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Maximal Operators Defined by Rearrangement Invariant Banach Function Spaces

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"Maximal Operators Defined by Rearrangement Invariant Banach Function Spaces"

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Abstract

In this thesis we study the boundedness of a generalization of the Hardy-Littlewood maximal operator, involving rearrangement invariant Banach function space and indices of the spaces. We first consider a classical proof of boundedness of the Hardy-Littlewood maximal operator on rearrangement invariant Banach function spaces. After establishing necessary and sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator, we consider a generalization of the Hardy-Littlewood maximal operator introduced by C. Pérez. We investigate and slightly improve the known sufficient conditions under which this more general maximal operator is bounded on a rearrangement invariant Banach function space. After which we search and find necessary conditions for boundedness in a general setting. In the final section we study Boyd indices and fundamental indices, especially how they are related to boundedness of the more general maximal operator. We also introduce weak fundamental indices and investigate some of their properties and uses. Finally we show how under certain assumptions we can state equivalent necessary and sufficient conditions for boundedness on Lorentz spaces $L^{p,q}$ and Orlicz spaces L^{Ψ}.

Introduction

The Hardy-Littlewood maximal operator is a classical operator used in real and harmonic analysis. We know a lot about the classical operator, for example its boundedness from L^1 into weak L^1, or the fact that it is bounded on L^p for $1 < p \leq \infty$. A well known proof by G.G. Lorentz & T. Shimogaki states sufficient and necessary conditions for boundedness on rearrangement invariant Banach function spaces. In Chapter 2 we will investigate this proof for a better understanding of the Hardy-Littlewood maximal operator.

There are a lot of generalizations of the Hardy-Littlewood maximal operator, like the fractional maximal operator or a maximal operator based on a general set function [6]. One such generalization is a maximal operator based on different rearrangement invariant norms, studied in [8]. They give sufficient conditions for boundedness on rearrangement invariant Banach function spaces. The result yields a condition for boundedness on weighted L^p spaces. In this paper we investigate this generalization further, in hopes of gaining a better understanding and finding more results.

Boyd indices prove useful in the theorem for boundedness of the Hardy-Littlewood maximal operator. They will also be useful when studying the more general maximal operator. They will however not be enough. Another index of rearrangement invariant space is the fundamental index, which was shown to not always be equal to the Boyd indices by T.Shimogaki. These indices not only help us state the conditions for boundedness in a more readable fashion, they also improve the intuitive understanding of the results. We will state the sufficient and necessary conditions for boundedness of the general maximal operator in terms of Boyd, fundamental and weak fundamental indices. The weak fundamental indices are introduced in section 2.3, where we discuss and explore their properties.

Contents

Chapter 1

Preliminaries

Notation

- Let (R, μ) be a measure space.
- $L(R, \mu)$ is the set of μ-measureable functions.
- $L^0(R, \mu) \subseteq L(R, \mu)$ is a set of μ-a.e. finite functions.
- For any function space L containing functions mapping into $[-\infty, \infty]$, we denote the subset of functions mapping into $[0, \infty]$ as $L^+ \subseteq L$.
- We use $\mathbb{R}^+$ as $(0, \infty)$.
- We use $\mathscr{B}(X, Y)$ to denote the set of operators bounded from X into Y, for convenience we write $\mathscr{B}(X, X) = \mathscr{B}(X)$.
- We use $\int_0^\infty f^*(s)\mathrm{d}\varphi(s) = \lim_{s\downarrow 0} f^*(s)\varphi(s) + \int_0^\infty f^*(s)\varphi'(s)\mathrm{d}s$ for a Riemann-Stieltjes integral.
- We say a function $f : R \to \mathbb{R}$ is locally in a Banach function space X when for all compact sets $E \subseteq R$, $f\chi_E$ is in X. We denote the collection of function that are locally in X by X_{loc}. Additionally, we say f is locally integrable if $f \in L^1_{loc}$.

In this chapter we first introduce the reader to Banach function spaces, discussing under which assumptions we work in this paper, and stating some fundamental theorems. In second section we introduce the reader to rearrangements and rearrangement invariant Banach function spaces. We investigate some useful tools like the elementary maximal operator, the Luxemburg representation theorem and the fundamental function. Along the way we also give examples of rearrangement invariant Banach function spaces.

1.1 Banach function spaces

Definition 1.1.1. *Suppose $\|\cdot\|_X$ is a norm and let*

$$X = \{f \in L^0(R,\mu) : \|f\|_X < \infty\}$$

Then $(X, \|\cdot\|_X)$, or just X, is called a Banach function space, if the following properties hold for all $f, g, f_n, (n = 1, 2, \ldots)$, in $L^0(R,\mu)$ and all measurable subsets E of R.

(i) (the lattice property) If $|g| \leq |f|$ μ-a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.

(ii) (the Fatou property) Suppose $f_n \in X, f_n \geq 0, (n = 1, 2, \ldots)$, and $f_n \uparrow f$ μ -a.e. If $f \in X$, then $\|f_n\|_X \uparrow \|f\|_X$ whereas if $f \notin X$, then $\|f_n\|_X \uparrow \infty$.

(iii) Every indicator function χ_E of a set E of finite measure belongs to X.

(iv) To each set E of finite measure there corresponds a constant $0 < C_E < \infty$ such that

$$\int_E |f| d\mu \leq C_E \|f\|_X$$

Note that, if we were to replace $L^0(R,\mu)$ with $L(R,\mu)$, we would find that $X \subseteq L^0$ (see for example [1, Lemma I.1.4]). Intuitively this makes sense, as we are only excluding f that are valued ∞ on positive measure sets. From Fatou's property, we get

Corollary 1.1.2 (Fatou's lemma). *If $f_n \in X, (n = 1, 2, \ldots), f_n \to f$ μ-a.e. and $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$.*

Fatou's lemma is then used to show that X is complete ([1, Theorem I.1.6]).

Example 1.1.3. Some familiar Banach function space are the Lebesgue spaces associated with $1 \leq p \leq \infty$, denoted by L^p. Let

$$\|f\|_{L^p} = \begin{cases} \left(\int_R |f|^p d\mu\right)^{1/p}, & (1 \leq p < \infty) \\ \operatorname*{ess\,sup}_R |f|, & (p = \infty) \end{cases} \qquad f \in L^0(R, \mu)$$

Indeed, property (i) follows from the linearity of the integral, (ii) from the monotone convergence theorem, (ii) follows from the fact that charactaristic functions on finite measure space have a finite integral and (iv) follows from Hölder's inequality.

Before we continue with rearrangements, it is useful to first consider some properties of general Banach function space. We first look at the associate space of a Banach function space for a useful representation of the norm. Later, the associate space will be useful in getting a better understanding of fundamental functions, which can tell us a lot about the Banach function spaces they belong to (see Definition 1.2.13).

Definition 1.1.4. *Let X be a Banach function space. The associate space of X, denoted by X', is also a Banach function space. Its norm is given by:*

$$\|g\|_{X'} = \sup\left\{\int_R |fg|\mathrm{d}\mu : f \in X, \|f\|_X \leq 1\right\}$$

In the case of L^p spaces, when $1 \leq p < \infty$, $(L^p)' = L^{p'}$ is the associate space, with p' satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. A classical inequality that goes hand in hand with the definition of the associate space is Hölders inequality:

$$\int_R |fg|\mathrm{d}\mu \leq \|f\|_X \|g\|_{X'}$$

Though Hölder's inequality is very useful and well known in analysis, we won't be using it all too often in this paper.

Theorem 1.1.5. *Every Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'', and in that case* $\|f\|_X = \|f\|_{X''}$

For a proof, see [1, Theorem I.2.7]. This gives us

$$\|f\|_X = \sup\left\{\int_R |fg|\mathrm{d}\mu : g \in X, \|g\|_{X'} \leq 1\right\}$$

which is a useful representation, especially for proving the Luxemburg representation theorem. Finally, we consider separability and the absolutely continuity of the norm, which is a useful property, as it implies density of the simple functions in some classes of function space.

Definition 1.1.6. *A Banach function space X is said to have absolutely continuous norm if for all $f \in X$, $\|f_n\|_X \to 0$ for every sequence $\{f_n\}_{n=1}^{\infty}$ satisfying $f_n \downarrow 0$ μ-a.e.*

Example 1.1.7. A familiar space that does not have an absolutely continuous norm is L^∞. Indeed, we see that for $f_n = \chi_{E_n}$, where E_n has positive but finite measure for all n, such that $f_n \downarrow 0$ μ-a.e. Then we have $\|\chi_{E_n}\|_{L^\infty} = 1$ for all n.

We call a Banach space X *separable* when it contains a countable subset that is dense. For Banach function spaces defined on a separable measure, separability coincides with having an absolutely continuous norm (for proof, see [1, Corollary I.5.6]). Note that for our purposes, only the first condition will matter, as we only use separability when discussing spaces defined on subspaces of $\mathbb{R}^d$ with the Lebesgue measure, which are indeed separable.

Theorem 1.1.8. *Let X be a Banach function space on a separable measure space (R, μ). If X is separable, then the simple functions are dense in X. That is, for all $f \in X$ and $\varepsilon > 0$ there exists a simple function $g \in X$ such that $\|f - g\|_X < \varepsilon$.*

This is a result of [1, Theorem I.3.11] and the fact that X has an absolute continuous norm when it is separable. Going back to example 1.1.7, we see that indeed a function like $f = \chi_R$ with R of infinite measure. Then if f_n is a sequence of simple functions f_n, we have $\mu(\{f_n > 0\}) < \infty$. Then $\|f_n - f\|_{L^\infty} \geq 1$, thus there does not exist a sequence of simple functions that approaches f in L^∞.

1.2 Rearrangement invariant Banach function spaces

Now we are finally ready to introduce rearrangements, rearrangement invariant norms and their properties. From this point on we will assume (R, μ) to be a σ-finite measure space. We say a measure space is σ-finite when all element of its σ-algebra, including R itself, can be represented as a countable union of sets of finite measure. The following definition will be useful for making “rearranging” a function a more rigorous process.

Definition 1.2.1. *The distribution function μ_f of a function f in $L^0(R, \mu)$ is given by*

$$\mu_f(\lambda) = \mu\big(\{x \in R : |f(x)| > \lambda\}\big), \quad (\lambda \geq 0)$$

Note that μ_f only depends on $|f|$, similar to norms of Banach function spaces. Observe that we do allow for μ_f to be $+\infty$, as we are working in a σ-finite measure space rather than a finite one.

Definition 1.2.2. *Two functions $f \in L^0(R, \mu)$ and $g \in L^0(S, \nu)$ are said to be equimeasurable if they have the same distribution functions, i.e.*

$$\mu_f(\lambda) = \nu_g(\lambda), \quad (\lambda \geq 0)$$

Note that we require equality for all $\lambda \geq 0$, but due to the right continuity of the distribution this is equivalent to being equal a.e. To better understand the distribution of a function we introduce some basic properties in the following proposition:

Proposition 1.2.3. *Suppose $f, g, f_n, (n = 1, 2, \dots)$, belong to $L^0(\mathbb{R}, \mu)$, let $A, B \subseteq R$ be disjoint and let $a \in \mathbb{R} \setminus \{0\}$. The following hold:*

$$\mu_f \text{ is non-negative, decreasing and right continuous on } [0, \infty)$$

$$|g| \leq |f| \text{ a.e.} \implies \quad \mu_g \leq \mu_f$$

$$\mu_{af}(\lambda) = \mu_f\left(\frac{\lambda}{|a|}\right), \quad (\lambda \geq 0)$$

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2), \quad (\lambda_1, \lambda_2 \geq 0)$$

$$|f| \leq \liminf_{n\to\infty} |f_n| \ \mu\text{-a.e.} \implies \quad \mu_f \leq \liminf_{n\to\infty} \mu_{f_n} \tag{1.1}$$

$$\mu_{f\chi_{A\cup B}}(\lambda) = \mu_{f\chi_A}(\lambda) + \mu_{f\chi_B}(\lambda) \quad (\lambda \geq 0) \tag{1.2}$$

We only give a proof for (1.2), for the other properties a proof is given in [1, Proposition II.1.3].

Proof.

$$\begin{aligned}
\mu_{f\chi_{A\cup B}}(\lambda) =& \mu(\{x \in R : |f\chi_{A\cup B}| > \lambda\}) \\
=& \mu(\{x \in R : |f\chi_A| + |f\chi_B| > \lambda\}) \\
=& \mu(\{x \in A : |f(x)| > \lambda\} \cup \{x \in B : |f(x)| > \lambda\}) \\
=& \mu(\{x \in A : |f(x)| > \lambda\}) + \mu(\{x \in B : |f(x)| > \lambda\}) \\
=& \mu_{f\chi_A}(\lambda) + \mu_{f\chi_B}(\lambda)
\end{aligned}$$

□

Example 1.2.4. To understand how a distribution function works, it will help to compute the distribution function of a nonnegtive simple function f. Let

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

be such that E_i are pairwise disjoint and all a_i distinct and such that $a_1 > a_2 > \cdots > a_n > 0$. Then for $a_1 \leq \lambda$ we have $\mu_f(\lambda) = 0$, but for $a_2 \leq \lambda < a_1$ we have that $\mu_f(\lambda) = \mu(E_1)$. Similarly we find for $a_3 \leq \lambda < a_2$ that $\mu_f(\lambda) = \mu(E_1) + \mu(E_2)$. Then we have

$$\mu_f(\lambda) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \mu(E_j) \right) \chi_{[a_{i+1}, a_i)}(\lambda) = \sum_{i=1}^{n} \mu(E_i) \chi_{[0,a_i)}(\lambda)$$

where $a_{n+1} = 0$.

Definition 1.2.5. *Suppose f belongs to $L^0(R, \mu)$. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by*

$$f^*(t) = \inf \left\{ \lambda : \mu_f(\lambda) \leq t \right\}, \quad (t \geq 0) \tag{1.3}$$

We make use of the convention that $\inf \varnothing = \infty$. However, when $f \in L^0(R, \mu)$ we have that f is finite μ-a.e. Thus $f^*(t) = \infty$ can only happen for $t = 0$, when f is an element of a Banach function space. An interesting identity for equation (1.3) follows from the right continuity of μ_f, the decreasing property of μ_f and the definition of distribution functions:

$$f^*(t) = \sup\{s : \mu_f(s) > t\} = \lambda_{\mu_f}(t) \tag{1.4}$$

where λ is the Lebesgue measure.

Example 1.2.6. Let f be as in example 1.2.4, and let $m_i = \sum_{j=1}^{i} \mu(E_i)$. By definition we find $f^*(t) = 0$ when $t \geq m_n$, $f^*(t) = a_n$ when $m_n \geq t > m_{n-1}$. We find

$$f^*(t) = \sum_{i=1}^{n} a_i \chi_{[m_{i-1}, m_i)}(t).$$

This makes sense intuitively, as we now have that $f^*(t) = a_1$ for $0 \leq t < |E_1|$, etc. such that f^* is indeed a decreasing function on $\mathbb{R}^+$ such that it is equimeasurable with f.

Another interesting example to build some intuition is $f(x) = 1 - \frac{1}{t+1}$ for $t \in (0, \infty)$. Its distribution function is simple to compute: $\mu_f(\lambda) = \infty$ for $0 \leq \lambda < 1$ and $\mu_f(\lambda) = 0$ for $1 < \lambda$. Then $f^*(t) = 1$ for $t > 0$. This example shows that rearranging a function may throw away some information of a function. However, we will study exactly those spaces for which such information is irrelevant.

Some more interesting properties of rearrangements will be introduced in the next proposition.

Proposition 1.2.7. *Suppose* f, g, *and* $f_n, (n = 1, 2, \ldots)$, *belong to* $L^0(R, \mu)$, *let* a *be any scalar and let* $h(t) : [0, \infty) \to [0, \infty)$ *be a strictly increasing function that vanishes in 0. The decreasing rearrangement* f^* *is a non-negative, decreasing, right-continuous function on* $[0, \infty)$. *Furthermore*

$$|g| \leq |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad g^* \leq f^* \tag{1.5}$$

$$(af)^* = |a| f^*$$

$$(f+g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2), \quad (t_1, t_2 \geq 0) \tag{1.6}$$

$$|f| \leq \liminf_{n\to\infty} |f_n| \ \mu\text{-a.e.} \quad \Rightarrow \quad f^* \leq \liminf_{n\to\infty} f_n^* \tag{1.7}$$

in particular,

$$|f_n| \uparrow |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad f_n^* \uparrow f^* \tag{1.8}$$

$$\mu_f\left(f^*(t)\right) \leq t, \quad \left(f^*(t) < \infty\right)$$

$$f \text{ and } f^* \text{ are equimeasurable} \tag{1.9}$$

$$\left(|f|^p\right)^* = (f^*)^p, \quad (0 < p < \infty) \tag{1.10}$$

$$(fg)^*(t) \leq f^*(t) g^*(t) \tag{1.11}$$

$$\forall t \in [0, \infty), \exists \lambda \in [0, \infty) : h(t) f^*(t) \leq \lambda h(\mu_f(\lambda)) \tag{1.12}$$

Proof. Proofs for (1.5)-(1.10) can be found in [1, Proposition II.1.7], (1.11) can be found at [5, pg.67, 10°].

