Time-Varying Biorthogonal Filter Banks: A State-Space Approach

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***Abstract*— Using state-space representations of biorthogonal time-varying filter banks, it is possible to come up with a theory for the transitional behavior between two stationary filter banks. The transition interval depends on the size of the common subspace spanned by the controllability operators of the initial and final filters on the decomposition sides, and the common subspace spanned by the observability operators of the filters on the reconstruction sides. When the respective operators span the same spaces, we can derive conditions under which the transition between the filter banks can be so controlled that both the decomposition and the reconstruction functions gently embrace. For such filters, the transition interval can be made arbitrarily short. If it is zero, then the special case of instantaneous transition is reached.**

***Index Terms*—Biorthogonal filter bank, impulse response (vector), filter weight vector, controllability operator, observability operator, state-space realization (map), stationary filter, transition filter.**

I. Introduction

DUE TO THE fact that filter banks appear in various forms and for various reasons in a wide range of applications, they have become the center of attraction for many researchers. People have started putting a considerable amount of work into the time-varying aspects of filter banks. To facilitate their investigations, most researchers concentrate on the study of transitions between two stationary filter banks. The reason for such considerations is partly that most filter banks tend to operate for considerable durations compared to their lengths, and can be considered stationary at the time of transition.

Generally, when transiting from one stationary filter bank to another without violating biorthogonality, the transition may be either: a) instantaneous or b) not instantaneous. The second category is studied by a number of authors [1]–[4]. An equal number of transition filters both on the decomposition and reconstruction sides are considered in [1]–[3], whereas in [4], unequal transition segments are studied. In [2], orthogonal projectors are used to complete the basis for the sequences generated by one side-bounded orthogonal filter bank and then, the results are directly, but not minimally, extended to the case of transition between filter banks.

The first category—switching between two stationary filter banks without any transition—is considered in [5], where it is shown that if the filters under question are related in a particular way, instantaneous transition both on the decomposition and reconstruction sides is possible.

This paper is motivated by the absence of an underlying theory that relates the above two categories. We show that by using state-space representation it is possible to come up with a comprehensive characterization of the transition filters. Moreover, the state-space approach enables us to give a clear-cut analysis of transition times and transition behavior. Unlike in [2], we first consider the general case of minimal transition between biorthogonal filter banks and then extend the idea to the special cases of: a) starting up a filter and b) terminating a running filter.

In Section II, we use state-space techniques to find the transition filters that give optimal transition duration both on the decomposition and reconstruction sides. Based on the results, we give classification of transition behaviors. In Section III, we discuss conditions under which the so-called lapped and blocked transitions are generated. We also indicate how we can refine the filter characteristics in the transition. Section IV presents a special set of filter banks that allows instantaneous transitions between filters which are reported in [5]. In Section V, we extend these results by allowing two-sided controlled transition with arbitrary transition segments. Finally, in Section VI, we give illustrative examples.

A. Biorthogonal Filter Banks

A biorthogonal filter bank (E, R) is a linear expansion of a sequence or a signal. Thus, if $u \in l_2^{P(n)}(Z)$,[1] then $u = \sum \langle u, e(n) \rangle r(n)$, where $e(n)$ and $r(n)$ are the rows and columns[2] of bounded matrix operators E and R, respectively, with $RE = I$, the identity operator, and $\langle ., . \rangle$ an inner product operator. We will assume throughout that $e(n)$ and $r(n)$ are of finite lengths, and that E is lower triangular (causal) and R upper triangular (anticausal). If the filter bank is stationary, then E and R are (block) Toeplitz operators characterized by their central row e and central column r, respectively. In this

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[1] That is, u is a finite energy sequence or column vector with $P(n)$-dimensional column vector entries $u(n)$, $n \in Z$.

[2] More precisely, $e(n)$ and $r(n)$ are block rows and block columns, respectively. We omit the adjective "block" throughout this paper for readability purposes.

case, we write $(E(e), R(r))$ instead of (E, R):

$$E(e) = \begin{bmatrix} \ddots & & \\ & \boxed{e} & \\ & \boxed{e} & \\ & & \ddots \end{bmatrix} \quad R(r) = \begin{bmatrix} \ddots & \boxed{r} & & \\ & & \boxed{r} & \\ & & & \ddots \end{bmatrix}. \quad (1)$$

In the context of filter banks, the rows of E are called *filter weight vectors* and the columns of R are called *impulse response vectors*. In the stationary case, they have the property that with $F(x)$, the Fourier transform of x, $F(e)$, and $F(r)$ are uniform spectral decompositions of the baseband.

In this paper, we find biorthogonal filter banks giving signal expansions of the form

$$u = \sum_{n=-\infty}^{n_o-1} \langle u, e1(n)\rangle r1(n) + \sum_{n=n_o}^{n_f-1} \langle u, e(n)\rangle r(n) + \sum_{n=n_f}^{+\infty} \langle u, e2(n)\rangle r2(n)$$

where $n_f \geq n_o$, and $e1(n)$ and $e2(n)$ are all shifted versions of $e1$ and $e2$, respectively (and similarly for $r1(n)$ and $r2(n)$). In other words, the behavior on the segments $(-\infty, n_o - 1]$ and $[n_f, +\infty)$ is stationary, whereas the segment $[n_o, n_f - 1]$ is the transition region on which $e(n)$ and $r(n)$ have doubly indexed entries. Typically, the objective is to have a small transition segment supporting a smooth transition between the stationary segments.

B. State-Space Representation

Instead of using input–output maps E and R, the filter bank can also be represented by state-space realizations. For the decomposition part E, the state-space realization at time-instant n maps presents input $u(n) \in l_2^{P(n)}(Z)$ and present state $x(n) \in l_2^{N(n)}(Z)$ to present output $y(n) \in l_2^{Q(n)}(Z)$ and next state $x(n+1) \in l_2^{N(n+1)}(Z)$. Thus, [denoting by $m(n)$] this map gives us

$$m(n) : \begin{bmatrix} x(n) \\ u(n) \end{bmatrix} \rightarrow \begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix}$$

where the $(N(n+1)+Q(n)) \times (N(n)+P(n))$ matrix[3] $m(n)$ is explicitly written as

$$m(n) = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}. \quad (2)$$

This is shown in Fig. 1. Let x represent the state sequence $\{x(n)\}$, $n \in (-\infty, \infty)$. Likewise, let $y = \{y(n)\}$ and $u = \{u(n)\}$. Then we can write

$$\begin{bmatrix} Zx \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix}$$

[3]For a maximally decimated system, we have $N(n+1) + Q(n) = N(n) + P(n)$, i.e., $m(n)$ is square.

