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DOI

[10.4064/sm180424-31-10](https://doi.org/10.4064/sm180424-31-10)

Publication date

2023

Document Version

Final published version

Published in

Studia Mathematica

Citation (APA)

Cioica-Licht, P. A., Cox, S. G., & Veraar, M. C. (2023). Stochastic integration in quasi-Banach spaces. *Studia Mathematica*, 269(1), 1-64. <https://doi.org/10.4064/sm180424-31-10>

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Stochastic integration in quasi-Banach spaces

by

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Abstract. We develop a stochastic integration theory for processes with values in a quasi-Banach space. The integrator is a cylindrical Brownian motion. The main results give sufficient conditions for stochastic integrability. They are natural extensions of known results in the Banach space setting. We apply our main results to the stochastic heat equation where the forcing terms are assumed to have Besov regularity in the space variable with integrability exponent $p \in (0, 1]$. The latter is natural to consider for its potential application to adaptive wavelet methods for stochastic partial differential equations.

1. Introduction. In this paper we extend the stochastic integration theory developed in [71, 74] from Banach spaces to *quasi*-Banach spaces. This means that we assume that the complete metric ρ on the state space E of the integrands is not necessarily induced by a norm but by an r -norm $\|\cdot\|$ for some $r \in (0, 1]$, i.e., $\rho(x, y) = \|x - y\|^r$, $x, y \in E$, where $\|\cdot\|: E \rightarrow [0, \infty)$ is a function obeying all the properties of a norm except the triangle inequality; instead, the *r-triangle inequality*

$$\|x + y\|^r \leq \|x\|^r + \|y\|^r, \quad x, y \in E,$$

holds. The Banach space setting corresponds to $r = 1$. See Section 2.1 for a precise definition of quasi-Banach spaces and their relationship to r -Banach spaces.

Our main motivation comes from the numerical analysis of stochastic partial differential equations (SPDEs, for short), where quasi-Banach spaces occur naturally in the form of Besov spaces: It is well-known from approximation theory that, under suitable assumptions, the optimal convergence rate of wavelet or finite element approximation methods is determined by

2020 *Mathematics Subject Classification*: Primary 60H05; Secondary 46A16, 60B11.

Key words and phrases: stochastic integration, quasi-Banach space, γ -radonifying operator, decoupling, Besov spaces, stochastic heat equation.

Received 24 April 2018; revised 20 October 2018.

Published online 19 December 2022.

the regularity $\alpha > 0$ of the target function $f: \mathcal{O} \rightarrow \mathbb{R}$ in the scale

$$(1.1) \quad B_{\tau, \tau}^{\alpha}(\mathcal{O}), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \alpha > 0,$$

of Besov spaces [8, 28, 37, 38]; here, $\mathcal{O} \subseteq \mathbb{R}^d$ is a Lipschitz domain and $p > 1$ describes the $L_p(\mathcal{O})$ -norm that is used to measure the error. Thus, to answer the question of optimal convergence rates, an analysis of the regularity in the scale (1.1) is required. However, for $\alpha > d(p-1)/p$ (note that the focus is on large values of α), the Besov spaces $B_{\tau, \tau}^{\alpha}(\mathcal{O})$ from (1.1) are not Banach spaces anymore, but merely quasi-Banach spaces. In the setting of SPDEs, this means that the regularity analysis in the Besov spaces from this scale cannot be carried out simply by using one of the abstract approaches to SPDEs, like, for instance, the semigroup approach or the variational approach, as they are all developed in the Banach space framework. By constructing a stochastic integral in quasi-Banach spaces we deliver the first building block towards a semigroup approach to SPDEs in quasi-Banach spaces, thus, in particular, to a regularity analysis in Besov spaces from the scale (1.1). For more background information on the relationship between regularity, optimal convergence rates and adaptivity in the context of SPDEs, we refer to the introduction of [22] and the survey article [21]; see also [20].

The main challenges for the extension of Itô's stochastic integral from Banach to quasi-Banach spaces come from the fact that the latter are not locally convex. As a consequence, we cannot use any convexity and duality arguments. In particular, it may happen that E has trivial dual (e.g. if $E = L^r(0, 1)$ for some $r \in (0, 1)$, see e.g. [84, Section 1.47]). On the other hand there are important cases (e.g. the Besov spaces from the scale (1.1) with $\alpha > d(p-1)/p$) where the dual space is not norming for E , but still separates the points of E (see, e.g., [87, Theorem 2.11.3]). In order to keep things as general as possible we will avoid duality arguments where possible. In several instances we will present further results in case the dual space is separating.

Another difficulty caused by the lack of convexity is that there is no analogue of the Riemann integral, of the Bochner integral, or of the Pettis integral for quasi-Banach-space-valued functions; see [1] for more details on this topic. Some integration theory especially developed for r -Banach spaces can be found in [95]. It is worth mentioning that, as a consequence of Corollary 3.17, we obtain some sufficient conditions for integrability in r -Banach function spaces as well; see also Remark 3.19.

