Hedging interest rate risk for pension schemes: Optimization and effectiveness

The case of the Netherlands

by

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If you think in terms of a year, plant a seed;
if in terms of ten years, plant trees;
if in terms of 100 years, teach the people.

Confucius

Abstract

Efficiently managing hedging portfolios on behalf of pension funds is key in achieving the target hedging strategy, which can significantly impact coverage ratios. A new optimization approach to fixed income portfolio management for pension funds is proposed that finds interest rate risk hedging strategies while incorporating additional requirements. These are relevant requirements for pension funds such as country allocations, low transaction costs and reasonable investment costs. In doing so, pension fund regulations and common practices are investigated in a rigorous mathematical framework. The hedging strategies are shown to perform well when back-testing. In addition, simulation of the interest rate and cash flows in a Defined Benefits pension scheme displays the good performance of the strategies. These strategies are further tailored to specific pension funds by considering the trade-off between yield and risk, which could contribute to increasing a pension fund's coverage ratio. Alternatively, a procedure is also proposed to generate more diversified albeit less optimal hedging portfolios using the optimization approach.

Keywords — Pension fund, Pension scheme, Defined Benefits, Optimization, Interest rate risk, Hedging, Key rate duration, Simplex

Preface

When I first came across this potential thesis subject, I was enthused by the prospect of writing a thesis on a subject that has both mathematical depth and societal relevance. I knew little about pensions — not strange considering I probably had 50 years of working left before retirement — and thought hedging of interest rate risk was very complicated. In the months since I have learned a great deal about the pension world — I am happy I spent this time researching the topic.

This thesis has been submitted for the degree of Master of Science in Applied Mathematics at Delft University of Technology, in the Netherlands. The research has been done under the acadamic supervision of Dr. Ir. L.A. Grzelak and Prof. Dr. Ir. C.W. Oosterlee. The work has been done at Kempen Capital Management under the supervision of Ir. A.C.A.G. Pochet and Drs. J.M. van Boxtel, MBA. The Client Solutions department of Kempen Capital Management performs asset management activities for pension funds.

I would not have been able to achieve these results without help from a number of people. As supervisors at the company, Alexandre Pochet and Job van Boxtel were always willing to help and gave me freedom in choosing a direction for the project. The many discussions with Alexandre — sometimes aberrations — were invaluable and I will not forget the talks with Job. Also, I would like to thank my other colleagues and in particular Tsz and Bas for helping me throughout the project.

I would also like to thank Lech Grzelak for his help throughout the project, pushing me in new direction after intense discussions that motivated me to keep working. Also, I would like to thank Kees Oosterlee for his excellent guidance and close involvement with the project. Finally, I would like to thank my family and friends for the continued support over these months.

Aizo Kroon
Delft, December 2019

Contents

Acronyms

ABP	Algemeen Burgerlijk Pensioenfonds. 9–11, 71, 82, 107
AFM	Autoriteit Financiële Markten. 78
ALM	Asset-Liability Management. 3, 19
AOW	Algemene Ouderdomswet. 8, 66, 73, 74
CBS	Centraal Bureau voor de Statistiek. 59, 64, 65, 67, 118, 124, 132–134
CDF	Cumulative Distribution Functino. 96, 98
CPI	Consumer Price Index. 3, 58–61, 132
DB	Defined Benefits. 8, 49, 50, 67, 90
DC	Defined Contribution. 49
DNB	De Nederlandsche Bank. 9, 27, 44, 50, 67, 68, 71–73, 80, 135, 139
ECB	European Central Bank. 61, 67
EONIA	Euro OverNight Index Average. 22
ESG	Environmental, Social and Governance. 19
ESTER	Euro Short-Term Rate. 22
EU	European Union. 61, 79, 94, 95
EURIBOR	Euro Interbank Offered Rate. 16, 22
HJM	Heath-Jarrow-Morton. 28
HMD	Human Mortality Database. 55
LIBOR	London Interbank Offered Rate. 16, 17, 22
LP	Linear Programming. 85, 86, 109
MiFID	Markets in Financial Instruments Directive. 78
MVEV	Minimaal Vereist Eigen Vermogen. 9, 10
OECD	Organisation for Economic Co-operation and Development. 7
SVD	Singular Value Decomposition. 56, 57
UBS	Union Bank of Switzerland. 128
UFR	Ultimate Forward Rate. 11, 14, 27
USA	United States of America. 62
VEV	Vereist Eigen Vermogen. 9, 10, 67
YTM	Yield to Maturity. 77, 80, 101, 102, 129

1

Introduction

The financial situation of pension schemes has changed in recent years as a result of decreasing interest rates. The present value of future pension liabilities increased while the assets did not necessarily exhibit the same increase in value. Coverage ratios have consequently declined and in the Netherlands heated discussions have taken place on the feasibility and ultimately the future of the pension system. For the first time, people have had to worry that their pensions might be cut. A hedging portfolio plays an integral role in minimizing interest rate risks — the portfolio is constructed such that that any change in the present value of the liabilities due to interest rate changes is accompanied by a similar change in the present values of the assets. This thesis concerns itself with finding an optimal hedging portfolio for pension schemes.

Research on the asset allocation of pension funds has been focused on overarching topics. Scholars such as Cairns et al. (2000), Battocchio and Menoncin (2004), Dondi (2005), Cairns et al. (2006), and Horvath et al. (2018) have researched the optimal asset allocations to certain asset classes in various levels of detail, giving very relevant results for Asset-Liability Management (ALM). However, these results do not yield a readily usable hedging portfolio.

Research on the future liabilities of pension funds by Van Rooij et al. (2004), Draper, Armstrong, et al. (2007), Michielsen et al. (2015), and De Waegenaere et al. (2018) has been focused on the expected cash flows of pension funds as a whole rather than the actual dynamics. Dondi (2005) proposes a more detailed model for pension fund liabilities, but volatilities are also not considered while this is very relevant for estimating risks. Still, extensive research is available on the processes underlying pension fund liabilities such as mortality (Lee and Carter 1992; Van Berkum et al. 2016), the Consumer Price Index (CPI) (Jarrow and Yildirim 2003) and notably salaries (Carriere and Shand 1998). However, researchers have not yet proposed such an inclusive model for future pension liabilities.

The importance of hedging interest rate risk to the coverage ratios of pension schemes has been shown by Kroon et al. (2017) — thus finding an optimal hedging portfolio given a specific pension fund is a relevant research topic from a practical point of view. There is ample research on the interest rate and interest rate derivatives. Brigo and Mercurio (2007) describe relevant assets and interest rate models, one of which is the model by Hull and White (1990). The discount curve — that determines the present value of future pension liabilities — has been researched by Hagan

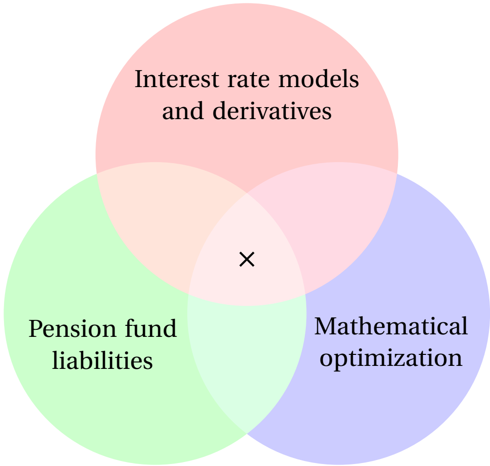


Figure 1.1: The main research fields of this thesis.

and West (2008). Oosterlee and Grzelak (2019) additionally describe the hedging of interest rate derivatives and future payments in detail. However, these results are not directly applicable to pension funds given the specific set of requirements that pension funds have.

Finding an optimal asset allocation for a pension fund to specific assets lies on the interface of these fields — this intersection is the territory of this thesis (Figure 1.1). By making use of research on mathematical optimization due to scholars such as Dantzig et al. (1955) and Boyd and Vandenberghe (2004), an optimization approach may be applied to this asset allocation problem. The field of multi-objective optimization (Zadeh 1963; Marler and Arora 2010) is of particular interest. This problem lies within the realm of fixed income portfolio management for pension funds and is the topic of this thesis.

The hedging of interest rate risk for pension funds is not innovative by itself. Using interest rate swaps and (government) bonds a hedging portfolio is generally constructed by pension fund managers. The sensitivity to interest rate changes is then usually calculated using the *key rate duration* first proposed by Ho (1992), which resembles a similar result due to Reitano (1992). This has been described in more detail by scholars such as Nawalkha, Soto, and Beliaeva (2005).

In the Netherlands it is particularly common to hedge interest rate risks, because pension funds with defined future benefits are commonplace and well-funded. This thesis is therefore also focused on the case of the Netherlands, but results should also be applicable to other countries.

Even though interest rate risks are already hedged by practitioners, there is room for improvements. A rigorous mathematical framework in which all relevant assets and pension fund dynamics and requirements are described is not readily available. Future pension liabilities and measures of interest rate sensitivity could be formulated accurately in such a framework, which may result in improved hedging portfolios. These portfolios could be improved further — pension fund managers often construct a hedging portfolio based on general hedging strategies without incorporating the specific requirements of pension funds initially. These are taken into account only at a later stage. This disintegrated approach does not necessarily yield the globally optimal hedging portfolio for a specific pension fund which might be possible using a more quantitative approach.

In this thesis such a quantitative approach to fixed income portfolio management is introduced by means of an optimization problem. In doing this, a rigorous mathematical framework is employed to model relevant assets such as fixed-rate bonds and interest rate swaps. This is integrated with the dynamics of a pension fund and the expected future liabilities arising from this. The interest rate risk of pension funds is measured in this framework and related to industry practice.

The main objective of this thesis is finding a hedging strategy tailored to pension funds in the Netherlands. A micro-approach on specific assets is taken rather than the macro-approach on asset classes that has been researched extensively. The result should be a hedging portfolio that can be readily implemented by pension funds.

The expected future pension liabilities are in turn extensively modeled to allow proper testing of this optimization approach.

The main hypotheses of this thesis is thus that a hedging portfolio can be calculated in an automated fashion that performs well and is preferable to hedging portfolios that are currently in use.

This thesis starts by describing the pension world in Chapter 2: the functioning of pension funds (in the Netherlands) is discussed in detail and relevant assets are introduced. The interest rate is then introduced in Chapter 3 from a practical perspective: the expected discount function is estimated from market information (Section 3.1), the sensitivity to interest rate changes is calculated (Sections 3.3 and 3.5) and a preliminary hedge is constructed (Section 3.4).

The results are then used to define the dynamics of a pension fund in Chapter 4, which results in a definition of the stochastic pension liability process (Section 4.6). It is possible to simulate this process but the expected value can be calculated analytically as well. A hedging strategy tailored to pension funds is introduced in Section 2.4 — the strategy is described in detail (Sections 5.1 and 5.2) and then thoroughly tested (Sections 5.3 and 5.4).

There are a number of interesting research topics that could be investigated further — these next steps are outlined in Chapter 6. Concluding remarks are made in Chapter 7 and an attempt is made to answer one important question: how can these results be used to provide financial security for the (future) elderly?

2

The pension world

Pension funds exist all over the world. This thesis is concerned with the Dutch pension world, where 380 pension funds were active as of June 2019[1]. These Dutch pension funds had €1,442 billion in assets. This is a very large amount, especially when compared internationally. Looking at the members of the Organisation for Economic Co-operation and Development (OECD), which consists of developed countries, Dutch pension funds are very well-funded. In 2018, Dutch funds possessed 5.49% of the pension assets in these countries, while just 1.36% of the people living in OECD countries reside in the Netherlands[2]. Evidently, research into the Dutch pension world is very relevant.

It should be noted that regulations and practices in the pension worlds of other countries could be significantly different than in the Netherlands. However, it should be possible to adapt the findings of this thesis to other countries.

Recently, there have been numerous discussions and negotiations on the future of Dutch pensions. These discussions have generated a lot of media attention, because many Dutch citizens should expect 5–10% pension cuts under current regulations (Wolzak 2019a). In the last year alone, coverage ratios of four of the five largest pension funds in the Netherlands have dropped from 103–106% to 88–93%. Politicians are therefore discussing 'solutions' to this problem, such as setting a higher (artificial) interest rate for discounting, using pension contributions to pay out current pensions, and changing the minimum required coverage ratio. In this thesis, current regulations (as of September 2019) have been used, and potential changes have not been taken into account unless stated otherwise.

[1] Data on Dutch pension funds is published quarterly by the regulator at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-kwartaal/dataset/54946461-ebfb-42b1-9479-fa56b72d6b1a/.

[2] Data on pension wealth of its members has been published by the OECD at https://stats.oecd.org/Index.aspx?DataSetCode=EDU_DEM.

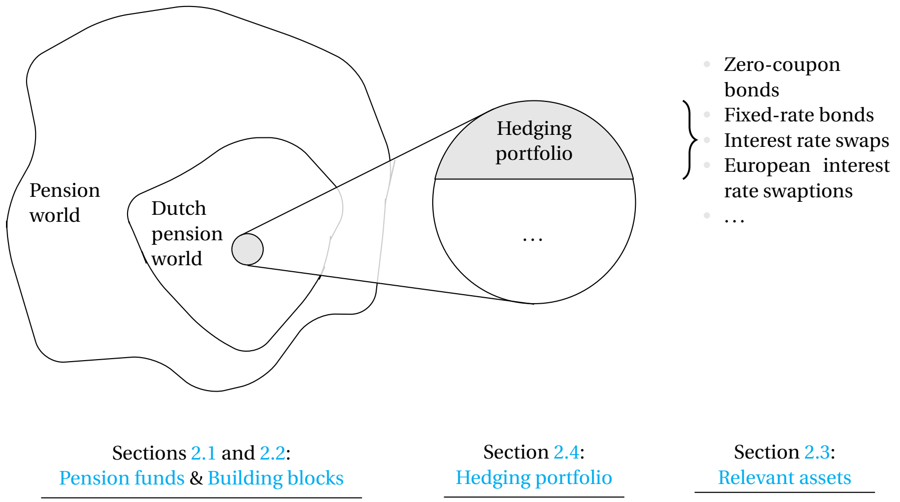


Figure 2.1: Structure of Chapter 2.

In this chapter, the inner workings of pension funds and their place in the pension world will first be described in Section 2.1. Then, building blocks for modeling pension funds will be stated in Section 2.2. This will be used to model assets that are relevant in the Dutch pension world in Section 2.3. Finally, a portfolio of these assets called the hedging portfolio will be described in Section 2.4. This hedging portfolio aims to decrease the risks that a pension fund is sensitive to, particularly the interest rate risks. The structure of this chapter is illustrated in Figure 2.1.

2.1. Pension funds

To model pension funds and their cash flows, customs and regulations are very relevant. For the Netherlands, this is described well by the government (Rijksoverheid n.d.). There are three pillars that together form the Dutch pension system. The first pillar is called the Algemene Ouderdomswet (AOW) — this is the base income and is provided by the government for all individuals once they reach a certain age. The *second pillar* is a supplementary pension via an employer. Employee and employer both pay a contribution to a pension fund, that invests the money on the employee's behalf and pays a pension when that individual reaches a certain age. Furthermore, there is a third pillar that consists of individual investments and savings — this is unrelated to pension funds and depends on an individual's choices.

This thesis concerns itself with the second pillar of the Dutch pension system, which is the realm of pension funds. Within the second pillar, many different types of pension schemes exist. Under all of these schemes, the pension fund receives (monthly) contributions while an individual is working, and pays out (monthly) benefits after an individual has reached the pension age. However, the rules and structures of these payments differ. The focus of this thesis is the most common scheme in the Netherlands: a Defined Benefits (DB) scheme. The intricacies of pension schemes and specifically the DB scheme are discussed in detail in Chapter 4. They are omitted here, because they are not relevant for this chapter.

It is now relevant to look at the balance sheet of a pension fund. All payments, contributions and benefits change the balance sheet of a pension fund. The evolution of the underlying financial

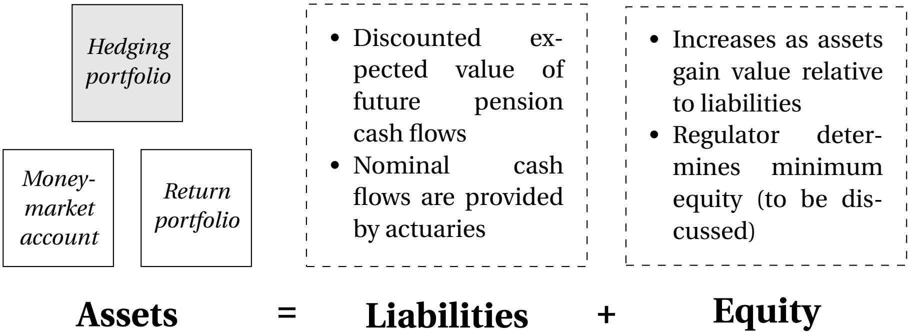


Figure 2.2: Simplified balance sheet of a pension fund. The hedging portfolio (gray box) is the focus of this thesis.

instruments — both liabilities and assets — also significantly impacts the balance sheet. This determines the financial position of a fund and thereby the capacity for continued benefit payments in the future. If the financial position of a fund is sufficiently solid, pension benefits might even be increased.

It is important to bear the main goal of a pension fund in mind: paying pensions to its participants. The purpose of the invested money is thus to enable payments of these future cash flows. In addition, a pension fund aims to index the pension to inflation as well — the pension should increase in nominal terms over time. To understand the investment decisions of a pension fund, it is relevant to study a (simplified) balance sheet of a pension fund.

Figure 2.2 gives an overview of a pension fund balance sheet. Assets are on the left side of the balance: these are the accumulated contributions of pension fund participants. The liabilities are the present value of future pension benefits. Finally, there is equity of the pension fund: excess money that is used to ensure the health of the pension fund and that may be used to increase pension benefits.

The coverage ratio, the main indicator of a pension fund's financial health, is given by the fraction assets/liabilities. These assets and liabilities are *market values* calculated in line with a number of regulations by De Nederlandsche Bank (DNB) . In practice, the 12-month average of this fraction is used in the Netherlands (called the *beleidsdekkingsgraad*) to assess the financial health. The financial health of a pension fund is assessed by looking at this number.

It is relevant to look at an example, to understand the implications of each part of the balance sheet.

Example 2.1 (ABP) *The Algemeen Burgerlijk Pensioenfonds (ABP) is the largest pension fund in the Netherlands with €399.0 billion in assets as of 2018 year-end (Algemeen Burgerlijk Pensioenfonds 2019, p. 10). It had €411.0 billion in liabilities on its balance sheet and therefore a negative equity of €-12.0 billion. This resulted in a coverage ratio of 97%. The 12-month average coverage ratio was 104% (Algemeen Burgerlijk Pensioenfonds 2019, p. 18), which was used to determine to pension fund's strategy. Based on these numbers, one might expect a positive financial outlook for the pension fund.* △

The regulator *De Nederlandsche Bank* determines the required equity (Vereist Eigen Vermogen (VEV)) and minimum required equity (Minimaal Vereist Eigen Vermogen (MVEV)) of a pension

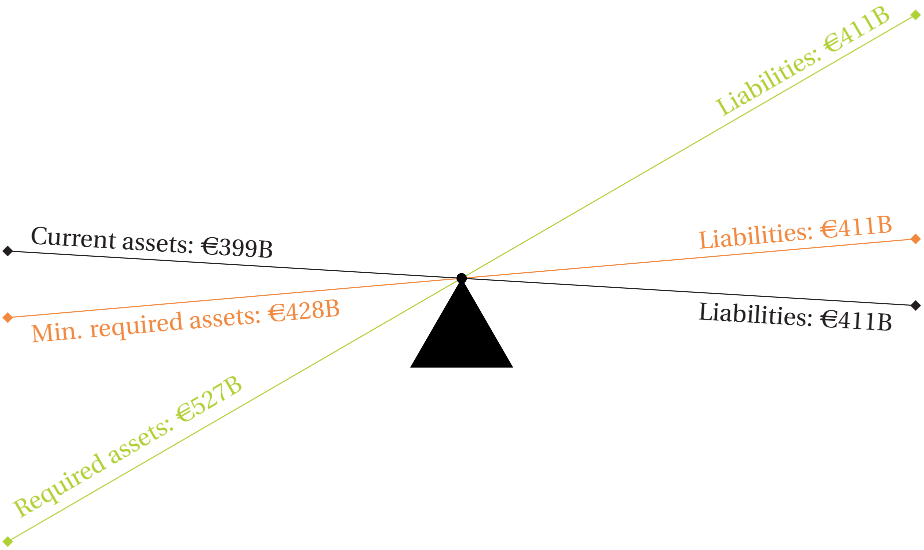


Figure 2.3: Balance of the ABP pension fund. In black the current balance, in orange the regulatory minimum required balance, and in green the required balance.

fund based on the risks associated with its assets and liabilities. The VEV yields a *required coverage ratio* that is usually 120 – 130%, and the MVEV similarly yields a *minimum required coverge ratio* that is usually 104 – 105%. If the 12-month average coverage ratio is lower than the minimum required coverage ratio for five years, the pension fund must mandatorily implement measures so that its coverage ratio is at least the required minimum within six months (De Nederlandsche Bank 2015a). Similarly, if this 12-month average coverage ratio is lower than the required coverage ratio at any point, the fund must implement measures so that its coverage ratio increases to the required level within 10 years (De Nederlandsche Bank 2019). These measures usually entail decreasing pension benefits, both the promised benefits of working and the current benefits of retired people.

These complex regulations are illustrated in an example.

Example 2.2 (ABP — continued) *For the ABP, the regulator decided on a required equity of €115.8 billion based on the risks related to the ABP's holdings, such as interest rate risk and foreign exchange risk (Algemeen Burgerlijk Pensioenfonds 2019, p. 111). This yields a required coverage ratio of 128%. In addition, the regulator set a minimum required coverage ratio of 104% (Algemeen Burgerlijk Pensioenfonds 2019, p. 10). At 97%, the current coverage ratio is much lower than this.*

The ABP submitted a recovery plan when the 12-month average coverage ratio became lower than 128%. Despite this, the pension fund's 12-month average year-end coverage ratio became lower than the required minimum in 2016. If this does not improve until the end of 2020, measures must be implemented to immediately increase the coverage ratio (Algemeen Burgerlijk Pensioenfonds 2019, p. 180). Figure 2.3 shows the (im)balance of this pension fund. △

There are risks related to both the pension fund liabilities and assets. These risks are important because they determine the regulatory rules, but even more so simply because they could have an impact on the financial status of the pension fund. This could negatively impact the ability of the pension fund to pay pensions to its participants. Because of this importance, the asset side of a pension fund's balance sheet is usually split in a number of portfolios with different targets. The fund invests a certain amount in a return portfolio: a (potentially) risky portfolio which has as sole target attaining a high return. Next, there is a hedging portfolio: a portfolio that attempts to hedge certain risks associated with the pension fund in the best possible manner. In practice, these are risks associated with the liabilities. Finally, a small part of the assets is a money-market account that is used to make the day-to-day payments. Figure 2.2 shows these portfolios as well on the left-hand side. The hedging portfolio is the focus of this thesis.

Kroon et al. (2017) have researched the risks related to a pension fund's coverage ratio, and find that interest rate risk is the most important type of risk. Interest rate risks arise because future pension benefits are defined in nominal terms. If the interest rate decreases, more money is required now to pay the future liabilities and the value of the liabilities on the balance sheet increases. This results in a lower coverage ratio, unless this risk is hedged by assets whose value increases as well. Although the regulator has introduced an Ultimate Forward Rate (UFR), which ensures that long-term interest rate remain artificially high — at least in comparison with current market levels —, interest rate changes still have a significant effect on a pension fund's hedging ratio.

The fund that Kroon et al. (2017) consider attempts to hedge 70% of its interest rate risk. They find the expected value and volatility of a pension fund's coverage ratio through a simulation procedure. This volatility is then decomposed into risk factors (Kroon et al. 2017, p. 11), which gives the result that interest rate risk and equity risk are the most important risk factors, (potentially) along with high-yield credit risk. Note that this is the risk remaining after the fund has attempted to hedge 70% of its interest rate risk. These researchers find that interest rate risk still accounts for 20–40% of the risks. If the interest rate risk would not have been hedged at all, the contribution to the total risks would have been much larger.

In addition, the Dutch regulator also stresses the importance of pensions funds' interest rate risk in a research paper (De Nederlandsche Bank 2015b).

Historically, coverage ratios and interest rates have been connected as well. Figure 2.4 shows the coverage ratios and interest rates over a 12-month period. Because pension fonds usually hedge (part of) their interest rate risks, one would expect an evident albeit not one on one relation. Up to 2012, it is clearly seen that interest rate and coverage ratio are strongly related. However, in September 2012 the UFR methodology was introduced for pension funds (dashed gray). Although there is evidently still a relation between both series, its presence has become less obvious. This is mainly because the UFR is partly cosmetic in the long end of the discount curve and ensures that long-term interest rates are significantly higher than the market rates.

Because interest rate risk is a pension fund's most important risk, it is usually the focus of the hedging portfolio. The ABP, for instance, has a "vastrentendewaarden" portfolio which strives to hedge 25–50% of the pension fund's interest rate risk (Algemeen Burgerlijk Pensioenfonds 2019, p. 29).

With a well-constructed hedging portfolio, the risks of the pension fund are minimized to ensure pension security in the future.

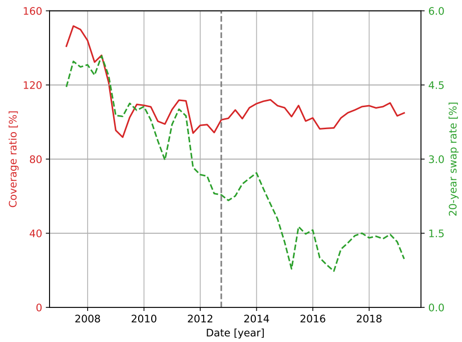


Figure 2.4: The coverage ratio of all pension funds in the Netherlands (solid red, left) and 20-year swap rate (dashed green, right). The datasets have been added to Appendices C.2 and C.6.

Finding the best hedging portfolio is the focus of this thesis. The focus is furthermore on hedging using sovereign risk — i.e. government bonds. The reason for this is that pension funds in practice hedge a large part of their interest rate risk via sovereign bonds. This thesis thus contributes to hedging risks — specifically interest rate risk — in the best possible way. In addition, certain strategic and risk beliefs are included. Often, this is (partly) captured in a country allocation, which contains a view on risks associated with countries and the target exposure to these countries. Besides these beliefs, another target is the minimization of costs such as transaction costs and investment costs, effectively maximizing profits.

2.2. Building blocks

To model pension funds and their asset allocation, a number of concepts are required. The financial world may largely be captured by mathematical concepts such as probability theory, statistics and time series — many of these concepts are also required for pension funds. This section describes building blocks such as the money-market account, value of assets, and pension fund cash flows. These concepts are in later chapters employed to build a full model.

All stochastic quantities in this thesis are defined on a *probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, with Ω the sample space, $\mathscr{F}$ the events and $\mathbb{P}$ the real-world probability measure. Often, the *natural filtration* $\mathscr{F}_t$ associated to a process is used, which contains behavior of this process up to time t and may be seen as a “model of the flow of public information” related to this process (Shreve 2004, pp. 51–53).

It is assumed that the financial market is *complete*, i.e. there are no transaction costs, there is perfect information for all investors and a price exists for all assets in all possible states of the universe (Brigo and Mercurio 2007, p. 26). In addition, it is assumed that the financial market is *arbitrage-free*, i.e. no profit can be made without taking risk (Brigo and Mercurio 2007, pp. 23–26). In an arbitrage-free market it is impossible to invest zero today and in exchange agree on a future cash flow X, with:

$$X \geq 0 \qquad \text{and} \qquad \mathbb{P}(X > 0) > 0.$$

In such a scenario, you would never make a loss while making a profit with a probability greater than zero. Using these two assumptions, the first and second fundamental theorems of asset pricing imply that there exists a unique risk-neutral measure $\mathbb{Q}$ that is equivalent to the real-world measure $\mathbb{P}$. In fact, this risk-neutral measure has as numéraire the money-market account, which will be defined shortly. Both these probability measures will be used throughout this work.

In reality, these assumptions usually do not hold at all times: minor price deviations may sometimes be observed in the market. However, overall these are realistic assumptions, which is why it is market practice to make these assumptions. The reader is referred to Delbaen and Schachermayer (2006) for a detailed discussion on the topic.

The first building block that will be described is the money-market account or bank account. This models an investment without risks, whose value changes with a dynamic risk-free rate $r(t)$ in the market. This is a risk-less investment and may be defined (Brigo and Mercurio 2007, pp. 2–3):

Definition 2.1 (Money-market account) *Let $M(t)$ be the value of a money-market account at time $t \geq 0$, defined by the differential equation:*

$$\mathrm{d}M(t) = r(t)M(t)\,\mathrm{d}t.$$

Assume that $M(0) = 1$. The money-market account is now fully defined and:

$$M(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right).$$

△

The quantity $r(t)$ is the instantaneous rate at which the money-market account's value changes. It is usually called the *instantaneous spot rate* or *short-rate*, and may be both a deterministic or a stochastic process. If it is a stochastic process, the money-market account is also a stochastic process. In Chapter 3, the short-rate process is described in detail.

If the interest rate process $r(t)$ is deterministic, the time value of money may be quantified using this function $M(t)$. Given two times t and T (with $T \geq t$), the value of the money-market account is known. Since this is an investment without risks, the discount factor between times t and T is then $M(t)/M(T)$. In the general case, the short-rate is a stochastic process, and the (future) discount factor at a time t_0 is therefore the expected value $\mathbb{E}^{\mathbb{Q}}\left[M(t)/M(T) \,\middle|\, \mathscr{F}_{t_0}\right]$. This general concept of a stochastic short-rate is used in this thesis.

One of the most important elements of a pension fund is of course the pension liabilities. These pension liabilities are (expected) future cash flows, consisting of the contributions that the pension fund will receive and the pension benefits it is obliged to pay to its retired members. These payments occur at fixed times, usually once every month. This leads to a definition of the pension cash flow:

Definition 2.2 (Pension cash flow) *Let $T_1, \ldots T_n$ be the payment times of a pension fund. The random variables $L(T_1), \ldots, L(t_n) : \Omega \to \mathbb{R}$ are the payments of the pension fund. Note that these payments can be both positive (a payment to the fund) or negative (a payment by the fund). These payments form the pension cash flow.* △

Finding a probability distribution for these random variables is not an easy task. This is undertaken in Chapter 4. In addition, analytical results for the expected values are also derived in that chapter.

With the future pension cash flow defined it is relevant to consider its value. Observe that the pension cash flow is a sum of payments so that each payment $L(T_i)$ may be valued separately. Therefore, a methodology to calculate the value of some contingent claim in the future with payoff $H(T)$ is required. This is done using the risk-neutral measure:

Corollary 2.1 (Value) *Let $H(T)$ be the pay-off of a contingent claim at time T. Then the value of this claim at time t_0 is:*

$$V(t_0) = M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}H(T)\,\middle|\,\mathcal{F}_{t_0}\right].$$

△

Proof. Because it has been assumed that the financial market is arbitrage-free and complete the Second Fundamental Theorem of Asset Pricing states that a unique risk-neutral probability measure exists with the money-market account as numéraire. The result then follows. □

Using this valuation formula, the value of the pension cash flow may be calculated. This concept is also used to calculate the value of assets in the next section.

There have been numerous discussions — mostly amongst politicians — on whether the present value of future pension liabilities should indeed be calculated in this way (Wolzak 2019b). An artificially higher rate such as the UFR could be used instead so that the present value of a pension fund 's liabilities is lower. However, in this thesis future pension liabilities are discounted as any other contingent claim.

2.3. Relevant assets

In this section assets that are relevant in the pension world will be defined. Assets are not physical assets but rather a contract for future payments. It has been explained that the interest rate plays an important role for pension funds, and that a fund aims to hedge against interest rate changes. Therefore, the assets are *interest rate derivatives*. All assets are defined consistent with literature such as Brigo and Mercurio (2007, pp. 4–22) and Oosterlee and Grzelak (2019, pp. 325–328, 359–375).

The payments of most assets depend on a time difference during which payments can accrue. Often referred to as a *year fraction*, *day-count convention* or *accrual factor*, a measure for time difference is required before assets are described.

Definition 2.3 (Accrual) *Suppose two times t and T are given, representing two times in years since since some starting time t_0. Let $\tau(t,T)$ be the accrual factor between times t and T:*

$$\tau(t,T) = T - t.$$

△

Many day-count conventions exist that define $\tau(t,T)$ differently. Henrard (2012, pp. 5–7) gives an overview of these conventions such as Actual/365, Actual/360 and 30/360. However, because this thesis is not primarily concerned with day-count conventions, a simple measure has been chosen. Definition 2.3 corresponds to the *Actual/365 L* convention.

The pay-offs and values of assets are given in this section. Under some interest rate models analytical formulas exist for certain interest rate derivatives. These will be stated when those models are discussed.

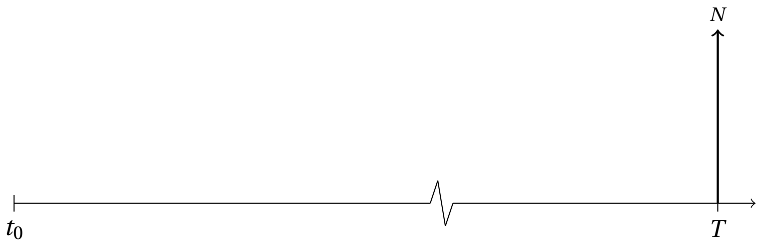


Figure 2.5: Payments of a zero-coupon bond with notional N and maturity T.

2.3.1. Zero-coupon bond

A zero-coupon bond is an interest rate derivative with a single payment at maturity and no intermediate payments:

Definition 2.4 (Zero-coupon bond) *A zero-coupon bond with notional N and maturity T is an asset with pay-off $H(t)$:*

$$H(t) = \begin{cases} N, & t = T, \\ 0, & \textit{otherwise}. \end{cases}$$

△

The payments of a zero-coupon bond are sketched in Figure 2.5.

From the pay-off, the value of the zero-coupon bond at time t_0 is calculated using Corollary 2.1:

$$V(t_0) = M(t_0) N \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{M(T)} \middle| \mathscr{F}_{t_0} \right].$$

The notional is frequently omitted — in such cases, it may be assumed that $N = 1$. In that case the value of a zero-coupon bond at time t_0 with notional 1 and maturity T is often represented by $P(t_0, T)$.

$$P(t_0, T) = M(t_0) \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{M(T)} \middle| \mathscr{F}_{t_0} \right] = \text{discount factor}. \tag{2.1}$$

2.3.2. Fixed-rate bond

A fixed-rate bond is an interest rate derivative with multiple interest payments, and one payment of the notional at maturity:

Definition 2.5 (Fixed-rate bond) *A fixed-rate bond with notional N, fixed rate r and payment times $T_1, \ldots, T_m$ is an asset with pay-off $H(t) = 0$ if $t \notin \{T_1, \ldots, T_m\}$ and otherwise:*

$$H(T_i) = \begin{cases} r\tau_i N, & i \in \{1, \ldots, m-1\}, \\ r\tau_m N + N, & i = m, \end{cases}$$

with $\tau_i = \tau\left(T_{i-1}, T_i\right), \forall i \in \{1, \ldots, m\}$. △

The payments of a fixed-rate bond are sketched in Figure 2.6.

Note that this fixed-rate bond is a sum of m individual payments. Therefore, the value $V_i(t_0)$ of each individual payment at T_i may be calculated from the pay-off $H(T_i)$ using the value of a zero-

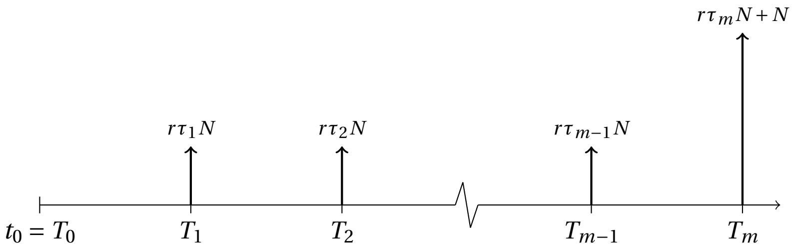


Figure 2.6: Payments of a fixed-rate bond with notional N, rate r and payment times $T_1, \ldots, T_m$.

coupon bond with notional 1:

$$V_i(t_0) = \begin{cases} P(t_0, T_i) r\tau_i N, & i \in \{1, 2, \ldots, m-1\}, \\ P(t_0, T_m)(r\tau_m N + N) & i = m, \end{cases}$$

Combining these payments gives the value of (all payments of) a fixed-rate bond:

$$V(t_0) = \sum_{i=1}^{m} V_i(t_0) = \left(\sum_{i=1}^{m} P(t_0, T_i) r\tau_i N \right) + P(t_0, T_m) N.$$

Observe that a fixed-rate bond with $r = 0$ is equivalent to a zero-coupon bond, since there is then just one non-zero cash flow: a payment of N at maturity.

2.3.3. Interest rate swap

A plain vanilla interest rate swap is an interest rate derivative where one party makes periodical fixed payments to another party that in return makes periodical floating payments. Interest rate swaps are often used to secure interest rate dependent cash flows.

The floating payments depend on an interbank rate. One of the most important interbank rates is the London Interbank Offered Rate (LIBOR), which is fixed every day in London[3]. However, other similar interbank rates such as the Euro Interbank Offered Rate (EURIBOR). The LIBOR is defined, but in its place any interbank rate may be used.

Definition 2.6 (LIBOR) *The LIBOR at time t from T_{i-1} to T_i is:*

$$\ell(t, T_{i-1}, T_i) = \frac{P(t, T_{i-1}) - P(t, T_i)}{P(t, T_i)}.$$

Observe that the LIBOR is defined in terms of the price of a zero-coupon bond. △

There are two parties in an interest rate swap, that each pay one leg and receive the other leg. To distinguish between these cases, the words payer and receiver are used. An *interest rate swap payer* receives the floating leg and pays the fixed leg while an *interest rate swap receiver* pays the floating legs and receives the fixed leg. It is now possible to define an interest rate swap:

[3] The LIBOR is calculated by asking a number of banks to estimate what they would be charged if they would borrow money from other banks. The LIBOR is then calculated from these estimates by averaging these values while disregarding the lowest and highest values. In this thesis a clear mathematical definition is used instead that does not take the exact procedure of banks into account.

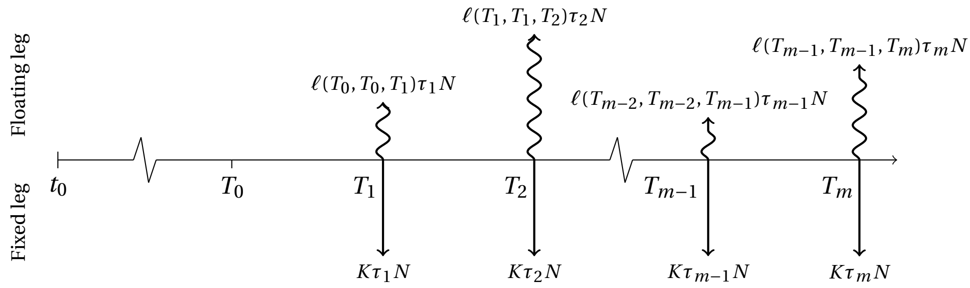


Figure 2.7: Payments of an interest rate swap with notional N, fixed rate K, payment times $T_1,\dots,T_m$, and first reset time T_0.

Definition 2.7 (Interest rate swap) *An interest rate swap with notional N, fixed rate K, payment times $T_1,\dots,T_m$*[4] *and first reset time T_0 is an asset with pay-off $H(t)=0$ if $t\notin\{T_1,\dots,T_m\}$ and otherwise:*

$$H(T_i)=\alpha\left(\ell(T_{i-1},T_{i-1},T_i)-K\right)\tau_iN,$$

with:

$$\alpha=\begin{cases}1, & \textit{for a payer swap,}\\ -1, & \textit{for a receiver swap,}\end{cases}\qquad \textit{and}\qquad \tau_i=\tau\left(T_{i-1},T_i\right),\forall i\in\{1,\dots,m\}.$$

△

Observe that the LIBOR is first calculated at T_0, and then also at the time points $T_1,\dots,T_{m-1}$. This explains the name 'first reset time' for T_0. The payments of the two legs in an interest rate swap are sketched in Figure 2.7.

The value of a payer swap at time t_0 is:

$$V(t_0)=M(t_0)N\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{m}\frac{1}{M(T_i)}\left(\ell(T_{i-1},T_{i-1},T_i)-K\right)\tau_i\,\middle|\,\mathscr{F}_{t_0}\right],\tag{2.2}$$

which Oosterlee and Grzelak (2019, pp. 361–362) rewrite using a measure change and the definition of the LIBOR to:

$$V(t_0)=N\left(P(t_0,T_0)-P(t_0,T_m)\right)-NK\sum_{i=1}^{m}\tau_iP(t_0,T_i).\tag{2.3}$$

From Definition 2.7 it is immediately clear that the value of a receiver swap is calculated by multiplying the value of the corresponding payer swap by -1.

In practice, a swap's value is always equal to 0 when it is issued. This is ensured by changing K: a value for K is calculated so that the swap's value is equal to zero.

In practice a number of slightly different types of interest rate swaps exist. One of these special types of interest rate swaps is a *zero-coupon interest rate swap*. This is a swap with only one payment time at maturity, i.e. $T_1=T_m$. It will be used later to calibrate a stochastic model.

[4] In the market, payment times for fixed and floating legs usually do not coincide. All results may be adapted to this, but equal payment times are assumed for the purpose of simplicity.

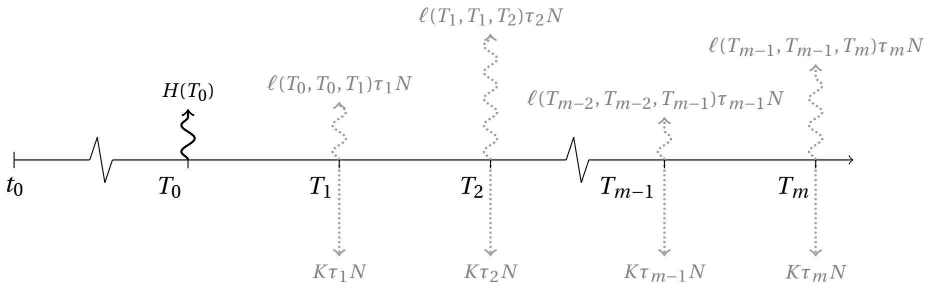


Figure 2.8: Pay-off of a European interest rate swaption with maturity T_0 (solid black). In addition, the payments of the underlying payer swap with notional N, fixed rate K, payment times $T_1,\ldots,T_m$ and first reset time T_0 have been added as well (dotted gray).

2.3.4. European interest rate swaption

A European interest rate swaption is a contract on an underlying derivative: an interest rate swap. A swaption is an option on a swap. This means that the holder has the right (but not the obligation) to enter into a swap with pre-established characteristics at some time in the future:

Definition 2.8 *Let S be a payer swap with notional N, fixed rate K, payment times $T_1,\ldots,T_m$ and first reset time T_0. Let $V^S(t)$ be the value of this swap S (Equation (2.2)). A European interest rate swaption with maturity T_0 is an asset that gives the right to enter into S at time T_0, and thus has pay-off $H(T_0)$:*

$$H(T_0) = \max\left(V^S(T_0), 0\right).$$

△

The pay-off of a European interest rate swaption and the payments of the underlying swap have been added to Figure 2.8.

Observe that the pay-off has been defined in terms of the swap's value at maturity, because the option will only be exercised if this value is positive at maturity. This optionality explains the maximum in this formula. Using the value of a swap from Equation (2.3), its pay-off may be rewritten so that:

$$H(T_0) = \max\left(N\left(1 - P(T_0, T_m)\right) - NK\sum_{i=1}^{m} \tau_i P(T_0, T_i), 0\right),$$

with $\tau_i = \tau\left(T_{i-1}, T_i\right), \forall i \in \{1,\ldots,m\}$.

The value of a European interest rate swaption at time t_0 can now be derived from this rewritten pay-off $H(T_0)$:

$$V(t_0) = M(t_0) N \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_0)} \max\left(1 - P(T_0, T_m) - K\sum_{i=1}^{m} \tau_i P(T_0, T_i), 0\right) \,\middle|\, \mathscr{F}_{t_0}\right].$$

Figure 2.9: Process that a pension fund uses to determine its asset allocation. The hedging portfolio is one of the resulting portfolios.

2.4. Hedging portfolio

The inner workings of a pension fund have been explained, and interest rate risk has been shown to be the most important risk of a pension fund (Section 2.1). It was stated that these risks are hedged in a *hedging portfolio*. In this section the position of the hedging portfolio in a pension fund's process is explained, and the hedging portfolio is described using the assets from Section 2.3.

A number of studies and decisions are made before a pension fund hedges its risks. This process is described before stating the hedging portfolio mathematically.

First, the risk appetite and ambitions of a pension fund are researched in an ALM study. The purpose of an ALM study is to make a high-level plan of matching assets you have (or buy) with your future liabilities. This requires calculation of the future pension cash flows, an assessment of the need and possibility of taking investment risks under the pension fund's targets, and the implications of this on the pension fund's required equity. This then results in long-term views on the pension fund's future liabilities, (financial) risks, and allocation of money to investment categories.

Secondly, the results of the ALM study are used to formulate a *strategic policy*. This implements the result of the ALM study[5] by setting broad strategic targets and policies. This could specifically include: allowed asset classes, hedging ratios, return and risk profiles, liquidity guidelines, counter-party rules, and Environmental, Social and Governance (ESG) guidelines.

After these two steps the pension fund's money has been divided into different portfolios, that each have their own targets and rules. One of the subportfolios is the hedging portfolio. The general idea of the hedging portfolio is that all pension benefits are payable from this portfolio.

This is an iterative process, and the results are frequently updated. A summary of this process has been added to Figure 2.9. After this extensive process, finding a hedging portfolio is a well-defined problem.

Such a hedging portfolio is a combination of assets that minimizes risks. Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a collection of assets with pay-offs $H_1, \dots, H_n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ (Section 2.3). The hedging portfolio is then a combination of these assets that minimizes the risks. The portfolio may contain an asset repeatedly, or not at all.

This hedging portfolio may be defined mathematically, using the concept of a portfolio:

[5] In practice, a tactical asset allocation with a 1–3 years time horizon is made as well. This is a more specific short-term version of the strategic policy, but does not materially change the process.

Definition 2.9 (Portfolio) *Let* $\mathscr{A} = \{A_1, \ldots, A_n\}$ *be a collection of assets with pay-offs* $H_1, \ldots, H_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$. *Let* $x_i \in \mathbb{R}$[6] *be the position of portfolio* P *in asset* A_i. *Then* P *is a portfolio with pay-off:*

$$H(t) = \sum_{i=1}^{n} H_i(t).$$

△

Since a portfolio is simply a linear combination of assets, the value of a portfolio is also a linear combination of the values of these assets.

In Chapter 5, a procedure will be described to actually find this hedging portfolio.

[6]It will later be described in Section 5.1 that in fact $x_i \in \mathbb{R}_{\geq 0}$, since shorting is not allowed.

3

Interest rate and associated risks

In Definition 2.1, the money-market account was described. It was defined to represent an investment without risks, and was then used to value assets and pension liabilities. The money-market account $M(T)$ in effect represents the amount of money in a bank account at time T, when there is $M(t)$ in this same bank account at time t (with $t < T$ and no intermediate payments being made). However, it was not explained how to find a formula for this money-market function, so that it may actually be used. The price of a risk-less zero-coupon bond was found to depend solely on this money-market function in Subsection 2.3.1. However, neither the money-market account nor the zero-coupon bond with price $P(t_0, t)$ is frequently traded in the market with sufficient liquidity (Hagan and West 2008, p. 1). Deducing these quantities from the financial market is therefore not elementary but at the same time very important.

In this chapter, a 'bootstrapping procedure' is described to derive the prices of risk-less zero-coupon bonds from instruments that are traded in the market (Section 3.1). After this bootstrapping, a stochastic model for the interest rate process is described in Section 3.2. Then, the sensitivity of instruments to interest rate changes — i.e. changes in the money-market account — is described in Sections 3.3 and 3.5. This is very relevant for a pension fund, because it describes their risks. These measures of risks are used in Section 3.4 to hedge a pension fund's interest rate risks.

It is important to understand which instruments should be used to deduce the interest rate before progressing any further. Moreover, the target results of such a procedure should be clarified.

Recall from Definition 2.1 that the money-market account is a risk-less investment that is defined in terms of the short rate. So, one would expect that a risk-free rate is used to define the money-market account. However, what is the risk-free rate? Since this 'rate' is not frequently traded in the market, like zero-coupon bonds, it must somehow be derived. This can be done using interest rate derivatives. From instruments that are traded in the market, the price of a zero-coupon bond $P(t_0, t)$ with any maturity should be derived. This then defines the money-market account as well. This curve of zero-coupon bond prices is often called the *yield curve*, *zero curve* or curve of *discount factors*. This curve is estimated in Section 3.1.

However, a question remains: which specific instruments — i.e. which interest rate derivatives — should be used as inputs for this bootstrapping procedure? Swaps are frequently used because of their liquidity — this is also done in Section 3.1. But other instruments may be used as well.

Smolenaers et al. (2009) have researched the interest rate in the context of pensions. They researched the best instruments for bootstrapping in the case of pension funds. The pension world is indeed a special case, because of the importance of long-term rates. They state four principles for the resulting yield curve that can be used to assess whether the correct instruments were selected (Smolenaers et al. 2009, pp. 12–13):

Risk-free The yield curve should be in line with cash flows that are entirely risk-free, i.e.: the return of a risk-free investment should be equal to the return implied by the yield curve.

Observable It should be possible to derive the yield curve instantly and frequently from the market, for all relevant future time points without room for interpretation.

Tradeable It should be possible to hedge interest rate risk in the market using the instruments that were used to construct the yield curve.

Robust The yield curve should not be very sensitive to market disruptions.

Based on these principles, Smolenaers et al. (2009, p. 7) describe that at the end of the last century government bonds were thought to be the best instruments for bootstrapping by pension funds. However, at the beginning of this century the number of (long-term) government bonds that were offered decreased. In addition, the liquidity of interest rate swaps had increased from the end of the last century and a wide range of swaps with various (long-term) maturities became available. Furthermore, the counter-party risk of interest rate swaps was (partly) mitigated through collateralisation. This resulted in changing preferences. The researchers thus reached the conclusion that interest rate swaps should be used for bootstrapping in the context of pension funds.

