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ABSTRACT

The non-linear vibrations of thin-walled cylindrical shells with initial geometric imperfections are studied using Donnell type non-linear shallow-shell equations in which the appropriate damping and inertial terms are introduced. The initial geometric imperfections are modelled with an axisymmetric and an asymmetric trigonometric function. The simply-supported boundary conditions and the circumferential periodicity condition are satisfied. Two vibration modes are assumed though only one of them is directly excited. Galerkin's procedure is used to obtain two coupled non-linear differential equations for the modal amplitudes. The approximate frequency-amplitude relationships for various initial imperfections, dampings and excitations are calculated from these two equations using the method of averaging. The stability of the solutions are studied using the method of slowly varying parameters.
The results indicate that both the initial geometric imperfections and the amount of damping have quite strong effects on the nonlinear vibration of the shells. The results obtained so far provide qualitatively good explanations for the available experimental results.

LIST OF SIMBOLS

A	Nondimensional amplitude function of the assumed driven mode
$A_1, A_2, \dots, A_{24}$	Coefficients in equation (11), defined in Appendix C of Ref [14]
$A_o(t)$	Slowly varying amplitude function of driven mode
$\bar{A}$	Average value (over one period) of $A_o(t)$
$A(\tau)$	Nondimensional generalized coordinate associated with A; see equation (22-1)
B	Nondimensional amplitude function of the assumed companion mode
$B_o(t)$	Slowly varying amplitude function of companion mode
$\bar{B}$	Average value (over one period) of $B_o(t)$
$B(\tau)$	Nondimensional generalized coordinate associated with B; see equation (22-2)
C	Nondimensional amplitude function of the assumed vibration mode
c	Damping coefficient
E	Young's modulus
F	Total stress function $(= \hat{F} + \hat{\hat{F}})$
$\hat{F}$	Stress function of the fundamental state
$\hat{\hat{F}}$	Stress function of the dynamic state
F_c	Generalized excitation function; see equation (12-2)
F_D	Generalized excitation function; see equation (12-1)
$\bar{F}_D$	Average value of F_D (over one period)
F_{SD}	Generalized excitation function in equation (21), F_D/β_2
h	Wall thickness of shell
L	Length of the shell
$\ell_i, \ell_k, \ell_m, \ell_\ell, \ell_n, \dots$	Normalized axial and circumferential wave parameters, defined in Appendix B of Ref [14]
L_D, L_H, L_Q	Linear differential operators, defined in Appendix A of Ref [14]
L_{NL}	Nonlinear differential operator, $L_{NL}(S,T) = S,_{xx}T,_{yy} - 2S,_{xy}T,_{xy} + S,_{yy}T,_{xx}$
$Q(x,y)$	Spatial distribution of the radial load; see equation (13)
$q(x,y,t)$	Radial loading applied to the surface of the cylinder
R	Radius of the shell
t	Time
W	Total radial displacement, $W = \hat{W} + \hat{\hat{W}}$
$\bar{w}$	Initial radial imperfection
$\hat{W}$	Radial displacement of the fundamental state
$\hat{\hat{W}}$	Radial displacement of the dynamic state
x,y	Axial and circumferential coordinates on the median surface, respectively
$\alpha_1, \dots, \alpha_3$	Coefficients in equation (18); defined by Eq. (18)
$\bar{\alpha}_1, \dots, \bar{\alpha}_{13}$	Coefficients in equation (12-1), defined in Appendix A, of Ref [23]

$\hat{\alpha}_1, \ldots, \hat{\alpha}_7$	Coefficients in equation (20), defined in Appendix B of Ref [23]
$\alpha_{s1}, \ldots, \alpha_{s7}$	Coefficients in equation (21-1), defined in Appendix A of Ref [21]
$\bar{\alpha}_{s1}, \ldots, \bar{\alpha}_{s6}$	Coefficients in equation (23-1), defined in Appendix B of Ref [21]
$\beta_1, \ldots, \beta_{10}$	Coefficients in equation (16) - (20), defined in Appendix C of Ref [14]
$\bar{\beta}_1, \ldots, \bar{\beta}_{10}$	Coefficients in equation (12-2), defined in Appendix A of Ref [23]
$\hat{\beta}_1, \ldots, \hat{\beta}_7$	Coefficients in equation (20-3), defined in Appendix B of Ref [23]
$\beta_{s1}, \ldots, \beta_{s6}$	Coefficients in equation (21-2), defined in Appendix A of Ref [21]
$\bar{\beta}_{s1}, \ldots, \bar{\beta}_{s6}$	Coefficients in equation (23-2), defined in Appendix B of Ref [21]
γ	Nondimensional damping coefficient
γ_s	Percentage of critical damping
$\delta_{n\ell}$	Kronecker delta function, $\begin{cases} 0 & n \neq \ell \\ 1 & n = \ell \end{cases}$
ε	Small parameter defined in Ref [3], $\left(\frac{n^2 h}{R}\right)^2$
ν	Poisson's ratio
ξ	Aspect ratio defined in Ref [3], $\frac{m\pi/L}{n/R}$
$\bar{\xi}_1$	Nondimensional amplitude of the axisymmetric imperfection
$\bar{\xi}_2$	Nondimensional amplitude of the asymmetric imperfection
$\hat{\xi}_o, \hat{\xi}_1, \hat{\xi}_2$	Nondimensional amplitudes of the radial displacement for the fundamental solution
$\bar{\rho}$	Specific mass density, defined in Appendix B of Ref [14]
τ	Nondimensional time, $(\tau = \bar{\omega}_{mn} t)$
ϕ	Phase angle of the driven mode
$\bar{\phi}$	Average value of ϕ
ψ	Phase angle of the companion mode
$\bar{\psi}$	Average value of ψ
$\zeta(\tau)$	Small perturbation in the amplitude of the driven mode
$\zeta_1(\tau)$	Slowly varying component of $\zeta(\tau)$; see equation (24-1)
$\zeta_2(\tau)$	Slowly varying component of $\zeta(\tau)$; see equation (24-1)
$\eta(\tau)$	Small perturbation in the amplitude of the companion mode
$\eta_1(\tau)$	Slowly varying component of $\eta(\tau)$; see equation (24-2)
$\eta_2(\tau)$	Slowly varying component of $\eta(\tau)$; see equation (24-2)
χ_1	Frequency parameter of the driven mode, $(\chi_1 = \Omega_s \tau + \bar{\phi})$
χ_2	Frequency parameter of the companion mode $(\chi_2 = \Omega_s \tau + \bar{\psi})$

ω	Vibration frequency
$\bar{\omega}_{mn}$	Natural frequency of the imperfect shell (Linear theory), $\sqrt{\frac{1}{2}\ \frac{\beta_2 E}{\bar{\rho}R^2}}$
Ω	Nondimensional frequency, $\omega R\sqrt{\frac{2\bar{\rho}}{E}}$
Ω_s	Nondimensional frequency, $\omega/\bar{\omega}_{mn}$
Δ	Difference of the phase angles, $\phi - \psi$
$\bar{\Delta}$	Average value of Δ
[M]	Matrix used in the stability analysis; defined in Appendix C of [21]
[N]	Matrix used in the stability analysis, defined in Appendix C of [21]
$\{\varphi\}$	Column matrix in equation (25); see equation (25)
$(\ )_{,x}$	Differentiation with respect to the variables following the komma

INTRODUCTION

Thin-walled circular cylindrical shells play an important role in modern industrial applications such as in aerospace engineering, off-shore structures, oil storage tanks etc. Considerable research efforts have been devoted in the past to the stress and buckling analysis of such structures. In recent years the emphasis has been shifting towards the study of the dynamic characteristics of preloaded shell structures.

The first paper that dealt with nonlinear vibrations of shells was the pioneering work of Reissner in 1955. Numerous research papers have been published since then. A detailed historical review was presented in [1].

An important contribution to the theory of nonlinear shell vibrations was made by Evensen in 1964, who introduced for the first time the companion mode in the nonlinear vibration analysis of rings [2], and then in the analysis of shells [3]. Evensen used the well-known Donnell shallow-shell equations. Galerkin's procedure and the method of averaging were used in order to get approximate solutions. Evensen's results indicated for the first time that there were two possible responses in the nonlinear vibration of rings, as well as shells. One is a single mode response similar to a simple mass-nonlinear spring system. The other is a coupled-mode response although only one mode is driven directly. Contrary to the results available at the time Evensen's results show that non-linearity of vibration is either softening or hardening, depending upon the aspect ratio of shell response, rather than always hardening.

A more meticulous analysis was performed by Bleich and Ginsberg for infinitely long cylindrical shells in 1970 [4]. Ginsberg subsequently extended this analysis to shells of finite length in 1973 [5]. In their analysis a special perturbation procedure was used. Many of the deficiencies in Evensen's analysis were corrected. Their results for undamped cases are qualitatively similar to those of Evensen. But for damped cases the agreement is not good. The "gap" phenomenon shown in Evensen's results for rings were not found in Bleich and Ginberg's analysis. Bleich and Ginberg's results showed that the damping has a pronounced influence on the response.

An alternative approach to the problem was taken by Chen who applied a systematic perturbation procedure to the Donnell shallow-shell equations in 1972 [6]. Chen's results for companion mode are quite similar to those by Ginsberg but for driven mode response the agreement is not satisfactory. This was attributed to the large damping used in Chen's analysis [7].

One of the conclusions that can be drawn from the results of previous studies is that although some basic characteristics on the nonlinear vibration behaviour of shells (such as the coupled mode response) have been both derived analytically and verified experimentally, there are other aeras where still considerable disagreement exists between results obtained by different procedures and between theoretical predictions and experimental evidence.

Initial geometric imperfections have been accepted as the main degrading factor of the buckling load in shells [8]. It is natural to suggest that initial geo-metric imperfections should have a simular influence on the vibration behaviour of shells. Considerable effort has been devoted in the past few years to establish a correlation between buckling and vibration problems. A summary of the work available before 1983 can be found in an article by Singer [9].

Rosen and Singer [10, 11] analyzed the frequency behaviour of isotropic cylinders with a single axisymmetric respectively a single asymmetric imperfection mode. Their work was later extended to closely stiffened shells, with either a single axisymmetric or a single asymmetric imperfection mode. The solutions were based on a linearized version of Koiter's non-linear equations with inertia terms included. This linearization was performed assuming the amplitudes of vibrations to be infinitesimal which considerably simplifies the analysis while retaining the imperfection terms [10, 11, 12].

Watawala and Nash studied the influence of a single asymmetric imperfection on the nonlinear undamped free and forced vibration problem of simply supported isotropic shells in 1982 [13, 14]. The procedure and vibration mode shape used in their analysis are quite similar to Evensen's approach except that Watawala and Nash did not include the companion mode.

A comparison of Watawala and Nash's nonlinear results with the linearized results obtained by Rosen and Singer shows (see Fig. 1) that if the circumferential wave form of the vibration mode is the same as that of the initial imperfection then for small values of the asymmetric imperfection $\bar{\xi}_2$ the predictions of the two approaches disagree. Rosen and Singer predict a 'hardening behaviour',* whereas Watawala and Nash come up with a 'softening behaviour'.** The reason for this disagreement can be attributed to the fact that Rosen and Singer's vibration mode does not satisfy the circumferential periodicity condition.

Hol analyzed the vibration behaviour of axially compressed shells with an initial geometric imperfection consisting of one axisymmetric and one asymmetric imperfection mode. The Donnell equations for non-shallow orthotropic cylinders were used. The classical simply supported boundary conditions were satisfied. The full nonlinear equations were used for the fundamental state. The approximate solution for the dynamic state was obtained based on linearized equations [15]. As can be seen from Fig. 1, for the case when the circumferential wave form of the dynamic response is the same as that of the imperfection also Hol's linearized solution predicts initially a softening behaviour with increasing asymmetric imperfection amplitude $\bar{\xi}_2$, but its agreement with Watawala and Nash's curve is not quite satisfactory.

The objective of the present analysis is to investigate the effect of both single and combined initial geometric imperfection modes on the vibration behaviour of shells. The emphasis is placed on the influence of initial geometric imperfection on the coupled-mode response for which no solution is as yet available. In addition, the authors intend to investigate discrepancies between the results of earlier studies and to get a reasonable explanation, if it is possible. The Donnell nonlinear differential equations for axially compressed stiffened shells are used. The equations of the fundamental state are solved by a procedure similar to the one used by Hol. The dynamic mode shape selected is a three-term function similar to the one used by Evensen. In the ensueing analysis Galerkin's method and the method of averaging [16] are used in sequence to obtain the amplitude-frequency relationships for various dampings, excitations and imperfections. The stability of solutions then are studied by employing the method of slowly varying parameters [17].

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* 'hardening behaviour' means the natural frequencies increase with increasing the amplitudes of the imperfection.