Our results show that, other than the deterministic integrals mentioned above, large parts of the stochastic integration theory developed in [71, 74] do not rely on the local convexity of the space. One may wish to go even further and study stochastic integration in metric vector spaces in general.

However, although many important developments on the geometry of metric spaces have been made in recent years in the Ribe program (see the survey [68] and references therein), at the moment it seems that too little is known about the specific part of metric geometry which we need.

There is an extensive literature on stochastic integration in Banach spaces: see e.g. [11, 12, 13, 23, 24, 27, 29, 30, 31, 36, 62, 64, 66, 70, 71, 74, 75, 76, 80, 82, 83, 93]. In this article we deal with Gaussian noise, more specifically, we aim to define the stochastic integral of an $\mathcal{L}(H, E)$ -valued process with respect to a cylindrical Wiener process W_H , where H is a separable Hilbert space and E is a separable quasi-Banach space. As in the Banach space case, a deterministic function is stochastically integrable if (and only if) it is γ -radonifying; see [71, 74] and Section 4 below. Regarding stochastic integrands, as already mentioned above, our approach is similar in spirit to the stochastic integration theory in Banach spaces with the UMD property that was developed in [71] (a survey can be found in [73]). However, as explained in Section 5.1 below, the UMD property does not make sense in the quasi-Banach space setting. A different (one-sided) type of decoupling is used, which results in one-sided estimates for the stochastic integral. This approach was previously explored in [24] in the Banach space setting.

Most of the results in this paper are formulated in terms of r -Banach spaces instead of quasi-Banach spaces (see Section 2.1 for precise definitions). However, due to the Aoki–Rolewicz theorem (see [4, 81]), this is not a restriction. It only makes our presentation easier to read and all relevant results can be translated back to the quasi-Banach space setting if required. We refer to Section 2.1 for more details.

This paper is organized as follows: In Section 2 we discuss necessary preliminaries on quasi-Banach spaces and r -Banach spaces, measurability, random sums and Gaussian random variables. We give a rather detailed account in this first part of the paper, emphasizing conceptual and notational clarity. In Section 3 we extend the theory of γ -radonifying operators to the r -Banach space setting. For this we will follow the presentation in [45, Chapter 9] and indicate those instances where changes are required. An important step is to obtain the (right) ideal property for γ -radonifying operators in r -Banach spaces. The key trick is a simple lemma on Gaussian sums which can be proved without convexity and duality arguments (see Proposition 3.2).

Section 4 is devoted to the characterization of stochastic integrability for non-random elements in terms of γ -radonifying operators. Further equivalences are obtained in the case that E has a separating dual in Theorem 4.12 and Proposition 4.13. In this case we obtain an r -Banach space analogue of [74, Theorem 2.5]; in particular, the stochastic integral of deterministic integrands can be seen as a stochastic Pettis integral. In Section 5 we consider

the more complicated setting of random integrands. After discussing the UMD property and one-sided decoupling inequalities, we show how decoupling can be used to obtain sufficient conditions for stochastic integrability. L^p -estimates for stochastic integrals can be found in Theorems 5.13 and 5.16. The special case where E has a separating dual is considered in Corollary 5.18 and this result is the natural analogue of [71, Theorems 3.6 and 5.9].

In Section 6 we illustrate how our integration theory can be used to study the stochastic heat equation on \mathbb{R}^d . The forcing terms are assumed to have Besov regularity in space and the integrability exponents are allowed to be any numbers in $(0, \infty)$. The main result of this section is Theorem 6.2 which is a space-time regularity result for the stochastic heat equation. Some of the technical estimates are proved in an appendix. In particular, in Section B we present several pointwise estimates for convolutions with heat kernels which are required for estimating the Besov norm.

Before we start our presentation, let us fix some notation.

NOTATION. All vector spaces in this article are assumed to be real unless stated otherwise. For mappings $A, B: \mathcal{P} \rightarrow \mathbb{R}$, we write ' $A(p) \lesssim B(p)$ ' to express ' $\exists C \in (0, \infty) \forall p \in \mathcal{P}: A(p) \leq CB(p)$ ', and given a further parameter set \mathcal{Q} and mappings $A, B: \mathcal{P} \times \mathcal{Q} \rightarrow \mathbb{R}$ we write ' $A(p, q) \lesssim_q B(p, q)$ ' to express ' $\forall q \in \mathcal{Q} \exists C_q \in (0, \infty) \forall p \in \mathcal{P}: A(p, q) \leq C_q B(p, q)$ '. Moreover, ' $A(p) \approx B(p)$ ' means ' $A(p) \lesssim B(p)$ and $B(p) \lesssim A(p)$ ', and ' $A(p, q) \approx_q B(p, q)$ ' means ' $A(p, q) \lesssim_q B(p, q)$ and $B(p, q) \lesssim_q A(p, q)$ '.

