomes

$$p(x, t) - p_0 = -\frac{1}{4\pi} \int_{\infty} \frac{\partial^2}{\partial x_i^2} \left(\frac{p^{(0)}}{r} \right) dy - \frac{1}{4\pi c_0^2} \int_{\infty} \frac{\sigma''}{r} dy \quad (\text{B15})$$

This may be expanded as (with $\frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r} \right) = \frac{\partial^2}{\partial y_i^2} \left(\frac{1}{r} \right) = \nabla^2 \frac{1}{r}$)

$$p(x, t) - p_0 = -\frac{1}{4\pi c_0^2} \int_{\infty} \frac{\sigma''}{r} dy - \frac{1}{4\pi} \left\{ \int_{\infty} p^{(0)} \nabla^2 \left(\frac{1}{r} \right) dy + \int_{\infty} 2 \frac{\partial p^{(0)}}{\partial x_i} \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) dy + \int_{\infty} \frac{1}{r} \frac{\partial^2 p^{(0)}}{\partial x_i^2} dy \right\} \quad (\text{B16})$$

The last two integrands are evaluated as follows:

$$\frac{\partial p^{(0)}}{\partial x_i} = -\frac{1}{c_0} \frac{x_i - y_i}{r^2} \left(\frac{\partial p^{(0)}}{\partial t} \right)_{\hat{t}} \quad \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) = -\frac{x_i - y_i}{r^3}$$

$$\frac{\partial}{\partial x_i} \frac{(x_i - y_i)}{r} = \frac{\delta_{ii}}{r} - \frac{(x_i - y_i)^2}{r^3} = \frac{2}{r}$$

$$\therefore \frac{1}{r} \frac{\partial^2 \phi^{(0)}}{\partial x_i^2} = \frac{1}{r} \left\{ \frac{1}{c_0^2} \frac{\partial^2 \phi^{(0)}}{\partial t^2} \frac{(x_i - y_i)^2}{r^2} - \frac{2}{c_0} \frac{\partial \phi^{(0)}}{\partial t} \frac{1}{r} \right\}$$

$$= \frac{1}{c_0^2} \frac{1}{r} \frac{\partial^2 \phi^{(0)}}{\partial t^2} - \frac{2}{c_0} \frac{\partial \phi^{(0)}}{\partial t} \frac{1}{r^2}$$

$$2 \frac{\partial \phi}{\partial x_i} \frac{\partial (1/r)}{\partial x_i} = \frac{2}{c_0} \frac{\partial \phi^{(0)}}{\partial t} \frac{(x_i - y_i)^2}{r^3} = \frac{2}{c_0} \frac{\partial \phi^{(0)}}{\partial t} \frac{1}{r^2}$$
(B17)

Thus Eq. (B16) reduces to

$$\rho(\underline{x}, t) - \rho_0 = -\frac{1}{4\pi} \int_{\infty} \rho^{(0)} \nabla^2 \left(\frac{1}{r} \right) dy - \frac{1}{4\pi c_0^2} \int_{\infty} \frac{\partial^2 \phi^{(0)}}{\partial t^2} \frac{1}{r} dy - \frac{1}{4\pi c_0} \int_{\infty} \frac{\sigma''}{r} dy$$
(B18)

It may be shown* that $-(1/4\pi) \nabla^2 (1/r)$ has the properties of a three-dimensional δ -function. The integral thus reduces to the final form

$$\rho(\underline{x}, t) - \rho_0 = \rho^{(0)} - \frac{1}{4\pi c_0^2} \int_{\infty} \frac{\partial^2 \phi^{(0)}}{\partial t^2} \left(\frac{1}{r} \right) dy - \frac{1}{4\pi c_0} \int_{\infty} \frac{\sigma''}{r} dy$$
(B9)

* For a flow possessing a potential ϕ the divergence theorem yields

$$\int \nabla^2 \phi dV = \int \frac{\partial \phi}{\partial n} dS$$

where \underline{n} is the normal to dS taken out of the fluid. Choose ϕ as $1/r$ and take S as a sphere of radius r , where $r = |\underline{x} - \underline{y}|$.

$$-\frac{1}{4\pi} \int \nabla^2 \left(\frac{1}{r} \right) dV = \left(\frac{1}{4\pi r^2} \right) 4\pi r^2 = 1$$

But $\nabla^2 (1/r) = 0$ for $r \neq 0$. Thus $-(1/4\pi) \nabla^2 (1/r)$ exhibits the properties of a three-dimensional δ -function: we can infer that

$$-\frac{1}{4\pi} \int f(\underline{y}, \hat{t}) \nabla^2 \left(\frac{1}{r} \right) dV = [f(\underline{y}, \hat{t})]_{r=0} = f(\underline{x}, t)$$

This completes the transformation of Eq. (B7) into the form of Eq. (B9), proving their equivalence.

B2 Tabular Comparison of the Quadrupole and Simple-Source Solutions

It may be of interest to exhibit a number of alternative forms for the quadrupole and simple-source solutions for flow noise in parallel columns including both near-field and far-field terms. The equations will be restricted to the absence of heat or mass addition and of body forces and will apply to a low-speed flow in an unbounded fluid. In the quadrupole formulations the stress tensor T_{ij} will be approximated by its incompressible-flow value $T_{ij}^{(0)}$.

QUADRUPOLE	SIMPLE SOURCE
$p = p(x, t) \quad (\text{B19})$ <p>The starting point is an integral in terms of simple "divergence" sources, Eq. (B20). Transformations employing the divergence theorem yield the quadrupole form Eq. (B21). Equations (B22) and (B23) are expanded forms of Eq. (B21).</p>	$p = p(x, t) \\ = p_0 + p^{(0)} + p^{(1)} \quad (\text{B19}')$ <p>where</p> <ul style="list-style-type: none"> p_0 = constant ambient pressure $p^{(0)}$ = dominates in the near-field ($2\pi r \ll \lambda$) $p^{(1)}$ = dominates in the far-field ($2\pi r \gg \lambda$) <p>In each of the following the first integral is $p^{(0)}$ and the second integral is $p^{(1)}$.</p>
$p - p_0 = \frac{1}{4\pi} \int_V \left(\frac{\partial^2 T_{ij}^{(0)}(y, \hat{t})}{\partial y_i \partial y_j} \right) \frac{1}{r} dy \quad (\text{B20})$ <p>where $r = \underline{x} - \underline{y}$ and $\hat{t} = t - r/c$.</p>	$p - p_0 = \frac{1}{4\pi} \int_V \frac{\partial^2 T_{ij}(y, t)}{\partial y_i \partial y_j} \frac{1}{r} dy \\ - \frac{1}{4\pi c^2} \int \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_{y, \hat{t}} \frac{1}{r} dy \quad (\text{B20}')$
$p - p_0 = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}^{(0)}(y, \hat{t})}{r} dy \quad (\text{B21})$	$p - p_0 = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{T_{ij}(y, t)}{r} dy \\ - \frac{1}{4\pi c^2} \int \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_{y, \hat{t}} \frac{1}{r} dy \quad (\text{B21}')$

QUADRUPOLE	SIMPLE SOURCE
$ \begin{aligned} p - p_0 &= \frac{1}{4\pi} \int_V T_{ij}^{(0)}(y, \hat{t}) \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \\ &+ \frac{1}{2\pi} \int_V \frac{\partial T_{ij}^{(0)}(y, \hat{t})}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{1}{r}\right) dy \\ &+ \frac{1}{4\pi} \int_V \frac{1}{r} \frac{\partial^2 T_{ij}^{(0)}(y, \hat{t})}{\partial x_i \partial x_j} dy \quad (B22) \end{aligned} $	$ \begin{aligned} p - p_0 &= \frac{1}{4\pi} \int_V T_{ij}^{(0)}(y, t) \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \\ &- \frac{1}{4\pi c^2} \int_V \frac{1}{r} \left(\frac{D^2 p^{(0)}}{Dt^2}\right)_{y, \hat{t}} dy \quad (B22') \end{aligned} $
$ \begin{aligned} p - p_0 &= \frac{1}{4\pi} \int_V T_{ij}^{(0)}(y, \hat{t}) \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \quad (\sim r^3) \\ &+ \frac{1}{4\pi c} \int_V \left(\frac{\partial T_{ij}^{(0)}}{\partial t}\right)_{y, \hat{t}} r \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \quad (\sim r^2) \\ &+ \frac{1}{4\pi c^2} \int_V \left(\frac{\partial^2 T_{ij}^{(0)}}{\partial t^2}\right)_{y, \hat{t}} \frac{(x_i - y_i)(x_j - y_j)}{r^3} dy \quad (\sim r^{-1}) \end{aligned} $	$ \begin{aligned} p - p_0 &= \frac{1}{4\pi} \int_V T_{ij}^{(0)}(y, t) \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \quad (\sim r^3) \\ &- \frac{1}{4\pi c^2} \int_V \left(\frac{D^2 p^{(0)}}{Dt^2}\right)_{y, \hat{t}} \frac{1}{r} dy \quad (\sim r^{-1}) \end{aligned} $
<p>where $\frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) = \frac{3(x_i - y_i)(x_j - y_j)}{r^5} - \frac{\delta_{ij}}{r^3}$ (B23)</p> <p>Eq. (B23) is essentially Eq. (9) of Ref. 41. (B23')</p>	

Near Field Approximation. - In the near field, characterized by* $2\pi r \ll$ a typical wavelength λ , those terms of Eqs. (B23) and (B23') which are $\sim r^{-1}$ may be neglected in comparison with those $\sim r^{-2}$ and $\sim r^{-3}$.

$$\begin{aligned}
 p - p_0 &= \frac{1}{4\pi} \int_V \left(T_{ij}^{(0)} + \frac{r}{c} \frac{\partial T_{ij}^{(0)}}{\partial t} \right)_{y, \hat{t}} \\
 &\quad \times \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \quad (B24)
 \end{aligned}$$

$$\begin{aligned}
 p - p_0 &= \frac{1}{4\pi} \int_V \left(T_{ij}^{(0)} \right)_{y, t} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r}\right) dy \\
 &\quad - \frac{1}{4\pi c^2} \int_V \left(\frac{D^2 p^{(0)}}{Dt^2}\right)_{y, \hat{t}} \frac{1}{r} dy \quad (B24')
 \end{aligned}$$

* See footnote next page

Now compare the time t at the point \underline{x} at which p is measured and the earlier time \hat{t} at the point y at which the sound was emitted: the transit time $t - \hat{t}$ is small in the near field. Thus we may approximate $T_{ij}^{(0)}(y, t)$ by two terms of a Taylor series as

$$T_{ij}^{(0)}(y, t) \approx T_{ij}^{(0)}(y, \hat{t}) + \left(\frac{\partial T_{ij}^{(0)}}{\partial t} \right)_{y, \hat{t}} (t - \hat{t}) \quad (\text{B25})$$

$$\approx \left(T_{ij}^{(0)} + \frac{r}{c} \frac{\partial T_{ij}^{(0)}}{\partial t} \right)_{y, \hat{t}} \quad (\text{B26})$$

To this order of accuracy it is seen that Eqs. (B24) and (B24') are equivalent.

