The Poincare-Wirtinger Inequality

on Smooth and Bounded Domains in $\mathbb{R}^d$

by

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Abstract

In this thesis we study for which domain types the Poincare inequality holds for all functions having continuous first derivative. We first consider the classical Poincare inequality, which we prove holds for a very large class of open sets in $\mathbb{R}^d$. We then constructively prove that bounded, open, and connected domains in $\mathbb{R}^d$, which also possess a smooth C^1-boundary, must satisfy the Poincare-Wirtinger inequality. We do this in six successive steps.

First, we show that an arbitrary open rectangle in $\mathbb{R}^d$ must satisfy the inequality. Second, we prove that a C^1-diffeomorphism with a sufficient condition, between a set which satisfies the inequality and an open, bounded and connected set implies the open, bounded and connected set also satisfies the Poincare-Wirtinger inequality. Third, we show that there exists such a C^1-diffeomorphism between a domain in the class of open rectangles with one face distorted by a C^1-function and another domain in the class of arbitrary open rectangles in $\mathbb{R}^d$. Fourth, we show the class of all open rectangles with one face distorted by a C^1-function satisfies the Poincare-Wirtinger inequality. Fifth, we show the union of non-disjoint open sets which satisfy the inequality in turn also satisfies the Poincare-Wirtinger inequality. Lastly, we cover the open, bounded and connected domain with a C^1-boundary by a collection of rectangles from the classes of open rectangles with one face distorted by a C^1-function and arbitrary open rectangles to show that the domain satisfies the Poincare-Wirtinger inequality.

Finally, we extend our function space to the first-order Sobolev space and show that we can directly extend our results to this function space.

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Introduction

We first start by motivating the Poincare inequality. To this end, let us temporarily abandon mathematical rigour in favour of building our intuition of the inequality.

Let D be a nonempty open subset of $\mathbb{R}^d$. Let us take a look at the Poisson problem with Dirichlet boundary conditions. This is nothing but

$$-\Delta u = f \text{ in } D \quad \text{and} \quad u = 0 \text{ on } \partial D \tag{1.1}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

Since $u = 0$ on the boundary, it seems reasonable to consider the class of differentiable functions which tend to 0 as $x \to \partial D$. We can then multiply the Poisson equation by some test function φ. Integrating over D gives us

$$\int_D (\Delta u)\varphi \mathrm{d}x = -\int_D f\varphi \mathrm{d}x \tag{1.2}$$

and we can then further perform an integration by parts on the left integral of (1.2) to obtain

$$\int_D (\Delta u)\varphi \mathrm{d}x = \int_{\partial D} (\nabla u)\varphi \mathrm{d}x - \int_D \nabla u \cdot \nabla\varphi \mathrm{d}x \tag{1.3}$$

where $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$ is the gradient.

Then, because of the boundary conditions we impose on u and φ, the integral over the boundary vanishes and so we can combine (1.2) and (1.3) to find

$$\int_D \nabla u \cdot \nabla\varphi \mathrm{d}x = \int_D f\varphi \mathrm{d}x. \tag{1.4}$$

We go through all this trouble because we want to define the concept of a weak solution. A function u is said to be a weak solution of the Poisson problem as presented in (1.1) if it satisfies (1.4) for all choices of test functions φ. The advantage of working with weak solutions, as opposed to classical solutions, is that (1.4) deals with only the first derivative and both integrals can be interpreted as inner products. Furthermore, a classical solution does not necessarily always exist and so there are cases where we must work with weak solutions.

It then remains to prove the existence and uniqueness of u. This is where the Poincare inequality comes into the picture as it is the main tool used for this proof. The inequality has the form

$$\|f\|_{L^p(D)} \leq k\|\nabla f\|_{L^p(D)} \tag{1.5}$$

for k some constant, f differentiable and $\|\cdot\|_{L^p(S)}$ the L^p-norm. We give much more rigorous definitions of these concepts in Chapter 2, but for now, it is sufficient to understand that the Poincare inequality bounds a function by its gradient, multiplied by some constant.

However, the Poincare inequality does not hold for all choices of $D \subseteq \mathbb{R}^d$, in fact it does not hold for $D = \mathbb{R}^d$. The problem then becomes to show for which domains it does hold and this is an entire field of mathematics. For the Poincare inequality with Dirichlet boundary conditions, we do not actually have to do much work. In fact, we give a proof in Chapter 2 that, for $r > 0$, (1.5) holds for the class of sets with form $D = (-r, r) \times \mathbb{R}^{d-1}$; this is immediately a quite broad class of open sets in $\mathbb{R}^d$.

The problems then arise when we repeat the previous derivation on the Poisson problem, but this time take Neumann boundary conditions in (1.1). In Chapter 2, we will see that this introduces an extra term in the norm of the left hand side of (1.5) and this makes it much harder to show that D satisfies the Poincare inequality.

In this thesis, we go about proving that for D open, bounded, connected and possessing a smooth boundary, the Poincare inequality with Neumann boundary conditions is satisfied. This proof already exists in the available literature, see [10, p. 383], however all such proofs strive to derive a contradiction and as a result do not tell us much about D. Instead, we give a different style of proof and show that we can cover D by a finite union of sets which satisfy the Poincare inequality and that this implies that D also satisfies the Poincare inequality.

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Preliminaries

This chapter gives the preliminary knowledge necessary for the results of Chapter 3. Section 1 is an introduction to relevant topics in real analysis and multi-dimensional calculus, which can be skipped by a reader familiar with these subjects. Note however that much of the notation used in later sections of the thesis is introduced and defined here.

Specifically, we refresh the reader on metric and normed vector spaces and define the concept of a C^1-boundary. We then move on to compact sets and prove the finite sub-cover property as well as the Heine-Borel theorem. Next, we refresh the reader on the notion of the integral and prove the monotone convergence theorem, after which we define the product measure and prove the Fubini-Tonelli theorem. Following this, we define L^p-spaces and prove Hölder's and Minkowski's inequalities. Finally, we define the derivative in $\mathbb{R}^d$, state the change of variables theorem, define diffeomorphic mappings, and state the inverse function theorem.

In Section 2, we give the definition of the Poincare inequality. We first visit the classic Poincare inequality and prove it for the class of sets with form $D = (-r, r) \times \mathbb{R}^{d-1}$, after which we move on to the Poincare-Wirtinger inequality; the form of the Poincare inequality we use in Chapter 3.

2.1. Real Analysis and Calculus in $\mathbb{R}^d$

We start at the foundation of real analysis.

2.1.1. Metric and Normed Vector Spaces

A metric space is the couple (M, ρ) consisting of a set M and a metric $\rho : M \times M \to [0, \infty)$ defined on M.

We define the diameter of a nonempty $A \subset M$ by $\mathrm{diam}(A) = \sup\{a, b \in A : \rho(a, b)\}$. We also define the distance between a point $x \in M$ and a set $A \subset M$ as $\rho(x, A) = \inf\{a \in A : \rho(x, a)\}$. Similarly, we define the distance between any two sets $A, B \subset M$ as $\rho(A, B) = \inf\{a \in A, b \in B : \rho(a, b)\}$.

A normed vector space is the couple $(V, \|\cdot\|)$ consisting of a vector space V and a norm $\|\cdot\| : V \to [0, \infty)$ defined on V.

Example 2.1.1. Let $V = \mathbb{R}^d$, we define the Euclidean norm on $\mathbb{R}^d$ as

$$\|x\|_2 = \left(\sum_{i=1}^{d} |x_i|^2 \right)^{\frac{1}{2}} \tag{2.1}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We always refer to the Euclidean norm when $x \in \mathbb{R}^d$ and denote it by $|x|$.

Example 2.1.2. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator. We define the operator norm by

$$\|A\|_{\mathrm{op}} = \sup_{|h| \le 1} |Ah|. \tag{2.2}$$

We want to work with the operator norm because of its useful properties, which we state and prove in Lemma 2.1.1 and Lemma 2.1.2.

Lemma 2.1.1. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator and denote by λ its largest eigenvalue. Then $\|A\|_{\text{op}} \geq |\lambda|$. Furthermore, if A is orthogonally diagonalisable, then $\|A\|_{\text{op}} = |\lambda|$.*

Proof. Let v_λ be a corresponding unit eigenvector of λ. Then we have

$$\|A\|_{\text{op}} \geq |Av_\lambda| = |\lambda v_\lambda| = |\lambda||v_\lambda| = |\lambda|. \tag{2.3}$$

Suppose now that A is diagonalisable with respect to an orthonormal basis $\{v_1, \dots, v_d\}$ with corresponding eigenvalues $\{\lambda_1, \dots, \lambda_d\}$. Then we can express any $h \in \mathbb{R}^d$ by $h = c_1 v_1 + \cdots + c_d v_d$, hence

$$Ah = c_1\lambda_1 v_1 + \cdots + c_d \lambda_d v_d \leq \lambda(c_1 v_1 + \cdots + c_d v_d) \tag{2.4}$$

and we have

$$\|A\|_{\text{op}} = \sup_{|h|\leq 1} |Ah| \leq \sup_{|h|\leq 1} |\lambda h| = |\lambda|. \tag{2.5}$$

We combine (2.3) and (2.5) to obtain $|A| = |\lambda|$. □

Lemma 2.1.2. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator and let A^{T} be its transpose. Then $\|A\|_{\text{op}} = \left\|A^{\mathrm{T}}\right\|_{\text{op}}$.*

Proof. We know that for all $x, y \in \mathbb{R}^d$

$$y^{\mathrm{T}} A x = (x^{\mathrm{T}} A^{\mathrm{T}} y)^{\mathrm{T}}. \tag{2.6}$$

Then we can express the operator norm as

$$\|A\|_{\text{op}} = \sup_{|x|\leq 1} |Ax| = \sup_{|x|\neq 0} \frac{|Ax|}{|x|} = \sup_{|x|\leq 1, |y|\leq 1} \left|\langle y, Ax\rangle\right|. \tag{2.7}$$

Combining (2.6) and (2.7), we find

$$\|A\|_{\text{op}} = \sup_{|x|\leq 1, |y|\leq 1} \left|\langle y, Ax\rangle\right| = \sup_{|x|\leq 1, |y|\leq 1} \left|y^{\mathrm{T}} A x\right| = \sup_{|x|\leq 1, |y|\leq 1} \left|(x^{\mathrm{T}} A^{\mathrm{T}} y)^{\mathrm{T}}\right| = \sup_{|x|\leq 1, |y|\leq 1} \left|\langle x, A^{\mathrm{T}} y\rangle\right| = \left\|A^{\mathrm{T}}\right\|_{\text{op}}.$$ □

We move on to the definition of the boundary of a set. Denote the open ball around x with radius $\epsilon > 0$ by $B_\epsilon(x)$.

Definition 2.1.1. Let (M, ρ) be a metric space and let $A \subset M$. We say a point $x \in M$ is a boundary point of A if and only if

$$B_\epsilon(x) \cap A \neq \emptyset \quad \text{and} \quad B_\epsilon(x) \cap A^c \neq \emptyset \tag{2.8}$$

for all $\epsilon > 0$. We denote the set of all boundary points of A by ∂A.

Suppose D is any open and bounded set in $\mathbb{R}^d$. It then follows immediately from the definition of open sets that $D \cap \partial D = \emptyset$. While this makes sense mathematically, when visualised in $\mathbb{R}^2$ for example, this property can be quite counter-intuitive since we physically observe that there is a line where D 'stops'. If, at every point on the boundary, we locally parameterise this line by a continuously differentiable function, then we arrive at the definition of a C^1-boundary.

Definition 2.1.2. Let $D \subset \mathbb{R}^d$ be open and bounded. We say ∂D is a C^1-boundary if for each point $x \in \partial D$ there exists an $r > 0$ and C^1-function $\gamma_x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, after relabelling and reorienting the axes if necessary, we have

$$D \cap B_r(x) = \{y \in B_r(x) | y_d > \gamma_x(y_1, \dots, y_{d-1})\}. \tag{2.9}$$

Intuitively, we say ∂D is C^1 if we can can press a C^1-function against D such that, after moving and rotating D in space if necessary, any point $x \in \partial D$ is 'above' γ_x.

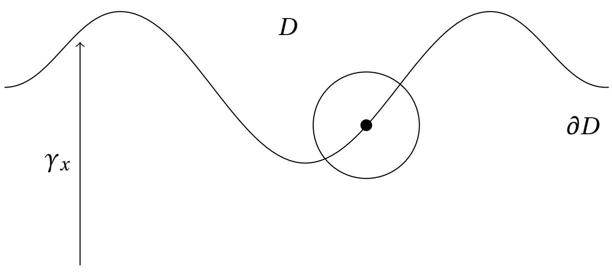

Figure 2.1: Open Ball Around $x \in \partial D$.

Note that γ_x does not parameterise the entire boundary, only the boundary located in the neighbourhood of x.

2.1.2. Compactness

We now introduce the notion of compactness.

Definition 2.1.3. Let (M,ρ) be a metric space and let $\epsilon > 0$. A set $A \subseteq M$ is said to be totally bounded if there exists finitely many points $x_1,\dots,x_n \in M$ such that $A \subset \cup_{i=1}^n B_\epsilon(x_i)$. We say (M,ρ) is complete if every Cauchy sequence in M converges to a point in M. Furthermore, we say (M,ρ) is compact if it is both complete and totally bounded.

Before we continue, we need to prove the nested set theorem for complete metric spaces as we will need it later. First, recall that for a metric space (M,ρ), a set $A \subset M$ is said to be closed if and only if all convergent sequences $(x_n)_{n\ge1} \in A$ converge to some point $x \in A$.

Theorem 2.1.3. (Nested Set Theorem) *Let (M,ρ) be a metric space. Let $F_1 \supset F_2 \supset F_3 \supset \cdots$, be a decreasing sequence of nonempty, closed sets in M with diam$(F_n) \to 0$. Then (M,ρ) is complete if and only if $\cap_{n=1}^\infty F_n \neq \emptyset$.*

Proof. For our purpose, we only require, and so only prove, the forward implication. Suppose M is complete and let $(F_n)_{n\ge1}$ be such a sequence as in the theorem's statement. Since each F_n is nonempty, we can find some $x_n \in F_n$ for all $n \ge 1$. By the assumption on diam(F_n), we have for all $y \in F_n$

$$\rho(x_n, y) \le \mathrm{diam}(F_n). \tag{2.10}$$

Let $\epsilon > 0$. Then we can find some $N \ge 1$ such that $m,n \ge N$ implies

$$\rho(x_n, x_m) \le \rho(x_n, y) + \rho(x_m, y) < \mathrm{diam}(F_N) + \mathrm{diam}(F_N) < \epsilon. \tag{2.11}$$

for $y \in F_n$, where we assume without loss of generality that $m < n$. Then $(x_n)_{n\ge1}$ is a Cauchy sequence and so $x_n \to x$ for some point $x \in M$. However, all F_n are closed. Hence $x_n \to x \in F_n$ for all $n \ge 1$. Then $x \in \cap_{n=1}^\infty F_n \neq \emptyset$. □

The reverse implication follows directly from the Bolzano-Weierstrass theorem on infinite and totally bounded subsets of M. For a detailed proof, an interested reader is referred to [2, p. 95].