If E, F are (quasi-)Banach spaces, then we write $E \hookrightarrow F$ if E embeds isomorphically into F , and we write $E \approx F$ if E and F are isomorphic as (quasi-)Banach spaces. We use the common notation $\langle \cdot, \cdot \rangle: E \times E^* \rightarrow \mathbb{R}$ for the dual form, i.e., $\langle x, x^* \rangle := x^*(x)$, for $x^* \in E^*$ and $x \in E$. The notation $\langle \cdot, \cdot \rangle_H$ is used to denote the inner product on a Hilbert space H .

In addition, throughout this article, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and \mathbb{E} is the corresponding expectation operator. Moreover, $(\varepsilon_n)_{n \in \mathbb{N}}$ denotes a *Rademacher sequence*, i.e., a sequence of independent and identically distributed $\{-1, 1\}$ -valued random variables with $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$ for $n \in \mathbb{N}$. Furthermore, $(\gamma_n)_{n \in \mathbb{N}}$ denotes a *Gaussian sequence*, i.e., a sequence of independent real-valued centered standard Gaussian random variables. $\mathcal{B}(S)$ denotes the Borel σ -algebra on a topological space S .

2. Preliminaries

2.1. Quasi-Banach spaces and r -Banach spaces. As explained in the introduction, in this article we develop a stochastic integration theory for processes taking values in a quasi-Banach space. More precisely, we consider processes taking values in an r -Banach space, $r \in (0, 1]$, but this is essentially the same thing—see Remark 2.4 below. In order to guarantee conceptual

clarity for the reader familiar only with the Banach space setting, we recall some basic facts about r - and quasi-Banach spaces in this section. Details can be found, e.g., in [51, 54].

2.1.1. r -Banach spaces

DEFINITION 2.1. Let E be an \mathbb{R} -vector space and let $r \in (0, 1]$. A mapping $\|\cdot\|: E \rightarrow \mathbb{R}$ is called an r -norm if for all $\lambda \in \mathbb{R}$, $x, y \in E$,

- (1) $\|\lambda x\| = |\lambda| \|x\|$;
- (2) $\|x\| = 0$ if and only if $x = 0$;
- (3) $\|x + y\|^r \leq \|x\|^r + \|y\|^r$ (r -triangle inequality).

In this case we call $(E, \|\cdot\|)$ an r -normed space, and if E is complete with respect to the metric $\rho_{\|\cdot\|}: E \times E \rightarrow \mathbb{R}$ induced by $\|\cdot\|$ (i.e., $\rho_{\|\cdot\|}(x, y) = \|x - y\|^r$ for $x, y \in E$), we call $(E, \|\cdot\|)$ an r -Banach space.

Note that a 1-norm is in fact a norm. In general, r -normed spaces are not locally convex. As a consequence, fundamental theorems such as the Hahn–Banach theorem do not hold in general. However, results relying mainly on completeness arguments still hold in the r -Banach space setting. These are, e.g., the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem together with their numerous consequences; see, e.g., [54] for details.

For $(E_1, \|\cdot\|_1)$ an r_1 -Banach space and $(E_2, \|\cdot\|_2)$ an r_2 -Banach space, $r_1, r_2 \in (0, 1]$, we write $\mathcal{L}(E_1, E_2)$ for the space of bounded linear operators from E_1 to E_2 , endowed with the r_2 -norm

$$\|R\|_{\mathcal{L}(E_1, E_2)} := \sup_{x \in E_1 \setminus \{0\}} \frac{\|Rx\|_2}{\|x\|_1}, \quad R \in \mathcal{L}(E_1, E_2).$$

Note that, as in the Banach space setting, boundedness of linear operators is equivalent to their continuity. We write $E^* := \mathcal{L}(E, \mathbb{R})$ for the topological dual of an r -Banach space $(E, \|\cdot\|)$ endowed with the norm $\|\cdot\|_{E^*}$. Note that the adjoint $R^* \in \mathcal{L}(E_2^*, E_1^*)$ of a linear bounded operator $R \in \mathcal{L}(E_1, E_2)$ can be defined in the usual way. However, since the Hahn–Banach theorem fails to hold, in general, we only have $\|R^*\|_{\mathcal{L}(E_2^*, E_1^*)} \leq \|R\|_{\mathcal{L}(E_1, E_2)}$.

Given a metric space (M, d) and a set $\Gamma \subset \mathcal{C}(M; \mathbb{R})$, we say that Γ separates the points of M if for every pair $x, y \in M$ with $x \neq y$ there exists a $\varphi \in \Gamma$ such that $\varphi(x) \neq \varphi(y)$. We also say that Γ is separating (for M). If E is a Banach space, then E^* separates the points of E by the Hahn–Banach theorem. For r -Banach spaces the situation is more complicated, as the following examples demonstrate.