(In the light of the last result, reexamination of Eq. (B23) shows that the r^{-2} dependence is only apparent: the first two integrals jointly possess an r^{-3} space dependence in terms of the source strength at time \hat{t}).

Far Field Approximation. - In the far field, characterized by $2\pi r \gg$ a typical wavelength λ , those terms of Eqs. (B23) and (B23') which are $\sim r^{-1}$ dominate over those $\sim r^{-2}$ (see last paragraph) and $\sim r^{-3}$; the latter (the near-field terms) have attenuated to negligible proportions:

$$p - p_0 = \frac{1}{4\pi c^2} \int_V \left(\frac{\partial^2 T_{ij}^{(0)}}{\partial t^2} \right)_{y, \hat{t}} \frac{(x_i - y_i)(x_j - y_j)}{r^3} dy \quad (\text{B27})$$

$$p - p_0 = \frac{-1}{4\pi c^2} \int \left(\frac{D^2 p^{(0)}}{D t^2} \right)_{y, \hat{t}} \frac{1}{r} dy \quad (\text{B27}')$$

* This characterization has a precise meaning with respect to the acoustic disturbance from a single volume element dy a distance r from the observer. When referred to a volume integral, as above, an observer may be in the near field of some elements and the far field of others unless $2\pi V^{1/3} \ll \lambda$, where V is the volume. However, this condition is probably much too stringent in view of the incoherence of the disturbances from volume elements separated by more than a correlation length L . A sufficient condition for a point \underline{x} to be regarded as located in the near field of a turbulence volume V would appear to be $L \ll \lambda$ together with requirement that \underline{x} lie within V or a distance $\ll \lambda$ outside V . (Since the boundary of V is only vaguely defined the last condition is rather vague).

If the condition $2\pi r \gg \lambda$ is supplemented by $r \gg V^{1/2}$, then variations in y are unimportant in the integrand except as they figure in the retarded time \hat{t} . This yields the further simplifications

$$p - p_0 = \frac{1}{4\pi c^2} \frac{x_i x_j}{x^3} \int \left(\frac{\partial^2 T_{ij}^{(0)}}{\partial t^2} \right)_{y, \hat{t}} dy \quad (B28)$$

$$p - p_0 = \frac{1}{4\pi G^2 x} \int \left(\frac{D^2 \phi^{(0)}}{Dt^2} \right)_{y, \hat{t}} dy \quad (B28')$$

Eqs. (B28) and (B28') are compared in Section 2.3 of the main text by means of a momentum balance.

B3 Effects of Bodies in the Flow or Bounding Surfaces

Solution in Terms of $\phi^{(0)}$. - When a fluid is bounded internally, as by immersed bodies, or externally, Eq. (B7) must be supplemented by surface integrals (cf. Eq. (B3)):

$$p(x, t) - p_0 = -\frac{1}{4\pi} \int_V \left(\frac{\partial^2 \phi^{(0)}}{\partial y_i^2} \right)_{\hat{t}} \frac{1}{r} dy - \frac{1}{4\pi c^2} \int_V \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy + \frac{1}{4\pi} \int_S \frac{\partial}{\partial x_i} \left(\frac{\phi}{r} \right)_{\hat{t}} l_i dS + \frac{1}{4\pi} \int_S \left(\frac{\partial \phi}{\partial y_i} \right)_{\hat{t}} \frac{l_i dS}{r} \quad (B29)$$

By virtue of Eq. (B14) and the definition of $\phi^{(1)}$, $p = p_0 + \phi^{(0)} + \phi^{(1)}$, this is

$$p(x, t) - p_0 = -\frac{1}{4\pi c^2} \int_V \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy - \frac{1}{4\pi} \left\{ \int_V \frac{\partial^2}{\partial x_i^2} \left(\frac{\phi^{(0)}}{r} \right)_{\hat{t}} dy + \int_S \frac{\partial}{\partial x_i} \left(\frac{\phi^{(0)}}{r} \right)_{\hat{t}} l_i dS + \int_S \left(\frac{\partial \phi^{(0)}}{\partial y_i} \right)_{\hat{t}} \frac{l_i dS}{r} \right\} + \frac{1}{4\pi} \left\{ \int_S \frac{\partial}{\partial x_i} \left(\frac{\phi^{(0)} + \phi^{(1)}}{r} \right)_{\hat{t}} l_i dS + \int_S \left(\frac{\partial (\phi^{(0)} + \phi^{(1)})}{\partial y_i} \right)_{\hat{t}} \frac{l_i dS}{r} \right\} \quad (B30)$$

$$p(x, t) - p_0 = -\frac{1}{4\pi c^2} \int_V \left(\frac{\sigma''}{r} \right)_{\hat{t}} dy + \frac{1}{4\pi} \int_V \frac{\partial^2}{\partial x_i^2} \left(\frac{\phi^{(0)}}{r} \right)_{\hat{t}} dy + \frac{1}{4\pi} \left\{ \int_S \frac{\partial}{\partial x_i} \left(\frac{\phi^{(1)}}{r} \right)_{\hat{t}} l_i dS + \int_S \left(\frac{\partial \phi^{(1)}}{\partial y_i} \right)_{\hat{t}} \frac{l_i dS}{r} \right\} \quad (B31)$$

The first integral is the same as the right hand side of Eq. (B15) except for the finite volume of integration V . The analysis thereafter is

applicable on replacing ∞ by V . The result is for point \underline{x} within V ,

$$p(\underline{x}, t) - p_0 = p^{(0)} - \frac{1}{4\pi c^2} \int_V \left(\frac{D^2 p^{(0)}}{Dt^2} \right)_t \frac{1}{r} dy + \frac{1}{4\pi} \left\{ \int_S \frac{\partial}{\partial x_i} \left(\frac{p^{(0)}}{r} \right)_t l_i dS + \int_S \left(\frac{\partial p^{(0)}}{\partial y_i} \right)_t \frac{l_i dS}{r} \right\} \quad (\text{B32})$$

where $p^{(0)}$ is a solution of the incompressible flow equation,

$$-\nabla^2 p^{(0)} = \left[\frac{\partial T_{ij}^{(0)}}{\partial y_i \partial y_j} \right]_{u_i=0} \quad (\text{B33})$$

(This is Eq. (A21) in the absence of body forces F_i or mass addition m .)

If the surfaces are fixed (or if the surface elements move in their own plane) $l_i u_i = 0$. This can be employed in the momentum equation which reads in the acoustic approximation

$$l_i \frac{\partial p^{(0)}}{\partial y_i} = \rho_0 l_i \frac{\partial u_i^{(0)}}{\partial t} \quad (\text{B34})$$

to yield $l_i \frac{\partial p^{(0)}}{\partial y_i} = 0$ on S . Thus for such surfaces the third integral of Eq. (B32) vanishes.

Value of $p^{(0)}$. - The general solution for the incompressible part of the pressure field in the presence of bounding surfaces S is given by

$$p^{(0)} = \frac{1}{4\pi} \int_V \frac{1}{r} \left(\frac{\partial^2 T_{ij}^{(0)}}{\partial y_i \partial y_j} \right)_t dy + \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \left(\frac{p^{(0)}}{r} \right)_t l_i dS + \frac{1}{4\pi} \int_S \left(\frac{\partial p^{(0)}}{\partial y_i} \right)_t \frac{l_i dS}{r} \quad (\text{B35})$$

By procedures similar to those leading to Eq. (B14) this may be transformed to

$$p^{(0)} = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left(\frac{T_{ij}^{(0)}}{r} \right)_t dy + \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \frac{1}{r} (T_{ij}^{(0)} + p^{(0)} \delta_{ij})_t l_j dS \\ + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial}{\partial y_i} (T_{ij}^{(0)} + p^{(0)} \delta_{ij})_t l_i dS \quad (\text{B36})$$

Equation (B36) is equivalent to Curle's equation following (2.14) (Ref. 6) except for the use above of the current time t in place of the retarded time \hat{t} in the integrands. With this difference all his later steps apply. Thus

$$p^{(0)} = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left(\frac{T_{ij}^{(0)}}{r} \right)_t dy + \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \frac{1}{r} (\rho u_i^{(0)} u_j^{(0)} + p_{ij}^{(0)})_t l_j dS \\ + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial}{\partial y_i} (\rho u_i^{(0)} u_j^{(0)} + p_{ij}^{(0)})_t l_i dS \quad (\text{B37})$$

where $p_{ij}^{(0)} = \tau_{ij}^{(0)} + p^{(0)} \delta_{ij}$ is the stress tensor giving the stress acting in the $-i$ direction on a fluid surface element with normal in the j direction. Following Curle (Ref. 6)

$$l_i \frac{\partial}{\partial y_i} (\rho u_i^{(0)} u_j^{(0)} + p_{ij}^{(0)}) = -l_i \frac{\partial}{\partial t} (\rho u_i^{(0)}) \quad (\text{B38})$$

and if there is zero normal velocity at the surface S ; that is, if S is fixed or if its motion is everywhere parallel to the surface then

$$l_i u_i = 0 \quad (\text{B39})$$

Therefore Eq. (B37) reduces

$$p^{(0)} = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left(\frac{T_{ij}^{(0)}}{r} \right)_t dy + \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \frac{1}{r} (p_{ij}^{(0)})_t l_j dS \quad (\text{B40})$$

$$= \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left(\frac{T_{ij}^{(0)}}{r} \right)_t dy - \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \left(\frac{P_i^{(0)}}{r} \right)_t dS \quad (\text{B41})$$

where

$$P_i^{(0)} = -l_j p_{ij}^{(0)} \quad (\text{B42})$$

is the i component of the force exerted by unit area of the surface S on the fluid.

Physical Interpretation. - The resultant pressure field due to a flow with bounding surfaces is given by Eq. (B32). The part $p^{(0)}$ (Eqs. (B41) and (B42)) is the "incompressible" field (essentially the near-field) that dominates in and near the unsteady flow, and the part $p^{(a)}$ is the compressible field that dominates at large distances. In the absence of bounding surfaces $p^{(a)}$ is given by

$$- \frac{1}{4\pi c_0^2} \int \frac{D^2 p^{(0)}}{Dt^2} \frac{1}{r} dy \quad (\text{B43})$$

It is evident that this same integral in Eq. (B32) represents the primary radiation field $p^{(0)}$ due to the unsteady flow. The first surface integral, from its form and from the fact it depends on $p^{(0)}$ rather than $p^{(a)}$, represents the reflection and diffraction (scattering) of the primary sound from the bounding surfaces. The second integral (which vanishes if S is fixed) accounts for additional radiation if the surface is vibrating (this would supplement that which would arise from changes in $p^{(0)}$ due to such vibration: see last integral of Eq. (B37).