** 'softening behaviour' means the natural frequencies decrease with increasing the amplitudes of the imperfection.

ANALYSIS

1. Governing Equations

The mathematical model of the vibrating imperfect stiffened cylindrical shell was arrived at by introducing the appropriate terms for the imperfections, the inertia andthe damping into the nonlinear Donnell type orthotropic imperfect shell equations.

The model consists of the following two coupled partial differential equations in the unknown functions W and F:

$$L_H(F) - L_Q(W) = -\frac{1}{R} W_{,xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W}) \tag{1}$$

$$L_Q(F) + L_D(W) = +\frac{1}{R} F_{,xx} + L_{NL}(F, W + \bar{W}) - \bar{\rho}h W_{,tt} - chW_{,t} + q \tag{2}$$

where W is total response of both fundamental and dynamic states.

F is Airy stress function;
L_H, L_Q, L_D, L_{NL} are differential operators, defined in the List of Symbols;
$\bar{\rho}$ is special material density defined in the List of Symbols;
c is damping coefficient;
h is thickness of shell and
q is lateral external excitation

Notice that W is positive inward in present analysis.
The basic idea for obtaining solutions of equations (1) and (2) is, as shown in [18], the assumption that the displacement W and the stress function F in the shell during its vibrations under axial compressed load and lateral excitation can be expressed as a linear superposition of two independent states of displacement and stress, namely the fundamental, static, geometrically nonlinear state due to the imperfections of shells and the application of a static compressive load, and the dynamic nonlinear state due to small but not infinitesimal vibrations about the fundamental state.
Therefore W and F can be expressed as follows

$$W = \hat{W} + \hat{\hat{W}} \tag{3}$$

$$F = \hat{F} + \hat{\hat{F}} \tag{4}$$

where $\hat{W}$, $\hat{F}$ and $\hat{\hat{W}}$, $\hat{\hat{F}}$ represent the displacement and stress function of fundamental and dynamic state, respectively. Upon substitution of equations (3) and (4) into equations (1) and (2), one obtains two sets of differential equations governing the fundamental and the dynamic state, respectively. The governing equations of the fundamental state are

$$L_H(\hat{F}) - L_Q(\hat{W}) = -\frac{1}{R} \hat{W}_{,xx} - \frac{1}{2} L_{NL}(\hat{W}, \hat{W} + 2\bar{W}) \tag{5-1}$$

$$L_Q(\hat{F}) + L_D(\hat{W}) = \frac{1}{R} \hat{F}_{,xx} + L_{NL}(\hat{F}, \hat{W} + \bar{W}) \tag{5-2}$$

The equations governing the dynamic state are

$$L_H(\hat{\hat{F}})-L_Q(\hat{\hat{W}}) = -\frac{1}{R}\hat{\hat{W}},_{xx} - L_{NL}(\hat{W}, \hat{\hat{W}} + 2\bar{W}) \tag{6-1}$$

$$L_Q(\hat{\hat{F}})+L_D(\hat{\hat{W}}) = \frac{1}{R}\hat{\hat{F}},_{xx} + L_{NL}(\hat{F}, \hat{\hat{W}}) + L_{NL}(\hat{\hat{F}}, \hat{W} + \bar{W})$$

$$+ L_{NL}(\hat{\hat{F}}, \hat{\hat{W}}) - ch\hat{\hat{W}},_t - \bar{\rho}h\hat{\hat{W}},_{tt} + q \tag{6-2}$$

For the present analysis a two-term approximation for the imperfections is used, which contains both an axisymmetric and an asymmetric component. In mathematical form

$$\bar{W} = \bar{\xi}_1 h \cos \ell_i x + \bar{\xi}_2 h \sin \ell_k x \cos \ell_n y \tag{7}$$

Where the $\bar{\xi}_1$ and $\bar{\xi}_2$ are nondimensional amplitudes of axisymmetric and asymmetric imperfections, respectively. The ℓ_i, ℓ_k and ℓ_n are normalized wave numbers defined in the List of Symbols.

The displacement mode for the fundamental state is assumed as

$$\hat{W} = \hat{\xi}_o h + \hat{\xi}_1 h \cos \ell_i x + \hat{\xi}_2 \sin \ell_k x \cos \ell_n y \tag{8}$$

This choice of the static response mode reflects the fact proven by several authors [10] that the effect of initial geometric imperfections is strongest when the response mode is affine to the inital imperfection mode.

The vibration mode shape similar to the one used by Watawala and Nash is used in the present analysis,

$$\hat{\hat{W}} = Ah \sin \ell_k x \cos \ell_\ell y + Bh \sin \ell_k x \sin \ell_\ell y + Ch \sin \ell_m^2 x \tag{9}$$

Where, A, B and C are the time-dependent amplitude functions.

After applying the periodicity requirement (see Ref [14]) equation (9) becomes

$$\hat{\hat{W}} = Ah \sin \ell_k x \cos \ell_\ell y + Bh \sin \ell_k x \sin \ell_\ell y$$

$$+ \ell_\ell^2 \frac{Rh}{4} [A^2 + B^2 + 2A\, \delta_{n,\ell} (\hat{\xi}_2 + \bar{\xi}_2)] \sin \ell_m^2 x \tag{10}$$

where $\delta_{n,\ell} = \begin{cases} 0 & n \neq \ell \\ 1 & n = \ell \end{cases}$ is Kronecker delta function.

According to the notation of Evensen's paper [3] the first term is called driven mode and the second term is called companion mode. It is noted that the above mode shape satisfies all the boundary conditions of a simply supported shell, except the moment-free condition at the ends. Therefore the mode shape used herein has boundary conditions that lie somewhere between simply supported and clamped ends.
Details of the solution of the fundamental state have been published. The detailed process is shown elsewhere. For the sake of brevity they will not be repeated here. The interested reader can refer to [15]. Only the analysis for the dynamic state is presented in the present paper.

2. Application of Galerkin's Procedure

Before the Galerkin's procedure can be applied, the stress function $\hat{\hat{F}}$ must be determined. Substituting equations (7), (8) and (10) into the compatibility equation (6-1) and then solving for $\hat{\hat{F}}$ one obtains the following particular solution:

$$\begin{aligned}\hat{\hat{F}} = {} & A_1 \cos \ell_{2\ell}y + A_2 \sin \ell_{2\ell}y + A_3 \cos \ell_{n+\ell}y + A_4 \sin \ell_{n+\ell}y + A_5 \cos \ell_{2k}x \\ & + A_6 \cos \ell_{2k}x \cos \ell_{n-\ell}y + A_7 \cos \ell_{2k}x \sin \ell_{n-\ell}y + A_8 \cos \ell_{n-\ell}y + A_9 \sin \ell_{2m+k}x \cos \ell_{\ell}y \\ & + A_{10} \sin \ell_{2m+k}x \cos \ell_n y + A_{11} \sin \ell_{2m-k}x \cos \ell_{\ell}y + A_{12} \sin \ell_{2m-k}x \cos \ell_n y \\ & + A_{13} \sin \ell_{2m+k}x \sin \ell_{\ell}y + A_{14} \sin \ell_{2m-k}x \sin \ell_{\ell}y + A_{15} \cos \ell_{2k}x \cos \ell_{n+\ell}y \\ & + A_{16} \cos \ell_{2k}x \sin \ell_{n+\ell}y + A_{17} \sin \ell_{n-\ell}y + A_{18} \sin \ell_k x \cos \ell_{\ell}y + A_{19} \sin \ell_k x \sin \ell_{\ell}y \\ & + A_{20} \cos \ell_{2m}x + A_{21} \sin \ell_{k+i}x \cos \ell_{\ell}y + A_{22} \sin \ell_{k-i}x \cos \ell_{\ell}y \\ & + A_{23} \sin \ell_{k+i}x \sin \ell_{\ell}y + A_{24} \sin \ell_{k-i}x \sin \ell_{\ell}y \qquad (11)\end{aligned}$$

where the $A_1, A_2, \ldots, A_{24}$ are functions of the time-dependent amplitudes A and B, the imperfection terms $\bar{\xi}_1$ and $\bar{\xi}_2$, and the fundamental response $\hat{\xi}_1$ and $\hat{\xi}_2$. These functions are given in Appendix C of [14].

At this state in the analysis, the equations (7), (8), (10) and (11) are substituted into the equation (6-2), and then Galerkin's procedure is used. This procedure yields two coupled nonlinear differential equations for A(t) and B(t)

$$\bar{\alpha}_1 \frac{d^2A}{dt^2} + \bar{\alpha}_2 \frac{dA}{dt} + \bar{\alpha}_3 A + \bar{\alpha}_4 \frac{d^2C}{dt^2}\left[A + \delta_{n,\ell}(\bar{\xi}_2 + \hat{\xi}_2)\right] + \bar{\alpha}_5 \frac{dC}{dt}\left[A + \delta_{n,\ell}(\bar{\xi}_2 + \hat{\xi}_2)\right]$$
$$+ \bar{\alpha}_6 A^2 + \bar{\alpha}_7 (A^2 + B^2) + \bar{\alpha}_8 (A^2 - B^2) + \bar{\alpha}_9 A^3 + \bar{\alpha}_{10} (A^2 + B^2) A$$
$$+ \bar{\alpha}_{11} (A^2 + B^2) A^2 + \bar{\alpha}_{12} (A^2 + B^2)^2 + \bar{\alpha}_{13} (A^2 + B^2)^2 A = F_D \tag{12-1}$$

$$\bar{\beta}_1 \frac{d^2B}{dt^2} + \bar{\beta}_2 \frac{dB}{dt} + \bar{\beta}_3 B + \bar{\beta}_4 \frac{d^2C}{dt^2} + \bar{\beta}_5 B \frac{dC}{dt} + \bar{\beta}_6 AB + \bar{\beta}_7 A^2 B + \bar{\beta}_8 (A^2 + B^2) B$$
$$+ \bar{\beta}_9 (A^2 + B^2) AB + \bar{\beta}_{10} (A^2 + B^2)^2 B = F_C \tag{12-2}$$

where the $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{13}$ and $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{10}$ are coefficients which are defined in Appendix A of Ref [23]. F_D and F_C are generalized dynamic forces. They are obtained by evaluating the integrals involving the external excitation $q(x,y,t)$ and Galerkin's weighting functions. In the present study q is assumed to be fixed in space and harmonic in time:

$$q(x,y,t) = Q(x,y) \cos \omega t \tag{13}$$

where $Q(x,y)$ is assumed to be symmetric with respect to y and have zero average value. In this case $F_C(t)$ is identical to zero and F_D is

$$F_D = 2 \int_o^L \int_o^{2\pi R} \frac{Q(x,y)\{\cos \ell_\ell y \sin \ell_K x + \frac{\ell_\ell^2 Rh}{2} [A + \delta_{n,\ell}(\bar{\xi}_2 + \hat{\xi}_2)] \sin \ell_m^2 x\} \cos \omega t}{\pi RL} dxdy \tag{14}$$

3. Application of the Method of Averaging

The coupled nonlinear differential equations (12-1) and (12-2) cannot yet be solved exactly. An approximate solution can be obtained by the procedure known as the Method of Averaging. The unknown functions $A(t)$ and $B(t)$ are taken to be of the form

$$A(t) = A_o(t) \cos (\omega t + \phi) \tag{15-1}$$

$$B(t) = B_o(t) \sin (\omega t + \psi) \tag{15-2}$$

where ϕ and ψ are the phase angle of driven and companion modes, respectively.