Let (S, \mathcal{A}, μ) be a measure space and $r \in (0, \infty)$. The space $L^r(S, \mathcal{A}, \mu)$ of all equivalence classes of $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable functions $f: S \rightarrow \mathbb{R}$ such

that

$$\|f\|_{L^r(S)}^r := \int_S |f|^r d\mu < \infty,$$

is a prominent example of a $\min\{r, 1\}$ -Banach space (see [26], where these r -Banach spaces were analyzed in detail for the first time for $0 < r < 1$). We simply write $L^r(\mathcal{O})$ if μ is the Lebesgue measure on a Borel-measurable subset \mathcal{O} of \mathbb{R}^d and $\ell^r(\mathbb{N})$ if μ is the counting measure on \mathbb{N} . For $0 < r < 1$, the topological duals of these spaces differ dramatically, depending on the structure of the underlying measure space (S, \mathcal{A}, μ) . Indeed, for $0 < r < 1$, the dual of $L^r(0, 1)$ is trivial, whereas the sequence space $\ell^r(\mathbb{N})$ has a very rich dual that is isomorphic to the space of all bounded sequences; see [26, 54]. In particular, $\ell^r(\mathbb{N})^*$ separates the points of $\ell^r(\mathbb{N})$.

Another example of an r -Banach space, particularly relevant from the point of view of approximation theory, has already been mentioned in the introduction: for $d \in \mathbb{N}$, $\mathcal{O} \subseteq \mathbb{R}^d$ and $p \in (1, \infty)$ the scale of Besov spaces $\{B_{r,r}^\alpha(\mathcal{O}) : \alpha, r \in (0, \infty), 1/r = \alpha/d + 1/p\}$ appears naturally in the context of non-linear approximation. If $\alpha > d(p-1)/p$ and $r \in (0, \infty)$ satisfy $1/r = \alpha/d + 1/p$, then $r < 1$, whence $B_{r,r}^\alpha(\mathcal{O})$ is an r -Banach space but not a Banach space. Note however that for such α, r the dual $B_{r,r}^\alpha(\mathcal{O})^*$ separates the points of $B_{r,r}^\alpha(\mathcal{O})$. This follows from the fact that $B_{r,r}^\alpha(\mathcal{O})$ is continuously embedded in $L^p(\mathcal{O})$ (see, e.g., [89, Theorem 1.73(i)] for the case $\mathcal{O} = \mathbb{R}^d$, which immediately implies the general case if one uses the definition by restriction; see [89, Section 1.11]).

2.1.2. Quasi-Banach spaces

DEFINITION 2.2. Let E be an \mathbb{R} -vector space. A mapping $\|\cdot\| : E \rightarrow \mathbb{R}$ is called a *quasi-norm* if there exists a $C \in (0, \infty)$ such that for all $\lambda \in \mathbb{R}$, $x, y \in E$,

- (1) $\|\lambda x\| = |\lambda| \|x\|$;
- (2) $\|x\| = 0$ if and only if $x = 0$;
- (3) $\|x + y\| \leq C(\|x\| + \|y\|)$ (quasi-triangle inequality).

In this case we call $(E, \|\cdot\|)$ a *quasi-normed space*. Moreover, we write $\mathcal{O}_{\|\cdot\|}$ for the *standard topology induced by $\|\cdot\|$* , i.e., $\mathcal{O}_{\|\cdot\|}$ is the collection of all subsets $U \subseteq E$ such that for all $x \in U$ there exists an $\varepsilon \in (0, \infty)$ such that $B_{x,\varepsilon}^{\|\cdot\|} := \{y \in E : \|x - y\| < \varepsilon\} \subseteq U$. If E is complete with respect to $\mathcal{O}_{\|\cdot\|}$, we call $(E, \|\cdot\|)$ a *quasi-Banach space*.

REMARK 2.3. Note that, unlike r -norms, a quasi-norm $\|\cdot\|$ is not necessarily continuous and an ‘open ball’ $B_{x,\varepsilon}^{\|\cdot\|}$ is not necessarily contained in the standard topology $\mathcal{O}_{\|\cdot\|}$; see, for instance, [4].

REMARK 2.4. Every r -normed space is a quasi-normed space as an r -norm fulfills all the requirements from Definition 2.2 with $C = 2^{\frac{1-r}{r}}$. Conversely, the Aoki–Rolewicz theorem (see [4, 81]) guarantees that, given a quasi-normed space $(E, \|\cdot\|)$, there exists an $r \in (0, 1]$ and an r -norm $\|\cdot\|$ that is equivalent to $\|\cdot\|$. In particular, $\mathcal{O}_{\|\cdot\|} = \mathcal{O}_{\|\cdot\|, \|\cdot\|}$, where the latter coincides with the standard topology on the metric space $(E, \rho_{\|\cdot\|, \|\cdot\|})$; see also Definition 2.1. Moreover, $\mathcal{O}_{\|\cdot\|}$ is Hausdorff, compatible with the vector space operations and the unit ball $B_{0,1}^{\|\cdot\|, \|\cdot\|}$ is a bounded neighborhood of the origin. Thus, from a topological point of view, a quasi-Banach space endowed with its standard topology is a locally bounded F -spaces, i.e., a complete locally bounded metrizable topological vector space. Conversely, on any locally bounded topological vector space (E, \mathcal{O}) , there exists a quasi-norm $\|\cdot\|$ such that $\mathcal{O} = \mathcal{O}_{\|\cdot\|}$ [54]. Moreover, if $(E, \|\cdot\|)$ is separable, then $(E, \mathcal{O}_{\|\cdot\|})$ is a Polish space, i.e., a complete metrizable topological space containing a dense countable subset.