Finally, it should be remarked that there are different types of swaps. In Subsection 2.3.3, an interest rate swap was described in terms of the LIBOR. However, it was mentioned that this rate could be substituted by any other rate such as the EURIBOR. Before the credit crunch crisis in 2007, the EURIBOR was considered the risk-free rate in the eurozone and deduced in a simple bootstrapping procedure. However, since then different markets have led to a changing consensus, so that the Euro OverNight Index Average (EONIA) rate is now considered the risk-free rate. Interest rate swaps are still dependent on the EURIBOR though. Bianchetti (2008) explains these considerations in more detail, and describes a solution in the form of 'dual-curve bootstrapping'. In this current thesis, a simpler single curve procedure is used. To further complicate matters, a transition to a new rate called the Euro Short-Term Rate (ESTER) is under way. This is described well by Schrimpf and Sushko (2019) but not considered in this thesis.

Throughout this chapter multiple related representations of the interest rate are used. Besides the money-market account $M(t)$ and discount function $P(t_0, t)$, the *instantaneous interest rate* $r(t)$ and discrete (annual) interest rate or *yield* $\bar{r}(t)$ are used. These are defined as in Brigo and Mercurio (2007, pp. 6–8).

Definition 3.1 (Instantaneous rate) *Let $P(t_0, t)$ be the price of a zero-coupon bond at current time t_0 paying 1 at maturity t. Then the instantaneous rate is:*

$$r(t) = -\frac{\log P(t_0, t)}{t - t_0}.$$

△

Definition 3.2 (Yield) *Let $r(t)$ be the instantaneous rate. Then the discrete (annual) interest rate or yield is:*

$$\bar{r}(t) = \exp[r(t)] - 1.$$

△

Note that these definitions define relationships between all representations of the interest rate.

A sufficient amount of background information is now known, so that relevant results for interest rates can be calculated in the next sections.

3.1. Bootstrapping — deducing the zero curve

In this section, the price of a zero coupon bond $P(t_0, t)$ is estimated from instruments traded in the market. Recall from Section 2.2 that these prices are equal to the discount factors. To be specific, the function is constructed from interest rate swaps.

In essence, a bootstrapping procedure is a mapping from instruments traded in the market to nodes on the discount curve. These nodes define the discount curve and are called the *spine points*. Let:

$$\Pi = \{\pi_1, \ldots, \pi_n\} \qquad \text{: The instruments traded in the market,}$$
$$\Omega = \{(t_1, p_1), \ldots, (t_n, p_n)\} \qquad \text{: The spine points, with } p_i = P(t_0, t_i).$$

The bootstrapping procedure is then a mapping:

$$\Pi \mapsto \Omega.$$

It is assumed without loss of generality that the t_i are ordered, so that $t_1 \leq \ldots \leq t_n$. Using an interpolation scheme these spine points in fact determine the entire discount curve — this will be specified later.

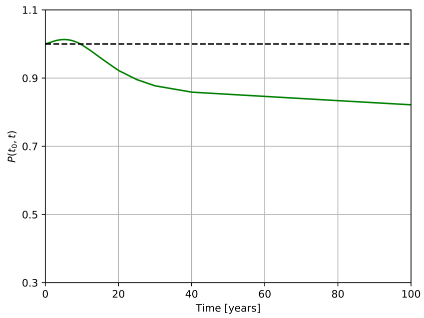
Remark 3.1 *Many scholars use the word 'bootstrapping' to describe a procedure that derives the spine points through forward substitutions: in an iterative process the first spine point is calculated, then the second spine point, etc. In this thesis the word* bootstrapping *is used as in Hagan and West (2008): to denote any mapping $\Pi \mapsto \Omega$ that determines the spine points from instruments that are traded in the market. Although the latter procedure is equivalent to the former under certain interpolation schemes, this is not necessarily the case.* △

Figure 3.1 gives an example of a bootstrapped discount curve. In this example the set of instruments Π consists of 17 interest rate swaps that are specified in Table 3.1.

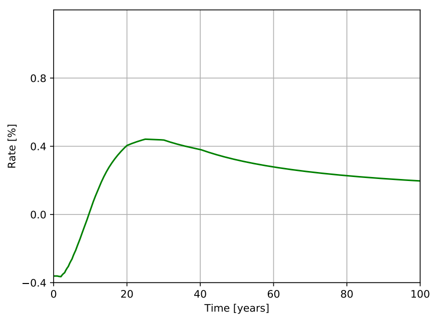
It is very important that this discount curve is accurate, because stochastic models for the interest rate use it as an input and it is the basis for the hedging of interest rate risks. Hagan and West (2008, p. 5) describe criteria to judge the discount curve. A bootstrapping procedure performs well if it yields satisfactory results on the following topics:

1. The yield curve should price back all instruments that were used to construct the curve;
2. Forward rates should be positive and continuous;
3. Interpolation should be local: a small change of the curve should not affect nodes that are far removed from this change;
4. Forward rates should be stable;
5. Hedges using the curve should be local: a product should not be hedged using many products, and certainly not many 'far away' products.

These five criteria should be used to assess the bootstrapping methodology.



(a) Discount curve (solid green) and the line $P(t_0, t) = 1$ (dashed black).



(b) Yearly interest rate curve.

Figure 3.1: Result of bootstrapping using 31 October 2019 swap rates.

Maturity	Fixed rate	Maturity	Fixed rate
1Y	-0.36%	10Y	0.02%
2Y	-0.36%	12Y	0.14%
3Y	-0.34%	15Y	0.27%
4Y	-0.30%	20Y	0.39%
5Y	-0.26%	25Y	0.43%
6Y	-0.21%	30Y	0.43%
7Y	-0.16%	40Y	0.38%
8Y	-0.10%	50Y	0.32%
9Y	-0.04%		

Table 3.1: Fixed rates of interest rate swaps with a one-year frequency, so that the values of the swaps are zero. Quotes from Bloomberg as of 31 October 2019.

3.1.1. Bootstrapping algorithm

An algorithm to find the results from Figure 3.1 will now be described. This is done using quotes of interest rate swaps, but the results may be extended to other interest rate derivatives. The Newton-Raphson method is employed to construct the discount curve, as done in Oosterlee and Grzelak (2019, pp. 367–371).

Suppose $\pi_1, \ldots, \pi_n$ are interest rate swaps. In addition — for a swap π_i — let N_i be the notional, K_i the fixed rate, $T_{i,0} = t_0$ the first reset time and $T_{i,1}, \ldots, T_{i,m_i}$ the payment times.

Let $V_i(t_0)$ be the theoretical value of swap π_i at time t_0 and $\hat{V}_i(t_0)$ its market value at that same time. Equation (2.3) specified that the value of a payer swap[1] at time t_0 is given by:

$$V_i(t_0) = N_i \left(P(t_0, T_{i,0}) - P(t_0, T_{i,m_i}) \right) - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} P(t_0, T_{i,j}),$$

with $\tau_{i,j} = T_{i,j} - T_{i,j-1}$. The bootstrapping procedure aims to find the function $P(t_0, t)$, such that all $V_i(t_0)$ are equal to their market value $\hat{V}_i(t_0)$.

To solve this problem, the value of a swap is rewritten. Observe that $P(t_0, T_{i,0}) = P(t_0, t_0) = 1$. In addition, recall that the discount curve $P(t_0, t)$ is defined by the spine points. So, $P(t_0, t)$ may be substituted by the function $P(\mathbf{p}; t)$ of the spine points $\mathbf{p} = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix}$.

This function $P(\mathbf{p}; t)$ interpolates between the spine points. Therefore, all t must be smaller than t_n. To ensure this — and to considerably simplify calculations — the spine points are generally set to the maturity of the instruments. So, set $t_i = T_{i,m_i}$. After substitution, the value of the interest rate swap is:

$$V_i(t_0) = V_i(\mathbf{p}; t_0) = N_i - N_i P(\mathbf{p}; T_{i,m_i}) - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} P(\mathbf{p}; T_{i,j}).$$

It is relevant to consider an example, in which the discount curve is constructed from interest rate swaps.

Example 3.1 *Suppose one would like to construct a discount curve up to year two based on interest rate swaps that are traded in the market. Let π_1, π_2 be such swaps with maturities $1, 2$ (years), notional 1[2] and 1-year frequencies. Set $t_0 = 0$ and set the spine points to the maturities of the swaps:*

$$t_1 = 1 \qquad \textit{and} \qquad t_2 = 2.$$

In Table 3.1, the rates of these swaps were given:

$$K_1 = -0.36\% \qquad \textit{and} \qquad K_2 = -0.36\%.$$

Recall from Subsection 2.3.3 that $\hat{V}_i(t_0) = 0$. The theoretical value of swap π_1 is:

$$\begin{aligned} V_1(t_0) &= 1 - P(t_0, t_1) - (-0.0036)\,[P(t_0, t_1)] \\ &= 1 - (1 - 0.0036) P(t_0, t_1). \end{aligned}$$

Since a $P(t_0, t_1)$ must be found such that $V_1(t_0) = \hat{V}_1(t_0) = 0$, it follows that:

$$P(t_0, t_1) = 1.0036,$$

[1] Payer swaps are assumed, but the result may be found in the same way for receiver swaps. In fact, results will be the same when the swap's market value is equal to zero, which is generally the case when they are issued (Subsection 2.3.3).

[2] Note that a different notional than 1 would not change the results.

which is also the result in Figure 3.1. Continuing using this result and that $\hat{V}_2(t_0) = 0$:

$$\begin{aligned} V_2(t_0) &= 1 - P(t_0,t_2) - (-0.0036)\left[P(t_0,t_1) + P(t_0,t_2)\right] \\ &= 1 - (1-0.0036)P(t_0,t_2) + 0.0036P(t_0,t_1) \\ \implies \quad P(t_0,t_2) &= 1.0073. \end{aligned}$$

A similar procedure may be used to find additional spine points. △

Observe that interpolation is not necessary in this example because the payment times of all swaps coincide with the spine points. However, this is usually not the case. It is therefore relevant to generalize this procedure to all cases using the Newton-Raphson algorithm.

As has been stated, the aim of the bootstrapping procedure is to find $P(t_0,t)$ such that all swap values $V_i(t_0)$ equal their market values $\hat{V}_i(t_0)$. Define:

$$y_i(\mathbf{p}) = V_i(\mathbf{p};t_0) - \hat{V}_i(t_0). \tag{3.1}$$

Observe that $V_i(\mathbf{p};t_0) = \hat{V}_i(t_0) \iff y_i(\mathbf{p}) = 0$. If there is no arbitrage in the market, $\mathbf{y}(\mathbf{p}) = \begin{bmatrix} y_1(\mathbf{p}) & \dots & y_n(\mathbf{p}) \end{bmatrix} = \mathbf{0}$. So, spine points $\mathbf{p}$ must be found such that $\mathbf{y}(\mathbf{p}) = \mathbf{0}$. It is possible to find this $\mathbf{p}$ using multi-dimensional Newton-Raphson iterations with the scheme:

$$\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - J^{-1}\left(\mathbf{p}^{(k)}\right)\mathbf{y}\left(\mathbf{p}^{(k)}\right), \tag{3.2}$$

where $J^{-1}\left(\mathbf{p}^{(k)}\right)$ denotes the inverse Jacobian of $\mathbf{y}\left(\mathbf{p}^{(k)}\right)$. Of course, some initial solution should be chosen. A reasonable value could be $\mathbf{y}^{(0)} = \mathbf{1}$, since the price of a zero-coupon bond with notional one is — at least for short maturities — close to 1.

However, the problem is not fully solved yet. In Equation (3.2) the $n \times n$ Jacobian matrix J is required, which is not always trivial to find. Its elements are $J_{ij} = \partial y_i / \partial p_j$ so these values should be derived:

$$\begin{aligned} \frac{\partial y_i}{\partial p_j} &= \frac{\partial}{\partial p_j}\left[V_i(p_1,\dots,p_n;t_0) - \hat{V}_i(t_0)\right] \\ &= \frac{\partial}{\partial p_j}\left[N_i - N_i P\left(p_1,\dots,p_n;T_{i,m_i}\right) - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(p_1,\dots,p_n;T_{i,j}\right)\right] \\ &= -N_i \frac{\partial}{\partial p_j} P\left(p_1,\dots,p_n;T_{i,m_i}\right) - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} \frac{\partial}{\partial p_j} P\left(p_1,\dots,p_n;T_{i,j}\right) \\ &= -N_i \mathbb{1}_{i=j} - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} \frac{\partial}{\partial p_j} \underbrace{P\left(p_1,\dots,p_n;T_{i,j}\right)}_{\text{approximate numerically}} . \end{aligned} \tag{3.3}$$

Note that this Jacobian matrix J is a lower triangular matrix by the ordering of the swaps. The Jacobian could of course have been approximated numerically as well. However, these analytical expressions are more accurate while they still do not require any assumptions (and connected loss of flexibility). Therefore, this is considered the better approach.

Observe that the result of the value of the interest rate swap has not been used in these derivations. So it would also be possible to use the same procedure to bootstrap a curve using other interest rate derivatives.

From Equation (3.3) it is evident that the Jacobian depends on the employed interpolation scheme through the function $P(\mathbf{p}; T_{i,j})$. Hagan and West (2008, p. 3) stress the importance of a close connection between the interpolation method and the bootstrapping procedure. They explain this strong relation by noting that bootstrapping is performed with insufficient data. This data is completed by the interpolation method in a well-defined scheme. However, such an interpolation scheme has not been defined yet. Hagan and West (2008) discuss numerous interpolation methods, including one they call "raw interpolation". The idea is to interpolate linearly on the (natural) logarithm of the discount factors: $\log P(t_0, t) = -r(t)t$.

To derive the interpolation, suppose $P(t_0, t)$ should be found. Let t_l and t_r be the spine points directly left and right of t — clearly $t_l \leq t \leq t_r$ by the ordering of the spine points. Then $r(t)t$ is interpolated using the scheme:

$$r(t)t = \frac{t - t_l}{t_r - t_l} r(t_r) t_r + \frac{t_r - t}{t_r - t_l} r(t_l) t_l. \tag{3.4}$$

This is a linear interpolation scheme. By using the relation $P(t_0, t) = \exp(-r(t)t)$, the required value has been found.

Hagan and West (2008) show that by using this interpolation method, all instruments used to construct the curve are priced back to their market values and forward rates are positive and continuous (criteria 1 and 2). They call it a "very attractive method". In fact, this interpolation scheme is also recommended by the Dutch regulator DNB[3]. However, Hagan and West (2008) also propose a method called "monotone convex", which they state should be "the method of choice for interpolation". In their paper, they show this method actually compares favorable on all specified criteria (criteria 1–5). However, this method has not been researched further because interpolation schemes are not the main topic of this thesis.

Sometimes, *extrapolation* of the curve beyond the last spine point is required. A methodology similar to the interpolation scheme is used, where the function $\log P(t_0, t)$ is extrapolated linearly from the last two spine points. So, suppose $P(t_0, t)$ must be calculated, with $t > t_n$. Then:

$$r(t)t = r(t_n)t_n + \frac{r(t_n) - r(t_{n-1})}{t_n - t_{n-1}}(t - t_n).$$

This is also the methodology DNB proposes if the UFR curve is not used.

3.2. A stochastic model for the interest rate process

In this section, a stochastic model for the short rate is described. The short rate defines the money-market account (Definition 2.1). One of the main reasons for modeling the short rate is to find results for the money-market account and thereby calculate the price of a zero-coupon bond. One of the main inputs is the discount function from Section 3.1 so that the model is calibrated to the market expectations. The model should allow for the simulation of paths and it should be possible to calculate several relevant expected values analytically.

A specific requirement is that it should be possible to analytically calculate the price of a zero-coupon bond at the current t_0 issued at t and maturing at T (with obviously $t \leq T$), i.e.:

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)} \,\middle|\, \mathscr{F}_{t_0}\right]. \tag{3.5}$$

[3] In 2017, DNB published a statement on interest rate curve construction (De Nederlandsche Bank 2017), referencing a methodology using the UFR from 2015 (De Nederlandsche Bank 2015c). This latter document states that an earlier methodology for constructing the yield curve (De Nederlandsche Bank 2005) remains in effect, which uses constant forward rates.

In this section, the Hull-White model is described: a 1-factor model that fits in the Heath-Jarrow-Morton (HJM) framework.

Many classes of models exist to model the interest rate. 1-factor and 2-factor short-rate models are often used, models have been developed in the HJM framework, and market models are a common practice. An overview of well-known models is given by Brigo and Mercurio (2007, pp. 51–442), with the 1-factor models being the simplest class. More complex models are able to capture additional behavior. 2-factor models for instance generally have a higher precision, and allow for correlations between times (Brigo and Mercurio 2007, pp. 137–138). Furthermore, market models such as the log-normal forward-LIBOR model and the log-normal forward-swap model are defined with the purpose of agreeing with well-established pricing formulas for some derivative products traded in the market. Despite these advantages, it has been decided to use a (relatively simple) 1-factor model for the sake of simplicity. The impact of this simplification should be small because the model is only used to price non-exoctic interest rate derivatives that additionally do not depend on correlation of the interest rates across time (Brigo and Mercurio 2007, pp. 138–139, 195–196).

In the class of 1-factor models, a suitable model should be chosen. Of the 1-factor short-rate models, only two are able to model negative rates (Brigo and Mercurio 2007, p. 55). This is very relevant in the current financial climate, which can also be seen in the curve implied from the swap rate (Figure 3.1). The two remaining 1-factor models under consideration are then the Vasicek model and the Hull-White model. In fact, it is possible for both of these models to analytically price bonds and options on bonds. However, the Hull-White model allows for perfect calibration to the market forward rate so that there is no arbitrage (Brigo and Mercurio 2007, p. 72). For this reason, the Hull-White model is selected as the interest rate model.

The Hull-White model will now be specified. This will be done using Oosterlee and Grzelak (2019, pp. 333–350) and Brigo and Mercurio (2007, pp. 71–80), which is based on work by (among others) Hull and White (1990). The model is defined by a stochastic differential equation:

$$\mathrm{d}r(t) = \lambda\left(\theta(t) - r(t)\right)\mathrm{d}t + \eta\,\mathrm{d}W_r^{\mathbb{Q}}(t), \tag{3.6}$$

with mean-reversion parameter λ, volatility parameter η and $W_r^{\mathbb{Q}}(t)$ a Brownian motion.

The parameters λ, η and function $\theta(t)$ should be defined. The former two parameters will be calibrated to swaptions traded in the market in Subsection 3.2.2. The function $\theta(t)$ defines the Hull-White model in the HJM framework, and is calibrated to the instantaneous forward rate $\hat{f}(t,T)$ that is observed in the market:

$$\theta(t) = \frac{1}{\lambda}\frac{\partial}{\partial t}\hat{f}(t_0,t) + \hat{f}(t_0,t) + \frac{\eta^2}{2\lambda^2}\left[1 - \exp(-2\lambda t)\right]. \tag{3.7}$$

The market instantaneous forward rate $\hat{f}(t_0,t)$ may be calculated from the market prices of zero-coupon bonds $\hat{P}(t_0,t)$:

$$\hat{f}(t_0,t) = -\frac{\partial}{\partial t}\log\hat{P}(t_0,t). \tag{3.8}$$

This function $\hat{P}(t_0,t)$ is found using the bootstrapping procedure in Section 3.1, where it was denoted by $P(t_0,t)$.

Thus, a model for the short rate $r(t)$ has been described. However, the goal was to find (expected values of) the money-market account. Recall that a relation between these quantities was given in Definition 2.1:

$$M(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right). \tag{3.9}$$

Using this result, the quantity in Equation (3.5) can be defined in terms of the short-rate, i.e. the price of a zero-coupon bond with notional 1 (Equation (2.1)) can be stated:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}\,\bigg|\,\mathscr{F}_{t_0}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\,\bigg|\,\mathscr{F}_{t_0}\right)\right]. \tag{3.10}$$

So, although the Hull-White model is a short-rate model the expected value in Equation (3.5) lies within the realm of this model. In Subsection 3.2.1, an analytical expression for the expected value is found.

It is now also possible to implement simulation of $r(t)$. Suppose the discount function should be simulated for the time period $[t_0, t_n]$. First, the time interval should be discretized into:

$$t_0, t_1, \ldots, t_n,$$

with $t_i - t_{i-1} = \Delta t, \forall i \in \{1,\ldots,n\}$. Using an Euler discretization scheme, $r(t_{i+1})$ can then be calculated recursively:

$$r(t_{i+1}) = r(t_i) + \lambda\left(\theta(t_i) - r(t_i)\right)\Delta t + \eta\cdot\Delta W_r^{\mathbb{Q}}(t_i). \tag{3.11}$$

Observe that $\Delta W_r^{\mathbb{Q}}(t_i) \sim N(0,\Delta t)$ since it is a Brownian motion. Also, the only unknowns in the definition of $\theta(t)$ (Equation (3.7)) are $f(0,t)$ and its derivative — these quantities can be approximated easily using central/forward finite differences[4] and Equation (3.8). Finally, $r(t_0)$ is also required. This follows from the relation $r(t) = f(t,t)$.

Thus, it has been explained how the short-rate process is simulated. In addition, the relation of the short-rate process with both the money-market account (Equation (3.9)) and the price of a zero-coupon bond (Equation (3.10)) has been stated. Not only is it possible to simulate the short-rate, the simulated paths of the money-market account and the zero-coupon bond price can also be deduced.

In Subsection 3.2.1, an analytical expression for $P(t,T)$ (as in Equation (3.5)) is found. Finally, the mean-reversion and volatility parameters are calibrated to instruments traded in the market in Subsection 3.2.2.

3.2.1. An analytical expression

The expected value given in Equation (3.5) is calculated so that it can be used in subsequent chapters. This value can be estimated from simulation results by averaging the paths. However, an analytical result can be derived as well under the Hull-White model.

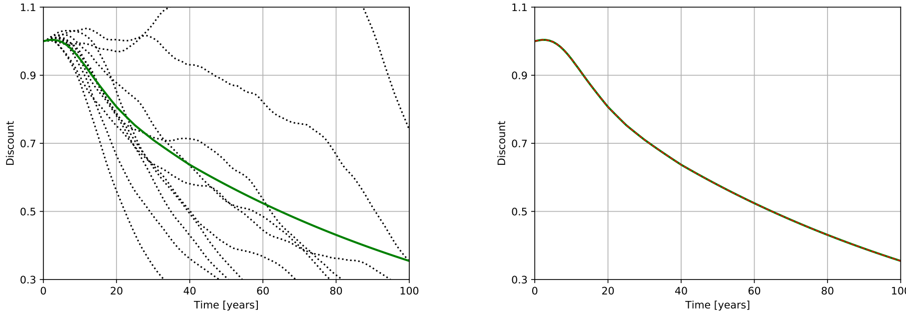
In Equation (3.10) it was already given that:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}\,\bigg|\,\mathscr{F}_{t_0}\right] = \left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)\,\bigg|\,\mathscr{F}_{t_0}\right]. \tag{3.12}$$

Brigo and Mercurio (2007, p. 75) find an analytical expression for this expected value under the Hull-White model:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}\,\bigg|\,\mathscr{F}_{t_0}\right] = A(t,T)\exp\left(-B(t,T)r(t)\right), \tag{3.13}$$

[4]Central differences should be used if possible. The error when using finite differences is of order h^2, while it is of order h when using forward differences (with bump/spacing h). However, this is not possible for $t < h$, so in those cases forward differences should be used.



(a) The analytical result $P(t_0, T)$ (solid green, Equation (3.13)) and the realized discount $\int_{t_0}^{T} r(s)\,\mathrm{d}s$ for 10 Monte-Carlo paths (dotted black, Equation (3.11)).

(b) The analytical result $P(t_0, T)$ (solid green, Equation (3.13)) and the market discount function $\hat{P}(t_0, T)$ (red dashed, Section 3.1).

Figure 3.2: Plots of the analytical results, simulations and market values. Curve is calibrated to 31 October 2019 market data which resulted in $\eta = 0.30\%, \lambda = 0.00\%$.

with:

$$B(t,T) = \frac{1}{\lambda}\left[1 - \exp(-\lambda(T-t))\right],$$

$$A(t,T) = \frac{\hat{P}(t_0,T)}{\hat{P}(t_0,t)} \cdot \exp\left(B(t,T)\hat{f}(t_0,t) - \frac{\eta^2}{4\lambda}\left(1 - \exp(-2\lambda(t-t_0))\right)B^2(t,T)\right).$$

A brief explanation of this result has been added to Appendix A.4.

In Figure 3.2 the analytical discount function $P(t_0, T) = \mathbb{E}^{\mathbb{Q}}\left[M(t_0)/M(T)\right]$ is compared to simulation results and the market discount function $\hat{P}(t_0, T)$.

3.2.2. Calibration of mean-reversion and volatility

The mean-reversion parameter λ and the volatility parameter η have been used in the description of the Hull-White model, but values have not been defined explicitely. These values can be chosen based on literature, however, a better procedure would be to find these values from instruments traded in the market. This calibration is performed in this section.

First, the instruments that are used in such a calibration process must be chosen. Because the bootstrapped discount curve from Section 3.1 does not contain any information on mean-reversion and volatility, different instruments should be used. Russo and Torri (2019, p. 277) explain that the Hull-White model is usually calibrated to caps, floors or swaptions. The reason for this is that these interest rate derivatives are very liquid and contain the required volatility information. Moreover, swaptions are preferred over the other derivatives, because they implicitly quantify the correlation between different points of the discount curve. It will now be detailed how the Hull-White model is calibrated to swaptions.

Gurrieri et al. (2009) discuss several calibration strategies. On a high level, they distinguish between strategies that model the parameters in a constant way, and strategies that model these parameters in a time-dependent way (for instance through piecewise constant functions). They conclude there is no ground for preferring either method (Gurrieri et al. 2009, p. 33), so the simpler method of constant parameters has been selected.

Furthermore, they stress the difference between strategies that fix the mean-reversion parameter and calibrate only the volatility parameter, strategies that calibrate first the mean-reversion and then (in a second step) the volatility, and strategies that calibrate both parameters at the same time. Because the mean-reversion has a significant influence on the values of derivatives (Gurrieri et al. 2009, pp. 11–12), both parameters should be calibrated. The two remaining strategies both perform well, so for simplicity the latter strategy (optimizing both parameters simultaneously) has been selected.

It will now be explained how to perform the actual calibration procedure. In Section 3.1, the curve was calibrated to instruments so that the instrument values *using the curve* exactly equal the market values in a 'bootstrapping' methodology. If parameters of the Hull-White model are time-dependent, a similar strategy could be used for the Hull-White model. However, both parameters are constant. In addition, Gurrieri et al. (2009, p. 12) have found that such a procedure yields parameters that are rather sensitive to changes in the calibration instruments and often exhibit big spikes between times. Therefore, a different methodology is used.

Clearly, the goal is to find parameters η, λ such that the analytical value is closest to the market value. So, given swaptions $\pi_1, \ldots, \pi_n$ with analytical present values $V_i(t_0)$ and market values $\hat{V}_i(t_0)$, the relative differences should be minimized:

$$\sum_{i=1}^{n} \left(\frac{V_i(t_0)}{\hat{V}_i(t_0)} - 1 \right)^2. \tag{3.14}$$

To minimize this sum of squares, an analytical formula for $V_i(t_0)$ should be found. Based on Jamshidian (1989), scholars such as Gurrieri et al. (2009, pp. 4–5) and Brigo and Mercurio (2007, pp. 75–78) are able to find this quantity under the Hull-White model. It is assumed that the notional equals 1, which does not make a difference for calibration purposes.

To reiterate, the 'missing link' is finding the value $V^{\pi}(t_0)$ of some European interest rate swaption π. Suppose π is a payer swaption with maturity T_0 on a payer swap with strike K and payment times $T_1, \ldots, T_m$. Then the value of this payer swaption may be written in terms of the values of m put options on zero-coupon bonds:

$$V^{\pi}(t_0) = \sum_{j=1}^{m} c_j \cdot V^{\psi_j}(t_0), \tag{3.15}$$

where V^{ψ_j} is the value of a put option ψ_j with maturity T_0 on a zero-coupon bond that pays its coupon X_j at time T_j, and:

$$\begin{aligned} c_j &= \begin{cases} K\tau_j, & j \in \{1, \ldots, m-1\} \\ K\tau_j + 1, & j = m \end{cases} \\ \tau_j &= T_j - T_{j-1} \\ X_j &= A_1(T_0, T_j) \exp\left(-B(T_0, T_j) r^*\right). \end{aligned}$$

In this last equation, $A_1(T_0, T_j)$ and $B(T_0, T_j)$ are defined as before in Subsection 3.2.1, and the quantity r^* satisfies the equation:

$$\sum_{j=1}^{m} c_j A_1(T_0, T_j) \exp\left(-B(T_0, T_j) r^*\right) = 1.$$

This equation may be solved using the Newton-Raphson algorithm.

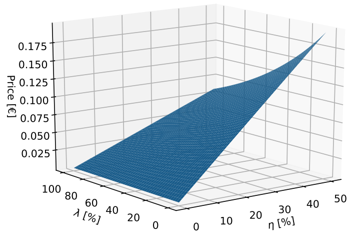


Figure 3.3: Value of an at the money payer swaption for different Hull-White parameters: $\lambda \in (0\%, 100\%], \eta \in (0\%, 50\%)$. The swaption has notional 1, option maturity in one year and swap maturity in two years.

However, one essential part of Equation (3.15) should still be specified: the value of a put option on a zero-coupon bond $V^{\Psi^j}(t_0)$. Under the Hull-White model, this may be priced using an analytical result as well (Brigo and Mercurio 2007, p. 76):

$$V^{\Psi^j}(t_0) = X_j P(t_0, T_0)\Phi(-h) - P(t_0, T_j)\Phi(-h + \eta'),$$

with:

$$\eta' = \eta\sqrt{\frac{1 - \exp(-2\lambda(T_0 - t_0))}{2\lambda}} B(T_0, T_j)$$
$$h = \frac{1}{\eta'} \log \frac{P(t_0, T_j)}{P(t_0, T_0)X_j} + \frac{\eta'}{2}.$$

It is now known how to price swaptions analytically under the Hull-White model. Figure 3.3 shows analytical swaption values using different Hull-White parameters. When calibrating the model using swaption values from the market, multiple surfaces of this kind are combined to find the optimal model parameters.

Using Equation (3.14), it is possible to calibrate the mean-reversion and volatility parameters of the Hull-White model to the market. This has been done by finding η, λ so that nine European interest rate swaptions (as in Subsection 2.3.4) are equal to their market values. Calibration of the Hull-White model to these swaptions as of 31 October 2019 (added in Appendix C.5) yields:

$$\eta = 0.30\% \qquad \text{and} \qquad \lambda = 0.00\%.$$

These results are in line with the calibration performed by Russo and Torri (2019, p. 287). With the mean-reversion almost equal to zero, the Hull-White model effectively reduces to a random walk. This is evident from Equation (3.6). Even though the volatility is very small as well, the interest rate can deviate significantly from market consensus represented by the discount curve found in Section 3.1. Research by scholars such as Van den End (2011) have indeed found only very limited historical evidence for the existence of a mean-reversion effect in interest rates.

Using these results the model has been fully calibrated and can be applied to interest rate modeling.

3.3. Sensitivity to interest rate changes

In Section 2.3, a number of interest rate derivatives were described. From those definitions it is evident that the value of those assets depends on the (expected) interest rate. So if the interest rate changes, the value of those assets changes as well. This change in value as a result of a change in the interest rate is investigated in this section — this is the *sensitivity* to interest rate changes.

It is relevant to investigate the relation between an interest rate derivative and the interest rate. The interest rate directly drives the money-market account $M(t)$ or the discount function that can be derived from it. However, in Section 2.3 it was observed that these are not financial instruments that are traded in the market. So, if the impact of changes in the money-market account on the value of an asset would be quantified, it would be difficult to relate this to changes in market conditions.

In Section 2.3, a discount function (and thereby money-market account function) was derived from interest rate swaps — instruments that *are* traded in the market. Since the money-market account is effectively a function of those swaps via the bootstrapping procedure, it is only natural to investigate the impact that changes in these swaps (that were used in bootstrapping) have on the value of an interest rate derivative. This the topic of this section.

Before mathematically formalizing this concept, it is relevant to look at an example.

Example 3.2 *Define three interest rate swaps π_1, π_2, π_3 with yearly payment times, values $V_1(t_0), V_2(t_0), V_3(t_0)$ at current time t_0, and:*

$$\begin{aligned} &\textit{Maturities}: \quad T_1 = 1, \qquad T_2 = 2, \qquad T_3 = 3, \\ &\textit{Fixed rates}: \quad K_1 = 1.0\%, \quad K_2 = 1.5\%, \quad K_3 = 2.5\%^{5}. \end{aligned}$$

The discount function $P(t_0, t)$ may be derived from these instruments by bootstrapping (as in Section 3.1).

Let X be a zero-coupon bond with value $V(t_0)$ paying $N = 10$ at maturity $T = 3$. Assume there is no risk associated with the pay-off. The bootstrapping procedure yields $P(t_0, T) \approx 0.9278$, so that the present value of this bond is:

$$V(t_0) = NP(t_0, T) \approx 10 \cdot 0.9278 \approx 9.2779.$$

Suppose the market sentiment changes, and the two-year swap is updated — now $K_2 = 1.6\%$. This clearly changes the value of the bond[6]*:*

$$K_2 = 1.6\% \implies V(t_0) \approx 9.2784,$$

a change in the value of 0.0005. *Similarly, the three-year swap rate could be updated*[7]*:*

$$K_3 = 2.6\% \implies V(t_0) \approx 9.2497,$$

a change in the value of -0.0282. *It is clear that the value of the bond changes when the underlying interest rate derivatives change.* △

[5] The fixed rates of interest rate swaps are not based on observed values, but merely for illustrative purposes.
[6] With $M(T) \approx 1.0778$ (unchanged due to rounding).
[7] With $M(T) \approx 1.0811$.

In this example, the impact of a 0.1% change in the fixed rate of the underlying swap was calculated. It should be noted that these are just two scenarios. This concept can be generalized mathematically by considering the partial derivatives of the asset's value to the value of the underlying.

To see this, let X be an interest rate derivative with value $V(t_0)$ at current time t_0. Let $\pi_1,\dots,\pi_n$ be interest rate derivatives with values $V_1(t_0),\dots,V_n(t_0)$ that are used in the bootstrapping procedure. Assume these π_i are interest rate swaps with fixed rate K_i. In Section 3.1 swaps were used for bootstrapping — therefore it is assumed that the instruments $\pi_1,\dots,\pi_n$ are also interest rate swaps. However, results may be extended to other interest rate derivatives.

The sensitivity to interest rate changes is now captured by n partial derivatives:

$$\frac{\partial V(t_0)}{\partial K_1},\dots,\frac{\partial V(t_0)}{\partial K_n}.$$

Often, these partial derivatives are called *delta's* and represented by:

$$\Delta = \begin{bmatrix}\Delta_1 & \dots & \Delta_n\end{bmatrix} = \begin{bmatrix}\frac{\partial V(t_0)}{\partial K_1} & \dots & \frac{\partial V(t_0)}{\partial K_n}\end{bmatrix}. \tag{3.16}$$

This Δ describes the sensitivity of the asset to interest rate changes. In fact, the change in $V(t_0)$ that was seen in Example 3.2 may be approximated through this Δ. To see this, suppose that the rate K_i of swap π_i was changed by δ yielding a new value $\tilde{V}(t_0)$ of X. Then a first-order Taylor expansion gives:

$$\begin{aligned}\tilde{V}(t_0) &= V(t_0) - \frac{\partial V(t_0)}{\partial K_i}\delta + \mathcal{O}\left(\delta^2\right)\\ \implies \tilde{V}(t_0) - V(t_0) &= -\frac{\partial V(t_0)}{\partial K_i}\delta + \mathcal{O}\left(\delta^2\right).\end{aligned} \tag{3.17}$$

It is by itself interesting to know how the value of an asset changes when the situation in the financial market changes. Moreover, this is also an integral concept in hedging the risk of interest rate changes, which could help one avoid significant losses. If two assets have opposite Δ a value decrease of the one assets would be accompanied by a value increase of the other asset. Combined the result would then be (approximately) no change in value, which is the goal when hedging. This concept is discussed in more detail in Section 3.4.

The derivatives $\partial V(t_0)/\partial V_i(t_0)$ can in fact be calculated analytically for some specific types of interest rate derivatives. This will be done in the following subsections. The concepts introduced in Section 3.1 play a pivotal role in this and notation from that section will be used.

It will be observed that the results depend not only on the instruments under consideration, but also on the interpolation scheme that is used when bootstrapping. This again shows the integral role of bootstrapping in valuing interest rate derivatives.

3.3.1. Zero-coupon and fixed-rate bond

Since bonds are important interest rate derivatives, it is very relevant to calculate the Δ of bonds. This will be done in this subsection which naturally requires some notation. To this end let ψ be a fixed-rate bond (as in Section 2.3) with notional N, rate r, payment times $S_1, \ldots, S_m$ and value $U(t_0)$ at current time t_0.

Remark 3.2 (Z-spread) *Thus far all zero-coupon and fixed-rate bonds have been priced without taking the issuer into account: the risk of payments has been assumed to be equal to the risk of the instruments used for bootstrapping, i.e. essentially risk-free. Pension funds often use government and government-related bonds that are sometimes perceived as more risky by the market. A constant spread called the* z-spread *is therefore introduced so that the market price equals the theoretical vaue.*

Let $\hat{U}(t_0)$ be the market value of ψ at t_0 and $U_i(t_0)$ the theoretical value at t_0 of its individual payments at times $S_1, \ldots, S_m$ (as in Subsection 2.3.2). Define:

$$U(t_0) = \sum_{i=1}^{m} \mathrm{e}^{-zS_i} U_i(t_0),$$

with the z-spread $z \in \mathbb{R} : U(t_0) = \hat{U}(t_0)$. Observe that this result can be adapted easily to zero-coupon bonds. This z-spread allows modeling of real-life bonds. △

A number of theoretical results are required before the Δ can be derived. First, a theorem due to Spivak (1965) is repeated:

Theorem 3.1 (Inverse Function Theorem) *Suppose that $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing $\mathbf{a}$, and $\det \mathbf{f}'(\mathbf{a}) \neq 0$. Then there is an open set V containing $\mathbf{a}$ and an open set W containing $\mathbf{f}(\mathbf{a})$ such that $\mathbf{f}\colon V \to W$ has a continuous inverse $\mathbf{f}^{-1}\colon W \to V$ which is differentiable and for all $\mathbf{y} \in W$ satisfies:*

$$\left(\mathbf{f}^{-1}\right)'(\mathbf{y}) = \left[\mathbf{f}'\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right]^{-1}.$$

○

Proof. For a proof the reader is referred to Spivak (1965, Theorem 2–11). □

Based on this theorem it can be found that the inverse Jacobian that was used for bootstrapping (Equation (3.2)) actually contains relevant information.

Corollary 3.1 *The inverse Jacobian $J^{-1}(\mathbf{p})$ of $\mathbf{y}(\mathbf{p})$ is:*

$$J^{-1}(\mathbf{p}) = \left(\frac{\partial(y_1, \ldots, y_n)}{\partial(p_1, \ldots, p_n)}\right)^{-1} = \begin{bmatrix} \frac{\partial p_1}{\partial V_1} & \frac{\partial p_1}{\partial V_2} & \cdots & \frac{\partial p_1}{\partial V_n} \\ \frac{\partial p_2}{\partial V_1} & \frac{\partial p_2}{\partial V_2} & \cdots & \frac{\partial p_2}{\partial V_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_n}{\partial V_1} & \frac{\partial p_n}{\partial V_2} & \cdots & \frac{\partial p_n}{\partial V_n}. \end{bmatrix}$$

△

Proof. Let $\mathbf{y}(\mathbf{p})$ be as in Equation (3.1). First it must be asserted that $\mathbf{y}(\mathbf{p})$ is continuously differentiable $\forall \mathbf{p} > \mathbf{0}$. Note that it suffices to show that $P(\mathbf{p}; t)$ is continuously differentiable with respect to the p_i $\forall t \geq t_0$ since:

$$\frac{\partial y_i}{\partial p_l} = \frac{\partial V_i}{\partial p_l} = N_i \frac{\partial P(\mathbf{p}; T_{i,m_i})}{\partial p_l} - N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} \frac{\partial P(\mathbf{p}, T_{i,j})}{\partial p_l}.$$

This depends on the interpolation schemed that is defined. Observe that these functions are indeed continuously differentiable for the interpolation scheme proposed in Equation (3.4). In addition, recall that with this interpolation scheme the Jacobian $J(\mathbf{p}) = \mathbf{y}'(\mathbf{p})$ is a lower-triangular matrix by Equation (3.3). Hence $\mathbf{y}'(\mathbf{p})$ is invertible and $\det \mathbf{y}'(\mathbf{p}) \neq 0$. Thus all conditions of the Inverse Function Theorem have been satisfied and $\mathbf{y}\colon V \to W$ has a continuous inverse $\mathbf{y}^{-1}\colon W \to V$ so that $\forall \mathbf{w} \in W$:

$$\frac{\partial (p_1, \ldots, p_n)}{\partial (y_1, \ldots, y_n)}(\mathbf{w}) = \left(\mathbf{y}^{-1}\right)'(\mathbf{w}) = \left(\mathbf{y}'\left(\mathbf{y}^{-1}(\mathbf{w})\right)\right)^{-1} = \left(\frac{\partial (y_1, \ldots, y_n)}{\partial (p_1, \ldots, p_n)}\right)^{-1} \left(\mathbf{y}^{-1}(\mathbf{w})\right).$$

The result follows by noting that $\hat{V}(t_0)$ is a constant and thus $\forall i, j \in \{1, \ldots, n\} : \partial p_i / \partial y_j = \partial p_i / \partial V_j$. □

In Corollary 3.1, it was shown that the inverse Jacobian of $\mathbf{y}(\mathbf{p})$ is made up of partial derivatives of the p_i with respect to the V_i (the value of the swap π_i). However, the Δ was defined in terms of derivatives with respect to K_i, the fixed rate of a swap. This relation is another building block that should be found.

Lemma 3.1 *Consider swap π_i. Define $\tau_{i,j} = \tau(T_{i,j-1}, T_{i,j})$ as in Definition 2.3. Then:*

$$\frac{\partial p_l}{\partial K_i} = \frac{\partial p_l}{\partial V_i} \left(-N_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right)\right).$$

△

Proof. The result follows from substitution of the value of a swap (Equation (2.3)):

$$\begin{aligned}
\frac{\partial p_l}{\partial K_i} &= \frac{\partial p_l}{\partial V_i} \frac{\partial V_i}{\partial K_i} \\
&= \frac{\partial p_l}{\partial V_i} \frac{\partial}{\partial K_i} \left(-N_i K_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right)\right) \\
&= \frac{\partial p_l}{\partial V_i} \left(-N_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right)\right).
\end{aligned}$$

□

Observe that it follows from Lemma 3.1 that:

$$\frac{\partial U}{\partial K_i} = \frac{\partial U}{\partial V_i} \left(-N_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right)\right).$$

This result will be used to relate $\partial U / \partial K_i$ and $\partial U / \partial V_i$.

Another result is required before the Δ can be found. Note that the value of a fixed-rate bond (Subsection 2.3.2) is actually a linear function of certain discount factors. Assets of which the value can be written in such a form have an interesting property. This is an application of a well-known mathematical property known as the *linearity of differentation* or the *rule of linearity*.

Lemma 3.2 *Let U be the value of an interest rate derivative. Suppose that U is a linear function of zero-coupon bond prices $P(t_0, s_1), \dots, P(t_0, s_m)$ for some $s_1, \dots, s_m$. Then:*

$$\sum_{j=1}^{m} P(t_0, s_j) \frac{\partial U(t_0)}{\partial P(t_0, s_j)} = U(t_0).$$

△

Proof. If $U(t_0)$ is a linear function of bond prices:

$$U(t_0) = \sum_{j=1}^{m} x_j P(t_0, s_j),$$

for some $x_j \in \mathbb{R}$. Then:

$$\frac{\partial U(t_0)}{\partial P(t_0, s_j)} = x_j,$$

and thus:

$$\sum_{j=1}^{m} P(t_0, s_j) \frac{\partial U(t_0)}{\partial P(t_0, s_j)} = \sum_{j=1}^{m} P(t_0, s_j) x_j = U(t_0).$$

□

This lemma shows how the value of an asset can be split up into its dependence on the zero-coupon bond price and the zero-coupon bond price itself. Using this result it is in fact possible to derive an analytical representation of the Δ of a fixed-rate bond.

Theorem 3.2 *The derivative of a fixed-rate bond ψ with respect to the rate K_i of swap π_i (that was used in bootstrapping) is:*

$$\frac{\partial U}{\partial K_i} = \sum_{l=1}^{n} \left(\sum_{j=1}^{m} \underbrace{\frac{\partial U}{\partial P(t_0, S_j)}}_{\text{Payment}} \underbrace{\frac{\partial P(t_0, S_j)}{\partial p_l}}_{\text{Interpolation}} \right) \underbrace{\frac{\partial p_l}{\partial K_i}}_{\text{Bootstrapping}},$$

where $\partial P(t_0,S_j)/\partial p_l$ can be approximated numerically and:

$$\frac{\partial U}{\partial P(t_0, S_j)} = \begin{cases} e^{-zS_j} r\tau\left(S_{j-1}, S_j\right) N, & j < m \\ e^{-zS_m} r\tau\left(S_{m-1}, S_m\right) N + e^{-zS_m} N, & j = m. \end{cases}$$

$$\frac{\partial p_l}{\partial K_i} = \frac{\partial p_l}{\partial V_i} \left(-N_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right) \right)$$

○

Proof. First, let:

$$x_j = \begin{cases} e^{-zS_j} r\tau\left(S_{j-1}, S_j\right) N, & j < m \\ e^{-zS_m} r\tau\left(S_{m-1}, S_m\right) N + e^{-zS_m} N, & j = m. \end{cases}$$

Note that $U(t_0) = \sum_{j=1}^{m} P(t_0, S_j)x_j$ (Subsection 2.3.2). Using Lemma 3.2 it then follows that:

$$\begin{aligned}\frac{\partial U}{\partial p_l} &= \frac{\partial}{\partial p_l} \sum_{j=1}^{m} P(t_0, S_j) \frac{\partial U}{\partial P(t_0, S_j)} \\ &= \sum_{j=1}^{m} \frac{\partial U}{\partial P(t_0, S_j)} \frac{\partial P(t_0, S_j)}{\partial p_l}.\end{aligned}$$

The main result then follows using the chain rule:

$$\begin{aligned}\frac{\partial U}{\partial K_i} &= \sum_{l=1}^{n} \frac{\partial U}{\partial p_l} \frac{\partial p_l}{\partial K_i} \\ &= \sum_{l=1}^{n} \left(\sum_{j=1}^{m} \frac{\partial U}{\partial P(t_0, S_j)} \frac{\partial P(t_0, S_j)}{\partial p_l} \right) \frac{\partial p_l}{\partial K_i}.\end{aligned}$$

To complete the proof, the result of the two quantities $\partial U / \partial P(t_0, S_j)$ and $\partial p_l / \partial K_i$ should be specified. The former follows directly from the (linear) definition of $U(t_0)$. As to the latter, this follows directly from Lemma 3.1. Thus, the theorem has been proven. □

Note that it is not novel that $\partial P(t_0, S_j) / \partial p_l$ should be approximated using numerical techniques — this was also done to calculate the Jacobian in Equation (3.3). Also, one might think that this theorem does not fully solve the problem, since the quantity $\partial p_l / \partial V_i$ is present in the formula. However, using Corollary 3.1 it is known that this quantity is an element in the inverse Jacobian $J^{-1}(\mathbf{p})$, which was calculated already in Section 3.1.

So, it is now possible to calculate the sensitivity of fixed-rate bonds to interest rate changes.

In addition, Theorem 3.2 may also be used to calculate the Δ of a zero-coupon bond. To see this, recall from Subsection 2.3.2 that a fixed-rate bond with $r = 0$ is equivalent to a zero-coupon bond. So, suppose $\bar{\psi}$ is a zero-coupon bond with notional $\bar{N}$, payment time $\bar{S}$, z-spread $\bar{z}$ and value $\bar{U}(t_0)$ at current time t_0. Then:

$$\frac{\partial \bar{U}(t_0)}{\partial K_i} = \sum_{l=1}^{n} \left(e^{-\bar{z} S_m} \bar{N} \frac{\partial P(t_0, \bar{S})}{\partial p_l} \right) \frac{\partial p_l}{\partial K_i}, \tag{3.18}$$

where $\partial P(t_0, \bar{S}) / \partial p_l$ is approximated numerically as before, and $\partial p_l / \partial K_i$ is calculated as in Theorem 3.2.

It is now possible to calculate the sensitivity of both fixed-rate bonds and zero-coupon bond semi-analytically. To illustrate this methodology, Example 3.2 is revisited and approached in this new way.

Example 3.3 *Recall Example 3.2, where the change in value of a zero-coupon bond after changes in the interest rate was calculated. It is possible to reach (approximately) the same results using Theorem 3.2. This is done for the fixed rate change of the three-year swap from* $K_3 = 2.5\%$ *to* $K_3 = 2.6\%$.

In Example 3.2 it was assumed that there is no risk associated with the pay-off — i.e. this zero-coupon bond has no counterparty risk. This same assumption was made for the interest rate swaps that were used to calculate the discount curve. Thus the z-spread of the zero-coupon bond is $z = 0$.

The partial derivatives $\partial P(t_0, \bar{S}) / \partial p_l$ *in Equation (3.18) can be calculated using central differences. Using Theorem 3.2, the partial derivatives* $\partial p_l / \partial K_i$ *are derived from the inverse Jacobian that was* *used in bootstrapping. This yields*[8]*:*

$$\frac{\partial P(t_0,\bar{S})}{\partial p_1} = 0, \qquad \frac{\partial P(t_0,\bar{S})}{\partial p_2} = 0, \qquad \frac{\partial P(t_0,\bar{S})}{\partial p_3} = 1,$$
$$\frac{\partial p_1}{\partial K_3} = 0, \qquad \frac{\partial p_2}{\partial K_3} = 0, \qquad \frac{\partial p_3}{\partial K_3} \approx 2.8180.$$

From this, it follows that:

$$\frac{\partial \bar{U}(t_0)}{\partial K_i} \approx 28.1803,$$

which, using a Taylor expansion (Equation (3.17)), gives a change in value:

$$\tilde{V}(t_0) - V(t_0) \approx -0.0282.$$

Observe that this is the same result as in Example 3.2, as should be expected. △

3.3.2. Interest rate swap

As in the case of bonds it is also relevant to calculate the Δ of interest rate swaps. This is the topic of this subsection.

Let ψ be an interest rate swap (as in Section 2.3) with notional N, fixed rate K_i, payments times $S_1,\ldots,S_m$ and value $U(t_0)$ at current time t_0. Assume that ψ is equal to its market value. The sensitivity of $U(t_0)$ to changes in K_i (the fixed rate of a swap that was used for bootstrapping) can be calculated in a manner similar to such calculations for fixed-rates bonds. A theorem can be stated that closely resembles the theorem for fixed-rate bonds (Theorem 3.2).