Substituting equations (15-1) and (15-2) into equations (12-1) and (12-2) and then applying the Method of Averaging yields after some regrouping the following four simultaneous, nonlinear, normalized algebraic equations for $\bar{A}$, $\bar{B}$, $\bar{\psi}$, $\bar{\phi}$:

$$-\Omega^2\bar{A}\left\{1+\beta_1\left[\bar{A}^2-\bar{B}^2\cos 2\bar{\Delta}+2\delta_{n,\ell}(\bar{\xi}_2+\hat{\xi}_2)^2\right]\right\}+\beta_2\bar{A}-\gamma\Omega\bar{A}\bar{B}\beta_1\sin 2\bar{\Delta}$$
$$+\beta_3\bar{A}^3+2\beta_4\bar{A}\bar{B}^2\left(1-\frac{1}{2}\cos 2\bar{\Delta}\right)+\beta_5\bar{A}\left[5\bar{A}^4+4\bar{A}^2\bar{B}^2\left(\frac{3}{2}-\cos 2\bar{\Delta}\right)\right.$$
$$\left.+2\bar{B}^4\left(\frac{3}{2}-\cos 2\bar{\Delta}\right)\right]=\bar{F}_D\cos\bar{\phi} \qquad (16\text{-}1)$$

$$\left\{\bar{A}\bar{B}^2\left[\beta_1\Omega^2-\beta_4-2\beta_5(\bar{A}^2+\bar{B}^2)\right]\right\}\sin 2\bar{\Delta}-\Omega\gamma\left\{2\bar{A}+\beta_1\left[\bar{A}^3-\bar{A}\bar{B}^2\cos 2\bar{\Delta}\right.\right.$$
$$\left.\left.+4\delta_{n,\ell}(\hat{\xi}_2+\bar{\xi}_2)^2\right]\right\}=\bar{F}_D\sin\bar{\phi} \qquad (16\text{-}2)$$

$$-\Omega^2\bar{B}\left\{1+\beta_6(\bar{B}^2-\bar{A}^2\cos 2\bar{\Delta})\right\}+\beta_7\bar{B}+\Omega\gamma\bar{A}^2\bar{B}\beta_6\sin 2\bar{\Delta}+\beta_9\bar{B}^3$$
$$+2\beta_8\bar{A}^2\bar{B}\left(1-\frac{1}{2}\cos 2\bar{\Delta}\right)+\beta_{10}\bar{B}\left\{5\bar{B}^4+\left[4\bar{A}^2\bar{B}^2+2\bar{A}^4\right]\left(\frac{3}{2}-\cos 2\bar{\Delta}\right)\right\}=0 \qquad (16\text{-}3)$$

$$\left\{\bar{A}^2\bar{B}\left[\beta_6\Omega^2-\beta_8-2\beta_{10}(\bar{A}^2+\bar{B}^2)\right]\right\}\sin 2\bar{\Delta}+\Omega\gamma\left\{2\bar{B}+\beta_6\left[\bar{B}^3-\bar{A}^2\bar{B}\cos 2\bar{\Delta}\right]\right\}=0 \quad (16\text{-}4)$$

where $\Omega = \omega R\sqrt{\frac{2\bar{\rho}}{E}}$, is the generalized nondimensional frequency;

$\gamma = cR\sqrt{\frac{1}{2\bar{\rho}E}}$, is the generalized nondimensional damping;

$\bar{\Delta} = \bar{\phi} - \bar{\psi}$, is the "average difference of phase angles";

$$\bar{F}_D = \frac{4R\int_o^L\int_o^{2\pi R} Q(x,y)\,\{\sin \ell_k x \cos \ell_\ell y + \frac{\ell_\ell^2 Rh}{2}\,[A+\delta_{n,\ell}(\hat{\xi}_2+\bar{\xi}_2)]\,\sin \ell_m^2 x\}\,dxdy}{\pi LE}$$

, is the generalized average excitation;

and $\beta_1, \beta_2, \ldots, \beta_{10}$ are coefficients which are given in the Appendix C of Ref [14].

The average values $\bar{A}$, $\bar{B}$, $\bar{\phi}$ and $\bar{\psi}$ can be calculated by solving these four simultaneous nonlinear algebraic equations for given $\bar{F}_D$, γ and Ω.

The analysis is carried out for two separate cases.

CASE 1 Single mode response ($\bar{A} \neq 0$, $\bar{B} = 0$)

If $\bar{B} = 0$, then the system of equations (16-1) - (16-4) reduces to the following two equations,

$$- \Omega^2 \bar{A} \left\{ 1 + \beta_1 \bar{A}^2 + 2\beta_1 \delta_{n,\ell} \ (\bar{\xi}_2 + \hat{\xi}_2)^2 \right\} + \beta_2 \bar{A} + \beta_3 \bar{A}^3 + 5\beta_5 \bar{A}^5 = \bar{F}_D \cos \bar{\phi} \quad (17\text{-}1)$$

$$- \Omega\gamma \left\{ 2\bar{A} + \beta_1 \bar{A}^3 + 4\beta_1 \delta_{n,\ell} \ (\hat{\xi}_2 + \bar{\xi}_2)^2 \right\} = \bar{F}_D \sin \bar{\phi} \quad (17\text{-}2)$$

A single equation governing the amplitude frequency relationship of the single mode response is obtained by first squaring both equations, then adding them and finally using the identity

$$\sin^2 \bar{\phi} + \cos^2 \bar{\phi} = 1$$

This yields

$$\alpha_1 \Omega^4 + \alpha_2 \Omega^2 + \alpha_3 = 0 \quad (18)$$

where

$$\alpha_1 = \bar{A} \left\{ 1 + \beta_1 \ [\bar{A}^2 + 2\delta_{n,\ell} \ (\hat{\xi}_2 + \bar{\xi}_2)^2] \right\}^2$$

$$\alpha_2 = - 2\bar{A}^2 \left\{ 1 + \beta_1 \ [\bar{A}^2 + 2\delta_{n,\ell} \ (\hat{\xi}_2 + \bar{\xi}_2)^2] \right\} \left\{ \beta_2 + \beta_3 \bar{A}^2 + 5\beta_5 \bar{A}^4 \right\}$$
$$+ \gamma^2 \bar{A}^2 \left\{ 2 + \beta_1 \bar{A}^2 + 4\beta_1 \delta_{n,\ell} (\hat{\xi}_2 + \bar{\xi}_2)^2 \right\}^2$$

$$\alpha_3 = \bar{A}^2 \left\{ \beta_2 + \beta_3 \bar{A}^2 + 5\beta_5 \bar{A}^4 \right\}^2 - \bar{F}_D^2$$

It is obvious that one can obtain not only the damped response for various damping and excitation levels but also the free vibration or undamped response if one lets the associate damping and or excitation terms vanish from equation (18).

CASE 2 Coupled-mode response ($A \neq 0$, $B \neq 0$)

For the undamped case the terms related to damping are identically equal to zero. Then the system of equations (16-1) - (16-4) reduces to the following two equations:

$$-\Omega^2\bar{A}\left\{1+\beta_1\left[\bar{A}^2-\bar{B}^2+2\delta_{n,\ell}(\hat{\xi}_2+\bar{\xi}_2)^2\right]\right\}+\beta_2\bar{A}+\beta_3\bar{A}^3+\beta_4\bar{A}\bar{B}^2$$
$$+\beta_5\bar{A}\left\{5\bar{A}^4+2\bar{A}^2\bar{B}^2+\bar{B}^4\right\}=\bar{F}_D \qquad (19\text{-}1)$$

$$-\Omega^2\bar{B}\left\{1+\beta_6\left[\bar{B}^2-\bar{A}^2\right]\right\}+\beta_7\bar{B}+\beta_9\bar{B}^3+\beta_8\bar{A}^2\bar{B}$$
$$+\beta_{10}\bar{B}\left\{5\bar{B}^4+2\bar{A}^2\bar{B}^2+\bar{A}^4\right\}=0 \qquad (19\text{-}2)$$

The amplitude-frequency relationships of undamped coupled-mode response for various amplitudes of initial geometric imperfections and excitations can be obtained by solving these two nonlinear algebraic equations simultaneously.

For the damped coupled-mode response a direct simultaneous solution of equations (16-1) through (16-4) is too cumbersome. A further simplification can be carried out as follows. Initially one solves equations (16-3) and (16-4) for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in terms of $\bar{A}$ and $\bar{B}$, respectively. Then one uses the identity $\sin^2 2\bar{\Delta}+\cos^2 2\bar{\Delta}=1$, which results in a single equation with the unknowns $\bar{A}$ and $\bar{B}$.

Next one backsubstitutes for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in equations (16-1) and (16-2) and then uses the identity $\sin^2\bar{\phi}+\cos^2\bar{\phi}=1$, which yields a second equation with the unknowns $\bar{A}$ and $\bar{B}$.

The amplitude-frequency relationship of damped coupled response then can be obtained by solving these two nonlinear algebraic equations simultaneously for given damping, imperfection and excitation. The equations can be expressed in the following form

$$\Big\langle -\Omega^2\bar{A}\left\{1+\beta_1\left[\bar{A}^2-\bar{B}^2\cos 2\bar{\Delta}+2\delta_{n,\ell}(\hat{\xi}_2+\bar{\xi}_2)^2\right\}+\beta_2\bar{A}-\gamma\Omega\bar{A}\bar{B}\beta_1\ \sin 2\bar{\Delta}+\beta_3\bar{A}^3\right.$$
$$\left.+2\beta_4\bar{A}\bar{B}^2(1-\tfrac{1}{2}\cos 2\bar{\Delta})+\beta_5\bar{A}\left[5\bar{A}^4+4\bar{A}^2\bar{B}^2(\tfrac{3}{2}-\cos 2\bar{\Delta})+2\bar{B}^4(\tfrac{3}{2}-\cos 2\bar{\Delta})\right]\Big\rangle^2\right.$$
$$+\Big\langle\left\{\bar{A}\bar{B}^2\left[\beta_1\Omega^2-\beta_4-2\beta_5(\bar{A}^2+\bar{B}^2)\right]\right\}\sin 2\bar{\Delta}-\Omega\gamma\left\{2\bar{A}+\beta_1\left[\bar{A}^3-\bar{A}\bar{B}^2\cos 2\bar{\Delta}\right.\right.$$
$$\left.\left.+4\delta_{n,\ell}(\hat{\xi}_2+\bar{\xi}_2)^2\right]\right\}\Big\rangle^2-\bar{F}_D^2=0 \qquad (20\text{-}1)$$

and

$$\hat{\alpha}_1\bar{B}^{12}+\hat{\alpha}_2\bar{B}^{10}+\hat{\alpha}_3\bar{B}^8+\hat{\alpha}_4\bar{B}^6+\hat{\alpha}_5\bar{B}^4+\hat{\alpha}_6\bar{B}^2+\hat{\alpha}_7=0 \qquad (20\text{-}2)$$

or

$$\hat{\beta}_1\bar{A}^{12}+\hat{\beta}_2\bar{A}^{10}+\hat{\beta}_3\bar{A}^8+\hat{\beta}_4\bar{A}^6+\hat{\beta}_5\bar{A}^4+\hat{\beta}_6\bar{A}^2+\hat{\beta}_7=0 \qquad (20\text{-}3)$$

where

$$\sin 2\bar{\Delta} = -\gamma\Omega\Big\{2\Big[\beta_6\Omega^2 - \beta_8 - \beta_{10}(4\bar{B}^2 + 2\bar{A}^2)\Big] + \beta_6\Big[\beta_7 - \Omega^2 + \bar{B}^2(\beta_9 - \beta_8) + 2\beta_8\bar{A}^2 + \beta_{10}(\bar{B}^4 + 4\bar{A}^2\bar{B}^2 + 3\bar{A}^4)\Big]\Big\} / \Big\{\bar{A}^2\beta_6^2\gamma^2\Omega^2 + \Big[\beta_6\Omega^2\bar{A} - \beta_8\bar{A} - 2\beta_{10}(\bar{A}^3 + \bar{A}\bar{B}^2)\Big]^2 - 2\beta_{10}\bar{B}^2\bar{A}^2\Big[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)\Big]\Big\}$$

$$\cos 2\bar{\Delta} = \Big\{\Omega^2\gamma^2\beta_6(2 + \beta_6\bar{B}^2) + \Big[\Omega^2(1 + \beta_6\bar{B}^2) - \beta_7 - \beta_9\bar{B}^2 - 2\beta_8\bar{A}^2 - \beta_{10}(5\bar{B}^4 + 6\bar{A}^2\bar{B}^2 + 3\bar{A}^4)\Big]\Big[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)\Big]\Big\} / \Big\{(\bar{A}\beta_6\gamma\Omega)^2 + \bar{A}^2\Big[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}\bar{A}(\bar{A}^2 + \bar{B}^2)\Big]^2 - 2\beta_{10}\bar{A}^2\bar{B}^2\Big[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)\Big]\Big\}$$

$\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_7$ and $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_7$ are functions of $\bar{A}$ or $\bar{B}$ respectively and are given in Appendix B of [23].

One can get solutions for driven mode $\bar{A}$ by solving equations (20-1) and (20-2) simultaneously for given amplitudes of campanion mode $\bar{B}$ or get solutions for companion mode $\bar{B}$ from equations (20-1) and (20-3) for given amplitudes of driven mode $\bar{A}$.