For (1.12), we use the identity in equation (1.4) to get

$$f^*(t) = \sup\{\lambda : \mu_f(\lambda) > t\} \tag{1.13}$$

Then assume that there exists a $t \in [0, \infty)$ such that $h(t) f^*(t) > \lambda h(\mu_f(\lambda))$ for all $\lambda \in [0, \infty)$, we will show this leads to a contradiction. we find that

$$\begin{aligned} h(t) f^*(t) &> \sup\{\lambda h(\mu_f(\lambda)) : \lambda \in [0, \infty)\} \\ &\geq \sup\{\lambda h(\mu_f(\lambda)) : \mu_f(\lambda) > t, \lambda \in [0, \infty)\} \end{aligned} \tag{1.14}$$

Now, if (1.14) is equal to 0, our assumption on h tells us that there is no $\lambda > 0$ such that $\mu_f(\lambda) > t$. By (1.13) we find that $f^*(t) = 0$, which is a contradiction. Now assume that (1.14) is not equal to 0, then we get:

$$\begin{aligned} h(t) f^*(t) &\geq \sup\{\lambda h(\mu_f(\lambda)) : \mu_f(\lambda) > t\} \\ &> \sup\{\lambda h(t) : \mu_f(\lambda) > t\} = h(t) f^*(t) \end{aligned}$$

Note that the last inequality is strict due to $h(t)$ being strictly increasing. This is again a contradiction, concluding the proof. ☐

Consider an inequality by Hardy & Littlewood,

$$\int_R |fg|\mathrm{d}\mu \leq \int_0^\infty f^*(s)g^*(s)\mathrm{d}s \tag{1.15}$$

for a proof, see [1, Theorem II.2.2]. An intuitive way to look at the inequality would be to see the difference between the two as how "resonant" two functions are. The more resonant they are, the smaller the difference. In this analogy the right side would be like a forced perfect resonance by rearranging both functions to have their peaks at the same point. This idea becomes more rigorous in the following definition:

Definition 1.2.8. *A σ-finite measure space (R, μ), is said to be resonant if, for each f, g in $L^0(R, \mu)$, the following identity holds*

$$\int_0^\infty f^*(t)g^*(t)\mathrm{d}t = \sup\left\{\int_R |f\tilde{g}|\,\mathrm{d}\mu : \tilde{g} \in L^0(R, \mu),\ \mu_g = \mu_{\tilde{g}}\right\}.$$

Trivial examples like a measure space with zero measure are clearly resonant. It is hard to see when a space in general is resonant. We will not delve into the proofs of the following results, as they are not relevant to this paper. They can be found in [1, Section II.2]. We will be using [1, Theorem II.2.7], which states: a σ-finite measure space is resonant if and only if it is one of the following two types:

(i) nonatomic;

(ii) completely atomic, with all atoms having equal measure.

An atom is a single element $x \in R$ with positive measure $\mu(\{x\}) > 0$, a nonatomic space is a space without atoms. An example of a resonant measure space which we will be using a lot, is $\mathbb{R}^d$ with the Lebesgue measure. In our pursuit for defining more general maximal function, we first introduce an elementary maximal function.

Definition 1.2.9. *Let f belong to $L^0(R, \mu)$. We call f^{**} the maximal function of f defined by*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad (t > 0)$$

We will later find that the boundedness of this maximal function is equivalent to that of the Hardy-Littlewood maximal function (see 2.1.7). Some elementary properties of $f \to f^{**}$ are the following:

Proposition 1.2.10. *Suppose f, g, and $f_n, (n = 1, 2, \cdots)$, belong to $L^0(R, \mu)$, and let a be any scalar. Then f^{**} is non-negative, decreasing and continuous on $(0, \infty)$. Furthermore, the following properties hold:*

$$\begin{aligned}
f^{**} \equiv 0 &\iff f = 0 \ \mu\text{-a.e.} \\
f^* &\leq f^{**} \\
|g| \leq |f| \ \mu\text{-a.e.} &\implies g^{**} \leq f^{**} \\
(af)^{**} &= |a| f^{**} \\
|f_n| \uparrow |f| \ \mu\text{-a.e.} &\implies f_n^{**} \uparrow f^{**}.
\end{aligned}$$

A proof is given in [1, Proposition II.3.2]. Applying the inequality in (1.15) to functions f and $g = \chi_E$ we find that

$$\frac{1}{\mu(E)} \int_E |f| \mathrm{d}\mu \leq f^{**}(\mu(E))$$

for all measurable sets E. If we have a resonant measure space we may take the supremum over sets E of equal measure to find equality:

$$\sup_{E:\mu(E)=t} \frac{1}{\mu(E)} \int_E |f| \mathrm{d}\mu = f^{**}(t)$$

For a proof, see [1, Proposition II.3.3]. Then on a resonant measure space (R, μ) we find for t in the range of μ that $f \to f^{**}$ is subadditive:

$$(f+g)^{**}(t) = \sup_{|E|=t} \frac{1}{\mu(E)} \int_E |f+g| \mathrm{d}\mu \leq \sup_{|E|=t} \frac{1}{\mu(E)} \int_E |f|+|g| \mathrm{d}\mu \leq f^{**}(t)+g^{**}(t) \tag{1.16}$$

Now that the rearrangements of functions have been well-defined and understood, we are ready to define the spaces we will use throughout this paper.

Definition 1.2.11. *Let X be a Banach function space. We say X is rearrangement invariant(r.i.), when equimeasurable functions are equal in norm. That is,*

$$\lambda \geq 0, \ \mu_g(\lambda) = \mu_f(\lambda) \quad \implies \quad \|f\|_X = \|g\|_X.$$

r.i. Banach function spaces is a very general class of Banach function spaces, including L^p, Lorentz spaces[1] $L^{p,q}$ and Orlicz spaces L^Ψ. We will

[1]Note that we use a slightly different norm from the usual norm. This is because the usual norm is actually only a quasi-norm and does thus not define a Banach function space.

introduce latter two later. From [1, Proposition II.4.2] we find that, when X is r.i., X' is also *r.i.* and the associate representation of the norm is equal to the following representation:

$$\|f\|_X = \sup\left\{\int_0^\infty f^*(s)g^*(s)\mathrm{d}s : \|g\|_{X'} \leq 1\right\}.$$

As a result, we find that on r.i. spaces that:

$$\int_R |fg|\mathrm{d}\mu \leq \int_0^\infty f^*(s)g^*(s)\mathrm{d}s \leq \|f\|_X\|g\|_{X'}$$

for $f \in X$ and $g \in X'$. The following result is the final norm representation we will discuss in this paper. It shows that every r.i. norm may be represented as an equivalent norm on $\mathbb{R}^+$. We will use this representation frequently.

Theorem 1.2.12 (Luxemburg representation theorem)**.** *Let X be a r.i. Banach function space over a resonant measure space (R,μ). Then there is a (not necessarily unique) r.i Banach function space $\bar{X}$ over $(\mathbb{R}^+,\lambda)$ such that*

$$\|f\|_X = \|f^*\|_{\bar{X}}, \quad (f \in L^0(R,\mu))$$

Furthermore, if $\bar{Y}$ is any r.i. Banach function space over $(\mathbb{R}^+,\lambda)$ which represents X, in the sense that

$$\|f\|_X = \|f^*\|_{\bar{Y}}, \quad (f \in L^0(R,\mu))$$

then the associate space X' of X is represented in the same way by the associate space $\bar{Y}'$ of $\bar{Y}$, that is,

$$\|g\|_{X'} = \|g^*\|_{\bar{Y}'}, \quad (g \in L^0(R,\mu))$$

We refer to [1, Theorem II.4.10] for a proof. An interesting result of the Luxemburg representation theorem is that r.i. spaces over a resonant measure space (R,μ) are determined by r.i. spaces over $(\mathbb{R}^+,\lambda)$. Note that, the space $\bar{X}$ is unique when restricted to $[0,\mu(R))$. Thus, X is represented uniquely if R has infinite measure, like $\mathbb{R}^d$. One could also say that X on (R,μ) is generated by $\bar{X}$. A well known representation that is of this form is of $L^p(R,\mu)$ for $1 \leq p < \infty$:

$$\int_R |f|^p\mathrm{d}\mu = \int_0^\infty (f^*(t))^p\mathrm{d}\mu = \|f^*\|_{L^p}^p \tag{1.17}$$

For a nonnegative simple function $f = \sum_{k=1}^{K} a_k \mu(E_k)$ with a_j in decreasing order, we indeed have:

$$\int_R |f|^p \mathrm{d}\mu = \sum_{k=1}^{K} a_k^p \mu(E_k) = \sum_{k=1}^{K} a_k \lambda\big([\mu(E_{k-1}), \mu(E_k)\big) = \int_0^\infty (f^*(t))^p$$

where $E_0 = \varnothing$ and λ is the Lebesgue measure. Using the monotone convergence theorem and properties (1.7), (1.1) then gives the result for all $f \in L^p(R, \mu)$. For L^∞ we have a more obvious representation:

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in R} |f(x)| = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0).$$

A useful way to characterize r.i. Banach function spaces is by how their norm acts on characteristic function. Especially because characteristic functions on sets of equal measure are equal in norm.

Definition 1.2.13. *Let X be a r.i. Banach function space on a resonant measure space (R, μ), then for t in the range of μ we have*

$$\varphi_X(t) := \|\chi_A\|_X, \quad \mu(A) = t$$

is called the fundamental function of X. The function is well defined on X as it is r.i., so when we have $A, B \subset R$ for which $\mu(A) = \mu(B)$ we get that χ_A and χ_B are equimeasurable so that $\|\chi_A\|_X = \|\chi_B\|_X$.

Some useful things we know about the fundamental function are as follows:

Proposition 1.2.14. *Let X be a r.i. Banach function space over a resonant measure space (R, μ). Then the fundamental function φ_X of X satisfies:*

$$\varphi_X(t)\varphi_{X'}(t) = t, \quad (0 < t < \infty) \tag{1.18}$$

φ_X is increasing; $\varphi_X(t) = 0$ iff $t = 0$

$$t \mapsto t^{-1}\varphi_X(t) \text{ is decreasing} \tag{1.19}$$

φ_X is continuous, except perhaps at the origin.

$$\frac{\varphi_X(t)}{t} \geq \frac{\partial}{\partial t}\varphi_X(t) \text{ for a.e. } t > 0 \tag{1.20}$$

Proof. We only prove property (1.20), for the proof of the other properties we refer to [1, Theorem II.5.2, Corollary II.5.3]. Note that φ_X is a monotone continuous function and therefore differentiable almost everywhere on $(0, \infty)$, see [10, Chapter 7]. Using property (1.19) we get

$$0 \geq \frac{\partial}{\partial t}\left[\frac{\varphi_X(t)}{t}\right] = \frac{\left(\frac{\partial}{\partial t}\varphi_X(t)\right)t - \varphi_X(t)}{t^2}$$

then since $t > 0$ this is equivalent to:

$$0 \geq \left(\frac{\partial}{\partial t}\varphi_X(t)\right)t - \varphi_X(t)$$
$$\frac{\varphi_X(t)}{t} \geq \frac{\partial}{\partial t}\varphi_X(t)$$

Note that we have the inequality for exactly the t where $\varphi_X(t)$ is differentiable, which is exactly where we will be using it.

□

Two r.i. Banach function spaces can be equivalent in the sense that they have the same functions, but not have equal norm. For example, if $\|f\|_X = 2\|f\|_Y$, then $f \in X \iff f \in Y$. Two spaces and their norms are called equivalent when the spaces contain exactly the same functions. We only say two spaces are equal when their norms are equal. Using the following lemma, we find that even though a r.i. Banach norm isn't necessarily such that its fundamental function is concave, there always exists an equivalent r.i. Banach norm such that it has a concave fundamental function. Note that we do know the fundamental is always quasi-concave [1, Corollary II.5.3].

Lemma 1.2.15. *Let X be a r.i. Banach function space over $(\mathbb{R}^+, \lambda)$. Then X can be equivalently renormed with a r.i. norm such that the resulting fundamental function is concave.*

For a proof, see [1, Proposition II.5.11]. Using The Luxemburg representation theorem, we find that any r.i. Banach function space over a resonant measure space may be renormed in such a way.

Definition 1.2.16. *Let X be a r.i. Banach function space over $(\mathbb{R}^+, \lambda)$ and suppose X has been renormed as in Lemma 1.2.15 so that its fundamental function φ_X is concave. The Lorentz spaces $\Lambda(X)$ and $M(X)$ are defined as follows. The space $M(X)$ consists of f in $L^0(\mathbb{R}^+, \lambda)$ such that the following norm is finite:*

$$\|f\|_{M(X)} = \sup_{0<t<\infty} \left\{f^{**}(t)\varphi_X(t)\right\}$$

The space $\Lambda(X)$ consists of all f in $L^0(\mathbb{R}^+, \lambda)$ for which

$$\|f\|_{\Lambda(X)} = \int_0^\infty f^*(s)d\varphi_X(s) = \lim_{s\downarrow 0}\varphi_X(s)f^*(s) + \int_0^\infty f^*(s)\varphi_X'(s)\mathrm{d}s \quad (1.21)$$

is finite.

Theorem 1.2.17. *Let X be a r.i. Banach function space over $\left(\mathbb{R}^+, \lambda\right)$ and, suppose X has been renormed to have concave fundamental function φ_X. Then the Lorentz spaces $\Lambda(X)$ and $M(X)$ are r.i. Banach function spaces and each has fundamental function equal to φ_X. Furthermore*

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$$

and each of the embeddings has norm 1

For a proof, see [1, Theorem II.5.13]. Due to the fact that these Lorentz spaces only depend on the fundamental function of X we sometimes write $M_{\varphi_X} = M(X)$ and $\Lambda_{\varphi_X} = \Lambda(X)$.

Remark 1.2.18. On the definition of the norm of $\Lambda(X)$. We note that when φ_X is continuous on $[0, \infty)$, we don't have to consider the limit. This limit is only necessary for when φ_X is discontinuous in 0. Then the derivative φ_X' only represents φ_X on $(0, \infty)$. The integral on the right side of (1.21) acts as $\|f\|_{\Lambda(X)} = \int_0^\infty f^*(s) d\tilde{\varphi}_X(s)$, where $\tilde{\varphi}_X$ is equal to φ_X on $(0, \infty)$ and $\tilde{\varphi}_X(0) = \varphi_X(0+)$.
A good example of when the limit is nonzero is L^∞, where the Riemann-stieltjes integral is defined as follows: a partition P_n of $[0, n]$ is of the form $P_n = \{0 = x_0, x_1, \cdots, x_k\}$ with $0 < |x_i - x_{i+1}| < \frac{1}{n}$, then define

$$S(P_n, f^*, \varphi_X) : \sum_{i=1}^{k} f^*(x_i)[\varphi_X(x_i) - \varphi_X(x_{i-1})]$$

then we have that

$$\int_0^\infty f^*(s) \mathrm{d}\varphi_X(s) = \lim_{n\to\infty} S(P_n, f^*, \varphi_X)$$

Then since $\varphi_{L^\infty}(t) = \chi_{(0,\infty)}$ we find that

$$\begin{aligned}\int_0^\infty f^*(s) \mathrm{d}\varphi_X(s) &= \lim_{n\to\infty} S(P_n, f^*, \varphi_X) \\ &= \lim_{n\to\infty} f^*(x_1)\chi_{(0,\infty)}(x_1) \\ &= \lim_{s\downarrow 0} f^*(s)\varphi_{L^\infty} \\ &= \lim_{s\downarrow 0} f^*(s)\varphi_{L^\infty} + \int_0^\infty f^*(s)\varphi_{L^\infty}'(s)\mathrm{d}s\end{aligned}$$

where the last equality is due to $\varphi_{L^\infty}'(s) = 0$ on $(0, \infty)$.

Definition 1.2.19. *Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. For $f \in L^0(R, \mu)$ we define:*

$$\|f\|_{p,q} = \begin{cases} \left\{\int_0^\infty [t^{1/p} f^{**}(t)]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} & (1 \leq q < \infty) \\ \sup\limits_{0<t<\infty} \{t^{1/p} f^{**}(t)\} & (q = \infty) \end{cases}$$

Then $L^{p,q}(\mathbb{R}^d)$ with its norm defined by $\|\cdot\|_{p,q}$ is a r.i. Banach function space.