Theorem 3.3 *The derivative of an interest rate swap ψ with respect to the rate K_i of swap π_i (that was used in bootstrapping) is:*

$$\frac{\partial U}{\partial K_i} = \sum_{l=1}^{n}\left(\sum_{j=1}^{m} \underbrace{\frac{\partial U}{\partial P(t_0,S_j)}}_{Payment} \underbrace{\frac{\partial P(t_0,S_j)}{\partial p_l}}_{Interpolation}\right) \underbrace{\frac{\partial p_l}{\partial K_i}}_{Bootstrapping},$$

where $\partial P(t_0,S_j)/\partial p_l$ can be approximated numerically and:

$$\frac{\partial U}{\partial P(t_0,S_j)} = \begin{cases} -K\tau(S_{j-1},S_j)N, & j<m \\ -K\tau(S_{j-1},S_j)N - N, & j=m. \end{cases}$$
$$\frac{\partial p_l}{\partial K_i} = \frac{\partial p_l}{\partial V_i}\left(-N_i \sum_{j=1}^{m_i} \tau_{i,j} P\left(t_0, T_{i,j}\right)\right)$$

○

Proof. First, let:

$$x_j = \begin{cases} -K\tau(S_{j-1},S_j)N, & j<m \\ -K\tau(S_{j-1},S_j)N - N, & j=m. \end{cases}$$

Note that $U(t_0) = \sum_{j=1}^{m} P(t_0,S_j)x_j$ (Subsection 2.3.3). The proof can then be completed as in Theorem 3.2. □

[8] Finite differences are used when indicated, with 'bump' size 10^{-10}.

The interest rate risk associated with an interest rate swap can now be assessed. If $\psi \in \Pi$ it can be shown that its sensitivity is in many cases equal to zero. Numerical approximations are then no longer required.

Lemma 3.3 *Suppose* $\psi = \pi_j \in \Pi$. *Then:*

$$i \neq j \implies \frac{\partial \psi}{\partial K_i} = 0.$$

△

Proof. Derive $P(t_0, t)$ by bootstrapping from $\Pi = \{\pi_1, \dots, \pi_n\}$. Recall from Section 3.1 that a *necessary condition* of a bootstrapped curve is that the values of all swaps π_i are equal to their market values:

$$V_1(t_0) = \hat{V}_1(t_0) \quad \cdots \quad V_n(t_0) = \hat{V}_n(t_0).$$

Now let $\bar{\pi}_i$ be an interest rate swap with notional N_i, payment times $T_{i,1}, \dots, T_{i,m_i}$ (both equal to those of π_i) and fixed rate $\bar{K}_i$. Let $\bar{V}_i(t_0)$ be its value at current time t_0. Again derive $\bar{P}(t_0, t)$ from $\bar{\Pi} = \{\pi_1, \dots, \bar{\pi}_i, \dots, \pi_n\}$ using bootstrapping. Then it should hold that:

$$V_j(t_0) = \hat{V}_j(t_0),\ j \neq i \quad \text{and} \quad \bar{V}_i(t_0) = \hat{V}_i(t_0).$$

So the value of swaps $\pi_j, j \neq i$ does not change when K_i is updated and thus:

$$\frac{\partial \pi_j}{\partial K_i} = 0,\ i \neq j,$$

from which the result follows. □

This is in fact a particularly important result for hedging using interest rate swaps. The importance of Lemma 3.3 will be seen later.

3.4. Hedging interest rate risk

A theoretical approach to *delta hedging* is described in this section. The idea of hedging is to minimize the risks of an investor, i.e. a pension fund, and to eliminate those risks altogether if possible. Interest rate risk is being hedged in this section because it is the major type of risk a pension fund is exposed to. It is therefore very relevant in the context of this thesis.

There are in practice many customs and regulations that a pension fund takes into account. These intricacies are outside the scope of this section and will be discussed in Chapter 5. A relatively simple case is first discussed and then later extended to a model that describes the situation of a pension fund.

Assume an investment universe consisting of an asset ψ with value $V_\psi(t)$ that makes one payment at maturity s_m and a derivative ξ of this asset with value $V(t, V_\psi)$. Consider the case of an investor that owns ξ (in quantity 1). In addition, the investor has a bank account with $B(t)$ money at time t and $B(t_0) = b$ initially. At the current time t_0, the value of the investor's assets is:

$$V(t_0, V_\psi) + B(t_0) = V(t_0, V_\psi) + b. \tag{3.19}$$

The investor would like to hedge its position in ξ so that it is no longer exposed to the risk of a change in value of ψ. A change in $V_\psi(t)$ would then no longer have any impact on the value of the investor's assets.

The easiest approach to do this would have the investor sell its position in ξ. This would result in an inflow into the bank account. The investor then clearly no longer has the unwanted exposure and is agnostic of value changes in ψ. This is called a *back-to-back* transaction. Unfortunately, it is often not possible to perform this hedging strategy either because no party is willing to buy the asset at a good price (insufficient liquidity) or there is simply no market for the asset. The latter is the case for a pension fund, which clearly cannot sell its pension liabilities.

Therefore, an alternative hedging strategy should be envisaged. This can actually be done by trading in the underlying instrument. To do this, consider the following portfolio with some position $x(t) \in \mathbb{R}$ invested in ψ:

$$\Lambda(t, V_\psi) = V(t, V_\psi) + x(t) V_\psi(t).$$

Recall that the objective of the hedging strategy is that the value of the investor's assets does not change when the value of ψ changes. This is the case when the partial derivative of $\Lambda(t, V_\psi)$ with respect to $V_\psi(t)$ is equal to zero, i.e.:

$$\frac{\partial \Lambda(t, V_\psi)}{\partial V_\psi(t)} = \frac{\partial V(t, V_\psi)}{\partial V_\psi(t)} + x(t) = 0 \quad \Longrightarrow x(t) = -\frac{\partial V(t, V_\psi)}{\partial V_\psi(t)}.$$

Suppose the investor decides to use this hedging strategy. Instead of holding only asset ξ and some money in the bank account at t_0, the investor buys ψ in the amount of $x(t_0)$ using the money in the bank account. Thus, the total value of the investor's assets at time t_0 would then be:

$$\Lambda(t_0, V_\psi) + B(t_0) = \underbrace{V(t_0, V_\psi) + x(t_0) V_\psi(t_0)}_{\text{Investment}} + \underbrace{b - x(t_0) V_\psi(t_0)}_{\text{Bank account}},$$

with $x(t_0)$ defined as before. As one expects, this has not changed compared to the original value of the assets in Equation (3.19). By investing according to this strategy at t_0 the investor mitigates its risk to changes in the value of the underlying.

Although this hedging strategy would work reasonable well, an investor's exposure would in practice not be hedged in its entirety. Consider a first-order Taylor expansion to see this:

$$\Lambda(t, V_\psi + \delta) = \Lambda(t, V_\psi) - \frac{\partial \Lambda(0, V_\psi)}{\partial V_\psi} \delta + \mathcal{O}\left(\delta^2\right) = \Lambda(t, V_\psi) + \mathcal{O}\left(\delta^2\right).$$

It is clear from this Taylor approximation that the hedge is not exact and that the error increases if the change in value of the underlying increases. This can be solved by considering higher-order derivatives. In practice it is often difficult to do this and this approach is therefore also not taken in this thesis. Still, the described strategy is a good *static hedge*: the investor hedges its position at t_0 and does not have to perform any other trades after that.

Alternatively, a *dynamic hedge* can be performed. The position $x(t)$ in the underlying instrument would then be updated at frequent time periods to ensure the hedge is still accurate. This is called *rebalancing*. Consider rebalancing times $s_0, \ldots, s_m$ with $s_0 = t_0$. At each time the position in the underlying asset would be updated to $x(t) = -\partial V(t, V_\psi) / \partial V_\psi(t)$ so that the hedge is again accurate. This of course has implications on the bank account that can be captured in a recursive formula:

$$B(s_i) = \underbrace{B(s_{i-1}) \frac{M(s_i)}{M(s_{i-1})}}_{\text{Interest payments}} + \underbrace{\left(x(s_{i-1}) - x(s_i)\right) V_\psi(s_i)}_{\text{Rebalancing}}, \quad \forall i \in \{1, \ldots, m-1\}.$$

At maturity of ψ the bank account is:

$$B(s_m) = \underbrace{B(s_{m-1})\frac{M(s_m)}{M(s_{m-1})}}_{\text{Interest payments}} + \underbrace{x(s_{m-1})V_\psi(s_m)}_{\text{Underlying}} + \underbrace{V(s_m, V_\psi)}_{\text{Payment of }\psi}.$$

If the hedge is good, time periods are sufficiently small and there are no transaction costs, $B(s_m)$ should approximate $M(s_0)/M(s_m)\left(B(t_0)+V(t_0,V_\psi)\right)$.

Rebalancing is discussed throughout when applicable. In Chapter 5 a static hedge is proposed that may additionally be used for rebalancing. The goal of the remainder of this section is finding a static hedge for the liabilities of a pension fund.

3.4.1. Hedging pension cash flows

The liabilities of a pension fund should be defined so that they can be hedged. Let ξ be the cash flow of a pension fund with value $V(t)$ at time t as defined in Definition 2.2. Its value depends on the yield curve[9] as is evident from Corollary 2.1. Recall from Section 3.1 that the yield curve is uniquely determined by the instruments that were used to bootstrap this curve. Therefore $V(t)$ in turn depends on these bootstrapping instruments.

Let (as in Section 3.1) $\Pi = \{\pi_1, \ldots, \pi_n\}$ be interest rate swaps traded in the market with rates $K_1, \ldots, K_n$ and values $V_1(t), \ldots, V_n(t)$. Let $\Omega = \left\{(t_1, p_1), \ldots, (t_n, p_n)\right\}$ (with $p_i = P(t_0, t_i)$) be the spine points derived from these instruments using the bootstrapping procedure.

In a way similar to the simpler case, a portfolio $\Lambda(t)$ should be found whose derivatives with respect to the underlying instruments are equal to zero[10]:

$$\frac{\partial\Lambda(t)}{\partial V_1(t)} = 0 \quad \cdots \quad \frac{\partial\Lambda(t)}{\partial V_n(t)} = 0.$$

Then the risk of a change in value of any of the underlying instruments has been hedged.

To derive such a portfolio let:

$$\Lambda(t) = V(t) + x_1(t)V_1(t) + \ldots + x_n(t)V_n(t).$$

It is elementary to see that:

$$\frac{\partial\Lambda(t)}{\partial V_i(t)} = \frac{\partial V(t)}{\partial V_i(t)} + x_i(t_0) = 0 \quad \Longrightarrow x_i(t) = -\frac{\partial V(t)}{\partial V_i(t)}, \tag{3.20}$$

since Lemma 3.3 gives that:

$$\frac{\partial V_i(t)}{\partial V_j(t)} = 0, \text{ if } i \neq j.$$

Because the objective is finding a *static hedge*, the portfolio should be determined at t_0. Therefore set $x(t) = x(t_0)$. The problem of hedging the pension fund cash flow ξ has thus been reduced to finding the partial derivatives:

$$\frac{\partial V(t_0)}{\partial V_1(t_0)}, \ldots, \frac{\partial V(t_0)}{\partial V_n(t_0)}.$$

[9] In Section 2.2 it was mentioned that there are ongoing discussions on whether the value of future pension liabilities should indeed (solely) depend on the yield curve. As was mentioned previously, the prevailing methodology that admits this dependence is used in this thesis.

[10] For brevity the values of the portfolio and the pension fund cash flow are not explicitly written as a function of the underlying, i.e.: $V(t)$ is used rather than $V(t, V_1, \ldots, V_n)$.

This problem may in fact be solved by making use of results from Section 3.3. To see this, note that a pension cash flow with payment times $T_1,\ldots,T_m$ and payments $L(T_1),\ldots,L(T_m)$ is — from a purely economical perspective and disregarding political decisions on pension valuations — in fact equivalent to m zero-coupon bonds with notionals $N_1 = L(T_1) \cdots N_m = L(T_m)$ and maturities $T_1,\ldots,T_m$. Let $U_1(t),\ldots,U_m(t)$ be the values of these bonds. Then:

$$V(t) = \sum_{j=1}^{m} U_j(t) \quad \Longrightarrow \frac{\partial V(t)}{\partial V_i(t)} = \sum_{j=1}^{m} \frac{\partial U_j(t)}{\partial V_i(t)}.$$

The problem has thus been reduced to finding the partial derivatives of the values of zero-coupon bonds with respect to the value of an underlying swap.

In Section 3.3 this problem of finding the required partial derivatives was already solved for zero-coupon bonds. Using Equation (3.18) $\partial U_j(t)/K_i$ can be calculated and using Lemma 3.1 $\partial U_j(t)/V_i(t)$ can be derived from this. It is thus known how to find the weights $x_1(t_0),\ldots,x_m(t_0)$ that complete the static hedge $\Lambda(t)$.

This hedging technique that uses the underlying interest rate swaps can be illustrated by an example.

Example 3.4 *Recall Examples 3.2 and 3.3, where the sensitivity of a zero-coupon bond maturing in three years was calculated. Assume that the swaps have notional $N_i = 10$. Suppose one would like to hedge the interest rate risk of this zero-coupon bond using swaps. The Δ of the bond are calculated using Subsection 3.3.1:*

$$\frac{\partial U}{\partial K_i} \approx -0.2356, \quad \frac{\partial U}{\partial K_i} \approx -0.4711, \quad \frac{\partial U}{\partial K_i} \approx 28.1803.$$

From this the hedge is derived using Equation (3.20) and Lemma 3.1. This gives:

$$x_1(t_0) \approx -0.0238, \quad x_2(t_0) \approx -0.0240, \quad x_3(t_0) \approx 0.9756.$$

Using Subsection 3.3.2 the sensitivity of the proposed swaps holdings is calculated:

$$\begin{aligned}
&x_1(t_0)\frac{\partial U}{\partial K_1} \approx 0.2356, && x_1(t_0)\frac{\partial U}{\partial K_2} \approx 0.0000, && x_1(t_0)\frac{\partial U}{\partial K_3} \approx 0.0000,\\
&x_2(t_0)\frac{\partial U}{\partial K_1} \approx 0.0000, && x_2(t_0)\frac{\partial U}{\partial K_2} \approx 0.4711, && x_2(t_0)\frac{\partial U}{\partial K_3} \approx 0.0000,\\
&x_3(t_0)\frac{\partial U}{\partial K_1} \approx 0.0000, && x_3(t_0)\frac{\partial U}{\partial K_2} \approx 0.0000, && x_3(t_0)\frac{\partial U}{\partial K_3} \approx -28.1803.
\end{aligned}$$

Note that $\forall j \in \{1,2,3\} : \partial U/\partial K_j + \sum_{i=1}^{3} x_i(t_0)\partial V_i/\partial K_j \approx 0$. Thus this is indeed a perfect delta hedge. $\triangle$

This hedge is in fact *unique* if one only considers hedges using these interest rate swaps. However, other hedges can be constructed using different instruments. This is done in Chapter 5.

3.5. Alternative measures of interest rate sensitivity

In Section 3.3 the sensitivity of the value of an interest rate derivative to changes in the interest rate was described. A change in the interest rate was modeled as a change in the fixed rate of one of the swaps that were used in the bootstrapping procedure. This is a change that can actually be observed in the market so this is a realistic model. Alternatively, the sensitivity to the change in yield — the implied yearly rate of return — of a bond maturing at the same time as one of the swaps can be calculated. This approach will be formalized in this section and will also be related to common practices in the pension world. Inspiration has been drawn from work by Coleman (2011) and Maio and de Jong (2015) who also touch upon this subject.

At first it might seem strange to calculate the sensitivity to a value that is not an input to the model. A potential problem could be the consistency of the model. However, it should be noted that there is a bijection between the yield and the corresponding swap rate: a change in the yield also changes the fixed rate of the corresponding swap. This finding is formally reiterated later and is essential to the results.

A question that remains is why practitioners calculate these sensitivities — with respect to the yield — rather than using the 'logical' sensitivity calculations. From the perspective of Section 3.3 such an alternative sensitivity measure only entails additional work without clear benefits. However, DNB regularly publishes a discount curve that should be used by pension funds[11]. By using this discount curve pension funds are compliant with the regulator without having to invest resources in creating a discount curve themselves. Therefore pension funds generally do not implement a bootstrapping procedure. This poses a problem in quantifying the result of changes in the swap rates — calculating sensitivity as described in Section 3.3 is impossible. The relation between the yield and the discount curve is well-known so calculating the sensitivity with respect to yield *is possible*. This explains the popularity of such sensitivities in the pension world.

This description of an alternative measure of interest rate sensitivity is now formalized mathematically.

Let (as before) $\pi_1, \ldots, \pi_n$ be interest rate swaps with values $V_1(t_0), \ldots, V_n(t_0)$ and fixed rates $K_1, \ldots, K_n$. Via bootstrapping (Section 3.1) spine points $(t_1, p_1), \ldots, (t_n, p_n)$ are derived from these interest rate swaps that determine the discount function $P(t_0, t)$ through interpolation. In addition define the *yield* using the relations in Definitions 3.1 and 3.2:

$$y_i = \bar{r}(t_i) = P(t_0, t_i)^{-\frac{1}{t_i - t_0}} - 1 = p_i^{-\frac{1}{t_i - t_0}} - 1. \tag{3.21}$$

Observe that this is equivalent with the well-known relation $p_i = \left(1 + y_i\right)^{-(t_i - t_0)}$.

Now consider an interest rate derivative ψ with value $U(t_0)$. In Section 3.3 the partial derivative $\partial U / \partial K_i$ was found. The partial derivative $\partial U / \partial y_i$ should be found in this section. This is in fact rather straightforward using the earlier results.

Proposition 3.1 *Consider* $\partial U / \partial K_i$ *and* $\partial U / \partial y_i$. *Then:*

$$\frac{\partial U}{\partial y_i} = \sum_{l=1}^{n} \frac{\partial U}{\partial K_l} \frac{\partial K_l}{\partial p_i} \frac{\partial p_i}{\partial y_i}.$$

△

[11] The discount curve is published by DNB at https://statistiek.dnb.nl/downloads/index.aspx#/details/nominale-rentetermijnstructuur-pensioenfondsen-zero-coupon/dataset/ed15534f-eab3-4862-a68e-f33effa78d6a.

Proof. Note that using the chain rule:

$$\frac{\partial K_l}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial K_l}{\partial p_j}\frac{\partial p_j}{\partial y_i} = \frac{\partial K_l}{\partial p_i}\frac{\partial p_i}{\partial y_i},$$

since $\partial p_j / y_i = 0$ if $j \neq i$. The final result can be found using the chain rule as well:

$$\frac{\partial U}{\partial y_i} = \sum_{l=1}^{n} \frac{\partial U}{\partial K_l}\frac{\partial K_l}{\partial y_i} = \sum_{l=1}^{n} \frac{\partial U}{\partial K_l}\frac{\partial K_l}{\partial p_i}\frac{\partial p_i}{\partial y_i}.$$

□

Observe that $\partial K_l / \partial p_i$ is an element of the Jacobian that was used for bootstrapping in Section 3.1 and thus known. In addition observe that $\partial p_i / \partial y_i$ can be calculated in a straightforward manner from Equation (3.21):

$$\frac{\partial p_i}{\partial y_i} = -\left(t_i - t_0\right)^{-(t_i - t_0)-1}.$$

Thus the required result has been found.

In Section 3.4 it was explained how the interest rate risk of ψ could be hedged if those risks are defined as the partial derivatives $\partial U / \partial K_i$, i.e. the sensitivity to changes in the fixed rate of swaps. It was shown how a portfolio with value $\Lambda(t_0)$ could be constructed with $\partial \Lambda / \partial K_i = 0, \forall i \in \{1, \ldots, n\}$. A relevant question is: does a portfolio that hedges well in the one sensitivity measure also perform well if the other sensitivity measure is used? This is in fact the case as will be shown in the next proposition.

Proposition 3.2 *Let ψ be a portfolio with value $\Lambda(t_0)$ at current time t_0. Then:*

$$\frac{\partial \Lambda}{\partial K_i} = 0 \iff \frac{\partial \Lambda}{\partial y_i} = 0.$$

△

Proof. Suppose $\partial U / \partial K_i = 0$. Then using Proposition 3.1:

$$\frac{\partial U}{\partial y_i} = \sum_{l=1}^{n} 0 \cdot \frac{\partial K_l}{\partial p_i}\frac{\partial p_i}{\partial y_i} = 0.$$

Suppose now that $\partial U / \partial y_i = 0$. It can be proven similar as in Proposition 3.1 that:

$$\frac{\partial U}{\partial K_i} = \sum_{l=1}^{n} \frac{\partial U}{\partial y_l}\frac{\partial y_l}{\partial p_i}\frac{\partial p_i}{\partial k_i}.$$

The result then follows as in the inverse case. □

So it may be concluded from this proposition that hedging under either measure is equivalent. This is an important result because it is thereby known that this alternative measure can actually be used.

3.5.1. Sensitivity calculation with dissimilar interpolation methods

In the industry an alternative sensitivity measure called *key rate duration* is frequently used. The idea is to measure the change in value of an instrument after changing the yield at some specific time point. This is similar to the partial derivative $\partial V/\partial y_i$ that has already been described but details are slightly different.

In classical hedging strategies it was assumed that yield curve changes would be parallel. Reitano (1992) concluded that this assumption has a significant impact and that such strategies still leave substantial risks unhedged. He proposed an alternative method that measures risks related to non-parallel yield curve changes through "partial durations". A similar approach was proposed by Ho (1992) that measures interest rate risks through *key rate durations*. This is described well by Nawalkha and Soto (2009, pp. 264–293).

The key rate durations of an asset can be calculated in three steps:

Selecting key rate First, the key rates should be chosen. The interest rate sensitivity is calculated as the change in value at or around these time points. Assume for now that the key rates are equal to the spine points, i.e. the key rates are equal to the maturities of the swaps that were used in the bootstrapping procedure.

Interpolating yield changes Secondly, the yield changes at all relevant time points should be calculated after a shock of the yield at one of the key rates. These "relevant time points" are the payment times of the asset under consideration. An interpolation scheme is defined to determine these changes: the change in yield is decreased linearly until it is zero at the neighboring key rates. This determines the changes in yield at all time points.

Calculating value change Finally, the change in value of the asset after a shock at a key rate should be calculated. This can for instance be done by discounting future payments using the new implied discount curve. The change in value is then the difference of this value with the original value.

These steps should be repeated for all key rates to find a vector of the key rate durations. Under this model the vector describes the sensitivity of an asset to interest rate changes.

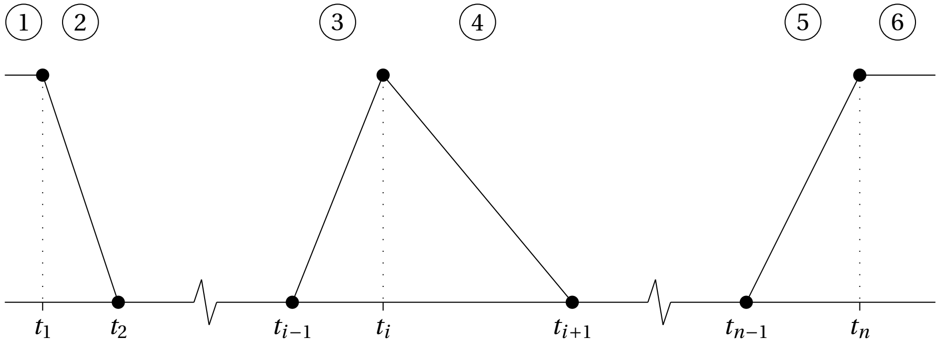


Figure 3.4: Yield curve changes as specified in Equation (3.22) with shocks at t_1, t_i and t_n — ranges have been marked with circled numbers.

The interpolation scheme is essential in these calculations and therefore described in more detail. To calculate the i-th key rate the yield is shocked by a certain amount δ at key rate $t_i \in \{t_1, \ldots, t_n\}$. The shock is then decreased linearly until it is zero at the neighboring key rates. At the first and last key rates, the yield change is kept constant respectively left and right of the key rate. Let $\bar{r}(t)$ be the yield of a bond maturing at t, as in Definition 3.2. Then the new yield of such a bond at time t after a shock of size δ at key rate t_i is:

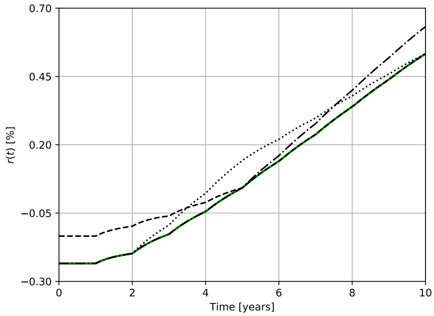
$$\bar{r}_i(t) = \bar{r}(t) + \delta I_i(t),$$

with the interpolation scheme:

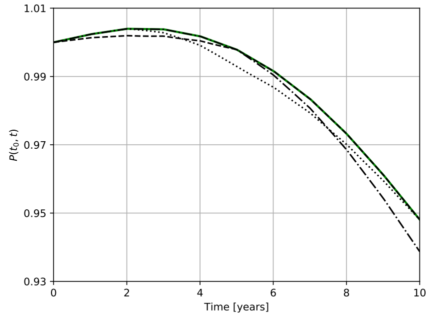
$$I_i(t) = \begin{cases} 1, & t_i = t_1, \quad t \leq t_i & \text{①} \\ \dfrac{t_2 - t}{t_2 - t_1}, & t_i = t_1, \quad t_i < t < t_{i+1} & \text{②} \\ \dfrac{t - t_{i-1}}{t_i - t_{i-1}}, & t_1 < t_i < t_n, \quad t_{i-1} < t \leq T_i & \text{③} \\ \dfrac{t_{i+1} - t}{t_{i+1} - t_i}, & t_1 < t_i < t_n, \quad t_i < t < t_{i+1} & \text{④} \\ \dfrac{t - t_{n-1}}{t_n - t_{n-1}}, & t_i = t_n, \quad t_{i-1} < t < t_i & \text{⑤} \\ 1, & t_i = t_n, \quad t \geq T_i & \text{⑥} \\ 0, & \text{otherwise.} \end{cases} \tag{3.22}$$

This interpolation scheme is illustrated in Figure 3.4. The impact of these changes on the instantaneous interest rate $r(t)$ and discount curve $P(t_0, t)$ have been added to Figure 3.5. One of the main reasons for employing this interpolation scheme is that the sum of all key rate durations is equal to the change in value resulting from a shock at all key rates. This is evident from the definition of the scheme. The importance of this property for pension funds will be seen in Chapter 5.

In essence key rate durations are very similar to the sensitivities that have been described in the beginning of this section. For an infinitesimal δ the i-th key rate duration resembles the partial derivative $\partial U / \partial y_i$: in both cases the sensitivity to yield changes is calculated. It should be noted there is a major difference in the interpolation between spine points: whereas the previous methods all use one consistent interpolation scheme, in the case of key rate duration the interpolation



(a) Instantaneous rate $r(t)$ as in Definition 3.1.



(b) Discount $P(t_0, t)$ as in Equation (2.1).

Figure 3.5: Impact of the key rate duration interpolation scheme from Equation (3.22) on the curve bootstrapped for Figure 3.1. Shocks of 10 basis points ($\delta = 0.001$) have been added at key rates $t = 2$ (dashed), $t = 5$ (dotted) and $t = 10$ (dash-dotted).

scheme used for bootstrapping and calculating sensitivities differs. This different interpolation scheme was not a conscious decision: Ho (1992, p. 3) in fact state that "linear interpolation ... is used for reasons of simplicity". Nonetheless, this methodology is frequently used in the industry.

It is possible to calculate key rate durations in the described framework by using this new interpolation function in Theorems 3.2 and 3.3 and Proposition 3.1. This is an important finding that relates an important concept in the pension world — key rate durations — to the mathematical concepts of interest rate sensitivity.

Thus far it has been assumed that the key rates coincide with the spine points. This is often not the case: a bootstrapped curve is used and then the best key rates are chosen. This is also the case in Figure 3.5. In fact scholars have extensively researched the best key rates independently of the instruments that were used to bootstrap the curve. For more information on this the reader is referred to Nawalkha, Soto, and Beliaeva (2005, pp. 281–282) and Ho (1992, p. 3). The results in this section can be extended to this general case by considering the impact that a change in yield at any time point has on the spine points. However, this exercise has not been done because it is not the main topic of this thesis.

Some scholars such as Hagan and West (2008, p. 13) suggest to perturb the discount curve $P(t_0, t)$ directly. They find good hedging results using sensitivities with respect to discount factors rather than yields. In a more recent paper Hagan (2015) claims to improve results further by introducing wave-shaped perturbations of the instantaneous forward rate. This might warrant further research but has not been included in this work — it has been decided to adhere to industry practice instead.

Hagan and West (2008, p. 13) in fact not only propose to shock the discount curve rather than the yield curve, but they also assess hedging results with rectangular shocks instead of triangular shocks. The key rate duration methology of Ho (1992) makes uses of triangular shocks — this can be seen clearly in Figure 3.4. The alternative interpolation scheme using rectangular shocks is described in Appendix A.5. The rectangular interpolation method performs best tests by Hagan and West (2008) but has not been studied extensively by other scholars. It is therefore not investigated in more detail in this thesis.

4

Pension fund cash flows

The main problem that this thesis aims to solve, is finding a portfolio of assets that hedges risks associated with liabilities. Actuaries provide numbers for the cash flows of liabilities in future years that may be used in calculations. However, this should not simply be taken as a given since it is also important to understand what these liabilities entail, and how the expected values are calculated. If this is unknown a pension fund manager cannot quantify the risks and faces an uncertain set of risky scenarios. This could result in monetary losses. This is already an important reason to research the modeling of liabilities. In addition, such research could also lead to more efficient solutions. For these reasons the modeling of pension liabilities is researched in this chapter.

In Section 2.1 it was explained that pension funds operate in the second pillar of the three-pillar Dutch pension system. In the second pillar of the pension system, however, there are again different types of pensions. An important distinction should be made between *Defined Contribution (DC)* and *Defined Benefits (DB)* pension schemes. In a DC scheme the pension fund invests the contribution for an individual and pays out a pension based on the investment returns: the size of the pension is variable. In a DB scheme, the size of the eventual pension is guaranteed while the pension fund takes on the risk of lower investment returns. In some cases the employer takes on this risk as well. Mixed schemes also exist, consisting of both a DB and a DC scheme.

One caveat in a DB scheme is that the benefit is usually defined in nominal terms and indexation may be decided by the pension fund. Although pension funds in the Netherlands generally aim to index pensions (Van Rooij et al. 2004, p. 5), a DB pension is therefore not as secure as one would think. Still, this is different from a DC scheme, which helps individuals save money and enables sharing of the long-life risk while the risk of low investment returns is taken on by employees rather than the pension fund as in a DB scheme.

In the Netherlands, DB schemes are still dominant. Although DC schemes have in recent years gained popularity mainly driven by the low coverage ratios of pension funds, most pension schemes are still DB schemes. Table 4.1 shows the popularity of the different pension types in 2018. The detailed 12-year historical dataset has been added to Appendix C.6.2. Because of the importance of DB schemes it has been decided to model a DB pension fund. Another reason for modeling a DB scheme is that a model for a DB pension scheme is easily adapted to a DC scheme, which does not hold vice versa (Draper, Armstrong, et al. 2007, pp. 95–96).

Type	Number of schemes	Pension liabilities
Defined benefit	77%	94%
Defined contribution	12%	1%
Mixed	11%	5%
Other	0%	0%

Table 4.1: Popularity of pension schemes in the Netherlands in 2018. The full dataset (by the DNB) has been added to Appendix C.6.2.

A distinction should be made between average and final wage DB schemes. The former pays a percentage of the average wage during an individual's entire career, while the latter pays a percentage of the final wage. In the Netherlands, most pension plans aim to pay 70% of a person's average wage in yearly pension — modeling such a pension fund will also be the aim of this chapter.

To summarize, this chapter focuses on finding a model for an average wage defined benefits scheme that aims to pay 70% of the average wage over an individual's career. If such a model is know, models for the alternative schemes can be derived as in Draper, Armstrong, et al. (2007).

The goal is to find a mathematical formulation for the cash flows of a pension fund that can be used for simulation purposes. It should ideally also be possible to calculate the expected value of these cash flows theoretically. Many quantities are required before these cash flows can be defined. Figure 4.1 gives an overview of the different parts of the model — these parts will be described in this chapter. The expected value of these processes has been added to Figure 4.2.

In this chapter, various processes are described related to pension fund cash flows. This will be done on a discretization of time — define times $t_0, t_1, t_2, \dots$ with t_0 the current time. Assume without loss of generality that these times are ordered, so $t_0 \le t_1 \le t_2 \le \dots$. It should be observed that this discretization is not necessarily a simplification: wages and pensions are also paid at discrete times (usually every month).

[1] Full-Time Equivalent: factor representing whether the employee works more than full-time (> 1) or less than full-time (< 1).

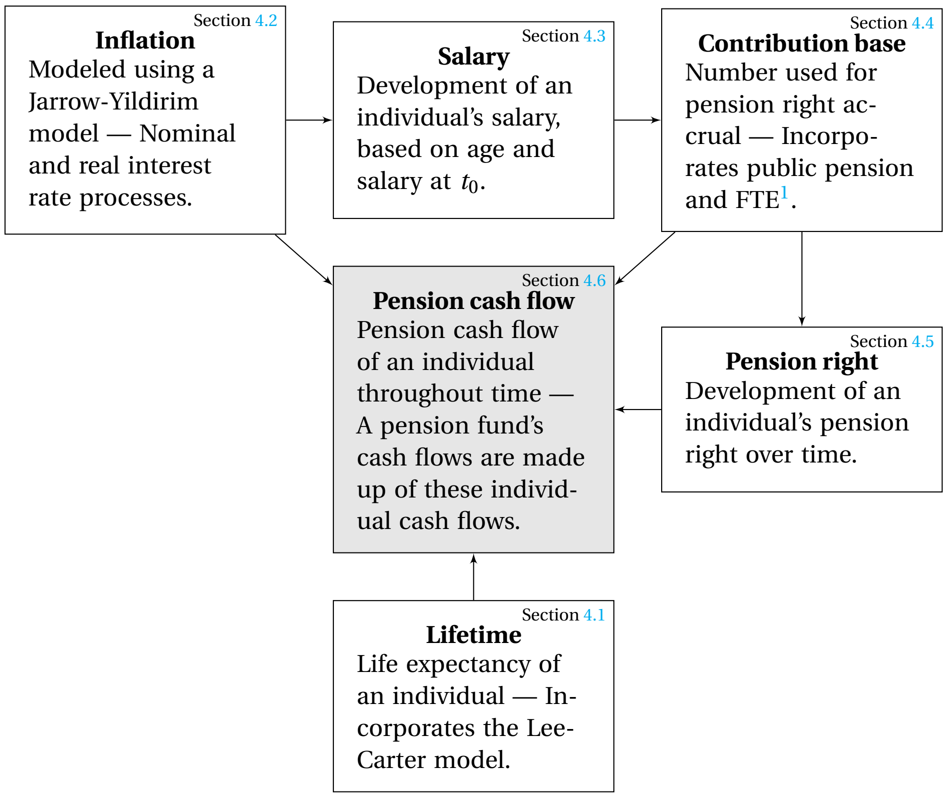
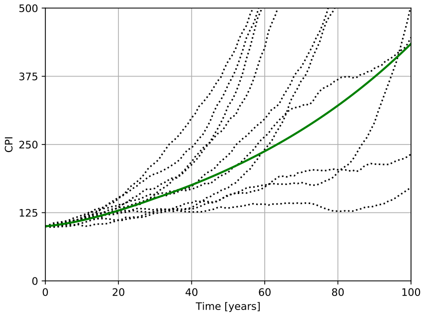
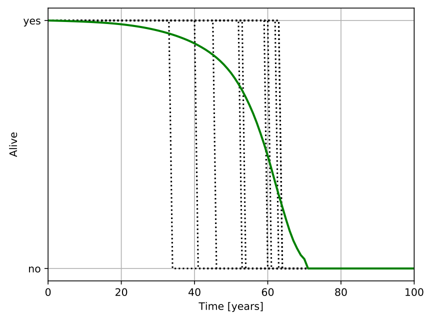


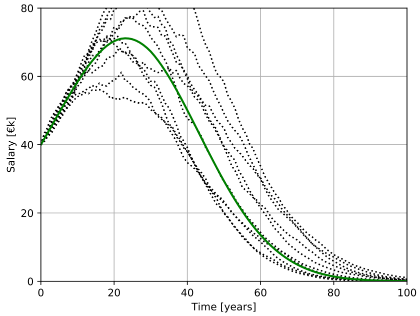
Figure 4.1: Overview of the processes employed to model the pension liabilities. The main goal is a proper model for the 'Pension cash flow' (gray box).



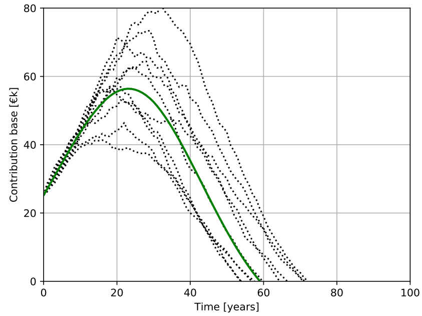
(a) Inflation process (Section 4.2).



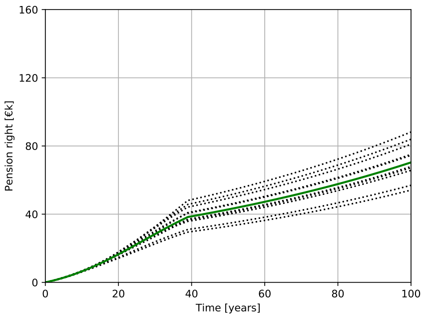
(b) Lifetime process (Section 4.1).



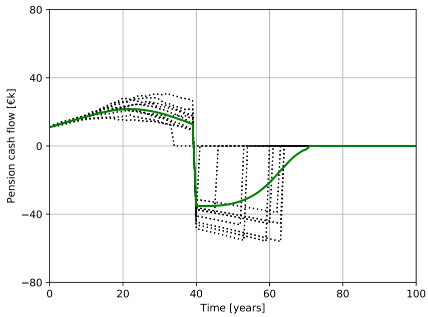
(c) Salary process (Section 4.3).



(d) Contribution base process (Section 4.4).



(e) Pension right process (Section 4.5).



(f) Pension cash flow process (Section 4.6).

Figure 4.2: Analytical expected value (solid green) and 10 Monte-Carlo paths (dotted black) for the process that are described in this chapter. Results are for a 30-year individual with a gross salary of €40,000 working full-time without initial pension right (at $t = 0$).

4.1. Lifetime process

The dynamics of a pension fund are such that an individual (and his or her employer) pay(s) a contribution and in return this individual receives a pension *until death*. Therefore, life expectancy is clearly an important part of any pension cash flow model. An indicator process $A_i(t_j) \in \{0,1\}$ will be defined that indicates whether person i is alive at time t_j.

The indicator $A_i(t_j)$ can clearly be defined in terms of the life expectancy (or time of death) of a person i. Let τ_i be the (stochastic) time of death of a person i. Then:

$$A_i(t_j) = \mathbb{1}_{\tau_i \geq t_j}. \tag{4.1}$$

Note that a person's death does not depend on future events. Thus this τ_i is in fact a stopping time as defined by Shreve (2004, pp. 340–345), since $\forall t \in \mathbb{R}_{\geq t_0} : \{\tau_i \leq t_n\} \in \mathscr{F}_n$.

The probability distribution of τ_i should be found. This may be related to the conditional probability of τ_i, which is the 1-period survival probability of a person. It is useful to find a relation with the conditional probability, because many models exist that can model the 1-period survival probability while models for the conditional probability itself are not commonplace. To find this relation, note that $\tau_i \geq t_j \wedge \tau_i \geq t_{j-1} \implies \tau_i \geq t_j$ by the ordering of the t_i. Therefore it follows by the Kolmogorov definition of the conditional probability[2] that $\forall j \in \mathbb{N} \setminus \{0\}$:

$$\mathbb{P}(\tau_i \geq t_j) = \mathbb{P}(\tau_i \geq t_j \mid \tau_i \geq t_{j-1}) \cdot \mathbb{P}(\tau_i \geq t_{j-1}).$$

In this it has been assumed that $\mathbb{P}(\tau_i \geq t_j) > 0)$. This is a reasonable assumption since there is no predetermined time of death as long as all people are alive at t_0[3]. So it is assumed that $\mathbb{P}\left(\tau_i \geq 0\right) = 1$, which is clearly a sensible assumption that may easily be checked. Applying the previous formula repeatedly yields:

$$\begin{aligned} \mathbb{P}(\tau_i \geq t_j) &= \left(\prod_{k=1}^{j} \mathbb{P}\left(\tau_i \geq t_k \mid \tau_i \geq t_{k-1}\right)\right) \cdot \mathbb{P}\left(\tau_i \geq t_0\right) \\ &= \prod_{k=1}^{j} \mathbb{P}\left(\tau_i \geq t_k \mid \tau_i \geq t_{k-1}\right). \end{aligned} \tag{4.2}$$

This gives the required relation between the conditional probability of τ_i and the 1-period survival probability.

Observe that the τ_i may have different probability distributions. To define the probability distribution of τ_i on the discretized time points it now suffices to define the conditional probability on these points. This conditional probability is in fact equal to the 1-period survival probability of an individual, which may be deduced easily from the 1-year mortality rate. A model for the 1-year mortality rate will now be introduced.

4.1.1. The Lee-Carter model

Lee and Carter (1992) introduced a stochastic model for mortality rates that allows for survival improvements, which yielded surprisingly good results when it was introduced. It does not integrate detailed data such as medical influences on mortality change, but still the observed mortality change is predicted well without detailed assumptions (Lee and Carter 1992, p. 659). In the model,

[2] Given probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $A, B \in \mathscr{F}$ with $\mathbb{P}(B) > 0$, then: $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

[3] It will later be assumed that all people die at age 100 with certainty. However, this does not pose a problem because an upper bound is then set on j.

the probability of survival is a function of age and time, and the model parameters are fitted to historical data.

Several scholars have proposed extensions and potential improvements to this model. For an overview of these extensions, the reader is referred to Van Berkum et al. (2016). Extensions under discussion are for instance dividing the population in groups (cohorts) based on some shared characteristics and decreasing the number of parameters by finding functional representations (Van Berkum et al. 2016, p. 581).

The Lee-Carter model is in fact frequently applied to model mortality in the context of pensions (Pelsser et al. 2016, p. 24; De Waegenaere et al. 2018, p. 51). An important characteristic in that context is that the mortality rate is always nonnegative under the model (Lee and Carter 1992, p. 660). For these reasons, this relatively simple original Lee-Carter model has been taken as the starting point for the modeling of survival/mortality predictions. It is defined by (Lee and Carter 1992, pp. 660, 663; Pelsser et al. 2016, p. 24):

$$\begin{aligned} \log m(x,t_j) &= \alpha(x) + \beta(x)\kappa(t_j) \\ \mathrm{d}\kappa(t) &= \mu_\kappa \,\mathrm{d}t + \sigma_\kappa dW_\kappa^{\mathbb{P}}(t), \end{aligned} \tag{4.3}$$

with:

$$\begin{aligned} m(x,t_j) &: \text{Central death rate for age } x \text{ in } (t_{j-1},t_j], \\ W_\kappa^{\mathbb{P}}(t) &: \text{Standard Brownian motion}^4. \end{aligned}$$

The three parameters of the Lee-Carter model each have their own function. Lee and Carter (1992, p. 660) describe the roles these parameters have:

- $\boldsymbol{\alpha(x)}$ defines how mortality changes with age. This shape itself is described by $\exp(\alpha(x))$.
- $\boldsymbol{\beta(x)}$ defines the sensitivity of mortality rates to changes in $\kappa(t)$ at certain ages: does the mortality rate decline slowly or rapidly in response to a changing $\kappa(t)$? This link is also present in the partial derivate $\partial \log m(x,t)/\partial t = \beta(x)\frac{\mathrm{d}k}{\mathrm{d}t}$.
- $\boldsymbol{\kappa(t)}$ may be considered as the underlying time trend of the mortality process. A decreasing $\kappa(t)$ implies that people are living longer.

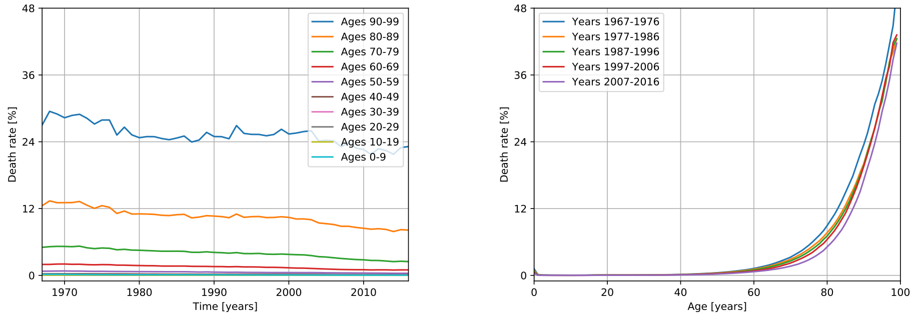
Observe that the parameters $\alpha(x)$ and $\beta(x)$ remain constant with time. So, all time dependence of the mortality process should be fully captured by the process $\kappa(t)$.

The parameters $\alpha(x), \beta(x), \mu_\kappa, \sigma_\kappa$ should be found from historical data. For some older ages there is insufficient data, so it is generally assumed that every person has died at a certain terminal age. A procedure to calibrate the model to historical data through additional constraints and a singular value decomposition will be described shortly.

Using this Lee-Carter model, the survival probability from t_{j-1} to t_j is found by taking the expectation of the central death rate. This expectation is easily calculated since $m(x,t_j)$ is log-normally distributed. This expected value should thus be substituted in Equation (4.2) to complete the equation (using the obvious relation between the 1-year mortality and survival probabilities):

$$\begin{aligned} \mathbb{P}\left(\tau_i \geq t_j \mid \tau_i \geq t_{j-1}\right) &= 1 - \mathbb{E}^{\mathbb{P}}\left[m(a_i(t_{j-1}),t_j) \mid \mathscr{F}_{t_0}\right] \\ &= 1 - \exp\left(A + \frac{B}{2}\right), \end{aligned} \tag{4.4}$$

[4] The Brownian motion $W_\kappa^{\mathbb{P}}(t)$ is independent of such processes that have been defined earlier, for instance in Chapter 3. This is the case for all Brownian motions in this chapter.



(a) Death rate of age groups over a 50-year period.

(b) Death rate in time periods for different ages.

Figure 4.3: Data of the Netherlands in the period 1967–2016 from the Human Mortality Database.

with $a_i(t_{j-1})$ the age of person i at time t_{j-1} and:

$$
\begin{aligned}
A &= \mathbb{E}\left[\log m(x,t_j) \mid \mathscr{F}_{t_0}\right] = \alpha(a_i(t_{j-1})) + \beta(a_i(t_{j-1}))\left[\kappa(t_0) + \mu_\kappa(t_j - t_0)\right],\\
B &= \operatorname{Var}\left(\log m(x,t_j)\right) \qquad = \beta^2(a_i(t_{j-1}))\sigma_\kappa^2(t_j - t_0).
\end{aligned}
$$

The process has now been described so that simulation is possible. In addition, a theoretical result for the expected value of the process can be found. Using Equations (4.1) and (4.2), this is:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[A_i(t_j) \,\middle|\, \mathscr{F}_{t_0}\right] &= \mathbb{P}(\tau_i \geq t_j)\\
&= \prod_{k=1}^{j} \mathbb{P}\left(\tau_i \geq t_k \mid \tau_i \geq t_{k-1}\right).
\end{aligned}
\tag{4.5}
$$

The parameters of the model have not been defined yet. The goal is now to estimate parameters $\alpha(x), \beta(x), \kappa(t_j)$ that fit the model to historical mortality data. When this is done, the parameters $\mu_\kappa, \sigma_\kappa$ may be inferred from the $\kappa(t_j)$, and are then used to forecast $\kappa(t_j)$ for future times.

Historical mortality data has been found in the Human Mortality Database (HMD). This is available for many countries. For calibration of this model, the mortality data of the Netherlands in the period 1967–2016 has been taken ($T = \{1967, \ldots, 2016\}$, 50 years). Sufficient data was available for ages 0–99, so it is implicitly assumed that every person will die with certainty when they become 100 years old ($X = \{0, \ldots, 99\}$, 100 ages). Figure 4.3 shows plots that summarize the mortality rate and the exposure to death in this dataset. For more details on this dataset, the reader is referred to Appendix C.1.

It is important to distinguish between death rates from the model and death rates from the HMD. Therefore, the model output is still denoted by $m_{x,t}$, and the data from the HMD is denoted by $\hat{m}(x,t)$.

There is one problem with historical data: how to deal with data anomalies? There could be special events in the dataset that are not a part of a longer trend. (Lee and Carter 1992, p. 663) present the case of an influenza epidemic in the United States in 1918, and decide to treat it as a “highly unusual event whose inclusion in the series would inappropriately influence the results”. However, there is no consensus among researchers on coping with data anomalies. In addition, determining events for exclusions is generally not a clear-cut case. Therefore, all data has been included in the calibration of the model.

The historical data that will be used to calibrate the model is now known. Thus, it should at this point be possible to calibrate the model. This will be done in a way similar to Lee and Carter (1992, pp. 661–662).

First, it should be observed that there exist an infinite number of solutions to the model described in Equation (4.3). It is in fact possible to find other solutions in a specific way.