STABILITY ANALYSIS

To study stability of the solutions, equations (12) are rewritten as follows:

$$\frac{d^2A}{d\tau^2} + 2\gamma_s \frac{dA}{d\tau} + A + \frac{3}{8}\varepsilon\left[\left(\frac{dA}{d\tau}\right)^2 + A\frac{d^2A}{d\tau^2} + \left(\frac{dB}{d\tau}\right)^2 + B\frac{d^2B}{d\tau^2} + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\frac{d^2A}{d\tau^2}\right]$$

$$\cdot\left[A + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\right] + \frac{3}{4}\gamma_s\varepsilon\left[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\frac{dA}{d\tau}\right]\left[A + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\right]$$

$$+ \alpha_{s1}A^2 + \alpha_{s2}B^2 + \alpha_{s3}A^3 + \alpha_{s4}AB^2 + \alpha_{s5}(A^2 + B^2)A^2 + \alpha_{s6}(A^2 + B^2)^2$$

$$+ \alpha_{s7}(A^2 + B^2)^2 A = F_{SD} \tag{21-1}$$

$$\frac{d^2B}{d\tau^2} + 2\gamma_s\frac{dB}{d\tau} + \beta_{s1}B + \frac{3}{8}\varepsilon B\left[\left(\frac{dA}{d\tau}\right)^2 + A\frac{d^2A}{d\tau^2} + \left(\frac{dB}{d\tau}\right)^2 + B\frac{d^2B}{d\tau^2}\right.$$

$$\left. + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\frac{d^2A}{d\tau^2}\right] + \frac{3}{4}B\gamma_s\varepsilon\left[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\hat{\xi}_2 + \bar{\xi}_2)\frac{dA}{d\tau}\right]$$

$$+ \beta_{s2}AB + \beta_{s3}A^2B + \beta_{s4}B^3 + \beta_{s5}(A^2 + B^2)AB + \beta_{s6}(A^2 + B^2)^2 B = 0 \tag{21-2}$$

where, $\Omega_s = \omega/\bar{\omega}_{mn}$

$\bar{\omega}^2_{mn} = \frac{1}{2}\frac{\beta_2 E}{\rho \bar{R}^2}$ – natural frequency of free vibration of the imperfect shell (Linear Theory)

$\gamma_s = \frac{c}{2\bar{\rho}\bar{\omega}_{mn}}$

$\tau = \bar{\omega}_{mn}\, t$

$F_{SD} = F_D/\beta_2$

The coefficients $\alpha_{s1}, \alpha_{s2}, \ldots, \alpha_{s7}$ and $\beta_{s1}, \beta_{s2}, \ldots, \beta_{s6}$ are defined in [21].

As a test of the stability of the response, the following small perturbations $\zeta(\tau)$ and $\eta(\tau)$ are introduced:

$$A(\tau) = \bar{A}\cos\chi_1 + \zeta(\tau) \tag{22-1}$$

$$B(\tau) = \bar{B}\sin\chi_2 + \eta(\tau) \tag{22-2}$$

where

$$\chi_1 = \Omega_s\tau + \bar{\phi}$$

$$\chi_2 = \Omega_s\tau + \bar{\psi}$$

These expressions are then substituted into equations (21-1) and (21-2). Only the first order terms in the perturbations are retained. This procedure results in two coupled differential equations for $\zeta(\tau)$ and $\eta(\tau)$, namely:

$$\bar{\alpha}_{s1}\frac{d^2\zeta}{d\tau^2} + \bar{\alpha}_{s2}\frac{d\zeta}{d\tau} + \bar{\alpha}_{s3}\zeta + \bar{\alpha}_{s4}\frac{d^2\eta}{d\tau^2} + \bar{\alpha}_{s5}\frac{d\eta}{d\tau} + \bar{\alpha}_{s6}\eta = 0 \qquad (23\text{-}1)$$

$$\bar{\beta}_{s1}\frac{d^2\eta}{d\tau^2} + \bar{\beta}_{s2}\frac{d\eta}{d\tau} + \bar{\beta}_{s3}\eta + \bar{\beta}_{s4}\frac{d^2\zeta}{d\tau^2} + \bar{\beta}_{s5}\frac{d\zeta}{d\tau} + \bar{\beta}_{s6}\zeta = 0 \qquad (23\text{-}2)$$

where the coefficients $\bar{\alpha}_{s1}$, $\bar{\alpha}_{s2}$, ... , $\bar{\alpha}_{s6}$ and $\bar{\beta}_{s1}$, $\bar{\beta}_{s2}$, ... , $\bar{\beta}_{s6}$ are defined in [21].

No closed form solutions of these equations are known. However, approximate solutions can be obtained by using the Method of Slowly Varying Perturbations [17]. In this method, the perturbations $\zeta(\tau)$ and $\eta(\tau)$ are assumed to have the following form

$$\zeta(\tau) = \zeta_1(\tau)\cos\chi_1 + \zeta_2(\tau)\sin\chi_1 \qquad (24\text{-}1)$$

$$\eta(\tau) = \eta_1(\tau)\sin\chi_2 + \eta_2(\tau)\cos\chi_2 \qquad (24\text{-}2)$$

Based on the premise that the functions ζ_1, ζ_2, η_1 and η_2 must vary slowly in the vicinity of stable points, one obtains upon substitution four simultaneous differential equations which can be expressed in matrix form as follows:

$$[M]\{\dot{\varphi}\} = [N]\{\varphi\} \qquad (25)$$

where $\{\varphi\} = [\zeta_1, \zeta_2, \eta_1, \eta_2]^T$

The constant coefficient matrices [M] and [N] are defined in [21].

Equation (25) is then solved by using $\varphi = \varphi_o e^{\lambda\tau}$. This leads to a standard eigenvalue problem for determining the λ's:

$$|N - \lambda M| = 0 \qquad (26)$$

It is quite obvious that if any of the roots of equation (26) has a positive real part, then the corresponding perturbations increase exponentially with time. This means that the associated steady-state solution is not stable.

DISCUSSION OF NUMERICAL RESULTS

The equations derived in this paper are quite general. They can be used to investigate the nonlinear vibration behaviour of orthotropic or isotropic shells, shells with or without axial compressive load, perfect or imperfect shells. However, in this paper numerical results for isotropic shells only are presented.

Two different shell geometries are considered in the numerical analysis that follows, which makes comparison with existing results possible. One is the shell used by Evensen, called ES herein. Its characteristic data are

$$\varepsilon = \left(\frac{n^2 h}{R}\right)^2 = 0.01$$

$$\xi = \frac{\pi R/n}{L/K} = 0.1 \quad \text{, aspect ratio of response}$$

$$\nu = 0.3$$

while the other one is called WN (the one used by Watawala and Nash). Its characteristic data are

$$\frac{h}{R} = \frac{1}{720}\,,\; \frac{L}{R} = \frac{2}{3}\,,\; \nu = 0.272$$

Notice that in the following numerical analysis we let $m = k$, thus satisfying the coupling condition between the axisymmetric and the asymmetric modes. We considered three separate cases.

CASE 1 Comparison with Evensen [14]

As a necessary check on the correctness of the computer program based on the rather general equations derived in the present study, the cases of free vibration and undamped response of both single and coupled modes for the various perfect shells used by Evensen were considered in detail. The amplitude-frequency relationships were calculated using equations (17) and (19). We were able to recover all the results presented by Evensen [14].

CASE 2 Undamped single mode response

In the first place, the free vibrations of WN shell were considered. Only asymmetric imperfections were taken. The classical axisymmetric buckling mode has $i_{c\ell} = 10$ half waves. To satisfy the coupling condition $i = 2k$, the classical asymmetric buckling mode is chosen as $k = 1$, $\ell = 25$. We assume that inital imperfections have the same shape as these classical axisymmetric and asymmetric buckling modes. In Fig. 1 the variation of natural frequency of vibration is plotted as a function of the asymmetric imperfection amplitude $\bar{\xi}_2$. Notice that the amplitudes of (nonlinear) vibration range from 0.001 to 0.9. Curves showing the results of Rosen and Singer [10], Hol [15] and Watawala and Nash [13] are also included.

It is evident that if the amplitude of (nonlinear) vibration is less than about 0.8 (say) for a shell with a small asymmetric imperfection ($\bar{\xi}_2 < 0.8$, say) the nonlinearity is softening. However, the nonlinearity will become hardening if the amplitude of imperfection is greater than this critical value. This phenomenon has been found in all cases with $n = \ell$. Hol's linearized vibration results agree well with the current nonlinear solutions if the amplitude of (nonlinear) vibration is very small (0.001 say) and if the imperfection is small ($\bar{\xi}_2 < 0.25$, say). Rosen and Singer's linearized vibration results indicate incorrectly a hardening behaviour. As mentioned earlier their vibration mode does not satisfy the circumferential periodicity condition.

The discrepancy between the current results and the one published by Watawala and Nash is harder to explain, since both papers make use of similar solution procedures. It appears that there might be a mistake in the derivation of the equations published by Watawala and Nash, since their equations cannot be reduced to Evensen's equations.

Figures 2 and 3 indicate the relationships between the frequency of vibration and the circumferential wave number ℓ for various values of imperfection amplitudes $\bar{\xi}_2$ at a very small and at a quite large amplitude of vibration, respectively. Results of the present analysis and those of [13], are in good agreement except when $n = \ell$. It is same problem as that shown in Fig. 1. Studying carefully Figures 2 and 3, one can see that the nonlinear vibration behaviour of a shell with an asymmetric imperfection depends upon the amplitude of vibration, the amplitude of imperfection and on the coupling between vibration mode and imperfection mode.

The influence of an asymmetric imperfection $\bar{\xi}_2$ on vibration for the case of $n \neq \ell$ is shown in Fig. 4. The behaviour of vibration is hardening, and it is only slightly influenced by the amplitude of vibration $\bar{A}$.

Figure 5 displays the undamped response of the WN shell to an excitation of $\bar{F}_D = 0.001$ for various asymmetric imperfection amplitudes $\bar{\xi}_2$. It is obvious that the increasing imperfection amplitudes will alter the behaviour of vibration from softening to hardening.

Further investigations have shown that the presence of axisymmetric or of combined axisymmetric and asymmetric imperfections produce effects similar to those described above for a single asymmetric imperfection mode. They can alter the degree of nonlineariry or the behaviour of the vibration depending on the amplitude of imperfection and on the satisfaction of certain coupling conditions.

The presence of an axial compressive load producing a prestressed fundamental state resulted in a reduction of the response frequency and in some cases in a change of the nonlinear behaviour. Because of spatial limitation these results have been omitted here. The interested reader can find them in Reference [20].

CASE 3 Undamped and damped coupled mode response

Starting values for a Newton-Raphson produce were obtained by cross-plotting some initial results.

Using the dimensions of the ES shell the wave numbers of vibration are chosen such that
a. They satisfy the accuracy requirement of Donnell's equations, namely the circumferential wave number should be greater than 4;
b. They would constitute lower order modes which can be excited easily into nonlinear region to make experimental verification possible.

In present analysis the mode $i = 10$, $\ell = 5$ was selected.

The first series of computations are meant to illustrate the influence of an asymmetric and an axisymmetric imperfection mode on the undamped coupled mode response. Thus Fig. 6 shows typical plots of single and coupled mode responses of the perfect ES shell for a certain excitation level. The influence of an asymmetric or of an axisymmetric imperfection are shown in Figs. (7) and (8), respectively. As can be seen both axisymmetric and asymmetric imperfections have strong influence when the nonlinear coupling conditions $i = 2k$ and $n = \ell$ are satisfied. In other cases the influence becomes weak.

The second series of computations is for damped single and coupled mode responses of the imperfect ES shell. Figs. 10-20 indicate the numerical results for this case. In order to facilitate the understanding discussion of these numerical results are divided into four categories.

a. Influence of Damping and Excitation

Figures 10-12 show the amplitude-frequency relationships of perfect shell for different values of damping and excitation. As can be seen from Fig. 6 and Figs. 10-11 damping has a pronounced influence on the nonlinear vibration behaviour. Thus increasing damping can quickly round off the coupled-mode response peak. Similar results have been found by Ginsberg [5].
An interesting fact can be deduced from results shown in Figures 10a, 12 and 13, namely, that increasing excitation or decreasing damping can disrupt the stability of the coupled-mode response peak. This fact has been found by Chen in his carefull experiment fifteen years ago, but has not been predicted by theoretical analysis before.

b. Influence of Asymmetric Imperfections

The response-frequency relationships of a shell with asymmetric imperfections are shown in Figs. 14-16, where the damping and excitation are kept constant. It can be observed that the asymmetric imperfection has a strong influence on the coupled-mode response if the coupling condition $n = \ell$ is satisfied. Increasing the amplitude of the asymmetric imperfection can quickly decrease the region where the coupled-mode response occurs. The asymmetric imperfections also changes the stability characteristics of the coupled-mode response. But, as can be seen the influence of the asymmetric imperfections on the shape of single mode response-frequency curve is minimal.

c. Influence of Axisymmetric Imperfections

Figures 17-19 display the amplitude-frequency relationships of a shell with axisymmetric imperfections. As can be seen, if the coupling condition $i = 2k$ is satisfied then also the axisymmetric imperfections have a strong influence on the nonlinear vibration behaviour of shells. Notice that the left bifurcation point is now on the lower branch of the associated single mode response curve rather than on the upper branch as in the case of an asymmetric imperfection. In addition the axisymmetric imperfections change the stability characteristics of the coupled mode response curves.

d. Influence of Combined Imperfections

Because of lack of space only one amplitude-frequency relationship for $\bar{\xi}_1 = -0.02$ and $\bar{\xi}_2 = 0.02$ is shown in Fig. 20. The coupling conditions are $i = 2k$ and $n = \ell$. As can be seen the shape of curve is quite similar to the case of $\bar{\xi}_1 = -0.02$ and $\bar{\xi}_2 = 0.0$, except for a smaller region of coupled-mode response.