2.2. Measurability. In this section we recall some standard results on measurability of functions with values in separable metric spaces (M, d) . Clearly, all these results apply to separable r -Banach spaces.

Let $\mathcal{B}(M)$ denote the Borel σ -algebra on (M, d) . Furthermore, let (S, \mathcal{A}) be an arbitrary measure space. A function $f: S \rightarrow M$ is called $\mathcal{A}/\mathcal{B}(M)$ -measurable (or simply *measurable*) if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(M)$. It is called *strongly \mathcal{A} -measurable* (or simply *strongly measurable*) if it is the pointwise limit of a sequence of \mathcal{A} -simple functions; recall that a function is called *\mathcal{A} -simple* if it is $\mathcal{A}/\mathcal{B}(M)$ -measurable and attains at most finitely many values. On separable metric spaces, these notions of measurability coincide; see, e.g., [90, Proposition I.1.9], where the following statement can be found.

PROPOSITION 2.5. *Let (S, \mathcal{A}) be a measure space and let (M, d) be a separable metric space. Given a function $f: S \rightarrow M$, the following are equivalent:*

- (i) *f is $\mathcal{A}/\mathcal{B}(M)$ -measurable.*
- (ii) *f is strongly \mathcal{A} -measurable.*
- (iii) *There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of countably valued measurable functions such that $f = \lim_{n \rightarrow \infty} f_n$ uniformly in S .*

If the metric space is complete, then measurability can also be characterized in the following way; see, e.g., [90, Theorem I.1.2 and Proposition I.1.10].

PROPOSITION 2.6 (Pettis' theorem). *Let (S, \mathcal{A}) be a measure space, let (M, d) be a Polish space and let $\Gamma \subseteq \mathcal{C}(M, \mathbb{R})$ separate the points of M . Then*

$$\mathcal{B}(M) = \sigma(\varphi : \varphi \in \Gamma),$$

and for a given function $f: S \rightarrow E$, the following are equivalent:

- (i) f is measurable.
- (ii) For all $\varphi \in \Gamma$, $\varphi \circ f$ is measurable.

EXAMPLE 2.7. Let E be a separable r -Banach space with separating dual E^* . Then

$$\mathcal{B}(E) = \sigma(x^* : x^* \in E^*)$$

and a function $f: S \rightarrow E$ is (strongly) measurable if and only if $\xi \mapsto \langle f(\xi), x^* \rangle$ is measurable for every $x^* \in E^*$.

Let $(E, \|\cdot\|_E)$ be a separable r -Banach space for some $r \in (0, 1]$, and let (S, \mathcal{A}, μ) be a σ -finite measure space. We let $L^0(S; E)$ be the space of all equivalence classes of $\mathcal{A}/\mathcal{B}(E)$ -measurable functions. For $p \in (0, \infty)$ we define $\|\cdot\|_{L^p(S; E)}: L^0(S; E) \rightarrow [0, \infty]$ by $\|f\|_{L^p(S; E)} = (\int_S \|f\|_E^p d\mu)^{1/p}$, and we set

$$L^p(S; E) = \{f \in L^0(S; E) : \|f\|_{L^p(S; E)} < \infty\}.$$

Note that for $p \in (0, \infty)$, $L^p(S; E)$ is a $\min\{p, r\}$ -Banach space. However, a definition of a Bochner integral in analogy with the Banach space setting is not possible on general r -Banach spaces; see [1]. In particular, in general the fact that $f \in L^1(S; E)$ does not mean that $\int_S f d\mu$ makes sense in the usual way of Lebesgue/Bochner type integrals.

2.3. Random sums. Throughout this section E denotes a separable r -Banach space, where $r \in (0, 1]$ is fixed. We present results on the coincidence of different convergence types for sums of independent (symmetric) r -Banach-space-valued random variables. In the finite-dimensional case, these results are often referred to as Lévy's theorem, whereas the generalization to Banach spaces is attributed to Kiyoshi Itô and Makaki Nisio due to their seminal paper [46].