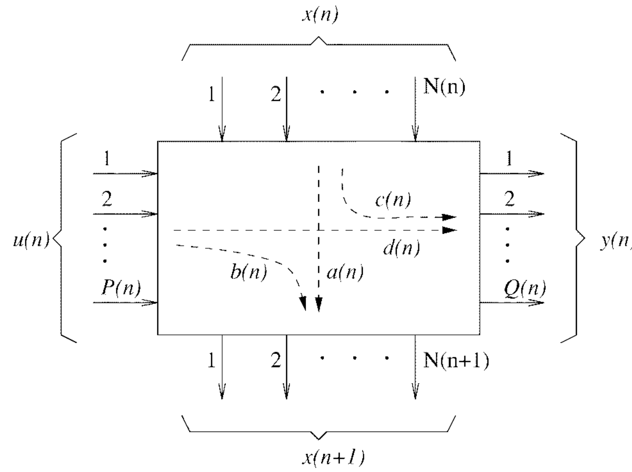


Fig. 1. The state-space realization at time-instant n for the decomposition part of a $Q(n)$-channel filter bank.

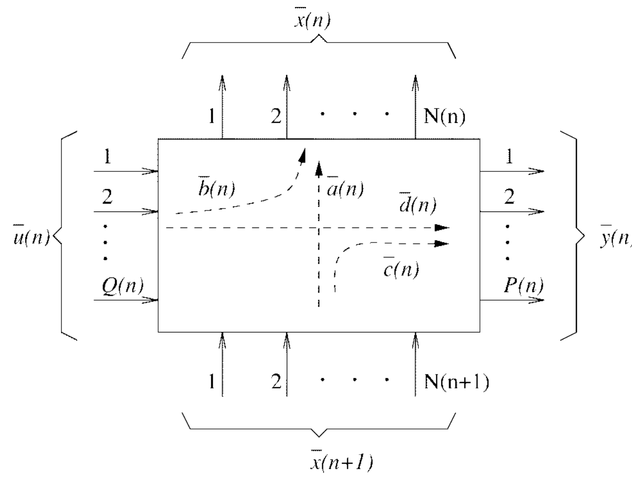


Fig. 2. The state-space realization at time-instant n for the reconstruction part of a $Q(n)$-channel filter bank.

where Z is the unitary shift operator

$$Z = \begin{bmatrix} \ddots & 1 & & & \\ & 0 & 1 & & \\ & 0 & 0 & 1 & \\ & 0 & 0 & 0 & 1 \\ & & & & \ddots \end{bmatrix} \quad (3)$$

and M is a multiband matrix referred to as a state-space realization operator [5]. For the reconstruction part R, we have *mutatis mutandis* similar relations.[4] In order to distinguish R from E, we add an overbar to the symbols when related to R: $\bar{m}(n), \bar{a}(n), \bar{b}(n), \bar{c}(n), \bar{d}(n)$. $\bar{m}(n)$ maps the next state and the present input to the present output and the present state, as shown in Fig. 2. The state-space realization operator on the reconstruction side is then $\bar{M}$.

In the state realization domain, the "biorthogonality" property $RE = I$ transfers to $\bar{M}M = I$ or equivalently, $\bar{m}(n)m(n) = 1$[5], for all n.

[4]Recall that R is an anticausal map, so "next" becomes "previous" and Z is to be replaced by Z^T.

[5]Where 1 stands for the identity matrix of appropriate order.

Let $\overleftarrow{u(n)}$ denote the strict past inputs, i.e., the inputs in the interval $(-\infty, n-1]$ and let $\overrightarrow{y(n)}$ denote the future outputs, i.e., the outputs in the interval $[n, \infty)$. The matrix $\mathcal{C}(n)$ then defined by

$$\begin{aligned}\mathcal{C}(n) = [\cdots a(n-1)a(n-2)b(n-3) \\ a(n-1)b(n-2) \quad b(n-1)]\end{aligned} \tag{4}$$

maps the past inputs $\overleftarrow{u(n)}$ to the present state $x(n)$ and the matrix $\mathcal{O}(n)$ defined by

$$\mathcal{O}(n) = \begin{bmatrix} c(n) \\ c(n+1)a(n) \\ c(n+2)a(n+1)a(n) \\ \vdots \end{bmatrix} \tag{5}$$

maps the present state $x(n)$ to the future outputs $\overrightarrow{y(n)}$.[6] From these definitions, we obtain the following relations [6] from M to E:

$$e(n) = [c(n)\mathcal{C}(n) \quad d(n)] \quad h(n) = \begin{bmatrix} d(n) \\ \mathcal{O}(n)b(n) \end{bmatrix} \tag{6}$$

where $e(n)$ and $h(n)$ are the nth row and column of E, respectively. Similar relations hold between R and $\bar{M}$. From here on, we will assume maximally decimated filter banks.

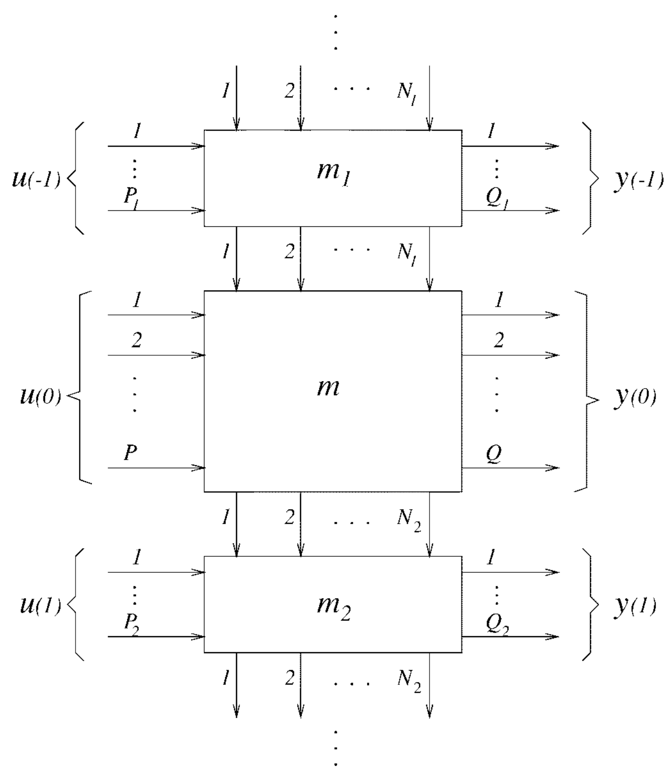


Fig. 3. Time-varying system-representation transition between two filters.