Further studies have indicated that the influence of combined initial imperfections on the vibration behaviour of the shell are insignificant unless the coupling conditions $i = 2k$ and $n = \ell$ are satisfied.

The application of an axial compressive load well below the initial buckling load reduces the frequencies of response but does not alter the vibration mode shapes. For further details the reader should consult Reference [23].

CONCLUSIONS

The nonlinear flexural vibrations of perfect and imperfect damped and undamped thin-walled cylindrical shell are analyzed by using Donnell's nonlinear shell equations. The numerical solutions are obtained by applying Galerkin's method and by using the Method of Averaging. The study yields the following conclusions:

1. A good agreement between present and Ginsberg's [5] analysis is obtained. The "gap" found in Evensen's analysis [2], which is the major difference between Evensen and Ginsberg is not found in the present analysis. In the author's opinion the "gap" resulted because Evensen neglected the negative values of cos $2\bar{\Delta}$ inappropriately in his ring analysis. Therefore, one can now say that no qualitative difference exists between the results of the different solution procedures, such as Galerkin's method (by Evensen [2,3] and present analysis), small parameter perturbation method (by Chen [6]), special perturbation procedure (by Ginsberg [5]).
2. The general characteristics of damped response of perfect shells found by Ginsberg are confirmed by present analysis, namely
   - the damping has a pronounced influence on the coupled-mode response. Increasing damping can completely eliminate coupled-mode response peaks,
   - damped response of a perfect shell can be devided into five regions, as shown in Fig. 10. In the region (3) both single mode and coupled-mode response are unstable. The coupled mode response peak in the region (4) is stable,
   - the single mode response between two bifurcation points is unstable.

   One of the extra results obtained by present analysis is that the stability of coupled-mode response in region (4) is not always stable. It depends on the amount of damping or excitation, as shown in Figs. 11 and 12.
3. Initial geometric imperfections have a significant influence on the free or forced vibration of the single mode under certain coupling conditions. As one can see from Figs. 1 and 5 the presence of initial geometric imperfections alter not only the degree of nonlinearity but also the basic behaviour. The "critical value", to be exact, the "critical region" of the amplitudes of imperfections has been found in the present analysis. The results indicate that the size of critical region also depends on the circumferential wave number and on the amplitude of vibration [14].
4. Initial geometric imperfections have also a considerable influence on the coupled-mode response under certain coupling conditions. In the undamped case, the unequal effect of imperfections on the driven mode and on the companion mode makes their response curves shift towards each other. In the damped case, the general influence of imperfections is quite similar to that of damping. That is increasing amplitudes of imperfections can quickly eliminate the coupled-mode response. In addition, the presence of initial geometric imperfection changes the stability characteristics of the solutions.

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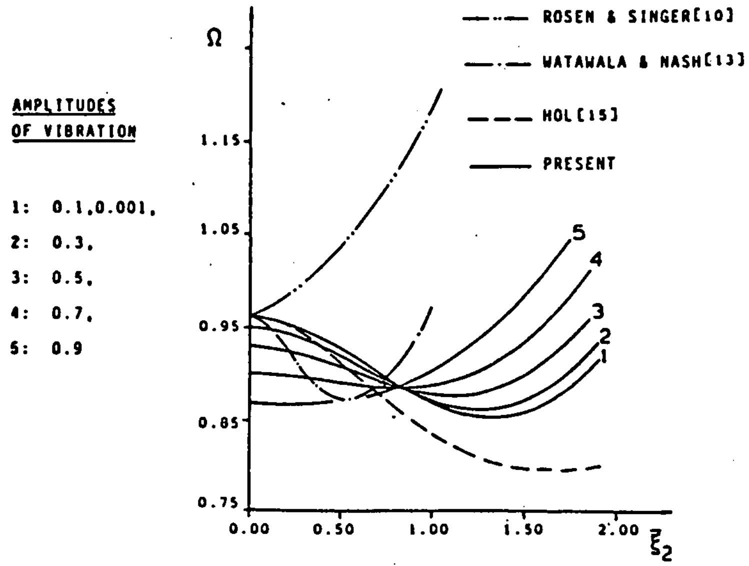

Fig.1 FREQUENCY OF FREE VIBRATION VS AMPLITUDE OF IMPERFECTION FOR DIFFERENT VALUES OF AMPLITIDE OF VIBRATION.

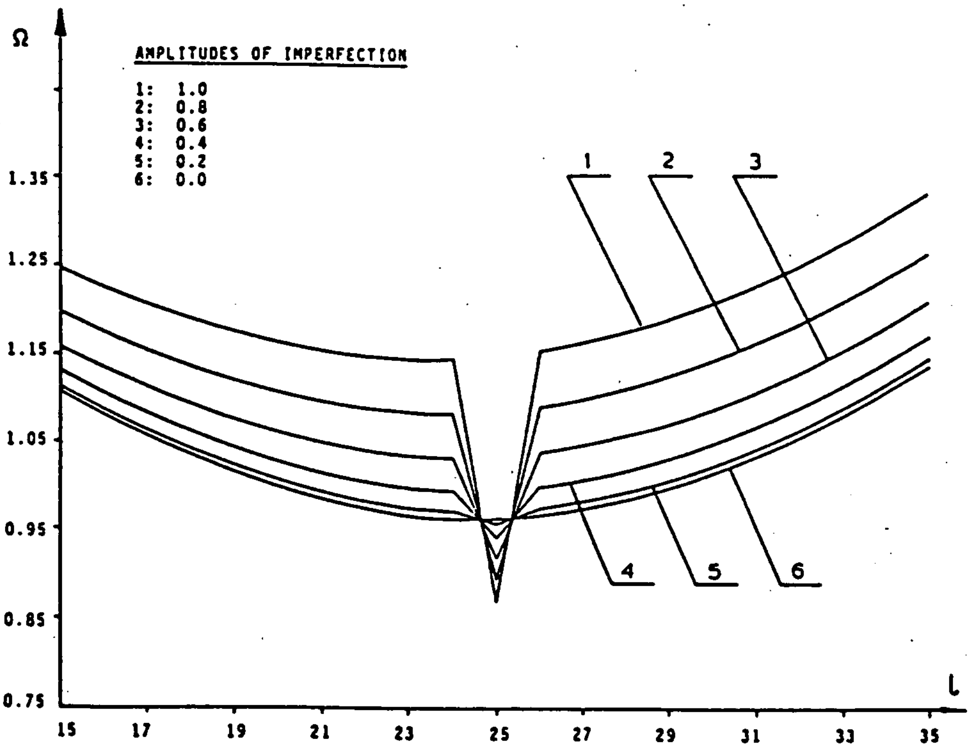

Fig.2 FREQENCY OF FREE VIBRATION VS CIRCUMFERENTIAL WAVE NUMBER FOR DIFFERENT VALUES OF AMPLITUDE OF IMPERFECTION. AMPLITUDE OF VIBRATION $\bar{A}$ = 0.001.

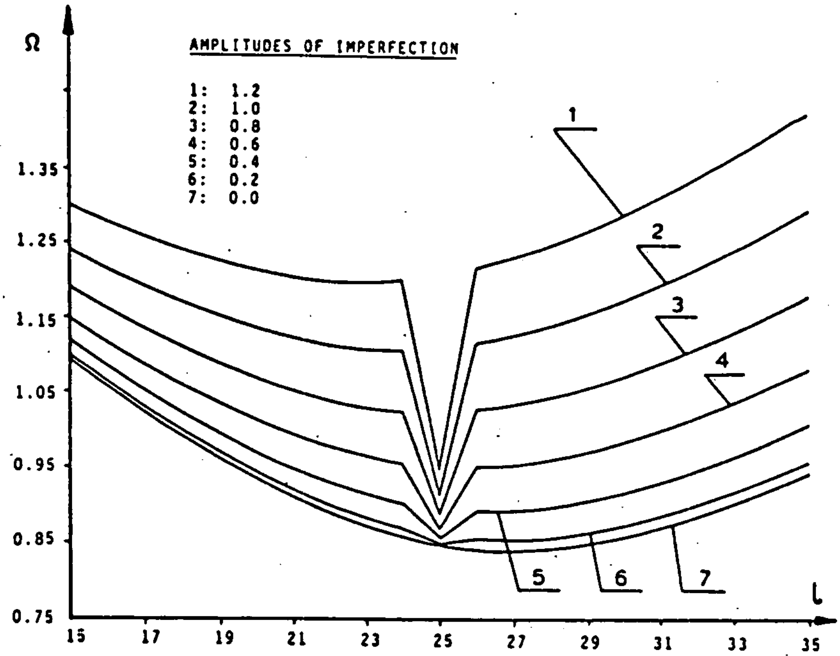

Fig.3 FREQUENCY OF FREE VIBRATION VS CIRCUMFERENTIAL WAVE NUMBER FOR DIFFERENT VALUES OF AMPLITUDE OF IMPERFECTION. AMPLITUDE OF VIBRATION $\bar{A} = 1.0$.

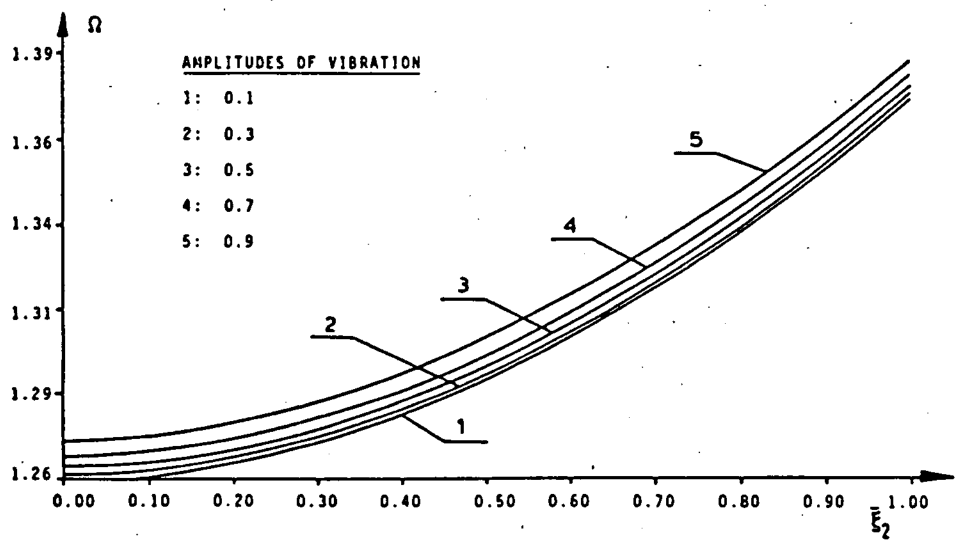

Fig.4 FREQUENCY OF FREE VIBRATION VS AMPLITUDE OF IMPERFECTION FOR DIFFERENT VALUES OF AMPLITUDE OF VIBRATION.

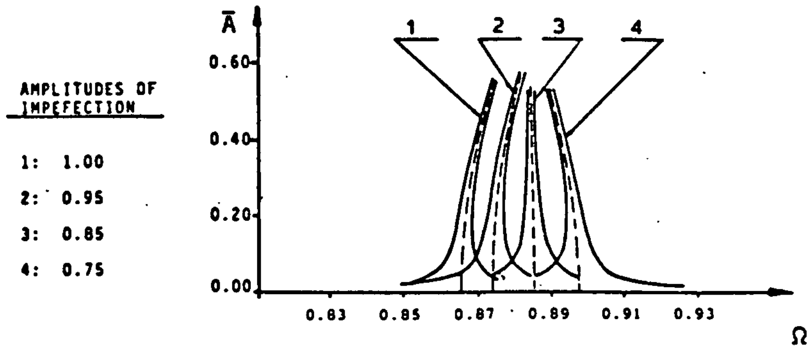

Fig.5 AMPLITUDE OF FORCED VIBRATION VS FREQUENCY OF EXCITATION FOR DIFFERENT VALUES OF AMPLITUDE OF IMPERFECTION. EXCITATION $\bar{F}_D = 0.001$.