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that a mapping $X: \Omega \rightarrow E$ is an E -valued random variable (or, simply, a random variable) if X is strongly measurable. In this case, we let $\mathbb{P}_X: \mathcal{B}(E) \rightarrow \mathbb{R}$ denote the distribution of X (i.e., $\mathbb{P}_X(B) = \mathbb{P}(X \in B)$, $B \in \mathcal{B}(E)$).

We start with some auxiliary results, which we frequently use in the sequel. The first one says that independence is preserved under convergence in distribution.

LEMMA 2.8. *Let (X, Y) be an $E \times E$ -valued random variable, and for all $n \in \mathbb{N}$ let (X_n, Y_n) be an $E \times E$ -valued random variable such that X_n and Y_n are independent. If $(X_n, Y_n) \rightarrow (X, Y)$ in distribution as $n \rightarrow \infty$, then X and Y are independent.*

Proof. It suffices to prove that for arbitrary open subsets $F, G \subseteq E$,

$$\mathbb{E}[\mathbb{1}_F(X)\mathbb{1}_G(Y)] = \mathbb{E}[\mathbb{1}_F(X)]\mathbb{E}[\mathbb{1}_G(Y)].$$

To this end, since the characteristic function on an open set can be approximated from below by a sequence of continuous bounded functions, we only need to check that for arbitrary $f, g \in \mathcal{C}_b(E)$,

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)].$$

This follows immediately from our assumption. ■

The following result shows that adding a symmetric independent random variable increases the L^p -norm (up to a constant). Recall that an E -valued random variable X is *symmetric* if X and $-X$ have the same distribution.

LEMMA 2.9. *Let X and Y be two E -valued random variables. If Y is symmetric and independent of X , then for all $p \in (0, \infty)$,*

$$(2.1) \quad \|X\|_{L^p(\Omega; E)} \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \|X + Y\|_{L^p(\Omega; E)}.$$

Proof. Follow the proof of [45, Proposition 6.1.5], using the quasi-triangle inequality instead of the triangle inequality. ■

The following example shows that the first inequality in Lemma 2.9 is sharp.

EXAMPLE 2.10. Consider, for $r \in (0, 1]$, $(E, \|\cdot\|) := (\ell_2^r, \|\cdot\|_{\ell_2^r})$ (i.e., the space \mathbb{R}^2 endowed with the r -norm $\|(x_1, x_2)\|_{\ell_2^r} := |x_1|^r + |x_2|^r$ for (x_1, x_2) in \mathbb{R}^2). Let $e_1 := (1, 0)$ and $e_2 := (0, 1)$ be the standard unit vectors. Let $x := e_1 + e_2$ and $y := e_1 - e_2$. Let $\varepsilon_1, \varepsilon_2$ be two independent $\{-1, 1\}$ -valued identically distributed random variables with $\mathbb{P}(\varepsilon_1 = -1) = \mathbb{P}(\varepsilon_1 = 1) = 1/2$. For all $p \in (0, \infty)$ one finds that $\mathbb{E}\|\varepsilon_1 x\|_{\ell_2^r}^p = \|x\|_{\ell_2^r}^p = 2^{p/r}$, whereas $\mathbb{E}\|\varepsilon_1 x + \varepsilon_2 y\|_{\ell_2^r}^p = \mathbb{E}\|(\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2)\|_{\ell_2^r}^p = 2^p$. Therefore, the constant $2^{\frac{1-(r \wedge p)}{r \wedge p}}$ in Lemma 2.9 is optimal for $p \geq r$. To see that the constant is also optimal for $p < r$ consider the \mathbb{R} -valued random variables $X := \varepsilon_1$ and $Y := \varepsilon_2$.

Note moreover that even in the Gaussian case one cannot, in general, omit the constant $2^{\frac{1-(r \wedge p)}{r \wedge p}}$ in (2.1). Indeed, let e_1 and e_2 be as above, let γ_1 and γ_2 be two independent standard normal Gaussian random variables and define $X := \frac{1}{4}\gamma_1(e_1 + e_2)$ and $Y := \frac{1}{16}\gamma_2(e_1 - e_2)$. Then $\mathbb{E}\|X\|_{\ell_2^{1/2}}^2 = 1$, whereas numerical integration yields $\mathbb{E}\|X + Y\|_{\ell_2^{1/2}}^2 \approx 0.987$.

For the remainder of this section we introduce the following:

SETTING 2.11. *Let $(X_j)_{j \in \mathbb{N}}$ be a sequence of independent E -valued random variables and for all $n \in \mathbb{N}$ let $S_n := \sum_{j=1}^n X_j$.*

The following version of Lévy's inequality about tail estimates holds in r -Banach spaces and plays an important role in our proofs. We refer to [24, Lemma 2.2] for a detailed proof.