II. Transition Between Filter Banks

Let $m_1(a_1, b_1, c_1, d_1)$ and $m_2(a_2, b_2, c_2, d_2)$ represent the state-space realizations of the decomposition parts of the stationary filter banks $(E_1(e_1), R_1(r_1))$ and $(E_2(e_2), R_2(r_2))$, respectively. The aim is to design an intermediate realization $m(a, b, c, d)$ such that the transitional output functions both on the decomposition and reconstruction sides are minimal.[7] Fig. 3 schematically depicts the state-space model of the transition process on the decomposition side. Let the time axis be such that m is a realization at $n = 0$, as shown in Fig. 3., i.e.,

$$m(n) = \begin{cases} m_1\colon & n \in (-\infty, -1] \\ m\colon & n = 0 \\ m_2\colon & n \in [1, +\infty). \end{cases}$$

Recall that the filter weight vector $e(n)$ at time-instant n is given by $e(n) = [c(n)\mathcal{C}(n) \;\; d(n)]$. This means that if the filter weight vectors for $n \geq 1$ have to be equal to the stationary values of the final filter, then the intermediate realization $m(a, b, c, d)$ should satisfy

$$\mathcal{C}_2 \overset{\text{lef}}{=} [a\mathcal{C}_1 \quad b] \tag{7}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ represent the controllability operators on the decomposition sides of the initial and the final stationary filter banks, respectively, and $\overset{\text{lef}}{=}$ stands for equality after disregarding possible zero columns on the left-most sides of the matrices. As the initial filter is assumed to have been operating stationarily up to $n = -1$, the filter weight vectors for $n < 0$ are unaffected by the intermediate realization m. This means that the filter behaves the same way as the initial stationary filter up to $n = -1$.

The reconstruction version of Fig. 3 can be obtained by reversing the directions of signal flows in it. For this reconstruction filter, the impulse response at time step n, which is the same as the nth column of the reconstruction map R, is given by

$$r(n) = \begin{bmatrix} \bar{\mathcal{O}}(n-1)\bar{b}(n) \\ \bar{d}(n) \end{bmatrix} \tag{8}$$

where $\bar{\mathcal{O}}(n)$ is the observability operator of $\bar{m}(n)$. Clearly, on the reconstruction side, for the impulse responses to agree with the respective stationary values before and after the transition, the intermediate realization $\bar{m}$ must satisfy

$$\bar{\mathcal{O}}_2 \overset{\text{top}}{=} \begin{bmatrix} \bar{\mathcal{O}}_1 \bar{a} \\ \bar{c} \end{bmatrix} \tag{9}$$

where $\bar{\mathcal{O}}_1$ and $\bar{\mathcal{O}}_2$ are the observability operators on the reconstruction sides of the initial and final stationary filter banks, respectively, and $\overset{\text{top}}{=}$ stands for equality after disregarding possible zero rows at the topmost positions of the matrices.

A. Solving for the Intermediate Realizations m and $\bar{m}$

Let L_1 and L_2 represent the lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Also, let $\mathcal{C}_2(:, 1:K_u)$ with $0 \leq K_u \leq \min(L_1, L_2)$ represent the first K_u columns of $\mathcal{C}_2$ such that $a\mathcal{C}_1 = [\mathbf{0} \;\; \mathcal{C}_2(:, 1:K_u)]$, where $\mathbf{0}$ is a zero matrix of appropriate dimensions.

[6] $\mathcal{C}(n)$ and $\mathcal{O}(n)$ are called the controllability and observability operators at time-instant n, respectively.

[7] There is one output vector for each state-space realization. The transition duration is measured by the dimension Q of the output vector of the intermediate realization m.

Then, if (7) has to be satisfied, we must have

$$a = [\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]\mathcal{C}_1^\dagger \tag{10}$$

$$b = \mathcal{C}_2(:, K_u + 1 : L_2). \tag{11}$$

Similar conditions for $\bar{a}$ and $\bar{c}$ in (9) are

$$\bar{a} = \bar{\mathcal{O}}_1^\dagger \begin{bmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1 : K_u, :) \end{bmatrix} \tag{12}$$

$$\bar{c} = \bar{\mathcal{O}}_2(K_u + 1 : L_2, :) \tag{13}$$

where $(\cdot)^\dagger$ is a pseudo-inverse operator. For a maximally decimated biorthogonal filter bank, $m\bar{m} = I$. With

$$m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \bar{m} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$$

this implies $a\bar{a} + b\bar{c} = I$. Substituting the values of a, $\bar{a}$, b, and $\bar{c}$ from (10)–(13), we get

$$[\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]\mathcal{C}_1^\dagger \bar{\mathcal{O}}_1^\dagger \begin{bmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1 : K_u, :) \end{bmatrix} + \mathcal{C}_2(:, K_u + 1 : L_2)\bar{\mathcal{O}}_2(K_u + 1 : L_2, :) = I. \tag{14}$$

If we replace the term $\mathcal{C}_1^\dagger \bar{\mathcal{O}}_1^\dagger$ in (14) with an identity matrix, the left-side terms reduce to $\mathcal{C}_2\bar{\mathcal{O}}_2$. However, for a maximally decimated final filter, $\mathcal{C}_2\bar{\mathcal{O}}_2 = I$. This means (14) is true if and only if

$$[\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]P_r \begin{bmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1 : K_u, :) \end{bmatrix} = [\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)] \begin{bmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1 : K_u, :) \end{bmatrix} \tag{15}$$

where $P_r = \mathcal{C}_1^\dagger \bar{\mathcal{O}}_1^\dagger = \bar{\mathcal{O}}_1 \mathcal{C}_1$ is a projector[8] to the column space of $\bar{\mathcal{O}}_1$ and to the row space of $\mathcal{C}_1$. Note that if $K_u = 0$ the above equation is trivially satisfied with $a, \bar{a} = [\,]$.[9] We address this in Section III. In the following, we present a proposition that puts conditions on the initial and final filters under which (15) is nontrivially satisfied.

Proposition II.1 Let $m(a, b, c, d)$ and $\bar{m}(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ represent the state-space maps of the transition system on the decomposition and reconstruction sides, respectively. Then, (7) and (9) are, respectively, satisfied by nonnull matrices a and $\bar{a}$ without violating biorthogonality if and only if there exists a $K_u > 0$ such that the rows of $[\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]$ are all in the space spanned by the rows of $\mathcal{C}_1$ and the columns of $\left[\begin{smallmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:) \end{smallmatrix}\right]$ are all in the space spanned by the columns of $\bar{\mathcal{O}}_1$.