(In cases where the surface vibration and any mean flow are sufficiently small to inhibit nonlinear coupling effects a simplification may be made: the vibratory contribution to $p^{(0)}$ is suppressed so that the last integral of Eq. (B32) represents the entire effect of vibration as a pure acoustic effect).

The effect of a rigid boundary does not appear explicitly in Eq. (B32) except in the scattering integrals. The effect is automatically accounted for in $\beta^{(o)}$, the "incompressible" part of the field. This is exhibited in Eqs. (B41) and (B42) where a part of the value of $\beta^{(o)}$ is contributed by a distribution of dipoles over the bounding surface S : the dipole strength $P_i^{(o)}$ represents the incompressible flow approximation to the force per unit area exerted by the surface on the fluid. (The "scattering" integral of Eq. (B32) provides a correction, with $-\beta^{(o)} l_i = P_i^{(o)}$ and $P_i^{(o)} + P_i^{(s)}$ being the corrected force).

B4 Remarks on the Correlation Volume

The length L defined by Eq. (2.30) is an average scale of the turbulent property σ . The asymmetry of the eddies may be accounted for by use of unequal scales along the three axes

$$L^3 = L_1 L_2 L_3 \quad (\text{B44})$$

implying the eddy correlation volume is a parallelepiped; however, definition of the individual L_1, L_2, L_3 poses a problem. A more precise - but more complicated - interpretation replaces the cube L^3 or parallelepiped $L_1 L_2 L_3$ by a volume whose curved bounding surface is given by

$$\text{radial distance } \hat{\xi} = \text{function of direction}$$

Working backwards from Eq. (2.30), with

$$d\xi = \xi^2 d\xi \sin\theta d\theta d\varphi \quad (\text{B45})$$

we obtain $\frac{1}{3} \hat{\xi}^3(\theta, \varphi) = \left[\int_0^\infty R(\xi, r; \gamma) \xi^2 d\xi \right]_{\theta, \varphi \text{ fixed}}$ (B46)

and $L^3 = \frac{1}{3} \int_0^{2\pi} \int_0^\pi \hat{\xi}^3(\theta, \varphi) \sin\theta d\theta d\varphi$ (B47)

The quantity $\hat{\xi}(\theta, \varphi)$ may be termed the "correlation radius" in the direction θ, φ . It is to be noted that $\hat{\xi}$ is unaltered by a reversal in direction. In the case of spherical symmetry ($\hat{\xi} = \text{constant}$)

$$L^3 = \frac{4}{3} \pi \hat{\xi}^3 \quad (\text{B48})$$

APPENDIX C

ESTIMATION OF RATIO $|\partial p^{(0)}/\partial t| / |u_i \partial p^{(0)}/\partial y_i|$.

We seek to estimate the ratio $(\partial p^{(0)}/\partial t)' / (u_i \partial p^{(0)}/\partial y_i)'$ = $(\partial p^{(0)}/\partial t)' / (u_i \partial p^{(0)}/\partial y_i)'$, where ()' designates the root mean square value. Now it can be shown that for a turbulent flow homogeneous (i. e., statistically uniform) in space and time

$$\left(\frac{\partial p^{(0)}}{\partial t}\right)' = \left[\frac{\partial^2}{\partial \tau^2} \langle p^{(0)} p^{(0)'}(\xi, \tau) \rangle_{AV} \right]_{\xi=\tau=0}^{1/2} \quad (C1)$$

$$\left(\frac{\partial p^{(0)}}{\partial y_i}\right)' = \left[\frac{\partial^2}{\partial \xi_1^2} \langle p^{(0)} p^{(0)'}(\xi, \tau) \rangle_{AV} \right]_{\xi=\tau=0}^{1/2} \quad (C2)$$

where $\langle p^{(0)} p^{(0)'} \rangle_{AV}$ is the two-point covariance with time-delay τ . For purposes of our estimate we shall ignore the complete violation of spatial homogeneity in or near a jet and assume

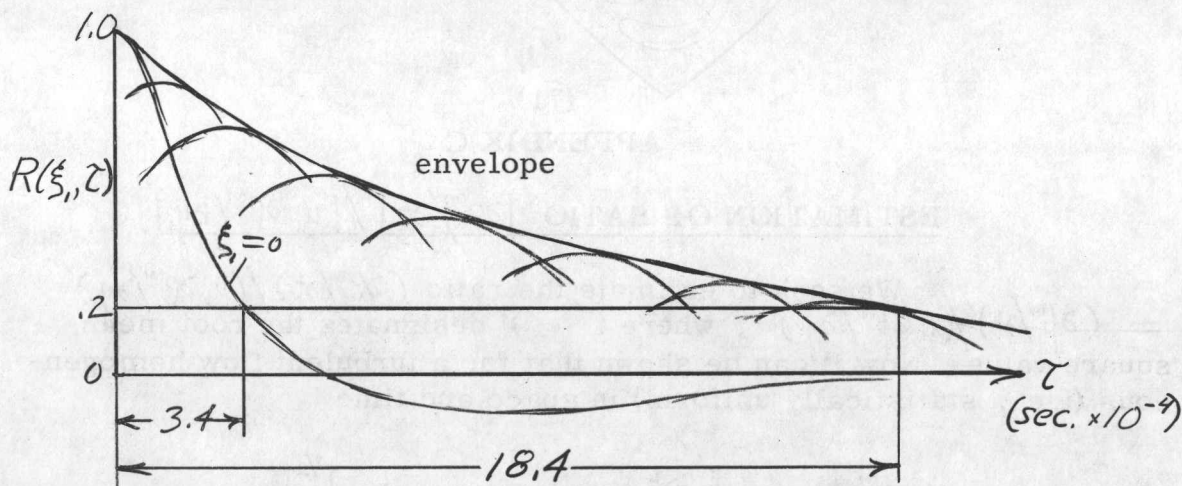
$$\frac{\left(\frac{\partial p^{(0)}}{\partial t}\right)'}{\left(u_i \frac{\partial p^{(0)}}{\partial y_i}\right)'} \sim \frac{\partial p^{(0)}/\partial t'}{(u)' (\partial p^{(0)}/\partial y_i)'} \quad (C3)$$

$$\sim \frac{\left[\frac{\partial^2}{\partial \tau^2} \langle p^{(0)} p^{(0)'} \rangle_{AV} \right]_{0,0}^{1/2}}{(u)' \left[\frac{\partial^2}{\partial \xi_1^2} \langle p^{(0)} p^{(0)'} \rangle_{AV} \right]_{0,0}^{1/2}} \quad (C4)$$

For the purpose of Eq. (C4), $\langle p^{(0)} p^{(0)'} \rangle_{AV}$ must be evaluated in a frame of reference moving with the turbulence pattern.

Equation (C4) will be applied to the correlation $R(\delta, \tau) \equiv \langle p^{(0)} p^{(0)'} \rangle_{AV} / \langle p^{(0)2} \rangle_{AV}$ for the near field of a jet as given in Fig. 20 of Ref. (14). (Note the notation change $\delta \rightarrow \xi_1$) In that figure the ordinate is $R(\delta, \tau)$ and the abscissa time delay τ . Curves are plotted for the separation values $\xi_1 = 0, 1.0, 1.5, 2.5, 4.5, 5.5, 7.0$ and 10 inches. As pointed out in Ref. (14), it is the envelope of these curves that constitutes the R vs. τ curve in a frame moving with the turbulence pattern. Moreover, it can be seen from Fig. 6 (right hand sketch) that

$$\left(\frac{\partial^2 R}{\partial \xi_1^2}\right)_{0,0} = \left(\frac{\partial^2 R}{\partial (U\tau)^2}\right)_{0,0}$$



Sketch 5. Space-time correlations near a jet (from Ref. 14)

Thus we express Eq. C4 as

$$\frac{(\partial \phi^{(0)}/\partial t)'}{(u_i \partial \phi^{(0)}/\partial y_i)'} \sim \frac{\left[\frac{\partial^2 R}{\partial \tau^2} \right]_{\text{envelope}}^{1/2}}{\frac{u'}{U} \left[\frac{\partial^2 R}{\partial \tau^2} \right]_{\xi_i=0}^{1/2}} \quad (\text{C5})$$

However, the curvature $\partial^2/\partial \tau^2$ is poorly defined near $\tau = 0$. Therefore we approximate the square root of the ratio of curvatures as the inverse ratio of decay times to some arbitrary fraction of $R(0)$, assuming the curves are similar. The ratio for $R \rightarrow 0.2$ (see Sketch 5) is $3.4/18.4$ or about $1/5$.* From Fig. 1 of Ref. (14) the value of u'/U may be taken as 0 ($1/100$) at the specified location $x/D = 1.5$, $y/D = 1.5$ outside the jet mixing region. Therefore Eq. (C5) yields

$$\text{jet near field} \quad \frac{(\partial \phi^{(0)}/\partial t)'}{(u_i \partial \phi^{(0)}/\partial y_i)'} \sim 0\left(\frac{1/5}{1/100}\right) = 0(20) \quad (\text{C6})$$

Actually values of most significance for the noise generation would be those within the mixing region. Here we have correlations of longitudinal velocity perturbation only, no pressure data. However, an equation of the form (E5), with $R = \langle u_i u_i \rangle_{av} / (u_i)'(u_i)'$ should still serve to provide an estimate because of the dynamical association of

* This ratio of decay times (which corresponds to the ratio "eddy" length/decay length) is a measure of the fluctuation parameter α of Eq. (3.1) and later equations.