For a proof of the last statement, see [1, Theorem IV.4.6]. For $1 < p < \infty$ we have that $L^{p,p}$ is equivalent to L^p(see [1, Lemma IV.4.5]). We also introduce the following

$$\rho_{p,q}(f) = \begin{cases} \left\{\int_0^\infty [t^{1/p} f^{*}(t)]^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} & (1 \leq q < \infty) \\ \sup\limits_{0<t<\infty} \{t^{1/p} f^{*}(t)\} & (q = \infty) \end{cases} \tag{1.22}$$

Note the subtle difference with the norm of $L^{p,q}$, we use f^* instead of f^{**}. By [1, Lemma IV.4.5], we get that

$$\rho_{p,q}(f) \leq \|f\|_{p,q} \leq \frac{p}{p-1}\rho_{p,q}(f).$$

$\rho_{p,q}$ is easier to work with, but it does not always define a norm.

$$\|f\|_{M(L^p)} = \sup_{0<t<\infty} \frac{\varphi_{L^p}(t)}{t} \int_0^t f^*(s)\mathrm{d}s = \sup_{0<t<\infty} t^{1/p} f^{**}(t) = \|f\|_{L^{p,q}}$$

It is a norm in the case that $1 \leq q \leq p < \infty$ (see [1, Theorem IV.4.3].

Example 1.2.20. For L^1, with $\varphi_{L^1}(t) = t$, it's easy to see that $\Lambda(L^1) = \overline{L^1} = M(L^1)$. Indeed, for $f \in L^0$:

$$\|f^*\|_{\Lambda(L^1)} = \int_0^\infty f^*(s)\mathrm{d}s = \|f^*\|_{\overline{L^1}} = \sup_{s>0} \int_0^s f^*(t)\mathrm{d}t = \|f^*\|_{M(L^1)} \tag{1.23}$$

For L^p with $1 < p < \infty$, we have $\varphi_{L^p} = t^{\frac{1}{p}}$,

$$\|f^*\|_{M(L^p)} = \sup_{s>0} s^{\frac{1}{p}-1} \int_0^t f^*\mathrm{d}s$$

Hence we find that for $f^* = t^{-\frac{1}{p}}\chi_{[1,\infty)}$ we have $f^* \in M(L^p)$ but $f^* \notin \overline{L^p}$, so that $M(L^p) \neq \overline{L^p}$ but rather $M(L^p) = L^{p,\infty}$. For $\Lambda(L^p)$ we find:

$$\|f\|_{\Lambda(L^p)} = \int_0^\infty f^*(t)\mathrm{d}(t^{1/p}) = \frac{1}{p}\int_0^\infty t^{1/p-1} f^*(t)\mathrm{d}t$$

We see that $\|\cdot\|_{\Lambda(L^p)}$ has an equivalent norm to $L^{p,1}$.

Chapter 2

Maximal Operators on Rearrangement Invariant Banach Function Spaces

Now that we have all the necessary knowledge of (r.i.) Banach function space, we are ready explore maximal functions. Of course we already saw the seemingly more elementary maximal function defined on rearrangements, but in practice it can be challenging to compute the rearrangement of a function. In this chapter we first look at the Hardy-Littlewood maximal operator, which is more naturally defined on functions.We also find how the boundedness of the Hardy-Littlewood operator is equivalent to that of f^{**} in r.i. Banach function spaces. Finally find necessary and sufficient conditions, as in [1, Theorem III.5.17], for when the Hardy-Littlewood maximal operator is bounded on a r.i. Banach function space defined on $\mathbb{R}^d$. In the next section we generalize the idea of a maximal operator, in which the Hardy-Littlewood operator is seen as a maximal operator defined using the L^1-norm. We find sufficient conditions for boundedness of these generalized maximal operators using a result from [8]. These sufficient conditions include the sufficient conditions found for the Hardy-Littlewood maximal operator in the first section. Additionally, we find necessary conditions. We then show under which hypotheses the necessary and sufficient conditions are equivalent.

2.1 Maximal operators

From here on, when we say X is a r.i. Banach function space on $\mathbb{R}^d$, we use the Lebesgue measure.

Definition 2.1.1. *Let f be a locally integrable function on $\mathbb{R}^d$. The Hardy-Littlewood maximal function Mf of f is defined by*

$$(Mf)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y, \quad x \in \mathbb{R}^d,$$

where cubes Q are assumed to have their sides parallel to the coordinate axes and of equal length. The operator $M : f \mapsto Mf$ is called the Hardy-Littlewood maximal operator

Notice that M is sublinear:

$$M(f+g) \leq Mf + Mg; \quad M(\lambda f) = |\lambda| Mf.$$

Clearly M is bounded on L^∞:

$$\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty} \left\| \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \mathrm{d}y \right\|_{L^\infty} = \|f\|_{L^\infty}.$$

However, for $f \neq 0$ μ-a.e., $(Mf)(x)$ never decays faster than $|x|^{-d}$:

$$(Mf)(x) \geq \frac{c}{|x|^{-d}} \quad (|x| > 1).$$

This shows that M is not bounded on L^1. The Lebesgue Differentiation theorem will be of use in bounding a function by a maximal operator:

Theorem 2.1.2 (Lebesgue's differentiation theorem)**.** *If f is a locally integrable function on $\mathbb{R}^d$, then*

$$\lim_{\substack{|Q| \to 0 \\ Q \ni x}} \frac{1}{|Q|} \int_Q \left| f(y) - f(x) \right| \mathrm{d}y = 0,$$

for almost every x in $\mathbb{R}^d$.

A proof can be found in [1, Lemma III.3.4]. The Lebesgue differentiation theorem is a central result in analysis, which we will be using to show a fundamental property of the maximal function Mf:

Corollary 2.1.3. *If f is locally integrable in $\mathbb{R}^d$, then*

$$|f(x)| \leq (Mf)(x),$$

for almost every x in $\mathbb{R}^d$.

Proof. Using Theorem 2.1.2 and the triangle inequality, we find

$$\lim_{\substack{|Q|\to 0 \\ Q\ni x}} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y = |f(x)|.$$

Taking the supremum instead of the limit gives us the result. $\square$

We would like to get an idea of the size of Mf relative to f for $f \in L^1(\mathbb{R}^d)$. To do that we will need the following Vitali covering lemma:

Lemma 2.1.4. *Let Ω be an arbitrary measurable subset of $\mathbb{R}^d$ of finite measure. Let $\mathscr{F}$ be a collection of cubes Q that covers Ω. Then there exist finitely many disjoint cubes, say $Q_1, Q_2, \ldots, Q_k$, from $\mathscr{F}$ such that*

$$\sum_{k=1}^{K} |Q_k| \geq 4^{-d}|\Omega|$$

For a proof, see [1, Lemma III.3.2]. Note that 4^{-d} is not the largest constant for which this statement is true, it works when we replace 4 with any constant $c > 3$. However, for our purposes, 4^{-d} is sufficient, as we require it to be a constant depending only on the dimension, d.

Theorem 2.1.5. *If f belongs to $L^1\left(\mathbb{R}^d\right)$, then*

$$t\,(Mf)^*\,(t) \leq 4^d \|f\|_{L^1}\,, \quad (t > 0) \tag{2.1}$$

Proof. We begin with $f \in L^1$ compactly supported, for which it is clear that, for $\lambda > 0$, the set $E_\lambda := \{x \in \mathbb{R}^d : (Mf)(x) > \lambda\}$ has finite measure. For each $x \in E_\lambda$ we find by definition of Mf that there exists a cube $Q_x \ni x$ such that

$$\lambda|Q_x| < \int_{Q_x} |f(y)|\mathrm{d}y. \tag{2.2}$$

Since is it clear that $\bigcup_{x\in E_\lambda} Q_x \supseteq E_\lambda$, Lemma 2.1.4 produces a finite sequence of disjoint cubes $Q_1, Q_2, \cdots, Q_k$ such that

$$\sum_{n=1}^{k} |Q_n| \geq 4^{-d}|E_\lambda|. \tag{2.3}$$

Hence, combining (2.3) and (2.2), we find

$$|E_\lambda| \leq 4^d \sum_{n=1}^{k} |Q_n| \leq \frac{4^d}{\lambda}\sum_{n=1}^{k}\int_{Q_n} |f(y)|\mathrm{d}y \leq \frac{4^d}{\lambda}\|f\|_{L^1} \tag{2.4}$$

Notice that $|E_\lambda|$ is the distribution function of Mf, thus by property (1.12) we get that for every $t > 0$ there exists a $\lambda > 0$ such that $t(Mf)^*(t) \leq \lambda|E_\lambda|$, which implies the required estimate. In the general case of an integrable function f, we may select an increasing sequence of non-negative simple functions $f_k \uparrow |f|$ a.e. Then the monotone convergence theorem yields $Mf_k \uparrow Mf$ a.e., property (1.8) then tells us that $(Mf_k)^* \uparrow (Mf)^*$. Since $\|f_k\|_{L^1} \uparrow \|f\|_{L^1}$, we see that the required estimate holds for f, completing the proof □

We claim that an interesting consequence of (2.1) is that Mf is bounded from L^1 into weak L^1, denoted by $L^{1,\infty}$. Its norm is usually defined by

$$\|f\|_{L^{p,\infty}} = \sup_{0<t<\infty} t\mu_f^{\frac{1}{p}}(t).$$

However, for our purposes we will use the following definition[1] :

$$\|f\|_{L^{p,\infty}} = \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t), \quad f \in L^0.$$

Then we indeed find:

$$\|Mf\|_{L^{1,\infty}} = \sup_{0<t<\infty} t(Mf)^*(t) \leq 4^d \|f\|_{L^1}$$

showing that $M : L^1 \to L^{1,\infty}$ is a bounded operator. Another interesting observation is

$$(Mf)^*(t) \leq \frac{4^d}{t} \int_0^\infty f^*(s)\mathrm{d}s,$$

revisiting the rough estimate in (2.4) gives rise to the suspicion that Mf is bounded by f^{**}. That suspicion is correct, in fact, Mf is an element of X exactly when f^{**} is an element of $\bar{X}$, which we will see in the theorem following another covering lemma.

Lemma 2.1.6. *Let Ω be an open subset of $\mathbb{R}^d$ with finite measure. Then there is a sequence of dyadic cubes $Q_1, Q_2, \ldots$, with pairwise disjoint interiors, that covers Ω and satisfies:*

1. $Q_k \cap \Omega^c \neq \varnothing, \quad (k = 1, 2, \cdots)$
2. $|\Omega| \leq \sum_{k=1}^\infty |Q_k| \leq 2^d |\Omega|$

[1] For more information on this definition see [1, Definition IV.4.1]

For a proof, see [1, Lemma III.3.2] on page 122. The reason that the the Vitali covering Lemma 2.1.4 wont suffice, or any finite covering lemma, is that we will need to cover an arbitrary set of finite measure entirely with cubes. A good example of a set of finite measure that we can't cover with finite sets of finite measure is

$$A = \bigcup_{i \in \mathbb{N}} A_i, \text{ with } A_i = [i, i + 2^{-i}).$$

A dyadic covering lemma gives us exactly the control we need to cover sets like that. It allows us to cover any set of finite measure with countably many cubes so that the union of the cubes has finite measure, additionally the cubes are also pairwise disjoint. Note that countability comes from the fact that a dyadic grid is a made up of a countable set of cubes.

Theorem 2.1.7.

$$(Mf)^*(t) \lesssim_d f^{**}(t) \lesssim_d (Mf)^*(t), \quad (t > 0)$$

for every locally integrable function f on $\mathbb{R}^d$

Proof. We fix $t > 0$, then for the left-hand inequality, we may assume that $f^{**}(t) < \infty$, else there would be nothing to prove. [1, Theorem II.6.2] gives for $t > 0$ and $f \in L^0(\mathbb{R}^d)$L

$$\inf_{f=g+h} \left\{ \|g\|_{L^1} + t\|h\|_{L^\infty} \right\} = tf^{**}(t).$$

Note that the infimum may depend on t, so for $t > 0$ and $\varepsilon > 0$ we know there exist functions $g_t \in L^1$, $h_t \in L^\infty$ such that $f = g_t + h_t$ and

$$\|g_t\|_{L^1} + t\|h_t\|_{L^\infty} \leq tf^{**}(t) + \varepsilon.$$

Then since $g_t + h_t = f$ for all $t > 0$ we may use the triangle inequality for rearrangements (1.6) and the sub-additivity of M, so that for all $s > 0$ we obtain:

$$(Mf)^*(s) \leq (Mg_t)^*\big(s/2\big) + (Mh_t)^*\big(s/2\big)$$

and then by Theorem 2.1.5 and the boundedness of M on L^∞, for all $s > 0$ we get

$$(Mg_t)^*\big(s/2\big) + (Mh_t)^*\big(s/2\big) \leq \frac{c}{s}\big(\|g_t\|_{L^1} + s\|h_t\|_{L^\infty}\big)$$

Setting $s = t$ gives

$$(Mf)^*(t) \leq tf^{**}(t) + \varepsilon$$

letting $\varepsilon \to 0$ gives the first of the inequalities. For the right-hand inequality we may again assume $(Mf)^*(t) < \infty$. We consider

$$\Omega := \{x \in \mathbb{R}^d : (Mf)(x) > (Mf)^*(t)\}$$

and take $x \in \Omega$. Then since $(Mf)(x) > (Mf)^*(t)$, we know that there exists a cube $Q_x \ni x$ such that

$$\frac{1}{|Q_x|} \int_{Q_x} |f(y)| \mathrm{d}y > (Mf)^*(t),$$

hence Ω is open. Then since Mf and $(Mf)^*$ are equimeasurable by (1.9), $|\Omega| < t$. Applying Lemma 2.1.6, we obtain $Q_1, Q_2, \cdots$ disjoint such that

1. $Q_k \cap \Omega^c \neq \varnothing, \quad (k = 1, 2, \cdots)$
2. $\sum_{k=1}^{\infty} |Q_k| \leq 2^d t$

With $F = \left(\bigcup_k Q_k\right)^c$, we set

$$g = \sum_k f\chi_{Q_k}, \quad h = f\chi_F$$

so that $f = g + h$. The sub-additivity of $f \to f^{**}$, (1.16), gives

$$\begin{aligned} f^{**}(t) \leq & g^{**}(t) + h^{**}(t) \\ = & \frac{1}{t} \int_0^t g^*(t) \mathrm{d}t + \frac{1}{t} \int_0^t h^*(t) \mathrm{d}t \\ = & \frac{1}{t} \int_0^\infty g^*(t) \mathrm{d}t + \|h\|_{L^\infty} \frac{1}{t} \int_0^t \mathrm{d}t \\ \leq & \frac{1}{t} \|g\|_{L^1} + \|h\|_{L^\infty}. \end{aligned} \tag{2.5}$$

Now, using that $Q_k \cap \Omega^c \neq \varnothing$, that is, there is a $x \in Q_k$ such that $x \notin \Omega$, we find that

$$\frac{1}{|Q_k|} \int_{Q_k} |f(y)| \mathrm{d}y \leq (Mf)^*(t), \quad (k = 1, 2, \cdots)$$

by the way that Ω is defined. Then the following

$$\|g\|_{L^1} = \sum_k \int_{Q_k} |f(y)| \mathrm{d}y \leq \sum_k |Q_k| (Mf)^*(t)$$

gives us

$$\|g\|_{L^1} \leq 2^d t (Mf)^*(t) \tag{2.6}$$

On the other hand, F is completely contained in Ω^c, so on F we have $Mf(x) \leq (Mf)^*(t)$. Then we have

$$\|h\|_{L^\infty} \leq \|f\chi_F\|_{L^\infty} \leq \big\|(Mf)\chi_F\big\|_{L^\infty} \leq (Mf)^*(t).$$

Putting this, (2.6) and (2.5) together finalizes the proof. $\square$

We will later see that this equivalence of boundedness is an essential piece in finding equivalent sufficient and necessary conditions for the boundedness on a r.i. Banach function space X. We define the following operator in preparation of finding those conditions.

Definition 2.1.8. *Let P_a, $0 < a \leq 1$, be the integral operator defined on $L^0(\mathbb{R}^+, \lambda)$ by*

$$\left(P_a f\right)(t) = t^{-a} \int_0^t s^a f(s) \frac{\mathrm{d}s}{s}, \quad (0 < t < \infty)$$

Note that for $a = 1$ we have an identity $f^{**}(t) = P_1 f(t)$. While P_a has more uses, we use it because it is such a generalization and due to the following lemma.