Proof: Biorthogonality is preserved if (14) is satisfied by the intermediate realization. For $K_u = 0$, the relation is always satisfied as long as the final filter is maximally decimated. Nevertheless, $K_u = 0$ means both a and $\bar{a}$ are null. Thus, if these have to be nonnull matrices, then K_u must be nonzero. This means that the relation given in (7) is nontrivially satisfied if we can find a $K_u > 0$ such that $a\mathcal{C}_1 = [\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]$. Since the matrix $a\mathcal{C}_1$ is formed by the linear combinations of the rows of $\mathcal{C}_1$, it always lies in the space spanned by the rows of $\mathcal{C}_1$. This means that (7) is satisfied by a nonnull matrix a, if the rows of $[\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]$, $K_u > 0$, are spanned by the rows of $\mathcal{C}_1$. The reverse is also true. That is to say, if there exists a $K_u > 0$ such that the rows of $[\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]$ are in the space spanned by the rows of $\mathcal{C}_1$, we can always express the former as linear combinations of the rows of the latter. In other words, there exists a nonnull matrix a for which $a\mathcal{C}_1 = [\mathbf{0} \quad \mathcal{C}_2(:, 1 : K_u)]$. With the same argument, it can be shown that (9) is satisfied by a nonnull matrix $\bar{a}$ if and only if the columns of $\left[\begin{smallmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:) \end{smallmatrix}\right]$ are spanned by the columns of $\bar{\mathcal{O}}_1$. □

Once $a, \bar{a}, b, \bar{c}$ are determined, we need to calculate the rest of the parameters $c, d, \bar{b}$ and $\bar{d}$ to complete the design. This can easily be done by generating equations from the requirements $\bar{m}m = I$ (biorthogonality) and $m\bar{m} = I$ (maximally decimation) as follows:

$$\begin{aligned} &\bar{a}a + \bar{b}c = I \quad && a\bar{b} + b\bar{d} = 0 \\ &\bar{a}b + \bar{b}d = 0 \quad && c\bar{a} + d\bar{c} = 0 \\ &\bar{c}a + \bar{d}c = 0 \quad && c\bar{b} + d\bar{d} = I \\ &\bar{c}b + \bar{d}d = I. && \end{aligned}$$

Note that given the solutions $m(a, b, c, d)$ and $\bar{m}(\bar{a}, \bar{b}, \bar{c}, \bar{d})$, $m'(a, b, tc, td)$ and $\bar{m}'(\bar{a}, \bar{b}s, \bar{c}, \bar{d}s)$ are also solutions, provided that $st = I$. In Section III-D, we will use this property to refine the behavior of the transition filters.

B. Minimality of the Intermediate Realization

We say the intermediate realization m is minimal if its output vector dimension Q is minimal. In the decomposition map E, this is equivalent to the number of transition filters. For example, in (17), Q would be the number of rows in the transition block e'_{12}.

Proposition II.2 Let a, $\bar{a}$, b, and $\bar{c}$ be given as in (10)–(13). The intermediate realizations m and $\bar{m}$ are then minimal if and only if there exists no K such that $K > K_u$ for which (15) is satisfied, where $0 \le K_u \le \min(L_1, L_2)$, with L_1 and L_2 as defined earlier.

Proof: From the definitions of a and b, we see that a is an $N_2 \times N_1$ matrix and b is an $N_2 \times (L_2 - K_u)$ matrix, where N_1 and N_2 are the state dimensions of the initial and final filters, respectively. For maximally decimated systems, m is square. This means that the dimension Q of the output vector must be $Q = L_2 + (N_1 - N_2) - K_u$. Here, all the quantities except K_u are fixed by the initial and the final filters. Thus, our only free parameter is K_u. This means that if there does not exist a $K > K_u$ for which (15) is satisfied, then the number of transitional output functions Q is minimal and, hence, m is minimal. In the same way, it is easy to show that $\bar{m}$ is also minimal. □

C. Classifying the Transitions

K_u is a measure that indicates the sizes of the common subspaces spanned by the controllability operators on the decomposition sides and by the observability operators on the reconstruction sides of the initial and final filters. Its value characterizes the transition behavior. This is summarized in

[8] This can easily be verified by showing $P_r = P_r^2$.

[9] Where $[\,]$ stands for a null matrix with arbitrary dimensions.

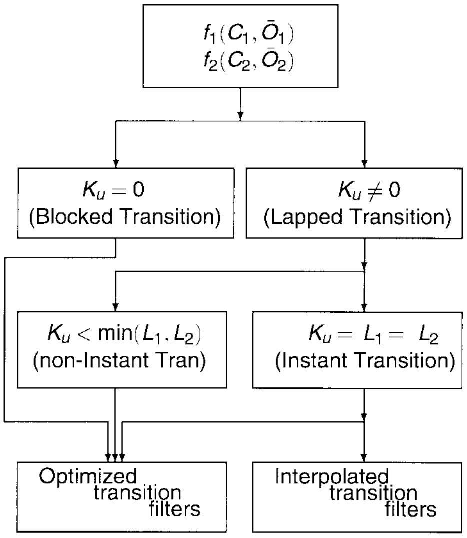


Fig. 4. Summary of the transition behaviors characterized by K_u.

Fig. 4. Depending on the value of K_u, the transition behavior ranges from instantaneous to blocked.

III. Blocked and Lapped Transitions

The meaning of $K_u = 0$ (or $a, \bar{a} = []$) is that during transition none of the states are transferred from the initial to final stationary filters. In such a case, the resulting transition is said to be a blocked (or a nonoverlapped) transition. This is equivalent to first terminating the initial filter and then starting the final one. Under this condition, the intermediate realization m can be split into two, as shown in Fig. 5.[10] The corresponding decomposition map $E_{a=0}$ then has the following structure:

$$E_{a=0} = \begin{bmatrix} \ddots & & & & & \\ & e_1 & & & & \\ & & e_1 & & & \\ & & & e'_{11} & 0 & \\ & & & 0 & e'_{22} & \\ & & & & & e_2 \\ & & & & & & e_2 \\ & & & & & & & \ddots \end{bmatrix}. \quad (16)$$

From the structure, we see that the initial and final filter banks are independently bounded from the bottom and top, respectively. Thus, we would like to do two things: firstly, we would like to terminate the initial filter with a minimum number of boundary filters and, secondly, we would like to start the final filter minimally.

A. Terminating the Initial Filter

We start off by assuming that there exists a null final filter. With this assumption, we can directly use the results of Section II-A. For the null final filter, we take $L_2 = 0$ and $N_2 = 0$. The dimensions of a, b, c, and d are then $0 \times N_1$, 0×0, $N_1 \times N_1$ and $N_1 \times 0$, respectively. This means that all the parameters except c are null, and we can write $m = c$. Obviously, if the biorthogonality has to be preserved, c must be an invertible matrix. Notice that the number of the boundary filters so determined are minimal because we cannot find a $K > K_u (= 0)$ which satisfies (15) (*Proposition II.2*).

[10] This case is considered in [2] for two-channel orthogonal filter banks.

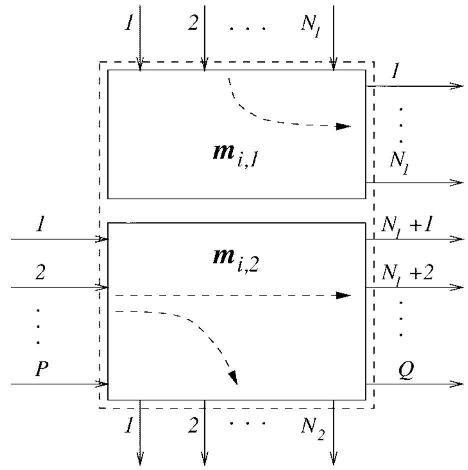


Fig. 5. Details of the intermediate state-space realization on the decomposition side for the case $a = 0$.