(73)

of velocity and pressure. (A factor of two in both frequencies and wave numbers cancels out). Applying this to Fig. 11 of Ref. 14, with u'/U taken $O(1/6)$ from Figs. 5 and 6, yields

$$\begin{array}{l} \text{jet mixing} \\ \text{region} \end{array} \frac{(\partial p^{(0)}/\partial t)'}{(u_i \partial p^{(0)}/\partial y_i)'} \sim O\left(\frac{1/4}{1/6}\right) = O\left(\frac{3}{2}\right) \quad (\text{C7})$$

APPENDIX D

SINGLE-FREQUENCY PATTERN OF SOURCES ALONG A LINE SEGMENT:
FAR-FIELD AUTOCORRELATION AND SPECTRUM

The line distribution of acoustic sources assumed in Sec. 3.4 has a source-strength covariance

$$\tilde{R}(\xi_1, \tau^+) = \langle \sigma^2 \rangle_{AV} e^{-a^2 (\xi_1 - U\tau^+)^2} \cos \omega_0 \tau^+ \quad (D1)$$

This is Eq. (3.3) with τ generalized to $\tau^+ = \tau + \tau'$: here $U\tau = M\xi_1 \cos \theta$ refers to a required time delay between source points and τ' is an arbitrary time delay between two pressure observations p and p' at the field point \underline{x} . In terms of \tilde{R} the pressure autocorrelation is (Eq. (3.7)):

$$\langle \tilde{p}\tilde{p}' \rangle(\tau', \underline{x}) = \frac{Y^3 a^{-2}}{16\pi^2 c^2 \lambda^2} \int_{-\infty}^{\infty} \tilde{R}(\xi_1, \tau^+) d\xi_1 \quad (3.7)$$

The following transformations will be helpful in evaluating the integral:

$$\begin{aligned} \xi_1 - U\tilde{\tau} &= \xi_1 (1 - M \cos \theta) - U\tau' \\ &= \xi_1 \textcircled{H} - U\tau' \\ &= \textcircled{H} z \end{aligned}$$

$$\therefore z = \xi_1 - U\tau' / \textcircled{H} \quad (D3)$$

$$\begin{aligned} \omega\tau^+ &= \beta\xi_1 + \omega_0\tau' \\ &= \beta z + \left(\omega_0 + \frac{\beta U}{\textcircled{H}}\right)\tau' \\ &= \beta z + \frac{\omega_0}{\textcircled{H}}\tau' \end{aligned} \quad (D4)$$

where

$$\left. \begin{aligned} \textcircled{H} &\equiv 1 - M \cos \theta \\ \beta &\equiv \frac{\omega_0 \cos \theta}{c_0} \end{aligned} \right\} \quad (D5)$$

Therefore

$$\langle \tilde{p}p' \rangle = \frac{a^{-2} Y^3 \langle \sigma^2 \rangle_{AV}}{16 \pi^2 c_0^2 x^2} \int_{-\infty}^{\infty} \left(\cos \beta z \cos \frac{\omega_0}{|\Theta|} \tau' - \sin \beta z \sin \frac{\omega_0}{|\Theta|} \tau' \right) e^{-a^2 \Theta^2 z^2} dz$$

The integral involving $\sin \beta z$ vanishes, being odd in z . The remaining integration yields the final result for the autocorrelation:

$$\langle \tilde{p}p' \rangle = \frac{a^{-3} Y^3 \langle \sigma^2 \rangle_{AV}}{192 \pi^{3/2} c_0^2 x^2 |\Theta|} e^{-\frac{\beta^2}{4a^2 \Theta^2}} \cos \frac{\omega_0}{|\Theta|} \tau' \quad (D6)$$

The frequency spectrum (spectral density) in the far field is given by the Fourier cosine transform of the autocorrelation (see Eq. (3.8)). Since

$$\frac{2}{\pi} \int_0^{\infty} \left(\cos \frac{\omega_0}{|\Theta|} \tau' ; \omega \right) = \delta \left(\omega - \frac{\omega_0}{|\Theta|} \right)$$

the spectrum function is

$$\frac{d\langle \tilde{p}^2 \rangle}{d\omega} = \frac{a^{-3} Y^3 \langle \sigma^2 \rangle_{AV}}{192 \pi^{3/2} c_0^2 x^2 |\Theta|} e^{-\frac{\beta^2}{4a^2 \Theta^2}} \delta \left(\omega - \frac{\omega_0}{|\Theta|} \right) \quad (D7)$$

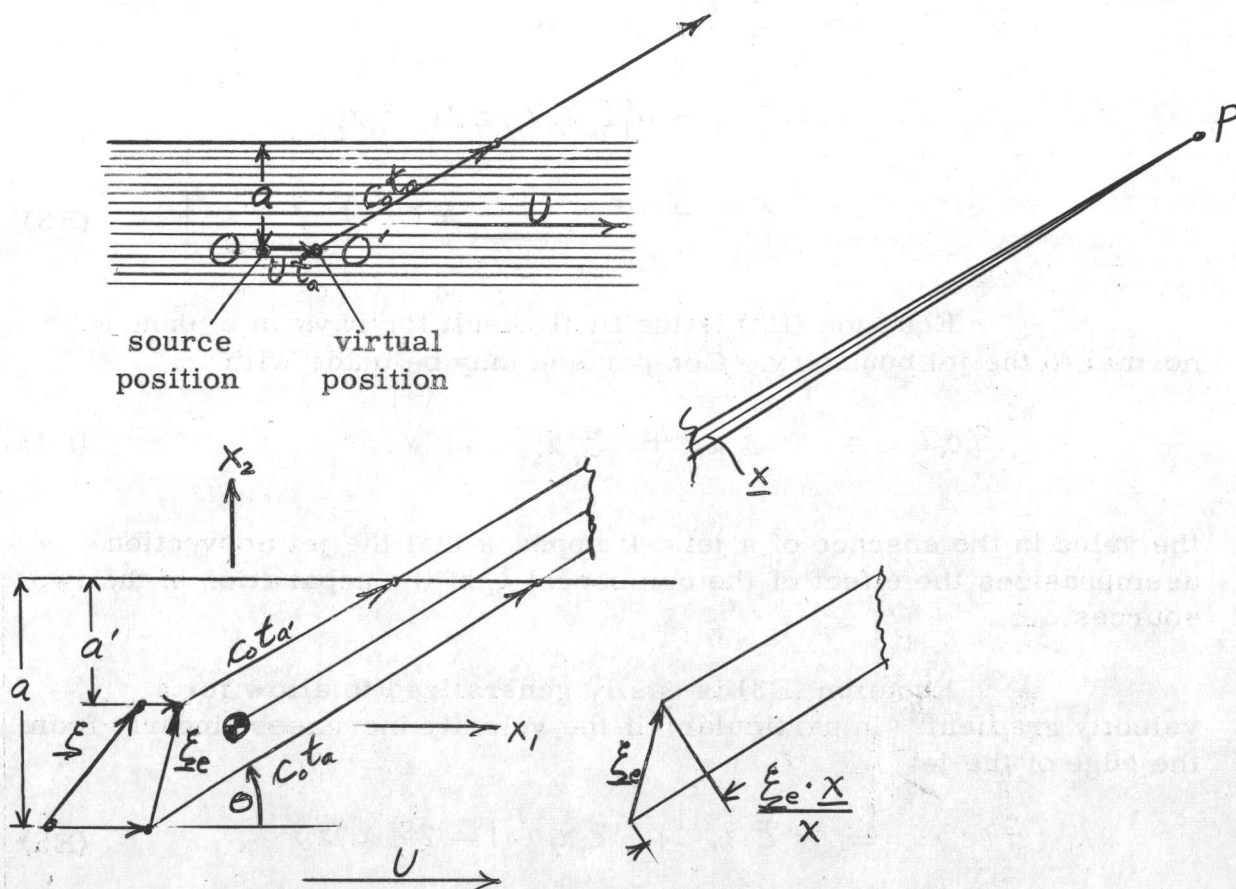
Eqs. (D6) and (D7) are the respective results of evaluating Eqs. (3.7) and (3.8) of the main text.

APPENDIX E

ROUGH ESTIMATION OF EFFECT OF JET FLOW ON TIME DELAY τ

As a first approximation to a real round jet the conical spreading and graduated velocity profile may be neglected: the jet is replaced by an infinite cylinder of diameter D (nozzle diameter) and uniform velocity U imbedded in fluid at rest. Such a model may serve for the approximation of the time delay τ for the case of sound rays that do not make too small an inclination to the flow direction.

For simplicity however we shall take a still cruder model, namely, a two-dimensional plane jet of width D . For this case we shall obtain an approximate value of τ . The situation is shown in Sketch 6. A species of ray acoustics will be assumed and refraction will be neglected. The main assumption is that the acoustic disturbance is governed by a linear superposition of convection and propagation.



Sketch 6. Sound rays from pair of sources in idealized plane jet.

By virtue of the convection an effective source separation ξ_e takes over the role of the actual source separation ξ . By geometry

$$\begin{aligned}\xi_e &= (\xi_{e1}, \xi_{e2}) \\ &= \xi_1 - U(t_a - t'_a), \xi_2 \\ &= \xi_1 - M \xi_2 x_2/x, \xi_2\end{aligned}\quad (\text{E1})$$

since

$$\left. \begin{aligned}t_a &= a/c_0 \sin \theta \approx ax_2/c_0 x \\ t'_a &= a'/c_0 \sin \theta \approx a'x_2/c_0 x\end{aligned} \right\} (\text{E2})$$

The time delay τ is approximately given by

$$\begin{aligned}\tau c_0 x &= \xi_e \cdot x \\ &= \xi_1 x_1 - M \xi_2 x_2 x_1/x + \xi_2 x_2 \\ &= \xi_1 x_1 + \xi_2 x_2 (1 - M x_1/x)\end{aligned}\quad (\text{E3})$$

Equation (E3) is the final result for rays in a plane normal to the jet boundary. Comparison may be made with

$$\tau c_0 x = \xi_1 x_1 + \xi_2 x_2, \quad (\text{E4})$$

the value in the absence of a jet. It appears that the jet convection deemphasizes the effect of the component ξ_2 of the separation of the two sources.

Equation (E3) is easily generalized to allow for a velocity gradient. In particular, if the velocity increases linearly from the edge of the jet

$$\tau c_0 x = \xi_1 x_1 + \xi_2 x_2 (1 - 2\bar{M} x_1/x) \quad (\text{E5})$$

where \bar{M} is the Mach number half-way between the two points. The velocity gradient therefore further deemphasizes ξ_2 .

It is not difficult to allow in addition for refraction - if the angular amount is known - but the crudeness of the present model would not seem to justify the additional complexity.

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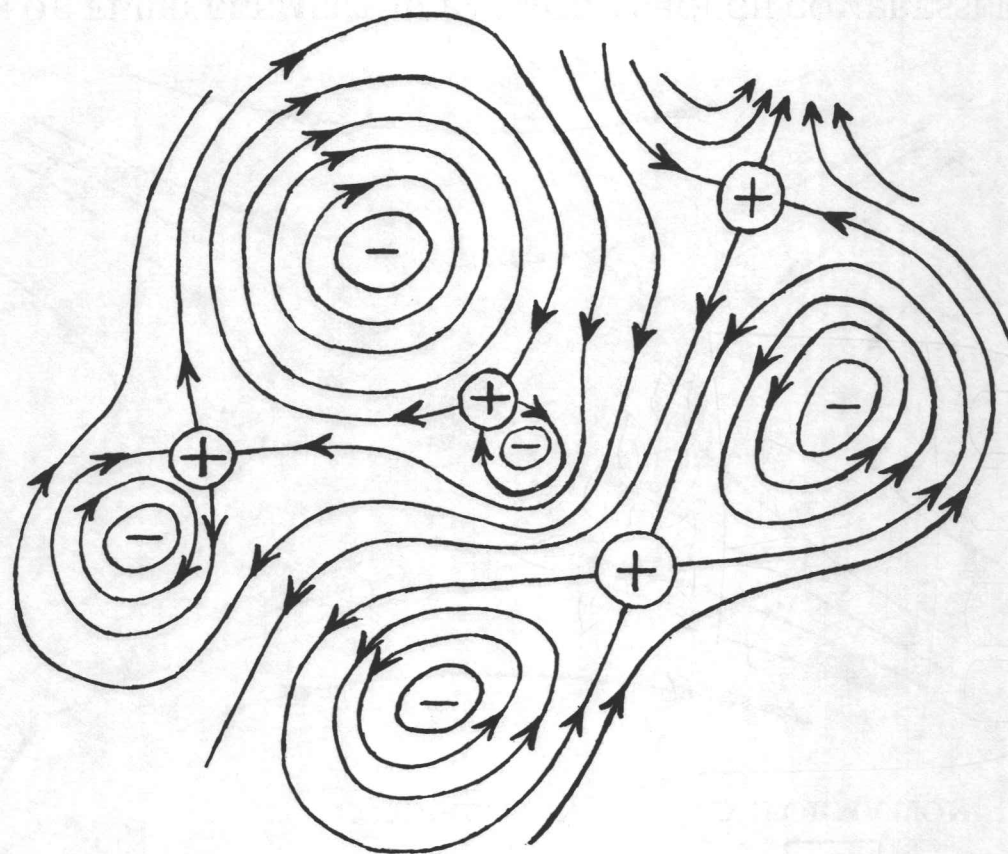


FIG. 1. RAREFIED \ominus AND COMPRESSED \oplus REGIONS IN AN EDDYING FLOW.