We call $\mathcal{G} = \{G_i\}_{i\in I}$ a covering of some set $A \subseteq M$ if $A \subset \cup_{i\in I} G_i$. We call such a covering an open covering if all G_i are open sets.

Theorem 2.1.4. *Let (M,ρ) be a metric space. M is compact if and only if every open covering $\mathcal{G}$ of M has finitely many $G_1,\dots,G_n \in \mathcal{G}$ such that $M \subset \cup_{i=1}^n G_i$.*

Proof. We only prove the forward implication. Suppose M is compact and suppose $\mathcal{G}$ is an open covering of M, but does not admit any finite sub-cover of M. We will derive a contradiction. Since M is totally bounded, we can cover M by finitely many closed sets of diameter at most 1. Then at least one of these sets, say A_1, cannot be covered by finitely many sets of $\mathcal{G}$. Note that $A_1 \neq \emptyset$ as this is an easy set to cover.

We know A_1 is totally bounded, because M is, hence it can be covered by finitely many closed sets of diameter at most $1/2$. Then at least one of these, say A_2, cannot be covered by finitely many sets of $\mathcal{G}$. Note again that $A_2 \neq \emptyset$ as this would be an easy set to cover.

We continue and obtain a decreasing sequence $A_1 \supset A_2 \supset A_3 \supset \cdots$, where A_n is closed, nonempty, has diam$(A_n) \le \frac{1}{n}$, and cannot be covered by finitely many sets of $\mathcal{G}$. Notice also that because M is complete, we find that $\cap_{n=1}^\infty A_n \neq \emptyset$ by the nested set theorem.

Let $x \in \cap_{n=1}^\infty A_n$. Since $\mathcal{G}$ is an open cover, $x \in G$ from some $G \in \mathcal{G}$. Since G is open, $x \in B_\epsilon(x) \subset G$ for some $\epsilon > 0$. Then for any n with $\frac{1}{n} < \epsilon$ we must have that $x \in A_n \subset B_\epsilon(x) \subset G$. Hence A_n is covered by a single set from $\mathcal{G}$ which is a contradiction to how we have defined A_n. □

Plainly speaking, Theorem 2.1.4 states that a set is compact if and only if every open covering of the set admits a finite sub-cover. We do not require the reverse implication for future results and so have omitted the proof for brevity. For a detailed proof, see [2, p. 112].

In Chapter 3, we work with open and bounded subsets of $\mathbb{R}^d$. It would be very convenient for us if such sets were compact. Unfortunately, they are not. Fortunately, the Heine-Borel theorem allows for an easy extension to a compact set.

Theorem 2.1.5. (Heine-Borel Theorem) *Let $D \subset \mathbb{R}^d$. Then D is compact if and only if D is closed and bounded.*

Proof. Suppose D is bounded. Then there is some $x \in \mathbb{R}^d$ and some $\epsilon > 0$ such that $D \subset B_\epsilon(x)$. It immediately follows that D is bounded if and only if it is totally bounded.

Suppose D is closed. Let $(x)_{n\geq 1} \in D$ be a Cauchy sequence. We know that $\mathbb{R}^d$ is a complete metric space, hence $x_n \to x$ for some $x \in \mathbb{R}^d$. It follows then that because D is closed, that $x \in D$. Hence D is complete. Conversely, suppose D is compact. Then D is complete. Hence all Cauchy sequences in D converge in D. Again, because $\mathbb{R}^d$ complete, D is then closed. □

The Heine-Borel theorem directly implies Lemma 2.1.6.

Lemma 2.1.6. *Let $D \subset \mathbb{R}^d$ be bounded. Then $D \cup \partial D$ is compact.*

Proof. By the Heine-Borel theorem it is sufficient to show that $D \cup \partial D$ is closed and bounded. Since D is bounded, we can write $D \subset B_r(x)$ for some $x \in D$ and $r > 0$. Now let $\epsilon > 0$ and define $l = r + \epsilon$. It is easy to see by the definition of ∂D that $\partial D \subset B_l(x)$ which implies $D \cup \partial D \subset B_l(x)$, because $l > r$, and thus $D \cup \partial D$ is bounded.

Since D is open, we have $D = \text{int}(D)$. Recall that $\overline{D} = \text{int}(D) \cup \partial D$, where $\overline{D}$ denotes the closure of D. Then $D \cup \partial D$ is a closed set and we are done. □

In this way, we can extend every bounded set $D \subset \mathbb{R}^d$ to a compact set $D \cup \partial D$. Such an extension is desirable because we can say much more about compact sets than bounded sets. Specifically, we can use Theorem 2.1.4 on $D \cup \partial D$ to extract a finite sub-cover from any open covering of $D \cup \partial D$. Then because $D \subset D \cup \partial D$, this same finite sub-cover also covers D.

2.1.3. Measure Spaces and the Fubini-Tonelli Theorem

The pair $(S, \mathscr{A})$ consisting of a set S and a σ-algebra $\mathscr{A} \in \mathscr{P}(S)$ is called a measurable space. We call a σ-additive function $\mu : \mathscr{A} \to [0, \infty)$ for which $\mu(\emptyset) = 0$ a measure. Then the triple $(S, \mathscr{A}, \mu)$ is called a measure space. The natural continuation of this definition is to somehow connect measure spaces through functions.

Definition 2.1.4. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. We call $f : X \to Y$ measurable if $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$.

The class of functions which are compositions of measurable functions are measurable.

Let us now revise our definition of the integral. A function $f : S \to \mathbb{R}$ is called a simple function if f is measurable and only takes finitely many values. We can express all simple functions as

$$f = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i} \tag{2.12}$$

where $x_1, \ldots, x_n \in \mathbb{R}$ are the distinct values which f can take and $A_i = \{s \in S : f(s) = x_i\}$. Notice that by this definition, it immediately follows that any simple function is a linear combination of characteristic functions.

We will define the integral first on simple functions, however in general we prefer to work with the class of all measurable functions. To this end, we want to somehow bridge these two classes of functions. Fortunately, it can be shown that we can pointwise approximate any measurable function with simple functions and we state this without proof.

Theorem 2.1.7. *Let $(S, \mathscr{A})$ be a measurable space. We can find for any measurable $f : S \to [0, \infty]$ a sequence of simple functions $(f_n)_{n\geq 1}$ with $0 \leq f_1 \leq f_2 \leq \ldots$ and $\lim_{n\to\infty} f_n(s) = f(s)$ for all $s \in S$.*

Moreover, let $\overline{\mathbb{R}} = [-\infty, \infty]$. We can extend Theorem 2.1.7 to any measurable function $f : S \to \overline{\mathbb{R}}$. Define the functions $f^+, f^- : S \to [0, \infty]$ by $f^+ = \max\{f, 0\}$ and $f^- = \min\{-f, 0\}$ and consider that for any f, we can write $f = f^+ - f^-$. Then the result follows from the linearity of limits. For a detailed proof of this and Theorem 2.1.7, an interested reader is referred to [15, p. 58]. Theorem 2.1.7 is a particularly useful result because it serves as the aforementioned bridge between the class of all simple functions and the class of all measurable functions.

We can now define the integral over $B \in \mathscr{A}$ for some simple function $f : (S, \mathscr{A}, \mu) \to [0, \infty)$ as

$$\int_B f \, \mathrm{d}\mu = \sum_{i=1}^{n} x_i \mu(B \cap A_i) \tag{2.13}$$

where $x_1,\dots,x_n$ are the values f can take and $A_1\dots,A_n$ disjoint sets in $\mathscr{A}$ as in (2.12). By Theorem 2.1.7 we can find for any measurable function $g:(S,\mathscr{A},\mu)\to[0,\infty]$, a sequence of simple functions $(g_n)_{n\geq1}$ with $0\leq g_n\uparrow g$. We then define the integral of g over $B\in\mathscr{A}$ as

$$\int_B g\mathrm{d}\mu=\lim_{n\to\infty}\int_B g_n\mathrm{d}\mu. \tag{2.14}$$

Finally, for any measurable function $h:(S,\mathscr{A},\mu)\to\overline{\mathbb{R}}$, we call h integrable if both $\int_B h^+\mathrm{d}\mu<\infty$ and $\int_B h^-\mathrm{d}\mu<\infty$. We then define the integral of h over $B\in\mathscr{A}$ as

$$\int_B h\mathrm{d}\mu=\int_B h^+\mathrm{d}\mu+\int_B h^-\mathrm{d}\mu. \tag{2.15}$$

Using these definitions, we can show that the limit and integral can often be interchanged. This is the monotone convergence theorem and we provide the proof from [11, p. 21].

Theorem 2.1.8. (Monotone Convergence Theorem) *Let $(f_n)_{n\geq1}$ be a sequence of measurable functions such that $0\leq f_n\uparrow f$. Then f is measurable and we have*

$$\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu=\int_S f\mathrm{d}\mu. \tag{2.16}$$

Proof. The first part of the theorem follows from the measurability of pointwise convergence. It remains to prove (2.16). By the monotonicity of the integral, we have

$$\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu\leq\int_S f\mathrm{d}\mu. \tag{2.17}$$

By Theorem 2.1.7, we can find a sequence of simple functions $(g_m)_{m\geq1}$ such that $0\leq g_m\uparrow f$. Choose some $m'\geq1$ and denote $g=g_{m'}$. Let $c\in(0,1)$ and set $E_n=\{s\in S:cg(s)\leq f_n(s)\}$ for all $n\geq1$. Then each E_n is measurable, $E_1\subset E_2\subset E_3\subset\dots$ and $\cup_{n\geq1}E_n=S$. We motivate the last equality. Suppose $s\in S$; if $f(s)=0$, then $s\in E_1$. If $f(s)>0$, then $cg(s)<f(s)$ since $c<1$, hence $s\in E_n$ for some $n\geq1$. Then by the monotonicity and linearity of the integral

$$c\int_{E_n}g\mathrm{d}\mu=\int_{E_n}cg\mathrm{d}\mu\leq\int_{E_n}f_n\mathrm{d}\mu\leq\int_S f\mathrm{d}\mu\leq\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu. \tag{2.18}$$

Now write $g=\sum_{i=1}^k x_i\mathbb{1}_{A_i}$ with $(x_i)_{i=1}^k$ the values g can take and $(A_i)_{i=1}^k$ disjoint sets in $\mathscr{A}$. Since $E_n\cap A_i\uparrow S\cap A_i$, we find that

$$\int_{E_n}g\mathrm{d}\mu=\sum_{i=1}^k x_i\mu(E_n\cap A_i)\to\sum_{i=1}^k x_i\mu(S\cap A_i)=\int_S g\mathrm{d}\mu. \tag{2.19}$$

Then $c\int_S g\mathrm{d}\mu\leq\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu$ and because $c\in(0,1)$ was arbitrary, $\int_S g_m\mathrm{d}\mu\leq\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu$ for all $m\geq1$. Letting $m\to\infty$, we obtain

$$\int_S f\mathrm{d}\mu\leq\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu\leq\int_S f\mathrm{d}\mu.$$ □

We now define so called jointly measurable functions. Consider two measurable spaces $(X,\mathscr{A})$ and $(Y,\mathscr{B})$. In general, we cannot simply assume that the Cartesian product $\mathscr{A}\times\mathscr{B}=\{Z\subset X\times Y:Z=A\times B,A\in\mathscr{A},B\in\mathscr{B}\}$ is a σ-algebra. Instead, consider $\sigma(\mathscr{A}\times\mathscr{B})$ to be the smallest σ-algebra generated by $\mathscr{A}\times\mathscr{B}$. That is

$$\sigma(\mathscr{A}\times\mathscr{B})=\cap\left\{\mathscr{C}:\mathscr{C}\textit{ is a }\sigma\textit{-algebra on }A\times B\textit{ and }\mathscr{A}\times\mathscr{B}\subset\mathscr{C}\right\}. \tag{2.20}$$

We denote this construction with the special notation $\mathscr{A}\otimes\mathscr{B}=\sigma(\mathscr{A}\times\mathscr{B})$ and call it the product σ-algebra.

It remains to construct a measure on $\mathscr{A}\otimes\mathscr{B}$. For this we define the tensor product. Let f and g be functions defined on X and Y respectively with both having values in $\mathbb{K}$ where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. The tensor product, $f\otimes g$, is the mapping $(x,y)\mapsto f(x)g(y)$ for all $(x,y)\in X\times Y$. We show in Lemma 2.1.9 that the tensor product inherits measurability from f and g.

Lemma 2.1.9. *Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be two measurable spaces. Suppose f is $\mathscr{A}$-measurable and g is $\mathscr{B}$-measurable. Then $f\otimes g$ is $\mathscr{A}\otimes\mathscr{B}$-measurable on $X\times Y$.*

Proof. Write

$$(f \otimes g)(x,y) = f(x)g(y) = (f \otimes \mathbb{1}_Y)(x,y)(\mathbb{1}_X \otimes g)(x,y). \tag{2.21}$$

Then it is sufficient to show measurability of $(f \otimes \mathbb{1}_Y)(x,y)$ and $(\mathbb{1}_X \otimes g)(x,y)$. For the first tensor product, we have for every set $D \in \mathbb{K}$

$$(f \otimes \mathbb{1}_Y)^{-1}(D) = \{(x,y) : f(x) \in D\} = f^{-1}(D) \times Y \in \mathscr{A} \otimes \mathscr{B}. \tag{2.22}$$

The measurability of the second tensor product is proved analogously. □

We can then define on the measurable space $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ the product measure $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ for all $A \times B \in \mathscr{A} \otimes \mathscr{B}$.

We go through the trouble of defining the product σ-algebra because of the Fubini-Tonelli theorem. In essence, this theorem allows us to switch the order of integration for non-negative and $\mathscr{A} \otimes \mathscr{B}$-measurable functions $f : X \times Y \to \mathbb{K}$.

Lemma 2.1.10. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces and let $f : X \times Y \to \mathbb{K}$ be $\mathscr{A} \otimes \mathscr{B}$-measurable, then*

i. *for all $x \in X$, the function $y \mapsto f(x,y)$ is measurable;*

ii. *for all $y \in Y$, the function $x \mapsto f(x,y)$ is measurable.*

Proof. Let $\mathscr{C}$ be the collection of all sets $C \in \mathscr{A} \times \mathscr{B}$ such that i. and ii. hold for $f = \mathbb{1}_C$. We know that the characteristic function is measurable, hence it can be shown that $\mathscr{C}$ is a σ-algebra containing every set of form $A \times B$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Then $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$ by (2.20).