B. Starting up the Final Filter

In this case, we assume that there exists a null initial filter and we take $L_1 = 0$ and $N_1 = 0$. This means that the dimensions of a, b, c, d are $N_2 \times 0$, $N_2 \times L_2$, $(L_2 - N_2) \times 0$, and $(L_2 - N_2) \times L_2$, respectively. The only nonnull matrices are, therefore, b and d, giving $m = \left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right]$. Note that b is fixed by the relation given in (11), and with $K_u = 0$ it becomes $b = \mathcal{C}_2$. d must be chosen in such a way that biorthogonality is preserved. Clearly, the boundary filters so determined are minimal because $K_u = 0$ is the maximum value for which (15) is satisfied.

C. Lapped Transition

When $K_u > 0$ (or $a, \bar{a} \neq []$), some of the states are transferred to the final filter and the resulting transition is said to be a lapped transition. This is because if we construct the rows of the decomposition map E of the time-varying system using (6), we get overlaps of the filter weight vectors at the transition as shown below:

$$E = \begin{bmatrix} \ddots & & & & \\ & e_1 & & & \\ & & e_1 & & \\ & & & e'_{12} & \\ & & & & e_2 \\ & & & & & e_2 \\ & & & & & & \ddots \end{bmatrix}. \quad (17)$$

The same can be said for $\bar{a}$ and the impulse responses of R.

If $K_u = L_2 = L_1$ and $N_2 = N_1 = N$, then a is an invertible $N \times N$ matrix with $\bar{a} = a^{-1}$. Such pairs of filter banks form a special class characterized by

$$\mathcal{C}_2 = T\mathcal{C}_1, \quad \text{and} \quad \bar{\mathcal{O}}_2 = \bar{\mathcal{O}}_1 S \tag{18}$$

for some invertible matrices T and S such that $ST = TS = I$. Filter banks under this category are considered in Sections IV and V.

D. Controlling the Behavior of the Transition and/or the Boundary Filters

In real applications, in addition to perfect reconstruction, good filtering behavior is required. Therefore, we would like to know if we can alter the behavior of the transition functions as desired without violating biorthogonality. Fortunately, we have some degree of freedom to do so.

Consider the state-space realization of the intermediate system shown in the middle part of Fig. 3. Cascading this system with an invertible $Q \times Q$ constant matrix t does not affect biorthogonality. This is actually equivalent to replacing the c and d parameters of the decomposition part of the intermediate realization with tc and td, respectively. If, in addition, t is optimized to give good desired impulse response transitions in R while maintaining smooth filter weight vector transition in E, we can improve the behavior of the transition filters without destroying biorthogonality. Note that due to the mixing of the filter coefficients of the upper and lower boundary filters by the cascaded operator t, we get a sort of "overlapped" transition for the case $a, \bar{a} = []$ as well [see (16)]. The matrix t could be determined using a wide range of optimization procedures. In all cases, we are simply exploiting the extra freedom we have in the parameters c and d on the decomposition side and the parameters $\bar{b}$ and $\bar{d}$ on the reconstruction side (see Section II-A). For further reading on optimization methods, the reader is advised to consult [2] and [7].

In the remaining part of this paper, we will show that if the two filters under question are such that (18) is satisfied, the transition between the filters can be made arbitrarily small. In the limiting case, we can instantaneously switch from the initial filter to the final one without any transition. For this subclass of filters, we first summarize (in the following section) results for the instantaneous transition case, which were presented in [5]. In Section V, we extend these results by introducing an interpolation method which allows a smooth transition between filter weight vectors in the decomposition part of the bank, and at the same time a smooth transition between impulse response vectors in the reconstruction part of the bank.

IV. Instantaneous Transition

Let $(E(e), R(r))$ be a stationary biorthogonal filter bank. Put $E(e) = [\frac{E_t}{E_b}]$ and $R(r) = [R_l \mid R_r]$ where E_t is the part of $E(e)$ above the central row e and R_l is the part of $R(r)$ to the left of the central column r.

Now, let $E_1(e_1) = [\frac{E_{1,t}}{E_{1,b}}]$, $E_2(e_2) = [\frac{E_{2,t}}{E_{2,b}}]$, and $R_1(r_1) = [R_{1,l} \mid R_{1,r}]$, $R_2(r_2) = [R_{2,l} \mid R_{2,r}]$. $R_1(r_1)E_1(e_1) = I$ and $R_2(r_2)E_2(e_2) = I$. The problem we want to address in this section is that given the above two filter banks

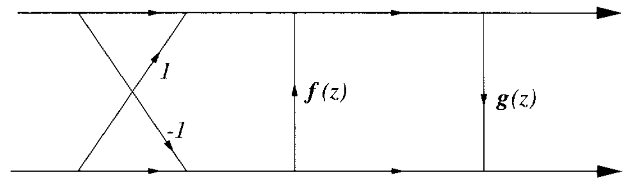


Fig. 6. A two-channel ladder filter bank.

$(E_1(e_1), R_1(r_1))$ and $(E_2(e_2), R_2(r_2))$, does there exist a filter bank (E_{12}, R_{12}) with

$$E_{12} = \left[\frac{E_{1,t}}{E_{2,b}}\right] \quad \text{and} \quad R_{12} = [R_{1,l} \mid R_{2,r}]$$

such that $R_{12}E_{12} = I$.

The answer is partly contained in the following proposition, a proof of which can be found in [5].

Proposition IV.1 Let $(E_1(e_1),\ R_1(r_1))$, and $(E_2(e_2),\ R_2(r_2))$ be two stationary biorthogonal filter banks with realization matrix pairs

$$(m_1(a_1, b_1, c_1, d_1), \bar{m}_1(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1))$$

and

$$(m_2(a_2, b_2, c_2, d_2), \bar{m}_2(\bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2))$$

respectively. If[11] $a_1 = a = a_2$, $b_1 = b = b_2$, $\bar{a}_1 = \bar{a} = \bar{a}_2$, $\bar{c}_1 = \bar{c} = \bar{c}_2$, $c_2 = t \times c_1$, $d_2 = t \times d_1$, $\bar{b}_2 = \bar{b}_1 \times s$, $\bar{d}_2 = \bar{d}_1 \times s$, and $s \times t = I$

$$\begin{aligned} I &= \begin{bmatrix} \bar{a} & \bar{b}_1 \\ \bar{c} & \bar{d}_1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c_1 & d_1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{a} & \bar{b}_1 \\ \bar{c} & \bar{d}_1 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & t \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c_1 & d_1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{a} & \bar{b}_2 \\ \bar{c} & \bar{d}_2 \end{bmatrix} \begin{bmatrix} a & b \\ c_2 & d_2 \end{bmatrix} \end{aligned}$$

then $([\frac{E_{1,t}}{E_{2,b}}], [R_{1,l} \mid R_{2,r}])$ is a biorthogonal filter bank with instantaneous filter weight vector transition in the decomposition part and instantaneous impulse response transition in the reconstruction part.