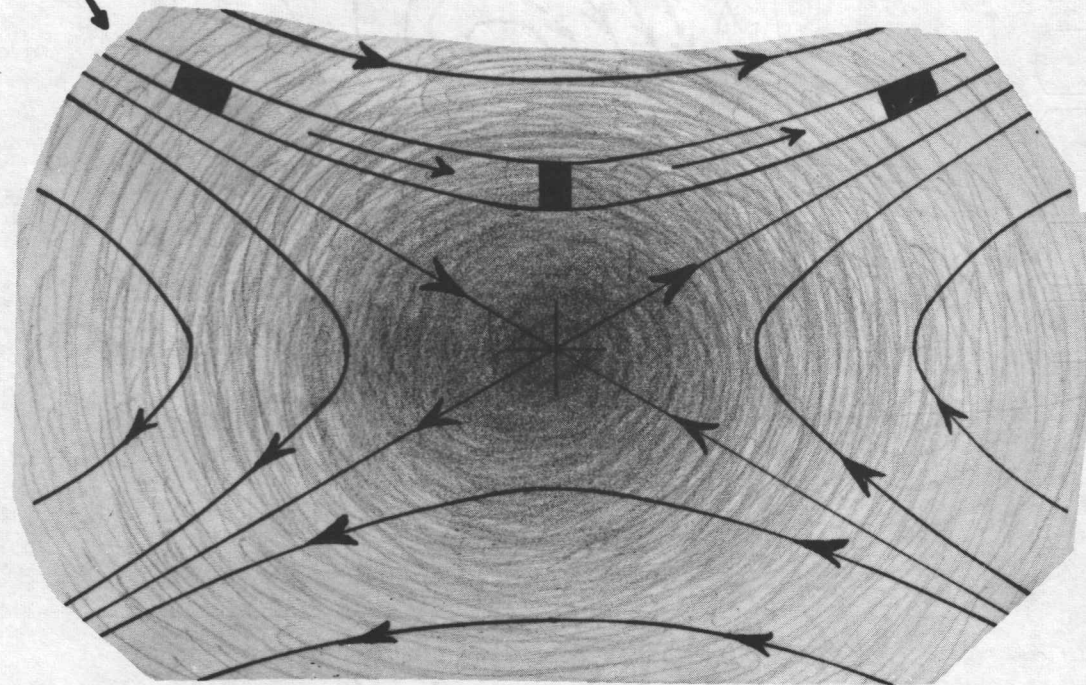
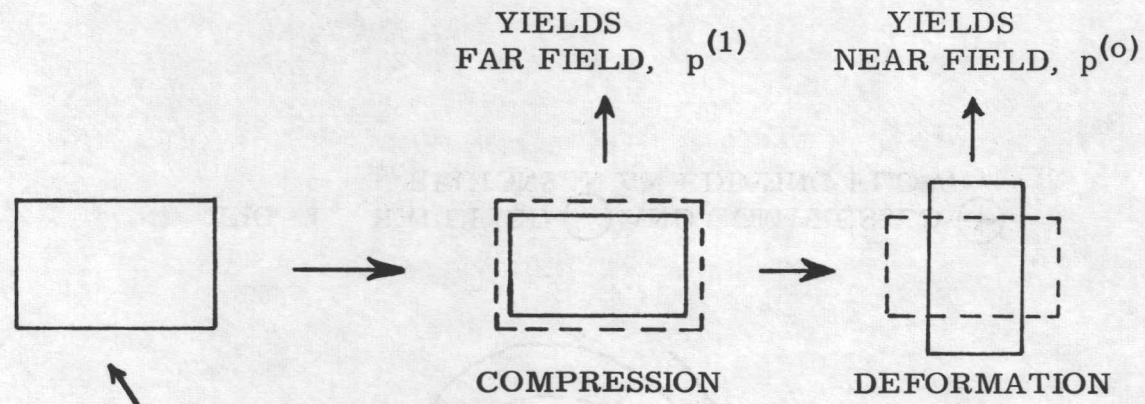


FIG. 2. DISTORTION OF FLUID ELEMENT IN PASSAGE THROUGH COMPRESSED REGION.

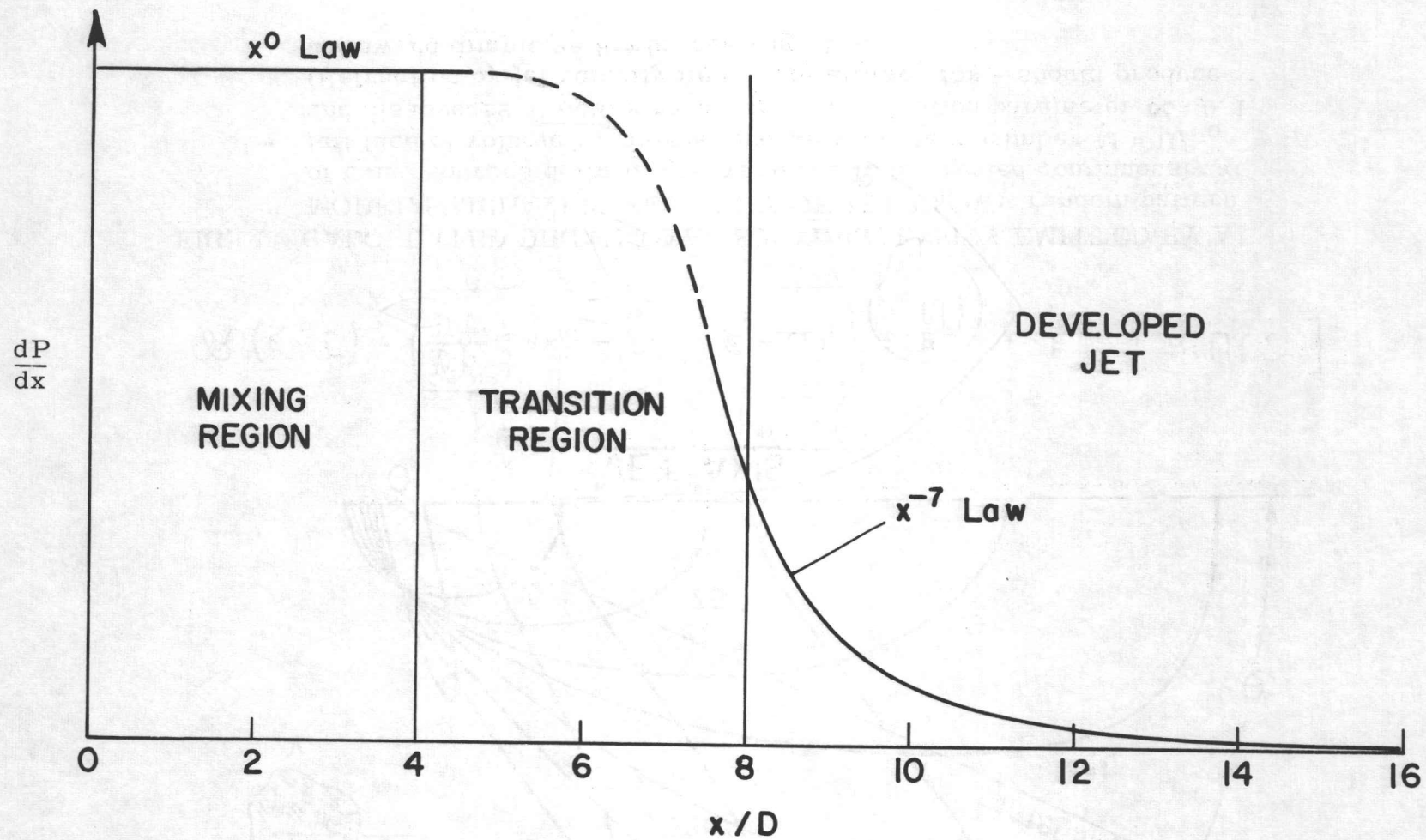
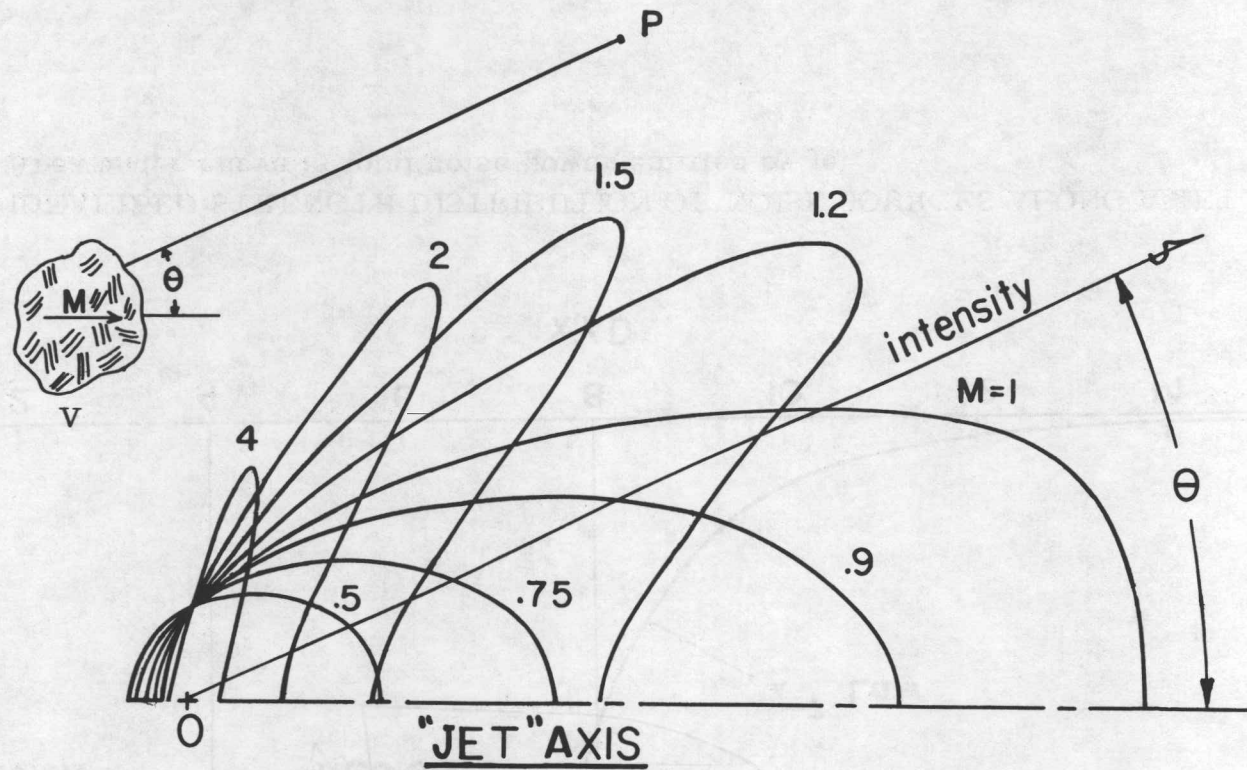