Now, let $f = \sum_{i=1}^n c_i \mathbb{1}_{C_i}$ be a simple function with disjoint sets $C_i \in \mathscr{A} \otimes \mathscr{B}$. Then, because $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$, i. and ii. hold by the linearity of measurable functions.

We extend this to f any positive, measurable function. Choose a sequence of simple functions as in Theorem 2.1.7, then i. and ii. follow from the pointwise approximation of positive, measurable functions by simple functions. Finally, let $f = f^+ - f^-$, then the result follows from the linearity of measurable functions. □

We say a measure space $(S, \mathscr{A}, \mu)$ is finite if $\mu(S) < \infty$. Moreover, we call a space σ-finite when we can find countably many $A_1, A_2, \dots \in \mathscr{A}$ with $\mu(A_n) < \infty$ for all $n \geq 1$ which satisfy $\cup_{n\geq 1} A_n = S$.

Theorem 2.1.11. (Fubini-Tonelli Theorem) *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \mathbb{K}$ be non-negative and $\mathscr{A} \otimes \mathscr{B}$-measurable, then*

i. *the non-negative function $y \mapsto \int_X f(x,y)\mathrm{d}\mu$ is measurable;*

ii. *the non-negative function $x \mapsto \int_Y f(x,y)\mathrm{d}\nu$ is measurable;*

iii.
$$\int_{X\times Y} f(x,y)\mathrm{d}(\mu \otimes \nu) = \int_X \left(\int_Y f(x,y)\mathrm{d}\nu \right) \mathrm{d}\mu = \int_Y \left(\int_X f(x,y)\mathrm{d}\mu \right) \mathrm{d}\nu. \tag{2.23}$$

Proof. Suppose that $\mu(X) = \nu(Y) = 1$ and let $\mathscr{C}$ be the collection of all sets $C \in \mathscr{A} \otimes \mathscr{B}$ such that all three statements hold for $f = \mathbb{1}_C$. We show $\mathscr{C}$ is a σ-algebra. All 3 conditions are trivial for $f = \mathbb{1}_\emptyset = 0$. Suppose i-iii. hold for some $C \in \mathscr{C}$. We write $\mathbb{1}_{C^c} = 1 - \mathbb{1}_C$. Then i. and ii. clearly hold for C^c by linearity of measurability and we have by the linearity of the integral

$$\begin{aligned}\int_{X\times Y} \mathbb{1}_{C^c}\mathrm{d}(\mu\otimes\nu) &= \int_{X\times Y} 1 - \mathbb{1}_C \mathrm{d}(\mu\otimes\nu) = 1 - \int_{X\times Y} \mathbb{1}_C \mathrm{d}(\mu\otimes\nu) \\ &= 1 - \int_Y \left(\int_X \mathbb{1}_C \mathrm{d}\mu \right) \mathrm{d}\nu = \int_Y \left(\int_X 1 - \mathbb{1}_C \mathrm{d}\mu \right) \mathrm{d}\nu \\ &= \int_Y \left(\int_X \mathbb{1}_{C^c} \mathrm{d}\mu \right) \mathrm{d}\nu.\end{aligned} \tag{2.24}$$

We can similarly derive the other equality in iii. and so iii. holds for $\mathbb{1}_{C^c}$. Suppose i-iii. hold for disjoint sets $C_1, C_2, \dots \in \mathscr{A} \otimes \mathscr{B}$ and let $C = \cup_{n\geq 1} C_n$. The monotone convergence theorem and linearity of the integral imply i-iii. all hold for $f = \mathbb{1}_C$. Hence $\mathscr{C}$ is a σ-algebra. Since i-iii. clearly hold for all $A \times B$ where $A \in \mathscr{A}$ and $B \in \mathscr{B}$, then $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$ by the definition of the product σ-algebra.

Now suppose μ and ν are finite. Then we can repeat the previous step on the normalised measures $\frac{\mu}{\mu(X)}$ and $\frac{\nu}{\nu(Y)}$ to find that i-iii. hold for all $C \in \mathscr{A} \otimes \mathscr{B}$. We can then extend this to the case where μ and ν are σ-finite. This follows from the monotone convergence theorem as in the last step we took to show that $\mathscr{C}$ was a σ-algebra in the first part of the proof.

Finally, we can extend this result to all measurable functions in much the same way as in the proof of Lemma 2.1.10. We extend to simple functions by taking linear combinations of characteristic functions. We extend to positive functions using the monotone convergence theorem. Finally, we extend to all measurable functions by taking a linear combination of positive functions and the result follows. □

The proofs of the Fubini-Tonelli theorem and Lemma 2.1.10 are partly sourced from [10, p. 672]. It is important to note that the Fubini-Tonelli theorem is not directly applicable to the Riemann integral. It can however be shown that for all continuous functions, the integral with respect to the Lebesgue measure corresponds to the Riemann integral. Moreover, f is Lebesgue integrable if and only if $\int_{-\infty}^{\infty} |f| \mathrm{d}x$ exists as an improper Riemann integral. As a direct result, we can apply the Fubini-Tonelli theorem to almost every Riemann integrable function.

Now that we have a better understanding of the integral, let us look at an example of a useful measure space and its respective integral.

Example 2.1.3. Let $S = \mathbb{N}$ and $\mathscr{A} = \mathscr{P}(S)$ as the σ-algebra. We can define the measure $\mu : \mathscr{A} \to [0,\infty)$ by

$$\mu(A) = \begin{cases} \#A & A \text{ is finite} \\ \infty & A \text{ is infinite} \end{cases}$$

for all $A \in \mathscr{A}$ where $\#A$ denotes the cardinality of A. This measure is called the counting measure. Then for any $f : S \to \mathbb{R}$ which is non-negative, we see that the integral is defined as

$$\int_S f \mathrm{d}\mu = \sum_{n=1}^{\infty} f(n). \tag{2.25}$$

2.1.4. L^p-Spaces and Hölder's Inequality

Let $(S, \mathscr{A}, \mu)$ be a measure space. We now give the definition of L^p-spaces.

Definition 2.1.5. For $p \in [1,\infty)$ let the space

$$L^p(S) = \left\{ f : S \to \mathbb{K} : f \text{ is measurable and } \int_S |f|^p \mathrm{d}\mu < \infty \right\} \tag{2.26}$$

be equipped with the norm

$$\|f\|_{L^p(S)} = \left(\int_S |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}. \tag{2.27}$$

Note that $L^1(S)$ is nothing but the class of integrable functions $f : S \to \mathbb{K}$. Note also that all functions $f \in L^p(S)$ are measurable and we have a positive function inside the integral norm. As a direct result of this, we can always apply the Fubini-Tonelli theorem when working with L^p-norms.

A very powerful result for L^p-spaces, which we use extensively in Chapter 3, is Hölder's inequality.

Lemma 2.1.12. (Hölder's Inequality) *Let $p, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(S)$ and $g \in L^q(S)$. Then $fg \in L^1(S)$ and $\|fg\|_{L^1(S)} \le \|f\|_{L^p(S)} \|g\|_{L^q(S)}$.*

For the proof of this result we first give the definition of convexity for functions and prove Young's inequality for products.

Let V be a real vector space. We say a set $X \subset V$ is convex if $(1-t)x + ty \in X$ for all $x, y \in X$ and $t \in [0,1]$. In $\mathbb{R}^d$, this translates to: any point on any straight line between any two points in X is itself in X.

Definition 2.1.6. Let X be a convex subset of a real vector space and let $f : X \to \mathbb{R}$ be a function. We say f is convex if for all $t \in [0,1]$ and all $x, y \in X$

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y). \tag{2.28}$$

It is a well known fact that a twice differentiable function is convex if and only if it has nonnegative second derivative.

Lemma 2.1.13. (Young's Inequality) *Let $a, b \in \mathbb{R}_{\geq 0}$ and $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.29}$$

Proof. The inequality clearly holds for $a = 0$ or $b = 0$, suppose then that $a > 0$ and $b > 0$. Let $t = \frac{1}{p}$, then $1 - t = \frac{1}{q}$. We know the mapping $x \mapsto -\ln(x)$ is convex because it has has second derivative $\frac{1}{x^2} \geq 0$. Hence by the convexity of our mapping

$$-\ln\left(ta^p + (1-t)b^q\right) \leq -t\ln\left(a^p\right) - (1-t)\ln\left(b^q\right) = -\ln(ab). \tag{2.30}$$

Multiplying our inequality by -1 and then exponentiating, we obtain

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

□

We are now ready to prove Hölder's inequality.

Proof. (Hölder's Inequality) We first treat two special cases. Suppose $\|f\|_{L^p(S)} = 0$, then $f = 0$ almost everywhere, hence the product $fg = 0$ almost everywhere, and thus $\|fg\|_{L^1(S)} = 0$. The second case, $\|g\|_{L^q(S)} = 0$, follows analogously.

Suppose then that $\|f\|_{L^p(S)} > 0$ and $\|g\|_{L^q(S)} > 0$. Denote $F = \frac{f}{\|f\|_{L^p(S)}}$ and $G = \frac{g}{\|g\|_{L^q(S)}}$, then $\|F\|_{L^p(S)} = \|G\|_{L^q(S)} = 1$. By Lemma 2.1.13, we have for all $s \in S$

$$|F(s)G(s)| \leq \frac{|F(s)|^p}{p} + \frac{|G(s)|^q}{q}. \tag{2.31}$$

Then, we integrate both sides over S to show that

$$\|FG\|_{L^1(S)} \leq \frac{\|F\|^p_{L^p(S)}}{p} + \frac{\|G\|^q_{L^q(S)}}{q} = \frac{1}{p} + \frac{1}{q} = 1 \tag{2.32}$$

and hence, after multiplying our inequality by $\|f\|_{L^p(S)}\|g\|_{L^q(S)}$, we obtain

$$\|fg\|_{L^1(S)} \leq \|f\|_{L^p(S)}\|g\|_{L^q(S)}.$$

□

Another well known inequality on L^p-spaces which we use is Minkowski's inequality.

Theorem 2.1.14. (Minkowski's Inequality) *Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in L^p(S)$. Then $f + g \in L^p(S)$ and $\|f + g\|_{L^p(S)} \leq \|f\|_{L^p(S)} + \|g\|_{L^p(S)}$.*

Proof. We first show that for $a, b \geq 0$

$$(a+b)^p \leq (2\max\{a,b\})^p = 2^p \max\{a^p, b^p\} \leq 2^p(a^p + b^p). \tag{2.33}$$

Then

$$\int_S |f+g|^p \mathrm{d}\mu \leq 2^p \int_S (|f|^p + |g|^p)\mathrm{d}\mu < \infty. \tag{2.34}$$

and hence $f + g \in L^p(S)$. We can rewrite the condition on p and q to $q = \frac{p}{p-1}$ such that $\|f^{p-1}\|_{L^q(S)} = \|f\|^{p-1}_{L^p(S)}$. Then by Hölder's inequality and the triangle inequality

$$\begin{aligned}
\|f+g\|^p_{L^p(S)} &= \int_S |f+g|^p \mathrm{d}\mu \\
&\leq \int_S |f||f+g|^{p-1}\mathrm{d}\mu + \int_S |g||f+g|^{p-1}\mathrm{d}\mu \\
&= \|f(f+g)^{p-1}\|_{L^1(S)} + \|g(f+g)^{p-1}\|_{L^1(S)} \\
&\leq \|f\|_{L^p(S)}\|(f+g)^{p-1}\|_{L^q(S)} + \|g\|_{L^p(S)}\|(f+g)^{p-1}\|_{L^q(S)} \\
&= \|f\|_{L^p(S)}\|f+g\|^{p-1}_{L^p(S)} + \|g\|_{L^p(S)}\|f+g\|^{p-1}_{L^p(S)}
\end{aligned} \tag{2.35}$$

Finally, dividing by $\|f+g\|_{L^p(S)}^{p-1}$, we obtain

$$\|f+g\|_{L^p(S)} \leq \|f\|_{L^p(S)} + \|g\|_{L^p(S)} \qquad \square$$

2.1.5. Calculus in $\mathbb{R}^d$

Let us now define the derivative of some function mapping points from $\mathbb{R}^d$ to $\mathbb{R}^{d'}$. First, we introduce the notion of partial differentiability.

Definition 2.1.7. Let D be contained in $\mathbb{R}^d$, let $f : D \to \mathbb{R}$ be a function, and let $a \in D$ be an interior point. Then f is called partially differentiable with respect to the $1 \leq i\text{th} \leq d$ variable x_i if

$$\lim_{h\to 0} \frac{f(a_1,\dots,a_i+h,\dots,a_d) - f(a_1,\dots,a_i,\dots,a_d)}{h} \tag{2.36}$$

exists.

If f is partially differentiable at a, we denote $\partial_i f(a)$ to be the partial derivative of f at a in the ith direction i.e. $\partial_i f(a) = \frac{\partial f(a)}{\partial x_i}$.

Definition 2.1.8. Let D be contained in $\mathbb{R}^d$, let $f : D \to \mathbb{R}^{d'}$ be a map, and let $a \in D$ be an interior point of D. Then the map f is called differentiable at a if there exists a linear map $L : \mathbb{R}^d \to \mathbb{R}^{d'}$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0 \tag{2.37}$$

If f is differentiable at a, the linear map L is unique. We then call it the differential of f at a and we denote this linear map by $df(a)$. However, (2.37) is in general an inconvenient definition to work with. To overcome this, we find another condition for differentiability in Theorem 2.1.15 and introduce more convenient notation.

Theorem 2.1.15. *Let D be contained in $\mathbb{R}^d$, let $f : D \to \mathbb{R}^{d'}$ be a map and $a \in D$ be an interior point. We write $f = (f_1,\dots,f_{d'})$.*

i. *The map f is differentiable at a if and only if all component functions $f_1,\dots,f_{d'}$ are differentiable at a.*

ii. *If f is differentiable at a, we call the column vector*

$$\nabla f(a) := \begin{bmatrix} \partial_1 f(a) \\ \vdots \\ \partial_d f(a) \end{bmatrix} \tag{2.38}$$

the gradient of f at a.

iii. *Furthermore, if f is differentiable at a, then we call the matrix of $df(a)$ the derivative of f at a, given by*

$$\begin{bmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \dots & \partial_d f_1(a) \\ \partial_1 f_2(a) & \partial_2 f_2(a) & \dots & \partial_d f_2(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_{d'}(a) & \partial_2 f_{d'} & \dots & \partial_d f_{d'} \end{bmatrix} \tag{2.39}$$

and we denote it by $f'(a)$.

The proof for i. follows from working in each coordinate of Definition 2.1.8. Furthermore, we will use linear maps from $\mathbb{R}^d \to \mathbb{R}^d$ in Chapter 3. In such a case, the determinant of $f'(a)$ is well-defined. We call it the Jacobian of f at a and we denote it by $J(a,f) := \det\big(f'(a)\big)$.

Let us also define the differentiable inverse.