In the above proposition we have tacitly assumed that the two banks $(E_1(e_1), R_1(r_1))$ and $(E_2(e_2), R_2(r_2))$ have the same number of channels. However, this is not a restriction. Indeed, if $m_1(a_1, b_1, c_1, d_1)$ is the realization matrix of, say, a p-channel filter, then a number of such realizations (say, r) in (time) succession can be merged (by eliminating intermediate states) to obtain a realization matrix $m_1(a_1, b_1, c_1, d_1)$ of an $r \times p$ channel filter. The realization matrix $m_2(a_2, b_2, c_2, d_2)$ in the proposition will then also characterize an $r \times p$ filter bank. See the example in Section VI.

Corollary IV.2 If two filter banks characterized by $(E_1(e_1),\ R_1(r_1))$ and $(E_2(e_2),\ R_2(r_2))$ are related according to (18), then they satisfy *Proposition IV.1*.

Proof: This can easily be shown by constructing $\mathcal{C}_1$, $\mathcal{C}_2$, $\bar{\mathcal{O}}_1$, and $\bar{\mathcal{O}}_2$ from the state-space parameters given in *Proposition IV.1*. □

[11] Up to similarity transformations.

V. Interpolated Transition

The matrix equation in the previous section expresses biorthogonality. The additional property is that the system is state stationary over all time, including the time instant of instantaneous transition. The transition is instantaneous because the transformation matrices t and s are constant. If, on the other hand, we let these matrices be time varying, then the transition will follow a certain trajectory, which we will have to control in the case when we want to enforce meaning on the trajectories of the filter weight vectors in the operator E as well as the impulse response vectors in the operator R. One can envisage several strategies to control the transition behavior, but we shall be confined to one of them—spiral interpolation—which has proven to be simple and satisfactory.

Thus, let $\{t(n)\}$ be the sequence of real $w \times w$ transformation matrices on the transition interval $[n_o, n_f]$, where $t(n_o) = I$ and $t(n_f) = t$ (t being, for example, the matrix t in *Proposition IV.1*.) Associated with this sequence is the sequence of inverse matrices $\{s(n)\}$, $s(n)t(n) = I$. Now let $t = g\lambda g^{-1}$ and $s = g\lambda^{-1}g^{-1}$ be the eigenvalue decompositions of t and s. The eigenvalues are either real λ_k or appear in conjugate pairs $(|\lambda_k|e^{j\theta_k}, |\lambda_k|e^{-j\theta_k})$.

Proposition V.1 Let for $i = 1, \cdots, w$, and $n_o \leq n \leq n_f$, $p_i(n)$ and $\gamma_i(n)$ be real and monotonically increasing functions from 0 at n_o to 1 at n_f. Put $t(n) = g[\mathrm{diag}(p_i(n)|\lambda_i| + (1 - p_i(n)))e^{j\gamma_i(n)\theta_i}]g^{-1}$. $t(n)$ is real. If $q_i(n) = \frac{p_i(n)|\lambda_i|}{p_i(n)|\lambda_i|+(1-p_i(n))}$, then $q_i(n)$ is monotonically increasing from 0 at n_o to 1 at n_f, $s(n) = g[\mathrm{diag}(q_i(n)|\lambda_i^{-1}| + (1 - q_i(n)))e^{j-\gamma_i(n)\theta_i}]g^{-1}$ is real, and $s(n) \times t(n) = I$. Moreover, the transition filters are such that the overall system remains biorthogonal.

Proof: The first part of the proposition is the result of complex algebra: given a complex vector $x \neq 0$ and two complex numbers $\alpha, \beta \neq 0$, the sum $\alpha x + \beta \mathrm{conj}(x)$ is real if and only if $\beta = \mathrm{conj}(\alpha)$, where $\mathrm{conj}(\cdot)$ stands for complex-conjugate operator. In our case, if the interpolation is made on the eigenvalues in such a way that conjugate pairs remain that way throughout the transition while keeping the eigenvectors unchanged, all the intermediate $t(n), s(n)$ $n_o \leq n < n_f$ will be real valued. Biorthogonality is preserved because of the fact that we keep $s(n) = t(n)^{-1}$ throughout the transition. □

VI. Illustrative Examples

A. Overlaped Transition

In this example, the transition behavior between two two-channel filter banks of lengths 18 and 10 are considered. The intermediate state-space realization has the property that corresponds to $K_u > 0$ (see Section III). The structures of both filters are as shown in Fig. 6. For the initial filter

$$f(z) = -0.1023z^{-2} + 0.223z^{-1} - 0.223z + 0.1023z^2$$

and

$$g(z) = -0.0945z^{-2} + 0.213z^{-1} - 0.213z + 0.0945z^2$$

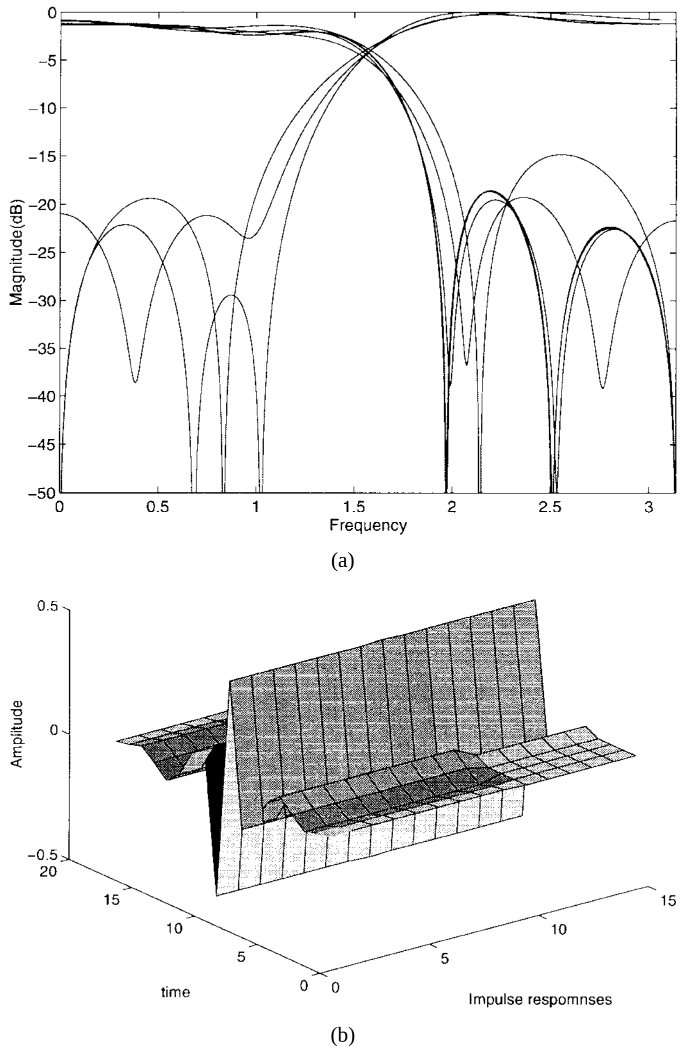


Fig. 7. Transition behaviors. (a) Spectra of the transition filters on the decomposition side. (b) Transitions in impulse responses of the second channel on the reconstruction side. The other channel transit in a likewise gentle way.