FIG. 3. IDEALIZED STRENGTH DISTRIBUTION OF NOISE SOURCES ALONG A JET.
Area under curve is total noise power emitted by jet.



$$\mathcal{R}(\underline{\xi}, \tau) = \left(\frac{D^2 p^{(a)}}{D t^2} \right)^2 \exp - \alpha^2 \left[(\xi_1 - U\tau)^2 + \xi_2^2 + \xi_3^2 + \alpha^2 U^2 \tau^2 \right]$$

FIG. 4. CALCULATED DIRECTIONAL SOUND INTENSITY EMITTED BY A MODEL SIMULATING FEATURES OF JET FLOW: random pattern of noise sources defined by covariance \mathcal{R} is created continuously at left face of volume V , moves through V at Mach number $M = U/c_0$, and disappears at downstream face. Fluctuation parameter $\alpha = 0.1$. (Refraction by jet velocity field - not allowed for - should produce an inward dimple as $\theta \rightarrow 0$: see Fig. 11).

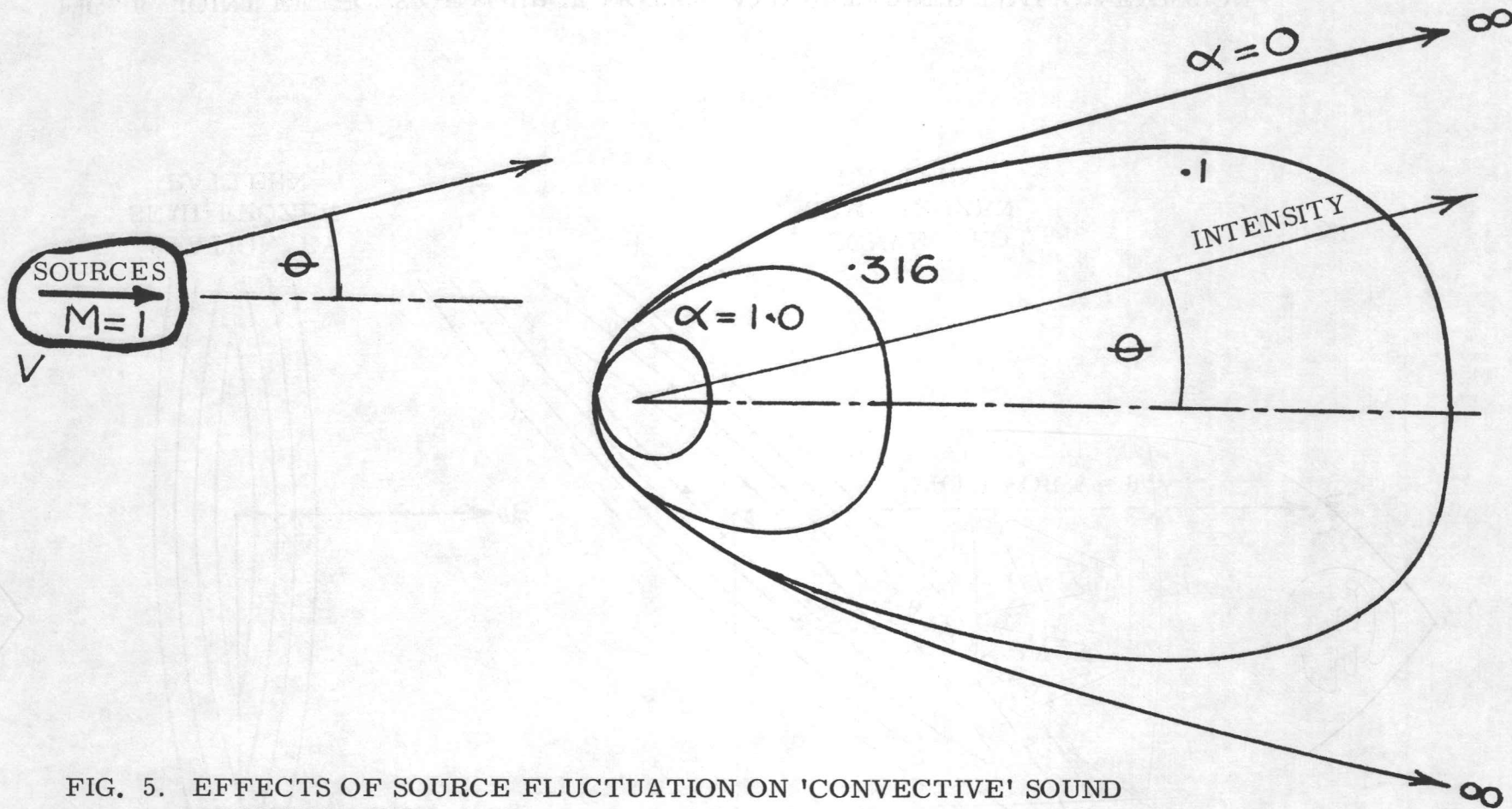


FIG. 5. EFFECTS OF SOURCE FLUCTUATION ON 'CONVECTIVE' SOUND AUGMENTATION: noise sources of Fig. 4 moving at sonic speed.

$\alpha = 0$ \sim fluctuation/convection = 0 ("frozen" pattern)

$\alpha = 0.1$ \sim fluctuation/convection = 0.1

$\alpha = 1.0$ \sim fluctuation/convection = 1.0

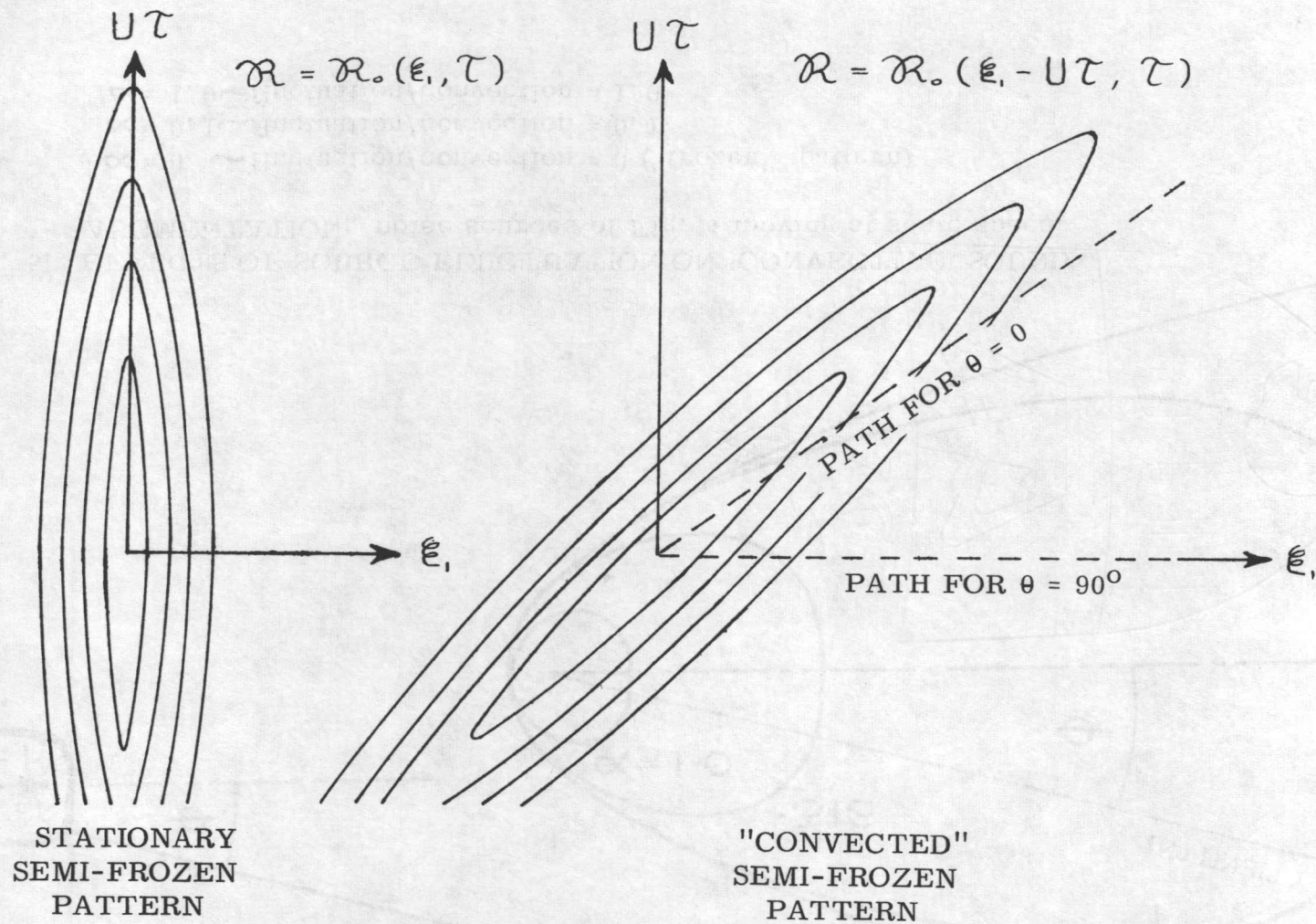


FIG. 6. JOINT EFFECTS OF SOURCE MOTION AND RETARDED TIME OF EMISSION.