Theorem 2.1.16. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^d$ be open and let $f : U \to \mathbb{R}^d$ be differentiable. Suppose that $J(a, f) \neq 0$ for $a \in U$, then there exists an open neighbourhood $V \subseteq U$ of a such that the map $f : V \to f(V)$ has differentiable inverse $g : f(V) \to V$ and for all $x \in V$, we have*

$$g'(f(x)) = (f'(x))^{-1}. \tag{2.40}$$

Essentially, the inverse function theorem allows us to interchange the derivative and inverse such that, for $f(a) = b$, we have $(f'(a))^{-1} = (f^{-1}(b))'$. For a proof, an interested reader is referred to [14, p. 35].

Let us now define a C^1-diffeomorphism.

Definition 2.1.9. Let D and D^* be contained in $\mathbb{R}^d$. We say they are homeomorphic if there exists a bijective map $f : D \to D^*$ such that both f and f^{-1} are continuous. We call such a mapping a homeomorphism. Furthermore, if f and f^{-1} are also differentiable, we call such a mapping a C^1-diffeomorphism.

It is immediate to see that a C^1-diffeomorphism is a stronger condition; all C^1-diffeomorphisms are homeomorphisms, but not all homeomorphisms are C^1-diffeomorphisms.

Theorem 2.1.17. (Change of Variables Theorem) *Let D and D^* be open and contained in $\mathbb{R}^d$. Suppose $g : D \to D^*$ is a diffeomorphism. Then for any measurable $f : D^* \to \mathbb{R}$ and any measurable set $A \subseteq D$*

$$\int_A f(g(x)) \left| J(x, g) \right| \mathrm{d}x = \int_{g(A)} f(y) \mathrm{d}y. \tag{2.41}$$

The proof for this theorem is beyond the scope of this paper, however observe that this theorem is a multi-dimensional variant of the one-variable substitution rule. For a proof of this theorem, an interested reader is referred to [3] and [13] .

2.2. Poincare Inequality

Let D be an open, bounded and connected set in $\mathbb{R}^d$. Define $C^1(D)$ as the space of all functions $f : D \to \mathbb{R}^d$ having continuous derivative. Define $C^1(\overline{D})$ as the space of all functions $f \in C^1(D)$ with ∇f having continuous extension to $\overline{D}$.

2.2.1. Classic Poincare Inequality

The Poincare inequality has two main forms, dependent on the conditions we set on our function. A domain D satisfies the classic Poincare inequality for $p \in (1,\infty)$ if for all $u \in C^1(\overline{D})$ with $u = 0$ on ∂D, there exists some constant $k > 0$ such that

$$\|u\|_{L^p(D)} \le k\|\nabla u\|_{L^p(D)}. \tag{2.42}$$

It is important to note that in Definition 2.1.5, we specify that $f \in L^p(S)$ is a function with image in $\mathbb{K}$, however we map u into $\mathbb{R}^d$. As a result, we are integrating over the Euclidean norm of u and ∇u.

We are specifically interested for which domains (2.42) holds. In Theorem 2.2.1, we show that we only need to bound D in one dimension.

Theorem 2.2.1. *Let D be contained in $R = (-r, r)\times\mathbb{R}^{d-1}$ for some $r > 0$ and let $p \in (1,\infty)$, then for any $u \in C^1(\overline{D})$ with $u = 0$ on ∂D, we have*

$$\|u\|_{L^p(D)} \le 2rp^{-\frac{1}{p}}\|\nabla u\|_{L^p(D)}. \tag{2.43}$$

Proof. Let $q \in (1,\infty)$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$. Let $s \in [-r, r]$ be arbitrary, then since $u(-r, x_2,\dots, x_d) = 0$, we have by Hölder's inequality

$$\begin{aligned}
|u(s, x_2,\dots, x_d)| &= \left|\int_{-r}^{s} \partial_1 u(t, x_2,\dots, x_d)\mathrm{d}t\right| \\
&\le \int_{-r}^{s} |\partial_1 u|\mathrm{d}t \\
&\le \left(\int_{-r}^{s} 1^q\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_{-r}^{s} |\partial_1 u|^p\mathrm{d}t\right)^{\frac{1}{p}} \\
&= (s+r)^{\frac{1}{q}} \left(\int_{-r}^{s} |\partial_1 u|^p\mathrm{d}t\right)^{\frac{1}{p}} \\
&\le (s+r)^{\frac{1}{q}} \left(\int_{-r}^{r} |\partial_1 u|^p\mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.44}$$

Let us rewrite our condition for q as $p = \frac{p}{q}+1$ for clarity. Then, raising to the power of p and integrating over R, we have by the Fubini-Tonelli theorem

$$\begin{aligned}
\|u\|_{L^p(R)}^p &= \int_R |u|^p\mathrm{d}x \\
&\le \int_R (s+r)^{\frac{p}{q}} \left(\int_{-r}^{r} |\partial_1 u(t, x_2,\dots, x_d)|^p\mathrm{d}t\right)\mathrm{d}x \\
&= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_{-r}^{r} (s+r)^{\frac{p}{q}} \int_{-r}^{r} |\partial_1 u|^p\mathrm{d}t\mathrm{d}s\mathrm{d}x_2\dots\mathrm{d}x_d \\
&= \int_{-r}^{r} (s+r)^{\frac{p}{q}}\mathrm{d}s \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_{-r}^{r} |\partial_1 u|^p\mathrm{d}t\mathrm{d}x_2\dots\mathrm{d}x_d \\
&= \left[\frac{1}{\frac{p}{q}+1}(s+r)^{\frac{p}{q}+1}\right|_{s=-r}^{r} \|\partial_1 u\|_{L^p(R)}^p \\
&\le \frac{(2r)^p}{p}\|\nabla u\|_{L^p(R)}^p
\end{aligned} \tag{2.45}$$

where the last inequality in (2.45) follows from the trivial inequality $\|\partial_i u\|_{L^p(R)} \le \|\nabla u\|_{L^p(R)}$ for all $1 \le i \le d$. Then raising our inequality to the power of $\frac{1}{p}$, we obtain (2.43) as desired. □

2.2.2. Neumann Boundary Conditions

Notice that the classic Poincare inequality imposes Dirichlet type boundary conditions on $u \in C^1(\overline{D})$. What happens if we use instead Neumann boundary conditions i.e. $\nabla u = 0$ on ∂D? This is the Poincare-Wirtinger inequality. For these boundary conditions, we see that the classic form of the Poincare inequality is not consistent for all $u \in C^1(\overline{D})$.

Example 2.2.1. Let $u \in C^1(\overline{D})$ be defined by $u(x) = c$ for some $c \in \mathbb{R}^d$. Then $\nabla u = 0$, but

$$\|u\|_{L^p(D)} = \left(\int_D |u|^p\right)^{\frac{1}{p}} = |c||D|^{\frac{1}{p}} \not\leq k\|\nabla u\|_{L^p(D)} \tag{2.46}$$

where $|D|$ is the Lebesgue measure of D.

To avoid such edge cases, we introduce a new value. Denote

$$u_D = \frac{1}{|D|}\int_D u(s)\mathrm{d}s \tag{2.47}$$

the 'average' of u. We say D satisfies the Poincare-Wirtinger inequality if for all $u \in C^1(\overline{D})$, there exists some constant $\kappa_p(D)$ such that

$$\|u - u_D\|_{L^p(D)} \leq \kappa_p(D)\|\nabla u\|_{L^p(D)}. \tag{2.48}$$

We now avoid the problem in Example 2.2.1 because $\|u - u_D\|_{L^p(D)} = 0$ for $u(x) = c$. Note the notation we use for our constant; $\kappa_p(D)$ is not a function, however it is always derived from D. Notice also that (2.48) does not concern itself with the location or orientation of D, only its size and shape; if some D centered on the origin satisfies (2.48), then so does the same D which is now centered on some arbitrary point in $\mathbb{R}^d$ and rotated by a likewise arbitrary angle.

We call domains which satisfy the Poincare-Wirtinger inequality p-Poincare and denote the collection of all such domains by $\mathscr{P}_p$. It turns out that we cannot so easily show (2.48) holds for some very large class of open sets as we just did in Theorem 2.2.1. In Chapter 3, we show that if D has a C^1-boundary, then $D \in \mathscr{P}_p$.

3

Poincare-Wirtinger Inequality

This chapter builds upon the preliminaries in Chapter 2 to constructively prove that an open, bounded and connected domain in $\mathbb{R}^d$ with a C^1-boundary is p-Poincare. Section 1 deals with the simplest case, we show an arbitrary open rectangle in $\mathbb{R}^d$ is p-Poincare. In Section 2, we prove that C^1-diffeomorphisms, satisfying a special condition, between p-Poincare domains and open, bounded and connected domains in $\mathbb{R}^d$ transmit the p-Poincare property. In Section 3, we construct an open rectangle with one face parameterised by a C^1-function and use results from the previous two sections to show such a rectangle is p-Poincare.

In Section 4 we combine the three previous sections to prove that an open, bounded and connected domain in $\mathbb{R}^d$ with a C^1-boundary is p-Poincare. We do this by constructing a finite covering consisting of domains from Section 1 and Section 3. Finally, in Section 5, we introduce the Sobolev space and extend the space of functions for which the central result of the thesis holds.

Denote D and D^* as open, bounded and connected subsets of $\mathbb{R}^d$. We write $|D|$ for the Lebesgue measure of D. We also denote p and q as the Hölder conjugates; $p, q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

3.1. Poincare-Wirtinger Inequality on an Open Rectangle

The arbitrary open rectangle is perhaps one of the simplest domains which we can prove is p-Poincare in $\mathbb{R}^d$. We define it as

$$\Omega = (0, l_1) \times \cdots \times (0, l_d) \subset \mathbb{R}^d \tag{3.1}$$

for $0 < l_i < \infty$ for all $1 \le i \le d$. Although we define Ω as lying on the origin of the plane, this placement is for convenience only. An equivalent definition of this shape is the arbitrary open rectangle generated by some point $x \in \mathbb{R}^d$. We define this rectangle by

$$\Omega_x = (x_1 - k_1, x_1 + k_1) \times \cdots \times (x_d - k_d, x_d + k_d). \tag{3.2}$$

with $0 < k_i < \infty$ for all $1 \le i \le d$. It is clear to see that Ω and Ω_x have the same shape and furthermore have the same size if $l_i = 2k_i$ for all $1 \le i \le n$. Hence, if $\Omega \in \mathscr{P}_p$, then $\Omega_x \in \mathscr{P}_p$ for all $x \in \mathbb{R}^d$. This is made even clearer when we consider that $\Omega = \Omega_{\frac{l}{2}}$; it is the open rectangle generated by $\frac{l}{2} = \left(\frac{l_1}{2}, \dots, \frac{l_d}{2}\right) \in \mathbb{R}^d$ with $k_i = \frac{l_i}{2}$ for all $1 \le i \le d$.

Before we can prove that Ω is p-Poincare, we prove a useful lemma.

Lemma 3.1.1. *For $(x_i)_{i=1}^n$, a finite sequence of points in $\mathbb{R}$, it holds that*

$$\left|\sum_{i=1}^n x_i\right|^p \le n^{\frac{p}{q}} \sum_{i=1}^n |x_i|^p \tag{3.3}$$

Proof. Let $S = \{1, \dots, n\}$ and μ be the counting measure. We work in the measure space $(S, \mathscr{P}(S), \mu)$. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(i) = |x_i|$ for all $1 \le i \le n$. Then f is clearly non-negative. Hence by the triangle inequality and Hölder's inequality

$$\begin{aligned} \left|\sum_{i=1}^{n} x_i\right| &\le \sum_{i=1}^{n} |x_i| = \|f\|_{L^1(S)} \le \|f\|_{L^p(S)} \|1\|_{L^q(S)} \\ &= \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} 1\right)^{\frac{1}{q}} = n^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \end{aligned} \tag{3.4}$$

and we arrive at (3.3) by raising our inequality to the power of p. □

We can now prove that $\Omega \in \mathscr{P}_p$.

Theorem 3.1.2. *Let $\Omega = (0, l_1) \times \cdots \times (0, l_d)$ be the d-dimensional rectangle with $l_i > 0$ for all $1 \le i \le d$. Let $l = \max_{1 \le i \le d} l_i$ and let $u \in C^1(\overline{\Omega})$, then*

$$\|u - u_\Omega\|_{L^p(\Omega)} \le d l \|\nabla u\|_{L^p(\Omega)} \tag{3.5}$$

Proof. We have by Hölder's inequality

$$\begin{aligned} |u - u_\Omega|^p &= \left| u(x) - \frac{1}{|\Omega|} \int_\Omega u(s) \mathrm{d}s \right|^p \\ &= \left| \frac{1}{|\Omega|} \int_\Omega u(x) - u(s) \mathrm{d}s \right|^p \\ &\le \left(\frac{1}{|\Omega|} \int_\Omega |u(x) - u(s)| \mathrm{d}s \right)^p \\ &\le \left(\frac{1}{|\Omega|} \left(\int_\Omega |u(x) - u(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \left(\int_\Omega 1 \mathrm{d}s \right)^{\frac{1}{q}} \mathrm{d}s \right)^p \\ &= \left(|\Omega|^{\frac{p}{q} - p} \int_\Omega |u(x) - u(s)|^p \mathrm{d}s \right) \\ &= \frac{1}{|\Omega|} \int_\Omega |u(x) - u(s)|^p \mathrm{d}s. \end{aligned} \tag{3.6}$$

Then, for $s, x \in \Omega$ arbitrary, we can draw a path, which travels parallel to the axes, between these two points. We know such a path exists because of the shape of Ω.