Lemma 2.1.9. *If P_a is a bounded operator on $\bar{X}$, that is*

$$\|P_a\|_{\mathscr{B}(\bar{X})} = \sup\left\{\|P_a f\|_{\bar{X}} : \|f\|_{\bar{X}} \leq 1\right\} < \infty,$$

then there exists $\varepsilon > 0$ such that $\|P_{a-\delta}\|_{\mathscr{B}(\bar{X})} < \infty$ for all $\delta < \varepsilon$

Proof. Let $\varepsilon > 0$ be such that $\varepsilon\|P_a\|_{\mathscr{B}(\bar{X})} < 1$. Then the operator $I - \varepsilon P_a$ belongs to $\mathscr{B}(\bar{X})$, is invertible, and since $\bar{X}$ is a Banach space the following Neumann series is convergent in the norm of $\mathscr{B}(\bar{X})$:

$$\left(I - \varepsilon P_a\right)^{-1} = \sum_{n=0}^{\infty} \varepsilon^n P_a^n.$$

The operator

$$T = P_a \left(I - \varepsilon P_a\right)^{-1} = \sum_{n=0}^{\infty} \varepsilon^n P_a^{n+1} \tag{2.7}$$

is therefore also in $\mathscr{B}(\bar{X})$. Take $f \in \bar{X}$, we claim that the iterate P_a^{n+1} of P_a may be written in the closed form

$$\left(P_a^{n+1} f\right)(t) = \int_0^1 f(st) \frac{(\log 1/s)^n}{n!} s^{a-1} ds \tag{2.8}$$

The proof proceeds by induction on n. The case $n = 0$ follows immediately from the definition of P_a, so suppose (2.8) holds for $n = 0, 1, 2, \ldots, N$. Then

$$\begin{aligned}\left(P_a^{N+1} f\right)(t) &= P_a\left(P_a^N f\right)(t) = \int_0^1 P_a^N f(rt) r^{a-1} dr \\ &= \int_0^1 \left(\int_0^1 f(srt) \frac{(\log 1/s)^N}{N!} s^{a-1} ds\right) r^{a-1} dr\end{aligned}$$

so making the change of variable $u = rs$, we have

$$\left(P_a^{N+1} f\right)(t) = \int_0^1 \left(\int_0^r f(ut) \frac{(\log r/u)^N}{N!} u^{a-1} du\right) \frac{dr}{r}$$

Interchanging the order of integration and making the change of variable $v = r/u$, we obtain

$$\begin{aligned}\left(P_a^{N+1} f\right)(t) &= \int_0^1 \left(\int_u^1 \frac{(\log r/u)^N}{N!} \frac{dr}{r}\right) f(ut) u^{a-1} du \\ &= \int_0^1 \left(\int_1^{1/u} \frac{(\log v)^N}{N!} \frac{dv}{v}\right) f(ut) u^{a-1} du \\ &= \int_0^1 \frac{(\log 1/u)^{N+1}}{(N+1)!} f(ut) u^{a-1} du\end{aligned}$$

This completes the induction and hence establishes (2.8) for all n. Combining (2.7), (2.8), and using the monotone convergence theorem, we obtain, for non-negative functions f in $\bar{X}$

$$(Tf)(t) = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(\varepsilon \log 1/s)^n}{n!}\right) f(st) s^{a-1} ds = \int_0^1 f(st) s^{a-\varepsilon-1} ds$$

By the usual device of splitting a function into its positive and negative parts we obtain this identity for all f in $\bar{X}$. Hence, $T = P_{a-\varepsilon}$. clearly it extends to $0 < \delta < \varepsilon$ as then we also have $\|P_a\|_{\mathscr{B}(\bar{X})} \delta < 1$, this concludes the proof. □

Note that, even if we find that P_a is bounded for all $a > b$, that does not necessarily mean P_b is bounded, since the ε found using this lemma depends on a. The use for this lemma will become clear in Theorem 2.1.14. We will want to define the upper Boyd index, before that we have to define the following operator:

Definition 2.1.10. *Let $s \in (0,\infty)$ and $f \in L^0(\mathbb{R}^d)$. Then the anti-dilation operator $D_s : L^0 \to L^0$ is defined by:*

$$D_s f(x) := f(sx).$$

The anti-dilation operator will be useful in obtaining the final result in this section, sufficient and necessary conditions for boundedness of the Hardy-Littlewood operator on r.i. Banach function spaces. Furthermore, in section 2.2 D_s will play a central role. It will therefore be useful to better understand the anti-dilation operator, so we prove some of its properties in the following proposition.

Proposition 2.1.11. *Let X, Y be a r.i. Banach function space on $(0,\infty)$ and $\mathbb{R}^d$ respectively. Let $c > 0$, f be in $L^0\big(\mathbb{R}^d\big)$ and $g \in X$, then*

$$\mu_{D_t f}(\lambda) = t^{-d}\mu_f(\lambda) \tag{2.9}$$
$$(D_t f)^* = D_{t^d} f^* \tag{2.10}$$
$$\|D_t\|_{\mathscr{B}(X)} \leq c \max\{1, \frac{1}{t}\} \tag{2.11}$$
$$(D_s g)^*(t) = D_s g^*(t) \tag{2.12}$$
$$\|D_{st}\|_{\mathscr{B}(Y)} \leq \|D_s\|_{\mathscr{B}(Y)} \|D_t\|_{\mathscr{B}(Y)} \tag{2.13}$$

Proof. We observe that

$$\mu_{D_t f}(\lambda) = \mu\Big(\{x \in \mathbb{R}^d : |f(tx)| > \lambda\}\Big) = \mu\Big(\{xt^{-1} \in \mathbb{R}^d : |f(x)| > \lambda\}\Big)$$

Then by the dilation property of the Lebesgue measure: $\mu\big(\{\delta x : x \in A\}\big) = \delta^d \mu(A)$ for $A \in \mathcal{B}(\mathbb{R}^d), \delta > 0$, we get:

$$\mu_{D_t f}(\lambda) = \mu\Big(\{xt^{-1} \in \mathbb{R}^d : |f(x)| > \lambda\}\Big) = t^{-d}\mu_f(\lambda)$$

giving us (2.9). Then for (2.10) we obtain

$$(D_t f)^*(s) = \sup\{\lambda : \mu_{D_t f}(\lambda) > s\} = \sup\{\lambda : \mu_f(\lambda) > st^d\} = D_{t^d} f^*$$

For (2.11) we use [1, Theorem III.2.2] , which tells us that X is an exact interpolation space between L^1 and L^∞, then since D_s is an admissible operator for all $s \in (0,\infty)$ we get

$$\|D_t\|_{\mathscr{B}(X)} \leq c \max\left\{\|D_t\|_{\mathscr{B}(L^1(\mathbb{R}^+))}, \|D_t\|_{\mathscr{B}(L^\infty(\mathbb{R}^+))}\right\}.$$

Clearly $\|D_t\|_{\mathscr{B}(L^\infty(\mathbb{R}^+))} = 1$, and

$$\int_0^\infty |D_t f(s)|\mathrm{d}s = \int_0^\infty |f(ts)|\mathrm{d}s = \frac{1}{t}\int_0^\infty |f(s)|\mathrm{d}s,$$

thus $\|D_t\|_{\mathscr{B}(X)} \leq c\max\big(1, \frac{1}{t}\big)$. The constant c comes from renorming X with a constant multiple of its original norm. For (2.12) we observe that g is a function on $\mathbb{R}$ that is 0 on $(-\infty, 0)$, and then simply apply property (2.10). For property (2.13) we point out an identity of the anti-dilation operator: $D_{st}f = D_s D_t f = D_t D_s f$. Then we may simply write

$$\begin{aligned}\|D_{st}\|_{\mathscr{B}(Y)} =& \sup\left\{\|D_s D_t f\|_Y : \|f\|_Y \leq 1\right\}\\ \leq& \sup\left\{\|D_s\|_{\mathscr{B}(Y)}\|D_t f\|_Y : \|f\|_Y \leq 1\right\} = \|D_s\|_{\mathscr{B}(Y)}\|D_t\|_{\mathscr{B}(Y)}\,.\end{aligned}$$

□

Definition 2.1.12. *Let X be a r.i. Banach function space on (R, μ), then the upper Boyd index is given by:*

$$\bar{\alpha}_X := \inf_{1<t<\infty} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t} = \lim_{t\to\infty} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t}.$$

For a proof of the last equality, see [1, Proposition III.5.13]. $\bar{\alpha}_X$ is one of two Boyd indices, for now we will only need the upper one. Inspection of the definition tells us that $\bar{\alpha}_X$ is "the largest" number a such that $t^a \lesssim \|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}$. In L^1 we find for example that $\bar{\alpha}_{L^1} = 1$, we even have $\|D_{t^{-1}}f^*\|_{\bar{X}} = t\|f^*\|_{\bar{X}}$, which is a property we don't have for every r.i. Banach function space X.

A more interesting case could be L^p. Recall that for $1 \leq p < \infty$ we have (1.17), then for $f \in L^p(\mathbb{R}^d)$:

$$\|D_{t^{-1}}f^*\|_{\overline{L^p}} = \left(\int_0^\infty f^*\big(s/t\big)^p\mathrm{d}s\right)^{\frac{1}{p}} = \left(\int_0^\infty tf^*(s)^p\mathrm{d}s\right)^{\frac{1}{p}} = t^{\frac{1}{p}}\|f^*\|_{\overline{L^p}}$$

Thus we get that $\bar{\alpha}_{L^p} = \frac{1}{p}$. For $p = \infty$, we get:

$$\|D_t f^*\|_{\overline{L^\infty}} = f^*(0) = \|f^*\|_{\overline{L^\infty}}\,, \quad (f \in L^\infty(\mathbb{R}^d)$$

Giving that $\bar{\alpha}_{L^\infty} = 0$.

Lemma 2.1.13. *For X a r.i. Banach function space on $\mathbb{R}^d$ with the Lebesgue measure. Let $0 < a \leq 1$, $0 < s < 1$ and $g \in X' : \|g\|_{X'} \leq 1$, then*

$$\int_0^\infty \big(D_s f^*(t)\big)g^*(t)\mathrm{d}t = \int_0^\infty f^*(st)g^*(t)\mathrm{d}t \leq as^{-a}\|P_a f^*\|_{\bar{X}}$$

Proof.

$$\begin{aligned}
\int_0^\infty f^*(st)g^*(t)\mathrm{d}t &= as^{-a}\left(\int_0^\infty f^*(st)g^*(t)\mathrm{d}t\right)\left(\int_0^s u^{a-1}\mathrm{d}u\right) \\
&\leq as^{-a}\int_0^s\left(\int_0^\infty f^*(ut)g^*(t)\mathrm{d}t\right)u^{a-1}\mathrm{d}u \\
&= as^{-a}\int_0^\infty g^*(t)\left(\int_0^s f^*(ut)u^a\frac{\mathrm{d}u}{u}\right)\mathrm{d}t \\
&\leq as^{-a}\int_0^\infty g^*(t)\left(\int_0^1 f^*(ut)u^a\frac{\mathrm{d}u}{u}\right)\mathrm{d}t \\
&= as^{-a}\int_0^\infty g^*(t)\left(t^{-a}\int_0^t y^a f^*(y)\frac{\mathrm{d}y}{y}\right)\mathrm{d}t \\
&= as^{-a}\int_0^\infty g^*(t)(P_af^*)(t)\mathrm{d}t \leq as^{-a}\|P_af^*\|_{\bar{X}}
\end{aligned}$$

in the first inequality we use that f^* is a decreasing function and therefore $f^*(ut) \geq f^*(st)$, in the second one the fact that $f^* \geq 0$ and $s < 1$. This is followed up by a change of variables, and finally we use the Luxemburg representation theorem in the last inequality ☐

Lemma 2.1.13 shows that there is a connection between the boundedness of P_a and the upper Boyd index, this will become more clear in the proof of the following theorem, where we make explicit use of this property.

Theorem 2.1.14. *Let X be a r.i. Banach function space on $\mathbb{R}^d$. P_a is a bounded operator on $\bar{X}$ if and only if $\bar{\alpha}_X < a \leq 1$.*

Proof. Suppose first that P_a is bounded on $\bar{X}$. Let f and g be functions in $\bar{X}$ and $\bar{X}'$ respectively, with

$$\|f\|_{\bar{X}} \leq 1, \quad \|g\|_{\bar{X}'} \leq 1. \tag{2.14}$$

Using (2.12), which says $(D_sf)^*(t) = D_sf^*(t)$, and Lemma 2.1.13 we get for $t > 1$

$$\int_0^\infty f^*(t^{-1}s)g^*(s)\mathrm{d}s \leq at^a\|P_a\|_{\mathscr{B}(\bar{X})}$$

Taking the supremum over all f and g satisfying (2.14) gives us

$$\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})} \leq at^{-a}\|P_a\|_{\mathscr{B}(\bar{X})}\,, \quad (t > 1)$$

Thus,

$$\frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t} \leq a + \frac{\log a\|P_a\|_{\mathscr{B}(\bar{X})}}{\log t} \to a$$

as $t \to \infty$ Now by Lemma 2.1.9 we find that there exists a $\varepsilon > 0$ such that

$$\bar{\alpha}_X = \lim_{t\to\infty} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X}}}{\log t} \leq a - \varepsilon < a$$

Now assume $a > \bar{\alpha}_X$ and let f, g be such that they satisfy (2.14). Then

$$\begin{aligned}
\left|\int_0^\infty (P_a f)(t) g(t) \mathrm{d}t\right| &\leq \int_0^\infty \left(t^{-a}\int_0^t |f(s)| s^{a-1}\mathrm{d}s\right)|g(t)|\mathrm{d}t \\
&= \int_0^\infty \left(\int_0^1 |f(st)| s^{a-1}\mathrm{d}s\right)|g(t)|\mathrm{d}t \\
&= \int_0^1 \left(\int_0^\infty \left|D_s f(t) g(t)\right| \mathrm{d}t\right) s^{a-1}\mathrm{d}s \\
&\leq \int_0^1 \|D_s f\|_{\bar{X}}\, s^{a-1}\mathrm{d}s \\
&\leq \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})}\, s^{a-1}\mathrm{d}s \\
&\leq \int_1^\infty \|D_{s^{-1}}\|_{\mathscr{B}(\bar{X})}\, s^{-a-1}\mathrm{d}s
\end{aligned}$$

Now, using $\lim_{t\to\infty} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t}$, we find that for $\varepsilon > 0$ such that $a - \bar{\alpha}_X > \varepsilon > 0$ there is a $\infty > T > 1$ such that for all $s > T$ we have $\|D_{s^{-1}}\|_{\mathscr{B}(\bar{X})} \leq s^{\bar{\alpha}_X + \varepsilon}$, giving

$$\leq \int_1^T \|D_{s^{-1}}\|_{\mathscr{B}(\bar{X})}\, s^{-a-1}\mathrm{d}s + \int_T^\infty s^{-a+\bar{\alpha}_X+\varepsilon-1}\mathrm{d}s < \infty$$

where, in the last step we know that the first integral is bounded since $\|D_{s^{-1}}\|_{\mathscr{B}(\bar{X})} \leq c\max\{1, t\}$, by (2.11). Taking the supremum over f and g satisfying (2.14), we find $\|P_a\|_{\mathscr{B}(\bar{X})} < \infty$

□

Theorem 2.1.15. *Let X be a r.i. Banach Function Space on $\mathbb{R}^d$. Then the Hardy-Littlewood maximal operator M is bounded on X if and only if the Boyd index $\bar{\alpha}_X < 1$.*

Proof. It follows from Theorem 2.1.7 that M is bounded on X if and only if P_1 is bounded on $\bar{X}$. Lemma 2.1.14 then concludes the proof □

Using the Boyd indices we found for L^p, $\bar{\alpha}_{L^p} = \frac{1}{p}$, we immediately find that M is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$. Similarly, by for example [1, Theorem IV.4.6], we get that the Boyd indices of $L^{p,q}$ are both equal to $\frac{1}{p}$. Thus M is bounded on $L^{p,q}(\mathbb{R}^d)$ for all $1 < p < \infty$ and $1 \leq q \leq \infty$.

2.2 Maximal operators based on r.i. spaces

From this chapter on, if we don't specify the measure on $\mathbb{R}^d$ or $\mathbb{R}^+$ we use the Lebesgue measure. When looking at the Hardy-Littlewood maximal operator, see that we may write

$$\begin{aligned}(Mf)(x) =& \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y \\ =& \sup_{Q \ni x} \frac{1}{|Q|} \int_{\mathbb{R}^d} |f(y)| \chi_Q(y) \mathrm{d}y \\ =& \sup_{Q \ni x} \big\| D_{l(Q)}(f\chi_Q) \big\|_{L^1}\end{aligned}$$

Where $l(Q)$ is the side-length of a cube Q. This is the essential inspiration for generalizing the maximal operator in the following way.

Definition 2.2.1. *Let Y be a r.i. Banach function space on $\mathbb{R}^d$. For $f \in Y_{loc}$ the maximal operator based on Y is given by*

$$M_Y f(x) := \sup_{Q \ni x} \|f\|_{Q,Y} = \sup_{Q \ni x} \big\| D_{l(Q)}(f\chi_Q) \big\|_Y$$

Where Q is any cube in $\mathbb{R}^d$ with its sides parallel to the axes and all side lengths equal.

This operator was first introduced by C. Pérez, and later studied further in [8] by W. Mastyło and C. Pérez. As before, M_Y is still a sub-linear operator:

$$M_Y(f+g) \leq M_Y f + M_Y g; \quad M_Y(\lambda f) = |\lambda| M_Y f, \quad f \in L^0(\mathbb{R}^d)$$

and for $f \in L^\infty(\mathbb{R}^d)$ we have:

$$\big\| M_Y(f) \big\|_{L^\infty} \leq \|f\|_{L^\infty} \big\| M_Y(\chi_{\mathbb{R}^d}) \big\|_{L^\infty} = \varphi_Y(1) \|f\|_{L^\infty}$$

so that all maximal operators based on r.i. Banach function spaces are bounded on $L^\infty(\mathbb{R}^d)$.