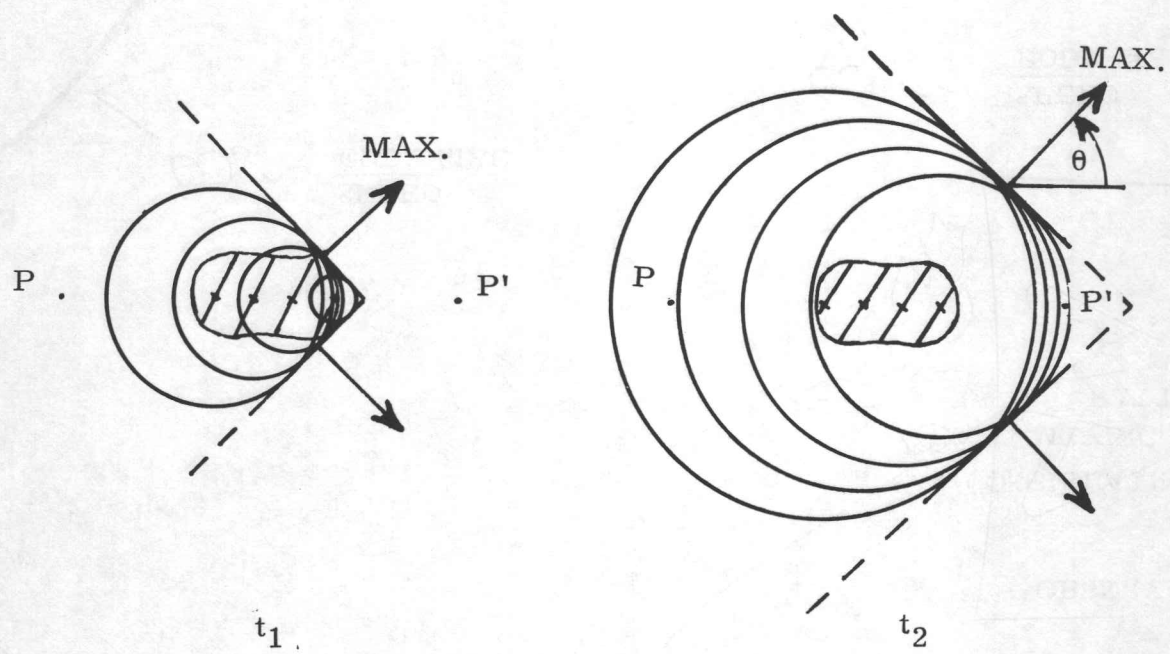


FIG. 7. PHYSICAL INTERPRETATION OF INTENSITY PEAK AT $M \cos \theta = 1$.

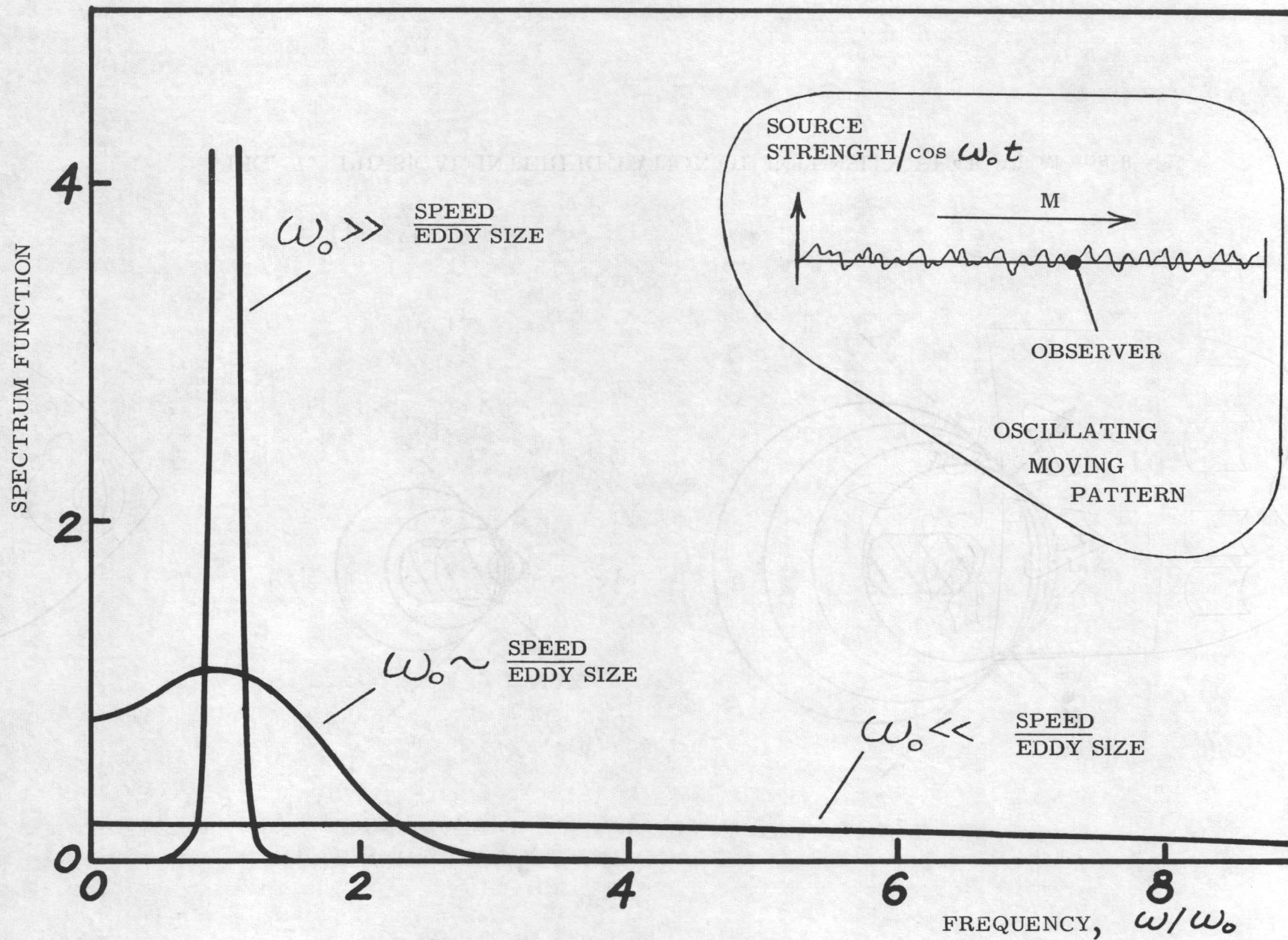


FIG. 8. CONVECTION BROADENING OF LINE SPECTRUM.

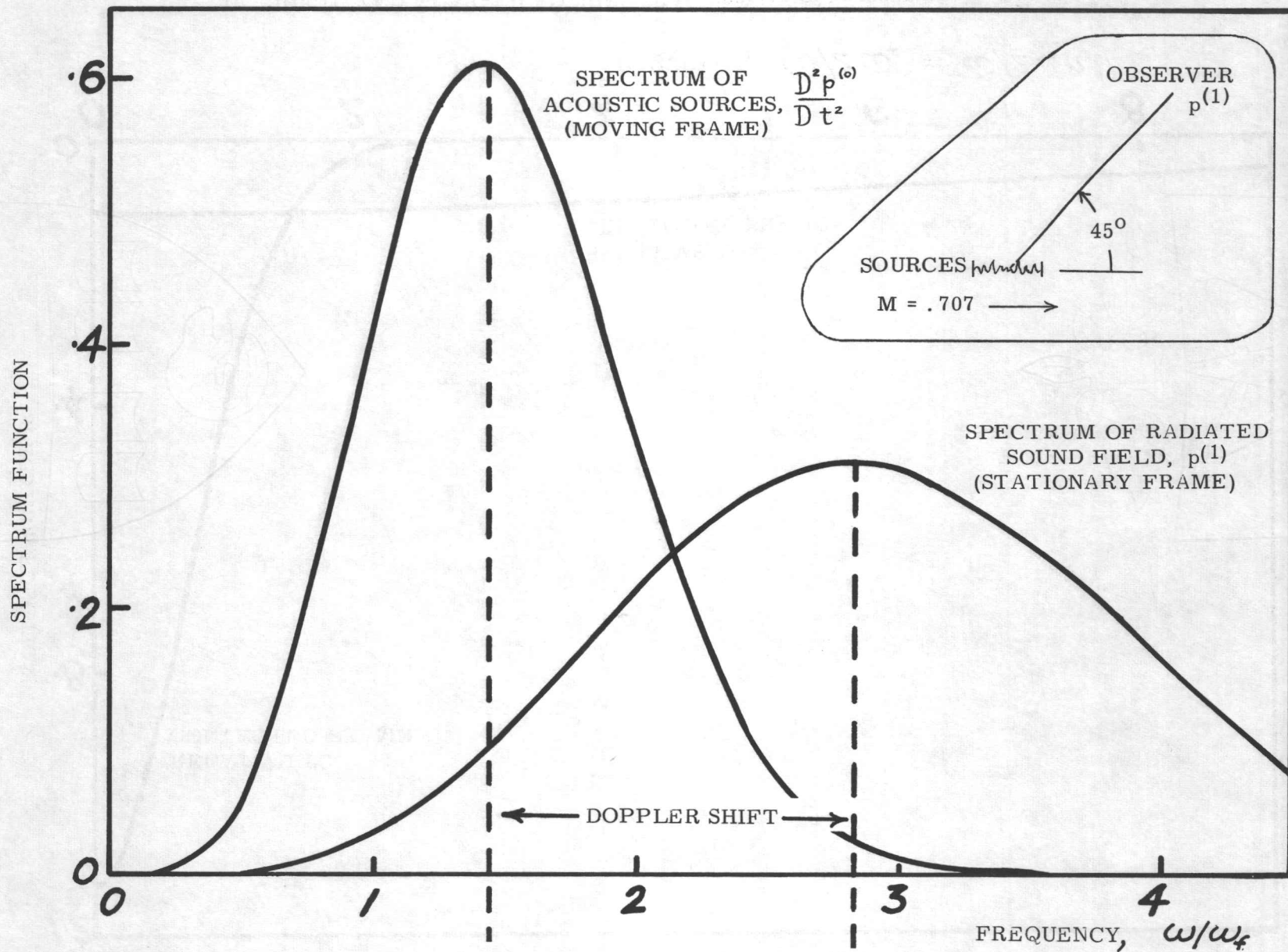


FIG. 9. CALCULATED EXAMPLE OF RADIATED PRESSURE SPECTRUM SHOWING DOPPLER SHIFT DUE TO SOURCE MOTION.