Proposition 4.1 *Let $\alpha(x), \beta(x), \kappa(t_j)$ be such that the defining equation of the Lee-Carter models holds (Equation (4.3)):*

$$\log m(x, t_j) = \alpha(x) + \beta(x)\kappa(t_j). \tag{4.6}$$

Then $\forall c \in \mathbb{R} \setminus \{0\}$:

$$\begin{aligned} \bar{\alpha}(x) &= \alpha(x), & \bar{\beta}(x) &= \frac{\beta(x)}{c}, & \bar{\kappa}(t_j) &= c\kappa(t_j), \quad \textit{and} \\ \hat{\alpha}(x) &= \alpha(x) - c\beta(x), & \hat{\beta}(x) &= \beta(x), & \hat{\kappa}(t_j) &= \kappa(t_j) + c, \end{aligned}$$

also solve Equation (4.6). △

Proof. Suppose $\alpha(x), \beta(x), \kappa(t_j)$ are defined such that Equation (4.6) holds. Then:

$$\bar{\alpha}(x) + \bar{\beta}(x)\bar{\kappa}(t_j) = \alpha(x) + \frac{\beta(x)}{c} c\kappa(t_j) = \alpha(x) + \beta(x)\kappa(t_j),$$

and thus Equation (4.6) still holds with $\bar{\alpha}(x), \bar{\beta}(x), \bar{\kappa}(t_j)$. Similarly:

$$\hat{\alpha}(x) + \hat{\beta}(x)\hat{\kappa}(t_j) = \alpha(x) - c\beta(x) + \beta(x)\kappa(t_j) + c\beta(x) = \alpha(x) + \beta(x)\kappa(t_j).$$

□

Lee and Carter (1992) ensure the solution is unique by imposing additional constraints:

$$\begin{aligned} \sum_{x \in X} \beta(x) &= 1, \\ \sum_{t_j \in T} \kappa(t_j) &= 0. \end{aligned} \tag{4.7}$$

Note that the solution *is* unique under these additional constraints. These constraints imply that $\alpha(x)$ must be equal to the average over time of $\log \hat{m}(x, t)$. Thus, the problem has simplified considerably and now only two unknowns $\beta(x)$ and $\kappa(t_j)$ must still be found.

Lee and Carter (1992, p. 661) propose to find a least squares solution, which can be done using a Singular Value Decomposition (SVD). To this end, let Z be a matrix with:

$$Z_{i,j} = \log \hat{m}(i, t_j) - \alpha(i).$$

Then using the SVD two vectors β, κ can be found such that the matrix defined by the product $\beta \cdot \kappa$ is an approximation of Z. Girosi and King (2007, pp. 6–10) have shown that this approximation by using only the first principal component generally works well for all-cause mortality. For the Netherlands 89% of variance was found to be explained by the model and this percentage is even higher for most other countries. Additionally, this is also a maximum likelihood estimator for this problem. Thus, the proposed SVD approach seems a good way to move forward.

Using the SVD, the parameters $\beta(x), \kappa(t_j)$ are easily estimated. In addition, the parameter $\alpha(x)$ was already defined previously. So, all values are now known. However, values still need to be normalized to ensure that the additional constraints (Equation (4.7)) are met. This is done using the transformations in Proposition 4.1. After this final step the model has been calibrated.

However, there is one caveat. If the total number of deaths in a year is calculated now using the model, this (most likely) will not be equal to the actual observed numbers of deaths. Lee and Carter (1992, p. 661) explain this effect by noting that death rates of young people (that are low) and older people (that are higher) have the same weight, while deaths of younger people make up a smaller amount of total deaths. They propose a second re-estimation stage to ensure the total number of deaths in a year is consistent. This is done by finding a new estimate of $\kappa(t_j)$ for each year while keeping the other parameters constant.

To formally describe this second stage, let $D(x, t_j)$ be the number of observed deaths and $E(x, t_j)$ the exposure to death — the number of people that could have died in the time period — in the time period $(t_{j-1}, t_j]$ for a person with age x. Then $\forall\, t_j \in T$, a $\kappa(t_j)$ should be found such that:

$$\sum_{x \in X} D(x, t_j) - \sum_{x \in X} E(x, t_j) \exp\Big(\alpha(x) + \beta(x)\kappa(t_j)\Big) = 0.$$

This procedure can be implemented using a Newton iteration scheme. It should be noted that the second stage is entirely optional; the model should also give good results without re-estimating this parameter.

It must be noted that this proposed second stage solution is controversial in literature and alternatives have been proposed. A reason for this is that the solution with the new $\kappa(t_j)$ no longer has the maximum likelihood properties. Also, Girosi and King (2007, pp. 26–28) show that $\kappa(t_j)$ does not necessarily have one solution if the $\beta(x)$ have different signs. However, there are not many datasets that have this characteristic since all-cause mortality has generally been decreasing worldwide. Some countries such as Hungary are an exception to this observation (Girosi and King 2007, p. 18), so an implementation should check the signs of the $\beta(x)$ and only re-estimate the $\kappa(t_j)$ if they are all equal.

All procedures that are required to estimate the parameters of the Lee-Carter model have now been described. Figure 4.4 shows the parameters resulting from the Dutch 1967-2016 dataset, including the re-estimation stage. The exact parameters can be found in Appendix A.1.

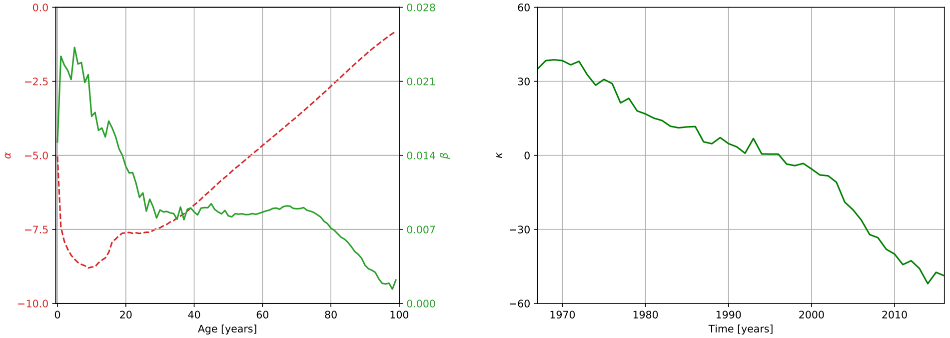
However, one thing still remains: the parameters μ_κ and σ_κ must still be estimated. This is required for forecasting the process $\kappa(t_j)$. Suppose $n+1$ values of $\kappa(t_j)$ are known:

$$\kappa(t_0), \kappa(t_1), \ldots, \kappa(t_n).$$

Then the unbiased maximum likelihood estimators of μ_κ and σ_κ are (Girosi and King 2007, p. 5):

$$\hat{\mu}_\kappa = \frac{\kappa(t_n) - \kappa(t_0)}{t_n - t_0},$$
$$\hat{\sigma}^2_\kappa = \frac{1}{t_n - t_0} \sum_{j=0}^{n-1} \Big(\kappa(t_{j+1}) - \kappa(t_j) - \hat{\mu}_\kappa\Big)^2 \text{[5]}.$$

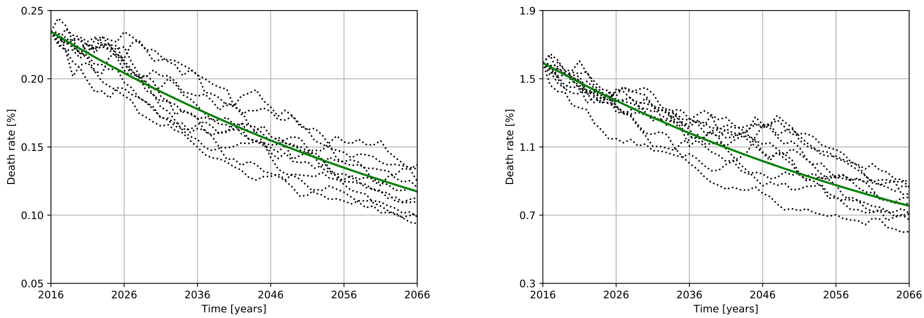
[5]This MLE of σ_κ assumes that $\forall\, j : t_j - t_{j-1} = 1$, which is a reasonable assumptions since equispaced data is under consideration.



(a) α (dashed red, left) and β (solid green, right).

(b) κ.

Figure 4.4: Parameters of the Lee-Carter model, calibrated to mortality data of the Netherlands in the years 1967-2016.



(a) Individuals aged 50.

(b) Individuals aged 70.

Figure 4.5: Development of the 1-year mortality rate under the Lee-Carter model in time. The expected value (solid green) and 10 Monte-Carlo paths (dotted black) have been added for two ages.

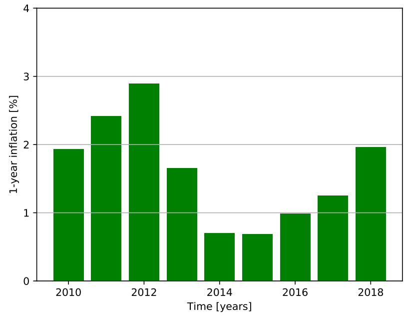
With this final definition, all procedures have been described that are required to calibrate the various parts of the Lee-Carter model. The expected value and Monte-Carlo paths of the process have been added to Figure 4.5.

4.2. Inflation process

Inflation is a concept known to many as ‘the reason why things become more expensive’. Rather than this heuristic it can also be explained in a rigorous mathematical framework — this will be done in this section. In later sections it will be shown that inflation actually impacts pension fund cash flows through its connection with salaries.

Inflation is defined as the change in the CPI over a certain time period. This CPI is a weighted average basket of consumer goods and popular services that consumers can buy. An increase of the CPI means that (most) products become more expensive — this is called *inflation*. A decrease of the CPI conversely means that (most) products become cheaper — this is called *deflation*. The

(a) General CPI as provided by the CBS — a weighted average of the CPI spending categories. The CPI has been normalized so that the *average CPI in 2015* is equal to 100 (standard used by the CBS).



(b) Inflation calculated from the CPI data that has been provided by the CBS. Inflation in a year has been defined as the increase of CPI in December relative to one year earlier[6].

Figure 4.6: Historical inflation in the Netherlands in the period 2010–2019. Data has been provided by the CBS and has been added to Appendix C.3

CPI is generally calculated by the national institute of statistics of a country. In the Netherlands this is done by the Centraal Bureau voor de Statistiek (CBS). In Figure 4.6 the CPI and corresponding yearly inflation have been added for a 10-year period.

Interest rate derivatives have been discussed extensively. Similarly, *inflation-linked derivatives* exist that depend on this CPI. Pension funds can use such derivatives to hedge against inflation increases. This could be necessary if the pension of the fund's participants must be increased with inflation. However, since this is generally not the case for pension funds in the Netherlands such derivatives are not discussed in detail in this thesis.

A model for the CPI is required to price these inflation-linked derivatives and to calculate an expectation of the CPI. Brigo and Mercurio (2007, pp. 643–693) discuss a number of inflation models and how to price inflation-linked derivatives using these models. Many of these discussed frameworks model the instantaneous nominal interest rate — as was done in Section 3.2 — and the real interest rate. This real interest rate is the inflation-adjusted interest rate. The CPI is then modeled as the exchange rate between the two rates, which Brigo and Mercurio (2007, p. 643) compare (in terms of mathematical constructs) with the exchange rate between foreign currencies.

One of the most well-known models for inflation modeling was originally proposed by Jarrow and Yildirim (2003). This model is also discussed by Brigo and Mercurio (2007, pp. 646–647) — these results are used to describe the inflation process in this chapter.

[6] An alternative way to measure inflation in a year is by calculating the increase in *average CPI* during a year. This definition is frequently used by the CBS but not as consistent with the framework employed in this thesis.

Let $n(t)$ be the instantaneous nominal rate, $r(t)$ the instantaneous real rate and $I(t)$ the CPI. Then the dynamics of the quantities under the Jarrow-Yildirim model are[7]:

$$\begin{aligned} \mathrm{d}n(t) &= \lambda_n(\theta_n(t) - n(t))\,\mathrm{d}t && + \eta_n\,\mathrm{d}W_n^{\mathbb{Q}}(t), \\ \mathrm{d}r(t) &= \lambda_r(\theta_r(t) - r(t))\,\mathrm{d}t && - \rho_{rI}\eta_r\eta_I\,\mathrm{d}t + \eta_r\,\mathrm{d}W_r^{\mathbb{Q}}(t), \\ \mathrm{d}I(t) &= (n(t) - r(t))\,I(t)\,\mathrm{d}t && + \eta_I I(t)\,\mathrm{d}W_I^{\mathbb{Q}}(t), \end{aligned}$$

with mean-reversion parameters λ_n, λ_r, volatility parameters η_n, η_r, η_I and Brownian motions $W_n^{\mathbb{Q}}(t), W_r^{\mathbb{Q}}(t), W_I^{\mathbb{Q}}(t)$ that are correlated with correlations parameters $\rho_{nr}, \rho_{nI}, \rho_{rI}$, i.e.:

$$\mathrm{d}W_a^{\mathbb{Q}}(t)\,\mathrm{d}W_b^{\mathbb{Q}}(t) = \rho_{ab}\,\mathrm{d}t, \quad a, b \in \{n, r, I\}.$$

Note that the nominal rate $n(t)$ is equal to the instantaneous interest rate as defined in Chapter 3 and the real rate $r(t)$ is *not equal* to this rate.

The methodology to model inflation has been described. It is interesting to observe that a negative inflation is also possible under this model since the CPI can decrease below 100 — this has become very relevant in recent years and the possibility of model negative inflation is an import characteristic of the model.

Before the inflation model can be used, the processes involved should be calibrated. Note that the nominal interest rate process was already calibrated to instruments that trade in the market in Section 3.2. The real interest rate process $r(t)$ can be calibrated in a similar way.

Let $\hat{P}_r(t_0, t)$ be the market price of a zero-coupon bond on the real interest rate and $\hat{f}_r(t_0, t)$ the corresponding instantaneous forward rate. Then $\theta_r(t)$ is (Brigo and Mercurio 2007, p. 647):

$$\theta_r(t) = \frac{1}{\lambda_r}\frac{\partial}{\partial t}\hat{f}_r(t_0, t) + \hat{f}_r(t_0, t) + \frac{\eta_r^2}{2\lambda_r^2}\left[1 - \exp(-2\lambda_r t)\right],$$

with again $\hat{f}_r(t_0, t) = -\partial/\partial t \log \hat{P}_r(t_0, t)$. However, $\hat{P}_r(t_0, t)$ is unknown. It can be derived from the price of a 'regular' zero-coupon bond on the nominal interest rate and the fixed rate of an inflation-indexed zero-coupon interest rate swap. In such a swap one party pays a fixed rate and the other party pays the inflation over the time period — this is similar to the interest rate swap described in Subsection 2.3.3 and described well by Brigo and Mercurio (2007, p. 648). Let $\hat{P}_n(t_0, t)$ be the market price of a zero-coupon bond on the nominal interest rate and $K(t)$ the fixed rate of an inflation-linked zero-coupon bond at time t. Brigo and Mercurio (2007, p. 651) then find that:

$$\hat{P}_r(t_0, t) = \hat{P}_n(t_0, t)(1 + K(t))^{t - t_0}.$$

Using this formula, the prices of zero-coupon bonds on the real interest rate can be calculated for liquid time points such as the spine points that were used for bootstrapping in Section 3.1 — values $K(t)$ at these liquid time points have been added to Table 4.2. For non-liquid points the curve should be interpolated or extrapolated as was done for the nominal interest rate in Section 3.1.

There are also a number of constants in the Jarrow-Yildirim model that should be calculated. In Section 3.2 the constants of the nominal interest rate process were calibrated using instruments that trade in the market. A similar procedure may be employed to calibrate the real interest rate

[7] This is in fact one specific parametrization of the Jarow-Yildirim model that is proposed by Brigo and Mercurio (2007, p. 646) to ease calculation of the prices of derivatives. For a full description of the model the reader is referred to Jarrow and Yildirim (2003).

Maturity	Fixed rate	Maturity	Fixed rate
1Y	0.71%	10Y	1.04%
2Y	0.74%	12Y	1.10%
3Y	0.79%	15Y	1.17%
4Y	0.84%	20Y	1.28%
5Y	0.88%	25Y	1.35%
6Y	0.91%	30Y	1.39%
7Y	0.94%	40Y	1.42%
8Y	0.97%	50Y	1.44%
9Y	1.01%		

Table 4.2: Fixed rates of inflation-linked zero-coupon swaps, so that the values of the swaps are zero. Quotes from Bloomberg as of 31 October 2019.

process using inflation-linked derivatives. One calibrates by ensuring that the values of these instruments under the model equal their market values. This is explained well by Brigo and Mercurio (2007, pp. 669–672).

This calibration is not performed because inflation is not the focus of this work. Moreover, proper calibration is not elementary. The parameters proposed by Jarrow and Yildirim (2003, p. 30) are therefore used instead:

$$\lambda_r = 4.34\%, \qquad \eta_r = 0.30\%, \qquad \eta_I = 0.87\%,$$
$$\rho_{nr} = 1.48\%, \qquad \rho_{nI} = 6.08\%, \qquad \rho_{rI} = -32.13\%.$$

A calibrated model for the CPI is now available. This can be used to simulate the processes involved as was done in Section 3.2. Moreover, the expected values of these processes can be calculated analytically. To see this, recall that $P_n(t,T)$ was calculated in Subsection 3.2.1. This result can be extended to zero-coupon bonds on the real interest rate as well (Brigo and Mercurio 2007, p. 652). Thus $P_n(t,T)$ and $P_r(t,T)$ are known analytically. The forward-looking CPI at time t_0 can then be expressed using these quantities (Brigo and Mercurio 2007, p. 645):

$$\mathbb{E}^{\mathbb{Q}}\left[I(t) \,\middle|\, \mathscr{F}_{t_0}\right] = I(t_0)\frac{P_r(t_0,t)}{P_n(t_0,t)}.$$

A note of caution should be added to this inflation model: it does not take contemporary monetary policies into account. The European Central Bank (ECB) wields a number of instruments that can impact inflation within the European Union (EU). The ECB in fact aims to “maintain inflation rates below, but close to, 2% over the medium term”. Although the model is calibrated to inflation expectations, this information on the inflation target has not been used explicitly in the model.

4.3. Salary process

Salaries are one of the main drivers of pension cash flows, because contributions and pension rights depend on the salaries. Therefore, scholars have studied the development of salaries in the context of pensions. These results will be discussed and dynamics for the salary process will be proposed. The salary process described in this section models the salary on a full-time basis — a correction fraction for part-time workers will be added in Section 4.4.

Cairns et al. (2000, pp. 135–136) and Cairns et al. (2006, p. 846) model the salary using a drift term, a standard Brownian motion for the relationship between salary increases and asset returns, and another Brownian motion (independent of the first) that incorporates non-hedgeable salary risks:

$$\mathrm{d}S(t) = S(t)\left[\big(r(t)+\mu(t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_S^{\mathbb{P}}(t) + \sigma_E\,\mathrm{d}W_E^{\mathbb{P}}(t)\right], \tag{4.8}$$

where $r(t)$ represents the short-rate, $\mu(t)$ represents the general salary drift and σ and σ_E quantify the non-hedgeable volatility and equity volatility respectively. Non-hedgeable salary risk could entail the risk of regulatory changes that impact salaries. Note that both types of risks are non-hedgeable *in the framework of this thesis*, but the risks associated with asset returns could be hedged in a more extensive framework.

Cairns et al. (2000, p. 136) stress the importance of the time-dependence of the salary process, because the level of salary increases changes with age. They state that a person's salary grows fast when they are young and that growth later in their life is much smaller. During the calibration of the process in Subsection 4.3.1 an analysis of historical data will actually show that these age-dependent effects exist. This time-dependence is indeed still present in the stochastic model proposed in the later work by Cairns et al. (2006, pp. 856–858).

Battocchio and Menoncin (2004, pp. 83–84) build on these ideas and model the stochastic interest rate in a different way:

$$\mathrm{d}S(t) = S(t)\big(r(t)+\mu\big)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W_S^{\mathbb{P}}(t) + \kappa_r\sigma_r\,\mathrm{d}W_r^{\mathbb{P}}(t) + \kappa_E\sigma_E\,\mathrm{d}W_E^{\mathbb{P}}(t).$$

where μ is a constant drift and κ_r and κ_E are measures of the salary process' relation with the interest rate and stock returns respectively. This model is clearly very similar to the earlier model in Equation (4.8).

It should be noted that these models define a relation between the risk-free rate (or equivalently money-market account) and salary growth so that a higher risk-free rate implies a higher salary growth. However, direct evidence of this relation is not readily available. From a conceptual perspective it is also not evident that a higher return on a bank account necessarily results in an accelerating salary growth. More natural would be a relation with inflation: if the prices of products increase one would expect that people also earn more money so that they are still able to afford these products. Similarly, salary increases would increase the costs of the resulting products, which would result in more inflation.

Researchers have attempted to assess whether a relation between inflation and salary changes exists and whether it can be quantified. Contemporary empirical results are inconclusive on a global scale and lead to different conclusions. Peneva and Rudd (2017) find that salary changes have very little effect on inflation in the United States of America (USA) and that such a relation might be absent altogether. Even though roughly two-thirds of the production costs of companies in the USA are comprised of labor costs, they find that salary changes have not had a material impact on the inflation level in the USA in recent years. Bobeica et al. (2019) reach a very different conclusion for the eurozone. In their study of four major European economies[8] they find a clear connection between the cost of labor and the inflation level.

For the purpose of modeling salaries in the framework of this thesis, Equation (4.8) is adapted. Based on the research of inflation, the interest rate process $r(t)$ as proposed in the models of Cairns et al. (2000), Cairns et al. (2006), and Battocchio and Menoncin (2004) is replaced by the instantaneous rate of inflation $i(t)$ that was calculated from the inflation $I(t)$ in Section 4.2. A strong

[8]The major European economies that have been studied are Germany, France, Italy and Spain.

relation is assumed, i.e. a one-on-one relation — the validity of this assumption is discussed in more detail in Section 4.7. Under the current model, the investment universe is relatively small and no investments in equity or even corporate bonds are allowed. Hedging using equity is thus impossible and therefore the equity risk in Equation (4.8) is modeled as a non-hedgeable risk. Thus the stochastic differential equation to model salary development becomes[9]:

$$\mathrm{d}S_j(t) = \left(i(t) + \mu_{S_j}(t)\right) S_j(t)\,\mathrm{d}t + \sigma_S S_j(t)\,\mathrm{d}W_S^{\mathbb{P}}(t). \tag{4.9}$$

Observe that one Brownian motion drives all salary volatilities. This Brownian motion is independent of the Lee-Carter Brownian motion $W_\kappa^{\mathbb{P}}$ and the Brownian motions driving the inflation process.

This salary process exhibits a two-fold trend: the trend is driven partly by market effects (through inflation) and partly by a time-dependent salary trend. Not only the previously described scholars have identified these effects — this is a common idea in the research of salary models. Scholars such as Dondi (2005, pp. 30–34) and Carriere and Shand (1998, pp. 189–192) describe this phenomenon as well. The former in fact gives empirical evidence for such a model.

A notable feature of the model is that salary develops according to a log-normal distribution as with any geometric Brownian motion. This ensures that a person's salary is always positive, which is clearly an important characteristic.

The dynamics of the salary process have now been set, so that it is possible to simulate the process. The required parameters will be found from historical data in Subsection 4.3.1. In addition to simulation, the expected value may also be calculated analytically.

To calculate the expected value, note that a theoretical solution for this geometric Brownian motion is:

$$\begin{aligned} S_j(t) = \underbrace{S_j(t_0)}_{\text{Current salary}} &\cdot \underbrace{\exp\left(\int_{t_0}^t i(u)\,\mathrm{d}u\right)}_{\text{Inflation factor}} \cdot \underbrace{\exp\left(\int_{t_0}^t \mu_{S_j}(u)\,\mathrm{d}u\right)}_{\text{Age-related factor}} \\ &\underbrace{\cdot \exp\left(-\frac{\sigma_S^2}{2}(t-t_0) + \sigma_S\left(W_S^{\mathbb{P}}(t) - W_S^{\mathbb{P}}(t_0)\right)\right)}_{\text{Random factor}}. \end{aligned} \tag{4.10}$$

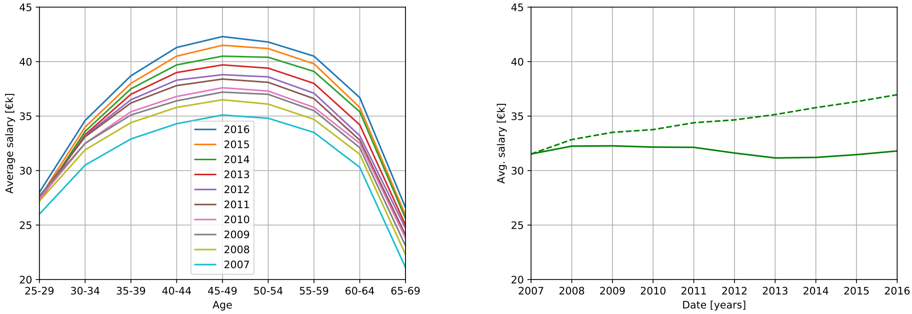
Observe that the factors are independent and the expected value of $S_j(t_0)$ is therefore a product of expected values. Note that $S_j(t_0)$ is a known deterministic quantity and the age-related factor is a to be calibrated deterministic function (see Subsection 4.3.1) — these factors can be calculated. Furthermore, the expected value of the random factor equals one, which follows from the log-normal distribution of the factor. In Section 4.2 the inflation expectation $\mathbb{E}^{\mathbb{Q}}\left[I(t)/I(t_0) \,\middle|\, \mathscr{F}_{t_0}\right]$ was found. This is taken as an approximation of the same expected value with respect to the real-world measure:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_{t_0}^t i(u)\,\mathrm{d}u\right) \,\middle|\, \mathscr{F}_{t_0}\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{I(t)}{I(t_0)} \,\middle|\, \mathscr{F}_{t_0}\right] \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{I(t)}{I(t_0)} \,\middle|\, \mathscr{F}_{t_0}\right].$$

The expected value can then be calculated analytically:

$$\mathbb{E}^{\mathbb{P}}\left[S_i(t) \,\middle|\, \mathscr{F}_{t_0}\right] = S_i(t_0) \cdot \mathbb{E}^{\mathbb{P}}\left[\frac{I(t)}{I(t_0)} \,\middle|\, \mathscr{F}_{t_0}\right] \cdot \exp\left(\int_{t_0}^t \mu_{S_i}(u)\,\mathrm{d}u\right). \tag{4.11}$$

[9] Subscripts have been added to clarify that salaries and some parameters differ per person. Subscript j is used for a person instead of i to avoid confusion with the instantaneous rate of inflation.



(a) Average salary of people in a number of age groups for various years.

(b) Average salary throughout time — nominal (dashed) and inflation-adjusted (solid).

Figure 4.7: Average salaries in the Netherlands for people aged 25–69 in the period 2007–2016. Source of the data is the CBS; more information can be found in Appendix C.4.

4.3.1. Calibration to historical data

In Equation (4.10) a theoretical solution of the salary process geometric Brownian motion was given. In this, the age-related factor $\exp\left(\int_{t_0}^{t} \mu_{S_i}(s)\,\mathrm{d}s\right)$ and parameter σ_S still had to be calibrated to historical data. This will now be done.

Battocchio and Menoncin (2004, p. 93) set $\mu_{S_i}(s) = \mu_S = 0.01$ and $\sigma_S = 1\%$. However, no substantiation for these values is given. Cairns et al. (2006, pp. 851, 854) set $\mu_{S_i}(s) = 0$ and $\sigma_S = 5\%$, stating that these values are consistent with salary data of the United Kingdom. They explain the zero drift by the observation that long-term salary increases are mainly driven by inflation. However, this overlooks the age-dependent effects that they describe in an earlier paper (Cairns et al. 2000, p. 136) and that are also decribed by scholars such as Dondi (2005, pp. 30–34) and Carriere and Shand (1998, pp. 189–192).

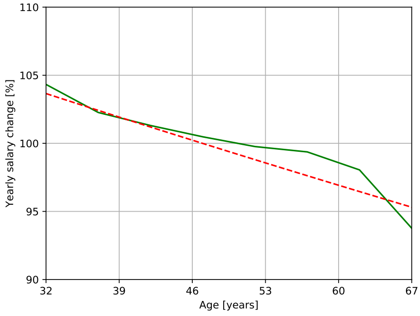
Dutch salary data is therefore studied, to assess whether age-dependent effects that scholars describe can also be identified in recent Dutch datasets and how this should be incorporated in the model.

This Dutch dataset in fact substantiates earlier statements in the case of the Netherlands. Figure 4.7a shows that salary changes exhibit age-dependent effects: a person's salary usually increases up to a certain age, and then decreases. So, salary increases are very much dependent on a person's age. Figure 4.7b shows that long-term inflation-adjusted average salaries of all ages do not exhibit a trend with time and stay rather constant. It is now possible to calibrate the model.

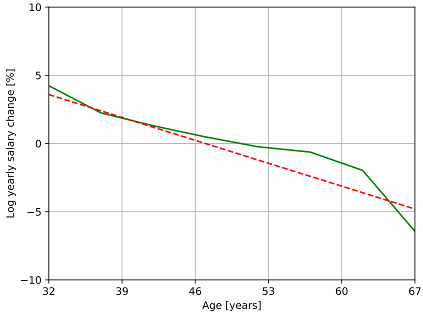
To calibrate the model, first the age-dependent effect will be separated from the fund's participants. The function $\mu_{S_i}(t)$ is defined for a single person in terms of time. However, since age-dependent effects are modelled, this trend may be captured by a general function in terms of age. To this end let $\nu(a)$ be the (age-dependent part of the) relative salary change from age $a-1$ to age a. Then:

$$\exp\left(\int_{t_{j-1}}^{t_j} \mu_{S_i}(s)\,\mathrm{d}s\right) = \nu(a_i(t_j)) \implies \exp\left(\int_{t_0}^{t_k} \mu_{S_i}(s)\,\mathrm{d}s\right) = \prod_{j=1}^{k} \nu(a_i(t_j)),$$

with $a_i(t_j)$ the age of person i at time t_j. The relative change is considered, to ensure the function is invariant from the salary itself. Figure 4.8 shows these salary changes in the most recent available



(a) Relative salary change in a year, represented by $\exp\left(\int_{t_0}^{t} \mu_{S_i}(s)\,\mathrm{d}s\right)$ in the process.



(b) Logarithm of the relative salary change in a year, represented by $\int_{t_0}^{t} \mu_{S_i}(s)\,\mathrm{d}s$ in the process.

Figure 4.8: Dynamics of age-dependent salary changes in the Netherlands in 2016. Data by the CBS (solid green) has been plotted alongside the model estimation (dashed red). More information on the CBS data can be found in Appendix C.4.

year available (2016). Note that the process has been discretized — since salaries are not increased countinously but at certain time intervals this discretization is very reasonable.

This function $v(a)$ would be approximated well by a linear function, judging by Figure 4.8. Looking at the process solution, in which the age-dependent part is an exponential, the logarithm of this factor could be described by a linear function. Therefore, c_1, c_2 should be found such that:

$$v(a) = \exp(c_1 \cdot a + c_2) \iff \log v(a) = c_1 \cdot a + c_2.$$

These parameters are estimated using a least squares method, which yields:

$$c_1 = -0.0024, \qquad c_2 = 0.1126,$$

with standard deviations $\sigma_{c_1} = 0.00015, \sigma_{c_2} = 0.00736$ and coefficient of determination $R^2 = 87\%$. These parameters are in fact rather stable throughout the years, as may be seen in Appendix A.2.

Combining these results, gives:

$$\exp\left(\int_{t_0}^{t_k} \mu_{S_i}(s)\,\mathrm{d}s\right) \approx \exp\left[c_1\left(\sum_{j=1}^{k} a_i(t_j)\right) + c_2 k\right].$$

Thus, the trend of the process has been fully determined.

The estimate of the age-dependent factor seems realistic: salary increases most early in one's career and decreases as one nears the pensionable age. This decrease is explained by the trend that people generally work less as they age, i.e. older people are more likely to work part-time. It should be noted that it is unrealistic that $v(a)$ tends to zero for large a: one would never have a 100% salary decrease in one year. However, this is not a problem because this only happens for very large a — in fact $\log v(a) = 50\% \iff a \approx 336$ in the calibrated model. Still a compounded salary increase would ensure the salary tends to zero for large a as one expects.

The volatility σ_S may be estimated as the sample standard deviation of the historical log returns, as is customary for a geometric Brownian motion. Because this is the volatility of the entire population — since there is one Brownian motion only — this may be estimated from the overall average salary. This yields $\sigma_S = 1.14\%$, which is similar to the volatility that Battocchio and Menoncin (2004) proposed ($\sigma_S = 1\%$).

A procedure has now been described to calibrate the model to historical data.

4.4. Contribution base process

In the Netherlands every individual receives a public (first pillar) pension from a certain age. Pensions in pillar two are supplementary to this pension. In an average wage defined benefits scheme, the pensions from these two pillars should be combined to give the target income. This is ensured through a contribution base, which is calculated by subtracting a 'franchise' from the gross salary. In addition, people working part-time should obviously receive less pension. For these reasons, the pension contribution of person i is not calculated based on solely the gross salary, but instead on a contribution base B_i over the interval $(t_{j-1}, t_j]$. This can be described mathematically, similar to Draper, Armstrong, et al. (2007, p. 38):

$$B_i(t_j) = \left(S_i(t_j) - f\right)\theta_i, \tag{4.12}$$

with:

$$\begin{aligned} S_i(t_j) &: \text{Gross salary of person } i \text{ over time } (t_{j-1}, t_j], \\ f &: \text{Franchise}, \\ \theta_i &: \text{Part-time correction fraction}, \theta \in [0,1]. \end{aligned}$$

The gross salary has been defined in Section 4.3 and the constant θ_i should be known to the pension fund. The constant f depends on the pension plan and the AOW a person will receive. For a plan that pays 70% of the average wage and a married individual, $f = 14,770$[10] in 2019.

In Section 4.3 it was stated that the salary of people on average decreases with age because older people generally work less. However, this effect is also modeled explicitly by the part-time correction fraction θ_i. This begs the questions why this fraction is not modeled as a time-dependent quantity that incorporates the age-dependent effects on θ_i. Unfortunately insufficient data is available to do this. Therefore θ_i is perceived as the *initial part-time correction fraction* and any time-dependent changes are modeled in the salary process.

The expected value is calculated using the expected value of the salary process:

$$\mathbb{E}^{\mathbb{P}}\left[B_i(t_j) \,|\, \mathscr{F}_{t_0}\right] = \left(\mathbb{E}^{\mathbb{P}}[S_i(t_j) \,|\, \mathscr{F}_{t_0}] - f\right)\theta_i. \tag{4.13}$$

[10] The franchise for a plan that pays 70% of the average wage is calculated by: $f = \frac{100}{70}(\text{AOW} + \text{holiday})$. Using the monthly AOW (€809.81) and holiday allowance (€51.75) of a married individual that have been published by the government (https://www.rijksoverheid.nl/ministeries/ministerie-van-sociale-zaken-en-werkgelegenheid/documenten/regelingen/2018/11/19/rekenregels-1-januari-2019-incl.-bijlage-i.1-en-i.2), this yields a franchise of €14,770.

4.5. Pension right process

Using the contribution base B_i of person i (Equation (4.12)), the pension right of a person at any time t will now be defined.

In an average wage defined benefits scheme, a person gains a fixed percentage of his/her contribution base in pension right every period (Draper, Armstrong, et al. 2007, p. 38). This pension right is the future yearly pension benefit — this amount is paid out from retirement until death. When the person is retired the pension right may still grow, but this is then no longer due to contributions but solely due to indexations. In line with Draper, Armstrong, et al. (2007), the pension right $P_i(t_j)$ of a person i at time t_j is defined by:

$$P_i(t_j) = \begin{cases} P_i(t_{j-1})\gamma(t_j) + \delta B_i(t_j), & t_j \leq T_i, \quad \text{(up to the pension)} \\ P_i(t_{j-1})\gamma(t_j), & t_j > T_i, \quad \text{(after the pension)} \end{cases} \tag{4.14}$$

with $P_i(t_0)$ known and:

$$\gamma(t_j): \text{Indexation over time } (t_{j-1}, t_j],$$
$$\delta: \text{Fixed contribution percentage}^{11},$$
$$T_i: \text{Pension time of person } i^{12}.$$

In addition to $\gamma(t_j)$, the notation $\gamma(t_j, t_k)$ is used to denote the total indexation over the time interval $(t_j, t_k]$.

The indexation would in reality depend on many factors, such as the coverage ratio of the fund and inflation. Because pension funds are not under an obligation to increase the pension rights, this number is very uncertain and it would not be realistic to use the inflation process. In fact pension funds are only allowed to index pension if their coverage ratio is above 110%. This is also the reason that only a minority of Dutch pension fund participants have had their pension indexed in recent years and that the average indexation in such cases is small, as can be seen in Table 4.3.

If one makes the assumption that the pension fund is *healthy* — i.e. its VEV is as required — it is realistic to expect indexation will happen. Therefore a constant indexation of 1% per year has been assumed so that $\gamma(t_j) = 1 + 0.01(t_j - t_{j-1})$. This number is lower than the inflation target of the ECB (close to 2%, European Central Bank (n.d.)[13]). Still, this indexation is significantly higher than actual indexations by pension funds in the Netherlands as can be seen in Table 4.3.

The contribution percentage δ determines the size of the pension right at the pensionable age. If δ is large, the pension right could be 100% of the average wage. In the Netherlands, a contribution percentage of $\delta = 1.75\%$ is common[15]. Assuming a working life of 40 years, this yields a pension right of 70% of the average salary since $(40 \cdot 1.75)\% = 70\%$.

[11] This δ is only constant if the discretization is equispaced: if $\exists \Delta t, \forall j : t_j - t_{j-1} = \Delta t$.

[12] The pension time T_i is a constant and not a random variable.

[13] The latest available full-year inflation in the Netherlands was 1.96% over the year 2018, which was published by the CBS. This number originates from a dataset that has been added to Appendix C.3 and was used in Sections 4.2 and 4.3 as well. Note that this is very similar to the target of the ECB.

[14] The full dataset can be found at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-jaar/dataset/78c1c804-0b65-4bbc-a5cc-df9cd75c9ded.

[15] According to data published by the DNB at https://statistiek.dnb.nl/downloads/index.aspx#/details/pensioenreglementen-naar-opbouwpercentage/dataset/90faddc7-1c14-4fea-b1ba-8ece114405f9, the great majority of DB pension schemes have a contribution percentage between 1.5% and 2.0%. Many pension schemes set $\delta = 1.75\%$, which has therefore been used.

Year	Funds with indexation	Participants with indexation	Average indexation weighted by participants
2014	51%	47%	0.55%
2015	39%	12%	0.44%
2016	26%	10%	0.61%
2017	43%	18%	0.47%
2018	53%	28%	0.55%

Table 4.3: Indexation statistics in the Netherlands calculated from a dataset provided by DNB[14]. Average indexation is the percentage in the case that indexation happens, i.e. only funds that have indexed pensions in the year contribute to the average.

The pension time T_i differs per person, but one would expect that the pension age is equal for all people. However, the pension age in the Netherlands is dependent on life expectancy and is therefore expected to rise in the future. There are discussions to change these regulations again, but this is still uncertain. In the model, the pension age has therefore been set to 69 years for all people.

An analytical result for the expected value will now be derived. The recursive formula for the pension right (Equation (4.14)) is easily rewritten into a direct function:

$$P_i(t_j) = P_i(t_0)\gamma(t_0, t_j) + \sum_{t=t_1}^{\min\{t_j, T_i\}} \delta B_i(t)\gamma(t, t_j).$$

Observe that $B_i(t_j)$ is the only stochastic quantity in this formula. Therefore, the expected value follows:

$$\mathbb{E}^{\mathbb{P}}\left[P_i(t_j) \,\middle|\, \mathscr{F}_{t_0}\right] = P_i(t_0)\gamma(t_0, t_j) + \sum_{t=t_1}^{\min\{t_j, T_i\}} \delta \mathbb{E}^{\mathbb{P}}\left[B_i(t) \,\middle|\, \mathscr{F}_{t_0}\right]\gamma(t, t_j). \tag{4.15}$$

4.6. Pension cash flow process

All quantities that are required to describe the pension cash flows have been given. These cash flows will now be defined using concepts from Draper, Armstrong, et al. (2007) and Van Rooij et al. (2004).

Consider cash flow $L_i(t_j)$ of a pension fund with respect to person i at time t_j. This cash flow consists of contributions that the fund receives, and benefits that the fund pays out. These cash flows are zero if the person is no longer alive.

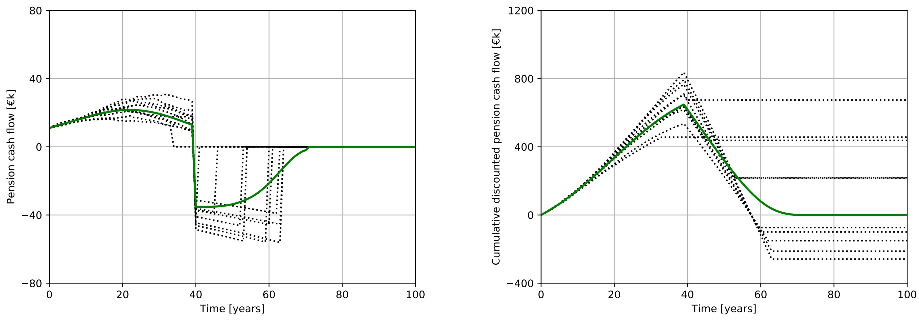
Therefore, the definition of the pension cash flow takes into account the pension age, the lifetime indicator, and the pensions rights:

$$L_i(t_j) = \underbrace{\epsilon(t_j)\left[\delta B_i(t_j)\right] \cdot A_i(t_j) \cdot \mathbb{1}_{t_j \leq T_i}}_{\text{contributions}} - \underbrace{P_i(t_j) \cdot A_i(t_j) \cdot \mathbb{1}_{t_j > T_i}}_{\text{benefits}}, \tag{4.16}$$

with:

$$\epsilon(t_j) : \text{Pension fund accrual factor at time } t_j.$$

One new factor in the formula is the accrual factor $\epsilon(t_j)$. This is required because the amount of money person i has to pay to receive the additional pension benefit $\delta B_i(t_j)$ is not simply this



(a) Pension cash flow process — repeated from Figure 4.2.

(b) Cumulative discounted yearly pension cash flow — the expected value equals zero while a profit/loss is made for individual paths.

Figure 4.9: Analytical expected value (solid green) and 10 Monte-Carlo paths (dotted black) for the pension cash flow process. Results are for a 30-year individual with a gross salary of €40,000 working full-time without initial pension right (at $t = 0$).

amount. The pension benefit is a yearly payout from the start of the pension to death, so more money should be paid for this right. The accrual factor is the amount of money that should be paid for one euro of pension benefit from the moment of retirement until death. This quantity should take the time value of money and expectations of the lifetime, salary and pension rights of individuals into account.

The accrual factor only depends on time and not on the person. In a defined benefits scheme there is generally one accrual factor for the entire fund (Draper, Armstrong, et al. 2007, p. 39): the percentage of the contribution base that should be paid is the same number for all members of the fund. In Subsection 4.6.1, this number is discussed in more detail and its expected value is calculated by looking at the expected value of future pension rights.

The expected value of the pension cash flow process of an individual may now be calculated:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[L_i(t_j) \mid \mathscr{F}_{t_0}] = \epsilon(t_j)\mathbb{E}^{\mathbb{P}}\left[\delta B_i(t_j) \mid \mathscr{F}_{t_0}\right]\mathbb{E}^{\mathbb{P}}\left[A_i(t_j) \mid \mathscr{F}_{t_0}\right] \cdot \mathbb{1}_{t_j \le T_i} \\ - \mathbb{E}^{\mathbb{P}}\left[P_i(t_j) \mid \mathscr{F}_{t_0}\right] \cdot \mathbb{E}^{\mathbb{P}}\left[A_i(t_j) \mid \mathscr{F}_{t_0}\right] \cdot \mathbb{1}_{t_j > T_i},\end{aligned} \tag{4.17}$$

by the independence of the processes $B_i(t_j)$, $A_i(t_j)$ and $P_i(t_j)$. In addition it has been used that $\epsilon(t_j)$ may be considered a scalar — in Subsection 4.6.1 $\epsilon(t_j)$ will be defined properly and this will follow from its definition.

This accrual factor $\epsilon(t_j)$ is of the utmost importance for a pension fund: if it is too low the pension fund might not be able to pay the pensions of all of its participants but if it is too high the pension fund might make a profit while there is then no longer a beneficiary to receive this profit. Therefore an accrual factor should be carefully calculated so that the cumulative discounted cash flows of the pension fund are equal to zero. The lifetime risk of people is then shared so that there is on average sufficient money — the main purpose of a pension fund. In Figure 4.9 it can be seen that this is indeed the case for the described model.

It now possible to find the main result of this chapter: the pension cash flows of an entire pension fund. These cash flows are easily derived from the pension cash flows of individuals — the cash flows of a pension fund as a whole are simply the total cash flows of all individual participants.

Suppose there are m people in the fund. The cash flows $L(t_j)$ of the entire fund are:

$$L(t_j) = \sum_{i=1}^{m} L_i(t_j). \tag{4.18}$$

Observe that it is trivial to calculate $\mathbb{E}^{\mathbb{P}}\left[L(t_j) \,\middle|\, \mathscr{F}_{t_0}\right]$ using the results of this chapter.

Now, the cash flows of a pension fund with an average wage defined benefits scheme have been fully described. In addition, both analytical and simulated results have been given.

4.6.1. Accrual factor

The accrual factor $\epsilon(t_j)$ determines the amount of money that needs to be paid now to receive a certain pension benefit from the pension age onwards. It is the cost of buying a one euro cash flow from the pension age until death. Clearly, this depends on mortality rates and expected discount factors (based on the money-market account). Van Rooij et al. (2004, p. 14) explain that the accrual factor in reality also depends on the coverage ratio (and thereby actual asset returns). If there is more money in the pension fund less money is required to fund the future benefits and thus lower contributions are accepted. However, in this model a constant asset return is simply assumed.

Multiple methodologies for calculating the accrual factor exist that differ significantly. This has been described by Michielsen et al. (2015); these results have been summarized here.

One method is *uniform accrual*, which is generally used in the Dutch pension system for defined benefit schemes. Under this scheme every person in the fund receives a *nominal* pension right as a percentage of the contribution base. In this case an old person receives the same nominal pension right (percentage-wise) as a young person: the accrual factor is equal for all people in the pension fund. Michielsen et al. (2015, p. 7) point out that this methodology does not take into account that the contributions of young people are paid before years of inflation and potential asset returns while this can only accumulate for a few years for older people. One could say this is an unfair method that transfers wealth from younger people in the fund to the older people. Particularly in the case of (large) age differences in the fund, the younger participants pay for the pension benefits of the older generation.

The alternative method, *actuarially fair accrual*, ensures that the pension right of a person equals the discounted value of the accrued rights. In this case the accrual factor differs per person. This scheme is closer to a defined contribution scheme and one could argue that this is the fairer scheme. However, the uniform accrual scheme is generally used. Therefore this scheme has been implemented and is described in the remainder of this chapter.

The results in this chapter could be adapted to an actuarially fair accrual scheme. The accrual factor would then differ per participant and would become more dependent on mortality and interest rates. The cash flow of the pension fund as a whole would not change and the results in Figures 4.2 and 4.9 would consequently remain unaltered.

The accrual factor will now be described. The goal is to find a factor $\epsilon(t_j)$ such that every person i in the fund pays:

$$\epsilon(t_j) \cdot \left[\delta B_i(t_j)\right],$$

to receive a pension right $\delta B_i(t_j)$, with δ fixed contribution percentage as defined in Section 4.5. This pension right will be paid in every period from retirement until death. Note that the factor $\epsilon(t_j)$ is the same for all people in the fund.

Let $X_i(t_j,t)$ be the benefit that person i receives at time t in exchange for contribution $\delta B_i(t_j)$ at time t_j:

$$X_i(t_j,t) = \mathbb{1}_{t_j \le T_i < t}\left[\delta B_i(t_j)\right] A_i(t)\gamma(t_j,t).$$

The indicator reflects the constraint that benefits are only built up before the pension starts and only paid out when the pension has started. The accrual factor may now be defined as the fraction of the present value of expected future pension benefits (of the entire fund) and total contributions, similar to Draper, Armstrong, et al. (2007, p. 39):

$$\epsilon(t_j) = \frac{\sum_{t=t_j}^{T} \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_j)}{M(t)}\sum_{i=1}^{n}\mathbb{E}^{\mathbb{P}}\left[X_i(t_j,t)\mid \mathcal{F}_{t_0}\right]\mid \mathcal{F}_{t_0}\right]}{\sum_{i=1}^{n}\mathbb{1}_{t_j\le T_i}\mathbb{E}^{\mathbb{P}}\left[\delta B_i(t_j)A_i(t_j)\mid \mathcal{F}_{t_0}\right]}, \tag{4.19}$$

with T the time at which the 1-period survival probability of all people in the pension fund equals zero with certainty [16], i.e.:

$$T \in \{t_1,t_2,\ldots\} : \mathbb{P}\left(A_i(T)=0\right) = 1, \forall i \in \{1,\ldots,n\}.$$

The expected values in Equation (4.19) can be calculated analytically using the results in this chapter. To see this, note that by the independence of the processes:

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{t_j\le T_i<t}\left(\delta B_i(t_j)\right)A_i(t)\gamma(t_j,t)\mid \mathcal{F}_{t_0}\right] = \mathbb{1}_{t_j\le T_i<t}\delta\,\mathbb{E}^{\mathbb{P}}\left[B_i(t_j)\mid \mathcal{F}_{t_0}\right]\mathbb{E}^{\mathbb{P}}\left[A_i(t)\mid \mathcal{F}_{t_0}\right]\gamma(t_j,t)$$

where the remaining expected values can be calculated using the results in Sections 4.1 and 4.4. The discounted value of this cash flow can then be calculated using the results from Section 3.1. Thus the numerator of Equation (4.19) can be calculated. In a similar way the denominator can be calculated.

The accrual factor is now fully defined. It should be noted that this accrual factor is a conditional expectation on the current time t_0 rather than the time at which it is used t_j. This is done because pension funds are in practice rather hesitant to (significantly) change the accrual factor because this would immediately impact the contribution that should be paid — both participants and their employers would be very unhappy with such a decision. Therefore the accrual factor is set in advance for the entire simulation period. This has the added benefit of considerably simplifying calculations.

One might have expected a factor for stock returns in this formula that incorporates expected returns in excess of the risk-free rate. Pension funds are indeed allowed to make other investments besides a bank account or bonds, but at the same time the regulator DNB prescribes that all future pension liabilities must be valued in a manner based on the risk-free rate. This has been explained in more detail in Section 2.2. The inclusion of an expected excess return of other asset classes in this accrual factor would thus have a negative impact on the coverage ratio of the pension fund, since the present value of the future pension liabilities would increase more than the contribution. The confidence of a pension in attaining additional results does not change this.

Some pension funds have indeed chosen an accrual factor that is lower than strictly required. The pension contributions of the ABP in 2018 for instance covered only 78% of the increase in pension

[16] Recall from Subsection 4.1.1 that it is assumed that every person dies at age 100 due to a lack of data.