We can obtain an analogous result to Theorem 2.1.3 for general M_Y quite easily:

Theorem 2.2.2. *Let $f \in Y_{loc}$, then there exists a $c > 0$ only dependent on the norm of Y such that*

$$Mf(x) \leq c M_Y f(x)$$

and in particular

$$|f(x)| \leq c M_Y f(x)$$

for almost every $x \in \mathbb{R}^d$.

Proof. Using [1, Theorem II.6.6] we find that for some $c > 0$, $\|\cdot\|_{L^1+L^\infty} \leq c\|\cdot\|_{\bar{Y}}$. By [1, Theorem II.6.4] we have that

$$\|f\|_{L^1+L^\infty} = \int_0^1 f^*(t)\mathrm{d}t.$$

Using (2.10) we find

$$\begin{aligned}
cM_Y f(x) &= \sup_{Q \ni x} c\left\| D_{l(Q)}\big(f\chi_Q\big)\right\|_Y \\
&= \sup_{Q \ni x} c\left\| D_{|Q|}\big(f\chi_Q\big)^*\right\|_{\bar{Y}} \\
&\geq \sup_{Q \ni x} \int_0^1 D_{|Q|}\big(f\chi_Q\big)^*(t)\mathrm{d}t \\
&= \sup_{Q \ni x} \frac{1}{|Q|}\int_0^{|Q|} \big(f\chi_Q\big)^*(t)\mathrm{d}t
\end{aligned}$$

Notice that $\big(f\chi_Q\big)^*(s)$ is 0 for $s > |Q|$. We refer to (1.17) for the equality in (2.15).

$$\begin{aligned}
&= \sup_{Q \ni x} \frac{1}{|Q|}\int_0^{\infty} \big(f\chi_Q\big)^*(t)\mathrm{d}t \\
&= \sup_{Q \ni x} \frac{1}{|Q|}\int_{\mathbb{R}^d} |f(y)|\chi_Q(y)\mathrm{d}y \qquad (2.15) \\
&= \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y = (M_{L^1}f)(x)
\end{aligned}$$

The first statement is then proved, as M_{L^1} is the Hardy-Littlewood maximal operator. The second statement follows from Theorem 2.1.3. □

To work towards the main result, we begin by looking for sufficient conditions for boundedness of M_Y on X. Firstly we view $\|f\|_{Q,Y}$ as a restriction of a function $F : \mathcal{B}\big(\mathbb{R}^d\big) \to \mathbb{R}^+$ to cubes, where F is such that $F(Q) = \|f\|_{Q,Y}$. We call a function that maps $\mathcal{B}(\mathbb{R}^d)$ into $\mathbb{R}^+$ a set function, where $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-algebra containing the open sets. To make use of the theorem presented in [6], we need the following definition.

Definition 2.2.3. *A set function F is called pseudo-increasing if there exists a $c > 0$ such that for any finite collection of pairwise disjoint cubes $\{Q_i\}$ with $Q = \bigcup_i Q_i$ we have*

$$\min_i F(Q_i) \leq cF(Q)$$

In the following lemma we use $MF : \mathbb{R}^d \to \mathbb{R}^+$ given by $MF(x) = \sup\limits_{Q \ni x} F(Q)$ as in [6]. The supremum is taken over cubes Q in $\mathbb{R}^d$ with sides parallel to the axes and all side lengths equal.

Lemma 2.2.4. *Let F be a pseudo-increasing set function with constant $c > 0$ as in the preceding definition. Then, for any $t > 0$*

$$(MF)^*(t) \le c \sup_{\infty > |E| > 4^{-d}t} F(E)$$

where the supremum is taken over all sets $E \in \mathcal{B}(\mathbb{R}^d)$ of finite measure $|E| > 4^{-d}t$.

A proof can be found in [6]. Note that the constant 4^{-d} originates from our Vitali covering, Lemma 2.1.4.

The following lemma follows [8, Lemma 3.4].

Lemma 2.2.5. *Let Y be a r.i. Banach function space on $\mathbb{R}^d$ such that its fundamental function is concave, Let E be the r.i. Banach function space on $\mathbb{R}^d$ generated by $\Lambda(Y)$. Then for $f \in E_{loc}$ we define the set function $F(A) := \left\| D_{|A|^{1/d}} f \chi_A \right\|_E$. Then F is pseudo-increasing with constant $c = 1$*

Proof. Let $\Omega = \{Q_i : 1 \le i \le n\}$ be a finite set of cubes in $\mathbb{R}^d$. Note that the following equation holds (see [5, formula (5.4)]):

$$\|f\|_{\Lambda(Y)} = \int_0^\infty \varphi_Y\big(\mu_f(s)\big)\mathrm{d}s, \quad f \in \Lambda(Y).$$

Let $A, B \subseteq \mathbb{R}^d$ be pairwise disjoint, using property (2.9) and the fact that f and f^* are equimeasurable we get

$$\begin{aligned} F(A \cup B) &= \left\| D_{\left(|A|+|B|\right)^{1/d}} f \chi_{A\cup B} \right\|_E = \left\| D_{|A|+|B|} (f\chi_{A\cup B})^* \right\|_{\Lambda(Y)} \\ &= \int_0^\infty \varphi_Y\left(\frac{\mu_{f\chi_{A\cup B}}(s)}{|A|+|B|}\right)\mathrm{d}s \\ &= \int_0^\infty \varphi_Y\left(\frac{\mu_{f\chi_A}(s)}{|A|}\frac{|A|}{|A|+|B|} + \frac{\mu_{f\chi_B}(s)}{|B|}\frac{|B|}{|A|+|B|}\right)\mathrm{d}s \end{aligned}$$

Note that, since A and B are disjoint we have $\mu_{f\chi_{A\cup B}} = \mu_{f\chi_A} + \mu_{f\chi_B}$ by (1.2). Now using that φ_Y is concave we find

$$
\begin{aligned}
&\geq \int_0^\infty \varphi_Y\left(\frac{\mu_{f\chi_A}(s)}{|A|}\right)\frac{|A|}{|A|+|B|} + \varphi_Y\left(\frac{\mu_{f\chi_B}(s)}{|B|}\right)\frac{|B|}{|A|+|B|}\mathrm{d}s \\
&= \frac{|A|}{|A|+|B|}\left\|D_{|A|}(f\chi_A)^*\right\|_{\Lambda(Y)} + \frac{|B|}{|A|+|B|}\left\|D_{|B|}(f\chi_B)^*\right\|_{\Lambda(Y)} \\
&= \frac{|A|}{|A|+|B|}F(A) + \frac{|B|}{|A|+|B|}F(B) \geq \min\{F(A), F(B)\}.
\end{aligned}
$$

Notice in the last step that we replace either $F(A)$ or $F(B)$ with the other, depending on which is smaller. Choosing $B = Q_1$ and $A = \Omega \setminus Q_1$ gives

$$
F(\Omega) \geq \min\big(F(Q_1), F(\Omega \setminus Q_1)\big)
$$

We may reapply the inequality finitely many times to find

$$
F(\Omega) \geq \min\Bigg(Q_1, \min\Big(F(Q_2), F\big(\Omega \setminus (Q_1 \cup Q_2)\big)\Big)\Bigg) \geq \cdots \geq \min_{1\leq i\leq n} F(Q_i).
$$

□

The following lemma will give us a result analogous to the left bound of Theorem 2.1.7 in a more general setting.

Lemma 2.2.6. *Let Y be a r.i. Banach function space on $\mathbb{R}^d$ with a concave fundamental function φ_Y. Let E be a r.i. Banach function space on $\mathbb{R}^d$ be generated by $\Lambda(Y)$, then for $f \in E_{loc}$ we have*

$$
(M_E f)^*(t) \leq \left\|(D_{4^{-d}t}f^*)\chi_{[0,1)}\right\|_{\Lambda(Y)} \tag{2.16}
$$

for every $t > 0$.

Proof. Let $F(A) = \left\|D_{|A|^{1/d}} f\chi_A\right\|_E$, then combining Lemma 2.2.4 and Lemma 2.2.5 we find

$$
(MF)^*(t) \leq \sup\left\{F(A) : \infty > |A| > \frac{t}{4^d}\right\} \tag{2.17}
$$

Since we have $(fg)^* \leq f^*g^*$ by (1.11) and $(D_a f)^* = D_{a^d} f^*$ by (2.10) for f on $\mathbb{R}^d$

$$
\begin{aligned}
F(A) &= \left\|D_{|A|^{1/d}} f\chi_A\right\|_E = \left\|D_{|A|}(f\chi_A)^*\right\|_{\Lambda(Y)} \\
&\leq \left\|D_{|A|}\big(f^*\chi_{[0,|A|)}\big)\right\|_{\Lambda(Y)} = \left\|\big(D_{|A|}f^*\big)\chi_{[0,1)}\right\|_{\Lambda(Y)}
\end{aligned} \tag{2.18}
$$

Combining the inequalities (2.18) and (2.17) and using the fact that $D_a f^*(t)$ is decreasing in a, we find

$$\begin{aligned}(M_E f)^*(t) = (MF)^*(t) \leq& \sup\left\{\left\|\left(D_{|A|}f^*\right)\chi_{[0,1)}\right\|_{\Lambda(Y)} : |A| > \frac{t}{4^d}\right\} \\ \leq& \left\|(D_{4^{-d}t}f^*)\chi_{[0,1)}\right\|_{\Lambda(Y)}\end{aligned}$$

□

This lemma is a small improvement over [8, Lemma 3.4], as we do not require $\varphi_Y(0+) = 0$. Recall from (1.23) that $\Lambda(L^1) = \overline{L^1}$. If we apply (2.16) to $Y = L^1(\mathbb{R}^d)$, then $E = L^1(\mathbb{R}^d)$ and we obtain the discussed, slightly sharper bound than the one given in Lemma 2.1.5:

$$(M_{L^1}f)^*(t) \leq 4^d \int_0^{4^{-d}t} f^*(s)\mathrm{d}s$$

instead of

$$(M_{L^1}f)^*(t) \leq 4^d \int_0^{\infty} f^*(s)\mathrm{d}s = 4^d\|f\|_{L^1}.$$

Note that, for Theorem 2.1.7 the rougher inequality was sufficient. In Theorem 2.2.8, however, we see that the sharper inequality makes a real difference as we only need the integral to be bounded around 0, but not as it tends to ∞. In the following lemma and proof we write $\left\|f(s)\right\|_{Y(s)} = \|f\|_Y$ for clarity.

Lemma 2.2.7. *Let X be a r.i. Banach function space on $\mathbb{R}^+$, then if ψ is an increasing continuous function on $[0,\infty)$ then*

$$\left\| \int_0^{\infty} D_s f^*(t)\mathrm{d}\psi(s) \right\|_{X(t)} \leq \int_0^{\infty} \left\|D_s f^*(t)\right\|_{X(t)} \mathrm{d}\psi(s) \qquad (2.19)$$

The following proof follows [5, Lemma II.4.7].

Proof. In the following inequality we use that $D_s f^*(t)$ is decreasing in s. Now for $\delta > 1$ we find:

$$\begin{aligned}\int_0^{\infty} D_s f^*(t)\mathrm{d}\psi(s) &= \sum_{k=-\infty}^{\infty} \int_{\delta^k}^{\delta^{k+1}} D_s f^*(t)\mathrm{d}\psi(s) \\ &\leq \sum_{k=-\infty}^{\infty} D_{\delta^k} f^*(t) \int_{\delta^k}^{\delta^{k+1}} \mathrm{d}\psi(s)\end{aligned}$$

Then, using the triangle inequality for norms, we get

$$\begin{aligned}
\left\| \int_0^\infty D_s f^*(t) \mathrm{d}\psi(s) \right\|_{X(t)} &\leq \sum_{k=-\infty}^{\infty} \left\| D_{\delta^k} f^*(t)[\psi(\delta^{k+1}) - \psi(\delta^k)] \right\|_{X(t)} \\
&= \sum_{k=-\infty}^{\infty} \left\| D_{\delta^k} f^*(t) \right\|_{X(t)} [\psi(\delta^{k+1}) - \psi(\delta^k)] \\
&= \sum_{k=-\infty}^{\infty} \left\| D_{\delta^k} f^*(t) \right\|_{X(t)} \int_{\delta^k}^{\delta^{k+1}} \mathrm{d}\psi(s).
\end{aligned}$$

Where in the first equality we use that ψ is increasing. Now using properties (2.11) and (2.13) of anti-dilations we get

$$\left\| D_{\delta^{-1}\delta^{k+1}} f^*(t) \right\|_{X(t)} \leq \delta \left\| D_{\delta^{k+1}} f^*(t) \right\|_{X(t)}$$

giving

$$\left\| \int_0^\infty D_s f^*(t) \mathrm{d}\psi(s) \right\|_{X(t)} \leq \delta \sum_{k=-\infty}^{\infty} \left\| D_{\delta^{k+1}} f^*(t) \right\|_{X(t)} \int_{\delta^k}^{\delta^{k+1}} \mathrm{d}\psi(s) \quad (2.20)$$

On the other side we have

$$\begin{aligned}
\int_0^\infty \left\| D_s f^*(t) \right\|_{X(t)} \mathrm{d}\psi(s) &= \sum_{k=-\infty}^{\infty} \int_{\delta^k}^{\delta^{k+1}} \left\| D_s f^*(t) \right\|_{X(t)} \mathrm{d}\psi(s) \\
&\geq \sum_{k=-\infty}^{\infty} \left\| D^{\delta^{k+1}} f^*(t) \right\|_{X(t)} \int_{\delta^k}^{\delta^{k+1}} \mathrm{d}\psi(s), \quad (2.21)
\end{aligned}$$

where we again use that $\left\| D_s f^*(t) \right\|_{X(t)}$ is decreasing in s. Combining the inequalities in (2.20) and (2.21) we find:

$$\left\| \int_0^\infty D_s f^*(t) \mathrm{d}\psi(s) \right\|_{X(t)} \leq \delta \int_0^\infty \left\| D_s f^*(t) \right\|_{X(t)} \mathrm{d}\psi(s)$$

since $\delta > 1$ is arbitrary we find that (2.19) is satisfied. □

This lemma is the last essential piece of the puzzle, as it helps us bound the operator found in Lemma 2.2.6 with the norm of f and $\|D_s\|_{\bar{X}}$ as a function in Λ_{φ_Y}.

The following theorem follows [8, Theorem 3.6] closely.