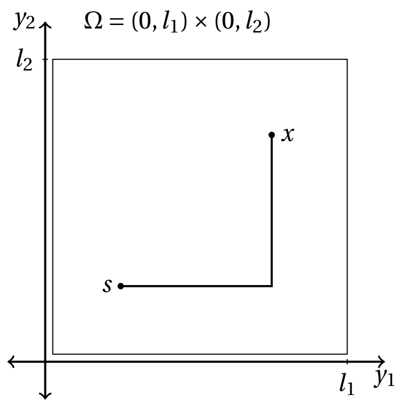

Figure 3.1: $s - x$ Path Visualisation in $\mathbb{R}^2$

We can further assume, without loss of generality, that our path walks along each dimension following the ordering of their indices. Walking along our path, we can apply the fundamental theorem of calculus on the first 1-dimensional line of our path to find

$$u(x_1, s_2, \ldots, s_d) = u(s_1, , \ldots, s_d) + \int_{s_1}^{x_1} \partial_1 u(t_1, s_2, \ldots, s_d) \mathrm{d}t_1 \tag{3.7}$$

Repeating the same idea on the second line of our path, we find

$$u(x_1,x_2,s_3\ldots,s_d) = u(s_1,\ldots,s_d) + \int_{s_1}^{x_1} \partial_1 u(t_1,s_2,\ldots,s_d)\mathrm{d}t_1 \\ + \int_{s_2}^{x_2} \partial_2 u(x_1,t_2,s_3\ldots,s_d)\mathrm{d}t_2. \tag{3.8}$$

Repeating this process such that we walk the entire path, we find

$$\begin{aligned} |u(x)-u(s)|^p &= \left|\sum_{n=1}^{d}\int_{s_n}^{x_n} \partial_n u(x_1,\ldots,x_{n-1},t_n,s_{n+1},\ldots,s_d)\mathrm{d}t_n\right|^p \\ &\leq \left(\sum_{n=1}^{d}\left|\int_{s_n}^{x_n}\partial_n u \mathrm{d}t_n\right|\right)^p \\ &\leq \left(\sum_{n=1}^{d}\int_{s_n}^{x_n}|\partial_n u|\mathrm{d}t_n\right)^p \\ &\leq \left(\sum_{n=1}^{d}\int_{0}^{l_n}|\partial_n u|\mathrm{d}t_n\right)^p \\ &\leq d^{\frac{p}{q}}\sum_{n=1}^{d}\left(\int_{0}^{l_n}|\partial_n u|\mathrm{d}t_n\right)^p \end{aligned} \tag{3.9}$$

where the last inequality in (3.9) follows from Lemma 3.1.1. By Hölder's inequality

$$\begin{aligned} |u(x)-u(s)|^p &\leq d^{\frac{p}{q}}\sum_{n=1}^{d}\left(\int_{0}^{l_n}|\partial_n u|\mathrm{d}t_n\right)^p \\ &\leq d^{\frac{p}{q}}\sum_{n=1}^{d}\left(\left(\int_0^{l_n}1^q\mathrm{d}t_n\right)^{\frac{1}{q}}\left(\int_0^{l_n}|\partial_n u|^p\mathrm{d}t_n\right)^{\frac{1}{p}}\right)^p \\ &= d^{\frac{p}{q}}\sum_{n=1}^{d} l_n^{\frac{p}{q}}\int_0^{l_n}|\partial_n u|^p\mathrm{d}t_n \\ &\leq d^{\frac{p}{q}}\sum_{n=1}^{d} l^{\frac{p}{q}}\int_0^{l_n}|\partial_n u|^p\mathrm{d}t_n. \end{aligned} \tag{3.10}$$

Integrating over $\Omega\times\Omega$ we have

$$\int_\Omega\int_\Omega |u(x)-u(s)|^p\mathrm{d}x \leq d^{\frac{p}{q}}\int_\Omega\int_\Omega\sum_{n=1}^{d} l^{\frac{p}{q}}\int_0^{l_n}|\partial_n u|^p\mathrm{d}t_n\mathrm{d}x \tag{3.11}$$

$$= d^{\frac{p}{q}}\sum_{n=1}^{d} l^{\frac{p}{q}}\int_\Omega\int_\Omega\int_0^{l_n}|\partial_n u|^p\mathrm{d}t_n\mathrm{d}x. \tag{3.12}$$

Let us zoom in on one element of (3.12), say element i. We see that our function $u(x_1,\ldots,x_{i-1},t_i,s_{i+1},\ldots,s_d)$ is independent of $\{s_1,\ldots,s_{i-1},s_i,x_i,x_{i+1},\ldots,x_d\}$. Then by the Fubini-Tonelli theorem

$$\begin{aligned} \int_\Omega\int_\Omega\int_0^{l_i}|\partial_i u|^p\mathrm{d}t_i\mathrm{d}x\mathrm{d}s &= \int_0^{l_d}\cdots\int_0^{l_1}\int_0^{l_d}\cdots\int_0^{l_1}\int_0^{l_i}|\partial_i u|^p\mathrm{d}t_i\mathrm{d}x_1\ldots\mathrm{d}x_d\mathrm{d}s_1\ldots\mathrm{d}s_d \\ &= l_i|\Omega|\int_0^{l_d}\cdots\int_0^{l_{i+1}}\int_0^{l_i}\int_0^{l_{i-1}}\cdots\int_0^{l_1}|\partial_i u|^p\mathrm{d}x_1\ldots\mathrm{d}x_{i-1}\mathrm{d}t_i\mathrm{d}s_{i+1}\ldots\mathrm{d}s_d \\ &\leq l|\Omega|\int_\Omega|\partial_i u|^p\mathrm{d}z \end{aligned} \tag{3.13}$$

where $\{z_1,\ldots,z_d\} = \{x_1,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_d\}$. Zooming back out, we have

$$\int_\Omega\int_\Omega|u(x)-u(s)|^p\mathrm{d}x\mathrm{d}s \leq d^{\frac{p}{q}}l^{\frac{p}{q}+1}|\Omega|\sum_{n=1}^{d}\int_\Omega|\partial_n u|^p\mathrm{d}z. \tag{3.14}$$

Combining (3.6) and (3.14), we have

$$
\begin{aligned}
\int_\Omega |u - u_\Omega|^p \mathrm{d}x &\leq \frac{1}{|\Omega|}\int_\Omega\int_\Omega |u(x)-u(s)|^p \mathrm{d}x\mathrm{d}s \\
&\leq d^{\frac{p}{q}} l^{\frac{p}{q}+1} \sum_{n=1}^{d} \int_\Omega |\partial_n u|^p \mathrm{d}z \\
&\leq d^{\frac{p}{q}} l^{\frac{p}{q}+1} \sum_{n=1}^{d} \int_\Omega |\nabla u|^p \mathrm{d}z \\
&= d^{\frac{p}{q}+1} l^{\frac{p}{q}+1} \int_\Omega |\nabla u|^p \mathrm{d}z.
\end{aligned}
\tag{3.15}
$$

Then, rewriting our condition for the Hölder conjugates as $\frac{p}{q}+1=p$ and exponentiating by $\frac{1}{p}$, we finally have

$$
\|u - u_\Omega\|_{L^p(\Omega)} \leq dl\|\nabla u\|_{L^p(\Omega)}. \qquad \square
$$

3.2. C^1-Diffeomorphism Theorem

In this section, we show that if $D \in \mathscr{P}_p$, then the existence of a C^1-diffeomorphism $f : D \to D^*$ with a special condition implies $D^* \in \mathscr{P}_p$. This is a desirable result because not all domains are so nicely shaped as Ω. As a result of this, it is often easier to show that an unwieldy domain is 'similar' to a nice domain which is p-Poincare than to show directly that the unwieldy domain is p-Poincare. Let us define the aforementioned sufficient condition which f must possess.

Definition 3.2.1. Let $f : D \to D^*$ be a C^1-diffeomorphism. We say f has L-bound if there exists some $L > 0$ such that for all $x \in D$ it holds that

$$\frac{1}{L}|h| \le \left|f'(x)h\right| \le L|h| \tag{3.16}$$

for all $h \in \mathbb{R}^d$.

Such a C^1-diffeomorphism has useful properties which we show.

Lemma 3.2.1. *Let $f : D \to D^*$ be a C^1-diffeomorphism with L-bound. Then $\left|J(x, f)\right| \ge L^{-d}$ for all $x \in D$.*

Proof. Let λ_i be eigenvalues of $f'(x)$ with corresponding unit eigenvectors v_{λ_i}. Then

$$\left|\lambda_i v_{\lambda_i}\right| = \left|f'(x)v_{\lambda_i}\right| \ge \frac{1}{L}\left|v_{\lambda_i}\right|. \tag{3.17}$$

Hence $|\lambda_i| \ge \frac{1}{L}$ and

$$\left|J(x, f)\right| = \left|\prod_{i=1}^{d} \lambda_i\right| = \prod_{i=1}^{d} |\lambda_i| \ge L^{-d} \tag{3.18}$$

for all $x \in D$. □

Theorem 3.2.2. *Let $f : D \to D^*$ be a diffeomorphism with L-bound. Then there exists some inverse, differentiable mapping $g : D^* \to D$ with L-bound.*

Proof. By Lemma 3.2.1, we see that $J(x, f) \neq 0$ for all $x \in D$. Since D open and bounded, the existence and differentiability of $g : D^* \to D$ directly follows from the inverse function theorem. Let $f(x) = y$, then $g'(y) = (f'(x))^{-1}$ by the inverse function theorem. Then we have for all $h \in \mathbb{R}^d$

$$|h| = \left|f'(x)(f'(x))^{-1}h\right| = \left|f'(x)g'(y)h\right| \le L\left|g'(y)h\right|. \tag{3.19}$$

Similarly, we have

$$|h| = \left|f'(x)(f'(x))^{-1}h\right| = \left|f'(x)g'(y)h\right| \ge \frac{1}{L}\left|g'(y)h\right|. \tag{3.20}$$

Combining (3.19) and (3.20), we have for all $y \in D^*$

$$\frac{1}{L}|h| \le \left|g'(y)h\right| \le L|h| \tag{3.21}$$

for all $h \in \mathbb{R}^d$ as desired. □

Before we continue, we prove one more Lemma which we will later need.

Lemma 3.2.3. *Let $A \subseteq D$. If $u \in L^p(D)$, then for all $c \in \mathbb{R}^d$*

$$\|u - u_A\|_{L^p(D)} \le 2\left(\frac{|D|}{|A|}\right)^{\frac{1}{p}} \|u - c\|_{L^p(D)}. \tag{3.22}$$

Proof. By Hölder's inequality

$$
\begin{aligned}
\|u_A - c\|_{L^p(D)} &= \left(\int_D \left|\frac{1}{|A|}\int_A u(s)\mathrm{d}s - c\right|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_D \left|\frac{1}{|A|}\int_A (u(s)-c)\mathrm{d}s\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&= |D|^{\frac{1}{p}}\left(\left|\frac{1}{|A|}\int_A (u(s)-c)\mathrm{d}s\right|^p\right)^{\frac{1}{p}} = \frac{|D|^{\frac{1}{p}}}{|A|}\left|\int_A (u(s)-c)\mathrm{d}s\right| \\
&\le \frac{|D|^{\frac{1}{p}}}{|A|}\int_A |u(s)-c|\mathrm{d}s = \frac{|D|^{\frac{1}{p}}}{|A|}\|u-c\|_{L^1(A)} \\
&\le \frac{|D|^{\frac{1}{p}}}{|A|}\|1\|_{L^q(A)}\|u-c\|_{L^p(A)} = \frac{|D|^{\frac{1}{p}}}{|A|}|A|^{\frac{1}{q}}\|u-c\|_{L^p(A)} \\
&= \left(\frac{|D|}{|A|}\right)^{\frac{1}{p}}\|u-c\|_{L^p(A)} \le \left(\frac{|D|}{|A|}\right)^{\frac{1}{p}}\|u-c\|_{L^p(D)}.
\end{aligned}
\tag{3.23}
$$

Since $A \subseteq D$, we have $\frac{|D|}{|A|} \ge 1$, hence we have by the triangle inequality

$$
\begin{aligned}
\|u - u_A\|_{L^p(D)} &\le \|u-c\|_{L^p(D)} + \|u_A - c\|_{L^p(D)} \\
&\le \left(\frac{|D|}{|A|}\right)^{\frac{1}{p}}\|u-c\|_{L^p(D)} + \|u_A - c\|_{L^p(D)} \\
&\le 2\left(\frac{|D|}{|A|}\right)^{\frac{1}{p}}\|u-c\|_{L^p(D)}.
\end{aligned}
\tag{3.24}
$$

as desired. □

Notice that for $D = A$, Lemma 3.2.3 translates to $\|u-u_D\|_{L^p(D)} \le 2\|u-c\|_{L^p(D)}$ for all $c \in \mathbb{R}^d$. We are now equipped with all the tools we need to prove the central theorem of this section.

Theorem 3.2.4. *Suppose that $D \in \mathscr{P}_p$ with Poincare constant $\kappa_p(D)$ and $f : D \to D^*$ is a C^1-diffeomorphism with L-bound. Then $D^* \in \mathscr{P}_p$ with Poincare constant $\kappa_p(D^*) = 2\kappa_p(D)L^{2\frac{d}{p}+1}$.*

Proof. Let $u \in C^1(\overline{D^*})$. For each $x \in D$, define $v : D \to \mathbb{R}$ by $v(x) = (u \circ f)(x)$. Denote $f'(x)^{\mathrm{T}}$ the transpose of $f'(x)$. Then

$$
\nabla v(x) = f'(x)^{\mathrm{T}}\nabla u(f(x))
\tag{3.25}
$$

and furthermore

$$
\left\|f'(x)\right\|_{\mathrm{op}} = \sup_{|h|\le 1}\left|f'(x)h\right| \le \sup_{|h|\le 1} L|h| = L.
\tag{3.26}
$$

Hence, $\left\|f'(x)^T\right\|_{\mathrm{op}} = \left\|f'(x)\right\|_{\mathrm{op}} \le L$ for all $x \in D$ by (3.26) and Lemma 2.1.2. Then by the change of variables theorem

$$
\begin{aligned}
\int_D |\nabla v(x)|^p \mathrm{d}x &= \int_D \left|f'(x)^{\mathrm{T}}\nabla u(f(x))\right|^p \mathrm{d}x \\
&\le L^p \int_D \left|\nabla u(f(x))\right|^p \mathrm{d}x \\
&\le L^{p+d}\int_D \left|\nabla u(f(x))\right|^p \left|(J(x,f)\right|\mathrm{d}x \\
&= L^{p+d}\int_{D^*}\left|\nabla u(y)\right|^p \mathrm{d}y.
\end{aligned}
\tag{3.27}
$$

By Lemma 3.2.1 and Theorem 3.2.2 we have $\left|J(y,f^{-1})\right| \ge L^{-d}$ for all $y \in D^*$. Then by the assumption that

$D \in \mathscr{P}_p$

$$
\begin{aligned}
\int_{D^*} \left|u(y) - v_D\right|^p \mathrm{d}y = \int_{D^*} \left|v(f^{-1}(y)) - v_D\right|^p \mathrm{d}y& \\
\leq L^d \int_{D^*} \left|v(f^{-1}(y)) - v_D\right|^p \left|J(x, f^{-1})\right| \mathrm{d}y& \\
= L^d \int_D |v(x) - v_D|^p \mathrm{d}x& \qquad (3.28)\\
\leq \kappa_p(D)^p L^d \int_D |\nabla v(x)|^p \mathrm{d}x& \\
\leq \kappa_p(D)^p L^{2d+p} \int_{D^*} \left|\nabla u(y)\right|^p \mathrm{d}y.&
\end{aligned}
$$

Hence

$$
\left(\int_{D^*} \left|u(y) - v_D\right|^p \mathrm{d}y\right)^{\frac{1}{p}} \leq \kappa_p(D) L^{2\frac{d}{p}+1} \left(\int_{D^*} \left|\nabla u(y)\right|^p \mathrm{d}y\right)^{\frac{1}{p}}. \qquad (3.29)
$$

Finally, we have by Lemma 3.2.3

$$
\|u - u_{D^*}\|_{L^p(D^*)} \leq 2\|u - v_D\|_{L^p(D^*)} \leq 2\kappa_p(D) L^{2\frac{d}{p}+1} \|\nabla u\|_{L^p(D^*)}. \qquad \square
$$

3.3. Open Rectangle with Smooth Face

Let us now return to our cuboid shape. We are working towards a finite covering of $D \cup \partial D$ for D having a C^1-boundary. It is clear to see that while Ω type cuboids are sufficient for covering the interior of D, problems arise when we try to cover the boundary. Let us then define a new sort of cuboid. This cuboid is very similar to Ω, except we let one face of its boundary be parameterised by a C^1-function. If we let $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ be a C^1-function, we define G as

$$G = (0, l_1) \times \cdots \times (0, l_{d-1}) \times (\gamma(x), l_d) \tag{3.30}$$

with $0 < l_i < \infty$ for all $1 \leq i \leq d$ and $x \in (0, l_1) \times \cdots \times (0, l_{d-1})$. For ease of writing, for $y \in \mathbb{R}^d$, we write $\gamma(y) := \gamma(y_1, \ldots, y_{d-1})$.