and the final filter has

$$f(z) = 0.223z^{-1} - 0.223z$$

and

$$g(z) = 0.246z^{-1} - 0.246z.$$

The intermediate state-space realization is constructed, as discussed in Section II-A (for these filters, it can be shown that $K_u = 4$ and $Q = 8$). The resulting transitional behavior is summarized in Fig. 7. From the plots, one can clearly see the smooth transition in the spectra of the decomposition filters and the gentle takeover in impulse responses on the reconstruction side.

B. Instantaneous Transition

Here, we demonstrate *Proposition IV.1* by considering a transition from a two-channel filter bank to a four-channel filter bank. The decomposition parts of the two banks are shown in Fig. 8, in stationary state. In the figure, $c0 = 0.2228, c1 =$

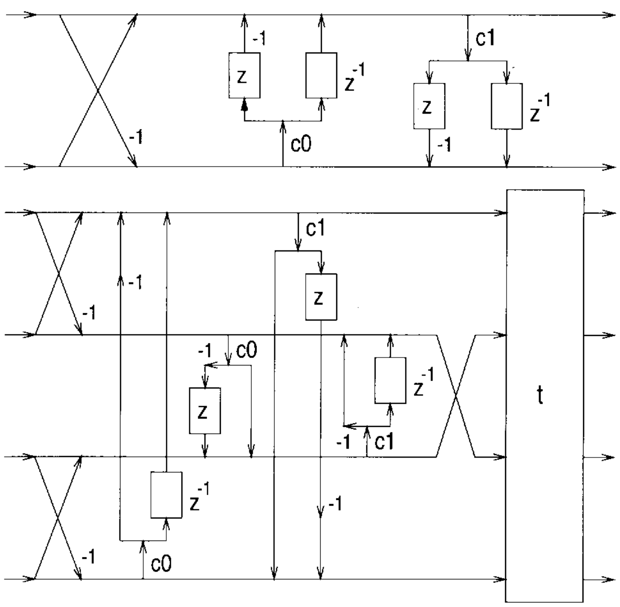


Fig. 8. A two-channel and four-channel decomposition part of two perfect reconstruction filter banks.

0.2465 and

$$t = \begin{bmatrix} 0.9547 & -0.9510 & -0.0935 & 0.0796 \\ 0.8909 & 0.9060 & -0.0908 & -0.1111 \\ -0.0555 & 0.0498 & 0.9248 & -0.9195 \\ -0.1315 & -0.0819 & 1.0889 & 1.0898 \end{bmatrix}. \quad (19)$$

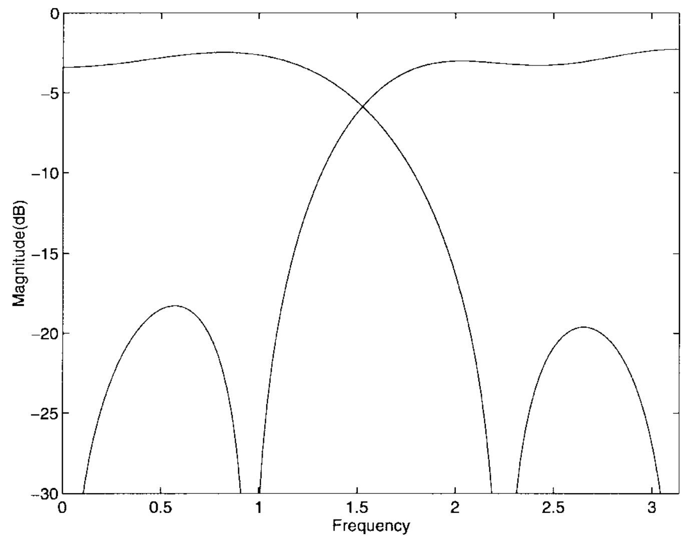
The Fourier transforms of the filter weight vectors of the two decomposition parts are shown in Fig. 9.

If in the second filter in Fig. 8 the matrix t is replaced by the identity, then the resulting flow graph is essentially twice the first filter. Thus, if t is taken away from the second filter, then it represents two time steps of the two-channel decomposition filter. This filter can be run for a while and then, say at $n = n_o$, t can be cascaded to the two two-channel filters and start running as a single four-channel decomposition filter. The takeover is instantaneous, i.e., the spectral characteristics switch instantly at $n = n_o$, from the top spectra in Fig. 9 to the bottom spectra. Moreover, the impulse responses of the reconstruction filters also have an instantaneous transition at $n = n_o$, as shown in Fig. 10, for the fourth channel of the two reconstruction filters.[12]

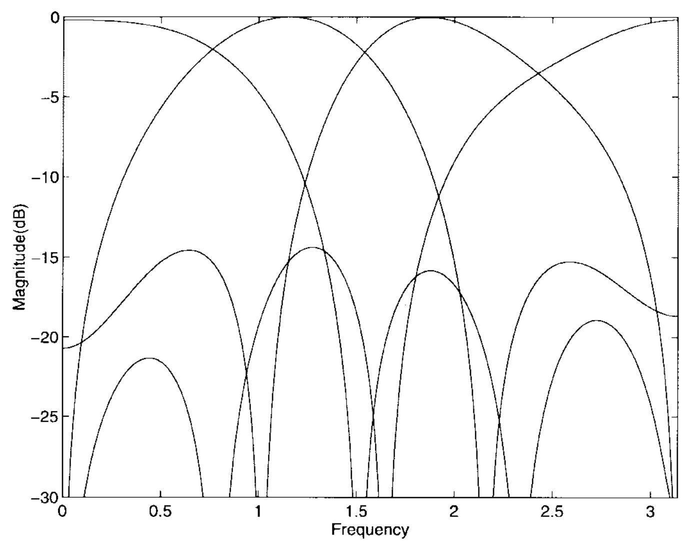
C. Interpolated Transition

Instead of instantly appending the constant matrix t given in (19), we now "spiral" along the matrix trajectory $t(n)$, as explained in *Proposition V.1*. $t(n)$ takes off at $t(n_o) = I$ and is constant $t(n_f) = t$ from $n = n_f$ on. Similarly, for the reconstruction filter: the input matrix starts off from $s(n_o) = I$ and spirals to end at constant $s(n_f) = s$ at time-instant $n = n_f$. In this example, the transition duration is arbitrarily chosen to be $\Delta n = n_f - n_o = 5$.