LEMMA 2.12 (Lévy's inequality for r -Banach spaces). *Assume Setting 2.11 and assume in addition that X_j is symmetric for all $j \in \mathbb{N}$. Then for $n \in \mathbb{N}$ and $t > 0$ one has*

$$(2.2) \quad \mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\|_E > t\right) \leq 2\mathbb{P}(\|S_n\|_E > 2^{1-1/r}t)$$

and

$$(2.3) \quad \mathbb{P}\left(\max_{1 \leq k \leq n} \|X_k\|_E > t\right) \leq 2\mathbb{P}(\|S_n\|_E > 2^{1-1/r}t).$$

Consequently, for all $p \in (0, \infty)$,

$$(2.4) \quad \max\left\{\mathbb{E}\left[\max_{1 \leq k \leq n} \|S_k\|_E^p\right], \mathbb{E}\left[\max_{1 \leq k \leq n} \|X_k\|_E^p\right]\right\} \leq 2^{1+p/r-p}\mathbb{E}\|S_n\|_E^p.$$

The following result generalizes Lévy's theorem on the convergence of sums of independent real-valued random variables to the r -Banach space setting.

PROPOSITION 2.13 (Lévy). *Assume Setting 2.11. The following are equivalent:*

- (a) $(S_n)_{n \in \mathbb{N}}$ converges almost surely to a random variable S .
- (b) $(S_n)_{n \in \mathbb{N}}$ converges in probability to a random variable S .
- (c) $(\mathbb{P}_{S_n})_{n \in \mathbb{N}}$ converges weakly to a probability measure μ .

In this case, the limits $S = \lim_{n \rightarrow \infty} S_n$ in (a) and (b) coincide and $\mathbb{P}_S = \mu$.

Proof. Let us first consider the equivalence of (a) and (b). In a first step, we assume that each random variable X_j is symmetric. Then the assertion can be proven in the same way as in [90, Theorem V.2.1] (which deals with the Banach space case), i.e., by using Lévy's inequality (2.2) from Lemma 2.12. In a second step, we can drop the symmetry assumption by using the same symmetrization technique as in the proof of [90, Theorem V.2.2]. Note that in the r -Banach space setting, the symmetrization inequality from [90, Proposition V.2.2] holds with $t/2$ replaced by $2^{-1/r}t$ on the right hand side.

The equivalence of (b) and (c) can be obtained directly without assuming the symmetry of the random variables, by repeating the arguments of the proof of [90, Theorem V.2.3]. One mainly has to use Prokhorov's theorem and the fact that the space of probability measures on a separable r -Banach space E , endowed with the convolution operator and the weak topology, forms a topological semigroup where the neutral element is the Dirac measure δ_0 ; see [90, Proposition I.4.5]. ■

Under reasonable integrability assumptions, the equivalent convergence types (a)–(c) from Proposition 2.13 turn out to be equivalent to convergence in L^p . The following proposition holds.

PROPOSITION 2.14 (Hoffmann-Jørgensen). *Assume Setting 2.11, and assume in addition that $0 < p < \infty$ and that $(S_n)_{n \in \mathbb{N}}$ converges almost surely to a random variable S . Then the following are equivalent:*

- (i) $\sup_{n \in \mathbb{N}} \mathbb{E}[\|S_n\|_E^p] < \infty$.
- (ii) $\mathbb{E}[\sup_{n \in \mathbb{N}} \|S_n\|_E^p] < \infty$.
- (iii) $\mathbb{E}[\|S\|_E^p] < \infty$.
- (iv) $\mathbb{E}[\|S\|_E^p] < \infty$ and $(S_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega; E)$ to S .

Conversely, if $(S_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega; E)$ to a random variable S , then $(S_n)_{n \in \mathbb{N}}$ converges almost surely to S .

Proof. First assume that each X_j is symmetric ($j \in \mathbb{N}$). Then, given the almost sure convergence of $(S_n)_{n \in \mathbb{N}}$, the equivalence of (i)–(iv) can be obtained as in [43, Theorems II.4.2(d) and II.5.4] by applying Lemma 2.12. In order to get rid of the symmetry assumption, one can use standard symmetrization techniques in a similar manner to the proof of Proposition 2.13.

Conversely, if $(S_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega; E)$, Markov's inequality yields convergence in probability. Thus, by Proposition 2.13, almost sure convergence follows. ■

We have already mentioned that the dual of a quasi-Banach space might be trivial. Obviously, in this case, we cannot expect that, e.g., the \mathbb{P} -a.s. convergence of $(\langle S_n, x^* \rangle)_{n \in \mathbb{N}}$ for all $x^* \in E^*$ implies the \mathbb{P} -a.s. convergence of $(S_n)_{n \in \mathbb{N}}$. However, we can guarantee this implication if we assume two things: Firstly, that E^* is rich enough to separate the points of E and, secondly, that each random variable X_j is symmetric for $j \in \mathbb{N}$. Note that the symmetry assumption is needed whenever we are dealing with infinite-dimensional spaces, even for Hilbert spaces; see [46, Remark after Theorem 4.1]. We can prove even more and obtain the following generalization of the so-called Itô–Nisio theorem from Banach to r -Banach spaces. Note that, in analogy to the Banach space setting, we can introduce the *Fourier transform* of a Borel probability measure on E to be the complex-valued function on E^* given by the formula

$$\hat{\mu}(x^*) := \int_E e^{-i\langle x, x^* \rangle} d\mu(x), \quad x^* \in E^*.$$

Standard arguments show that, if E^* separates the points of E , the Fourier transform is unique, i.e., if two Borel probability measures μ and ν on E have the same Fourier transform, then these measures coincide on $\mathcal{B}(E)$.