Similarly to Section 1, we can also equivalently define such a cuboid as being generated from some point $x \in \mathbb{R}^d$. Then let $\gamma_x : \mathbb{R}^{d-1} \to \mathbb{R}$ be a C^1-function, we define G_x as

$$G_x = (x_1 - k_1, x_1 + k_1) \times \cdots \times (x_{d-1} - k_{d-1}, x_{d-1} + k_{d-1}) \times (x_d - \gamma_x, x_d + k_d) \tag{3.31}$$

with $0 < k_i < \infty$ for all $1 \leq i \leq d$. It is clear to see that G and G_x have the same shape and furthermore have the same size for $l_i = 2k_i$ for all $1 \leq i \leq d-1$ and $l_d = k_d + 2\gamma_x$. Then as with Ω and Ω_x in Section 1, $G \in \mathscr{P}_p$ if and only if $G_x \in \mathscr{P}_p$ for all $x \in \mathbb{R}^d$.

Example 3.3.1. It is useful to visualise G in $\mathbb{R}^2$ and $\mathbb{R}^3$ to understand how it differs from Ω. Notice that one face is distorted by γ.

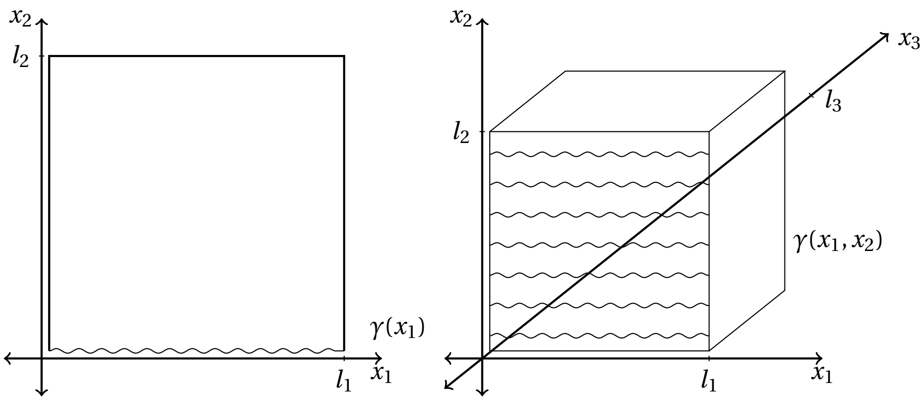

Figure 3.2: G in $\mathbb{R}^2$ and $\mathbb{R}^3$

We want to combine Theorem 3.1.2 and Theorem 3.2.4 to show $G \in \mathscr{P}_p$. To do this we construct an explicit C^1-diffeomorphism $f : \Omega \to G$ with L-bound.

However notice that the definition of a C^1-boundary speaks of an open ball around some point on the boundary, but we desire a cuboid form. We show every open ball in $\mathbb{R}^d$ contains an open rectangle.

Lemma 3.3.1. *Let $r > 0$ and let $B_r(x)$ be the open ball around some $x \in \mathbb{R}^d$. Then there exists an open rectangle R such that $R \subset B_r(x)$ and $x \in R$.*

Proof. Let $R = (x_1 - \frac{r}{d}, x_1 + \frac{r}{d}) \times \cdots \times (x_d - \frac{r}{d}, x_d + \frac{r}{d})$. Clearly $x \in R$, so it remains to show that $R \subset B_r(x)$. Let $y \in R$, then $y_i \in (x_i - \frac{r}{d}, x_i + \frac{r}{d})$ and

$$|x - y| \leq \sum_{i=1}^{d} |x_i - y_i| < \sum_{i=1}^{d} \frac{r}{d} = r. \tag{3.32}$$

Hence $y \in B_r(x)$ and this completes the proof. □

Now let us return to Definition 2.1.2. Let $x \in \partial D$ have some $r > 0$ and C^1-function $\gamma_x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, after relabelling and reorienting the axes if necessary, we have

$$D \cap B_r(x) = \{y \in B_r(x) : y_d > \gamma_x(y)\}. \tag{3.33}$$

By Lemma 3.3.1, we can find some open rectangle R in $B_r(x)$ with $x \in R$. Then let this rectangle be bisected by γ_x such that $R = R_1 \cup R_2$ with R_1 and R_2 both open rectangles with one face parameterised by γ_x. By our definition of γ_x, one of our rectangles has nonempty intersection with D, say R_1, and the other has empty intersection. Then this R_1 has the same shape as G_x.

We will prove later that the mapping $\Psi : \Omega \to G$ given by

$$\Psi(x_1,\ldots,x_d) = \left(x_1,\ldots,x_{d-1}, x_d + \frac{l_d - x_d}{l_d}\gamma(x_1,\ldots,x_{d-1})\right) \tag{3.34}$$

is a sufficient mapping for our purposes. However, as it stands, our definition of G is incomplete.

Example 3.3.2. For this example, we work in $\mathbb{R}^2$ for visual clarity. We have defined G by $(0,l_1)\times(\gamma(x_1),l_2)$. The problem which can then arise is that there is nothing in the definitions of G to ensure that $\gamma(x_1) < l_2$ for all $x_1 \in (0,l_1)$. We visualise this in Figure 3.3.

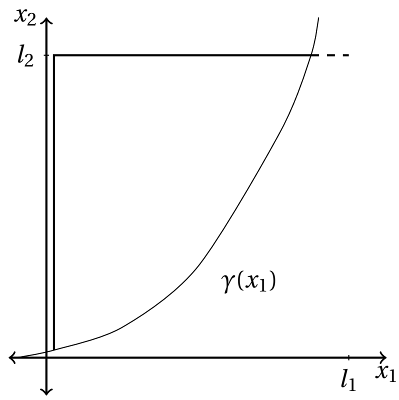

Figure 3.3: G with γ Leaving Through the 'Roof' of G

It is clear to see that there exists some small $\epsilon > 0$ such that $\gamma(l_1 - \epsilon) > l_2$; this is inconsistent with our definition of G. Additionally, $\Psi(l_1 - \epsilon, x_2) \notin G$ for all $x_2 \in (0,l_2)$ and so Ψ is not a mapping from Ω to G.

We want to somehow avoid the problem of Example 3.3.2. Notice that we can bound the growth of all $f \in C^1$ by $\|\nabla f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm. Then for $x \in (0,l_1)\times\cdots\times(0,l_{d-1})$ we have

$$\left|\gamma(x)\right| \le |(x_1,\ldots,x_{d-1})|\left\|\nabla\gamma\right\|_\infty \le |(l_1,\ldots,l_{d-1})|\left\|\nabla\gamma\right\|_\infty. \tag{3.35}$$

However, we are still not done because γ can be some constant function, say $\gamma(x) = c$ for $c \in \mathbb{R}$. Then clearly (3.35) does not hold for such a γ. We account for this edge case by moving and reorienting the axes such that $\gamma(\frac{l_1}{2},\ldots,\frac{l_{d-1}}{2}) = 0$. We can do this because, again, we are only interested in the size and shape of G. Hence, it is sufficient to make G small enough in the first $d-1$ dimensions to avoid the problem in Example 3.3.2.

Then for $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ a C^1-function, we redefine G as

$$G = (0,l_1)\times\cdots\times(0,l_{d-1})\times(\gamma(x),l_d) \tag{3.36}$$

for $0 < l_d < \infty$ and $l_1,\ldots,l_{d-1}$ such that $|(l_1,\ldots,l_{d-1})|\left\|\nabla\gamma\right\|_\infty < l_d$ and, after relabeling and reorienting the axes if necessary, $\gamma(\frac{l_1}{2},\ldots,\frac{l_{d-1}}{2}) = 0$.

Similarly, for some point $x \in \mathbb{R}^d$, let $\gamma_x : \mathbb{R}^{d-1} \to \mathbb{R}$ be a C^1-function. We define such a cuboid generated by $x \in \mathbb{R}^d$ as, after relabeling and reorienting the coordinate axes as in Definition 2.1.2 and then moving G_x such that x sits on the origin,

$$G_x = (-k_1,k_1)\times\cdots\times(-k_{d-1},k_{d-1})\times(\gamma_x,k_d) \tag{3.37}$$

with $0 < k_d < \infty$ and $k_1,\ldots,k_{d-1}$ such that $|(2k_1,\ldots,2k_{d-1})|\left\|\gamma_x\right\|_\infty < k_d$.

We can now show that Ψ is indeed a C^1-diffeomorphism with L-bound from Ω to G.

Theorem 3.3.2. *Let the mapping $\Psi : \Omega \to G$ be given by*

$$\Psi(x_1,\ldots,x_d) = \left(x_1,\ldots,x_{d-1}, x_d + \frac{l_d - x_d}{l_d}\gamma(x_1,\ldots,x_{d-1})\right). \tag{3.38}$$

Then Ψ is a C^1-diffeomorphism.

Proof. We first show Ψ is a bijection. Suppose $x, y \in \Omega$ such that $\Psi(x) = \Psi(y)$. Then clearly $x_i = y_i$ for $1 \le i \le d-1$ and hence $\gamma(x) = \gamma(y)$. Then

$$\begin{aligned} x_d + \frac{l_d - x_d}{l_d}\gamma(x) &= y_d + \frac{l_d - y_d}{l_d}\gamma(y) \\ \Rightarrow x_d + \cancel{l_d\gamma(x)} - \frac{x_d}{l_d}\gamma(x) &= y_d + \cancel{l_d\gamma(x)} - \frac{y_d}{l_d}\gamma(x) \\ \Rightarrow x_d\cancel{(l_d - \gamma(x))} &= y_d\cancel{(l_d - \gamma(x))} \\ \Rightarrow x_d &= y_d \end{aligned} \tag{3.39}$$

and Ψ is injective. Choose arbitrary $y \in G$. We look for $x \in \Omega$ such that $\Psi(x) = y$. Clearly $\Psi_i(x_i) = y_i \Leftrightarrow x_i = y_i$ for all $1 \le i \le d-1$, hence $\gamma(x) = \gamma(y)$ again. Then

$$\begin{aligned} x_d + \frac{l_d - x_d}{l_d}\gamma(x) &= y_d \\ \Rightarrow x_d(l_d - \gamma(x)) &= l_d y_d - l_d\gamma(x) \\ \Rightarrow x_d &= l_d\frac{y_d - \gamma(x)}{l_d - \gamma(x)} \end{aligned} \tag{3.40}$$

and it remains to check whether $x_d \in (0, l_d)$. By (3.35), we have $l_d - \gamma(x) > 0$ and $y_d - \gamma(x) > 0$. Then $x_d > 0$. Since $y_d < l_d$, we find that $\frac{y_d-\gamma(x)}{l_d-\gamma(x)} < 1$. Then $x_d = l_d\frac{y_d-\gamma(x)}{l_d-\gamma(x)} < l_d$ and $x_d \in (0, l_d)$. Hence $x \in G$ and Ψ is a bijective mapping.

We now check the differentiability of Ψ. We have, because γ is a C^1 function, for $a \in \Omega$

$$\partial_i\Psi(a) = \left(0\ldots,1,\ldots,0,\frac{l_d - a_d}{l_d}\partial_i\gamma(a)\right) \tag{3.41}$$

for all $1 \le i \le d-1$. Then also

$$\partial_d\Psi(a) = \left(0,\ldots,0,1 - \frac{1}{l_d}\gamma(a)\right) \tag{3.42}$$

and hence Ψ is differentiable by Theorem 2.1.15.

We now show $\Psi^{-1}: G \to \Omega$ is similarly bijective and differentiable. Denote $\Phi = \Psi^{-1}$. Clearly $\Phi_i(x) = \Psi_i(x)$ for all $1 \le i \le d-1$. We then derive an explicit expression for $\Phi_d(x)$

$$\begin{aligned} \Phi_d(x) + \frac{l_d - \Phi_d(x)}{l_d}\gamma(\Phi(x)) &= x_d \\ \Rightarrow \Phi_d(x) &= l_d\frac{x_d - \gamma(x)}{l_d - \gamma(x)}. \end{aligned} \tag{3.43}$$

Hence $\Phi: G \to \Omega$ is the mapping defined by

$$\Phi(x) = \left(x_1,\ldots,x_{d-1}, l_d\frac{x_d - \gamma(x)}{l_d - \gamma(x)}\right). \tag{3.44}$$

It is well know that the inverse mapping of a bijective mapping is itself bijective, hence Φ is bijective. Finally, because $\gamma(x)$ is a C^1 function, all component functions of Φ are clearly differentiable from which it follows that Φ is differentiable by Theorem 2.1.15. □

It remains to show that Ψ has L-bound. Note that we are not very interested in what L is precisely, merely that it exists. As it turns out however, the quickest way to check its existence is to derive it.