Coverage ratio of the contributions	Number of pension funds
40% – 80%	24
80% – 100%	82
100% – 120%	53
120% – 160%	25

Table 4.4: Numbers on whether pension funds in The Netherlands were able to finance new pension benefits from the corresponding contributions in 2018. If the coverage ratio of the contributions is greater than 100% the coverage ratio of the pension fund as a whole increases from the contribution; if it is smaller than 100% it decreases. DNB has published the full dataset[17] at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-jaar/dataset/78c1c804-0b65-4bbc-a5cc-df9cd75c9ded.

liabilities arising from these contributions, which directly resulted in lower coverage ratios (Algemeen Burgerlijk Pensioenfonds 2019, pp. 23–24). In Table 4.4 it can be seen that this is the case for the majority of Dutch pension funds. Therefore there have been discussions recently to allow the inclusion of expected excess returns in the calculations of pension liabilities as well, e.g. based on the 10-year average historical returns (Wolzak 2019b). The coverage ratio would then no longer decrease if expected returns are included in the calculation of the accrual factor. However, it has been decided to implement the 'conservative' approach in this thesis that does not include excess returns of risky asset classes.

4.7. Practical considerations and assumptions

Modeling of pension cash flows is a hard task. A lot of assumptions must be made and often not all required data is available. Therefore, actuaries usually estimate future pension cash flows for funds. They provide the expected value of the fund's cash flows $\mathbb{E}\left[L(t_j)\,|\,\mathscr{F}_{t_0}\right]$ (Equation (4.18)), with the t_j defined as the yearly discretization. However, actuaries do not describe the dynamics of the stochastic processes involved. The model of this chapter does fully describe these processes, which is a very useful contribution.

In the model of this chapter, some assumptions and simplifications have been made:

Benefits Solely old age pension benefits are modeled. In reality, such benefits account for only about two thirds of total benefit payments (Van Rooij et al. 2004). A significant amount (about 23%) is paid to *surviving relatives* (widows, young children), and smaller amounts are paid for *disablement pensions* (about 5%) and *early pensions* (about 5%). The latter could be modeled using an option pricing approach inspired by Chen et al. (2017). However, adding these different types of pensions is not an easy task. Therefore, these effects have now not been taken into account. Van Rooij et al. (2004) suggest to multiply any result by a constant factor (of 1.5, Van Rooij et al. (2004, p. 14)), however, this has not been done thus far.

Franchise The pension franchise f is constant and not increasing over time. More realistic would be for instance to relate it to inflation or the increases in wages.

Indexation An indexation function $\gamma(t)$ has been used in Sections 4.5 and 4.6. In reality indexation would not be determined by a deterministic function but would depend on asset returns and inflation. A pension fund will only index its liabilities if it has sufficient assets. An

[17] Six of the 190 pension funds in the dataset provided by DNB have been excluded, because the data was not representative and the coverage ratio of the contributions was outside the set ranges (either smaller than 40% or greater than 160%).

alternative to this elaborate approach would be to choose a realistic indexation parameter such as the average historical inflation or the yearly salary increase — this has been done now. However, this simple indexation methodology may be improved.

Correlation In this chapter is has been assumed that most processes are uncorrelated, even though this might not necessarily be the case in reality. Although the well-researched correlation between the interest rate and inflation has been taken into account, more research could be done into the potential correlations between for instance the salary and lifetime processes.

Costs All pension funds make costs for administrative tasks and asset management. In 2018, the median annual administrative costs of pension funds were around €300 per participant and the cost of asset management was around 0.4% of the money invested[18]. It is mathematically not complicated to incorporate these costs into the model, but these costs are at the moment not taken into account.

Income The income of all people in the fund is simply modeled using a salary process. People can in reality also stop working, at which point the salary process would no longer be an accurate representation of their income. Van Rooij et al. (2004, p. 10) take this effect into account by also modeling the participation of people in the labor market. Specifically, they incorporate that people work less each year after they become 28. This could explain (part of) the downward salary trend of older people. Regardless, the assumption has been made that the salary process itself represents the future income of people along with the part-time correction fraction θ_i (at time t_0).

Mortality The mortality rate is modeled in a simple way, while scholars have proposed many improvements (for instance in Van Berkum et al. (2016)). One simple improvement would be to model men and woman separately. However, none of this has been done.

New participants People joining the fund are not modeled. Building a reliable model for this is very challenging, since it is not elementary to model the behavior of people that could join the pension fund in the future. Furthermore, if one assumes that participants in a pension fund pay for their own future pension, future participants would not be an interesting addition to the pension cash flow model.

Pension age The time T_i at which a person reaches the pension age is set to a constant number, which is equal for all people in the fund. In reality, the age at which a person receives the first pillar pension (AOW) differs depending on the date of birth, and for every person born after October 1st, 1956 this date is currently uncertain (Belastingdienst 2019) but likely to increase. Michielsen et al. (2015) model this increasing age. In addition, pension funds may offer the possibility to start the pension earlier at a lower benefit. So, modeling the time T correctly is not an easy task. In this model, the pension age of each individual has been fixed.

Salary and inflation One of the main drivers of the salary is the *inflation*, while scholars such as Cairns et al. (2000), Battocchio and Menoncin (2004), and Cairns et al. (2006) model a relation between the interest rate and salary. Although this decision has been substantiated, more research could be beneficial. In addition, a one-on-one relation between salary and inflation changes has been assumed. Bobeica et al. (2019) find that this relation is usually smaller and in fact depends on the level of inflation: a higher inflation implies a stronger connection.

[18]DNB has published a dataset from which the costs can be calculated at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-jaar/dataset/78c1c804-0b65-4bbc-a5cc-df9cd75c9ded

Salary dynamics The salary process has been assumed to follow a geometric Brownian motion. These are also the dynamics that scholars such as Cairns et al. (2000), Battocchio and Menoncin (2004), and Cairns et al. (2006) assume for this process. However, with additional research it might be possible to further substantiate these dynamics for the salary process.

It is not elementary to order these assumptions and simplifications by priority: additional research is required to assess the potential error that they might cause. However, the first three points — *benefits, franchise and indexation* — are deemed the most significant. The suggestion by Van Rooij et al. (2004, p. 14) to multiply old age pension benefits by 1.5 shows the potential error due to this simplification. Additionally, the pension franchise is related to the average salary in the Netherlands via the height of the AOW — modeling it as a constant is thus a significant simplification. Indexation is particularly relevant in the Netherlands at this moment and a more elaborate inclusions of indexation dynamics could greatly benefit the applicability of the model.

Considering this rather large number of assumptions, one should realize that actual future pension cash flows could deviate significantly from model expectations, even if the volatility under the model is small. However, the same could be said for actuarial estimates of the expected value. A relevant question is therefore whether this uncertainty should be used in a procedure to find a hedging portfolio.

5

Hedging portfolio

Recall Section 3.4, where the interest rate risk of a pension fund was hedged using interest rate swaps. Notwithstanding the performance of such a hedging strategy, pension funds are in practice often not willing to acquire large amounts of interest rate swaps to hedge their risks. Some pension funds do not directly invest in swaps altogether — only to a lesser extent via externally managed funds. Instead pension funds often prefer to hedge (the majority of) their interest rate risk using government bonds. The reason for preferring government bonds over interest rate derivatives is two-fold: derivatives add complexity to a portfolio which many pension funds perceive as an operational risk and government bonds (potentially) provide a higher return than derivatives such as interest rate swaps. There is also a historical reason for using other asset classes besides swaps: pension funds have a long history in hedging using bonds and often mandate other parties for this service.

For these reasons pension funds often hedge a substantial part of their liabilities using bonds. The main aim of this chapter is constructing a hedge using bonds that performs well. Any remaining 'gaps' in this hedge may then be resolved using derivatives such as interest rate swaps as done in Section 3.4. The size of this gap is usually a strategic choice by the pension fund.

Hedging for pension funds is complicated by the additional constraints and preferences that pension funds generally have: simply finding a hedge using an investment universe of bonds does not suffice. Alternative (more general) measures of hedging performance for instance have to be used in addition to the ones described in Section 3.3. Moreover, more constraints on (among others) exposure to countries must be taken into account.

The complicating factors when hedging for pension funds are described in Section 5.1. This problem is then solved in Section 5.2 using an optimization approach. The performance of this solution is assessed in Section 5.3. The sensitivity of these results to parameter changes is investigated in Section 5.4. Sections 5.5 and 5.6 make use of the earlier results to assess the yield of a hedging portfolio and to find more diversified portfolios.

This chapter builds on the pension world as described in Chapter 2 and particularly makes frequent use of the assets that were described in Section 2.3. The interest rate risks are measured using the tools in Chapter 3. Chapter 4 is used to find the expected values of future pension liabilities, although these expected values are in practice usually provided by the pension funds and its

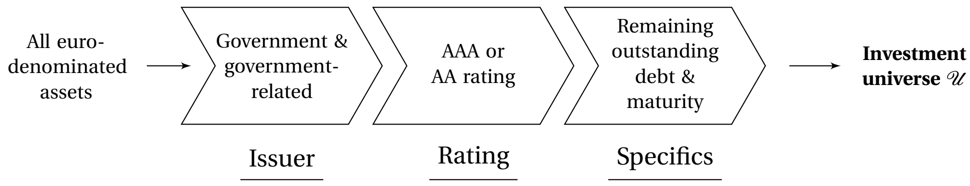


Figure 5.1: The process to find the investment universe $\mathscr{U}$.

actuaries. The pension fund processes nonetheless play an important role in assessing the algorithm performance through simulation in Subsection 5.3.2.

The results in this chapter provide pension funds and their fiduciary managers with a new approach to hedge interest rate risks in a way that is tailored to the pension fund industry.

5.1. Pension fund hedging

As has been stated in the introduction to this chapter, pension funds generally aim to hedge a substantial part of their interest rate risks using bonds rather than interest rate derivatives. A specific class of bonds is in fact selected. This already complicates finding a hedging strategy compared to Section 3.4. In addition to this, pension funds have specific targets and requirements that should be taken into account — these are described in this section.

One of the requirements that pension funds have is a specific *investment universe*. Generally only euro-denominated assets can be included in the hedging portfolios of pension funds in the Netherlands. This collection of assets is usually filtered further. A common example of such a filter is given in this section but some pension funds may deviate from this.

First, only government and government-related bonds are included — corporate bonds are deemed too risky and pension funds are not interested in adding interest rate derivatives to this part of the portfolio (as has been explained). Then the collection of assets is shrunk further by disregarding all bonds that do not have a AAA or AA credit rating — the probability of default of bonds with a lower credit rating is generally considered too high although some pension funds might accept lower rated bonds as well. Finally, the investment universe is found by keeping all bonds that have at least a certain minimum outstanding debt and whose remaining maturity falls within a defined range. The exact parameters differ per pension fund. This example process is illustrated in Figure 5.1.

This results in an investment universe $\mathscr{U}$ consisting of m assets:

$$\mathscr{U} = \{\psi_1, \ldots, \psi_m\}.$$

These assets $\psi_1, \ldots, \psi_m$ have values $U_1(t_0), \ldots, U_m(t_0)$ at the current time t_0.

Further mathematical notation is required to accurately describe the objectives and requirements that pension funds have for a hedging portfolio. Let P be a portfolio of assets in $\mathscr{U}$ with value $V(t_0)$ at current time t_0. The position of portfolio P in asset $\psi_j \in \mathscr{U}$ at time t_0 is denoted by $x(t_0)$. In this chapter P represents the portfolio of the proposed hedging strategy. In addition, there is a portfolio corresponding with the current hedging strategy: $\bar{P}$. Its position in $\psi_j \in \mathscr{U}$ at time t_0 is denoted by $\bar{x}(t_0)$.

The portfolios P and $\bar{P}$ aim to hedge against the liabilities of the pension fund. The value of liabilities at current time t_0 is $L(t_0)$. The value of the liabilities and proposed hedging portfolio can be combined as in Section 3.4:

$$\Lambda(t_0) = L(t_0) + V(t_0). \tag{5.1}$$

A vector notation of these quantities is frequently used. Therefore the following quantities should be defined:

$$\begin{aligned} \mathbf{x}(t_0) &= \begin{bmatrix} x_1(t_0) & \dots & x_m(t_0) \end{bmatrix}, \\ \bar{\mathbf{x}}(t_0) &= \begin{bmatrix} \bar{x}_1(t_0) & \dots & \bar{x}_m(t_0) \end{bmatrix}, \\ \mathbf{u}(t_0) &= \begin{bmatrix} U_1(t_0) & \dots & U_m(t_0) \end{bmatrix}. \end{aligned}$$

Often the functional is omitted from these vectors to simplify notation, e.g. $\mathbf{x}$ is used to denote $\mathbf{x}(t_0)$. The aim of this chapter is finding the optimal portfolio P in line with the preferences of a pension fund, i.e. finding the best $x_1(t_0), \dots, x_m(t_0)$.

In Figure 5.1 it can be observed that the investment universe $\mathscr{U}$ exists of bonds only. A measure of return will be required to quantify the targets of most pension funds. If one assumes that a bond is held until maturity and all payments are made on time, the internal rate of return can be calculated using the current market price. This is a well-known result that can be defined formally:

Definition 5.1 (Yield to Maturity (YTM)) *Let $\psi_j \in \mathscr{U}$ a fixed-rate bond with known* market value *$\hat{U}(t_0)$ at t_0, notional N, fixed rate r and payment times $S_1, \dots, S_k$. Then the yield to maturity of this bond is $y \in \mathbb{R}$ such that (Nawalkha, Soto, and Beliaeva 2005, p. 49):*

$$\hat{U}(t_0) = \mathrm{e}^{-yS_1} r\tau_1 N + \mathrm{e}^{-yS_2} r\tau_2 N + \dots + \mathrm{e}^{-yS_{k-1}} r\tau_{k-1} N + \mathrm{e}^{-yS_k}\left(r\tau_k N + N\right),$$

with $\tau_k = \tau(S_{k-1}, S_k)$ and the accrual function defined as in Definition 2.3. △

The discussion in this section focuses on the preferences of pension funds. Deciding these preferences is in practice usually a discussion between the pension fund, its fiduciary manager and other advisors. In this section this interaction is disregarded and instead only a 'pension fund' is referenced.

A pension fund decides on targets that are used to judge the quality of a proposed hedging strategy. These targets should encompass the quality of a hedge, the general risk appetite of the pension fund and the costs necessary to implement the hedge. The proposed targets in order of priority are:

Parallel changes Pensions funds are very interested in how well the hedging portfolio performs if the interest rate changes in a parallel way, i.e.: either the swap fixed rates or the yields (as discussed in Sections 3.3 and 3.5) change in a constant way throughout time. The performance of pension fund asset managers in terms of hedging is usually assessed first and foremost based on this measure. Therefore this is also considered the most important target in this section.

However, it should be noted that academics have criticized this measure of interest rate risk because it often does not yield an adequate hedge. Reitano (1992) for instance stated over two decades ago that a parallel yield curve shift "can disguise risk". Because of its shortcomings this measure is therefore used alongside other measures.

This concept of parallel interest rate changes can be stated in terms of the sensitivity measures that were introduced in Section 3.3. The goal is then to minimize:

$$\left|\sum_{i=1}^{n} \frac{\partial \Lambda(t_0)}{\partial K_i}\right|.$$

If the key rate duration methodology (Subsection 3.5.1) is used to calculate these Δ — as many pension funds do — this sum is equal to the *modified duration* or simply *duration*. These well-known concepts are explained in detail by Nawalkha, Soto, and Beliaeva (2005, pp. 20–25).

Interest rate sensitivity The difference between the interest rate sensitivity of the liabilities and assets should be minimal to ensure a reasonable hedge. In Section 3.4 a hedging strategy was found using swaps that perfectly hedged these interest rate risks: $\partial\Lambda(t_0)/\partial K_i = 0, \forall i \in \{1,\ldots,n\}$. However, this is often not possible when one is not allowed to use interest rate swaps. The sum of these absolute Δ is therefore minimized instead:

$$\sum_{i=1}^{n}\left|\frac{\partial\Lambda(t_0)}{\partial K_i}\right|.$$

This sum represents the total remaining Δ risk of the liabilities and hedging portfolio at all maturities — i.e. if it is zero the delta sensitivity with respect to all swaps is entirely hedged. The result is a hedging portfolio that hedges Δ risks as well as possible and ideally the result is a perfect delta hedge.

Transaction costs Pension funds generally aim to minimize transaction costs. Like any investor, pension funds strive to minimize the money lost due to transaction costs that 'leak' out of the portfolio. Figure 5.2 shows that 50% of the pension funds in the Netherlands annually pay 0.06%–0.14% in transaction costs: the goal is to minimize this number. But there are two more reasons specific to pension funds that explain why they strive to minimize the number of trades. First, the long time horizon of pension funds means that they do not have an inclination to sell quickly and instead prefer to passively keep their position. In addition, pension funds are under Markets in Financial Instruments Directive (MiFID) II regulations (that were introduced in 2018) obliged to report any transactions to the regulator Autoriteit Financiële Markten (AFM) (Autoriteit Financiële Markten n.d.). This leads to further costs when transactions are being made.

In Section 2.2 it was assumed that there are no transaction costs. This assumptions remains relevant in the *context of pricing*. However, transaction costs can be taken into account in the *context of optimization* because this does not violate the assumption that there are no transaction costs for pricing. Minimization of transaction costs can therefore be described in the framework of this section. Let $c \in \mathbb{R}_{\geq 0}$ the transaction costs for a transaction with monetary value 1. To minimize transaction costs the following quantity should then be minimized:

$$(|\mathbf{x}-\bar{\mathbf{x}}|\cdot\mathbf{u})\cdot c,$$

with (as before) $\mathbf{x}$ the positions in the portfolio P, $\bar{\mathbf{x}}$ the positions in the current portfolio and $\mathbf{u}$ the values of the assets. For institutional investors a realistic transaction cost is $c =$ 0.005%, i.e. half a basis point.

Country allocation Pension funds — like most investors — would like to be exposed to 'safe' counterparties that have a small credit risk. This is in part done by selecting only AA/AAA-rated bonds in the investment universe as was done in Figure 5.1. But bonds can have a good credit rating because of their characteristics while the issuer's rating is not as high. A negative event pertaining to the issuer could have a depressing impact on the value of a bond, even if the interest does not change. Or in the terminology of Remark 3.2: the *z-spread would increase*. These risks should be mitigated.

In addition, pension funds would like to have exposure in its hedging portfolio to countries that are related to its liabilities — i.e. a pension fund with Dutch participants only would

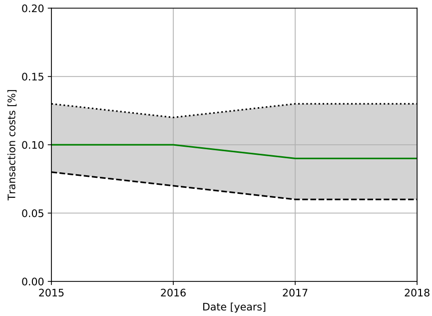


Figure 5.2: Transaction costs of the 50 largest pension funds (by participants in 2018) in the Netherlands as a percentage of total investments. Median (solid green), 25th percentile (dashed black) and 75th percentile (dotted black) have been plotted. More information on this dataset has been added to Appendix C.7.

not invest the majority of its money outside the EU. This is inadvisable both from a financial and societal point of view.

To incorporate these investment beliefs a country allocation is often set by a pension fund or its fiduciary manager. The country allocation in terms of the value of investments in the hedging portfolio should be close to the target country allocation, i.e. the difference between the country allocations should be minimized[1]. Let $b \in \mathbb{R}_{\geq 0}$ be the budget of this portfolio. Let $\mathscr{A} \subset \mathscr{U}$ be the assets in a certain country and $r \in [0,1]$ the target allocation to this country as a percentage of the budget. Then the goal is to minimize:

$$\left| \left(\sum_{\psi_j \in \mathscr{A}} x_j u_j \right) - rb \right|. \tag{5.2}$$

This quantity equals the difference between the assets in a certain country in portfolio P — represented by $\sum_{\psi_j \in \mathscr{A}} x_j u_j$ — and the target amount in this same country — represented by rb. Let $a \in \mathbb{R}^m$ be the indicator vector of $\mathscr{A}$ on $\mathscr{U}$, i.e. $a_j = 1$ if $\mathscr{U}_j \in \mathscr{A}$ and $a_j = 0$ otherwise. Then Equation (5.2) can be written in a vector notation as well so that the goal is to minimize:

$$|\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{a}) - rb|,$$

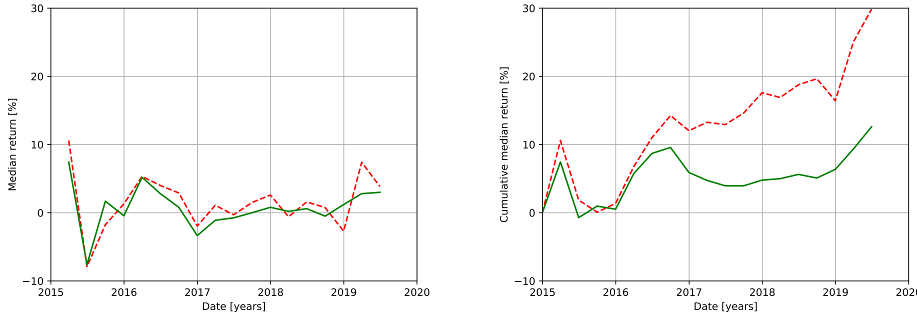
where the operator $\odot$ is used to denote the element-wise multiplication of the vectors — this operation is also known as the Hadamard product. A number of these country allocations are set by the pension fund.

Investment costs A pension fund would like to spend the least amount of money possible on its hedging portfolio. Any remaining money can be invested in a return portfolio that generally results in higher returns. The total investment costs should thus be minimized:

$$\mathbf{x} \cdot \mathbf{u}.$$

A hedging portfolio and a return portfolio have very different objectives: the aim of the former is offsetting any change in value of the liabilities — both positive and negative —

[1]Country allocations are in practice often also measured by the part of the portfolio's change in a value after a parallel change that can be attributed to a country. This has been implemented but is not discussed in detail in this thesis because it would unnecessarily complicate the results.



(a) Median quarterly return on all assets (dashed red, including the hedging portfolio) and on the hedging portfolio only (solid green).

(b) Cumulative median quarterly return on all assets (dashed red, including the hedging portfolio) and on the hedging portfolio only (solid green).

Figure 5.3: Median quarterly return of the 50 largest pension funds in the Netherlands. Data has been provided by DNB — more details have been added to Appendix C.7.

while the aim of the latter is simply maximizing returns. These higher returns of the return portfolio can also be observed when looking at the median returns of the 50 largest pension funds in the Netherlands over the last five years — this has been illustrated in Figure 5.3.

Investment return Given two portfolios that are identical on the five targets that have thus far been discussed, any investor would clearly prefer the portfolio with the higher return. Pension funds of course have the same preference. Actual return would be measured on a regular basis and would depend on changes in the z-spread. It is not elementary to model this so therefore the *yield to maturity* is taken as a proxy. The YTM of the portfolio is calculated as the weighted average of the YTM of the assets in the portfolio. This implicitly assumes that an investor is able to reinvest any proceeds at the same rate as before.

Let $\mathbf{y} \in \mathbb{R}^m$ be the yield to maturities of all assets in $\mathscr{U}$ given the current market price (as in Definition 5.1). Then the goal is to *maximize*:

$$\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{y}) .$$

Note that this is equivalent to minimizing $-\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{y})$.

The main objective of the pension fund for the hedging portfolio is mitigating risks at minimal costs — the return on these investments has a considerably lower priority. Any money that is not required for hedging the risks can be invested in a portfolio that is focused solely on return. If two potential hedging portfolios are performing equally well on all targets but investment return, a pension fund would of course prefer the one with the higher return. However, this is in practice seldom the case. The investment return target is therefore not considered in the general optimization algorithm. In Section 5.5 an alternative approach is proposed to consider the investment return.

In addition to the objectives set out for the hedging portfolio, a pension fund also sets a number of requirements that must be met by any hedging portfolio. These are practical requirements, strategic decisions and legal agreements:

Budget There is a maximum amount of money that can be spent on the hedging portfolio, i.e. the present value of the hedging portfolio should be less than or equal to the budget[2]. Let (as before) $b \in \mathbb{R}_{\geq 0}$ be the budget. Then:

$$V(t_0) = \mathbf{x} \cdot \mathbf{u} \leq b.$$

Note that this number b should include not only the current cash position but also the present value of the current hedging portfolio (if it exists) of the pension fund, since those assets can clearly be sold. A negative budget could in theory be possible: one could borrow money to buy the hedging portfolio, possibly using other assets such as equity for collateral. However, this is in practice not done and certainly not in line with the risk-averse behavior expected of pension funds.

No shorting A short position is not allowed in the hedging portfolio, i.e.:

$$\mathbf{x} \geq \mathbf{0}.$$

With a short position there is theoretically no upper limit on the potential loss. Pension funds are unwilling to take this risk. The reasons for rejecting interest rate derivatives discussed in the introduction to Section 2.4 also apply to the disallowance of short positions.

Individual asset At most a certain fraction of the budget may be invested in a single asset. Let $q \in (0,1]$ be this fraction. Then:

$$x_i u_i \leq qb, \ \forall i \in \{1, \ldots, m\}.$$

The quantity qb represents the maximum amount that may be invested in a single asset. This constraint ensures that the portfolio is sufficiently diversified.

Turnover If a current portfolio $\bar{\mathbf{x}}$ exists and the hedging portfolio is therefore not newly constructed, there is an upper limit on the number of changes that is allowed. Therefore a constraint is set on the turnover in the portfolio. In this fashion pension funds aim to ensure no large amounts of transaction costs are paid because they are not comfortable with doing so, as has been explained earlier in this section. Let t_{-1} be the time at which the hedging portfolio was last changed and let $p \in (0,1]$ be the allowed turnover as a fraction of the budget per time unit. Then the turnover constraint is:

$$\frac{|\mathbf{x} - \bar{\mathbf{x}}| \cdot \mathbf{u}}{2} \leq \tau(t_{-1}, t_0)\, pb,$$

with $\tau(t, T)$ defined as in Definition 2.3. The quantity $|\mathbf{x}-\bar{\mathbf{x}}|\mathbf{u}/2$ equals the turnover of P relative to the current portfolio. Note that a change in the portfolio usually entails selling some assets and buying other assets with the money that was freed up. The division by two ensures such an operation is not counted double. This should be less than or equal to the allowed turnover in the time period $(t_{-1}, t_0]$, which equals $\tau(t_{-1}, t_0)pb$.

Country allocations In addition to the country allocation target, pension funds often also set hard constraints on the country allocation. This could for instance encompass that at least 50% of the budget should be invested in the Netherlands, Germany and Austria. These hard allocations can also be used to define an allowed range around the country allocation target that was discussed earlier. Let (as before) $\mathscr{A} \subset \mathscr{U}$ be the assets in a certain country, $r \in [0,1]$

[2]Some pension funds addtionally also limit the cash position that may be held within the mandate — i.e. $\mathbf{x} \cdot \mathbf{u} \geq b - \bar{b}$ for some constant $\bar{b} \in (0, b]$. This constraint may be implemented in a similar manner but has not been added to this thesis.

the set allocation to this country as a percentage of the budget and $\smile \in \{\leq, =, \geq\}$. Then the country allocation constraint is:

$$\sum_{\psi_j \in \mathscr{A}} x_j u_j \smile rb.$$

It is again possible to rewrite this equation in a vector notation:

$$\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{a}) \smile rb,$$

where $\mathbf{a} \in \mathbb{R}^m$ again denotes the indicator vector of $\mathscr{A}$ on $\mathscr{U}$. In practice a significant number of such country allocation restrictions are set.

Country allocation restrictions are usually combined with the country allocation targets (that have been defined earlier). Two restrictions are added for each target so that the country allocation must be in a range centered around the target.

It should be noted that these restrictions impose a limit relative to the *budget* b rather than to the actual amount invested, while one might have expected the latter. However, pension funds usually define constraints in terms of the budget rather than the amount invested. In addition, this ensures that the right-hand side of the equation (or inequality) is constant which makes optimal solutions more stable.

Pension funds often define additional restrictions in addition to the ones described in this section. These are outlined — along with the described requirements — in the *investment mandate* the pension fund shares with its fiduciary manager. An example would be a minimal allocation to so-called 'green bonds': bonds for which the issuer commits itself to spending the proceeds on sustainable projects. Such extensions have not been added in this framework, but the author has not come across requirements that cannot be modeled similarly to one of the stated requirements.

One important risk that has been omitted so far is counterparty risk: the risk that a counterparty might default on its contractual obligations. Because only AA/AAA-rated government(-related) bonds are in the investment universe (Figure 5.1), this risk is assumed to be zero. This assumption is substantiated by historical data (S&P Global 2018), which states counterparty risk for such bonds is negligible. In addition, the country allocation decreases counterparty risk as well. Nonetheless, the model could be improved by incorporating counterparty risk.

In this chapter it is assumed that pension funds aim to fully hedge the risks — notably interest rate risks — associated with their pension liabilities. Many pension funds have a view on the market and willingly take interest rate risk. This is reflected by the *hedging ratio*: a number that represents the percentage of interest rate risks a pension fund hedges[3]. The ABP for instance hedged 25%[4] of its interest rate risk in 2017 and 2018 (Algemeen Burgerlijk Pensioenfonds 2019, pp. 36–37). With the decreasing interest rate over the last years, a hedging ratio below 100% has led to deteriorating coverage ratios. This effect can be seen in Figure 5.4.

[3]The definition of the hedging ratio may different per pension fund. It is most commonly defined as the risks hedged by a parallel yield curve change.

[4]One could observe that there is no pension fund with a hedging ratio of 25% and a similar coverage ratio as the ABP in Figure 5.4. However, since year end 2018 the ABP further decreased its hedging ratio to 22% in the second quarter of 2019. This figure can be seen in Figure 5.4.

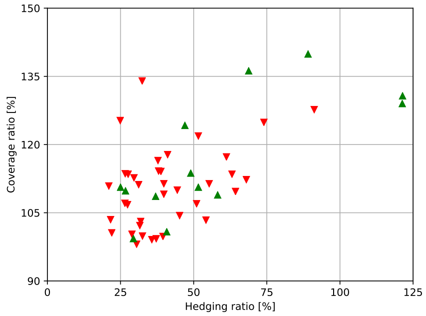


Figure 5.4: A comparison of the coverage ratios and hedging ratios of the 50 largest pension funds in the Netherlands as of the end of the second quarter in 2019. The majority of pension funds have decreased their hedging ratio in the year before (red downwards triangle) and some pension funds have increased their hedging ratio (green upwards triangle). None of these pension funds have kept the same hedging ratio. The dataset has been added to Appendix C.7.

5.2. Portfolio optimization

The targets and restrictions that were described in Section 5.1 can be stated mathematically as an *optimization problem*. This is in fact a minimization problem with a number of constraints.

A few quantities should be defined (in addition to the ones used in Section 5.1) before the optimization problem can be stated. Define:

$$\begin{aligned}
\Delta^{\psi} &\in \mathbb{R}^{n\times m} &\quad \text{with} \quad \Delta^{\psi}_{ij} &= \frac{\partial U_j(t_0)}{\partial K_i},\\
\Delta^{L} &\in \mathbb{R}^{n} &\quad \text{with} \quad \Delta^{L}_{i} &= \frac{\partial L(t_0)}{\partial K_i},\\
\Delta^{\Lambda} &\in \mathbb{R}^{n} &\quad \text{with} \quad \Delta^{\Lambda}_{i} &= \frac{\partial \Lambda(t_0)}{\partial K_i}.
\end{aligned}$$

The interest rate sensitivity of $\Lambda(t_0)$ can be stated using these quantities and the portfolio allocations $\mathbf{x}$:

$$\Delta^{\Lambda}_i = \frac{\partial \Lambda(t_0)}{\partial K_i} = \frac{\partial L(t_0)}{\partial K_i} + \sum_{j=1}^{m} \mathbf{x}_i(t_0)\frac{\partial U_j(t_0)}{\partial K_i} = \Delta^{L}_i + \Delta^{\psi}_{i*}\cdot\mathbf{x},$$

where Δ_{i*} denotes the i-th row of the matrix Δ. The partial derivatives with respect to K_i may be replaced with one of the alternative measures for interest rate sensitivity in Section 3.5. The vector Δ^{Λ} can equivalently be calculated using a matrix multiplication:

$$\Delta^{\Lambda} = \Delta^{L} + \Delta^{\psi}\mathbf{x}.$$

Note that parallel interest rate changes are then equal to $\sum_{i=1}^{n}\Delta^{\Lambda}_i = \Delta^{\Lambda}\cdot\mathbf{1}$, with $\mathbf{1}\in\mathbb{R}^n$ a vector of ones with size n[5].

Both the country allocation *targets* and *constraints* should be included in the optimization problem. Let l the number of country allocation targets and $\bar{l}$ the number of country allocation con-

[5]The vector $\mathbf{1}$ will be used throughout this thesis to denote a vector of ones. The size is often not mentioned explicitly if this is clear from the context.

straints. The country allocation targets are determined by l sets $\mathscr{A}_1, \ldots, \mathscr{A}_l \subset \mathscr{U}$ with corresponding budget fractions $r_1, \ldots, r_l$. These sets and fractions together set the country allocation targets. Similarly, the country allocation constraints are determined by $\bar{l}$ sets $\mathscr{A}_{l+1}, \ldots, \mathscr{A}_{l+\bar{l}}$ with corresponding budget fractions $r_{l+1}, \ldots, r_{l+\bar{l}}$. In addition, relations $\overset{l+1}{\smile}, \ldots, \overset{l+\bar{l}}{\smile} \in \{\leq, =, \geq\}$ are specified as in Section 5.1. Define the accompanying indicator vectors by $\mathbf{a}^1, \ldots, \mathbf{a}^{l+\bar{l}}$.

The optimization problem can now be stated. In Section 5.1 the targets and requirements of a hedging portfolio for pension funds have been described in detail. These requirements were formulated based on the portfolio allocation $\mathbf{x}$ — it is not challenging to add them in an optimization problem. The objective function poses more problems: how should the five described objectives be combined? The class of optimization problems that simultaneously take multiple objectives into account is commonly called *multi-objective optimization*.

Zadeh (1963) proposed to optimize a weighted sum of objectives for such optimization problems: the objective function of the optimization problem is set to a weighted sum of the individual targets. He then showed that this method yields a Pareto optimal solution if the weights are nonnegative, i.e. a solution such that no other solution exists that improves one of the targets without worsening any of the other targets. Marler and Arora (2010) researched this method in detail and concluded that it performs well for preferences that can be captured by a linear approximation, as long as the weights are selected in a certain way. The pension fund targets can be combined in this fashion because it is possible to compare the individual targets in a linear function — i.e. a pension fund usually describes the relative importance of different aspects of the hedging portfolio. Therefore this methodology is suited well to the problem under consideration. The weights are selected in Subsection 5.2.1 in line with this research.

Let $\alpha_1, \ldots, \alpha_5 \in \mathbb{R}_{\geq 0}$ be the optimization weights. The yield target is omitted as discussed in Section 5.1. The optimization problem can now be formulated using all notation that has been introduced — this has been added to Problem Formulation 5.1.

Problem Formulation 5.1 Portfolio optimization

$$\min_{\mathbf{x}} \quad \overbrace{\alpha_1 \left| \left(\Delta^L + \Delta^\Psi \mathbf{x} \right) \cdot \mathbf{1} \right|}^{\text{Parallel changes}} + \overbrace{\alpha_2 \left| \Delta^L + \Delta^\Psi \mathbf{x} \right| \cdot \mathbf{1}}^{\text{Interest rate sens.}} + \overbrace{\alpha_3 \left(|\mathbf{x} - \bar{\mathbf{x}}| \cdot \mathbf{u} \right) c}^{\text{Transaction costs}} + \overbrace{\alpha_4 \sum_{k=1}^{l} \left| \mathbf{x} \cdot \left(\mathbf{u} \odot \mathbf{a}^k \right) - r_k b \right|}^{\text{Country allocations}} + \overbrace{\alpha_5 \left(\mathbf{x} \cdot \mathbf{u} \right)}^{\text{Inv. costs}}$$

$$\begin{aligned}
\text{s.t.} \quad & \mathbf{x} \cdot \mathbf{u} && \leq b, && && \text{Budget} \\
& \mathbf{x} && \geq \mathbf{0}, && && \text{No shorting} \\
& \mathbf{x}_j \mathbf{u}_j && \leq q b, && \forall j \in \{1, \ldots, m\}, && \text{Individual asset} \\
& 1/2 |\mathbf{x} - \bar{\mathbf{x}}| \cdot \mathbf{u} && \leq \tau (t_{-1}, t_0)\, p b, && && \text{Turnover} \\
& \mathbf{x} \cdot \left(\mathbf{u} \odot \mathbf{a}^k \right) && \overset{k}{\smile} r_k b, && \forall k \in \{l+1, \ldots, l+\bar{l}\}. && \text{Country allocations}
\end{aligned}$$

A solution to this optimization problem should now be found to find the hedging portfolio. Note that all quantities except $\mathbf{x}$ and the $\alpha_1, \ldots, \alpha_5$ are known constants that have already been calculated. A model for the expected future pension liabilities was proposed in Chapter 4 and the sensitivity of these nominal payments to interest rate changes can be calculated using Section 3.3. Thus finding a solution to Problem Formulation 5.1 in fact consists of two problems: solving the mathematical optimization problem and determining weights $\alpha_1, \ldots, \alpha_5$. The latter task — finding the weights — is undertaken in Subsection 5.2.1. Attention will now be shifted to finding an optimal solution to the problem as given in Problem Formulation 5.1 under the assumption that the weights are known.

Solving Problem Formulation 5.1 is not elementary because there are (potentially) a large number of assets and restrictions. Finding the optimal solution through brute-forcing therefore quickly becomes infeasible. Methods to solve these *nonlinear optimization* problems exist but results are suboptimal (Boyd and Vandenberghe 2004, pp. 9–11). It is usually difficult to ensure that a global optimum is found rather than a local one and to execute the algorithm within a reasonable time. Boyd and Vandenberghe (2004, pp. 127–189) describe the class of *convex optimization* problems and its subset *linear optimization* problems. For these classes of optimization problems efficient algorithms *do* exist — Boyd and Vandenberghe (2004, pp. 561–630) describe the interior-point method to solve the former class of optimization problems. Linear optimization problems can additionally be solved using Linear Programming (LP) methods such as the Simplex method described by Dantzig et al. (1955).

These kinds of methods are fast and proven to reach the global optimum in a finite time. If the problem (in Problem Formulation 5.1) with its objective function and constraints can be rewritten into a problem that minimizes $\mathbf{c}\cdot\mathbf{x}$ subject to $D\mathbf{x} \le \mathbf{d}$ and $\mathbf{x} \ge \mathbf{0}$ (and optionally equality constraints) for some matrix D and vectors $\mathbf{c}$ and $\mathbf{d}$, LP methods can be used. Solving the problem would be simplified greatly. This rewriting will be done now. Observe that the main issue in doing this is related to the absolute values in the objective function and the turnover constraint.

First, note that the budget, no shorting, individual asset and country allocations constraints can easily be rewritten into the prescribed form. This is not elementary for the turnover constraint because it makes use of an absolute value. By introducing extra variables the constraint can be linearized. To see this, note that the constraint may equivalently be written as:

$$\sum_{j=1}^{m} \left| \left(\mathbf{x}_j - \bar{\mathbf{x}}_j \right) \mathbf{u}_j \right| \le 2\tau \left(t_{-1}, t_0 \right) pb,$$

by rewriting and by using that $\mathbf{u} \ge \mathbf{0}$. Equivalent constraints may be formulated without absolute values by introducing m new variables $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 & \dots & \mathbf{y}_m \end{bmatrix}$:

$$\sum_{j=1}^{m} \mathbf{y}_j = 2\tau \left(t_{-1}, t_0 \right) pb,$$

$$\mathbf{y}_j \ge \left| \left(\mathbf{x}_j - \bar{\mathbf{x}}_j \right) \mathbf{u}_j \right|.$$

This later constraint can in turn be substituted by two constraints without an absolute value:

$$\mathbf{y}_j \ge \left| \left(\mathbf{x}_j - \bar{\mathbf{x}}_j \right) \mathbf{u}_j \right| \iff -\mathbf{y}_j \le \left(\mathbf{x}_j - \bar{\mathbf{x}}_j \right) \mathbf{u}_j \le \mathbf{y}_j.$$

The result is an equality constraint and a number of inequality constraints. It has now been shown how the turnover constraint can be rewritten into the necessary format. Observe that $\mathbf{y} \ge \mathbf{0}$ by the constraints — this fact will be used later.

A similar methodology can be applied to the objective function. Consider the simplified optimization problem that only includes the first term of this function:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \alpha_1 \left| \left(\Delta^L + \Delta^\psi \mathbf{x} \right) \cdot \mathbf{1} \right| \\ \text{s.t.} \quad & D\mathbf{x} \le \mathbf{d}, \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

If the coefficient $\alpha_1 \geq 0$ (which it is by definition), this optimization problem can be restated into a problem with the same optimum (Boyd and Vandenberghe 2004, p. 294):

$$\begin{aligned} \min_{\mathbf{x}, z_1} \quad & \alpha_1 z_1 \\ \text{s.t.} \quad & z_1 \geq \left| \left(\Delta^L + \Delta^\psi \mathbf{x} \right) \cdot \mathbf{1} \right|, \\ & D\mathbf{x} \leq \mathbf{d}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

This is equivalent to:

$$\begin{aligned} \min_{\mathbf{x}, z_1} \quad & \alpha_1 z_1 \\ \text{s.t.} \quad & z_1 \geq \left(\Delta^L + \Delta^\psi \mathbf{x} \right) \cdot \mathbf{1}, \\ & z_1 \geq -\left(\Delta^L + \Delta^\psi \mathbf{x} \right) \cdot \mathbf{1}, \\ & D\mathbf{x} \leq \mathbf{d}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

This last result is easily rewritten into the prescribed form. This procedure may be repeated for all absolute values in the objective function of Problem Formulation 5.1. A number of additional variables is then obviously required to formulate the new optimization problem. Let $\mathbf{z} \in \mathbb{R}^{1+n+m+l}$ be a vector of these extra variables with clearly $\mathbf{z}_1 = z_1$. Each of the absolute values in the original objective function corresponds with a part of $\mathbf{z}$ — in many cases more than one element of $\mathbf{z}$ because the absolute values of vectors are used. To make this 'division', define the indices of these parts:

$$I_1 = \{1\}, \qquad I_2 = \{2, \ldots, 2+n-1\},$$
$$I_3 = \{2+n, \ldots, 2+n+m-1\}, \qquad I_4 = \{2+n+m, \ldots, 2+n+m+l-1\}.$$

Each part of $\mathbf{z}$ can then be referenced using its indices. Let $I \in \{I_1, \ldots, I_4\}$ a set of indices. Then $\mathbf{z}_I \in \mathbb{R}^{|I|}$ is a vector whose elements are the elements referenced by I, i.e. $\left[\mathbf{z}_k\right]_{k \in I}$. So the vector $\mathbf{z}$ is:

$$\mathbf{z} = \Big[\underbrace{\mathbf{z}_1}_{\mathbf{z}_{I_1}} \quad \underbrace{\mathbf{z}_2 \quad \cdots \quad \mathbf{z}_{2+n-1}}_{\mathbf{z}_{I_2}} \quad \underbrace{\mathbf{z}_{2+n} \quad \cdots \quad \mathbf{z}_{2+n+m-1}}_{\mathbf{z}_{I_3}} \quad \underbrace{\mathbf{z}_{2+n+m} \quad \cdots \quad \mathbf{z}_{2+n+m+l-1}}_{\mathbf{z}_{I_4}} \Big].$$

The rewritten constraints and objective function yield — using this notation — a new optimization problem of which the optimum is equal to the optimum of the original optimization problem. This rewritten optimization problem has been added to Problem Formulation 5.2.

It is easily checked that this new optimization problem *can* be rewritten into the prescribed form, i.e. matrix D and vectors $\mathbf{c}, \mathbf{d}$ exist such that Problem Formulation 5.2 is equivalent with minimizing $\mathbf{c} \cdot \mathbf{w}$ subject to $D\mathbf{w} \leq \mathbf{d}$ and $\mathbf{w} \geq \mathbf{0}$. In this new problem $\mathbf{w}$ is a vector that contains $\mathbf{x}$ and all additionally added variables:

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \in \mathbb{R}^{1+n+3m+l}$$

The problem that is now under consideration is a problem that is linear in both the objective function and constraints. Therefore the optimal solution to this optimization problem can be found using LP methods. This is done using the Simplex method, which is described well by Dantzig et al. (1955).

It should be noted that the restated problem is significantly more complex than the original problem in terms of the number of variables and constraints. The initial problem consisted of m variables and $m + \bar{l} + 2$ constraints (disregarding the default non-negativity constraint $\mathbf{x} \geq \mathbf{0}$). The

Problem Formulation 5.2 Portfolio optimization (rewritten from Problem Formulation 5.1)

$$
\begin{array}{llllr}
\min\limits_{\mathbf{x},\mathbf{z}} & \multicolumn{4}{l}{\overbrace{\alpha_1 \mathbf{z}_{I_1}}^{\text{Parallel changes}} + \overbrace{\alpha_2 \left(\mathbf{z}_{I_2} \cdot \mathbf{1}\right)}^{\text{Interest rate sens.}} + \overbrace{\alpha_3 \left(\mathbf{z}_{I_3} \cdot \mathbf{u}\right) c}^{\text{Transaction costs}} + \overbrace{\alpha_4 \left(\mathbf{z}_{I_4} \cdot \mathbf{1}\right)}^{\text{Country allocations}} \overbrace{+ \alpha_5 \left(\mathbf{x} \cdot \mathbf{u}\right)}^{\text{Inv. costs}}} \\
\text{s.t.} & \mathbf{x} \cdot \mathbf{u} & \leq b, & & \text{Budget} \\
& \mathbf{x} & \geq \mathbf{0}, & & \text{No shorting} \\
& \mathbf{x}_j \mathbf{u}_j & \leq qb, & \forall j \in \{1, \ldots, m\}, & \text{Individual asset} \\
& \mathbf{y} \cdot \mathbf{1} & = 2\tau \left(t_{-1}, t_0\right) pb, & & \text{Turnover (1)} \\
& \mathbf{y}_j & \geq \left(\mathbf{x}_j - \bar{\mathbf{x}}_j\right) \mathbf{u}_j, & \forall j \in \{1, \ldots, m\}, & \text{Turnover (2)} \\
& \mathbf{y}_j & \geq -\left(\mathbf{x}_j - \bar{\mathbf{x}}_j\right) \mathbf{u}_j, & \forall j \in \{1, \ldots, m\}, & \text{Turnover (3)} \\
& \multicolumn{2}{l}{\mathbf{x} \cdot \left(\mathbf{u} \odot \mathbf{a}^k\right) \overset{k}{\smile} r_k b,} & \forall k \in \{l+1, \ldots, l+\bar{l}\}, & \text{Country allocations (1)} \\
& \mathbf{z}_{I_1} & \geq \left(\Delta^L + \Delta^\Psi \mathbf{x}\right) \cdot \mathbf{1}, & & \text{Parallel changes (1)} \\
& \mathbf{z}_{I_1} & \geq -\left(\Delta^L + \Delta^\Psi \mathbf{x}\right) \cdot \mathbf{1}, & & \text{Parallel changes (2)} \\
& \mathbf{z}_{I_2} & \geq \left(\Delta^L + \Delta^\Psi \mathbf{x}\right), & & \text{Interest rate sens. (1)} \\
& \mathbf{z}_{I_2} & \geq -\left(\Delta^L + \Delta^\Psi \mathbf{x}\right), & & \text{Interest rate sens. (2)} \\
& \mathbf{z}_{I_3} & \geq \left(\mathbf{x} - \bar{\mathbf{x}}\right), & & \text{Transaction costs (1)} \\
& \mathbf{z}_{I_3} & \geq -\left(\mathbf{x} - \bar{\mathbf{x}}\right), & & \text{Transaction costs (2)} \\
& \mathbf{z}_k & \geq \left(\mathbf{x} \cdot \left(\mathbf{u} \odot \mathbf{a}^k - r_k b\right)\right), & \forall k \in \mathbf{z}_{I_4}, & \text{Country allocations (2)} \\
& \mathbf{z}_k & \geq -\left(\mathbf{x} \cdot \left(\mathbf{u} \odot \mathbf{a}^k - r_k b\right)\right), & \forall k \in \mathbf{z}_{I_4}, & \text{Country allocations (3)} \\
& \mathbf{z} & \geq \mathbf{0}. & & \text{Extra variables}
\end{array}
$$

rewritten problem is much more extensive:

$$n+2m+l+1 \text{ variables} \qquad \text{and} \qquad 2n+5m+2l+\bar{l}+4 \text{ constraints.}$$

Note that there are more constraints than variables in this rewritten problem, so looking at the dual problem does not decrease the problem dimensions Boyd and Vandenberghe (2004, pp. 223–231, 248–249). Still, there is now a linear optimization problem that can be solved if it admits a feasible solution — i.e. if the constraints allow for a solution. This is a significant improvement over Problem Formulation 5.1.

The weights $\alpha_1,\ldots,\alpha_5$ will be discussed next. Then the performance of the algorithm will be tested and more advanced applications will be described.

5.2.1. Determining optimization weights

The optimization problem has been stated formally. Both in the initial problem (Problem Formulation 5.1) and the restated problem (Problem Formulation 5.2), regularization parameters $\alpha_1,\ldots,\alpha_5$ are used. These weights evidently have a significant influence on optimization results — if $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_5 = 0$ all targets except parallel changes would for instance be disregarded.

One would usually like to calibrate such important parameters so that they are not arbitrarily chosen. This poses a problem for these parameters because there is no obvious dataset to which these parameters can be calibrated, i.e. no a priori target portfolio is known. One could select the weights so that the resulting portfolio is closest to the current portfolio — a similar exercise was done in Subsection 3.2.2 to calibrate the Hull-White process. But there is in this case no guarantee that parameters exist for which the result is reasonable similar to the current portfolio. More importantly, this would bias results towards the current strategy. The main goal of employing an optimization procedure is finding a better portfolio. Calibration with the aim of replicating the current portfolio would be opposed to this objective.

Alternatively, one could ask the client (the pension fund board or its fiduciary manager) to state the preference between two automatically generated portfolios. This yields a partial ordering of potential portfolios, from which the weights of the optimization problem can then be deduced. However, it is often difficult for a pension fund manager to determine the preference between portfolios, particularly without extensive quantitative analyses. This kind of procedure would also be rather complicated to implement.

A different way to choose the parameters $\alpha_1,\ldots,\alpha_5$ has been envisaged that does not involve calibration but instead aims to restate the weights in a manner more easily understood by clients.