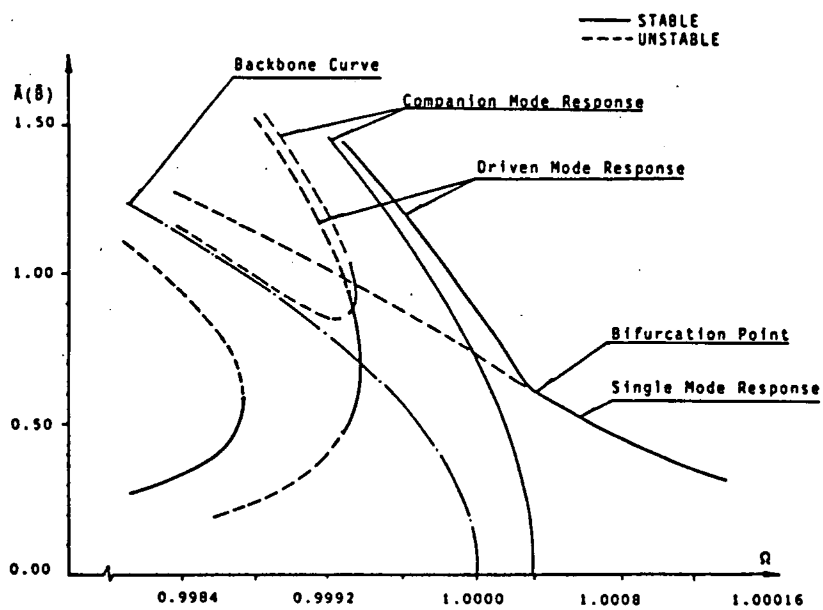

Fig.6 AMPLITUDE-FREQUENCY RELATIONSHIP OF UNDAMPED VIBRATION OF PERFECT SHELL. EXCITATION $\bar{F}_D = 2\times10^{-6}$.

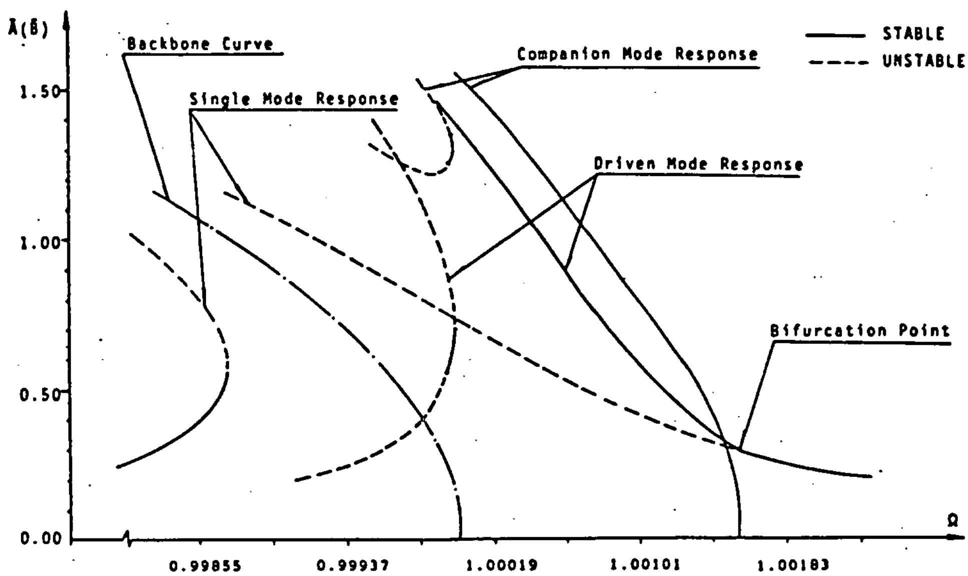

Fig.7 AMPLITUDE-FREQUENCY RELATIONSHIP OF UNDAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1$ = 0.00 AND $\bar{\xi}_2$ = 0.05. EXCITATION $\bar{F}_D = 2\times10^{-6}$.

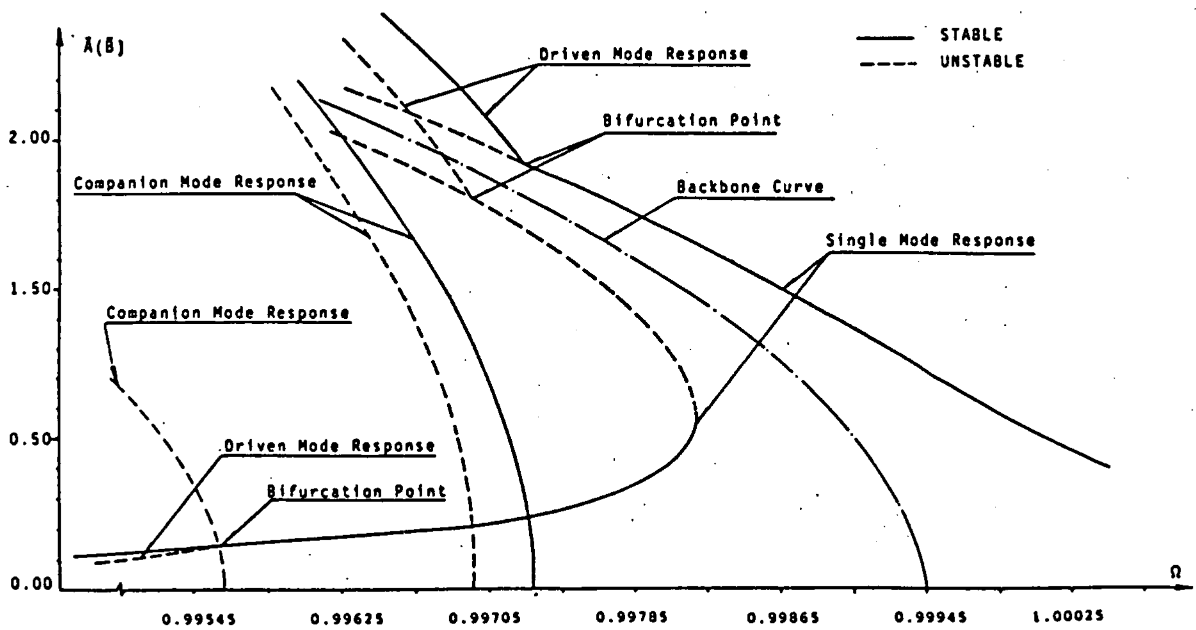

Fig.8 AMPLITUDE-FREQUENCY RELATIONSHIP OF UNDAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1$ = -0.04 AND $\bar{\xi}_2$ = 0.00. EXCITATION $\bar{F}_D = 2\times10^{-6}$.

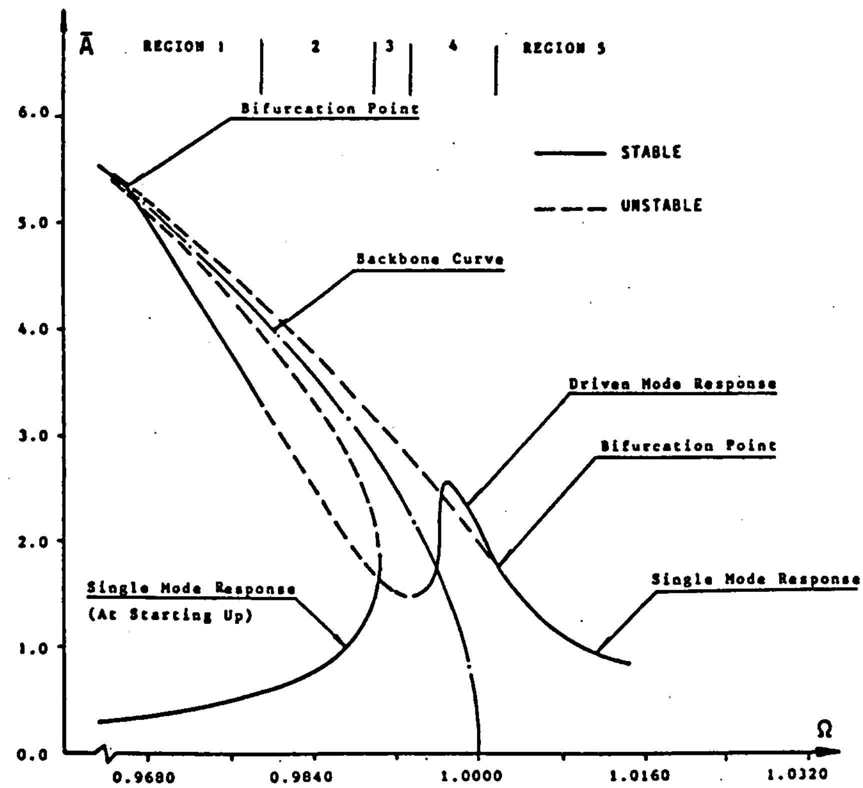

Fig.10a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL.
DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

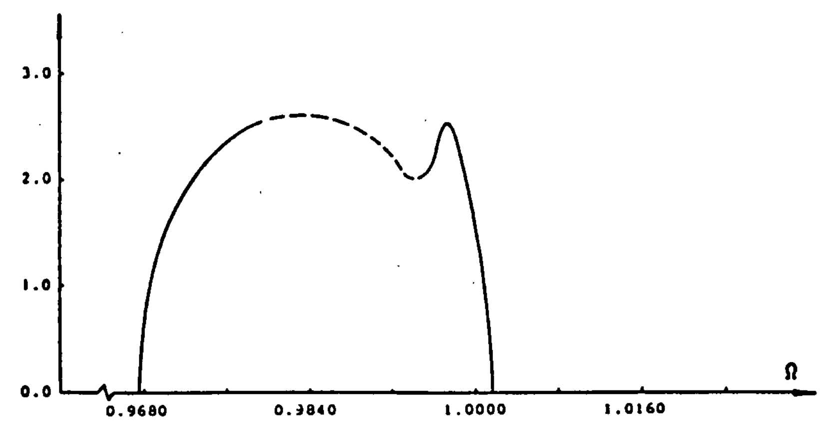

Fig.10b COMPANION MODE RESPONSE.

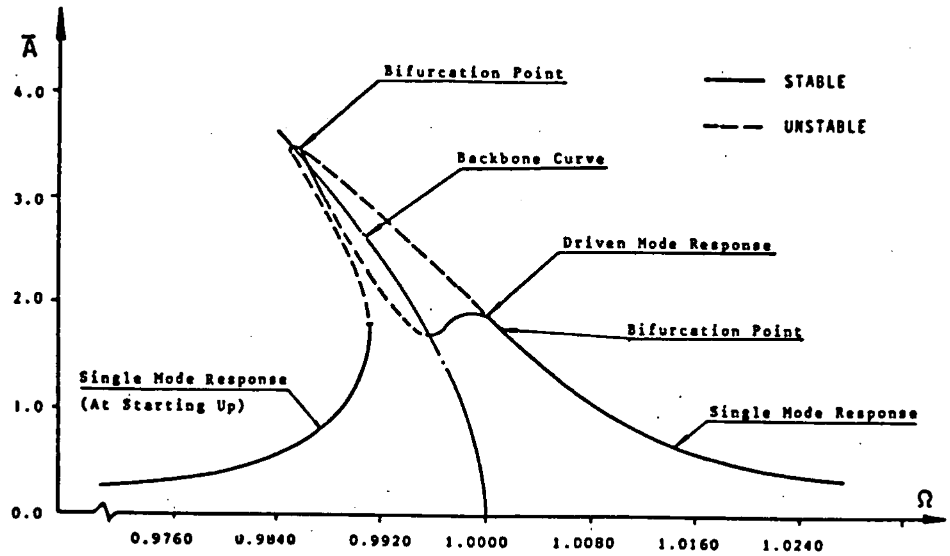

Fig.11a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL.
DAMPING $\gamma = 1.35\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

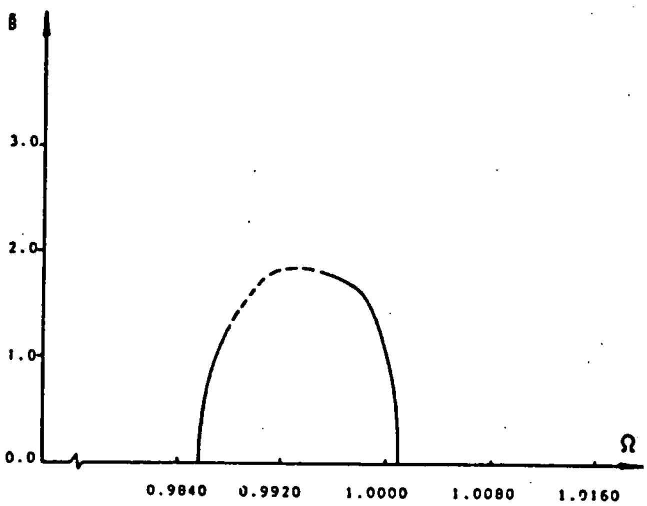

Fig.11b COMPANION MODE RESPONSE.