Theorem 2.2.8. *Let Y, X and E be r.i. Banach function spaces on $\mathbb{R}^d$, such that X is generated by $\bar{X}$ on $(0,\infty)$, E is generated by $\Lambda(Y)$ and*

$$\int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \, \mathrm{d}\varphi_Y(s) < \infty$$

Then the following statements hold:

1. *there exists a constant $C > 0$ such that*

$$\|M_Y f\|_X \leq C\|f\|_X, \quad f \in E_{loc} \tag{2.22}$$

2. *if $E_{loc} \cap X$ is dense in X, M_Y is bounded on X.*

Proof. 1. By Lemma 1.2.15 we find the Y may be equivalently renormed such that its fundamental function is concave, so without loss of generality we may assume it is concave. Then by Theorem 1.2.17 we have that for $f \in E$:

$$\|f\|_E = \|f^*\|_{\Lambda(Y)} \geq \|f^*\|_{\bar{Y}} = \|f\|_Y$$

So that we have $M_E f \geq M_Y$ and $(M_E f)^* \geq (M_Y f)^*$. By Lemma 2.2.6, for all $f \in E_{loc}$ we find

$$\begin{aligned}\|M_Y f\|_X = \left\|(M_Y f)^*\right\|_{\bar{X}} \leq \left\|(M_E f)^*\right\|_{\bar{X}} &\leq \left\|\int_0^1 D_t D_{4^{-d}} f^*(s) \mathrm{d}\varphi_Y(s)\right\|_{\bar{X}(t)} \\ &\leq \left\|\int_0^1 D_s D_{4^{-d}} f^*(t) \mathrm{d}\varphi_Y(s)\right\|_{\bar{X}(t)}\end{aligned}$$

We define $\tilde{\varphi}_Y(t)$ to be equal to $\varphi_Y(t)$ for all $t \in (0,\infty)$, and define $\tilde{\varphi}_Y(0) := \lim_{t\downarrow 0} \varphi_Y(t)$. Note that this limit exists since φ_Y is monotone and bounded near 0. Now $\tilde{\varphi}_Y$ is continuous on $[0,\infty)$ so that we may use Lemma 2.2.7

$$\begin{aligned}&= \left\|\lim_{s\downarrow 0} \varphi_Y(s)(D_{s4^{-d}} f^*(t)) + \int_0^1 D_s D_{4^{-d}} f^*(t) \mathrm{d}\tilde{\varphi}_Y(s)\right\|_{\bar{X}(t)} \\ &= \left\|\lim_{s\downarrow 0} \varphi_Y(s)(D_{s4^{-d}} f^*(t))\right\|_{\bar{X}(t)} + \left\|\int_0^1 D_s D_{4^{-d}} f^*(t) \mathrm{d}\tilde{\varphi}_Y(s)\right\|_{\bar{X}(t)}\end{aligned} \tag{2.23}$$

Notice that (2.22) is as follows:

$$\int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \, \mathrm{d}\varphi_Y(s) = \lim_{s\downarrow 0} \|D_s\|_{\mathscr{B}(\bar{X})} \, \varphi_Y(s) + \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \, \varphi_Y'(s) \mathrm{d}s < \infty.$$

Note that for every $n \in \mathbb{N}$ we have $\varphi_Y(1/n)(D_{\frac{1}{n}} D_{4^{-d}} f^*(t)) \in \bar{X}$ since $\varphi_Y(1/n)$ is a positive constant and D_s is a bounded operator by (2.11). Using Fatou's lemma we find:

$$
\begin{aligned}
\left\| \lim_{s \downarrow 0} \varphi_Y(s)(D_s D_{4^{-d}} f^*(t)) \right\|_{\bar{X}(t)} &= \left\| \liminf_{n \to \infty} \varphi_Y(1/n)(D_{\frac{1}{n}} D_{4^{-d}} f^*(t)) \right\|_{\bar{X}(t)} \\
&\leq \liminf_{n \to \infty} \left\| \varphi_Y(1/n)(D_{\frac{1}{n}} D_{4^{-d}} f^*(t)) \right\|_{\bar{X}(t)} \\
&\leq \|D_{4^{-d}}\|_{\mathscr{B}(\bar{X})} \liminf_{n \to \infty} \varphi_Y(1/n) \left\| D_{\frac{1}{n}} \right\|_{\mathscr{B}(\bar{X})} \left\| f^*(t) \right\|_{\bar{X}(t)} \\
&\leq \|D_{4^{-d}}\|_{\mathscr{B}(\bar{X})} \|f\|_X \lim_{s \downarrow 0} \varphi_Y(s) \|D_s\|_{\mathscr{B}(\bar{X})}
\end{aligned}
$$

By property Lemma 2.2.7 we find

$$
\begin{aligned}
\left\| \int_0^1 D_s D_{4^{-d}} f^*(t) \mathrm{d}\tilde{\varphi}_Y(s) \right\|_{\bar{X}(t)} &\leq \|D_{4^{-d}}\|_{\mathscr{B}(\bar{X})} \int_0^1 \left\| D_s f^*(t) \right\|_{\bar{X}(t)} \mathrm{d}\tilde{\varphi}_Y(s) \\
&\leq C \|f\|_X \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \mathrm{d}\tilde{\varphi}_Y(s)
\end{aligned}
$$

Combining these two bounds with (2.23) gives:

$$
\begin{aligned}
&\left\| \lim_{s \downarrow 0} \varphi_Y(s)(D_s D_{4^{-d}} f^*(t)) \right\|_{\bar{X}(t)} + \left\| \int_0^1 D_s D_{4^{-d}} f^*(t) \mathrm{d}\tilde{\varphi}_Y(s) \right\|_{\bar{X}(t)} \\
\leq &\|D_{4^{-d}}\|_{\mathscr{B}(\bar{X})} \|f\|_X \left(\lim_{s \downarrow 0} \varphi_Y(s) \|D_s\|_{\mathscr{B}(\bar{X})} + \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \mathrm{d}\tilde{\varphi}_Y(s) \right) \\
= &\|D_{4^{-d}}\|_{\mathscr{B}(\bar{X})} \|f\|_X \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \mathrm{d}\varphi_Y(s) < \infty.
\end{aligned}
$$

property (2.11) finishes the proof for all $f \in E_{loc}$.

2. Note that M_Y is a positive sublinear operator and so, for all $f, g \in Y_{loc}$, we have

$$
|M_Y f - M_Y g| \leq M_Y(f - g), \quad a.e.
$$

Since $E_{loc} \cap X$ is dense in X, the above estimate allows us to extend the inequality obtained in 1. for all $f \in X$ by density. □

Remark 2.2.9. Suppose X is separable, then by Theorem 1.1.8 we have that the simple functions are dense in X. Since E and X are both r.i. Banach function spaces, they contain the simple functions. Thus, $E_{loc} \cap X$ is dense in X.

Theorem 2.2.8 is a slight improvement over [8, Theorem 3.6] in that we don't require $\varphi_Y(0+) = 0$, which is due to the improvement in Lemma 2.2.5 over [8, Lemma 3.4]. Additionally, the requirement that $E_{loc} \cap X$ is dense is slightly weaker as well. An example of this would be $L^1 = Y$, which we discuss after Corollary 2.3.6. An interesting note is that, if M_Y is bounded on X, it is necessary to have $X \subseteq Y_{loc}$. Indeed, if $f \in L^0$ such that $f \notin Y_{loc}$, we find there exists a compact set A of positive measure such that $\|f\chi_A\|_Y = \infty$, then $M_Y f \notin L^0$ giving $M_Y f \notin X$.

Having established one of the main results, sufficient conditions for boundedness, we will now state a necessary condition in a general setting.

Theorem 2.2.10. *Let X, Y be r.i. Banach function spaces defined on $\mathbb{R}^d$. If M_Y is bounded on X, then*

$$\left\|\varphi_Y(\min\{1, \frac{1}{t}\})\right\|_{\bar{X}} < \infty$$

where $\varphi_Y(t)$ is the fundamental function of Y.

For convinience's sake we will write $\max\{|x_1|, \cdots, |x_d|\} = \|x\|_{\ell^\infty}$, $x \in \mathbb{R}^d$ in the following proof:

Proof. Let $f := \chi_{[-1,1)^d}$, then using (2.10) we find the following:

$$\begin{aligned}
\left\|D_{l(Q)}(f\chi_Q)\right\|_Y &= \left\|D_{|Q|}\left(\chi_{[-1,1)^d}\chi_Q\right)^*\right\|_{\bar{Y}} \\
&= \left\|D_{|Q|}\chi_{\left[0,\lambda\left([-1,1)^d \cap Q\right)\right)}\right\|_{\bar{Y}} \\
&= \left\|\chi_{\left[0,\lambda\left([-1,1)^d \cap Q\right)|Q|^{-1}\right)}\right\|_{\bar{Y}}
\end{aligned}$$

We now try to find an appropriate Q. Let Q_x be a closed cube such that one of its corners is the origin with side length $l(Q) = \|x\|_{\ell^\infty}$. Then, we require $x \in Q_x$. Note that this gives multiple but finite options for Q_x, any will suffice. For clarification: Say that $x = (2, 1)$ in $\mathbb{R}^2$, then there is only one closed cube such that it has one corner in the origin, side length 2 and contains x. If $x = (2, 0)$, the cube there are 2 closed cubes containing x.

If $\|x\|_{\ell^\infty} > 1$, then Q_x has one corner at the origin and a side length larger than 1, we find $|[-1, 1)^d \cap Q| = 1$, and

$$|[-1, 1)^d \cap Q_x||Q_x|^{-1} = |Q_x|^{-1} = \|x\|_{\ell^\infty}^{-d}.$$

If $\|x\|_{\ell\infty} \leq 1$ we find that $Q_x \subseteq [-1,1)^d$, then

$$|[-1,1)^d \cap Q_x||Q_x|^{-1} = |Q_x||Q_x|^{-1} = 1.$$

Then we find that

$$\left\|\chi_{\left[0,\lambda\left([-1,1)^d \cap Q_x\right)|Q_x|^{-1}\right)}\right\|_{\bar{Y}} \geq \left\|\chi_{\left[0,\min\left\{1,\,\|x\|_{\ell\infty}^{-d}\right\}\right)}\right\|_{\bar{Y}}$$

Taking the supremum over $Q \ni x$ gives

$$\begin{aligned}
M_Y f(x) &= \sup_{Q\ni x}\left\|\chi_{\left[0,\lambda\left([-1,1)^d \cap Q\right)|Q|^{-1}\right)}\right\|_{\bar{Y}} \\
&\geq \left\|\chi_{\left[0,\min\left\{1,\,\|x\|_{\ell\infty}^{-d}\right\}\right)}\right\|_{\bar{Y}} = \varphi_Y\left(\min\left\{1,\,\|x\|_{\ell\infty}^{-d}\right\}\right)
\end{aligned}$$

We define the following two functions

$$\begin{aligned}
&g : \mathbb{R}^d \to \mathbb{R}^+ \\
&g(x) := \varphi_Y\left(\min\left\{1,\,\|x\|_{\ell\infty}^{-d}\right\}\right) \\
&h : \mathbb{R}^+ \to \mathbb{R}^+ \\
&h(t) = \varphi_Y\left(\min\left\{1, t^{-1}\right\}\right)
\end{aligned}$$

Now by definition,

$$x \in \{x \in \mathbb{R}^d : g(x) > s\} \iff \|x\|_{\ell\infty}^d \in \{t \in \mathbb{R}^+ : h(t) > s\}.$$

Now notice $h(t)$ is decreasing and continuous on $(0,\infty)$, then

$$\{t \in \mathbb{R}^+ : h(t) > s\} = \big(0, \lambda(\{t \in \mathbb{R}^+ : h(t) > s\})\big).$$

Thus for $s > 0$

$$\begin{aligned}
\lambda_g(s) =& \lambda(\{x \in \mathbb{R}^d : g(x) > s\}) \\
=& \lambda\left(\left\{x \in \mathbb{R}^d : \|x\|_{\ell\infty}^d \in \{t \in \mathbb{R}^+ : h(t) > s\}\right\}\right) \\
=& \lambda\left(\left\{x \in \mathbb{R}^d : \|x\|_{\ell\infty}^d < \lambda\left(\{t \in \mathbb{R}^+ : h(t) > s\}\right)\right\}\right) \\
=& 2^d \lambda(\{t \in \mathbb{R}^+ : h(t) > s\}) = 2^d \lambda_h(s)
\end{aligned}$$

Using property (2.9) we find:

$$\lambda_g(s) = \lambda_{D_{2^{-d}}h}(s)$$

Thus, have that g and $D_{2^{-d}}h$ are equimeasurable, and thus:

$$M_Y(\chi_{[-1,1)^d})^*(t) \geq D_{2^{-d}}h(t)$$

Note the fact that for r.i. Banach function spaces we have $\chi_{[-1,1)^d} \in X$, then by the boundedness of M_Y on X we have that $\Big(M_Y(\chi_{[-1,1)^d})\Big)^* \in \bar{X}$, finally (2.11) concludes the proof. $\square$

Due to the abstract definition of a r.i. norm, it will be hard to formulate when exactly $\|\varphi_Y\|_{\bar{X}}$ is finite for a given r.i. Banach function space Y. It will however, be easier to formulate this condition for a given X: Suppose $1 < p < \infty$, then recall we have the r.i. norm of L^p in explicit form (1.17). Theorem 2.2.10 gives the following necessary condition for boundedness of M_Y on L^p:

$$\int_0^1 \varphi_Y(s)^p s^{-2} \mathrm{d}s = \int_1^\infty \varphi_Y(1/t)^p \mathrm{d}t < \infty$$

This is equivalent to the necessary condition for boundedness of M_Y on L^p shown in [8, Theorem 3.9]:

$$\int_0^c \frac{\varphi_Y(s)^p}{s^2} ds < \infty$$

In the next section we will show that under some assumptions, this necessary condition is equivalent to the sufficient conditions given in Theorem 2.2.8.

2.3 Indices on rearrangement invariant Banach function spaces

In this section we look at how the Boyd indices and fundamental indices effect boundedness of the generalized maximal operator defined in section 2.2. We also introduce weak fundamental indices and prove some useful properties. Finally we show under which assumptions we can formulate equivalent necessary and sufficient conditions for boundedness of a maximal operator M_Y on Lorentz spaces $L^{p,q}$, and Orlicz spaces L^Ψ. We begin by defining all indices, we also restate the definition of both Boyd indices for completeness's sake.

Definition 2.3.1. *Let X be an r.i. Banach function space on $\mathbb{R}^d$ and let $\Phi_X(t) = \sup\limits_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)}$, The upper and lower Boyd indices are given respectively by:*

$$\bar{\alpha}_X := \inf_{1<t<\infty} \frac{\log \|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t}, \text{ and, } \underline{\alpha}_X := \sup_{0<t<1} \frac{\log \|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t}$$

Secondly we define

$$\bar{\beta}_X := \inf_{1<t<\infty} \frac{\log \Phi_X(t)}{\log t}, \text{ and, } \underline{\beta}_X := \sup_{0<t<1} \frac{\log \Phi_X(t)}{\log t} \tag{2.24}$$

Which are called the upper and lower fundamental indices. Lastly, the upper and lower weak fundamental indices are given by:

$$\bar{\gamma}_X := \liminf_{t\to\infty} \frac{\log \varphi_X(t)}{\log t}, \text{ and, } \underline{\gamma}_X := \limsup_{t\downarrow 0} \frac{\log \varphi_X(t)}{\log t}$$

Notice that the weak fundamental indices are defined with lim sup and lim inf rather than the supremum and infimum themselves. This is due to the fact that φ_X lacks some of the properties that Φ_X and $\|D_t\|_{\mathscr{B}(\bar{X})}$ have. We will need the following definition to formulate some properties of the indices.

Definition 2.3.2. *We say a function $\varphi : (0,\infty) \to [0,\infty)$ is submultiplicative near 0 when there exist $c, \varepsilon > 0$ such that for all $\varepsilon > t > 0$ we have*

$$\varphi(st) \leq c\varphi(s)\varphi(t), \quad (s \in (0,\infty)).$$

Similarly, we say φ is submultiplicative for large t when there exist $T, c > 0$ such that for all $t > T$ we have

$$\varphi(st) \leq c\varphi(s)\varphi(t), \quad (s \in (0,\infty)).$$

A function is called submultiplicative if there is a $c > 0$ such that the inequality holds for all t. We can interchange submultiplicative with supermultiplicative when the same holds with the inequalities turned around.

We see that both Φ_Y and $\|D_t\|_{\mathscr{B}(\bar{X})}$ are submultiplicative. For fundamental functions φ_Y this is not necessarily the case. We first show some useful properties of Φ_Y and φ_Y:

Proposition 2.3.3. *Let X be an r.i. Banach function space on $\mathbb{R}^d$, then*

$$\Phi_X(ts) \leq \Phi_X(t)\Phi_X(s) \text{ for all } 0 < s, t < \infty \tag{2.25}$$

$$\Phi_X(t) = t\Phi_{X'}\left(\frac{1}{t}\right) \tag{2.26}$$

$$\varphi_X(t) \leq \varphi_X(1)\Phi_X(t) \tag{2.27}$$

$$\Phi_X(t) \leq c\varphi_X(t) \text{ near 0 or for large } t \text{ when } \varphi_X \text{ is submultiplicative with constant } c > 0 \text{ near 0 or for large } t \text{ respectively.} \tag{2.28}$$

Proof. We first note that

$$\Phi_X(st) = \sup_{0<u<\infty} \frac{\varphi_X(stu)}{\varphi_X(u)} = \sup_{0<u<\infty} \frac{\varphi_X(stu)}{\varphi_X(su)} \frac{\varphi_X(su)}{\varphi_X(u)} \leq \Phi_X(t)\Phi_X(s)$$

giving us property (2.25). Consider the following identity, which we obtain using (1.18):

$$\Phi_X(t) = \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} = \sup_{0<s<\infty} \frac{t\varphi_{X'}(s)}{\varphi_{X'}(st)} = t \sup_{0<s<\infty} \frac{\varphi_{X'}(\frac{s}{t})}{\varphi_{X'}(s)} = t\Phi_{X'}\left(\frac{1}{t}\right)$$

giving us (2.26)
Property (2.27) follows from:

$$\varphi_X(1)\Phi_X(t) = \varphi_X(1) \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} \geq \frac{\varphi_X(1)\varphi_X(t)}{\varphi_X(1)}$$

For property (2.28) we suppose φ_X is submultiplicative near 0 and for large t. Then there is a $T, c, \varepsilon > 0$ such that

$$\Phi_X(t) = \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} \leq \sup_{0<s<\infty} \frac{\varphi_X(s)c\varphi_X(t)}{\varphi_X(s)} = c\varphi_X(t), \quad \begin{pmatrix} \infty > t > T, \\ \varepsilon > t > 0 \end{pmatrix}$$

this shows both cases for the inequality in(2.28), concluding the proof. □

In the following proposition we will see what effect the submultiplicativity of φ_Y has on the indices. We also prove a lot of useful properties for the indices in general.