Lemma 3.3.3. *Let Ψ be as in Theorem 3.3.2. Then Ψ has L-bound.*

Proof. For $a \in \Omega$, we have by (3.41) and (3.42)

$$\Psi'(a) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \frac{l_d-a_d}{l_d}\partial_1\gamma(a) & \frac{l_d-a_d}{l_d}\partial_2\gamma(a) & \ldots & \frac{l_d-a_d}{l_d}\partial_{d-1}\gamma(a) & 1 - \frac{1}{l_d}\gamma(a) \end{bmatrix}. \tag{3.45}$$

Then for any $h \in \mathbb{R}^d$, we have

$$\Psi'(a)h = \begin{bmatrix} h_1 \\ \vdots \\ h_{d-1} \\ \frac{l_d - a_d}{l_d}\left(h_1\partial_1\gamma(a) + \cdots + h_{d-1}\partial_{d-1}\gamma(a)\right) + h_d\left(1 - \frac{1}{l_d}\gamma(a)\right) \end{bmatrix}. \tag{3.46}$$

Let us zoom in on the dth coordinate. We find

$$\begin{aligned} \left(h_1\partial_1\gamma(a) + \cdots + h_{d-1}\partial_{d-1}\gamma(a)\right) &= (h_1, \ldots, h_{d-1}) \cdot \nabla\gamma(a)^{\mathrm{T}} \le |(h_1, \ldots, h_{d-1})|\left|\nabla\gamma(a)\right| \\ &\le |(h_1, \ldots, h_{d-1})|\left\|\nabla\gamma\right\|_\infty \le l_d \frac{|(h_1, \ldots, h_{d-1})|}{|(l_1, \ldots, l_{d-1})|} \end{aligned} \tag{3.47}$$

where $\cdot$ denotes the dot product. Then, by the triangle inequality and (3.35), we have

$$\begin{aligned} \left|(\Psi'(a)h)_d\right| &= \left|\frac{l_d - a_d}{l_d}\left(h_1\partial_1\gamma(a) + \cdots + h_{d-1}\partial_{d-1}\gamma(a)\right) + h_d\left(1 - \frac{1}{l_d}\gamma(a)\right)\right| \\ &\le \left|(l_d - a_d)\frac{|(h_1, \ldots, h_{d-1})|}{|(l_1, \ldots, l_{d-1})|} + h_d\left(1 - \frac{1}{l_d}\gamma(a)\right)\right| \\ &\le 2l_d\frac{|(h_1, \ldots, h_{d-1})|}{|(l_1, \ldots, l_{d-1})|} + 2|h_d|. \end{aligned} \tag{3.48}$$

Denote $l = \frac{l_d}{|(l_1, \ldots, l_{d-1})|}$. By Young's inequality, we know $(a+b)^2 \le 2(a^2 + b^2)$ for all $a, b \ge 0$. Then taking the Euclidean norm of $\Psi'(a)h$, we have

$$\begin{aligned} \left|\Psi'(a)h\right| &= \sqrt{h_1^2 + \cdots + h_{d-1}^2 + ((\Psi'(a)h)_d)^2} \\ &\le \sqrt{h_1^2 + \cdots + h_{d-1}^2 + (2l|(h_1, \ldots, h_{d-1})| + 2|h_d|)^2} \\ &\le \sqrt{(1 + 8l^2)|(h_1, \ldots, h_{d-1})|^2 + 8h_d^2} \le \max\{\sqrt{1 + 8l^2}, 2\sqrt{2}\}|h|. \end{aligned} \tag{3.49}$$

It remains to find a lower bound for $\left|\Psi'(a)h\right|$. By Theorem 3.2.2, this is equivalent to finding an upper bound for $\left|\Phi'(a)h\right|$. Then we have for $a \in D$

$$\Phi'(a) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \frac{l_d(a_d - l_d)}{(l_d - \gamma(a))^2}\partial_1\gamma(a) & \frac{l_d(a_d - l_d)}{(l_d - \gamma(a))^2}\partial_2\gamma(a) & \ldots & \frac{l_d(a_d - l_d)}{(l_d - \gamma(a))^2}\partial_{d-1}\gamma(a) & \frac{l_d}{l_d - \gamma(a)} \end{bmatrix}. \tag{3.50}$$

Then for any $h \in \mathbb{R}^d$, we have

$$\Phi'(a)h = \begin{bmatrix} h_1 \\ \vdots \\ h_{d-1} \\ \frac{l_d(a_d - l_d)}{(l_d - \gamma(a))^2}\left(h_1\partial_1\gamma(a) + \cdots + h_{d-1}\partial_{d-1}\gamma(a)\right) + h_d\frac{l_d}{l_d - \gamma(a)} \end{bmatrix}. \tag{3.51}$$

Suppose $\gamma(a) \le 0$, then because $\gamma(x) < l_d$

$$\frac{l_d^2}{(l_d - \gamma(a))^2} \le \frac{(l_d - \gamma(a))^2}{(l_d - \gamma(a))^2} = 1. \tag{3.52}$$

Similarly, if $\gamma(a) \ge 0$, then

$$\frac{l_d^2}{(l_d - \gamma(a))^2} \le \frac{l_d^2}{l_d^2} = 1. \tag{3.53}$$

Combining (3.52) and (3.53), we find that $\left(\frac{l_d}{l_d-\gamma(a)}\right)^2 \leq 1$. Then zooming in on the dth coordinate, we further find by the triangle inequality and (3.35) that

$$\begin{aligned}
\left|(\Phi'(a))_d\right| &= \left|\frac{l_d(a_d-l_d)}{(l_d-\gamma(a))^2}\left(h_1\partial_1\gamma(a)+\cdots+h_{d-1}\partial_{d-1}\gamma(a)\right)+h_d\frac{l_d}{l_d-\gamma(a)}\right| \\
&\leq \left|\frac{l_d^2(a_d-l_d)}{(l_d-\gamma(a))^2}\frac{|(h_1,\ldots,h_{d-1})|}{|(l_1,\ldots,l_{d-1})|}\right|+\left|h_d\frac{l_d}{l_d-\gamma(a)}\right| \\
&\leq |a_d-l_d|\frac{|(h_1,\ldots,h_{d-1})|}{|(l_1,\ldots,l_{d-1})|}+|h_d| \\
&\leq 2l_d\frac{|(h_1,\ldots,h_{d-1})|}{|(l_1,\ldots,l_{d-1})|}+2|h_d|
\end{aligned} \tag{3.54}$$

Denote $l=\frac{l_d}{|(l_1,\ldots,l_{d-1})|}$ again. Then taking the Euclidean norm of $\Phi'(a)h$, we have again by Young's inequality

$$\begin{aligned}
\left|(\Phi'(a))_d\right| &= \sqrt{h_1^2+\cdots+h_{d-1}^2+((\Phi'(a)h)_d)^2} \\
&\leq \sqrt{h_1^2+\cdots+h_{d-1}^2+(2l|(h_1,\ldots,h_{d-1})|+2|h_d|)^2} \\
&\leq \sqrt{(1+8l^2)|(h_1,\ldots,h_{d-1})|+8h_d^2} \leq \max\{\sqrt{1+8l^2},2\sqrt{2}\}|h|.
\end{aligned} \tag{3.55}$$

Let $L=\max\{\sqrt{1+8l^2},2\sqrt{2}\}$, then Ψ has L-bound by (3.49) and (3.55). □

We now prove that $G\in\mathscr{P}_p$.

Theorem 3.3.4. *Let $\gamma:\mathbb{R}^{d-1}\to\mathbb{R}$ be a C^1-function and let G be defined as*

$$G=(0,l_1)\times\cdots\times(0,l_{d-1})\times(\gamma(x),l_d) \tag{3.56}$$

for $0<l_d<\infty$ and $l_1,\ldots,l_{d-1}$ such that $|(l_1,\ldots,l_{d-1})|\left\|\nabla\gamma\right\|_\infty<l_d$. Then $G\in\mathscr{P}_p$.

Proof. This is directly follows from the combined results of Theorem 3.1.2, Theorem 3.2.4, Theorem 3.3.2, and Lemma 3.3.3. □

3.4. Open and Bounded Domains with Smooth Boundary

In this section, we combine the results from previous sections to show that if D has a C^1-boundary, then it is indeed p-Poincare. We first prove a very important Lemma on the unions of Poincare domains.

Lemma 3.4.1. *Let D_1 and D_2 be open, bounded, and connected in $\mathbb{R}^d$. Suppose $D_1, D_2 \in \mathscr{P}_p$ with Poincare constant $\kappa_p(D_i)$ for $i = 1,2$ such that $D_1 \cap D_2 \neq \emptyset$. Then $D = D_1 \cup D_2 \in \mathscr{P}_p$ with Poincare constant*

$$\kappa_p(D) = \frac{4}{|D_1 \cap D_2|^{\frac{1}{p}}} \left(|D_1|\kappa_p(D_1)^p + |D_2|\kappa_p(D_2)^p\right)^{\frac{1}{p}}. \tag{3.57}$$

Proof. We apply Lemma 3.2.3 twice and use the monotonicity property of integrals, as well as the additive property of integrals for disjoint sets, to show that

$$\begin{aligned}
\int_D |u(x) - u_D|^p \mathrm{d}x &\leq 2^p \int_{D_1 \cup D_2} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x \\
&= 2^p \left(\int_{D_1} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x + \int_{D_2 \setminus D_1} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x \right) \\
&\leq 2^p \left(\int_{D_1} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x + \int_{D_2} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x \right) \\
&= 2^p \sum_{i=1}^{2} \int_{D_i} \left|u(x) - u_{D_1 \cap D_2}\right|^p \mathrm{d}x \\
&\leq 2^p \sum_{i=1}^{2} 2^p \frac{|D_i|}{|D_1 \cap D_2|} \int_{D_1 \cup D_2} \left|u(x) - u_{D_i}\right|^p \mathrm{d}x \\
&\leq \frac{2^{2p}}{|D_1 \cap D_2|} \sum_{i=1}^{2} |D_i| \int_{D_i} \left|u(x) - u_{D_i}\right|^p \mathrm{d}x.
\end{aligned} \tag{3.58}$$

Then by the assumption

$$\begin{aligned}
\|u - u_D\|_{L^p(D)} &\leq \frac{4}{|D_1 \cap D_2|^{\frac{1}{p}}} \left(\sum_{i=1}^{2} |D_i| \left\|u - u_{D_i}\right\|^p_{L^p(D_i)} \right)^{\frac{1}{p}} \\
&\leq \frac{4}{|D_1 \cap D_2|^{\frac{1}{p}}} \left(\sum_{i=1}^{2} |D_i| \kappa_p(D_i)^p \|\nabla u\|^p_{L^p(D_i)} \right)^{\frac{1}{p}} \\
&\leq \frac{4}{|D_1 \cap D_2|^{\frac{1}{p}}} \left(\sum_{i=1}^{2} |D_i| \kappa_p(D_i)^p \|\nabla u\|^p_{L^p(D)} \right)^{\frac{1}{p}} \\
&= \frac{4}{|D_1 \cap D_2|^{\frac{1}{p}}} \left(\sum_{i=1}^{2} |D_i| \kappa_p(D_i)^p \right)^{\frac{1}{p}} \|\nabla u\|_{L^p(D)}
\end{aligned} \tag{3.59}$$

as desired. □

With the Poincare constant explicitly written out, it is easy to see why we require there to be some overlap between two domains in order for their union to also be Poincare. If $D_1 \cap D_2 = \emptyset$, then $|D_1 \cap D_2| = 0$ and we divide by zero which we clearly cannot do.

Notice also that $A_1 \cap A_2 \neq \emptyset$ does not necessarily imply $|A_1 \cap A_2| \neq 0$ for arbitrary sets $A_1, A_2 \in \mathbb{R}^d$. For example, the Cantor set is a well known set with infinite points but Lebesgue measure equal to 0. However, for our case, a nonempty intersection does in fact imply nonzero Lebesgue measure. This is because D_1 and D_2 are open, hence their intersection is open.

If $x \in D_1 \cap D_2 \neq \emptyset$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset D_1 \cap D_2$. Then this open ball has nonzero Lebesgue measure, hence $|D_1 \cap D_2| > 0$ and it is sufficient to show that the intersection between two open, bounded, and connected sets, which are both in $\mathscr{P}_p$, is nonempty in order to be able to apply Lemma 3.4.1.

For $x, y \in \mathbb{R}^d$, we write $\rho(x, y)$ to mean $\rho(x, y) = \left|x - y\right|$, sometimes called the Euclidean distance, for the following proof.

Theorem 3.4.2. *Let D have a C^1-boundary. Then $D \in \mathscr{P}_p$.*

Proof. For all $x \in D$, let Ω_x be an open rectangle generated around x. We want to contain Ω_x in D, hence we define it as

$$\Omega_x = \big(x_1 - \rho(x,\partial D), x_1 + \rho(x,\partial D)\big) \times \cdots \times (x_d - \rho(x,\partial D), x_d + \rho(x,\partial D)). \tag{3.60}$$

We similarly define $\frac{\Omega_x}{2}$, also generated by $x \in D$, as

$$\frac{\Omega_x}{2} = \left(x_1 - \frac{\rho(x,\partial D)}{2}, x_1 + \frac{\rho(x,\partial D)}{2}\right) \times \cdots \times \left(x_d - \frac{\rho(x,\partial D)}{2}, x_d + \frac{\rho(x,\partial D)}{2}\right). \tag{3.61}$$

Then by Theorem 3.1.2, $\Omega_x \in \mathscr{P}_p$ and $\frac{\Omega_x}{2} \in \mathscr{P}_p$ for all $x \in D$.

For all $y \in \partial D$, let $\gamma_y : \mathbb{R}^{d-1} \to \mathbb{R}$ be a C^1-function such that, after relabeling and reorienting the axes if necessary, $G_y \cap D = \{z \in G_y : z_d > \gamma_y(z)\}$ as in Definition 2.1.2. Then further moving G_y in space so that y sits on the origin, we define

$$G_y = (-k_1, k_1) \times \cdots \times (-k_{d-1}, k_{d-1}) \times (\gamma_y(y), k_d) \tag{3.62}$$

with $0 < k_d < \infty$ and $k_1, \ldots, k_{d-1}$ such that $|(2k_1, \ldots, 2k_{d-1})| \|\gamma_y\|_\infty \leq k_d$. We similarly define $\frac{G_y}{2}$, also an open rectangle generated by $y \in \partial D$, as

$$\frac{G_y}{2} = (-k_1', k_1') \times \cdots \times (-k_{d-1}', k_{d-1}') \times \left(\gamma_y(y), \frac{k_d}{2}\right) \tag{3.63}$$

with $k_1', \ldots, k_{d-1}'$ such that $k_i' < k_i$ for all $1 \leq i \leq d$ and $\left|(2k_1', \ldots, 2k_{d-1}')\right| \|\gamma_y\|_\infty \leq \frac{k_d}{2}$. Note that G_y and $\frac{G_y}{2}$ both contain y and that by Theorem 3.3.4, $G_y \in \mathscr{P}_p$ and $\frac{G_y}{2} \in \mathscr{P}_p$ for all $y \in \partial D$.