A pension fund board or its fiduciary manager is often able to explain the relative importance of the different targets. Statements such as ‘a good delta hedge and minimal transaction costs are equally important but twice as important as the country allocation’ can be expected. These are *linear* relations, which Marler and Arora (2010, p. 860) found was an important prerequisite for using the weighted sum optimization method. It is not elementary to convert these statements in optimization weights because the terms in the objective function have not been normalized — i.e.: each term may have values in a very different range. Marler and Arora (2010, pp. 610–611) also stress the importance of normalization when weights are choosen a priori based on a comparative approach. The goal should therefore be to normalize the terms in the optimization problem.

Let $f(\mathbf{w})$ be the objective function of Problem Formulation 5.2. Recall that $w = \begin{bmatrix}\mathbf{x} & \mathbf{y} & \mathbf{z}\end{bmatrix}$. Define the individual terms of the objective function:

$$f_1(\mathbf{w}) = \mathbf{z}_{I_1}, \quad f_2(\mathbf{w}) = \mathbf{z}_{I_2}\cdot\mathbf{1}, \quad f_3(\mathbf{w}) = \left(\mathbf{z}_{I_3}\cdot\mathbf{u}\right)c, \quad f_4(\mathbf{w}) = \mathbf{z}_{I_4}\cdot\mathbf{1}, \quad f_5(\mathbf{w}) = \mathbf{x}\cdot\mathbf{u},$$

so that:

$$f(\mathbf{w}) = \alpha_1 f_1(\mathbf{w}) + \alpha_2 f_2(\mathbf{w}) + \alpha_3 f_3(\mathbf{w}) + \alpha_4 f_4(\mathbf{w}) + \alpha_5 f_5(\mathbf{w}).$$

Let $\alpha'_1, \ldots, \alpha'_5 \in \mathbb{R}_{\geq 0}$ the normalized weights. The goal is then to define the α_i in terms of the α'_i so that desirable portfolios have:

$$\alpha_i f_i(\mathbf{w}) \in \left[0, \alpha'_i\right], \quad \forall i \in \{1, \ldots, 5\}.$$

Weights should be defined in such a way that a portfolio with $\alpha_i f_i(\mathbf{w}) < 0$ cannot exist. Conversely, a portfolio with $\alpha_i f_i(\mathbf{w}) > \alpha'_i$ can exist but only if this is an undesirable portfolio that should be penalized.

The definition of α_i that normalizes the terms can be derived from the set of feasible values. This will be done one-by-one:

Parallel changes An upper bound of $f_1(\mathbf{w})$ can be derived from Δ^L and Δ^Ψ in combination with the 'individual asset' restriction. However, many solutions within this upper bound would be undesirable because the portfolio hedges very poorly against parallel changes. If this upper bound is used to normalize the parameter results would therefore be sub-optimal. Instead, portfolios that hedge poorly against parallel changes should be penalized for this so that they are only optimal when they perform well in other respects.

To achieve this the function $f_1(\mathbf{w})$ is normalized using the largest acceptable value of $f_1(\mathbf{w})$. Many pension funds define this value as a fraction of the absolute sensitivity of the liabilities to a parallel change. Let $h \in (0, 1]$ be this fraction — a common value is $h = 2.5\%$. Define:

$$\alpha_1 = \frac{1}{h\left|\Delta^L \cdot \mathbf{1}\right|} \alpha'_1.$$

Then — since $f_1(\mathbf{w}) \geq 0$ by definition — $\mathbf{w}$ gives a 'desirable' portfolio if and only if:

$$\alpha_1 f_1(\mathbf{w}) \leq \alpha'_1 \iff f_1(\mathbf{w}) \leq h\left|\Delta^L \cdot \mathbf{1}\right|.$$

Note that all quantities required to calculate $h\left|\Delta^L \cdot \mathbf{1}\right|$ are known.

Interest rate sensitivity Similar to the first term, an upper bound of the interest rate sensitivity term can be derived but this would yield unsatisfactory results. Instead, undesirable solutions should again be penalized. This is done once more by normalizing with respect to the risk of the liabilities:

$$\alpha_2 f_2(\mathbf{w}) = \frac{1}{h\left|\Delta^L \cdot \mathbf{1}\right|} \alpha'_2 f_2(\mathbf{w}).$$

It might seem strange to normalize using the sensitivity of the liabilities to a parallel change rather than the actual interest rate sensitivity of the liabilities. However, note that the former is defined as a sum of the later and the two quantities are overall closely connected.

Transaction costs A pension fund can incur maximum transaction costs by selling all assets and buying different assets for the entire budget. An upper bound of the transaction costs $f_3(\mathbf{w})$ can be stated:

$$f_3(\mathbf{w}) \leq (\bar{\mathbf{x}} \cdot \mathbf{u} + b)\, c.$$

The weight α_5 is then defined using this result:

$$\alpha_3 f_3(\mathbf{w}) = \frac{1}{(\bar{\mathbf{x}} \cdot \mathbf{u} + b)\, c} \alpha'_3 f_3(\mathbf{w}) \in \left[0, \alpha'_3\right].$$

Country allocation Observe that $\left|\mathbf{x}\cdot\left(\mathbf{u}\odot\mathbf{a}^k\right)-r_k b\right|\leq\max\left(r_k b,(1-r_k)b\right)$. The sum of these maxima would be an upper bound of $f_4(\mathbf{w})$. This upper bound can be tightened by considering also the country allocation restrictions that have been set. It has been mentioned that two restrictions are usually added for each target that ensure the resulting country allocation is within a certain range centered around the target. Assume this is indeed the case[6]. Let $\bar{r}\in(0,1]$ this range[7]. Then:

$$\left|\mathbf{x}\cdot\left(\mathbf{u}\odot\mathbf{a}^k\right)-r_k b\right|\leq\bar{r}b,\quad\forall k\in\{1,\dots,l\}.$$

Thus $f_4(\mathbf{w})\leq l\bar{r}b$ and the term can be normalized by defining α_4:

$$\alpha_4 f_4(\mathbf{w})=\frac{1}{l\bar{r}b}\alpha_4' f_4(\mathbf{w})\in\left[0,\alpha_4'\right].$$

It should be noted that the derived upper bound could still be too conservative if additional country allocations have been defined. That should not have a large impact and can easily be checked by a user.

Investment costs A pension fund can spend at most its budget on the hedging portfolio, i.e.: $f_r(\mathbf{w})\in[0,b]$. Normalization can thus be done using this upper bound:

$$\alpha_5 f_5(\mathbf{w})=\frac{1}{b}\alpha_5' f_5(\mathbf{w})\in\left[0,\alpha_5'\right].$$

After this exercise the normalized weights $\alpha_1',\dots,\alpha_5'$ have been set based on discussions with pension fund managers. This has resulted in the following weights:

$$\begin{aligned}\alpha_1'&=15, &\alpha_2'&=10, &\alpha_3'&=10,\\ \alpha_4'&=10, &\alpha_5'&=0.\end{aligned}\tag{5.3}$$

The sensitivity of the results to changes in these parameters is investigated in Section 5.4.

5.3. Performance of the hedging portfolio

The performance of a hedging portfolio can be tested in a number of ways. One would of course start by investigating whether the result is as expected. Are bonds with long maturities indeed bought if the pension fund expects long-term payments?

After passing this preliminary test, more rigorous testing is necessary. One way to do this is backtesting: would a proposed hedging portfolio have performed well historically? This is done for the 2007–2009 credit crisis in Subsection 5.3.1.

A second way to test performance is through simulation: how does the hedging portfolio perform when the processes involved are simulated? This is done in Subsection 5.3.2. The stochastic processes defined in Section 3.2 and Chapter 4 are essential in doing this.

In these performance measurements realistic example liabilities are vital. Such liabilities enable proper testing of the methodology. To this end, liabilities have been defined — the 75 years of cash flows have been added to Figure 5.5. One would expect these kinds of cash flows for a DB pension fund whose participants are for the most part 40–60 years of age: some pensions are already being paid but outflow will increase significantly in the coming years. Similar expected payments are described by Dondi (2005, pp. 107– 112).

[6]Results would not change significantly without the assumption that two restrictions are added for each country allocation target. A normalization factor could still be derived although it would likely be more complicated. This would mathematically not be an interesting extension.

[7]Note that $\bar{r}\leq\min\left(r_1,\dots,r_l\right)$ and $\bar{r}\leq 1-\max\left(r_1,\dots,r_l\right)$ should hold to ensure $r_k\pm\bar{r}\in[0,1],\forall k\in\{1,\dots,l\}$.

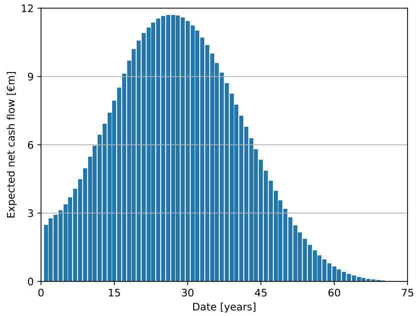


Figure 5.5: Example pension cash flows — each bar indicates the total expected cash flows of the pension fund in that year. This pension fund is expected to pay pension benefits over the next 75 years, with every year more benefit payments than contributions from participants that are still working. The exact values have been added to Appendix B.1.

5.3.1. Historical performance — Backtesting

Backtesting is done during the 2007–2009 credit crisis to assess how the proposed hedging procedure would have performed in that financially challenging time period. A discount curve was calculated from swap rates as of the end of September 2007 (Section 3.1). Additionally, an investment universe of 292 government bonds from Austria, Belgium, Finland, France, Germany and the Netherlands was downloaded as of this same date.

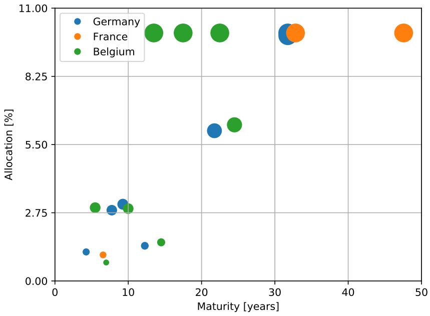
This investment universe was downloaded from Bloomberg and is not perfect. The dataset contains many 'stripped' bonds: fixed-rate bonds that have been split in multiple zero-coupon bonds (one for each payment). Although the liquidity of such derived bonds is generally lower, this should not have a material impact on the results. In reality a greater number of regular fixed-rate bonds would have existed in September 2007 but this data is unfortunately not readily available.

An optimal hedging portfolio is calculated for the pension cash flows of Figure 5.5. The budget is set to €128M and at most 10% of the budget may be invested in a single bond. No other restrictions have been set and no current portfolio or country allocation is specified. The result of this optimization procedure is a portfolio of 18 bonds and has been added to Figure 5.6.

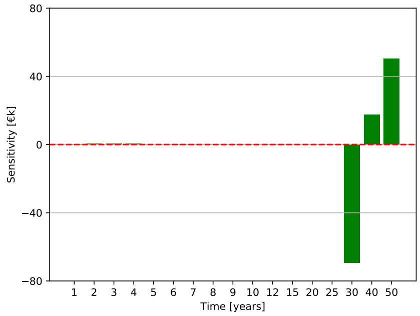
In Figure 5.6b it may be observed that this delta hedge is imperfect: sensitivities on the long end are not hedged perfectly, i.e. the net result is not equal to zero. A larger investment universe would solve this and this would therefore not have been an issue in 2007.

It is possible to test the performance of this hedging portfolio by considering post-September realizations of the interest rate and actual asset prices. Let $t_0, t_1, \ldots, t_{24} \in \mathbb{R}_{\geq 0}$ the measurement dates. Define these dates as the last working day in subsequent months — that is t_0 represents 28 September 2007, t_1 represents 31 October 2007, etc.

Define $T_1, \ldots, T_n$ as the payment times of the pension fund in amounts $L(t_1), \ldots, L(t_n)$. Let $V(t)$ be the present value of the remaining payments at time t. It is assumed that these payments are deterministic. Let $\psi_1, \ldots, \psi_{18}$ be the 18 selected fixed-rate bonds. Considering fixed-rate bond ψ_i, denote by $U_i(t)$ its value at time t, by $\hat{U}_i(t)$ its market value at time t and by $S_{i,1}, \ldots, S_{i,m_i}$ its payments times. Let x_i be the position of asset ψ_i in the hedging portfolio.



(a) Allocation of the hedging portfolio to 18 government bonds of varying countries. Each bond is represented by one dot whose size differs by its allocation.



(b) Sensitivity of the liabilities and hedging portfolio combined to interest rate changes (of 1 basis point). The net result (green bar) after hedging and the sensitivity to parallel changes in the interest rate (red dotted line) can be seen.

Figure 5.6: Portfolio to hedge the pension liabilities in Figure 5.5 as of September 2007.

The swap rates are known at times $t_0, \ldots, t_{24}$. Therefore $V(t)$ and $U_1(t), \ldots, U_{18}(t)$ can be calculated for any $t \in \{t_0, \ldots, t_{24}\}$. The market values of the assets $\hat{U}_1(t), \ldots, \hat{U}_{18}(t)$ are known at these time points as well. It is thus already possible to assess the value of the liabilities and hedging portfolio throughout time. However, coupons are received and pension payments are made. This money should be taken into account as well because it does not disappear. The cash position of the pension fund should therefore be registered at all time points.

Let $c(t)$ be the cash position at time t and assume $c(t_0) = 0$ — this is a reasonable assumption since any money not spent when the portfolio is constructed will be invested in a return portfolio. The cash position of the pension fund can be modeled using the payments of the liabilities and assets. Note that the pension fund receives interest on any cash position it has, and should pay interest if it borrows money. Borrowing is only allowed for comparative reasons and short positions in government bonds are still not possible. The cash position at time $t_i, i \in \{1, \ldots, 24\}$ is then:

$$c\left(t_i\right) = c\left(t_{i-1}\right) \mathbb{E}^{\mathbb{Q}}\left[\frac{M\left(t_i\right)}{M\left(t_{i-1}\right)}\right] - \sum_{j=1}^{n} \mathbb{1}_{t_{i-1} < T_j \leq t_i} L\left(T_j\right) + \sum_{k=1}^{18} \sum_{j=1}^{m_i} \mathbb{1}_{t_{i-1} < S_{k,j} < t_i} x_k H_k\left(S_{k,j}\right), \tag{5.4}$$

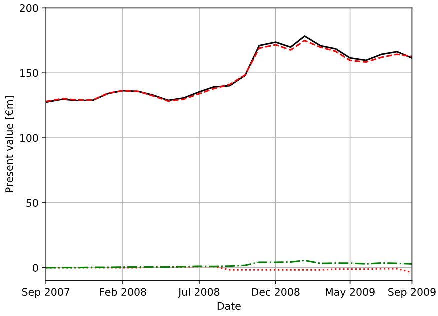
where $H_k\left(S_{k,j}\right)$ is the pay-off of fixed-rate bond ψ_k at its j-th payment time $S_{k,j}$ as defined in Subsection 2.3.2. Accrual between the measurement times is not taken into account, i.e. it is assumed that all payments happen at the end of the month. Although this is in fact a common day to make payments, this assumption could result in small errors.

Using the defined quantities the net result of the hedge can be calculated. This is equal to the net difference in value over time:

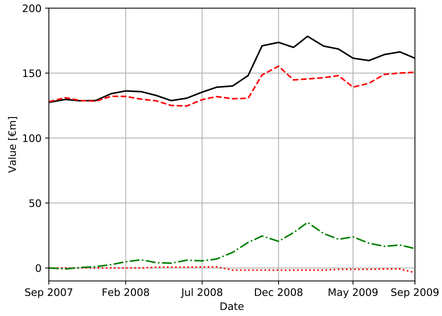
$$z(t_j) = V(t_j) - V(t_0) + \sum_{i=1}^{18} x_i U_i(t_j) - x_i U_i(t_0) + c(t_j) - c(t_0).$$

Additionally, the values of the assets in $z(t_j)$ may be substituted with the market values of the assets:

$$\hat{z}(t_j) = V(t_j) - V(t_0) + \sum_{i=1}^{18} x_i \hat{U}_i(t_j) - x_i \hat{U}_i(t_0) + c(t_j) - c(t_0).$$



(a) Performance of the hedging portfolio using historical interest rates. Liabilities value $-V(t)$ (solid black), hedging portfolio value $\sum_{i=1}^{18} x_i U_i(t)$ (dashed red), cash position $c(t)$ (dotted red) and net result $-z(t)$ (dashed dotted green) have been plotted.



(b) Performance of the hedging portfolio using historical interest rates and asset prices. Liabilities value $-V(t)$ (solid black), hedging portfolio market value $\sum_{i=1}^{18} x_i \hat{U}_i(t)$ (dashed red), cash position $c(t)$ (dotted red) and net result $-\hat{z}(t)$ (dashed dotted green) have been plotted.

Figure 5.7: Performance of the hedging portfolio of Figure 5.6 from September 2007 to September 2009.

These results have been plotted in Figure 5.7, where some quantities have been multiplied by -1 to enable a good interpretation of the results.

The net result in Figure 5.7a is almost equal to zero: the portfolio performs rather well. There are two explanations for the inaccuracies in this hedge. First, the delta sensitivities were not entirely hedged (as seen in Figure 5.6b) — the hedging portfolio does not fully hedge against long-term interest rate changes. Secondly, a small part of the hedging inaccuracy may arise because second-order sensitivities have not been taken into account.

The performance of this same hedging portfolio has worsened considerably in Figure 5.7b: the net result deviates significantly from zero. This is explained by changes in the z-spreads of the assets. In the framework of this thesis, the risk of a specific bond relative to the risk-free rate is modeled by its z-spread (Remark 3.2). This ensures that the theoretical and market values are equal. During the credit crisis the market changed its perception of sovereign risks: some countries such as Belgium and France were perceived as more risky. This led to increased z-spreads while the z-spread is constant in the framework of this thesis. Theoretical and market values are therefore no longer equal after t_0. This explains the results in Figure 5.7b.

Government bonds may be extended to incorporate z-spread changes. However, this has not been done in this thesis — instead the approach taken by pension funds is used. Pension funds frequently rebalance their portfolios and incorporate their strategic country beliefs in this procedure — this is both incorporated in Problem Formulation 5.2. Timely rebalancing ensures that the net value difference will not be as large as in Figure 5.7b — rebalancing would have been done in the meantime. Country allocations may also help to minimize this risk: Figure 5.8 shows how results improve slightly after setting prudent country allocations.

5.3.2. Performance using simulated processes

Simulation of the processes that drive a pension fund's present value may help in the assessment of hedging performance in multiple ways. One can calculate the impact that interest rate changes would have on both sides of the balance sheet — this was done in Subsection 5.3.1 in a backward-

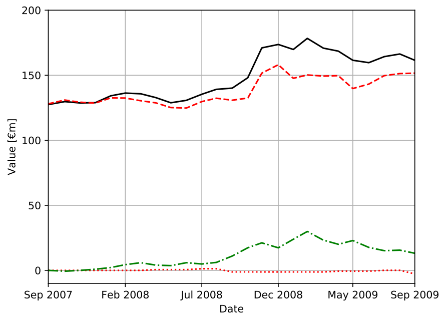


Figure 5.8: Performance of the hedging portfolio *with country allocation* using historical interest rates and asset prices. The target country allocation has been set to 10% in Belgium and France with an allowed deviation from this target of 5%. The allocation to the riskier countries in the EU decreases as a result of this in comparison with Figure 5.6. Liabilities value $-V(t)$ (solid black), hedging portfolio market value $\sum_{i=1}^{18} x_i \hat{U}_i(t)$ (dashed red), cash position $c(t)$ (dotted red) and net result $-z(t)$ (dashed dotted green) have been plotted.

looking manner but can additionally be done in a forward-looking manner. This will be done shortly. The other way to test the hedging portfolio through simulation is more involved: one can simulate all future cash flows. The net result — i.e. the remaining money — at termination of the pension fund gives information on the performance of the hedge. Results will also be given using this testing methodology.

Simulation of the interest rate

A hedging portfolio is calculated as of 31 October 2019 for the liabilities specified in Figure 5.5. The investment universe consists of 180 government bonds issued by Austria, Belgium, Finland, France, Germany and the Netherlands[8]. Bonds with outstanding debt below €500M have been omitted in this investment universe because of reduced liquidity and bonds with a maturity longer than 75 years have been removed from it because the pension fund makes its last (expected) payment in 75 years. Additionally, country allocations are set as specified in Table 5.1 — this reflects realistic country allocations of a pension fund in the Netherlands. The budget has been set to €400M[9] and at most 10% of this budget may be allocated to a single bond. This yields a hedging portfolio of 17 government bonds that is detailed in Figure 5.9.

The interest rate is then simulated using the Hull-White model given in Section 3.2 to test the performance of the hedging portfolio after interest rate changes. The value of the liabilities and the hedging portfolio can be calculated for each Monte-Carlo path. Combining this with the cash position yields the net result $z(t_j)$ for all times and paths. These quantities were defined in Subsection 5.3.1 and therefore not repeated here. The results have been added to Figure 5.10.

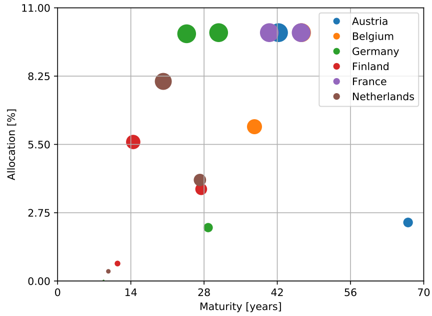
In Figure 5.10b five paths have been plotted. It is clear that the hedging portfolio performs well and is relatively invariant to interest rate changes. One does notice that the net result is not entirely

[8]The government bonds in the *Bloomberg Barclays Euro-Aggregate Index* have been downloaded and were then further filtered to fit the requirements in the manner described in Section 5.1. This index consists of euro-denominated bonds or bonds issued in one of the EU legacy currencies, with some specific rules and restrictions. More information and the dataset itself can be found at Barclays Live (https://live.barcap.com).

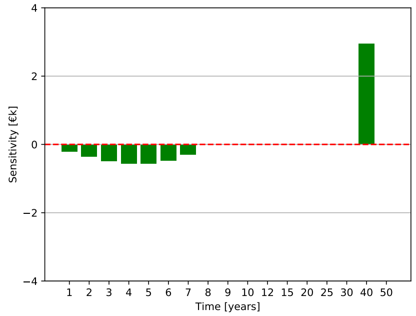
[9]In Subsection 5.3.1 a hedging portfolio for the same pension fund cash flows was found with a budget of €128M. This has been updated to €400M because the present value of the liabilities has increased as a result of the lower interest rates.

Country	Target allocation	Margin
Germany	27.5%	10.0%
The Netherlands	22.5%	10.0%
France	20.0%	10.0%
Austria	10.0%	10.0%
Belgium	10.0%	10.0%
Finland	10.0%	10.0%

Table 5.1: Proposed country allocation as of October 2019. The country allocation in Germany for instance must be in the range [17.5%, 37.5%] while the optimization algorithm aims to reach the target allocation of 27.5%. The majority of the exposure would be in Germany and the Netherlands — the EU countries with the highest credit rating.

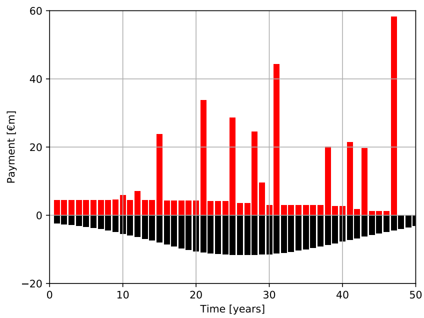


(a) Allocation of the hedging portfolio to 17 government bonds of varying countries. Each bond is represented by one dot whose size differs by its allocation.

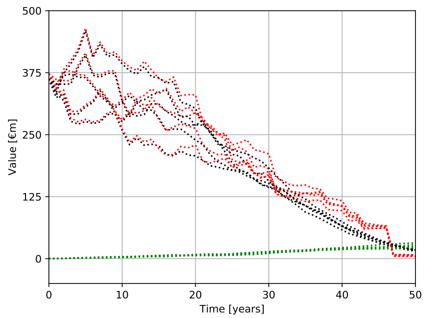


(b) Sensitivity of the liabilities and hedging portfolio combined to interest rate changes (of 1 basis point). The net result (green bar) and the sensitivity to parallel changes in the interest rate (red dotted line) can be seen.

Figure 5.9: Portfolio to hedge the pension liabilities in Figure 5.5 as of October 2019.

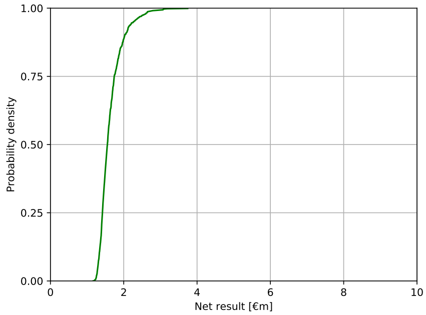


(a) Payments of the liabilities (black, negative) and hedging portfolio (red, positive). Note that coupon payments are frequently received and notional payments are received when a bond matures.

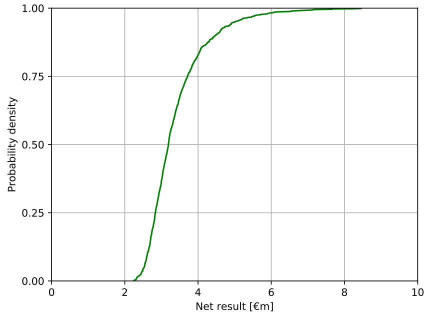


(b) Five Monte-Carlo paths of the liabilities value $-V(t)$ (dotted black), portfolio value $\sum_{i=1}^{17} x_i U_i(t)$ (dotted red) and net result $z(t)$ (dotted green). Note that there is a small downward drift in the net result.

Figure 5.10: Performance of the hedging portfolio of Figure 5.9b from October 2019 to October 2069.



(a) Empirical CDF of $z(t_5)$.



(b) Empirical CDF of $z(t_{10})$.

Figure 5.11: Empirical cumulated distribution function of the net result $z(t)$ estimated from 1,000 Monte-Carlo simulation. This uses the hedging portfolio of Figure 5.9 that hedges the liabilities of Figure 5.5.

equal to zero — it in fact shows an upward drift. There are two reasons for this trend. One is that the delta sensitivities of the portfolio are not entirely hedged (as seen in Figure 5.9b) because a parallel change in the discount curve is the most important for pension funds — the hedging strategy therefore aims to offset under- and overhedges. Additionally, the cash flows of the liabilities and hedging portfolio do not perfectly coincide (Figure 5.10a): at some times there is a positive cash position (which yields interest payments) and at other times money must be borrowed to pay the pension benefits — interest is then paid on this short-term debt. It should be noted that deviations are still small in comparison with the size of the liabilities.

Figure 5.11 shows the empirical Cumulative Distribution Functino (CDF) of the net result $z(t_j)$ in five years and 10 years calculated from 1,000 Monte-Carlo simulations. This shows that the volatility of the net result is small: the probability of a net position greater than €2.5M in five years for instance is around 2.8%. Thus it is clear that this hedging portfolio entails a viable asset allocation for a risk-averse pension fund.

It should be noted that the situation illustrated in Figure 5.10b would in reality never arise: pension funds would always rebalance their hedging portfolios at frequent intervals. This is done to ensure the hedging portfolio continues to track the liabilities well, but also because the expected liabilities generally change with time. As mentioned before, this model also allows for doing this. However, a good static hedge is the basis of a good dynamic hedge. It is therefore a good result that the portfolio would in principle not need rebalancing in the upcoming years — this sound performance of the static hedge is a solid result.

Simulation of the interest rate and pension fund dynamics

While only the interest rate was simulated in the previous result, it is also possible to simulate all cash flows of a pension fund. To this end a pension fund is defined so that its liabilities may be simulated using Chapter 4. The pension fund is small and consists of 93 people aged 50–80: three people per age. These people have earned 25% above the average salary for their age group throughout their career and all individuals that have not reached the pensionable age yet currently work full-time. Pension rights at this moment equal 70% of a person's average (past) salary indexed with inflation. This constructed dataset has been added to Appendix B.2.

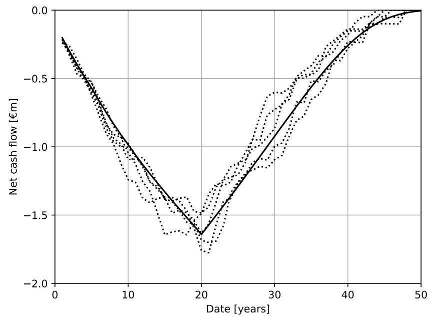


Figure 5.12: Expected value $\mathbb{E}^{\mathbb{P}}\left[L(t) \mid \mathscr{F}_{t_0}\right]$ and 10 Monte-Carlo paths of $L(t)$.

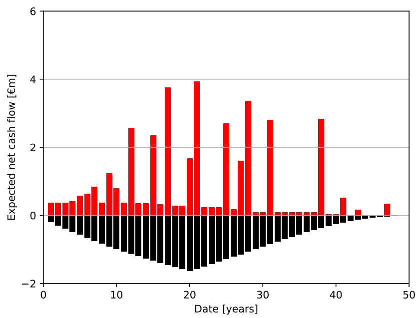


Figure 5.13: Payments of the liabilities (black, negative) and hedging portfolio (red, positive). Note that coupon payments are frequently received and notional payments are received when a bond matures.

The present value of the liabilities is (as of 31 October 2019) €35.7M and the pension fund has €40.0M in assets, meaning that its coverage ratio is 110.9%. The expected value of these pension liabilities and 10 Monte-Carlo paths throughout time have been added to Figure 5.12.

A hedging portfolio is calculated for these liabilities as of 31 October 2019. The investment universe of 180 government bonds of the earlier simulation is used again. The resulting hedging portfolio contains 23 government bonds with a total present value of €36.1M. Specifics on this hedging portfolio have been added to Appendix B.3. The yearly cash flows of this hedging portfolio and the yearly expected cash flows of the liabilities have been added to Figure 5.13.

These results may be used to calculate the variable of interest: the cash position of the pension fund throughout time. This cash position can be calculated for the next 50 years as in Equation (5.4). Rather than using the expected value of the liabilities, the cash position can be calculated for each of the Monte-Carlo paths. These results have been added to Figure 5.14.

This cash position oscillates around zero. This is a reasonable result: there is a great influx of cash when a bond matures but money must be borrowed (or liquidated from the return portfolio) to pay pension liabilities at other times. The expected value of the cash position when the pension fund is liquidated equals zero — this should be the case since a pension funds does not aim to make

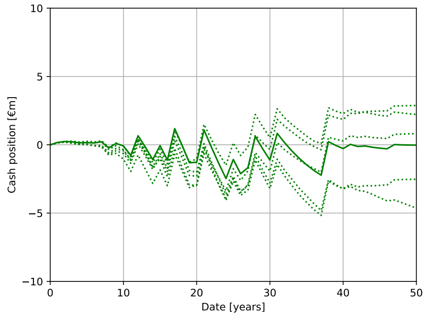
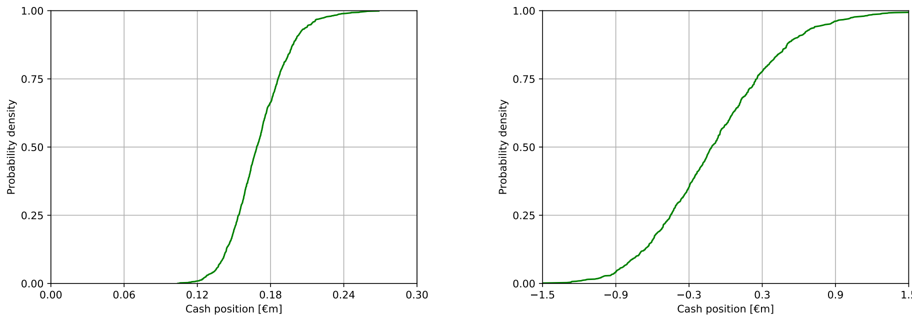


Figure 5.14: Cash position of the pension fund throughout time arising from the hedging portfolio — part of the budget not invested in the hedging portfolio is left out of the consideration. The average of 1,000 Monte-Carlo paths (solid green) and five Monte-Carlo paths (dotted green) have been added.



(a) Empirical CDF of $c(t_5)$.

(b) Empirical CDF of $c(t_{10})$.

Figure 5.15: Empirical cumulated distribution function of the cash position $c(t)$ estimated from 1,000 Monte-Carlo simulation. This uses the hedging portfolio calculated for the example pension fund (Appendix B.2) as of 31 October 2019.

a profit but at the same time should not make a loss either. However, there are quite some paths that deviate significantly from this expected value implying that a profit or loss could potentially be made. It is therefore relevant to look at the probability distribution of the cash position.

This can be done using the CDF. The CDF of $c(t_5)$ and $c(t_{10})$ may be estimated from the Monte-Carlo simulation. These results have been added to Figure 5.15. This shows that deviations in the coming 10 years are relatively small compared to the portfolio size.

The main conclusion that can be drawn from these simulations is two-fold. First, the potential risk is significant at long maturities and one should not use this hedging portfolio as a static hedge for such a long time. Secondly, the results are exciting: all inputs change but the hedge nonetheless remains rather adequate on a long maturity such as 50 years — the empirical expected cash position equals zero. This shows the potential of this methodology if rebalancing is done at frequent time periods.

5.4. Sensitivity to parameter changes

In Subsection 5.2.1 the weights in the optimization problem were normalized and set based on discussions with pension fund managers. The performance of the resulting hedging portfolio — that evidently depends on the values of these weights — was assessed in Section 5.3. In these analyses the weights were assumed to be certain and the sensitivity to changes in these weights was not investigated. This is done in this section. The weights in Equation (5.3) are used as the base on which further changes are made.

The hedging portfolio of Figure 5.9 — that hedges the liabilities of Figure 5.5 as of October 2019 — is taken as the base case. The sensitivity to changes in the optimization weights is then assessed in a numerical procedure. This is done by decreasing and increasing the weights (as long as they remain nonnegative) and calculating new hedging portfolios for these changed weights.

To formalize this notion, let $\alpha'_1, \ldots, \alpha'_5$ the current normalized weights. Let $h \in \mathbb{R}_{\geq 0}$ the radius of the sensitivity analysis, i.e. weights in the range

$$\left[\max\left(\alpha'_i - h, 0\right), \alpha'_i + h\right], \quad \forall i \in \{1, \ldots, 5\}$$

will be tested. The maximum arises because weights must be nonnegative. To actually perform the sensitivity analysis, the range is discretized into N time points. For each $i \in \{1, \ldots, 5\}$, N optimal hedging portfolios are then calculated by changing the i-th optimization weight while keeping the others unchanged. This is done for $h = 10$ and $N = 25$.

The change in the hedging portfolio as a result of transformed weights is measured as the difference in asset value as a percentage of the budget. Given asset values $\mathbf{u}$ and two hedging portfolios with allocations $\mathbf{x}$ and $\bar{\mathbf{x}}$, the difference between the portfolios is then:

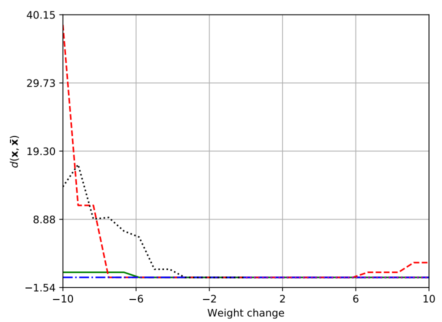
$$d\left(\mathbf{x}, \bar{\mathbf{x}}\right) = \frac{|\mathbf{x} - \bar{\mathbf{x}}| \cdot \mathbf{u}}{b}.$$

Note that $d\left(\mathbf{x}, \bar{\mathbf{x}}\right) \in [0\%, 200\%]$ and is maximal if $\mathbf{x}$ and $\bar{\mathbf{x}}$ share no assets and both portfolios spend the full budget. Numerical sensitivity analysis of $d\left(\mathbf{x}, \bar{\mathbf{x}}\right)$ has been added to Figure 5.16a.

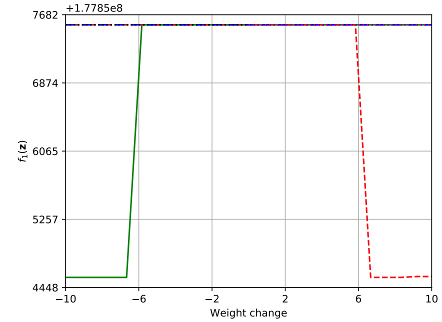
Alternatively, the impact that changes of the optimization weights have on the (terms of the) objective function may be calculated. This gives more insight in the relationship between the weights and the objective function. These results have been added to Figure 5.16.

It is striking that the sensitivities of $f_3(\mathbf{z})$ and $f_5(\mathbf{z})$ (Figures 5.16d and 5.16f) to weight changes are equal in shape. However, this is to be expected since no current portfolio was set and transaction costs are therefore a factor of the investment costs.

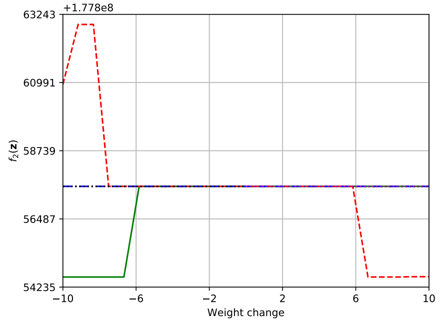
From the sensitivity analysis it is evident that the solution is very stable to changes in the optimization weights. Although the hedging portfolio does reflect the choice of weights, major differences only occur when very different weights are chosen. This shows that the methodology for choosing the $\alpha'_1, \ldots, \alpha'_5$ that was employed in Subsection 5.2.1 — a methodology that was not exact but based on expert opinions — is not problematic to reaching accurate and stable results.



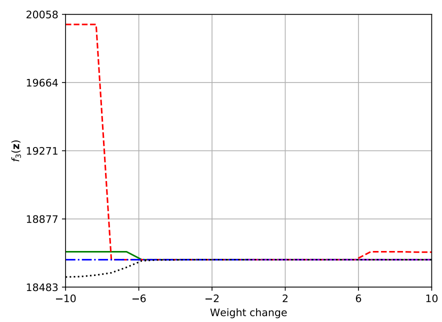
(a) Change of the portfolio $d(\mathbf{x}, \bar{\mathbf{x}})$.



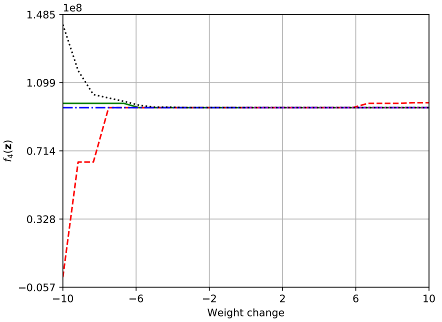
(b) Parallel change term $f_1(\mathbf{z})$ of the objective function.



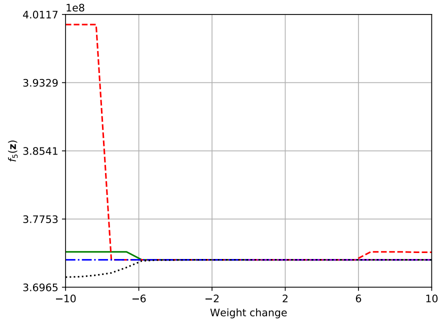
(c) Interest rate sensitivity term $f_2(\mathbf{z})$ of the objective function.



(d) Transaction costs term $f_3(\mathbf{z})$ of the objective function.



(e) Country allocations term $f_4(\mathbf{z})$ of the objective function.



(f) Investment costs term $f_5(\mathbf{z})$ of the objective function.

Figure 5.16: Sensitivity of the portfolio in Figure 5.9 to changes in the (normalized) optimization weights relative to the base weights of Equation (5.3). The sensitivities to changes in α'_1 (solid green), α'_2 (dashed red), α'_3 (dashed dotted blue), α'_4 (dotted black) and α'_5 (dotted magenta) have been plotted. The sensitivity to changes in α'_5 can only be calculated for positive changes because $\alpha'_5 = 0$ in Equation (5.3) and weights must be nonnegative.

5.5. Yield of the hedging portfolio

The expected return of the hedging portfolio has thus far not been taken into account in any way: the optimization procedure solely minimized the different risk measures and costs. 'Investment return' was introduced in Section 5.1 as one of the targets of pension funds. However, it was disconsidered in the final optimization problem (Problem Formulation 5.2) because combining risk measures and expected return is not elementary.

A different approach to incorporating yield is proposed in this section. This is done using the *yield to maturity* (Definition 5.1): the annual return one attains if one holds a bond until it matures. Rather than adding the YTM as another term in the objective function, a minimum yield constraint is added to the optimization problem. This ensures that the optimization algorithm gives the optimal portfolio — in the sense that the objective function of Problem Formulation 5.2 is minimal — with at least the requested YTM.

This idea will now be stated formally. The notation of Section 5.1 will be used to do this.

Let (as before) $\mathbf{y}$ the YTM of the assets in the investment universe. Suppose the portfolio's YTM should be at least $\bar{y} \in \mathbb{R}$. Then the following constraint should be added to find a portfolio whose YTM is at least $\bar{y}$:

$$\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{y}) \geq b\bar{y}.$$

Note that the YTM of the portfolio is calculated as the weighted average of the YTM of the assets — by doing this one implicitly assumes (as in Section 5.1) that one is able to reinvest the proceeds at the same rate of return as before. This is a linear constraint and it is therefore straightforward to add this constraint to Problem Formulation 5.2.

It is now possible to investigate how the optimal hedging portfolio changes as the YTM increases. This is done for the first results of Subsection 5.3.2, where a hedging portfolio (Figure 5.9) was calculated to hedge the risks associated with the pension liabilities of Figure 5.5 as of 31 October 2019.

A large number $N \in \mathbb{N}$ of different portfolios should be calculated with different minimum YTMs. First, calculate a solution without a YTM constraint. Let $\bar{y}_1$ the YTM of this initial portfolio. Secondly, choose an upper bound $\bar{y}_N$ of the YTM so that no portfolio exists that has at least this YTM. A reasonable value would be the maximum of the YTM of the assets:

$$\bar{y}_N = \max_{j \in \{1,\dots,m\}} (\mathbf{y}_i).$$

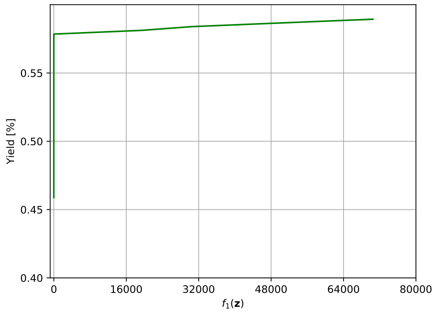
Thirdly, discretize the interval $[\bar{y}_1, \bar{y}_N]$ in N points, giving $\bar{y}_1, \dots, \bar{y}_N$ with $\bar{y}_1 < \dots < \bar{y}_N$.

Note that with this discretization $\forall i \in \{2, \dots, N\}$:

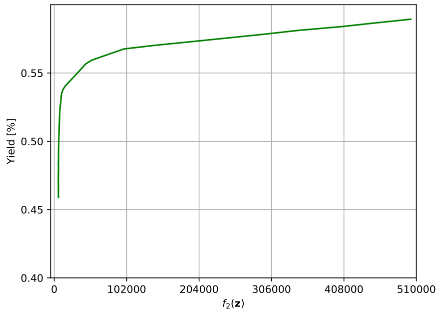
$$\mathbf{x} \cdot (\mathbf{u} \odot \mathbf{y}) \geq b\bar{y}_i \implies \mathbf{x} \cdot (\mathbf{u} \odot \mathbf{y}) \geq b\bar{y}_{i-1},$$

i.e. any solution that is feasible with minimum YTM $\bar{y}_i$ is also feasible with minimum YTM $\bar{y}_{i-1}$. Thus the constraints become progressively tighter as $\bar{y}_i$ increases. Because the Simplex algorithm only returns solutions that fulfill all constraints (Dantzig et al. 1955, p. 191), the objective function must therefore worsen as the minimum YTM is increased. The one exception to this is of course the case in which no feasible solution with the given YTM exists — subsequent optimization problems with even tighter constraints would then also be infeasible.

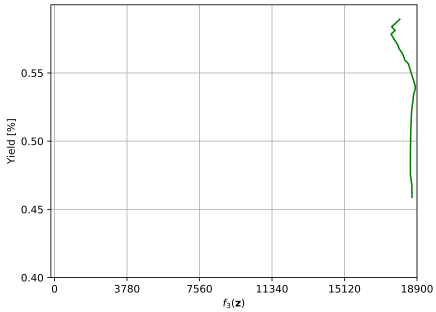
It is now possible to calculate the best portfolio for the pension fund given any yield. These results have been added to Figure 5.17. In this figure it can be clearly observed that each part of the objective function worsens relative to the initial optimum as the yield constraint is tightened. A pension



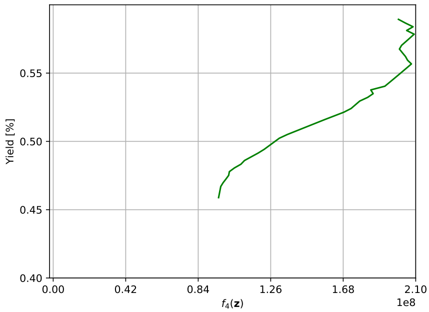
(a) Parallel change term $f_1(\mathbf{z})$ of the objective function.



(b) Interest rate sensitivity term $f_2(\mathbf{z})$ of the objective function.



(c) Transaction costs term $f_3(\mathbf{z})$ of the objective function.



(d) Country allocation term $f_4(\mathbf{z})$ of the objective function.

Figure 5.17: Objective function of optimal hedging portfolios at a number of yields. This hedges the liabilities of Figure 5.5. Weights have been set as proposed in Equation (5.3) — $f_5(\mathbf{z})$ has been omitted because $\alpha_5' = 0$.

fund might use this to find a portfolio with a higher yield that is still acceptable from a risk and cost perspective. In Figure 5.17 for instance, $f_1(\mathbf{z})$, $f_2(\mathbf{z})$ and $f_3(\mathbf{z})$ remain almost unchanged for higher-yielding portfolios. The country allocation does drift from the target but is still within the allowed range — otherwise there would be no feasible solution. Many pension funds would prefer the portfolio with 0.54% YTM over the initial portfolio as the objective function is very similar but yield improves from 0.46% to 0.54%.

These results are very useful in making an informed decision between YTM, risk and costs. A number of hedging portfolios have in fact been calculated by including yield. A comparison has then been made with the current hedging portfolios. The potential of this extension is evident from these results — they have been added to Appendix B.5.

5.6. Portfolio diversification

In Subsection 5.3.2 a hedging portfolio was calculated that hedges the liabilities given in Figure 5.5 as of 31 October 2019. This hedging portfolio consisted of 17 bonds. Some pension funds would like to hold a more diversified portfolio that is made up of a greater number of assets. This reduces the risk of unexpected behavior in the price of one of the bonds.

The hedging portfolio could also be used as a benchmark for existing pension fund managers, which would require a portfolio that consists of a greater number of assets as well. The managers would like to receive this larger number of bonds so that they can select bonds in line with their investment strategy.

Setting a minimum number of bonds is not possible with Problem Formulation 5.2. This possibility is therefore added in a different way: through an iterative procedure. An adapted version of the *bisection method* (Boyd and Vandenberghe 2004, pp. 145–146) is proposed to achieve this. Let $d_l \in (0,1]$ the target allocation per bond (as a percentage of the budget) so that there is a sufficient number of bonds in the portfolio, i.e. if the target is 40 bonds then $d_l = 2.5\%$. Let d_u the maximum allocation to a single bond in the current portfolio as a percentage of the budget, i.e.:

$$d_u = \max_{j \in \{1,\ldots,m\}} \left(\frac{\mathbf{x}_j \cdot \mathbf{u}_j}{b} \right),$$

with (as in Section 5.2) $\mathbf{x}_j$ the position of the hedging portfolio in one of the m assets and $\mathbf{u}_j$ the value of this asset.

If $d_u \leq d_l$ the target has already been achieved and no further calculations are required. However, if $d_u > d_l$ the maximum allowed allocation per asset should be decreased while the performance of the hedging portfolio should at the same time remain reasonable and not deteriorate significantly. A function $g(\mathbf{z})$ is defined — generally in terms of the $f_1(\mathbf{z}),\ldots,f_5(\mathbf{z})$ of Section 5.2 — that signifies when a hedging portfolio should be rejected:

$$g(\mathbf{z}) < 0 \implies \text{reject} \qquad \text{and} \qquad g(\mathbf{z}) \geq 0 \implies \text{accept}.$$

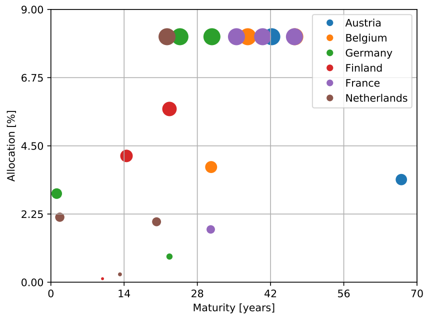
This function reflects the trade-off a pension fund would make between quality and diversification. In many cases this function only depends on the interest rate sensitivity, i.e. $g(\mathbf{z}) < 0 \iff \left|f_2(\mathbf{z})\right| > \epsilon$ for some $\epsilon \in \mathbb{R}_{\geq 0}$. The most diversified portfolio that is still accepted may then be found by repeatedly calculating optimal portfolios after decreasing the ‘individual asset’ restriction q (as in Problem Formulation 5.2), until a portfolio is rejected. The values of q should be chosen from the interval $[d_l, d_u]$.

This procedure can be implemented in an efficient manner using the bisection method. Note that this method may be used because the restrictions are only tightened as q decreases.

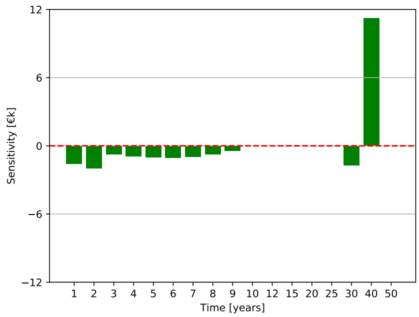
The procedure has been performed for the problem of Subsection 5.3.2 with $d_l = 5\%$ and:

$$g(\mathbf{z}) = 0.025 \left| \sum_{i=1}^{n} \frac{\partial L(t_0)}{\partial K_i} \right| - f_2(\mathbf{z}), \tag{5.5}$$

which compares the total unhedged delta risk with the delta risk of the liabilities. The result has been added to Figure 5.18. Instead of 17 bonds the hedging portfolio now consists of 20 bonds and the largest allocation to a single bond is 8.1% of the budget. As a consequence of this the delta sensitivities have deteriorated — this is clearly seen in Figure 5.18b. However, this portfolio is still deemed acceptable under Equation (5.5).



(a) Allocation of the hedging portfolio to 20 government bonds of varying countries. Each bond is represented by one dot whose size differs by its allocation.