Proposition 2.3.4. *Let X be an r.i. Banach function space on $\mathbb{R}^d$, then*

$$\underline{\beta}_{X'} = 1 - \bar{\beta}_X, \quad \bar{\beta}_{X'} = 1 - \underline{\beta}_X \tag{2.29}$$

$$\underline{\beta}_X = \lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} \tag{2.30}$$

$$\bar{\beta}_X = \lim_{t\to\infty} \frac{\log \Phi_X(t)}{\log t} \tag{2.31}$$

$$0 \le \underline{\alpha}_X \le \underline{\beta}_X \le \bar{\beta}_X \le \bar{\alpha}_X \le 1 \tag{2.32}$$

$$\bar{\alpha}_X = \lim_{t\to\infty} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t}, \quad \underline{\alpha}_X = \lim_{t\downarrow 0} \frac{\log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}}{\log t} \tag{2.33}$$

$$\textit{If } \varphi_X(t) \textit{ is submultiplicative near 0, then } \underline{\gamma}_X = \underline{\beta}_X \tag{2.34}$$

$$\textit{If } \varphi_X(t) \textit{ is submultiplicative for } t \textit{ large enough, } \bar{\gamma}_X = \bar{\beta}_X \tag{2.35}$$

$$\underline{\beta}_X \le \bar{\gamma}_X,\ \underline{\gamma}_X, \quad \bar{\gamma}_X,\ \underline{\gamma}_X \le \bar{\beta}_X \tag{2.36}$$

Proof. From (2.26) we immediately get

$$\frac{\log \Phi_X(t)}{\log t} = 1 - \frac{\log \Phi_{X'}\left(\frac{1}{t}\right)}{\log \frac{1}{t}}, \quad (0 < t < \infty) \tag{2.37}$$

Combining (2.37), (2.24) and the fact that $X'' = X$(Theorem 1.1.5) gives us both identities in (2.29).

For the next property we refer to [1, Lemma III.5.8], which states that an increasing subadditive function ω on $(-\infty, \infty)$ for which $\omega(0) = 0$ we have that $\omega(s)/s$ tends to a finite limit $a = \lim_{t\to\infty} \frac{\omega(s)}{s} = \inf_{s>0} \frac{\omega(s)}{s}$ Then due to the submultiplicativity of $\Phi_X(t)$ and the fact that $\Phi_X(1) = 1$ we get that $\log \Phi_X(e^t)$ is such a function and thus

$$\lim_{t\to\infty} \frac{\log \Phi_X(t)}{\log t} = \lim_{s\to\infty} \frac{\log \Phi_X(e^s)}{s} = \inf_{s>0} \frac{\log \Phi_X(e^s)}{s} = \inf_{t>1} \frac{\log \Phi_X(t)}{\log t}$$

This gives us (2.31). Then combining this result with (2.37) and the second equation in (2.29) we obtain (2.30).

For the fact that $0 \le \underline{\alpha}_X \le \bar{\alpha}_X \le 1$ we refer to [1, Proposition III.5.13] Now notice that we have

$$\Phi_X(t) = \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} = \sup_{0<s<\infty} \frac{\left\|D_{t^{-1}}\chi_{[0,s)}\right\|_{\bar{X}}}{\varphi_X(s)} \le \|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}$$

We find that $\log \Phi_X(t) \le \log\|D_{t^{-1}}\|_{\mathscr{B}(\bar{X})}$ since log is an increasing function on $\mathbb{R}^+$. Plugging this inequality into our definitions we get $\bar{\beta}_X \le \bar{\alpha}_X$ since

$\log t > 0$ for $t > 1$, similarly, $\underline{\alpha}_X \leq \underline{\beta}_X$ since $\log t < 0$ for $0 < t < 1$. Finally, using the submultiplicativity of $\Phi_X(t)$ we find $1 = \Phi_X(1) \leq \Phi_X(t)\Phi_X\left(\frac{1}{t}\right)$. Hence, for all $t > 1$,

$$\frac{\log \Phi_X\left(\frac{1}{t}\right)}{\log \frac{1}{t}} = \frac{\log\left(\frac{1}{\Phi_X\left(\frac{1}{t}\right)}\right)}{\log t} \leq \frac{\log \Phi_X(t)}{\log t}$$

Letting $t \to \infty$ and using both (2.30) and (2.31) then gives us $\underline{\beta}_X \leq \bar{\beta}_X$, giving us property (2.32). For property (2.33) we refer to [1, Proposisition III.5.13].

From now on in this proof we will use (2.30) and (2.31) interchangeably with the definition of the fundamental indices. To prove (2.34) we assume that $\varphi_X(t)$ is submultiplicative near 0 with $\varepsilon, c > 0$. Then, using properties (2.28) and (2.27), for $0 < t < \varepsilon$ we have

$$\Phi_X(t) = \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} \leq c\varphi_X(t) \leq c\varphi_X(1)\Phi_X(t).$$

Notice that for all $c > 0$ we have

$$\lim_{t\downarrow 0} \frac{\log c\Phi_X(t)}{\log t} = \lim_{t\downarrow 0} \frac{\log \Phi_X(t) + \log c}{\log t} = \lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} \tag{2.38}$$

, the same is true for φ_X and the lim sup. Then indeed

$$\begin{aligned}
\underline{\beta}_X =& \lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} \\
\geq& \limsup_{t\downarrow 0} \frac{\log \varphi_X(t)}{\log t} = \underline{\gamma}_X \\
\geq& \lim_{t\downarrow 0} \frac{\log c\varphi_X(1)\Phi_X(t)}{\log t} = \underline{\beta}_X
\end{aligned}$$

giving $\underline{\beta}_X = \underline{\gamma}_X$. The proof of (2.35) is similar, and will be left as an exercise for the reader. For (2.36), we first show $\bar{\beta}_X \geq \bar{\gamma}_X$ and $\underline{\beta}_X \leq \underline{\gamma}_X$. By (2.27) we obtain that

$$\begin{aligned}
\frac{\log \varphi_X(1)\Phi_X(t)}{\log t} &\leq \frac{\log \varphi_X(t)}{\log t}, \quad (0 < t < 1) \\
\frac{\log \varphi_X(1)\Phi_X(t)}{\log t} &\geq \frac{\log \varphi_X(t)}{\log t}, \quad (\infty > t > 1)
\end{aligned}$$

Using (2.38), we find that:

$$\underline{\beta}_X = \lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} = \limsup_{t\downarrow 0} \frac{\log \varphi_X(1)\Phi_X(t)}{\log t} \leq \limsup_{t\downarrow 0} \frac{\log \varphi_X(t)}{\log t} = \underline{\gamma}_X$$

Similarly we find

$$\bar{\gamma}_X = \liminf_{t\to\infty} \frac{\log \varphi_X(t)}{\log t} \leq \liminf_{t\to\infty} \frac{\log \varphi_X(1)\Phi_X(t)}{\log t} = \lim_{t\to\infty} \frac{\log \Phi_X(t)}{\log t} = \bar{\beta}_X$$

Now what is left to do is to show $\bar{\beta}_X \geq \underline{\gamma}_X$ and $\underline{\beta}_X \leq \bar{\gamma}_X$. Using $\varphi_X(t)\varphi_{X'}(t) = t$, (1.18), we find

$$\begin{aligned}\bar{\gamma}_X &= \liminf_{t\to\infty} \frac{\log \varphi_X(t)}{\log t} = \liminf_{t\downarrow 0} \frac{\log \varphi_X\left(\frac{1}{t}\right)}{\log \frac{1}{t}} \\ &= \liminf_{t\downarrow 0} \frac{\log\left(\frac{1}{\varphi_X\left(\frac{1}{t}\right)}\right)}{\log t} = \liminf_{t\downarrow 0} \frac{\log\left(t\varphi_{X'}\left(\frac{1}{t}\right)\right)}{\log t} \end{aligned} \tag{2.39}$$

Then using (2.26) we find

$$\lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} = \lim_{t\downarrow 0} \frac{\log\left(t\Phi_{X'}\left(\frac{1}{t}\right)\right)}{\log t}$$

now using that $\varphi_{X'}(1)\Phi_{X'}(t) \geq \varphi_{X'}(t)$ (2.27), the fact that log is increasing and that $\log t < 0$ for $0 < t < 1$ we find

$$\begin{aligned}\underline{\beta}_X &= \lim_{t\downarrow 0} \frac{\log \Phi_X(t)}{\log t} = \liminf_{t\downarrow 0} \frac{\log\left(t\Phi_{X'}\left(\frac{1}{t}\right)\right)}{\log t} \\ &\leq \liminf_{t\downarrow 0} \frac{\log\left(t\varphi_{X'}\left(\frac{1}{t}\right)\right)}{\log t} = \bar{\gamma}_X\end{aligned}$$

Using the same trick for $\underline{\gamma}_X$ as we did in (2.39) for $\bar{\gamma}_X$ we find

$$\underline{\gamma}_X = \limsup_{t\to\infty} \frac{\log\left(t\varphi_{X'}\left(\frac{1}{t}\right)\right)}{\log t}$$

and we similarly obtain

$$\begin{aligned}\bar{\beta}_X &= \lim_{t\to\infty} \frac{\log \Phi_X(t)}{\log t} = \limsup_{t\to\infty} \frac{\log\left(t\Phi_{X'}\left(\frac{1}{t}\right)\right)}{\log t} \\ &\geq \limsup_{t\to\infty} \frac{\log\left(t\varphi_{X'}\left(\frac{1}{t}\right)\right)}{\log t} = \underline{\gamma}_X\end{aligned}$$

Note that the inequality is the other way because $\log t > 0$ for $t > 1$. This concludes the proof.

$\square$

We see that we don't yet have $\underline{\gamma}_X \leq \bar{\gamma}_X$ when φ_X is not submultiplicative. Note that to obtain this φ_X need only be submultiplicative on one side for this inequality to hold. As if it were submultiplicative either near 0 or for large t, one of the bounds in (2.36) would yield equality by (2.34) or (2.35).

Lemma 2.3.5. *Let Y, X be r.i. Banach function spaces on $\mathbb{R}^d$ such that φ_Y is concave. Then*

$$\left\| \|D_t\|_{\mathscr{B}(\bar{X})} \chi_{[0,1)}(t) \right\|_{\Lambda(Y)} < \infty \implies \underline{\gamma}_Y \geq \bar{\alpha}_X \tag{2.40}$$

$$\left\| \|D_t\|_{\mathscr{B}(\bar{X})} \chi_{[0,1)}(t) \right\|_{\Lambda(Y)} < \infty \impliedby \underline{\gamma}_Y > \bar{\alpha}_X \tag{2.41}$$

Proof. For (2.40), consider

$$\begin{aligned}\lim_{t\downarrow 0} \|D_t\|_{\mathscr{B}(\bar{X})} \varphi_Y(t) &\leq \lim_{t\downarrow 0} \|D_t\|_{\mathscr{B}(\bar{X})} \varphi_Y(t) + \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \varphi_Y'(s) \mathrm{d}s \\ &= \left\| \|D_t\|_{\mathscr{B}(\bar{X})} \chi_{[0,1)}(t) \right\|_{\Lambda(Y)} < \infty\end{aligned}$$

Then by (2.32) we have that$\bar{\alpha}_X = \lim_{t\downarrow 0} \frac{\log \|D_t\|_{\mathscr{B}(\bar{X})}}{\log 1/t}$, giving

$$\lim_{t\downarrow 0} \|D_t\|_{\mathscr{B}(\bar{X})} \varphi_Y(t) = \lim_{t\downarrow 0} t^{\underline{\gamma}_Y - \bar{\alpha}_X} < \infty$$

so that indeed $\bar{\alpha}_X \leq \underline{\gamma}_Y$. By Proposition 1.2.14 we have that $\varphi_Y'(t) \leq \frac{\varphi_Y(t)}{t}$ for almost every t. Then,

$$\begin{aligned}\left\| \|D_t\|_{\mathscr{B}(\bar{X})} \chi_{[0,1)}(t) \right\|_{\Lambda(Y)} &= \lim_{t\downarrow 0} \|D_t\|_{\mathscr{B}(\bar{X})} \varphi_Y(t) + \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \varphi_Y'(s) \mathrm{d}s \quad . \\ &\leq \lim_{t\downarrow 0} t^{\underline{\gamma}_Y - \bar{\alpha}_X} + \int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \frac{\varphi_Y(s)}{s} \mathrm{d}s\end{aligned}$$

Then since

$$\lim_{t\downarrow 0} \frac{\log \|D_t\|_{\mathscr{B}(\bar{X})}}{\log \frac{1}{t}} = \bar{\alpha}_X < \underline{\gamma}_Y = \limsup_{t\downarrow 0} \frac{\log \varphi_Y(t)}{\log t}$$

we get that for $\varepsilon > 0$ with $\bar{\alpha}_X + \varepsilon < \underline{\gamma}_Y - \varepsilon$, there exists a $T < 1$ such that

$$\frac{\log \|D_t\|_{\mathscr{B}(\bar{X})}}{\log \frac{1}{t}} < \bar{\alpha}_X + \varepsilon < \underline{\gamma}_Y - \varepsilon < \frac{\log \varphi_Y(t)}{\log t}$$

for all $t < T$. Then

$$\int_0^1 \|D_s\|_{\mathscr{B}(\bar{X})} \frac{\varphi_Y(s)}{s} \mathrm{d}s \leq \int_0^T s^{\underline{\gamma}_Y - \varepsilon - (\bar{\alpha}_X + \varepsilon) - 1} \mathrm{d}s + \|D_T\|_{\mathscr{B}(\bar{X})} \int_T^1 \frac{\varphi_Y(1)}{s} \mathrm{d}s < \infty$$

Note that $\varphi_Y(t) \leq \varphi_Y(1)$ for $t \leq 1$ since φ_Y is increasing. This concludes the proof of (2.41). $\square$

These statements are close to being equivalent, however, we have the following counter examples to show that such an equivalence is false: $X = Y = L^\infty$ has that $0 = \bar{\alpha}_{L^\infty} \geq \underline{\gamma}_{L^\infty}$, and $\Lambda(L^\infty) = L^\infty$(see Remark 1.2.18). Then indeed

$$\left\| \|D_t\|_{\mathscr{B}(\bar{L^\infty})} \chi_{[0,1)}(t) \right\|_{\Lambda(L^\infty)} = \left\| \chi_{[0,1)} \right\|_{L^\infty} < \infty$$

showing that (2.40) cannot be improved to a strict inequality. On the other hand, for L^1 we have $\underline{\gamma}_{L^1} = 1 = \bar{\alpha}_{L^1}$ and by Example 1.23 we have

$$\left\| \|D_t\|_{\mathscr{B}(\bar{L^1})} \chi_{[0,1)}(t) \right\|_{\Lambda(L^1)} = \int_0^1 \frac{1}{t} \mathrm{d}t = \infty.$$

which shows us that the strict inequality in (2.41) cannot be relaxed.

Corollary 2.3.6. *Let Y, X and E be r.i. Banach function spaces on $\mathbb{R}^d$, with E generated by $\Lambda(Y)$. If $E_{loc} \cap X$ is dense in X and*

$$\underline{\gamma}_Y > \bar{\alpha}_X \text{ or } \underline{\beta}_Y > \bar{\alpha}_X, \tag{2.42}$$

M_Y is a bounded operator on X.

Proof. For $\underline{\gamma}_Y$ this follows directly from Lemma 2.3.5, Lemma 1.2.15 and Theorem 2.2.8. For $\underline{\beta}_Y$ this follows from (2.36) and the same argumentation. $\square$

For a space with submultiplicative φ_Y, like L^p, the inequalities in (2.42) are the same. However, in general we don't know if φ_Y is submultiplicative. The following is an example of how we can formulate equivalent sufficient and necessary conditions. The sufficient conditions given in Corollary 2.3.6 include the sufficient conditions we found in Theorem 2.1.15 as the particular case of $Y = L^1$, since we have

$$\overline{L^1}_{loc} \supseteq L^1 + L^\infty \supseteq \bar{X},$$

so that the density condition is always satisfied. Then indeed since $\underline{\gamma}_{L^1} = 1$ we again find that M_{L^1} is bounded on X if and only if $\bar{\alpha}_X < 1$.

Corollary 2.3.7. *Suppose* $1 < p < \infty$, $1 \leq q < \infty$ *and let* $L^{p,q}(\mathbb{R}^d)$ *be as in Definition 1.2.19. Let* Y, E *be r.i. Banach function spaces on* $\mathbb{R}^d$ *such that* φ_Y *is submultiplicative near 0 and* E *is generated by* $\Lambda(Y)$*. Then* M_Y *is bounded on* $L^{p,q}$ *if and only if* $\underline{\beta}_Y > \frac{1}{p}$.