[12] The reconstruction filters are not shown, as they are easily obtained by reversing the direction of signal flow from output to input in the filters from Fig. 8.



(a)



(b)

Fig. 9. Fourier transforms of filter weight vectors of two- and four-channel filters.

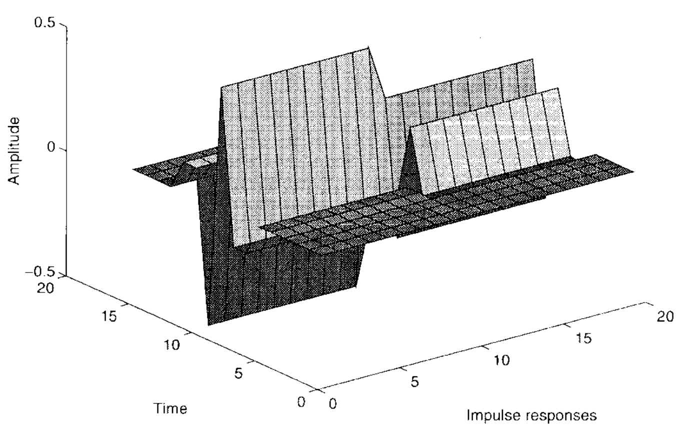


Fig. 10. Instantaneous switching in the impulse response corresponding to the fourth channel.

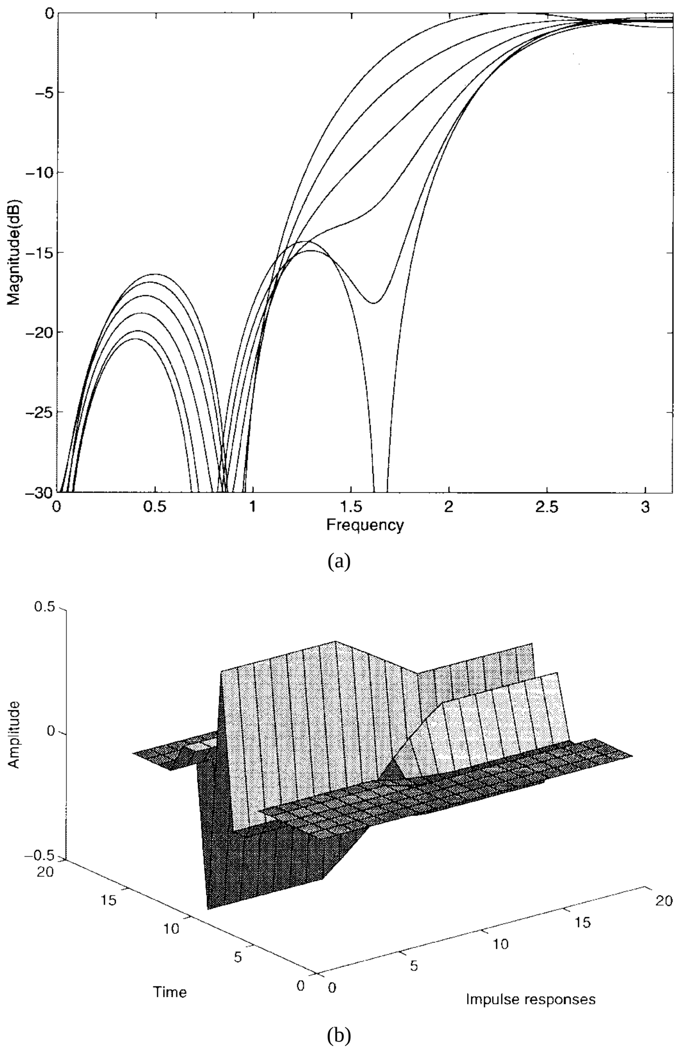


Fig. 11. Smooth transitions in the frequency and impulse responses corresponding to the fourth channel.

The smooth transitions of both the spectral characteristics at the decomposition side and the impulse responses at the reconstruction side are clearly seen in the plots shown in Fig. 11 for the fourth channel. The other channels transit in a likewise gentle way. As a final plot, the trajectories of the eigenvalues of $t(n)$ and $s(n)$ are shown in Fig. 12.

VII. Conclusion

Using the state-space representation of filter banks, we are able to come up with a comprehensive characterization of transitions between two stationary filter banks. The transition behavior is completely described by the common subspaces spanned by the controllability operators on the decomposition sides and by the observability operators on the reconstruction sides of the respective filters. Depending on the sizes of the common subspaces, the transition behavior ranges from instantaneous to blocked. Instantaneous transition is obtainable when the controllability/observability operators of the two filters span the same space. On the other hand, when the operators span disjoint spaces, the resulting transition is a blocked one. All other transition behaviors are in between the above two categories.

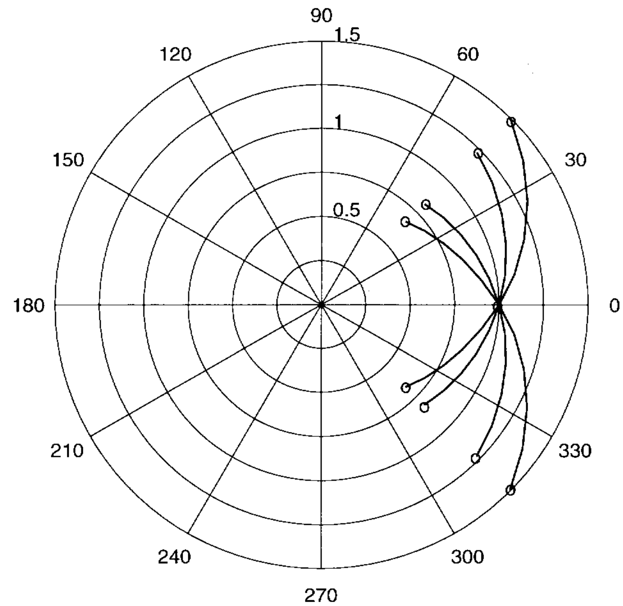


Fig. 12. Eigenvalue trajectories on the transition support. Initial values are all 1. End values are outside the unit circle for the decomposition filter and inside the unit circle for the reconstruction filter.

We are also able to show that under specific conditions the transition interval can be arbitrarily varied. For this class of filters, the transition filters are controlled in such a way that both the filter weight vectors and the impulse responses gently embrace.

Our approach makes use of two sets of equations generated from two major assumption: biorthogonality and maximally decimation. Further generalization can be obtained by considering oversampled filter banks. Finally, it is relevant to mention that these results can readily be applied to multidimensional filters by appropriately defining the state-space representations [8].

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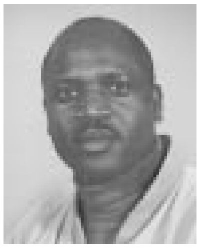

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