(b) Sensitivity of the liabilities and hedging portfolio combined to interest rate changes (of 1 basis point). The net result (green bar) after hedging and the sensitivity to parallel changes in the interest rate (red dotted line) can be seen.

Figure 5.18: Diversified portfolio to hedge the pension liabilities in Figure 5.5 as of September 2007.

6

Interpretation and discussion

A novel hedging strategy tailored to pension funds in the Netherlands has been proposed in this thesis. In doing so, a number of interesting results have been found. These results are in this chapter discussed by their merits and limitations. Additionally, areas for future research are proposed.

A pivotal result in the hedging of interest rate risk is naturally the measurement of the interest rate sensitivity, because this determines how a hedge is constructed. This thesis has shown that there is a discrepancy between the measurement of interest rate sensitivity in theory and in industry. The result found in Section 3.5 provides a connection between these different worlds. Although this is a relevant contribution, a full translation methodology from one measure to the other has not been given. This is not vital for the end result — it would not cause a hedge to be constructed in a different manner. However, such a result would be relevant for a greater understanding of the methodologies.

This interest rate sensitivity is used in Sections 5.1 and 5.2 — along with many other results — to formulate an optimization problem. This problem yields a hedging portfolio that takes the requirements and preferences of a pension fund into account. The importance of this result should not be overlooked: the automatic construction of such a hedging portfolio that fulfills all known targets is very relevant. A hedging strategy for pension funds that specifies an optimal asset allocation attributable to individual assets has hitherto not been discussed in literature.

One could argue that this is not a novel result since all mathematical tools were already available — pension funds have indeed been implementing hedging portfolios for quite some time. But the proposed hedging strategy finds an *optimal solution* given the specific objectives and requirements of a pension fund. This could lead to significant performance improvements both with respect to risk and return.

A number of relevant results have been found that substantiate the performance of the proposed hedging strategy.

First, the performance of the hedging portfolio was tested on a historical 2007–2009 dataset. This crisis testing showed the solid performance of the hedging portfolio when the interest rate changes. At the same time, it also indicated an important weakness of the hedging strategy: changes in the riskiness of countries are not taken into account, i.e. no view is taken on the volatility of the z-spread. An important component of this is a country's probability of default. One could argue that the assumption to disregard spread risk is a fair assumption when only government bonds are used in the hedging portfolio because the probability of default is then negligible. However, the crisis testing clearly showed that this risk should not be overlooked in such scenarios.

The backtesting result itself could also be criticized. The performance of the hedging strategy was tested during the 2007–2009 financial crisis — a global economic crisis well-known for the collapse of Lehman Brothers. Perhaps more relevant in the context of government bonds would be backtesting during the 2008–2014 sovereign crisis. This started with the collapse of the Icelandic banking system in October 2008. The beginning of this crisis is thus already part of the time period under consideration and the extent to which more insight could be gained from extending the testing horizon should therefore not be overestimated.

Secondly, a number of processes were simulated to test the performance of hedging portfolios. It was observed that many factors besides the interest rate impact the efficiency of the hedging portfolio. This integration of all 'unknowns' in one model is a relevant contribution that shows the complexity of managing a pension fund. Nonetheless, Section 4.7 details a number of improvements that can be made to these simulations. The explicit modeling of all types of pension benefits (such as surviving relatives and disablement pensions) could significantly improve the accuracy of the model, as would a non-constant franchise and having a realistic indexation parameter that depends on inflation and the pension fund's financial position. The accuracy of the current simulations should nonetheless suffice to test the performance of a hedging portfolio.

This testing using different types of simulations showed excellent results, particularly on short time periods. It should be pointed out that the static hedges become less accurate as time progresses. This is a common result in hedging interest rate risk and could be remedied by frequent rebalancing.

Thirdly, the sensitivity to changes in the optimization weights was assessed. These optimization weights are a drawback of the proposed approach because they are defined based on expert judgments and cannot be calibrated easily in a quantitative manner. This could be a major disadvantage of the optimization approach. However, sensitivity analysis of these weights showed the stability of the optimization result — i.e. the global optimum of the defined problem. Although this does not necessarily hold for all pension fund cash flows, these results nonetheless give evidence to the robustness of the proposed approach.

6.1. Future research

After the discussion of the current work it is relevant to consider potential next steps for research. It is tempting to look at the next steps from a theoretical perspective. One might then think of second-order interest rate sensitivities, more extensive interest rate, inflation and mortality models, and the inclusion of more exotic derivatives in a hedging portfolio. These would indeed be very interesting future research topics.

Nonetheless, more relevant directions for future research are those arising from a need in industry. An attempt is therefore made to state such topics in a sufficient level of detail.

In this work the focus has been on calculating a static hedge. The current portfolio is one of the inputs in Problem Formulation 5.2 and the optimization approach can thus also be used for *rebalancing*. Throughout this thesis rebalancing has been purported as an almost golden solution that is able to improve hedging results when a static hedge performs insufficiently. However, 'rebalancing' also raises a number of questions.

When does rebalancing actually improve performance, considering that transaction costs will have to be made? How should rebalancing be done — should all interest rate risk be hedged or only part of it? How often should rebalancing be done, i.e. what is the optimal rebalancing time? Or an alternative question could be asked: when should rebalancing be done? Some preliminary results of rebalancing have been added to Appendix B.4, but there is certainly potential for further research in this direction.

The results in this thesis have not been related to the coverage ratios of pension funds — a hedging strategy was determined based on a set of objectives and requirements that did not include the coverage ratio. But the *coverage ratio* is of paramount importance for a pension fund: this determines whether pensions can be paid and possibly even indexed. It could therefore be beneficial to envisage an alternative approach that explicitly aims to maximize the coverage ratio rather than employ the proxies that have been used in this thesis. This could be an interesting direction for further research.

In Examples 2.1 and 2.2 and Figure 2.3 the financial situation of the ABP was described. Its challenging financial situation was related to its decision to hedge only a minority of its interest rate risk. In Figure 5.4 this same relation between the *hedging ratio* and coverage ratio was identified in the 50 largest pension funds of the Netherlands. However, these results are biased: interest rates have been decreasing in recent years and pension funds with high hedging ratios have benefited from this. A relevant question would thus be: what is the best hedging ratio for a pension fund? This should take its financial situation and the prevailing market outlook into account.

In Subsection 5.3.1 the impact of a changing market view on the prices of government bonds was evident, i.e. the accuracy of the hedge decreased due to the changing z-spreads of government bonds. An aim of further research could be to model the spread risk of these bonds, which is related to the probabilities of default of counterparties. Furthermore, a thorough analysis could be done to decide on reasonable actions in such scenarios. The benefit of this research might not be evident in good financial times but it could be very advantageous in a future financial crisis.

Closely related to this is another potential venue for further research. One could quantify the likely error of the hedging portfolio relative to the pension liabilities. This *tracking error* was calculated by an external risk system in Appendix B.5 and (among others) incorporates correlations between issuers. It could even be possible to incorporate this risk measure in the optimization function to minimize not only the net interest rate sensitivity but also the expected error of the hedging portfolio.

A number of relevant contributions have been made. At the same time, this work gives rise to many questions that could be answered in future research. Given the societal relevance of this topic, the research directions arising from a practical need are particularly interesting. Additional steps in those fields have the potential to significantly improve the asset management of pension funds in the Netherlands.

7

Conclusions

Pension funds are commonly perceived as 'dusty' and research into the inner workings of pension schemes is not at the forefront of mathematical research. However, the topic is very relevant since vast amounts of money have been saved by people for their retirements — particularly in the Netherlands — and this capital should be invested wisely to ensure the financial security of (future) pensioners. While working on this thesis this field of research has become even more relevant, with new problems arising and heated political discussions taking place.

In this thesis a novel approach to interest rate risk hedging for pension funds was proposed. Although general hedging strategies of such risks are well-known, the proposed approach takes into account the practicalities and regulations that surround pension funds — specifically in the Netherlands. It was found that an LP problem could be formulated that aptly incorporates all constraints and objectives of pension funds. Extensive performance assessments by means of backtests and simulations manifested the validity of this hedging strategy.

The main objective of this thesis was finding a hedging strategy on a micro-level that would be attributable to individual assets. This target has indeed been achieved.

It was hypothesized in the introduction that a hedging portfolio could be constructed in an automated manner that outperforms conventional hedging portfolios. Research has shown that the proposed approach certainly performs very well. However, a comparative approach is challenging because data availability is limited. Appendix B.5 nonetheless indicates that the proposed hedging strategy indeed outperforms current portfolios.

In the course of finding this hedging strategy, a number of other significant contributions have been made to the field of pension research. First, 'the pension world' has been described in a mathematical framework that includes the practicalities involved. Particularly interesting is the discussion of interest rate sensitivities and the key rate duration — this provides a 'glue' between a practical methodology and theoretical concepts.

A second relevant contribution is the model for pension fund cash flows. With this work the theoretical foundations of scholars have been combined to give a model that is well-suited to the contemporary financial climate and regulatory situation. Of particular interest is the ability to model negative interest rates and deflationary monetary policies.

The third — and most important — contribution is the robust, dependable hedging strategy tailored to pension funds that has been shown to perform as desired, also in a stress scenario.

A number of future research topics have been outlined in Chapter 6 — the topic of rebalancing notably is highly interesting. Rebalancing is already done by pension funds and has been included in the hedging strategy of this thesis. But further research is warranted on how rebalancing should exactly be done. Such works might benefit from the model of this thesis.

Perhaps the most important contributions of this thesis are practical in nature. The proposed approach to the hedging of interest rate risks could be applied *in its current state* to pension funds — principally when the return extension is included. The resulting hedging portfolio could then be an important input to investment decisions.

An alternative use-case could be benchmark construction. Market benchmarks are not applicable to hedging portfolios because they do not take the structure of future pension liabilities into account. Competitive benchmarks could be created for pension funds tailored to their specific financial situation and strategic beliefs. The performance of their hedging portfolios — both in terms of risk and return — could be assessed through a comparison with the automatically constructed benchmark.

The practical implications of this thesis should not be overlooked. It therefore seems fitting to express the hope this thesis could — as stated in the introduction — contribute to the financial security of (future) pensioners.

Bibliography

Algemeen Burgerlijk Pensioenfonds. (2019). Jaarverslag 2018. Retrieved August 20, 2019, from http://jaarverslag.abp.nl/docs/ABP_JV_2018/pdfs/ABP_JV_2018_00_Totaal.pdf

Autoriteit Financiële Markten. (n.d.). Mifid ii - transactierapportages. Retrieved November 7, 2019, from https://www.afm.nl/nl-nl/professionals/onderwerpen/mifid-2/transactierapportages

Battocchio, P., & Menoncin, F. (2004). Optimal pension management in a stochastic framework. *Insurance: Mathematics and Economics*, *34*(1), 79–95.

Belastingdienst. (2019). Wanneer bereikt u de aow-leeftijd. Retrieved May 27, 2019, from https://www.belastingdienst.nl/wps/wcm/connect/bldcontentnl/belastingdienst/prive/werk_en_inkomen/pensioen_en_andere_uitkeringen/wanneer_bereikt_u_de_aow_leeftijd/wanneer_bereikt_u_de_aow_leeftijd

Bianchetti, M. (2008). Two curves, one price: Pricing & hedging interest rate derivatives decoupling forwarding and discounting yield curves. *One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves*.

Bobeica, E., Ciccarelli, M., & Vansteenkiste, I. (2019). The link between labor cost and price inflation in the euro area. Retrieved from https://www.ecb.europa.eu/pub/pdf/scpwps/ecb.wp2235~69b97077ff.en.pdf

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Brigo, D., & Mercurio, F. (2007). *Interest rate models-theory and practice: With smile, inflation and credit*. Springer Science & Business Media.

Cairns, A. J., Blake, D., & Dowd, K. (2000). *Optimal dynamic asset allocation for defined-contribution pension plans*. Pensions Institute.

Cairns, A. J., Blake, D., & Dowd, K. (2006). Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control*, *30*(5), 843–877.

Carriere, J. F., & Shand, K. J. (1998). New salary functions for pension valuations. *North American Actuarial Journal*, *2*(3), 18–26.

Chen, D. H., Beetsma, R. M., Broeders, D. W., & Pelsser, A. A. (2017). Sustainability of participation in collective pension schemes: An option pricing approach. *Insurance: Mathematics and Economics*, *74*, 182–196.

Coleman, T. (2011). A guide to duration, dv01, and yield curve risk transformations. doi:10.2139/ssrn.1733227

Dantzig, G. B., Orden, A., Wolfe, P. et al. (1955). The generalized simplex method for minimizing a linear form under linear inequality restraints. *Pacific Journal of Mathematics*, *5*(2), 183–195.

De Nederlandsche Bank. (2005). Vaststelling van de methodiek voor de rentetermijnstructuur. Retrieved May 9, 2019, from https://www.toezicht.dnb.nl/binaries/50-212329.pdf

De Nederlandsche Bank. (2015a). Minimaal vereist eigen vermogen. Retrieved September 13, 2019, from https://www.toezicht.dnb.nl/2/50-202132.jsp

De Nederlandsche Bank. (2015b). Renteafdekking van pensioenfondsen. Retrieved September 23, 2019, from https://www.dnb.nl/binaries/AV293139%5C%20TK%5C%20Bijlage_tcm46-334285.pdf

De Nederlandsche Bank. (2015c). Vaststelling rentetermijnstructuur pensioenfondsen vanaf 15 juli 2015. Retrieved May 9, 2019, from https://www.toezicht.dnb.nl/2/50-234026.jsp

De Nederlandsche Bank. (2017). Technische voorzieningen - pensioenfondsen. Retrieved May 9, 2019, from https://www.toezicht.dnb.nl/2/6/50-234100.jsp

De Nederlandsche Bank. (2019). Eisen aan herstelplan van pensioenfondsen. Retrieved September 13, 2019, from https://www.toezicht.dnb.nl/3/50-237496.jsp

De Waegenaere, A., Joseph, A., Janssen, P., & Vellekoop, M. (2018). Het delen van langlevenrisico. *Netspar Industry Paper*.

Delbaen, F., & Schachermayer, W. (2006). *The mathematics of arbitrage*. Springer Science & Business Media.

Dondi, G. A. (2005). *Models and dynamic optimisation for the asset and liability management of pension funds* (Doctoral dissertation, ETH Zurich).

Draper, N., Armstrong, A. et al. (2007). *Gamma; a simulation model for ageing, pensions and public finances*. Bureau for Economic Policy Analysis.

European Central Bank. (n.d.). Definition of price stability. Retrieved November 4, 2019, from https://www.ecb.europa.eu/mopo/strategy/pricestab/html/index.en.html

Girosi, F., & King, G. (2007). Understanding the lee-carter mortality forecasting method.

Gurrieri, S., Nakabayashi, M., & Wong, T. (2009). Calibration methods of hull-white model. *Available at SSRN 1514192*.

Hagan, P. S. (2015). Calculating delta risks and hedges via waves. *Wilmott*, *2015*(76), 56–59.

Hagan, P. S., & West, G. (2008). Methods for constructing a yield curve. *Wilmott Magazine, May*, 70–81.

Henrard, M. (2012). Interest rate instruments and market conventions guide. *OpenGamma Quantitative Research*,

Ho, T. S. (1992). Key rate durations: Measures of interest rate risks. *The Journal of Fixed Income*, *2*(2), 29–44.

Horvath, F., de Jong, F., & Werker, B. J. (2018). Dynamic asset liability management under model uncertainty. *SSRN Electronic Journal*. doi:10.2139/ssrn.3101345

Hull, J., & White, A. (1990). Pricing interest-rate-derivative securities. *The Review of Financial Studies*, *3*(4), 573–592.

Jamshidian, F. (1989). An exact bond option formula. *The journal of Finance*, *44*(1), 205–209.

Jarrow, R., & Yildirim, Y. (2003). Pricing treasury inflation protected securities and related derivatives using an hjm model. *Journal of Financial and Quantitative Analysis*, *38*(2), 337–358.

Kroon, E., Wouters, A., & de Carvalho, R. L. (2017). Decomposing funding-ratio risk: Providing pension funds with key insights into their liabilities hedge mismatch and other factor exposures. *The Journal of Portfolio Management*, *43*(4), 71–86.

Lee, R. D., & Carter, L. R. (1992). Modeling and forecasting us mortality. *Journal of the American statistical association*, *87*(419), 659–671.

Maio, M., & de Jong, J.-J. (2015). The geometry of interest rate risk. Retrieved from https://uglyduckling.nl/library_files/PRE-PRINT-UD-2014-02.pdf

Marler, R. T., & Arora, J. S. (2010). The weighted sum method for multi-objective optimization: New insights. *Structural and multidisciplinary optimization*, *41*(6), 853–862.

Michielsen, T. et al. (2015). *Cpb asset liability management model for pension analyses*. Bureau for Economic Policy Analysis.

Nawalkha, S. K., & Soto, G. M. (2009). Managing interest rate risk: The next challenge? *Journal of Investment Management*, *7*(3), 86–100.

Nawalkha, S. K., Soto, G. M., & Beliaeva, N. A. (2005). *Interest rate risk modeling: The fixed income valuation course*. John Wiley & Sons.

Oosterlee, C. W., & Grzelak, L. A. (2019). *Mathematical modeling and computation in finance*. doi:10.1142/q0236. eprint: https://worldscientific.com/doi/pdf/10.1142/q0236

Pelsser, A., Salahnejhad, A., & Van den Akker, R. (2016). Market-consistent valuation of pension liabilities. *Netspar Design Paper*, *63*.

Peneva, E. V., & Rudd, J. B. (2017). The passthrough of labor costs to price inflation. *Journal of Money, Credit and Banking*, *49*(8), 1777–1802.

Reitano, R. R. (1992). Non-parallel yield curve shifts and immunization. *Journal of portfolio management*, *18*(3), 36–43.

Rijksoverheid. (n.d.). Opbouw pensioenstelsel. Retrieved July 16, 2019, from https://www.rijksoverheid.nl/onderwerpen/pensioen/opbouw-pensioenstelsel

Russo, V., & Torri, G. (2019). Calibration of one-factor and two-factor hull–white models using swaptions. *Computational Management Science*, *16*(1-2), 275–295.

S&P Global. (2018). 2017 annual sovereign default study and rating transitions. Retrieved May 6, 2019, from https://www.spratings.com/documents/20184/774196/2017+Annual+Sovereign+Default+Study+And+Rating+Transitions.pdf

Schrimpf, A., & Sushko, V. (2019). Beyond libor: A primer on the new reference rates. *BIS Quarterly Review, March*.

Shreve, S. E. (2004). *Stochastic calculus for finance ii: Continuous-time models*. Springer Science & Business Media.

Smolenaers, J. G. A., Lijesen, S. C. M., Vos, R. M., Pelsser, A. A. J., Spauwen, A. J. F., Steenbeek, O. W., … Visser, E. P. (2009). Principes voor de rentetermijnstructuur. Retrieved October 3, 2019, from https://www.ag-ai.nl/view.php?action=view&Pagina_Id=451

Spivak, M. (1965). *Calculus on manifolds: A modern approach to classical theorems of advanced calculus*. Addison-Wesley Publishing Company.

Van Berkum, F., Antonio, K., & Vellekoop, M. (2016). The impact of multiple structural changes on mortality predictions. *Scandinavian Actuarial Journal*, *2016*(7), 581–603.

Van den End, J. W. (2011). Statistical evidence on the mean reversion of interest rates.

Van Rooij, M., Siegmann, A., & Vlaar, P. J. (2004). Palmnet: A pension asset and liability model for the netherlands.

Wilmoth, J. R., Andreev, K., Jdanov, D., Glei, D. A., & Riffe, T. (2007). Methods protocol for the human mortality database.

Wolzak, M. (2019a). 'pensioenkortingen? die komen er niet'. *Het Financieele Dagblad*, 10–11. Retrieved from https://fd.nl/achtergrond/1317080/pensioenkortingen-die-komen-er-niet

Wolzak, M. (2019b). Debat over rekenrente laait weer op door brief 'prominenten'. *Het Financieele Dagblad*, 8. Retrieved from https://fd.nl/economie-politiek/1320170/rekenrente-weer-in-brandpunt-pensioendebat-door-brief-prominenten

Zadeh, L. (1963). Optimality and non-scalar-valued performance criteria. *IEEE transactions on Automatic Control*, *8*(1), 59–60.

A

Further results and specifications

A number of relevant results have not been given in detail in this thesis. These are the parameters to the calibrated models, analytical results that are not essential and extensions of the current methodology that are not used at this moment. These results have been added to this appendix.

A.1. Lee-Carter model parameters

The Lee-Carter model has been calibrated to historical mortality data (Appendix C.1). The procecure to do this has been described in Subsection 4.1.1. The estimated values of α_x, β_x have been placed in Table A.1 and the values of κ_{t_j} have been placed in Table A.2.

x	α_x	β_x	x	α_x	β_x	x	α_x	β_x
0	-5.0370	0.0153	34	-7.2038	0.0085	67	-3.9907	0.0092
1	-7.4184	0.0234	35	-7.1119	0.0080	68	-3.8898	0.0092
2	-7.9039	0.0225	36	-7.0493	0.0091	69	-3.7976	0.009
3	-8.1696	0.0220	37	-6.9735	0.0079	70	-3.7025	0.009
4	-8.3717	0.0212	38	-6.8804	0.0089	71	-3.5985	0.009
5	-8.5070	0.0242	39	-6.7774	0.0090	72	-3.4993	0.0091
6	-8.6193	0.0226	40	-6.6753	0.0086	73	-3.3988	0.0088
7	-8.6826	0.0228	41	-6.5833	0.0084	74	-3.2964	0.0087
8	-8.7277	0.0209	42	-6.4751	0.0090	75	-3.1946	0.0086
9	-8.8039	0.0216	43	-6.3647	0.0091	76	-3.0895	0.0084
10	-8.7721	0.0177	44	-6.2634	0.0090	77	-2.9843	0.0082
11	-8.7606	0.0181	45	-6.1614	0.0094	78	-2.8814	0.0078
12	-8.6291	0.0164	46	-6.0519	0.0089	79	-2.7784	0.0075
13	-8.5385	0.0166	47	-5.9451	0.0086	80	-2.6663	0.0071
14	-8.4633	0.0157	48	-5.8383	0.0085	81	-2.5572	0.0069
15	-8.2776	0.0172	49	-5.7448	0.0088	82	-2.4545	0.0066
16	-7.9248	0.0166	50	-5.6513	0.0083	83	-2.3459	0.0063
17	-7.8291	0.0158	51	-5.5400	0.0082	84	-2.2387	0.0061
18	-7.7122	0.0146	52	-5.4377	0.0085	85	-2.1324	0.0058
19	-7.6278	0.014	53	-5.3488	0.0084	86	-2.0223	0.0054
20	-7.6204	0.0129	54	-5.2501	0.0085	87	-1.9184	0.0049
21	-7.6019	0.0123	55	-5.1529	0.0084	88	-1.8147	0.0046
22	-7.6326	0.0124	56	-5.0568	0.0084	89	-1.7113	0.0043
23	-7.6170	0.0113	57	-4.9536	0.0085	90	-1.6125	0.0036
24	-7.6372	0.0100	58	-4.8614	0.0084	91	-1.5067	0.0033
25	-7.6181	0.0105	59	-4.7737	0.0085	92	-1.4119	0.0031
26	-7.5959	0.0087	60	-4.6630	0.0086	93	-1.3192	0.0029
27	-7.5996	0.0099	61	-4.5697	0.0087	94	-1.2313	0.0023
28	-7.5355	0.0091	62	-4.4775	0.0088	95	-1.1415	0.0019
29	-7.4720	0.0081	63	-4.3761	0.0090	96	-1.0542	0.0018
30	-7.4465	0.0088	64	-4.2853	0.0090	97	-0.9669	0.0019
31	-7.3791	0.0086	65	-4.1831	0.0089	98	-0.8863	0.0014
32	-7.3306	0.0087	66	-4.0892	0.0091	99	-0.8192	0.0022
33	-7.2513	0.0086						

Table A.1: Lee-Carter model parameters α_x and β_x using the methodology from Subsection 4.1.1 — x is the age of a person.

t_j	κ_{t_j}	t_j	κ_{t_j}
1967	35.0108	1992	0.8503
1968	38.3949	1993	6.8215
1969	38.7347	1994	0.5669
1970	38.3822	1995	0.5119
1971	36.6739	1996	0.505
1972	38.0991	1997	-3.5372
1973	32.6205	1998	-4.1749
1974	28.4247	1999	-3.2736
1975	30.7785	2000	-5.509
1976	29.0694	2001	-7.9442
1977	21.2541	2002	-8.2942
1978	23.1036	2003	-10.9083
1979	18.0069	2004	-18.9459
1980	16.7834	2005	-22.0806
1981	15.0751	2006	-26.2371
1982	14.1116	2007	-32.1008
1983	11.7933	2008	-33.3738
1984	11.1677	2009	-38.073
1985	11.5124	2010	-39.999
1986	11.6718	2011	-44.2773
1987	5.4782	2012	-42.7043
1988	4.7148	2013	-45.8537
1989	7.1912	2014	-51.9995
1990	4.7582	2015	-47.4072
1991	3.4319	2016	-48.805

Table A.2: Lee-Carter model parameter κ_{t_j} using the methodology from Subsection 4.1.1 — t_j is the time in years.

A.2. Salary process age-dependence parameters

In Section 4.3 the salary model was calibrated to 2016 data by the CBS — this dataset has been added to Appendix C.4. The values of the age-dependence parameters c_1, c_2 has been added to Table A.3 for a number of years. This shows that these parameters have been very stable throughout time.

Year	c_1	c_2	R^2
2007	-0.0023	0.1018	79%
2008	-0.0022	0.0989	80%
2009	-0.0021	0.0973	82%
2010	-0.0021	0.0980	84%
2011	-0.0022	0.1013	87%
2012	-0.0022	0.1027	88%
2013	-0.0023	0.1083	87%
2014	-0.0024	0.1118	86%
2015	-0.0024	0.1135	87%
2016	**-0.0024**	**0.1126**	**87%**

Table A.3: Age-dependence parameters of the salary process calibrated to 2007–2016 data.

A.3. Lee-Carter mortality expected value

The Lee-Carter model was described in Subsection 4.1.1. In Equation (4.4) the expected value of the mortality rate was used — a derivation of this result is given in this appendix.

Lemma A.1 *Let $m(x, t_j)$ and $\kappa(t)$ as in Equation (4.3). Then:*

$$\mathbb{E}[m(x,t_j)] = \exp\left(\alpha(x) + \beta(x)\left(\kappa(t_0) + \mu_\kappa(t_j - t_0)\right) + \frac{\beta^2(x)\sigma_\kappa^2(t_j - t_0)}{2}\right).$$

△

Proof. Note that:

$$\begin{aligned}
\kappa(t_j) &= \kappa(t_0) + \int_{t_0}^{t_j} \mu_\kappa \,\mathrm{d}s + \int_{t_0}^{t_j} \sigma_\kappa \,\mathrm{d}W^\kappa(s) \\
&= \kappa(t_0) + \mu_\kappa(t_j - t_0) + \sigma_\kappa\left(W^\kappa(t_j) - W^\kappa(t_0)\right) \\
&\sim N\left(\kappa(t_0) + \mu_\kappa(t_j - t_0), \sigma_\kappa^2(t_j - t_0)\right).
\end{aligned}$$

This implies that:

$$\log m(x,t_j) \sim N\left(\alpha(x) + \beta(x)\left(\kappa(t_0) + \mu_\kappa(t_j - t_0)\right), \beta^2(x)\sigma_\kappa^2(t_j - t_0)\right).$$

So, $m(x, t_j)$ is log-normally distributed and:

$$\mathbb{E}[m(x,t_j)] = \exp\left(\alpha(x) + \beta(x)\left(\kappa(t_0) + \mu_\kappa(t_j - t_0)\right) + \frac{\beta(x)^2\sigma_\kappa^2(t_j - t_0)}{2}\right).$$

□

A.4. Hull-White model analytical results

In Section 3.2 the Hull-White model was described. The analytical value of zero-coupon bond $P(t,T)$ at time t_0 was stated in Subsection 3.2.1. In this appendix a bit more background is given to this analytical result. Still, it is not derived entirely.

Suppose one would like to calculate two expected values of the money-market account:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}\,\middle|\,\mathscr{F}_t\right]=\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}_t\right], \tag{A.1}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{M(T)}{M(t)}\,\middle|\,\mathscr{F}_t\right]=\left[\exp\left(\int_t^T r(s)\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}_t\right]. \tag{A.2}$$

For processes that start at time t_0, Brigo and Mercurio (2007, p. 75) show that the integral involved is normally distributed:

$$\int_t^T r(s)\,\mathrm{d}s\,\middle|\,\mathscr{F}_t \sim \mathscr{N}\Bigg(B(t,T)\,[r(t)-\alpha(t)]\,+\log\left(\frac{\hat{P}(t_0,t)}{\hat{P}(t_0,T)}\right)$$
$$+\frac{1}{2}\,[V(t_0,T)-V(t_0,t)]\,,V(t,T)\Bigg),$$

with:

$$B(t,T)=\frac{1}{\lambda}\,[1-\exp(-\lambda(T-t))]\,,$$
$$V(t,T)=\frac{\eta^2}{\lambda^2}\left[T-t+\frac{2}{\lambda}\exp(-\lambda(T-t))-\frac{1}{2\lambda}\exp(-2\lambda(T-t))-\frac{3}{2\lambda}\right]$$
$$\alpha(t)=\hat{f}(t_0,t)+\frac{\eta^2}{2\lambda^2}\left(1-\exp(-\lambda(t-t_0))\right)^2.$$

By the log-normality, the expected value of Equation (A.1) can be found:

$$\mathbb{E}\left[\frac{M(t)}{M(T)}\,\middle|\,\mathscr{F}_t\right]=\mathbb{E}\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}_t\right]$$
$$=\exp\Bigg(-B(t,T)[r(t)-\alpha(t)]-\log\left(\frac{\hat{P}(t_0,t)}{\hat{P}(t_0,T)}\right)$$
$$+\frac{1}{2}\,[-V(t_0,T)+V(t_0,t)+V(t,T)]\Bigg)$$
$$=A_1(t,T)\exp\left(-B(t,T)r(t)\right), \tag{A.3}$$

with:

$$A_1(t,T)=\frac{\hat{P}(t_0,T)}{\hat{P}(t_0,t)}\cdot\exp\left(B(t,T)\hat{f}(t_0,t)\right)$$
$$\cdot\exp\left(\frac{1}{2}\left(-V(t_0,T)+V(t_0,t)+V(t,T)\right)\right)$$
$$\cdot\exp\left(B(t,T)\frac{\eta^2}{2\lambda^2}\left(1-\exp(-\lambda(t-t_0))\right)^2\right).$$

Brigo and Mercurio (2007, p. 75) simplify this futher, yielding:

$$A_1(t,T)=\frac{\hat{P}(t_0,T)}{\hat{P}(t_0,t)}\cdot\exp\left(B(t,T)\hat{f}(t_0,t)-\frac{\eta^2}{4\lambda}\left(1-\exp(-2\lambda(t-t_0))\right)B^2(t,T)\right).$$

This result was given in Subsection 3.2.1.

Similarly, the other expected value (Equation (A.2)) may be found:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\frac{M(T)}{M(t)}\,\middle|\,\mathscr{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T r(s)\,\mathrm{d}s\right)\,\middle|\,\mathscr{F}_t\right]\\ &= A_2(t,T)\exp\left(B(t,T)r(t)\right),\end{aligned}$$

with:

$$\begin{aligned}A_2(t,T) = \frac{\hat{P}(t_0,t)}{\hat{P}(t_0,T)} &\cdot \exp\left(-B(t,T)\hat{f}(t_0,t)\right)\\ &\cdot \exp\left(\frac{1}{2}\left(V(t_0,T)-V(t_0,t)+V(t,T)\right)\right)\\ &\cdot \exp\left(-B(t,T)\frac{\eta^2}{2\lambda^2}\left(1-\exp(-\lambda(t-t_0))\right)^2\right).\end{aligned}$$

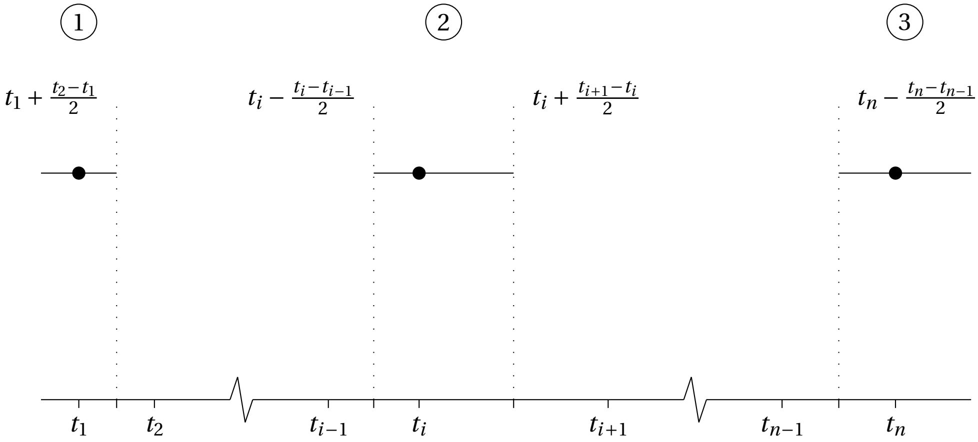


Figure A.1: Function specified in Equation (A.4) with shocks at t_1, t_i and t_n — ranges have been marked with circled numbers.

A.5. Rectangular interpolation for interest rate sensitivity

In Section 3.5 alternative measures of interest rate sensitivity were described. One of these measures are key rate durations. This required a certain triangular interpolation scheme. Scholars have proposed an alternative rectangular interpolation scheme. Mirroring Equation (3.22) and Figure 3.4, this rectangular scheme is defined by Equation (A.4) and illustrated in Figure A.1.

$$H_{T_i}(t) = \begin{cases} 1, & t_i = t_1, \quad t \leq t_i + \dfrac{t_{i+1} - t_i}{2} & \text{①} \\ 1, & t_1 < t_i < t_n, \quad t_i - \dfrac{t_i - t_{i-1}}{2} < t \leq t_i + \dfrac{t_{i+1} - t_i}{2} & \text{②} \\ 1, & t_i = t_n, \quad t_i - \dfrac{t_i - t_{i-1}}{2} < t & \text{③} \\ 0, & \text{otherwise.} \end{cases} \tag{A.4}$$

B

Pension fund setup and results

B.1. Example pension fund cash flows

In Chapter 5 example pension fund cash flows have been used. This was illustrated by Figure 5.5. The full dataset has been added to Table B.1.

Year	Expected payment	Year	Expected payment	Year	Expected payment
1	€2,480,000	26	€11,700,000	51	€2,800,000
2	€2,770,000	27	€11,700,000	52	€2,450,000
3	€2,910,000	28	€11,660,000	53	€2,140,000
4	€3,120,000	29	€11,570,000	54	€1,850,000
5	€3,380,000	30	€11,430,000	55	€1,590,000
6	€3,680,000	31	€11,240,000	56	€1,350,000
7	€4,050,000	32	€11,000,000	57	€1,140,000
8	€4,480,000	33	€10,710,000	58	€950,000
9	€4,970,000	34	€10,370,000	59	€780,000
10	€5,470,000	35	€10,000,000	60	€640,000
11	€5,950,000	36	€9,590,000	61	€520,000
12	€6,430,000	37	€9,160,000	62	€410,000
13	€6,910,000	38	€8,700,000	63	€320,000
14	€7,400,000	39	€8,230,000	64	€250,000
15	€7,930,000	40	€7,750,000	65	€190,000
16	€8,500,000	41	€7,270,000	66	€140,000
17	€9,110,000	42	€6,780,000	67	€100,000
18	€9,700,000	43	€6,290,000	68	€70,000
19	€10,200,000	44	€5,800,000	69	€50,000
20	€10,580,000	45	€5,330,000	70	€30,000
21	€10,890,000	46	€4,860,000	71	€20,000
22	€11,150,000	47	€4,410,000	72	€10,000
23	€11,360,000	48	€3,970,000	73	€10,000
24	€11,530,000	49	€3,560,000	74	€10,000
25	€11,650,000	50	€3,170,000	75	€0

Table B.1: Example pension cash flow.

B.2. Example pension fund participants

93 participants of a pension fund have been defined. These participants have earned 25% above the average salary for their age group throughout their career. Data on average salaries has been retrieved from the CBS and has been discussed in Section 4.3. All individuals under the pensionable age work full-time. The current pension rights of these individuals equals 70% of the average salary throughout their career, indexed with inflation. Relevant data on these pension fund participants has been added to Table B.2.

	Age	Salary [€ p.y.]	Contribution base [€ p.y.]	Avg. salary in career [€ p.y.]	Avg. contr. base in career [€ p.y.]	Pension right [€ p.y.]
1	50	53,941	39,171	47,721	32,951	23,066
2	50	53,941	39,171	47,721	32,951	23,066
3	50	53,941	39,171	47,721	32,951	23,066
4	51	53,941	39,171	47,721	32,951	23,066
5	51	53,941	39,171	47,721	32,951	23,066
6	51	53,941	39,171	47,721	32,951	23,066

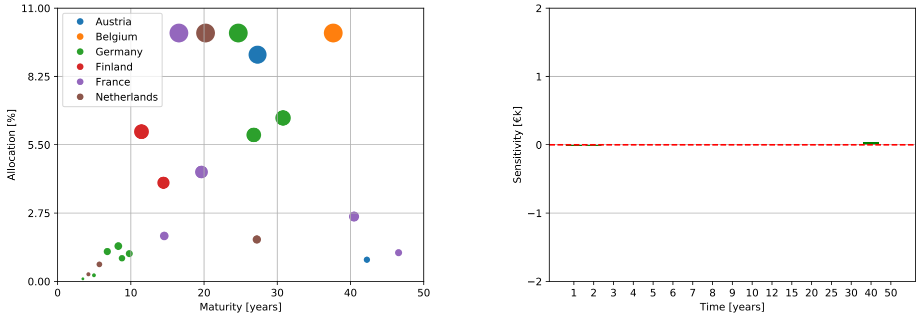
(continued on the next page)

	Age	Salary [€ p.y.]	Contribution base [€ p.y.]	Avg. salary in career [€ p.y.]	Avg. contr. base in career [€ p.y.]	Pension right [€ p.y.]
7	52	53,941	39,171	47,721	32,951	23,066
8	52	53,941	39,171	47,721	32,951	23,066
9	52	53,941	39,171	47,721	32,951	23,066
10	53	53,941	39,171	47,721	32,951	23,066
11	53	53,941	39,171	47,721	32,951	23,066
12	53	53,941	39,171	47,721	32,951	23,066
13	54	53,941	39,171	48,758	33,988	23,792
14	54	53,941	39,171	48,758	33,988	23,792
15	54	53,941	39,171	48,758	33,988	23,792
16	55	52,264	37,494	48,758	33,988	23,792
17	55	52,264	37,494	48,758	33,988	23,792
18	55	52,264	37,494	48,758	33,988	23,792
19	56	52,264	37,494	48,758	33,988	23,792
20	56	52,264	37,494	48,758	33,988	23,792
21	56	52,264	37,494	48,758	33,988	23,792
22	57	52,264	37,494	48,758	33,988	23,792
23	57	52,264	37,494	48,758	33,988	23,792
24	57	52,264	37,494	48,758	33,988	23,792
25	58	52,264	37,494	48,758	33,988	23,792
26	58	52,264	37,494	48,758	33,988	23,792
27	58	52,264	37,494	48,758	33,988	23,792
28	59	52,264	37,494	49,259	34,489	24,142
29	59	52,264	37,494	49,259	34,489	24,142
30	59	52,264	37,494	49,259	34,489	24,142
31	60	47,360	32,590	49,259	34,489	24,142
32	60	47,360	32,590	49,259	34,489	24,142
33	60	47,360	32,590	49,259	34,489	24,142
34	61	47,360	32,590	49,259	34,489	24,142
35	61	47,360	32,590	49,259	34,489	24,142
36	61	47,360	32,590	49,259	34,489	24,142
37	62	47,360	32,590	49,259	34,489	24,142
38	62	47,360	32,590	49,259	34,489	24,142
39	62	47,360	32,590	49,259	34,489	24,142
40	63	47,360	32,590	49,259	34,489	24,142
41	63	47,360	32,590	49,259	34,489	24,142
42	63	47,360	32,590	49,259	34,489	24,142
43	64	47,360	32,590	49,021	34,251	23,976
44	64	47,360	32,590	49,021	34,251	23,976
45	64	47,360	32,590	49,021	34,251	23,976
46	65	34,326	19,556	49,021	34,251	23,976
47	65	34,326	19,556	49,021	34,251	23,976
48	65	34,326	19,556	49,021	34,251	23,976
49	66	34,326	19,556	49,021	34,251	23,976
50	66	34,326	19,556	49,021	34,251	23,976
51	66	34,326	19,556	49,021	34,251	23,976

(continued on the next page)

	Age	Salary [€ p.y.]	Contribution base [€ p.y.]	Avg. salary in career [€ p.y.]	Avg. contr. base in career [€ p.y.]	Pension right [€ p.y.]
52	67	34,326	19,556	49,021	34,251	23,976
53	67	34,326	19,556	49,021	34,251	23,976
54	67	34,326	19,556	49,021	34,251	23,976
55	68	34,326	19,556	49,021	34,251	23,976
56	68	34,326	19,556	49,021	34,251	23,976
57	68	34,326	19,556	49,021	34,251	23,976
58	69	34,326	19,556	47,389	32,619	22,833
59	69	34,326	19,556	47,389	32,619	22,833
60	69	34,326	19,556	47,389	32,619	22,833
61	70	0	0	47,389	32,619	22,833
62	70	0	0	47,389	32,619	22,833
63	70	0	0	47,389	32,619	22,833
64	71	0	0	47,389	32,619	22,833
65	71	0	0	47,389	32,619	22,833
66	71	0	0	47,389	32,619	22,833
67	72	0	0	47,389	32,619	22,833
68	72	0	0	47,389	32,619	22,833
69	72	0	0	47,389	32,619	22,833
70	73	0	0	47,389	32,619	22,833
71	73	0	0	47,389	32,619	22,833
72	73	0	0	47,389	32,619	22,833
73	74	0	0	45,631	30,861	21,603
74	74	0	0	45,631	30,861	21,603
75	74	0	0	45,631	30,861	21,603
76	75	0	0	45,631	30,861	21,603
77	75	0	0	45,631	30,861	21,603
78	75	0	0	45,631	30,861	21,603
79	76	0	0	45,631	30,861	21,603
80	76	0	0	45,631	30,861	21,603
81	76	0	0	45,631	30,861	21,603
82	77	0	0	45,631	30,861	21,603
83	77	0	0	45,631	30,861	21,603
84	77	0	0	45,631	30,861	21,603
85	78	0	0	45,631	30,861	21,603
86	78	0	0	45,631	30,861	21,603
87	78	0	0	45,631	30,861	21,603
88	79	0	0	45,631	30,861	21,603
89	79	0	0	45,631	30,861	21,603
90	79	0	0	45,631	30,861	21,603
91	80	0	0	45,631	30,861	21,603
92	80	0	0	45,631	30,861	21,603
93	80	0	0	45,631	30,861	21,603

Table B.2: Participants of the example pension fund.



(a) Allocation of the hedging portfolio to 23 government bonds of varying countries. Each bond is represented by one dot whose size differs by its allocation.

(b) Sensitivity of the liabilities to interest rate changes. The net result (green bar) and the sensitivity to parallel changes in the interest rate (red dotted line) can be seen.

Figure B.1: Portfolio to hedge the pension liabilities of Appendix B.2 as of October 2019.

B.3. Additional hedging results

In Subsection 5.3.2 a hedging portfolio was calculated to hedge the pension liabilities in Appendix B.2 as of 31 October 2019. This hedging portfolio was not described in detail then but has instead been added here to Figure B.1.

B.4. Rebalancing

The results of Subsection 5.3.1 can be extended to include frequent rebalancing of the portfolio. In Figure 5.7b it was observed that the net result $\hat{z}(t)$ deviated considerably from zero. This can be remedied to some extent by rebalancing, as proposed as a next step in Chapter 6. A result that shows the potential of this has been added to this appendix.

Suppose that the portfolio is now rebalanced bi-annually, i.e. the initial portfolio is constructed at 28 September 2007 and rebalancing then takes place at 31 March 2008, 30 September 2008 and 31 March 2009. The hedging portfolio is then no longer constant — the portfolio weights x_i are a function of time. The performance of the hedging portfolio *with rebalancing* has been added to Figure B.2. This also takes transaction costs into account — both in the optimization algorithm when calculating a new portfolio and in the performance measurement by deducting transaction costs from the cash position.

It is clear that the performance of the hedging portfolio improves: the net result is closer to zero and the pension fund thus takes less risk. However, transaction costs have to be paid to accommodate this, which is not necessarily beneficial for the pension fund.

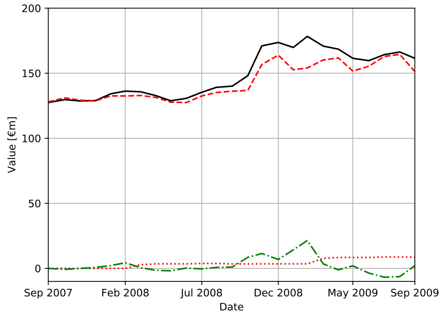


Figure B.2: Performance of the hedging portfolio using historical interest rates and asset prices. Liabilities value $-V(t)$ (solid black), hedging portfolio market value $\sum_{j=1}^{m} x_j(t)\hat{U}_j(t)$ (dashed red), cash position $c(t)$ (dotted red) and net result $-\hat{z}(t)$ (dashed dotted green) have been plotted. The investment universe consists of m assets.

B.5. An external assessment of hedging performance

In addition to the performance tests of Sections 5.3 and 5.4, the performance of the optimization algorithm has been assessed using an external risk tool. This is a tool by Union Bank of Switzerland (UBS) called *UBS Delta* that can measure the risk and performance of fixed income portfolios, while taking the liabilities of pension funds into account. More information can be found at https://www.ubs.com/microsites/ubs-delta/en/home.html.

The liabilities and current hedging portfolios of two pension funds are known: pension fund A with a present value of around €350M and pension fund B with a present value of around €100M. At most 10% of the budget may be invested in an individual government bond and country allocations have been set as in Table 5.1. Four hedging portfolios are calculated using the return-aware extension that was described in Section 5.5:

1. A hedging portfolio consisting of government bonds that strictly follows all mandate restrictions;
2. A hedging portfolio consisting of government bonds that takes additional freedom — for pension fund A this means that country allocations are more lenient and for pension fund B the delta sensitivities are not hedged fully;
3. A hedging portfolio of government bonds and government-related bonds that strictly follows all mandate restrictions, with at most 2% of the budget allocation to an individual government-related bond — in the case of pension B at most 15% of the budget may be allocated to government-related bonds;
4. A hedging portfolio of government bonds and government-related bonds that takes additional freedom — for pension fund A this means that country allocations are more lenient and for pension fund B this means that now 20% of the budget may be allocated to government-related bonds.

Portfolio	YTM	Risk
Current	-0.28%	0.94%
Government bonds (Portfolio 1)	-0.32%	1.09%
Government bonds, freedom (Portfolio 2)	-0.28%	1.07%
Government and government-related bonds (Portfolio 3)	-0.21%	1.01%
Government and government-related bonds, freedom (Portfolio 4)	-0.18%	0.98%

Table B.3: Return (as the YTM) and risk of the current portfolio of pension fund A and four newly proposed portfolios.

Portfolio	YTM	Risk
Current	-0.26%	2.90%
Government bonds (Item 1)	-0.08%	3.20%
Government bonds, freedom (Portfolio 2)	-0.02%	3.20%
Government and government-related bonds (Portfolio 3)	0.30%	2.50%
Government and government-related bonds, freedom (Portfolio 4)	0.30%	2.30%

Table B.4: Return (as the YTM) and risk of the current portfolio of pension fund B and four newly proposed portfolios.

The hedging portfolios have been calculated as of 31 October 2019 for pension fund A and 31 August 2019 for pension fund B. These results have been added to Tables B.3 and B.4.

The current portfolios are (in terms of investment universe and freedom) comparable with Portfolio 4 — they should thus also be compared with those proposed hedging portfolios. Table B.4 is a particularly good result that shows that the YTM can be improved at a lower risk level than that of the current portfolio.

C

Relevant data

C.1. Historical mortality data

Historical mortality data has been found in the Human Mortality Database (HMD) (https://www.mortality.org/), which originated from the Department of Demography at the University of California (Berkeley, USA) and the Max Planck Institute for Demographic Research (Rostock, Germany). From that database, death rates and exposures to death in The Netherlands have been downloaded for the period 1967–2016. The death rates and exposures to death may be downloaded readily from this source and are therefore not repeated here. Wilmoth et al. (2007) explain exactly how the death rate and exposure are defined in this database.

C.2. Historical interest rate

Historical data on the interest rate has been downloaded from Bloomberg. Because pension funds have long-term liabilities, the 20-year swap rate has been selected. The 20-year swap rate has been chosen because this is a liquid instrument and in line with a common duration for a pension fund. However, results would be similar if another swap rate has been selected. This dataset has been added to Table C.1 and is used in Figure 2.4.

Quarter	20-year swap rate	Quarter	20-year swap rate
2007Q1	4.46%	2013Q2	2.50%
2007Q2	4.98%	2013Q3	2.61%
2007Q3	4.86%	2013Q4	2.72%
2007Q4	4.91%	2014Q1	2.40%
2008Q1	4.70%	2014Q2	2.09%
2008Q2	5.09%	2014Q3	1.79%
2008Q3	4.69%	2014Q4	1.32%
2008Q4	3.88%	2015Q1	0.78%
2009Q1	3.86%	2015Q2	1.63%
2009Q2	4.13%	2015Q3	1.49%
2009Q3	3.99%	2015Q4	1.56%
2009Q4	4.06%	2016Q1	1.00%
2010Q1	3.79%	2016Q2	0.86%
2010Q2	3.37%	2016Q3	0.73%
2010Q3	2.98%	2016Q4	1.18%
2010Q4	3.70%	2017Q1	1.30%
2011Q1	4.01%	2017Q2	1.45%
2011Q2	3.87%	2017Q3	1.50%
2011Q3	2.83%	2017Q4	1.41%
2011Q4	2.69%	2018Q1	1.44%
2012Q1	2.65%	2018Q2	1.39%
2012Q2	2.30%	2018Q3	1.48%
2012Q3	2.28%	2018Q4	1.33%
2012Q4	2.16%	2019Q1	0.98%
2013Q1	2.25%		

Table C.1: Historical 20-year swap rate.