Proof. Notice first that by [4, Theorem 1.4.13] we find that simple functions are dense in $L^{p,q}$ for $0 < q < \infty$, since all r.i. Banach function spaces contain the simple functions we find that $E_{loc} \cap L^{p,q}$ is dense in $L^{p,q}$. As mentioned before, the Boyd indices of $L^{p,q}$ are given by $\underline{\alpha}_{p,q} = \bar{\alpha}_{p,q} = \frac{1}{p}$. By Corollary 2.3.6, $\underline{\beta}_Y > \bar{\alpha}_{p,q}$ is a sufficient condition for boundedness of M_Y on $L^{p,q}$. Now suppose that M_Y is bounded on $L^{p,q}$, then by Theorem 2.2.10 we find that $\left\|\varphi_Y(\min\{1, 1/t\})\right\|_{p,q} < \infty$. Recall for $\rho_{p,q}$, as in (1.22), we have

$$\rho_{p,q}(f) \leq \|f\|_{p,q} \leq \frac{p}{p-1}\rho_{p,q}(f).$$

Then since φ_Y is submultiplicative near 0, by (2.28) we find there is a $c > 0$ and $1 > \varepsilon > 0$ such that

$$\Phi_Y(t) \leq c\varphi_Y(t)$$

for all $0 < t < \varepsilon$. Then by definition of $\underline{\beta}_Y$ we get for all $0 < t < \varepsilon$ that

$$\frac{\log c\varphi_Y(t)}{\log t} \leq \frac{\log \Phi_Y(t)}{\log t} \leq \sup_{0<s<1} \frac{\log \Phi_Y(s)}{\log s} = \underline{\beta}_Y$$

We use that log is an increasing function and $\log t < 0$ when $t < 1$. Now see that

$$\begin{aligned}
\rho^q_{p,q}(\varphi_Y(\min\{1, 1/t\})) &= \int_0^\infty [t^{1/p}\varphi_Y(\min\{1, 1/t\})]^q \frac{\mathrm{d}t}{t} \\
&\geq \int_{\varepsilon^{-1}}^\infty [t^{1/p}\varphi_Y(1/t)]^q \frac{\mathrm{d}t}{t} \\
&\geq \int_{\varepsilon^{-1}}^\infty [t^{1/p}\frac{1}{c}(1/t)^{\underline{\beta}_Y}]^q \frac{\mathrm{d}t}{t} \\
&\geq \frac{1}{c^q}\int_{\varepsilon^{-1}}^\infty [t^{1/p-\underline{\beta}_Y}]^q \frac{\mathrm{d}t}{t}
\end{aligned}$$

The observation that this last integral diverges to ∞ when $\underline{\beta}_Y \leq \frac{1}{p}$ combined with the fact that $\rho_{p,q}(f) \leq \|f\|_{p,q}$ then concludes the proof for $q < \infty$. $\square$

Note that for $q = \infty$, a proof of this kind fails to hold, since

$$\sup_{0<s<\infty} t^{1/p-\underline{\beta}_Y} = 1$$

when $1/p = \underline{\beta}_Y$.

Lemma 2.3.8. *Let Y, X be r.i. Banach function spaces. If M_Y is a bounded operator on X, then the weak fundamental indices satisfy:*

$$\bar{\gamma}_X \leq \underline{\gamma}_Y$$

Proof. Using Theorem 2.2.10 and Theorem 1.2.17 we get that there exists some $c > 0$ such that

$$\begin{aligned}
\infty > c\Big\|\varphi_Y\big(\min\{1,t^{-1}\}\big)\Big\|_{\bar{X}} &\geq \Big\|\varphi_Y\big(\min\{1,t^{-1}\}\big)\Big\|_{M_{\varphi_X}} \\
&= \sup_{0<t<\infty} \frac{\varphi_X(t)}{t}\int_0^t \varphi_Y\big(\min\{1,s^{-1}\}\big)\mathrm{d}s \\
&\geq \sup_{0<t<\infty} \varphi_X(t)\varphi_Y\big(\min\{1,t^{-1}\}\big) \\
&\geq \lim_{t\to\infty} \varphi_X(t)\varphi_Y\big(t^{-1}\big) \geq \lim_{t\to\infty} t^{\bar{\gamma}_X-\underline{\gamma}_Y}.
\end{aligned}$$

Then indeed this limit is only finite when $\bar{\gamma}_X \leq \underline{\gamma}_Y$.

□

Lemma 2.3.8 may give more extensive results when combining it with Proposition 2.3.4. Note that the inequality in Lemma 2.3.8 cannot be improved to a strict inequality in the general case, for example take M_{L^∞} which is bounded on L^∞ itself. Then indeed $\bar{\gamma}_{L^\infty} = 0 = \underline{\gamma}_{L^\infty}$. For our final result, we introduce Orlicz spaces, a generalization of L^p.

Definition 2.3.9. *Let $\psi : [0,\infty) \to [0,\infty]$ be increasing and left-continuous, with $\psi(0) = 0$. Suppose on $(0,\infty)$ that ψ is neither identically equal to 0 or ∞, that is, there exists $x \in (0,\infty)$ such that $0 < \psi(x) < \infty$. Then the function Ψ[2] defined by*

$$\Psi(s) = \int_0^s \psi(u)\mathrm{d}u, \quad (s \geq 0)$$

is said to be a Young's function.

[2]Usually we use Φ to represent the Orlicz space, but to avoid confusion with Φ defined for the fundamental indices we use Ψ.

Definition 2.3.10. *Let Ψ be a Young's function. The Orlicz space $L^\Psi(\mathbb{R}^+)$, with its norm given by:*

$$\|f\|_{L^\Psi(\mathbb{R}^+)} = \inf\{k : \int_0^\infty \Psi(k^{-1}f(x))\mathrm{d}x \leq 1\}$$

In [9, Theorem 2.6.9] it is shown that this is a Banach function spaces, in [7, Page 120] it is shown that this norm is r.i. Then using [1, Theorem II.4.9] we find that the following norm is r.i. and produces a Banach function space:

$$\|f\|_{L^\Psi(\mathbb{R}^d)} := \|f^*\|_{L^\Psi(\mathbb{R}^+)} .$$

In the sense of the Luxemburg representation theorem we have $\overline{L^\Psi(\mathbb{R}^d)} = L^\Psi(\mathbb{R}^+)$. From now on we will use L^Ψ to denote a space on $\mathbb{R}^d$ and $\overline{L^\Psi}$ a space on $\mathbb{R}^+$. Consider the following:

Definition 2.3.11. *Let Ψ be a Young's function, then we call*

$$\Psi^{-1}(t) = \sup\{s : \Psi(s) \leq t\}$$

the right continuous inverse.

Now let $s_0 = \sup\{s : \Psi(s) = 0\}$ and $s_\infty = \inf\{s : \Psi(s) = \infty\}$ then by its definition we find that Ψ is strictly increasing and continuous on $[s_0, s_\infty)$. Then we find that:

$$s = \Psi^{-1}(t) \quad \iff \quad \Psi(s) = t, \quad (s_0 < s < s_\infty).$$

In other words, Ψ is bijective from (s_0, s_∞) to $(0, \infty)$. The following lemma gives a useful explicit form for the fundamental function of an Orlicz space.

Lemma 2.3.12. *Let Ψ be a Young's function, then the fundamental function of L^Ψ is given by*

$$\varphi_\Psi(t) = \frac{1}{\Psi^{-1}(1/t)}, \quad (0 < t < \infty)$$

Additionally, φ_Ψ is submultiplicative for large t when Ψ^{-1} is supermultiplicative near 0 and vice versa.

Proof.

$$\int_0^\infty \Psi(k\chi_{[0,t)}(s))\mathrm{d}s = \int_0^t \Psi(k)\mathrm{d}s = t\Psi(k)$$

Then we have by definition:

$$\begin{aligned}\varphi_\Psi(t) =&\left\|\chi_{[0,t)}\right\|_{L^\Psi} = \inf\{k : \Psi(k^{-1}) \le 1/t\}\\ =&\big(\sup\{k^{-1} : \Psi(k^{-1}) \le 1/t\}\big)^{-1}\\ =&\big(\sup\{k : \Psi(k) \le 1/t\}\big)^{-1} = \frac{1}{\Psi^{-1}(1/t)}.\end{aligned}$$

Furthermore, if φ_Ψ is submultiplicative for large t we have the following: there exists a $\infty > T > 0$ such that for all $t > T$ and all $s \in (0,\infty)$ we have

$$\Psi^{-1}\big(1/(st)\big) = \frac{1}{\varphi_\Psi(st)} \ge \frac{1}{c\varphi_\Psi(s)\varphi_\Psi(t)} = \frac{1}{c}\Psi^{-1}(1/s)\Psi^{-1}(1/t).$$

The other way around is shown the same way. □

Note that we still have $\varphi_\Psi(0) = 0$. For the final proof we will need the following theorem:

Theorem 2.3.13. *Let Ψ be a Young's function, and define:*

$$g(t) = \sup\left\{\frac{\Psi^{-1}(s)}{\Psi^{-1}(st)} : s \in (0,\infty)\right\}, \quad t \in (0,\infty)$$

Then the upper and lower Boyd indices of L^Ψ are given by:

$$\bar{\alpha}_\Psi = \lim_{s\to 0+} \frac{-\log g(s)}{\log s}, \quad \underline{\alpha}_\Psi = \lim_{s\to\infty} \frac{-\log g(s)}{\log s}$$

A proof is given in [2]. For our purposes we rewrite this to:

$$\bar{\alpha}_\Psi = \lim_{s\to 0+} \frac{-\log g(s)}{\log s} = \lim_{s\to 0+} \frac{\log g(s)}{\log 1/s} = \lim_{s\to\infty} \frac{\log g(1/s)}{\log s} \tag{2.43}$$

$$\underline{\alpha}_\Psi = \lim_{s\to\infty} \frac{-\log g(s)}{\log s} = \lim_{s\to\infty} \frac{\log g(s)}{\log 1/s} = \lim_{s\to 0+} \frac{\log g(1/s)}{\log s} \tag{2.44}$$

Then notice that $\Phi_{L^\Psi}(t)$ as defined in Definition 2.3.1 can be written as follows by Lemma 2.3.12:

$$\Phi_{L^\Psi}(t) = \sup_{0<s<\infty} \frac{\varphi_\Psi(st)}{\varphi_\Psi(s)} = \sup_{0<s<\infty} \frac{\Psi^{-1}(1/s)}{\Psi^{-1}(1/st)} = \sup_{0<s<\infty} \frac{\Psi^{-1}(s)}{\Psi^{-1}(s/t)} = g(1/t) \tag{2.45}$$

Corollary 2.3.14. *Let Ψ be a Young's function, then*

$$\bar{\alpha}_\Psi = \bar{\beta}_\Psi, \quad \underline{\alpha}_\Psi = \underline{\beta}_\Psi$$

Proof. It follows directly from combining Theorem 2.3.13 with equations (2.43), (2.44) and (2.45). $\square$

Now we are ready to present the theorem for boundedness on Orlicz spaces:

Theorem 2.3.15. *Let Y, E be r.i. Banach function spaces on $\mathbb{R}^d$ such that φ_Y is submultiplicative near 0 and E is generated by $\Lambda(Y)$. Let Ψ be a Young's function such that $\Psi^{-1}(t)$ is supermultiplicative near 0, $s_\infty = \infty$ and such that $E_{loc} \cap L^\Psi$ is dense in L^Ψ. Then M_Y is bounded on $L^\Psi(\mathbb{R}^d)$ if and only if $\underline{\beta}_Y > \bar{\alpha}_\Psi$.*

Proof. Sufficiency follows by applying Corollary 2.3.6. For necessity we use Lemma 2.2.10. Firstly, note that by Lemma 2.3.8 and properties (2.34) and (2.35) from Lemma 2.3.4, we have that $\bar{\beta}_\Psi \leq \underline{\beta}_Y$ is necessary. It will thus suffice to show that $\bar{\beta}_\Psi = \underline{\beta}_Y$ leads to M_Y not being bounded. Now by Lemma 2.3.12, $\varphi_\Psi(t)$ is submultiplicative for large t, thus we find that there are constants $c_1, c_2 > 0$ and $1 > \varepsilon_1, \varepsilon_2 > 0$ such that:

$$\Phi_Y(t) \leq c_1\varphi_Y(t), \quad \Phi_\Psi(s) \leq c_2\varphi_\Psi(s), \quad (0 < t < \varepsilon_1,\ 1/\varepsilon_2 < s < \infty)$$

Set $c = \max\{c_1, c_2\}$ and $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Since log is strictly increasing on $(0, \infty)$ and $\log t < 0$ for $t \in (0, 1)$ we obtain

$$\frac{\log c\varphi_Y(t)}{\log t} \leq \frac{\log \Phi_Y(t)}{\log t} \leq \sup_{0<s<1} \frac{\log \Phi_Y(s)}{\log s} = \underline{\beta}_Y \quad (0 < t < \varepsilon)$$

Similarly, for $t > 1$ we have $\log t > 0$ and thus:

$$\frac{\log c\varphi_\Psi(t)}{\log t} \geq \frac{\log \Phi_\Psi(t)}{\log t} \geq \inf_{s>1} \frac{\log \Phi_\Psi(s)}{\log s} = \bar{\beta}_\Psi, \quad (\varepsilon^{-1} < t < \infty)$$

We consider :

$$\begin{aligned}
\int_0^\infty \Psi\big(\varphi_Y(\min\{1, 1/t\})\big)\mathrm{d}t &= \int_0^1 \Psi(\varphi_Y(1))\mathrm{d}t + \int_1^\infty \Psi(\varphi_Y(1/t))\mathrm{d}t \\
&\geq \int_1^{\varepsilon^{-1}} \Psi(\varphi_Y(1/t))\mathrm{d}t + \int_{\varepsilon^{-1}}^\infty \Psi\bigg(\frac{c\varphi_Y(1/t)}{c}\bigg)\mathrm{d}t \\
&\geq \int_{\varepsilon^{-1}}^\infty \Psi\bigg(\frac{1}{ct^{\underline{\beta}_Y}}\bigg)\mathrm{d}t
\end{aligned}$$

In the last inequality, we use $c\varphi_Y(u) \geq u^{\underline{\beta}_Y}$ for $0 < u < \varepsilon < 1$ and that Ψ is increasing. Additionally we use that Ψ is a nonnegative function in both inequalities. Similarly we find that $c\varphi_\Psi(u) \geq u^{\bar{\beta}_\Psi}$ for $1 < \varepsilon^{-1} < u < \infty$, using Lemma 2.3.12 we obtain

$$u^{-\bar{\beta}_\Psi} \geq \frac{1}{c\varphi_\Psi(u)} = \frac{\Psi^{-1}(1/u)}{c}$$

Then we we get

$$\int_{\varepsilon^{-1}}^{\infty} \Psi\left(\frac{1}{ct^{\underline{\beta}_Y}}\right)\mathrm{d}t \geq \int_{\varepsilon^{-1}}^{\infty} \Psi\left(\frac{\Psi^{-1}(1/t)}{c^2}\right)\mathrm{d}t$$

Let $k \in \mathbb{R}^+$, then since $s_\infty = \infty$, we know that there exists an $\infty > \delta_k > 0$ such that $\Psi^{-1}(\delta_k) \geq ck$. Using the submultiplicativity of φ_Ψ for $t > \varepsilon^{-1}$ we find:

$$\frac{\Psi^{-1}(1/t)}{k} = \frac{1}{k\varphi_Y(t)} \geq \frac{1}{ck\varphi_Y(\delta_k t)\varphi_Y(\delta_k^{-1})} \geq \Psi^{-1}(1/t\delta_k)$$

giving

$$\int_{\varepsilon^{-1}}^{\infty} \Psi\left(\frac{\Psi^{-1}(1/t)}{k}\right)\mathrm{d}t \geq \int_{\varepsilon^{-1}}^{\infty} \frac{1}{\delta t}\mathrm{d}t = \infty$$

Then we find

$$\left\|\varphi_Y(\min\{1, 1/t\})\right\|_{\overline{L^\Psi}} \geq \left\|\chi_{[\varepsilon^{-1},\infty)}\Psi^{-1}(1/t)\right\|_{\overline{L^\Psi}} = \infty$$

□

By [3, Theorem 2.1] we find that we may drop the requirement of density for L^Ψ with $\underline{\beta}_\Psi > 0$, as L^Ψ is separable in this case.

Remark 2.3.16. In general we cannot yet state an if and only if statement, we can, however, state the following: In general, for two r.i. Banach function spaces on $\mathbb{R}^d$ X and Y, we know from Lemma 2.3.8 and Lemma 2.3.6 that for boundedness of M_Y on X, that $\underline{\gamma}_Y \geq \bar{\gamma}_X$ is necessary, and that $\underline{\gamma}_Y > \bar{\alpha}_X$ is sufficient given that for E_{loc} on $\mathbb{R}^d$, generated by $\Lambda(Y)$, $E_{loc} \cap X$ is dense in X.

While in many spaces we have $\bar{\gamma}_X = \bar{\beta}_X = \bar{\alpha}_X$, like $L^{p,q}$, these lemmas still don't lead to an if and only if statement. Indeed, as we saw in Theorem 2.3.15 and Corollary 2.3.7, we used Theorem 2.2.10 to show that, under some assumptions, boundedness of M_Y on restricted X is only attainable for Y such that $\underline{\gamma}_Y > \bar{\alpha}_X$.

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