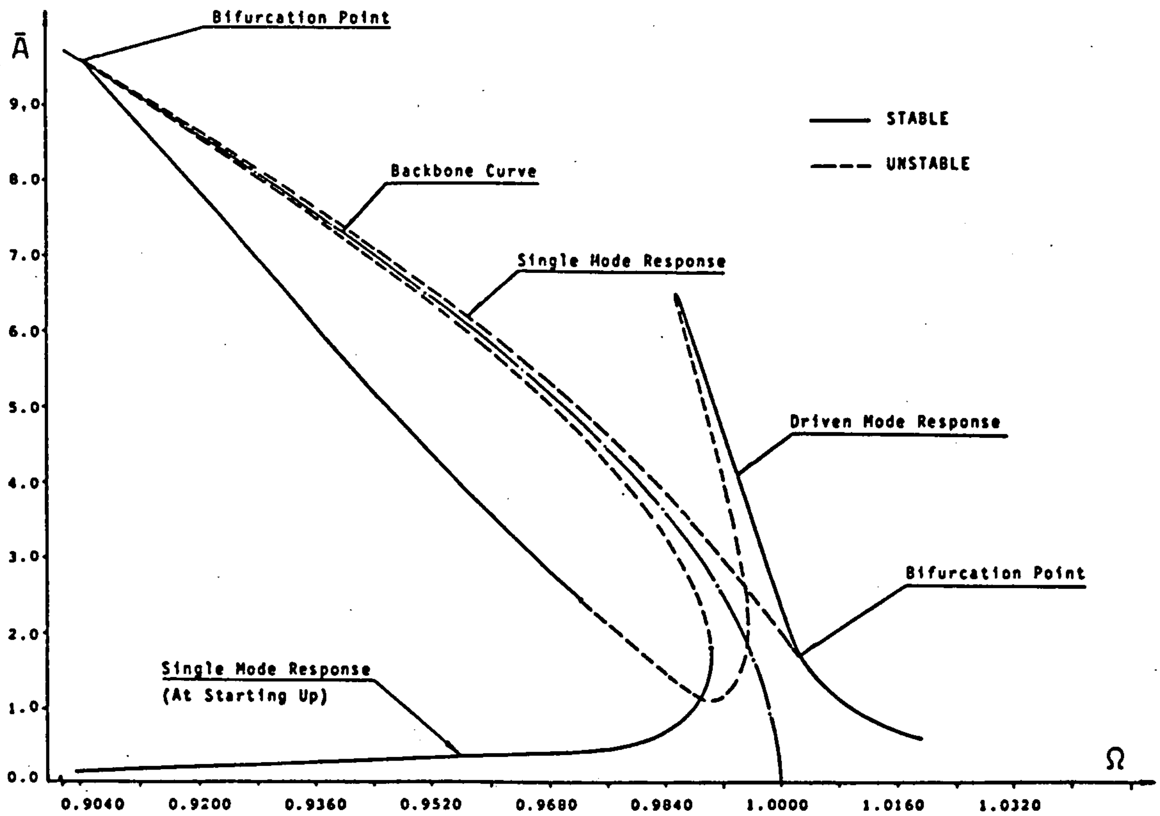

Fig.12a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. DAMPING $\gamma = 5\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

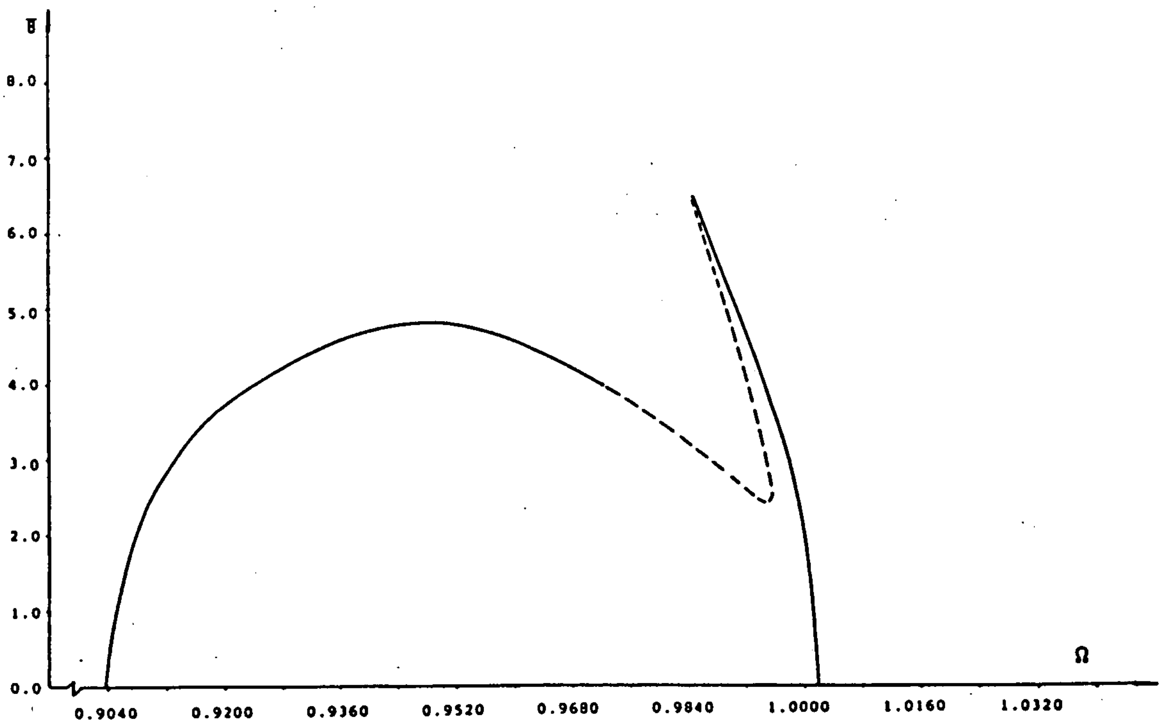

Fig.12b COMPANION MODE RESPONSE.

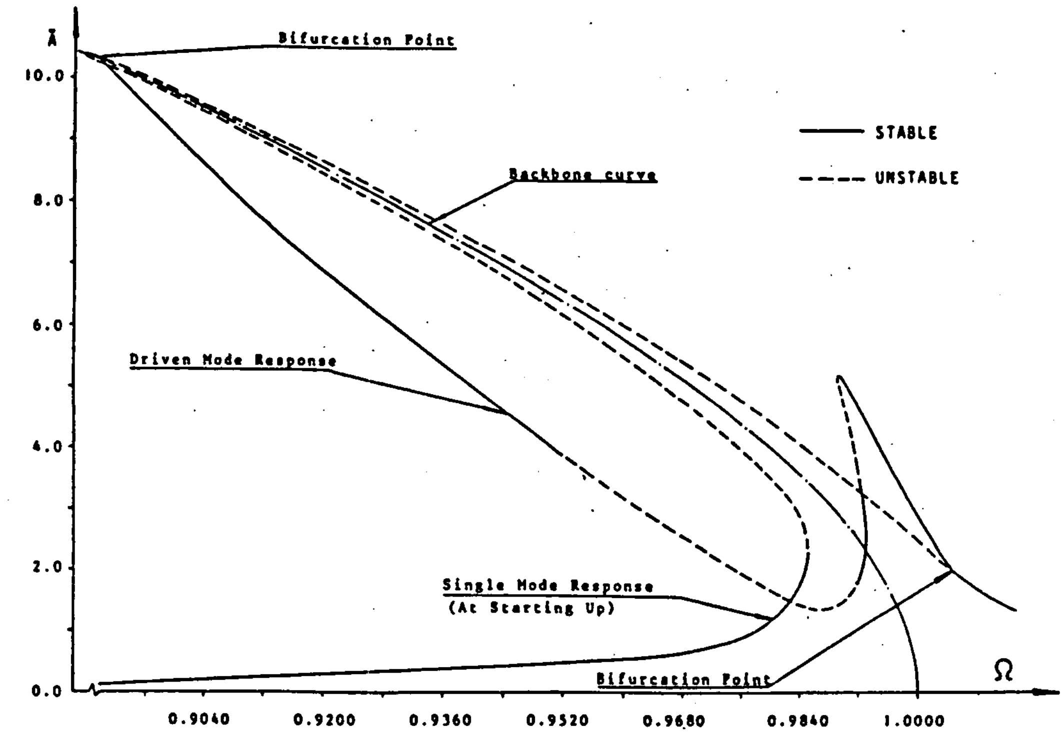

Fig.13a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 8.50\times10^{-5}$.

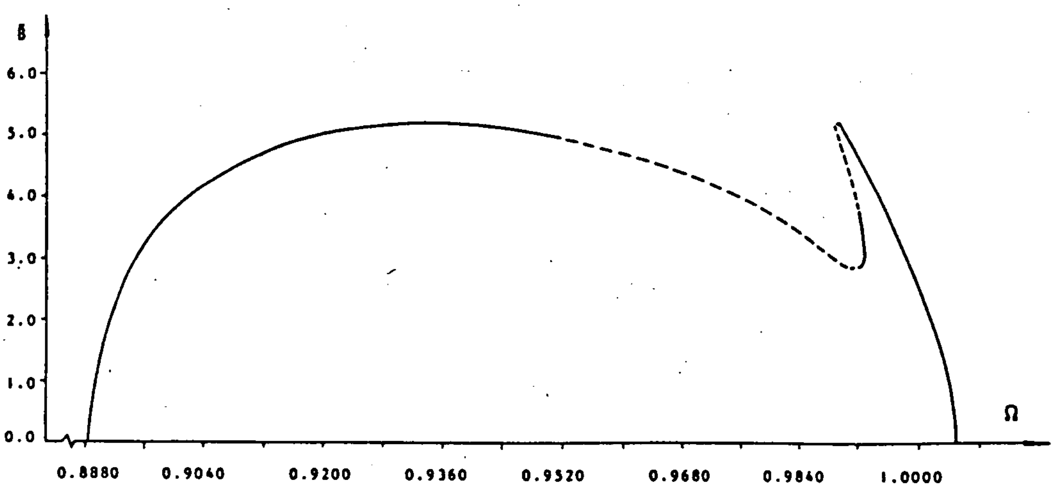

Fig.13b COMPANION MODE RESPONSE.

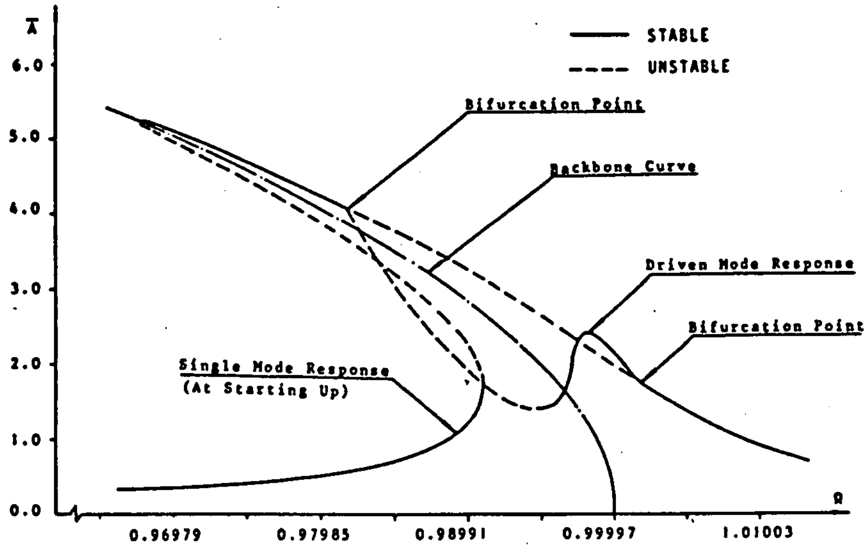

Fig.14a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = 0.00$ AND $\bar{\xi}_2 = 0.05$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

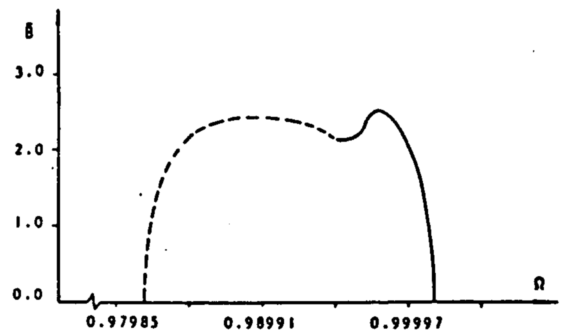

Fig.14b COMPANION MODE RESPONSE.