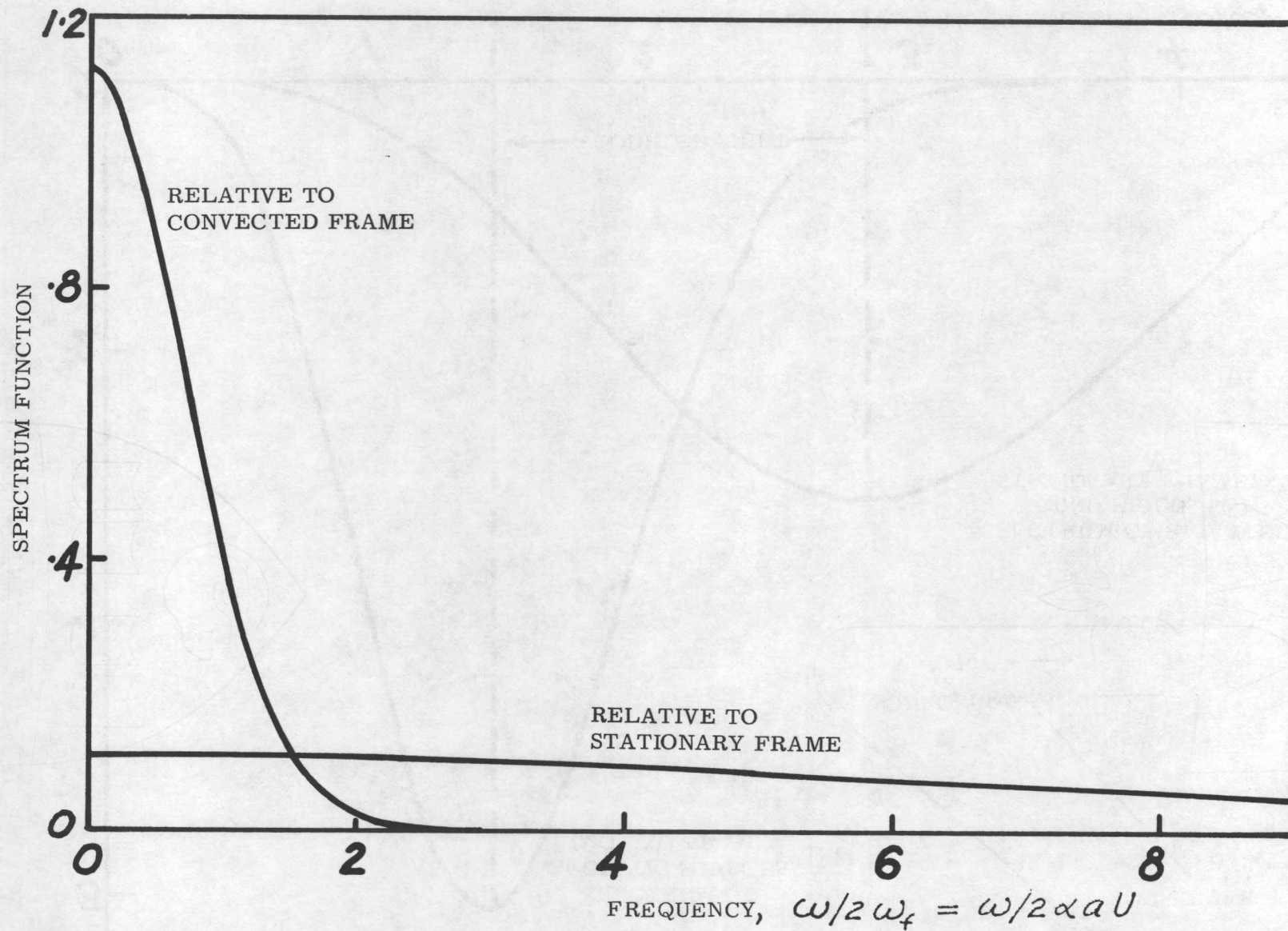


FIG. 10. IDEALIZED NEAR-FIELD SPECTRA: spectra of pressure field $p^{(0)}$ within line source pattern of Fig. 9 compatible with acoustic source strength being $\bar{D}^2 p^{(0)} / \bar{D} t^2$.

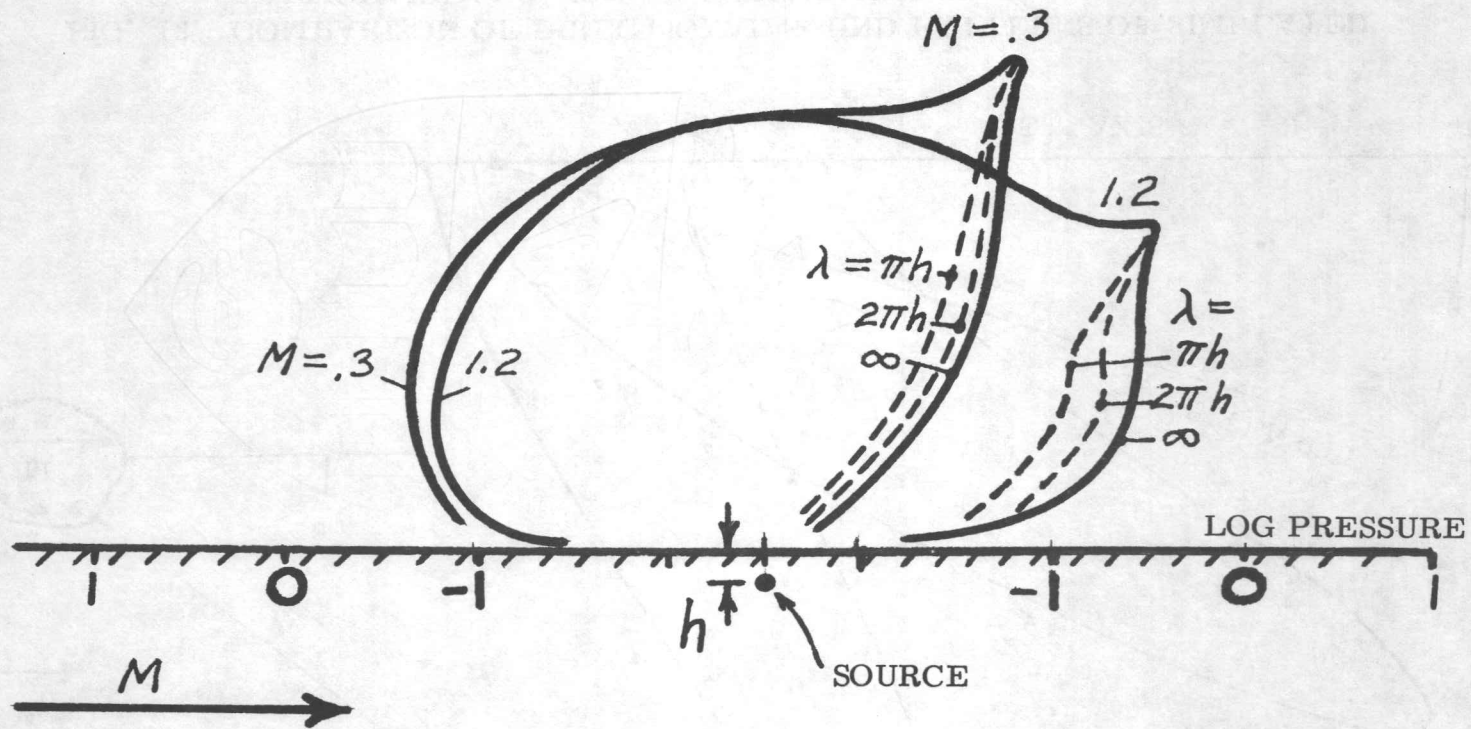


FIG. 11. IDEALIZED EXAMPLE OF REFRACTION OF SOUND BY A JET FLOW: transmitted directionality pattern for oscillating line source near a velocity discontinuity. (Supplied by P. Gottlieb from ms. of Ref. 34b).

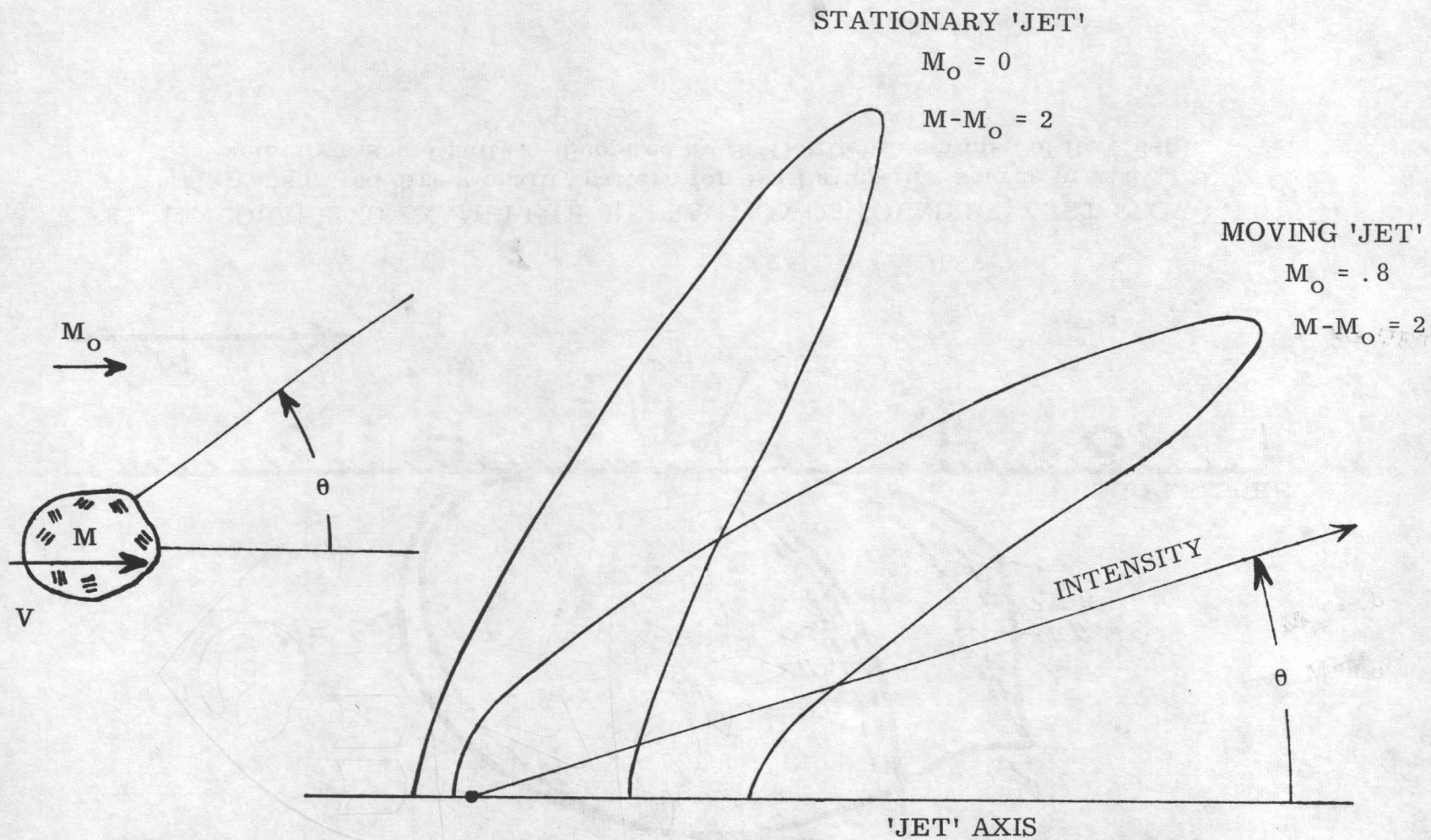


FIG. 12. COMPARISON OF DIRECTIONAL SOUND PATTERNS OF SIMULATED STATIONARY AND MOVING SUPERSONIC JETS: M = speed of convection of acoustic source pattern ("eddies") through volume V , M_0 = speed of external uniform stream (motion of jet). Other details as in Fig. 4.