Let $\mathscr{C} = \{x \in D : \frac{\Omega_x}{2}\} \cup \{y \in \partial D : \frac{G_y}{2}\}$, then $\mathscr{C}$ is an open covering of $D \cup \partial D$. We know $D \cup \partial D$ compact by Lemma 2.1.6, hence, by Theorem 2.1.4, $\mathscr{C}$ admits a finite sub-cover. That is, we can find finitely many $x_1, \ldots, x_n \in D$ and $y_1 \ldots, y_m \in \partial D$ such that

$$D \subset D \cup \partial D \subset \left(\cup_{i=1}^n \frac{\Omega_{x_i}}{2}\right) \cup \left(\cup_{j=1}^m \frac{G_{y_j}}{2}\right) \subset \left(\cup_{i=1}^n \Omega_{x_i}\right) \cup \left(\cup_{j=1}^m G_{y_j}\right). \tag{3.64}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m$, let

$$\mathscr{K} = \{x_i \in D : \Omega_{x_i}\} \cup \{y_j \in \partial D : G_{y_j}\} \text{ and } \frac{\mathscr{K}}{2} = \left\{x_i \in D : \frac{\Omega_{x_i}}{2}\right\} \cup \left\{y_j \in \partial D : \frac{G_{y_j}}{2}\right\}. \tag{3.65}$$

Since $\mathscr{K}$ is a finite, open covering of D, it remains to show that we cannot write it as $\mathscr{K} = \mathscr{F} \cup \mathscr{F}^*$ with $F \cap F^* = \emptyset$ for all $F \in \mathscr{F}$ and $F \in \mathscr{F}^*$, where we define $\mathscr{F}$ and $\mathscr{F}^*$ as, after reordering the indices if necessary,

$$\mathscr{F} = \{x_i \in D, 1 \leq i \leq n' : \Omega_{x_i}\} \cup \{y_j \in D, 1 \leq j \leq m' : G_{y_j}\} \tag{3.66}$$

and

$$\mathscr{F}^* = \{x_i \in D, 1 \leq i \leq n^* : \Omega_{x_i}\} \cup \{y_j \in D, 1 \leq j \leq m^* : G_{y_j}\}. \tag{3.67}$$

with $n' + n^* = n$ and $m' + m^* = m$. Suppose that we can write $\mathscr{K} = \mathscr{F} \cup \mathscr{F}^*$ with $F \cap F^* = \emptyset$ for all $F \in \mathscr{F}$ and $F \in \mathscr{F}^*$.

We claim that for any $F \in \mathscr{F}$, and therefore any $F^* \in \mathscr{F}^*$, that if $A \subset D \cup \partial D$ and $\frac{F}{2} \cap A = \emptyset$, we have $\rho(F, A) < \rho\left(\frac{F}{2}, A\right)$. This directly follows from $\frac{F}{2} \subsetneq F$ for all $F \in \mathscr{F}$ and the way in which we have define G_y with respect to the location of $D \cup \partial D$.

Let $\epsilon > 0$. We claim that because $\mathscr{K}$ is an open covering of $D \cup \partial D$ and D is connected, we can find some $F \in \mathscr{F}$ and $F^* \in \mathscr{F}^*$ such that $\rho(F, F^*) < \epsilon$. Suppose we cannot, i.e. there exists some $\epsilon > 0$ such that $\rho(F, F^*) \geq \epsilon$ for all $F \in \mathscr{F}$ and $F^* \in \mathscr{F}^*$. Then, because $D \cup \partial D$ is connected and the way by which we have constructed the sets of our open covering, there must be some $z \in D \cup \partial D$ such that $0 < \rho(z, F) + \rho(z, F^*) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. In other words, there is some $z \in D \cup \partial D$ which is located in the space between $\mathscr{F}$ and $\mathscr{F}^*$ and which is not in F or F^* for all $F \in \mathscr{F}$ and $F^* \in \mathscr{F}^*$. Hence $\mathscr{F} \cup \mathscr{F}^*$ is not an open covering of $D \cup \partial D$ which is a contradiction.

Suppose then that $\rho(z, F^*) \leq 2\rho(F, F^*)$ for some $z \in F$. If $z \in \frac{F}{2}$, then $\rho\left(\frac{F}{2}, F^*\right) \leq \rho(z, F^*) \leq \rho(z, F) + \rho(F, F^*) = \rho(F, F^*)$, but we must have $\rho(F, F^*) < \rho\left(\frac{F}{2}, F^*\right)$. Then we have a contradiction by the assumption on z. Hence

$z \in F \setminus \frac{F}{2}$. But since $\frac{\mathcal{K}}{2}$ is an open covering of $D \cup \partial D$, we must have that $z \in \frac{F_1}{2}$ for some $F_1 \in \mathcal{F}$. Then $\rho\left(\frac{F_1}{2}, F^*\right) \leq \rho(z, F^*) \leq \rho(z, F_1) + \rho(F_1, F^*) = \rho(F_1, F^*)$ which is a contradiction because we must have $\rho(F_1, F^*) < \rho\left(\frac{F_1}{2}, F^*\right)$. Hence $F \cap F^* \neq \emptyset$.

We have now shown we cannot write $\mathcal{K} = \mathcal{F} \cup \mathcal{F}^*$ with $F \cap F^* = \emptyset$ for all $F \in \mathcal{F}$ and $F \in \mathcal{F}^*$. Additionally, if $x \in K_1 \cap K_2 \neq \emptyset$ for some $K_1, K_2 \in \mathcal{K}$, then, because K_1 and K_2 open, there exists some $r > 0$ such that $B_r(x) \subset K_1 \cap K_2$. Hence $K_1 \cap K_2$ has positive Lebesgue measure. We have also shown that $K \in \mathcal{P}_p$ for all $K \in \mathcal{K}$. Then $\cup_{K \in \mathcal{K}} K \in \mathcal{P}_p$ by Lemma 3.4.1 which gives finally $D \in \mathcal{P}_p$.

□

3.5. Extending to Sobolev Spaces

Define $C_c^1(D)$ as the subspace of all $C^1(D)$-functions which have compact support. We define the support of a function f as the set

$$\operatorname{supp}(f) = \overline{\{x \in D : f(x) \neq 0\}}, \tag{3.68}$$

the closure of all points on which $f(x) \neq 0$. We say f has compact support if there exists a compact set $K \subset D$ such that $\{x \in D : f(x) \neq 0\} \subseteq K$; the set of all points on which $f \neq 0$ is contained in some compact set. Intuitively, we say f has compact support on a set D if it is zero on all points outside of some compact set $K \subset D$.

Denote $L^1_{\text{loc}}(D)$ as the space of all measurable functions $f : D \to \mathbb{R}^d$ for which $f : U \to \mathbb{R}^d$ is integrable for U any open set with compact closure contained in D. We call such functions locally integrable.

Then for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we define the order of α as $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We also denote $\partial^\alpha = \partial_1^{\alpha_1} \circ \cdots \circ \partial_d^{\alpha_d}$. For our purposes, we are only interested in $|\alpha| \leq 1$; if $|\alpha| = 0$, then $\partial^\alpha f = f$ and if $|\alpha| = 1$, then we have $\alpha_i = 1$ for some $1 \leq i \leq d$ and $\partial^\alpha f = \partial_i f$.

Definition 3.5.1. Let $f \in L^1_{\text{loc}}(D)$. A function $g \in L^1_{\text{loc}}(D)$ is said to be weak derivative of order $|\alpha| \in \mathbb{N}^d$ of f if for all $h \in C_c^1(D)$ we have

$$\int_D f(x)\partial^\alpha h(x)\mathrm{d}x = (-1)^{|\alpha|} \int_D g(x)h(x)\mathrm{d}x. \tag{3.69}$$

We then call f weakly differentiable if it has weak derivatives $\partial^\alpha f \in L^1_{\text{loc}}(D)$ for $|\alpha| \leq 1$.

We can use this definition of weak differentiation to define a new space of functions.

Definition 3.5.2. Let $1 \leq p < \infty$. The Sobolev space $W^{1,p}(D)$ is defined as the space of all functions $f \in L^p(D)$ which are weakly differentiable with $\partial^\alpha f \in L^p(D)$ for $\alpha \leq 1$. We equip this space with the norm

$$\|f\|_{W^{1,p}(D)} = \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_{L^p(D)} = \|f\|_{L^p(D)} + \|\nabla f\|_{L^p(D)}. \tag{3.70}$$

We have so far been working with $u \in C^1(\overline{D})$. However, we can use the previous sections as a foundation and extend our function space to $W^{1,p}(D)$. For this we need to first prove that $C^1(\overline{D})$ is dense in $W^{1,p}(D)$. We give an equivalent theorem without proof.

Theorem 3.5.1. *For any $f \in W^{1,p}(D)$, there exists a sequence of functions $(f)_{n \geq 1} \in C^1(\mathbb{R}^d)$ which, when we restrict their domain to D, satisfy* $\lim_{n\to\infty} \|f - f_n\|_{W^{1,p}(D)} = 0$.

For a proof, see [10, p. 367]. Recall that for (M, ρ) a metric space, a set $A \subset M$ is said to be dense in M if every point in M is a limit of a sequence from A, so the density of $C^1(\overline{D})$ in $W^{1,p}(D)$ directly follows from Theorem 3.5.1. We can rewrite the limit in Theorem 3.5.1 to a more useful form. Let $\epsilon > 0$. Then for all $f \in W^{1,p}(D)$ there exists some $g \in C^1(\overline{D})$ such that $\|f - g\|_{L^p(D)} < \epsilon$ and $\|\nabla(f - g)\|_{L^p(D)} < \epsilon$.

Theorem 3.5.2. *Suppose $D \in \mathscr{P}_p$ with Poincare constant $\kappa_p(D)$. Then for all $f \in W^{1,p}(D)$*

$$\|f - f_D\|_{L^p(D)} \leq 2\kappa_p(D)\|\nabla f\|_{L^p(D)}. \tag{3.71}$$

Proof. Let $\epsilon > 0$. Then by Theorem 3.5.1, we can find some $g \in C^1(D)$ such that

$$\|f - g\|_{L^p(D)} < \frac{\epsilon}{2} \tag{3.72}$$

and

$$\|\nabla(f - g)\|_{L^p(D)} < \frac{\epsilon}{2}. \tag{3.73}$$

We have by Minkowski's inequality

$$\begin{aligned}\left(\int_D |f - f_D|^p \mathrm{d}x\right)^{\frac{1}{p}} &= \left(\int_D |f - g + g - g_D + g_D - f_D|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \left(\int_D |f - g|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_D |g - g_D|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_D |g_D - f_D|^p \mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned} \tag{3.74}$$

We treat the first and third integral separately. By (3.72), we have for the first integral

$$\left(\int_D |f-g|^p \mathrm{d}x\right)^{\frac{1}{p}} = \|f-g\|_{L^p(D)} \leq \frac{\epsilon}{2}. \tag{3.75}$$

We can rewrite our condition for q to $\frac{p}{q} = p-1$ for clarity. Then by Hölder's inequality, we have for the third integral

$$\begin{aligned}
\int_D |f_D - g_D|^p \mathrm{d}x &= \int_D \left|\frac{1}{|D|}\int_D f(s)\mathrm{d}s - g_D\right|^p \mathrm{d}x = \frac{1}{|D|^{p-1}}\left|\int_D f(s) - g_D \mathrm{d}s\right|^p \\
&\leq \frac{1}{|D|^{\frac{p}{q}}}\left(\int_D |f(s)-g_D|\mathrm{d}s\right)^p \\
&\leq \frac{1}{|D|^{\frac{p}{q}}}\left(\left(\int_D |f(s)-g_D|^p \mathrm{d}s\right)^{\frac{1}{p}}\left(\int_D 1\mathrm{d}s\right)^{\frac{1}{q}}\right)^p \\
&= \int_D |f(s)-g_D|^p \mathrm{d}s = \int_D |f(s)-g(s)+g(s)-g_D|^p \mathrm{d}s.
\end{aligned} \tag{3.76}$$

Then by Minkowski's inequality and again (3.72)

$$\begin{aligned}
\left(\int_D |f_D - g_D|^p \mathrm{d}x\right)^{\frac{1}{p}} &\leq \left(\int_D |f(s)-g(s)+g(s)-g_D|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
&\leq \left(\int_D |f(s)-g(s)|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_D |g(s)-g_D|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
&\leq \frac{\epsilon}{2} + \left(\int_D |g(s)-g_D|^p \mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.77}$$

Combining (3.75) and (3.77), we have by the assumption on D

$$\left(\int_D |f-f_D|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \epsilon + 2\left(\int_D |g-g_D|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \epsilon + 2\kappa_p(D)\left(\int_D |\nabla g|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{3.78}$$

Let us now treat the integral separately for brevity. Then we have again by Minkowski's inequality

$$\begin{aligned}
\left(\int_D |\nabla g|^p \mathrm{d}x\right)^{\frac{1}{p}} &\leq \left(\int_D |\nabla g - \nabla f + \nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\int_D |\nabla g - \nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_D |\nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.79}$$

Then by (3.73)

$$\left(\int_D |\nabla g - \nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}} = \|\nabla g - \nabla f\|_{L^p(S)} = \|\nabla(f-g)\|_{L^p(S)} < \frac{\epsilon}{2}. \tag{3.80}$$

Combining everything, we have

$$\begin{aligned}
\|f-f_D\|_{L^p(D)} &= \left(\int_D |f-f_D|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \epsilon + 2\kappa_p(D)\left(\frac{\epsilon}{2} + \left(\int_D |\nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}}\right) \\
&= \epsilon(1+\kappa_p(D)) + 2\kappa_p(D)\|\nabla f\|_{L^p(D)} \\
&\leq 2\kappa_p(D)\|\nabla f\|_{L^p(D)}
\end{aligned} \tag{3.81}$$

as desired. □

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Bibliography

[1] Acosta, G. & Durán, R. G. (2004). An Optimal Poincaré Inequality in L^1 for Convex Domains. Proceedings of the American Mathematical Society, 132(1), 195–202. http://www.jstor.org/stable/1193994

[2] Carothers, N. L. (2000). Real Analysis. Cambridge: Cambridge University Press.

[3] De Guzman, M. (1980). A Change-of-Variables Formula Without Continuity. The American Mathematical Monthly, 87(9), 736–739. https://doi.org/10.2307/2321865

[4] Evans, L. C. (1998). Partial differential equations. Evans. American Mathematical Society.

[5] Farwig, R. & Rosteck, V. (2016). Note on Friedrichs' inequality in N-star-shaped domains. Journal of Mathematical Analysis and Applications, 435(2), 1514–1524. https://doi.org/10.1016/j.jmaa.2015.10.046

[6] Hurri, R. (1988). Poincare domains in $\mathbb{R}^n$. Annales Academiae Scientiarum Fennicae. Mathematica Dissertationes, 1988(71).

[7] Jacod, J. & Protter, P. (2004). Probability essentials. Springer Berlin Heidelberg.

[8] Krylov, N. V. (2008). Lectures on elliptic and parabolic equations in Sobolev spaces. American Mathematical Society.

[9] Maz'ya, V. (2011). Sobolev spaces with applications to elliptic partial differential equations. Springer.

[10] van Neerven, J. (2022). Functional Analysis. Cambridge University Press.

[11] Rudin, W. (1986). Real and complex analysis. McGraw Hill.

[12] Rudin, W. (1991). Functional analysis. McGraw-Hill.

[13] Schwartz, J. (1954). The Formula for Change in Variables in a Multiple Integral. The American Mathematical Monthly, 61(2), 81–85. https://doi.org/10.2307/2307790

[14] Spivak, M. (1965). Calculus on Manifolds. Addison-Wesley Publishing Company.

[15] Tao, T. (2011). An introduction to measure theory. American Mathematical Society.