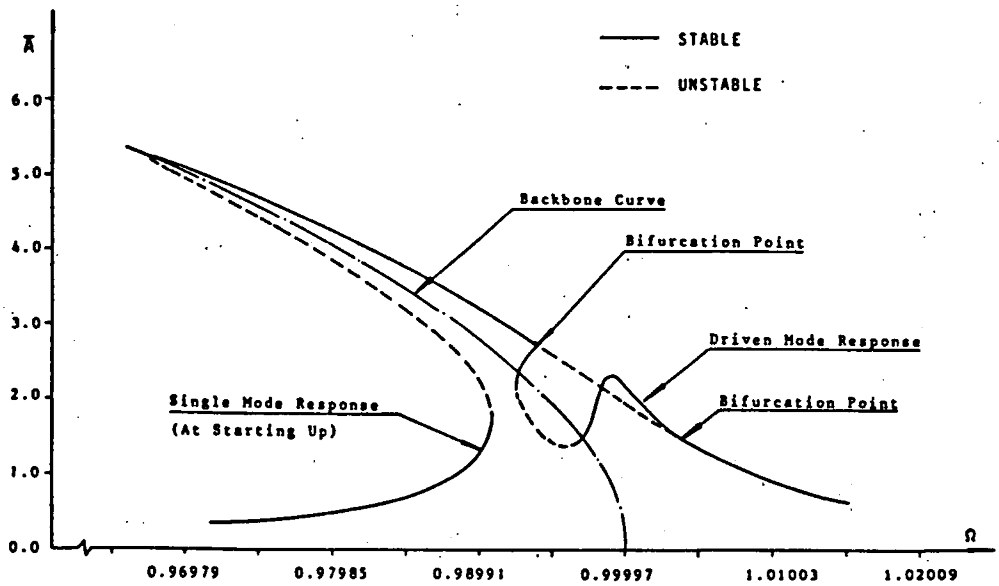

Fig.15a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = 0.00$ AND $\bar{\xi}_2 = 0.07$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

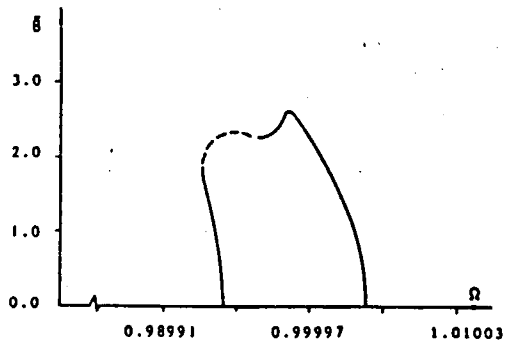

Fig.15b COMPANION MODE RESPONSE.

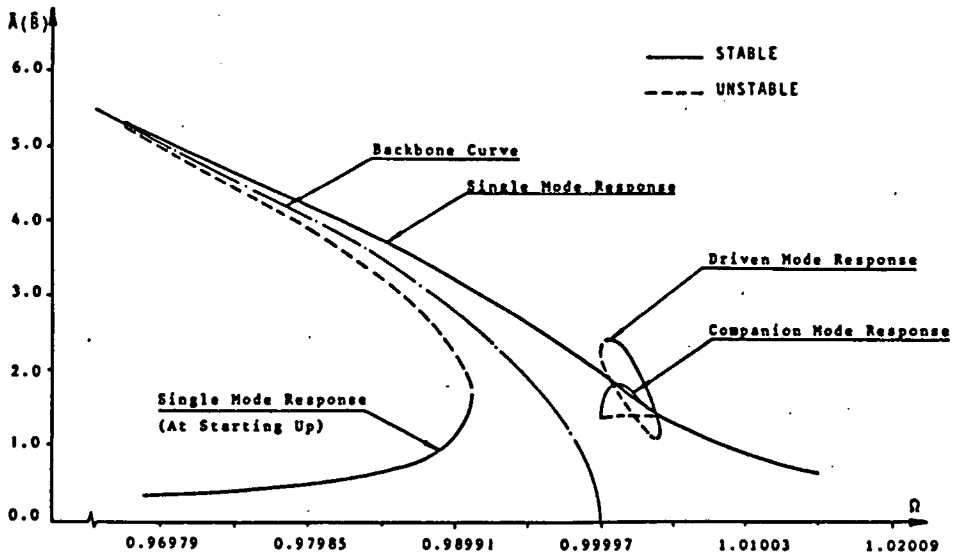

Fig.16 AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = 0.00$ AND $\bar{\xi}_2 = 0.10$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

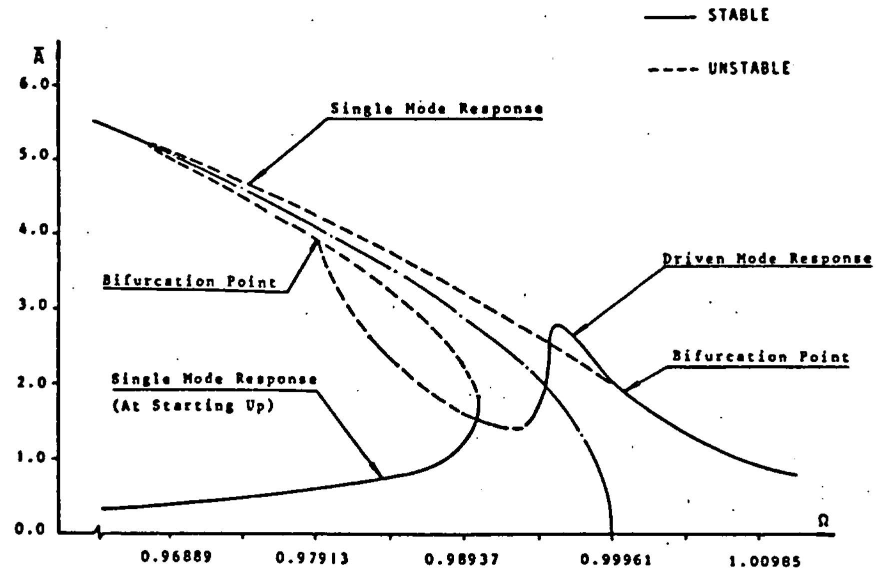

Fig.17a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = -0.02$ AND $\bar{\xi}_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

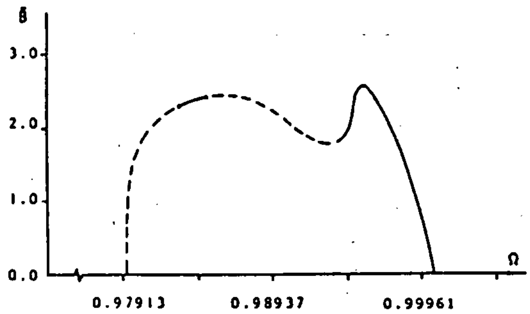

Fig.17b COMPANION MODE RESPONSE.

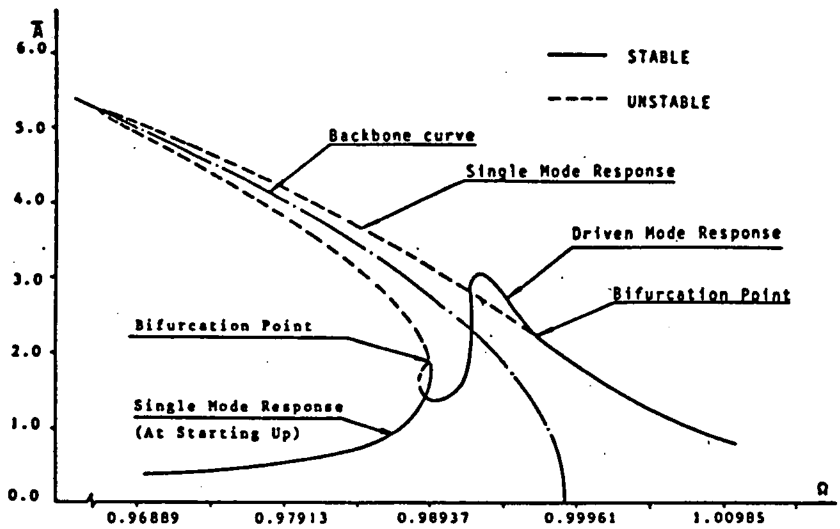

Fig.18a AMPLITUDE-FREQUENCY RELATIOSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = -0.06$ AND $\bar{\xi}_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

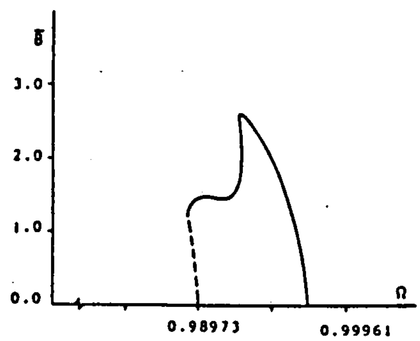

Fig.18b COMPANION MODE RESPONSE.

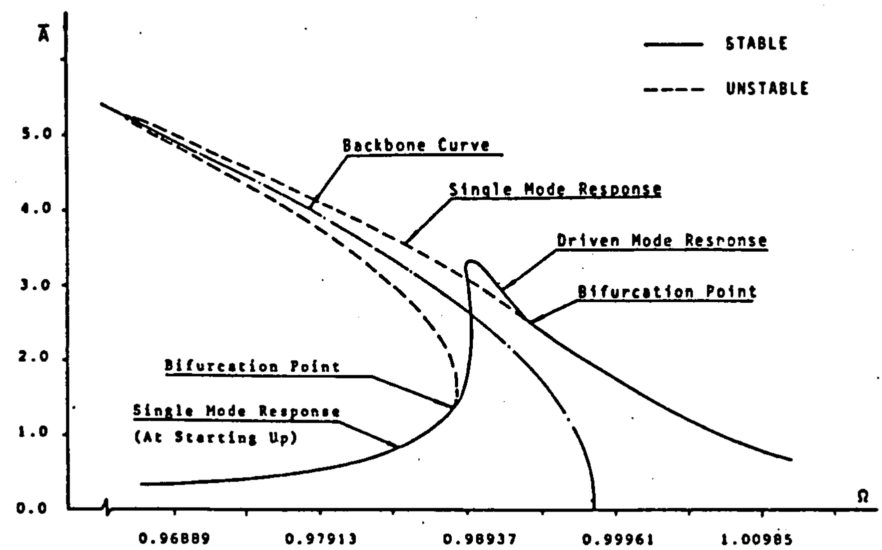

Fig.19a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1$ = -0.10 AND $\bar{\xi}_2$ =0.00. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

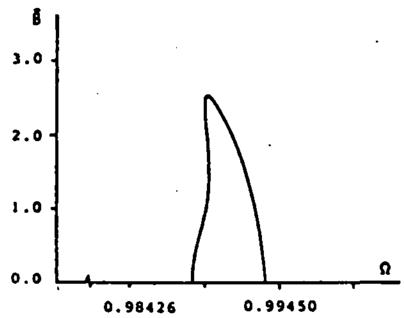

Fig.19b COMPANION MODE RESPONSE.

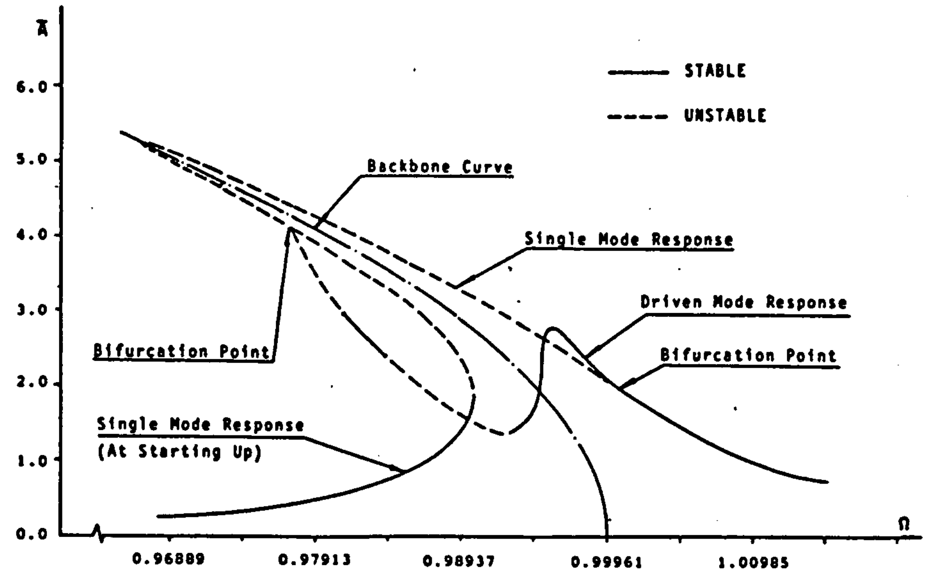

Fig.20a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\bar{\xi}_1 = -0.02$ AND $\bar{\xi}_2 = 0.02$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

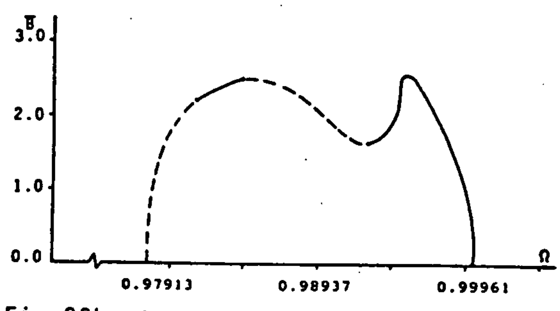

Fig.20b COMPANION MODE RESPONSE.

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