C.3. Historical inflation

Inflation in the Netherlands is measured by the CBS and regularly published in the form of the CPI. This data can be retrieved from https://opendata.cbs.nl/statline/#/CBS/nl/dataset/83131NED/table (13 November 2019). It has been used in this thesis to illustrate the historical inflation in Section 4.2 and to calibrate the salary process in Section 4.3 (by making adjustments for inflation). A subset of the data has been added to Table C.2.

Month	CPI	Yearly inflation	Month	CPI	Yearly inflation
December 2009	**90.08**	**1.1%**	November 2014	99.25	1.0%
January 2010	90.05	0.8%	**December 2014**	**99.05**	**0.7%**
February 2010	90.69	0.8%	January 2015	98.15	0.0%
March 2010	91.77	1.0%	February 2015	98.69	0.1%
April 2010	92.08	1.1%	March 2015	99.75	0.4%
May 2010	92.04	1.0%	April 2015	100.39	0.5%
June 2010	91.60	0.8%	May 2015	100.61	1.1%
July 2010	91.34	1.6%	June 2015	100.23	1.0%
August 2010	91.48	1.5%	July 2015	101.00	0.9%
September 2010	92.05	1.6%	August 2015	100.69	0.7%
October 2010	92.13	1.6%	September 2015	100.50	0.5%
November 2010	92.02	1.6%	October 2015	100.41	0.6%
December 2010	**91.82**	**1.9%**	November 2015	99.85	0.6%
January 2011	91.72	1.9%	**December 2015**	**99.73**	**0.7%**
February 2011	92.41	1.9%	January 2016	98.71	0.6%
March 2011	93.46	1.8%	February 2016	99.30	0.6%
April 2011	93.96	2.0%	March 2016	100.31	0.6%
May 2011	94.03	2.2%	April 2016	100.40	0.0%
June 2011	93.57	2.2%	May 2016	100.58	0.0%
July 2011	94.04	3.0%	June 2016	100.26	0.0%
August 2011	94.19	3.0%	July 2016	100.76	-0.2%
September 2011	94.53	2.7%	August 2016	100.88	0.2%
October 2011	94.57	2.6%	September 2016	100.57	0.1%
November 2011	94.28	2.5%	October 2016	100.83	0.4%
December 2011	**94.04**	**2.4%**	November 2016	100.49	0.6%
January 2012	93.99	2.5%	**December 2016**	**100.71**	**1.0%**
February 2012	94.70	2.5%	January 2017	100.35	1.7%
March 2012	95.78	2.5%	February 2017	101.09	1.8%
April 2012	96.17	2.4%	March 2017	101.44	1.1%
May 2012	96.03	2.1%	April 2017	101.98	1.6%
June 2012	95.57	2.1%	May 2017	101.65	1.1%
July 2012	96.20	2.3%	June 2017	101.37	1.1%
August 2012	96.32	2.3%	July 2017	102.11	1.3%
September 2012	96.72	2.3%	August 2017	102.27	1.4%
October 2012	97.29	2.9%	September 2017	102.03	1.5%
November 2012	96.89	2.8%	October 2017	102.17	1.3%
December 2012	**96.76**	**2.9%**	November 2017	102.00	1.5%
January 2013	96.78	3.0%	**December 2017**	**101.97**	**1.3%**
February 2013	97.54	3.0%	January 2018	101.82	1.5%
March 2013	98.58	2.9%	February 2018	102.31	1.2%
April 2013	98.67	2.6%	March 2018	102.47	1.0%
May 2013	98.69	2.8%	April 2018	103.11	1.1%
June 2013	98.32	2.9%	May 2018	103.42	1.7%
July 2013	99.15	3.1%	June 2018	103.10	1.7%
August 2013	99.03	2.8%	July 2018	104.28	2.1%
September 2013	99.08	2.4%	August 2018	104.40	2.1%
October 2013	98.80	1.6%	September 2018	103.95	1.9%
November 2013	98.31	1.5%	October 2018	104.32	2.1%
December 2013	**98.36**	**1.7%**	November 2018	104.07	2.0%
January 2014	98.15	1.4%	**December 2018**	**103.97**	**2.0%**
February 2014	98.57	1.1%	January 2019	104.05	2.2%
March 2014	99.39	0.8%	February 2019	104.97	2.6%
April 2014	99.85	1.2%	March 2019	105.37	2.8%
May 2014	99.51	0.8%	April 2019	106.08	2.9%
June 2014	99.24	0.9%	May 2019	105.94	2.4%
July 2014	100.05	0.9%	June 2019	105.84	2.7%
August 2014	99.98	1.0%	July 2019	106.90	2.5%
September 2014	99.96	0.9%	August 2019	107.37	2.8%
October 2014	99.84	1.1%	September 2019	106.70	2.6%

Table C.2: Historical inflation as published by the CBS. Year-end figures have been formatted in bold — these figures were used in calibrating the salary process of Section 4.3.

C.4. Population and salaries

In Section 4.3 the salary process was calibrated to average salaries. In this calibration process and the accompanying analyses, data provided by the CBS was used. A relevant subset of this data has been added to Tables C.3 and C.4. The full dataset can be found at https://www.cbs.nl/nl-nl/maatwerk/2018/18/inkomen-naar-leeftijd-branche-en-opleiding-2007-2015.

Age	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	Average
15–19	5,000	5,100	5,100	4,800	4,600	4,500	4,400	4,400	4,400	4,400	4,670
20–24	16,400	16,900	17,000	16,400	15,900	15,400	15,000	14,800	15,100	15,600	15,850
25–29	26,000	27,200	27,700	27,400	27,600	27,400	27,200	27,200	27,500	28,000	27,320
30–34	30,500	31,900	32,500	32,500	33,100	33,200	33,300	33,600	34,000	34,600	32,920
35–39	32,900	34,400	35,100	35,400	36,200	36,500	37,000	37,500	38,000	38,700	36,170
40–44	34,300	35,800	36,400	36,800	37,800	38,300	39,000	39,700	40,500	41,300	37,990
45–49	35,100	36,500	37,200	37,600	38,400	38,800	39,700	40,500	41,500	42,300	38,760
50–54	34,800	36,100	37,000	37,300	38,100	38,600	39,400	40,400	41,200	41,800	38,470
55–59	33,500	34,700	35,500	35,800	36,600	37,100	38,000	39,100	39,800	40,500	37,060
60–64	30,300	31,500	32,100	32,500	32,800	33,200	34,200	35,400	35,800	36,700	33,450
65–69	21,100	22,300	23,100	23,800	24,100	24,700	25,000	25,600	25,900	26,600	24,220
70–74	18,400	19,100	19,700	20,100	20,400	20,800	21,200	21,900	22,400	23,100	20,710
25–69	31,522	32,844	33,509	33,761	34,391	34,650	35,142	35,778	36,326	36,964	34,489

Table C.3: Average salary of people with a personal income in the Netherlands over the time period 2007–2016 by age. Only the averages of people aged 25–69 have been used in Section 4.3 for calibration. The average salaries for people aged 25–69 has been calculated as a weighted average using Table C.4.

Age	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
15–19	865,200	882,300	873,400	872,500	860,600	842,500	829,400	829,800	834,600	848,500
20–24	923,400	941,600	953,200	966,500	972,400	980,900	984,400	988,000	994,300	982,800
25–29	932,900	939,300	937,600	943,900	936,800	944,400	954,900	966,100	978,600	996,200
30–34	996,000	962,300	945,900	939,300	942,800	947,600	946,700	944,000	949,200	952,600
35–39	1,199,700	1,189,800	1,155,900	1,107,900	1,052,500	1,006,800	967,400	948,900	939,700	946,300
40–44	1,212,500	1,213,800	1,208,400	1,214,900	1,217,700	1,211,900	1,194,500	1,158,200	1,105,900	1,051,900
45–49	1,138,300	1,161,800	1,181,000	1,202,100	1,216,400	1,216,700	1,211,900	1,204,800	1,206,200	1,207,500
50–54	1,016,800	1,035,100	1,054,100	1,080,600	1,105,700	1,126,800	1,144,800	1,161,500	1,178,200	1,188,700
55–59	945,800	941,200	941,400	953,300	970,200	988,600	1,002,900	1,020,500	1,042,000	1,062,900
60–64	774,500	841,100	884,400	916,300	967,400	942,500	926,300	917,800	919,600	927,500
65–69	689,300	707,800	729,600	757,300	783,400	866,400	933,700	975,900	1,003,900	1,035,100
70–74	560,400	565,600	579,500	594,900	628,400	640,800	658,800	679,000	705,800	719,400
25–69	8,905,800	8,992,200	9,038,300	9,115,600	9,192,900	9,251,700	9,283,100	9,297,700	9,323,300	9,368,700

Table C.4: Number of people with a personal income in the Netherlands over the time period 2007–2016 by age.

C.5. Market values of European interest rate swaptions

The market value of a number of interest rate swaptions have been downloaded from Barclays Live (https://live.barcap.com/). These market values have been added to Table C.5. The notation '1Y5Y' is used to describe an option in one year on an interest rate swap maturing in 6 years $(1+5)$.

Interest rate swaption	Market value [€]
1Y1Y	0.0075
1Y5Y	0.0645
1Y10Y	0.1805
2Y1Y	0.0135
2Y5Y	0.1035
2Y10Y	0.2645
5Y1Y	0.0385
5Y5Y	0.2145
5Y10Y	0.4530

Table C.5: Market values of European interest rate swaptions of which the underlying interest rate swaps have a notional $N = 1000$.

C.6. Pension funds in the Netherlands

The regulator DNB publishes a large amount of data on pension funds in its online databank. A relevant subset of this data — both on the pension world in general and specific pension funds — has been added to this appendix.

C.6.1. Coverage ratios

Historical data on the coverage ratios of pension funds has been published by the regulator *De Nederlandsche Bank* on a quarterly basis since 2007. This may be found in the databank of the regulator at
https://statistiek.dnb.nl/downloads/index.aspx#/details/financi-le-positie-van-pensioenfondsen-kwartaal/dataset/fc8e7817-0884-4473-b822-62284b445278/resource/615b322a-9aa5-49b0-8a78-2c98b069059a
(17 September 2019). A relevant summary of this dataset has been added to Table C.6. This has been used in Figure 2.4.

Quarter	Coverage ratio	Quarter	Coverage ratio
2007Q1	140.9%	2013Q2	101.8%
2007Q2	151.8%	2013Q3	107.7%
2007Q3	149.9%	2013Q4	109.9%
2007Q4	144.0%	2014Q1	111.2%
2008Q1	132.3%	2014Q2	112.0%
2008Q2	136.0%	2014Q3	108.8%
2008Q3	121.3%	2014Q4	107.7%
2008Q4	95.5%	2015Q1	102.9%
2009Q1	91.8%	2015Q2	108.9%
2009Q2	102.5%	2015Q3	100.5%
2009Q3	109.5%	2015Q4	102.2%
2009Q4	109.0%	2016Q1	96.3%
2010Q1	108.2%	2016Q2	96.6%
2010Q2	100.4%	2016Q3	96.8%
2010Q3	98.9%	2016Q4	102.2%
2010Q4	106.7%	2017Q1	105.0%
2011Q1	111.8%	2017Q2	106.5%
2011Q2	111.4%	2017Q3	108.3%
2011Q3	94.0%	2017Q4	108.8%
2011Q4	98.2%	2018Q1	107.6%
2012Q1	98.6%	2018Q2	108.3%
2012Q2	94.3%	2018Q3	110.3%
2012Q3	101.2%	2018Q4	103.3%
2012Q4	102.0%	2019Q1	104.9%
2013Q1	106.5%		

Table C.6: Coverage ratio of Dutch pension funds.

C.6.2. Types of pension schemes

Historical data on the types of pension schemes has been published by the regulator *De Nederlandsche Bank* on a yearly basis since 2007. This may be found in the databank of the regulator at https://statistiek.dnb.nl/downloads/index.aspx#/details/pensioenovereenkomsten-jaar/dataset/d2c03ef8-1d7a-4132-bc31-35ab45588fdf (23 September 2019). A relevant subset of this dataset has been added to Tables C.7 and C.8.

Scheme type	Breakdown of pension scheme	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Defined Benefit	Final pay	189	168	163	149	141	106	100	81	54	41	35	32
Defined Benefit	Career-average pay	465	480	491	448	434	376	334	305	255	232	224	214
Defined Benefit	Mixed final and career-average pay	36	34	42	38	31	30	27	26	17	12	11	10
Defined Benefit	Level amounts	5	4	5	7	5	3	4	4	4	4	3	3
Defined Benefit	Other	10	10	14	6	7	11	8	6	6	7	5	6
Defined Contribution	Total	81	87	92	84	77	65	60	52	39	40	38	40
Capital agreement	Total	8	8	5	4	2	3	2	1	2	0	0	0
Mixed scheme types	Total	98	104	99	97	93	78	68	65	52	47	40	37
Other scheme types	Total	21	13	6	5	0	0	0	0	0	0	0	0
Total scheme types	Total	913	908	917	838	790	672	603	540	429	382	356	342

Table C.7: Number of pension funds by pension scheme type, in the period 2007–2018.

Scheme type	Breakdown of pension scheme	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Defined Benefit	Final pay	24,747	16,993	26,367	38,977	40,661	47,296	49,199	46,668	23,398	16,748	16,605	15,459
Defined Benefit	Career-average pay	415,007	412,933	504,351	510,040	592,402	700,861	773,314	753,281	948,613	1,024,908	1,132,569	1,113,552
Defined Benefit	Mixed final and career-average pay	15,371	14,439	18,309	18,705	21,273	24,816	25,407	26,900	22,695	20,949	22,386	20,559
Defined Benefit	Level amounts	54	47	61	137	1	127	354	322	364	373	385	381
Defined Benefit	Other	14,780	8,511	5,951	8,884	9,966	14,940	13,155	12,165	14,086	8,089	8,735	10,646
Defined Contribution	Total	5,959	4,818	6,186	7,862	8,475	10,733	11,856	9,225	10,272	9,629	8,922	9,219
Capital agreement	Total	192	328	221	236	4	89	95	9	12	0	0	0
Mixed scheme types	Total	19,295	33,559	28,709	33,645	32,108	30,821	32,900	30,762	45,666	55,244	58,979	62,568
Other scheme types	Total	1,574	847	175	256	0	0	0	0	0	0	0	0
Total scheme types	Total	496,980	492,474	590,331	618,743	704,891	829,684	906,282	879,333	1,065,107	1,135,941	1,248,581	1,232,384

Table C.8: Liabilities of pension funds by pension scheme type in millions of euro, in the period 2007–2018.

C.7. Individual pension funds

The DNB has published quarterly data on all pension funds in the Netherlands on its website since 2015. This dataset contains information on statistics such as a fund's financial position and its performance. It can be found at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-kwartaal/dataset/54946461-ebfb-42b1-9479-fa56b72d6b1a/resource/a4b6584f-09b7-498d-bce5-3ef12e966f87 (6 November 2019).

To generate Figure 5.4, historical data on both the coverage ratio and hedging ratio of individual pension funds in the Netherlands was required. This information can also be found in this dataset. A relevant summary that includes the coverage ratio and hedging ratio of the 50 largest pension funds (by total investments) has been added to Table C.9.

Additionally, the DNB has published yearly data on all pension funds in the Netherlands on its website since 2014 at https://statistiek.dnb.nl/downloads/index.aspx#/details/gegevens-individuele-pensioenfondsen-jaar/dataset/78c1c804-0b65-4bbc-a5cc-df9cd75c9ded. Since 2015 this dataset contains information on the transaction costs paid by pension funds. A relevant summary of this dataset — i.e. the transaction costs for the 50 largest pension funds by number of participants — has been added to Table C.12. Figure 5.2 has been generated from this data.

To generate Figure 5.3, historical data on the returns that pension funds in the Netherlands have achieved over the last years was required. This information can be found in the dataset provided by DNB as well. A relevant summary that includes the historical returns on all investments of the 50 largest pension funds in the Netherlands has been added to Table C.10. The historical returns of these funds on their hedging portfolios has been added to Table C.11.

[1] The 1-year change in hedging ratio was not present in the dataset but has been calculated from the hedging ratios of other years.

Pension fund	12-month average coverage ratio	Hedging ratio	Investments [€k]	1-year change in hedging ratio[1]
ABP	100.6%	22.0%	441,876,563	-4.5%
Zorg en Welzijn	99.9%	32.5%	225,207,077	-1.3%
Metaal en Techniek	100.8%	40.8%	82,380,712	0.7%
Bouwnijverheid	116.5%	37.8%	64,287,989	-8.2%
Metalektro, bedrijfstakpensioenfonds	99.8%	39.5%	52,595,816	-1.3%
ABN AMRO Bank	129.0%	121.3%	30,855,185	3.5%
Beroepsvervoer over de Weg	103.4%	54.2%	30,075,931	-2.6%
Shell	124.2%	47.0%	29,255,482	8.0%
ING	139.9%	89.1%	29,153,676	0.6%
PGB	107.1%	26.5%	28,783,299	-25.7%
Rabobankorganisatie	113.7%	49.0%	27,402,206	23.0%
Detailhandel	109.7%	64.3%	26,019,841	-1.1%
Philips	117.3%	61.2%	20,437,176	-0.6%
BPL	99.3%	29.4%	19,484,478	3.6%
Spoorwegpensioenfonds	112.7%	29.6%	18,067,088	-0.2%
Woningcorporaties	114.1%	38.8%	14,279,013	-9.6%
Medische Specialisten	124.9%	74.0%	11,794,451	-1.9%
KPN	117.8%	41.1%	11,651,624	-8.1%
Horecabedrijf	110.6%	51.6%	11,628,548	3.3%
Huisartsen	136.2%	68.8%	11,386,801	18.9%
KLM Algemeen	113.5%	27.6%	9,426,901	-8.0%
Werk en (re)Integratie	109.8%	26.7%	9,319,992	3.4%
KLM Vliegend Personeel	125.3%	24.9%	9,146,699	-1.0%
Hoogovens	113.6%	26.6%	9,128,172	-1.0%
PostNL	114.2%	38.0%	9,082,414	-11.5%
Achmea	121.9%	51.6%	8,377,654	-0.6%
UWV	103.1%	31.9%	8,164,220	-16.7%
Schilders-, Afwerkings- en Glaszetbedrijf	110.6%	25.0%	7,657,282	4.4%
DSM Nederland	106.8%	27.4%	7,314,321	-1.0%
Levensmiddelenbedrijf	100.3%	28.9%	6,592,931	-1.0%
Media PNO	103.5%	21.6%	6,505,754	-4.7%
Zorgverzekeraars	110.0%	44.4%	6,082,550	-5.0%
Schoonmaak- en Glazenwassersbedrijf	98.1%	30.5%	5,708,284	-2.6%
APF	108.6%	37.0%	5,562,713	1.3%
AHOLD	108.9%	58.2%	5,370,865	5.0%
IBM Nederland	130.7%	121.4%	5,301,360	32.2%
Kring Progress (Unilever)	134.0%	32.4%	5,240,485	-0.6%
Architectenbureaus	111.4%	55.3%	4,766,280	-3.6%
Bakkersbedrijf BPF	99.3%	37.2%	4,345,785	-1.7%
Openbaar Vervoer	109.1%	39.8%	4,302,895	-0.3%
Koopvaardij	113.5%	63.1%	4,289,147	-0.9%
Fysiotherapeuten	99.1%	35.7%	4,092,483	-3.5%
Wonen	104.4%	45.2%	3,850,708	-1.4%
SNS Reaal Groep	112.3%	68.0%	3,849,884	-1.8%
Heineken	110.9%	21.0%	3,770,752	-3.3%
TNO	111.4%	39.8%	3,711,043	-0.2%
Delta Lloyd	127.7%	91.2%	3,667,443	-0.4%
Mode-, Interieur-, Tapijt- en Textielindustrie	102.2%	31.6%	3,624,933	-2.4%
KLM-Cabinepersoneel	111.2%	31.2%	3,581,747	-2.8%
Meubelindustrie en Meubileringsbedrijven	107.0%	51.0%	3,542,944	-3.5%

Table C.9: Data on the 50 largest pension funds in the Netherlands (by total investments) as of (the end of) the second quarter in 2019.

Pension fund	2015Q1	2015Q2	2015Q3	2015Q4	2016Q1	2016Q2	2016Q3	2016Q4	2017Q1	2017Q2	2017Q3	2017Q4	2018Q1	2018Q2	2018Q3	2018Q4	2019Q1	2019Q2
ABP	8.8%	-4.3%	-3.1%	1.9%	2.2%	3.9%	2.7%	0.4%	2.0%	0.0%	1.9%	3.6%	-1.0%	2.3%	1.1%	-4.6%	7.3%	1.8%
Zorg en Welzijn	9.3%	-6.6%	-3.2%	1.1%	4.4%	4.3%	2.9%	-0.1%	0.5%	-0.8%	1.7%	3.7%	-0.6%	2.9%	1.3%	-3.9%	8.5%	3.8%
Metaal en Techniek	11.2%	-8.3%	-1.5%	1.8%	5.5%	4.7%	2.7%	-2.2%	0.9%	-0.6%	1.1%	2.6%	-0.6%	1.7%	0.9%	-1.8%	7.5%	4.8%
Bouwnijverheid	11.9%	-9.9%	-0.8%	1.0%	6.1%	4.9%	2.8%	-2.0%	1.2%	-0.5%	1.8%	3.7%	-0.3%	2.1%	1.2%	-2.7%	6.2%	2.5%
Metalektro, bedrijfs-takpensioenfonds	9.3%	-7.2%	-2.5%	2.2%	4.6%	3.8%	3.1%	-1.6%	1.3%	-0.6%	1.3%	2.6%	-0.9%	1.5%	1.2%	-2.7%	7.8%	4.6%
ABN AMRO Bank	12.9%	-11.9%	-0.1%	1.2%	8.3%	4.8%	2.0%	-2.9%	0.4%	-1.5%	0.7%	2.5%	-0.1%	1.1%	0.1%	0.2%	7.3%	5.6%
Beroepsvervoer over de Weg	13.7%	-12.9%	-1.1%	0.9%	9.3%	5.8%	3.3%	-5.3%	0.5%	-0.5%	1.2%	3.6%	-0.6%	1.3%	0.6%	-2.5%	10.3%	6.8%
Shell	7.3%	-1.6%	-2.8%	1.8%	1.1%	2.1%	2.3%	0.8%	2.3%	0.4%	1.4%	0.9%	-0.1%	0.3%	0.8%	-1.3%	5.0%	3.1%
ING	13.8%	-12.4%	1.1%	0.4%	7.4%	6.4%	1.4%	-4.6%	-1.6%	-0.6%	1.0%	2.1%	0.0%	2.4%	-0.4%	-0.8%	4.7%	3.8%
PGB	10.0%	-7.5%	-2.1%	1.8%	4.1%	3.9%	2.9%	-0.3%	2.0%	0.1%	1.9%	2.6%	-1.1%	2.1%	1.0%	-4.2%	8.2%	3.1%
Rabobankorganisatie	9.4%	-6.7%	-1.8%	0.8%	5.4%	3.0%	2.9%	-2.7%	0.1%	-0.1%	1.4%	2.3%	-0.2%	0.7%	0.4%	-3.3%	5.7%	4.3%
Detailhandel	15.2%	-13.8%	-1.4%	1.6%	10.5%	5.7%	3.8%	-6.4%	0.1%	-1.1%	0.9%	3.4%	0.2%	1.6%	0.4%	-1.6%	10.9%	7.7%
Philips	10.7%	-8.9%	-1.6%	1.1%	5.5%	5.1%	2.3%	-2.8%	0.5%	0.8%	1.4%	3.0%	-0.6%	1.4%	0.6%	-3.7%	7.1%	3.5%
BPL	11.9%	-8.1%	-2.2%	1.6%	4.6%	3.5%	3.2%	-0.9%	2.1%	-0.2%	1.7%	3.0%	-0.3%	1.7%	1.1%	-3.0%	8.1%	3.8%
Spoorwegpensioen-fonds	8.3%	-6.7%	-2.6%	1.2%	5.8%	4.5%	2.6%	-1.9%	1.8%	0.7%	1.7%	2.3%	0.0%	0.6%	1.5%	-2.8%	7.1%	4.0%
Woningcorporaties	10.6%	-7.8%	-1.7%	0.8%	6.0%	5.8%	1.9%	-1.2%	0.8%	-1.0%	1.5%	3.4%	-0.9%	2.9%	1.0%	-4.0%	8.6%	2.9%
Medische Specialisten	14.5%	-11.8%	-0.9%	1.3%	8.8%	4.9%	3.6%	-5.2%	0.8%	-0.1%	1.7%	2.8%	-0.7%	1.4%	0.2%	-2.6%	10.1%	6.7%
KPN	12.3%	-6.1%	-3.9%	2.4%	3.3%	3.5%	4.1%	-0.1%	2.3%	-0.3%	1.9%	3.3%	-1.2%	2.8%	1.3%	-4.6%	8.4%	3.3%
Horecabedrijf	17.6%	-14.7%	-1.8%	1.7%	7.9%	5.3%	3.1%	-4.3%	0.9%	-1.1%	1.6%	3.6%	-0.4%	2.8%	0.8%	-3.7%	10.5%	6.6%
Huisartsen	9.0%	-5.6%	-3.1%	1.6%	2.8%	4.0%	2.2%	-0.8%	0.9%	-0.3%	1.8%	3.1%	-0.5%	1.7%	0.1%	-2.7%	6.9%	4.5%
KLM Algemeen	8.8%	-5.4%	-3.7%	1.8%	3.3%	3.4%	3.3%	-0.9%	2.4%	0.8%	1.8%	2.6%	-0.8%	1.5%	1.3%	-3.7%	7.4%	3.0%
Werk en (re)Integratie	8.5%	-6.4%	-3.4%	2.7%	2.0%	2.9%	3.1%	-0.7%	3.2%	0.9%	2.1%	2.5%	-0.8%	0.5%	0.6%	-4.4%	7.4%	2.8%
KLM Vliegend Personeel	7.1%	-2.3%	-3.8%	1.7%	1.6%	2.8%	2.9%	0.4%	2.5%	0.6%	1.5%	2.2%	-0.9%	1.2%	0.7%	-3.1%	6.5%	1.8%
Hoogovens	7.9%	-4.0%	-2.9%	2.0%	2.0%	2.3%	3.2%	0.2%	3.0%	0.3%	1.4%	2.3%	-0.8%	2.1%	1.2%	-3.8%	7.4%	2.7%
PostNL	9.4%	-7.0%	-1.9%	1.6%	5.0%	3.1%	3.4%	-1.0%	1.3%	0.0%	1.5%	2.5%	-0.9%	2.2%	0.6%	-2.4%	6.4%	3.0%
Achmea	11.0%	-9.5%	-0.9%	0.9%	7.1%	4.0%	3.1%	-3.8%	0.8%	-0.2%	1.3%	2.7%	-0.2%	0.9%	0.8%	-3.1%	7.7%	5.0%
UWV	10.4%	-8.9%	-1.4%	0.4%	6.7%	5.1%	2.1%	-2.3%	0.9%	-0.8%	1.3%	2.0%	-0.8%	1.6%	0.3%	-1.6%	7.3%	3.7%
Schilders-, Afwerkings- en Glaszetbedrijf	13.4%	-12.2%	-3.3%	2.6%	2.4%	3.2%	4.0%	0.0%	2.9%	0.1%	1.7%	2.3%	-1.1%	1.9%	1.9%	-4.3%	8.2%	3.5%
DSM Nederland	9.5%	-7.3%	-2.8%	1.6%	2.6%	3.7%	2.4%	-1.4%	1.4%	0.5%	1.3%	2.0%	-1.2%	1.6%	0.5%	-2.6%	5.7%	2.3%
Levensmiddelenbedrijf	16.6%	-14.4%	-0.8%	1.2%	8.6%	3.4%	2.8%	-2.9%	0.8%	-0.2%	1.2%	2.7%	-0.5%	1.6%	0.6%	-2.6%	7.5%	4.4%
Media PNO	8.4%	-4.7%	-2.7%	3.0%	2.9%	3.6%	2.9%	-1.1%	2.5%	0.9%	2.1%	2.4%	-0.1%	2.4%	1.8%	-3.8%	7.0%	3.4%
Zorgverzekeraars	12.2%	-11.6%	0.0%	0.4%	8.0%	5.9%	3.4%	-5.8%	0.3%	0.0%	1.0%	2.8%	-0.3%	0.7%	0.6%	-1.3%	7.4%	4.6%
Schoonmaak- en Glazenwassersbedrijf	11.6%	-11.5%	0.1%	0.2%	5.3%	4.0%	2.6%	-2.8%	0.7%	0.1%	1.4%	2.2%	0.3%	1.7%	0.8%	-2.0%	7.1%	4.3%
APF	8.4%	-7.7%	-1.3%	0.7%	3.5%	2.6%	2.4%	-1.0%	1.2%	0.5%	2.2%	3.4%	-0.6%	1.1%	0.6%	-4.7%	6.9%	2.7%
AHOLD	11.8%	-12.0%	-0.8%	1.2%	7.7%	4.4%	3.5%	-4.9%	0.0%	-0.7%	0.9%	2.8%	-1.0%	0.9%	0.1%	-0.7%	9.7%	6.4%
IBM Nederland	9.8%	-7.5%	-1.4%	0.6%	5.3%	4.3%	2.0%	-2.5%	0.5%	-0.4%	1.3%	1.9%	-0.6%	2.2%	-0.2%	0.2%	4.4%	4.1%

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Pension fund	2015Q1	2015Q2	2015Q3	2015Q4	2016Q1	2016Q2	2016Q3	2016Q4	2017Q1	2017Q2	2017Q3	2017Q4	2018Q1	2018Q2	2018Q3	2018Q4	2019Q1	2019Q2
Kring Progress (Unilever)	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	2.8%	-1.5%	1.2%	0.9%	-4.4%	5.4%	2.0%
Architectenbureaus	9.8%	-8.0%	-1.9%	1.7%	6.9%	4.0%	3.3%	-2.5%	1.3%	-0.3%	1.6%	3.5%	-0.1%	1.1%	0.9%	-1.8%	8.0%	5.0%
Bakkersbedrijf BPF	13.2%	-10.8%	-1.4%	1.1%	5.0%	4.6%	2.8%	-2.6%	0.5%	-0.5%	1.5%	2.1%	-1.2%	1.6%	0.3%	-2.1%	7.1%	3.9%
Openbaar Vervoer	8.1%	-8.1%	-2.2%	1.1%	6.4%	4.8%	2.7%	-3.1%	1.1%	0.2%	1.5%	2.4%	0.2%	0.4%	1.1%	-2.4%	6.7%	4.0%
Koopvaardij	10.6%	-10.2%	0.3%	0.4%	5.6%	3.3%	1.7%	-2.7%	0.1%	-0.6%	0.9%	1.6%	-0.2%	1.4%	-0.1%	0.2%	6.0%	4.8%
Fysiotherapeuten	10.8%	-7.6%	-1.8%	1.5%	5.3%	3.5%	3.4%	-1.8%	1.2%	0.0%	1.5%	2.8%	0.1%	1.4%	0.8%	-1.7%	7.5%	4.9%
Wonen	13.5%	-13.2%	-1.5%	0.3%	7.1%	4.5%	3.7%	-2.7%	1.2%	-0.9%	1.7%	4.2%	-0.2%	1.6%	1.1%	-2.9%	9.3%	5.6%
SNS Reaal Groep	12.3%	-11.4%	-1.4%	0.0%	7.7%	4.2%	3.6%	-3.2%	-1.0%	-1.2%	1.2%	2.9%	-0.4%	0.5%	-0.3%	-0.3%	7.5%	5.6%
Heineken	9.7%	-4.8%	-3.8%	2.3%	1.1%	3.2%	2.0%	2.2%	2.1%	-0.7%	1.9%	2.4%	-1.0%	2.4%	1.3%	-3.9%	6.1%	2.1%
TNO	11.1%	-7.8%	-1.1%	1.3%	3.8%	4.0%	1.6%	-0.2%	1.1%	-0.4%	1.1%	2.1%	-0.3%	1.0%	0.6%	-2.4%	6.2%	3.3%
Delta Lloyd	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	-1.1%	-0.5%	1.2%	2.2%	0.0%	2.2%	-0.3%	-2.0%	8.4%	6.2%
Mode-, Interieur-, Tapijt- en Textielindustrie	9.9%	-7.3%	-1.5%	1.0%	4.1%	5.0%	2.3%	-1.1%	1.2%	-0.6%	1.3%	2.1%	-0.6%	1.8%	0.5%	-2.4%	7.4%	3.7%
KLM-Cabinepersoneel	9.8%	-6.4%	-4.0%	2.0%	3.7%	3.2%	3.8%	-0.8%	2.8%	0.7%	1.8%	3.0%	-1.1%	1.3%	1.2%	-4.7%	8.3%	3.4%
Meubelindustrie en Meubileringsbedrijven	10.8%	-12.8%	-1.1%	1.5%	6.8%	4.8%	3.3%	-3.0%	1.2%	-0.4%	1.8%	3.4%	-0.5%	1.1%	0.1%	-3.0%	8.2%	4.8%

Table C.10: Historical returns of the 50 largest pension funds in the Netherlands (by total investments) on all their investments. Numbers have been calculated from the dataset as the weighted average of the returns on the (mutually exclusive) investment classes.

Pension fund	2015Q1	2015Q2	2015Q3	2015Q4	2016Q1	2016Q2	2016Q3	2016Q4	2017Q1	2017Q2	2017Q3	2017Q4	2018Q1	2018Q2	2018Q3	2018Q4	2019Q1	2019Q2
ABP	4.0%	-4.0%	0.7%	0.0%	2.6%	2.3%	0.7%	-1.9%	-0.4%	-0.6%	-0.1%	0.5%	-0.2%	0.4%	-0.4%	0.7%	0.8%	0.7%
Zorg en Welzijn	4.8%	-6.1%	0.9%	-0.4%	4.2%	2.0%	0.9%	-2.9%	-1.1%	-0.8%	0.0%	0.8%	0.3%	0.6%	-0.6%	1.3%	2.9%	2.9%
Metaal en Techniek	6.9%	-7.1%	1.7%	-0.7%	6.1%	3.2%	0.8%	-4.5%	-1.3%	-0.9%	0.0%	0.8%	0.4%	0.6%	-0.6%	1.2%	3.3%	3.5%
Bouwnijverheid	9.7%	-10.8%	2.2%	-0.2%	5.7%	3.6%	1.0%	-3.9%	-0.9%	-1.3%	-0.4%	0.9%	-0.2%	0.7%	-0.6%	1.5%	1.9%	1.6%
Metalektro, bedrijfs-takpensioenfonds	5.9%	-6.5%	1.1%	-0.3%	5.2%	2.7%	0.9%	-4.0%	-1.1%	-0.9%	-0.1%	0.7%	0.1%	0.6%	-0.5%	1.1%	3.3%	3.0%
ABN AMRO Bank	9.5%	-11.1%	2.6%	-0.9%	8.2%	3.3%	0.8%	-4.7%	-1.5%	-1.2%	0.0%	1.4%	0.5%	1.0%	-1.2%	2.5%	4.7%	5.2%
Beroepsvervoer over de Weg	8.2%	-11.2%	2.1%	-0.8%	7.6%	3.0%	1.0%	-5.7%	-1.5%	-1.7%	-0.4%	1.4%	0.1%	1.2%	-0.7%	2.4%	5.5%	5.6%
Shell	9.3%	-3.9%	-0.2%	0.9%	-1.6%	0.9%	-0.4%	2.1%	-0.3%	-2.1%	0.4%	-0.1%	3.0%	4.6%	-3.3%	3.8%	6.4%	5.7%
ING	11.1%	-12.2%	2.6%	-0.9%	7.5%	5.5%	0.7%	-5.4%	-2.5%	-0.6%	0.7%	1.5%	0.3%	1.7%	-1.2%	0.3%	2.7%	3.2%
PGB	11.3%	-13.2%	3.1%	-0.9%	10.2%	5.5%	1.3%	-4.6%	-1.7%	-0.4%	0.4%	0.7%	0.4%	0.9%	-0.5%	1.3%	2.8%	1.9%
Rabobankorganisatie	6.3%	-6.7%	2.2%	-1.4%	5.8%	2.4%	0.6%	-3.4%	-1.5%	-0.7%	0.1%	0.6%	0.1%	0.4%	-0.4%	0.7%	1.3%	3.7%
Detailhandel	10.9%	-12.8%	3.0%	-1.0%	10.7%	3.9%	1.8%	-7.7%	-2.6%	-1.3%	-0.4%	1.7%	1.0%	0.8%	-1.1%	2.3%	5.8%	6.4%
Philips	9.1%	-8.5%	1.9%	-0.7%	6.4%	4.2%	0.1%	-4.8%	-1.8%	-0.4%	-0.4%	0.8%	0.1%	0.8%	-0.7%	1.2%	3.6%	3.2%
BPL	6.8%	-7.1%	0.8%	-0.3%	4.8%	2.5%	1.1%	-3.1%	-0.5%	-0.7%	0.1%	0.9%	0.2%	0.3%	-0.3%	0.9%	2.3%	2.9%
Spoorwegpensioen-fonds	13.5%	-16.2%	4.7%	-0.8%	4.4%	2.5%	0.7%	-2.8%	-0.9%	-0.2%	0.1%	0.6%	0.6%	0.1%	-0.3%	0.6%	1.9%	2.2%
Woningcorporaties	7.9%	-7.7%	1.8%	-0.1%	6.0%	4.2%	0.4%	-3.6%	-1.1%	-1.1%	-0.3%	0.7%	0.2%	0.9%	-0.6%	1.5%	1.0%	1.0%
Medische Specialisten	9.1%	-10.7%	2.4%	-0.9%	8.4%	3.6%	1.3%	-6.2%	-1.8%	-1.5%	-0.2%	1.2%	0.6%	1.1%	-0.9%	2.2%	4.6%	4.8%
KPN	7.1%	-4.3%	0.9%	-0.4%	3.9%	1.6%	1.2%	-2.8%	-0.7%	-0.8%	0.1%	1.0%	-0.2%	0.7%	-0.4%	1.0%	3.3%	2.8%
Horecabedrijf	12.8%	-13.8%	2.7%	-1.0%	8.5%	3.9%	0.7%	-5.9%	-2.0%	-1.6%	-0.2%	1.3%	0.8%	0.9%	-0.9%	1.8%	4.2%	5.1%
Huisartsen	4.0%	-4.7%	1.2%	-0.4%	3.8%	2.0%	0.4%	-2.7%	-0.9%	-0.5%	0.1%	0.5%	0.3%	0.1%	-0.9%	3.5%	2.6%	2.9%
KLM Algemeen	3.5%	-4.5%	0.6%	-0.5%	3.5%	2.0%	0.7%	-2.1%	-0.7%	-0.3%	0.1%	0.6%	0.2%	0.6%	-0.4%	0.7%	1.4%	1.4%
Werk en (re)Integratie	5.2%	-5.5%	0.7%	0.1%	2.1%	1.4%	0.4%	-1.2%	-0.3%	-0.1%	0.2%	0.3%	0.2%	0.2%	-0.2%	0.3%	1.2%	1.4%
KLM Vliegend Personeel	1.8%	-1.3%	0.2%	-0.4%	1.5%	1.4%	0.6%	-0.7%	-0.3%	-0.1%	0.2%	0.6%	-0.2%	0.5%	-0.4%	-0.2%	1.0%	0.6%
Hoogovens	3.1%	-3.3%	0.8%	-0.1%	2.8%	1.5%	0.7%	-2.1%	-0.3%	-0.3%	0.1%	0.5%	0.0%	0.5%	-0.2%	0.7%	1.6%	1.6%
PostNL	4.1%	-3.8%	0.6%	-0.5%	3.5%	1.6%	1.1%	-2.3%	-0.6%	-0.6%	0.3%	0.8%	-0.2%	1.0%	-0.1%	0.8%	3.2%	3.0%
Achmea	9.9%	-9.2%	2.0%	-0.5%	7.1%	3.0%	1.3%	-1.5%	-1.5%	-0.9%	0.0%	0.1%	0.4%	0.6%	-0.6%	1.2%	3.8%	3.9%
UWV	5.8%	-7.5%	1.9%	-0.5%	5.1%	2.3%	0.5%	-3.7%	-1.2%	-0.6%	0.0%	0.8%	0.2%	0.6%	-0.5%	1.3%	2.8%	2.5%
Schilders-, Afwerkings- en Glaszetbedrijf	10.0%	-11.7%	1.0%	-0.2%	3.0%	1.6%	0.9%	-2.4%	-0.5%	-0.5%	0.2%	0.5%	0.0%	0.4%	-0.3%	0.7%	2.0%	1.9%
DSM Nederland	5.2%	-6.1%	0.5%	-0.4%	2.9%	2.5%	0.8%	0.4%	-0.7%	0.3%	0.5%	0.7%	0.0%	0.5%	-0.4%	0.4%	1.5%	1.5%
Levensmiddelenbedrijf	12.6%	-13.7%	2.8%	-1.2%	9.1%	2.5%	0.4%	-4.5%	-1.6%	-0.8%	-0.3%	0.8%	0.5%	0.4%	-0.6%	1.2%	2.9%	3.3%
Media PNO	4.0%	-4.4%	0.9%	0.1%	3.9%	3.0%	0.6%	-2.2%	-0.5%	-0.9%	0.1%	0.3%	-0.2%	1.0%	0.0%	0.5%	2.8%	0.0%
Zorgverzekeraars	11.3%	-11.8%	1.8%	-0.2%	7.5%	5.7%	1.8%	-6.0%	-1.6%	-1.9%	-0.6%	1.1%	-0.6%	1.2%	-0.3%	1.7%	4.6%	3.7%
Schoonmaak- en Glazenwassersbedrijf	8.8%	-11.2%	2.7%	-0.8%	6.6%	3.4%	1.1%	-4.7%	-1.2%	-0.2%	0.0%	0.6%	0.7%	1.0%	-0.5%	1.4%	2.5%	3.3%
APF	5.6%	-7.4%	-7.4%	-0.1%	3.3%	1.7%	1.0%	-2.4%	-0.5%	-0.1%	0.3%	0.6%	0.1%	0.0%	-0.2%	0.4%	1.6%	2.3%
AHOLD	12.4%	-12.4%	1.3%	-0.7%	8.0%	2.6%	1.2%	-5.5%	-1.7%	-2.3%	-0.6%	1.5%	-0.9%	1.6%	-0.3%	2.2%	4.4%	4.7%
IBM Nederland	4.4%	-4.8%	1.0%	-0.4%	3.4%	3.8%	1.0%	-2.8%	-0.8%	-1.6%	-0.1%	0.8%	-0.6%	1.8%	-0.6%	1.1%	3.2%	3.3%

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Pension fund	2015Q1	2015Q2	2015Q3	2015Q4	2016Q1	2016Q2	2016Q3	2016Q4	2017Q1	2017Q2	2017Q3	2017Q4	2018Q1	2018Q2	2018Q3	2018Q4	2019Q1	2019Q2
Kring Progress (Unilever)	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	1.0%	-0.4%	0.5%	-0.2%	-0.5%	0.2%	0.5%
Architectenbureaus	6.9%	-8.0%	1.9%	-0.4%	6.5%	2.9%	1.2%	-4.4%	-1.2%	-0.9%	0.0%	1.4%	0.3%	0.7%	-0.6%	1.4%	4.2%	3.7%
Bakkersbedrijf BPF	7.4%	-8.4%	2.1%	-0.7%	6.5%	3.6%	0.6%	-4.7%	-1.3%	-1.0%	0.0%	0.9%	0.2%	0.8%	-0.6%	1.5%	3.3%	3.3%
Openbaar Vervoer	12.0%	-9.1%	-2.5%	1.9%	5.2%	3.0%	0.9%	-4.0%	-1.3%	-0.4%	0.2%	0.4%	0.9%	0.1%	-0.4%	0.7%	0.5%	1.7%
Koopvaardij	9.0%	-9.9%	1.9%	-0.5%	5.7%	2.7%	0.7%	-3.8%	-1.0%	-1.0%	0.2%	0.5%	0.2%	-0.8%	-0.8%	1.3%	3.7%	3.8%
Fysiotherapeuten	7.3%	-7.5%	1.5%	-0.9%	6.9%	3.1%	1.4%	-2.8%	-0.7%	-1.7%	-0.4%	1.1%	0.3%	0.4%	-0.5%	1.4%	2.6%	3.8%
Wonen	10.8%	-12.8%	2.1%	-0.7%	7.9%	3.2%	1.9%	-5.1%	-0.9%	-1.3%	0.0%	1.4%	0.3%	0.2%	-0.6%	1.7%	3.7%	4.3%
SNS Reaal Groep	9.6%	-10.9%	1.5%	-0.9%	7.7%	3.3%	1.9%	-4.0%	-3.0%	-1.4%	0.2%	1.8%	0.1%	0.6%	-0.8%	1.9%	4.6%	5.0%
Heineken	7.9%	-8.5%	3.0%	0.0%	0.0%	0.0%	0.3%	-1.7%	-0.4%	0.0%	-0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	1.1%	1.2%
TNO	7.5%	-7.4%	1.9%	-0.3%	4.5%	3.1%	0.2%	-2.5%	-0.6%	-0.1%	0.2%	0.8%	0.3%	0.5%	-0.3%	0.7%	2.4%	2.5%
Delta Lloyd	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-2.7%	-0.5%	0.3%	1.4%	0.9%	0.7%	-0.9%	1.5%	5.2%	5.3%
Mode-, Interieur-, Tapijt- en Textielindustrie	6.2%	-6.5%	1.7%	-0.7%	4.9%	3.4%	0.4%	-3.3%	-1.2%	-0.5%	0.3%	0.6%	0.4%	0.6%	-0.5%	0.8%	2.7%	2.7%
KLM-Cabinepersoneel	4.2%	-5.2%	0.6%	-0.6%	4.0%	1.8%	0.9%	-2.3%	-0.6%	-0.5%	0.1%	0.7%	0.0%	0.6%	-0.5%	0.7%	1.9%	1.8%
Meubelindustrie en Meubileringsbedrijven	8.7%	-12.1%	2.8%	-0.4%	7.2%	3.4%	1.1%	-4.6%	-1.3%	-0.9%	0.0%	1.4%	-0.4%	0.7%	-0.7%	1.7%	3.8%	3.9%

Table C.11: Historical returns of the 50 largest pension funds in the Netherlands (by total investments) on their hedging portfolios.

Pension fund	Transaction costs (2015)	Transaction costs (2016)	Transaction costs (2017)	Transaction costs (2018)	Participants (2018)
ABP	0.06%	0.06%	0.12%	0.10%	2,967,676
Zorg en Welzijn	0.09%	0.10%	0.10%	0.09%	2,796,300
Horecabedrijf	0.16%	0.17%	0.13%	0.12%	1,349,348
Personeelsdiensten	0.14%	0.11%	0.09%	0.08%	1,338,084
Metaal en Techniek	0.10%	0.08%	0.06%	0.07%	1,321,799
Detailhandel	0.12%	0.11%	0.08%	0.03%	1,216,452
Bouwnijverheid	0.10%	0.11%	0.14%	0.13%	783,477
BPL			0.06%	0.07%	672,808
Beroepsvervoer over de Weg	0.10%	0.12%	0.12%	0.08%	671,411
Metalektro, bedrijfstakpensioenfonds	0.08%	0.08%	0.06%	0.05%	631,327
Schoonmaak- en Glazenwassersbedrijf	0.13%	0.11%	0.08%	0.07%	546,548
Levensmiddelenbedrijf	0.08%	0.07%	0.07%	0.05%	337,078
PGB	0.06%	0.12%	0.10%	0.10%	321,537
Flexsecurity	0.09%	0.08%	0.06%	0.05%	297,702
Werk en (re)Integratie	0.16%	0.06%	0.13%	0.14%	202,759
Bakkersbedrijf BPF	0.08%	0.04%	0.06%	0.08%	155,882
Wonen	0.15%	0.14%	0.11%	0.54%	144,546
Mode-, Interieur-, Tapijt- en Textielindustrie	0.09%	0.14%	0.07%	0.06%	136,596
Vlees- en Vleeswarenindustrie en de Gemaksvoedingindustrie	0.10%	0.07%	0.09%	0.07%	114,119
Schilders-, Afwerkings- en Glaszetbedrijf	0.07%	0.09%	0.17%	0.12%	108,983
Meubelindustrie en Meubileringsbedrijven	0.07%	0.07%	0.09%	0.15%	105,622
Rabobankorganisatie	0.08%	0.13%	0.09%	0.09%	101,732
Philips	0.10%	0.11%	0.15%	0.15%	99,015
PostNL	0.03%	0.14%	0.19%	0.16%	95,415
Recreatie (SPR)	0.06%	0.11%	0.10%	0.01%	90,580
ABN AMRO Bank	0.05%	0.06%	0.07%	0.07%	88,718
AHOLD	0.12%	0.07%	0.08%	0.06%	86,815
Spoorwegpensioenfonds	0.12%	0.09%	0.08%	0.06%	75,287
Agrarische en Voedselvoorzieningshandel	0.32%	0.16%	0.11%	0.12%	74,453
Samenwerking / Slagersbedrijf	0.11%	0.07%	0.11%	0.04%	74,192
KPN	0.20%	0.03%	0.20%	0.26%	71,984
Kappersbedrijf	0.08%	0.06%	0.03%	0.02%	70,989
ING	0.03%	0.02%	0.03%	0.02%	69,733
Woningcorporaties	0.13%	0.14%	0.13%	0.13%	68,376
BPF Foodservice		0.09%	0.06%	0.03%	66,132
Particuliere Beveiliging	0.06%	0.24%	0.13%	0.10%	58,298
Media PNO	0.13%	0.11%	0.14%	0.15%	58,038
Medewerkers Apotheken	0.09%	0.12%	0.05%	0.12%	54,841
Koopvaardij	0.11%	0.04%	0.05%	0.04%	54,227
UWV	0.17%	0.11%	0.09%	0.09%	53,410
Architectenbureaus	0.07%	0.09%	0.10%	0.09%	47,254
Zoetwarenindustrie	0.13%	0.08%	0.18%	0.16%	46,404
Zorgverzekeraars	0.14%	0.16%	0.13%	0.23%	43,094
Achmea		0.12%	0.06%	0.09%	37,137
Fysiotherapeuten	0.12%	0.13%	0.14%	0.17%	35,740
Shell	0.11%	0.09%	0.09%	0.10%	33,609
APF	0.14%	0.12%	0.05%	0.12%	33,496
KLM Algemeen	0.28%	0.20%	0.19%	0.15%	33,181
Groothandel	0.05%	0.05%	0.04%	0.08%	32,431
Bouwmaterialen	0.09%	0.07%	0.07%	0.05%	31,725

Table C.12: Historical transaction costs of the 50 largest pension funds in the Netherlands (by number of participants) as a percentage of total investments.