PROPOSITION 2.15 (Itô–Nisio). *Assume Setting 2.11 and assume in addition that E^* separates the points of E and that X_j is symmetric for all $j \in \mathbb{N}$. Then statements (a)–(c) from Proposition 2.13 are equivalent to each of the following three statements:*

- (d) $(\mathbb{P}_{S_n})_{n \in \mathbb{N}}$ is uniformly tight.

(e) *There exists an E -valued random variable S such that*

$$\lim_{n \rightarrow \infty} \langle S_n, x^* \rangle = \langle S, x^* \rangle \quad \text{in probability for all } x^* \in E^*.$$

(f) *There exists a Borel probability measure μ on E such that*

$$\lim_{n \rightarrow \infty} \widehat{\mathbb{P}}_{S_n}(x^*) = \hat{\mu}(x^*) \quad \text{for all } x^* \in E^*.$$

Proof. These equivalences can be proven by following the lines of the proof of [46, Theorem 4.1]. Note that, when proving ‘(d) \Rightarrow (a)’, one needs [46, Theorem 3.2] saying that the uniform tightness of $(\mathbb{P}_{S_n})_{n \in \mathbb{N}}$ implies the existence of a sequence $(c_n)_{n \in \mathbb{N}} \subseteq E$ such that $S_n - c_n$ converges \mathbb{P} -almost surely. But this turns out to hold also in our setting, if one repeats the arguments from the proof given therein.

Also note that when adapting the proof of [46, Theorem 4.1] to prove ‘(f) \Rightarrow (e)’ we encounter a difficulty: the sequence $(z_n)_{n \in \mathbb{N}}$ in the proof of [46, Theorem 4.1] is constructed by means of the Hahn–Banach theorem. However, as E is a separable metric space, it is also Lindelöf. This can be used to show that if E^* separates the points of E , then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ in E^* which separates the points of E too; see, e.g., [44, Proposition B.1.11]. But this suffices for the proof; see also the implication ‘(f) \Rightarrow (e)’ presented in [90, Theorem V.2.4], as well as the proof of [90, Theorem IV.2.5]. ■

In what follows, we frequently make use of the following Kahane–Khinchin inequality for Rademacher and Gaussian sums. Recall from the ‘Notation’ part in the introduction that $(\varepsilon_n)_{n \in \mathbb{N}}$ denotes a Rademacher sequence, and $(\gamma_n)_{n \in \mathbb{N}}$ denotes a Gaussian sequence.

THEOREM 2.16 (Kahane–Khinchin inequality). *For all $p, q \in (0, \infty)$ there exists a constant $C_{p,q,r}$, depending only on the parameters p, q , and r , such that for all finite sequences $x_1, \dots, x_N \subseteq E$ we have*

$$(2.5) \quad \left(\mathbb{E} \left[\left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_E^p \right] \right)^{1/p} \leq C_{p,q,r} \left(\mathbb{E} \left[\left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_E^q \right] \right)^{1/q},$$

and

$$(2.6) \quad \left(\mathbb{E} \left[\left\| \sum_{n=1}^N \gamma_n x_n \right\|_E^p \right] \right)^{1/p} \leq C_{p,q,r} \left(\mathbb{E} \left[\left\| \sum_{n=1}^N \gamma_n x_n \right\|_E^q \right] \right)^{1/q}.$$

Proof. The inequality (2.5) for Rademacher sums can be proven by adapting Kahane’s original proof (see [50, Theorem 2.1]). The inequality (2.6) can be obtained from (2.5) by a (finite-dimensional) central limit argument as used, e.g., in the proof of [63, Corollary 4.8]; see also [59, p. 103]. ■

The Kahane–Khinchin inequalities guarantee that the convergence of Gaussian and Rademacher series in probability implies their L^p -convergence. The following holds.

PROPOSITION 2.17. *For a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$, the following are equivalent:*

- (i) *The series $\sum_{n=1}^{\infty} \gamma_n x_n$ converges in probability.*
- (ii) *The series $\sum_{n=1}^{\infty} \gamma_n x_n$ converges in $L^p(\Omega; E)$ for all $0 < p < \infty$.*
- (iii) *The series $\sum_{n=1}^{\infty} \gamma_n x_n$ converges in $L^p(\Omega; E)$ for some $0 < p < \infty$.*

These equivalences also hold if all instances of $\sum_{n=1}^{\infty} \gamma_n x_n$ above are replaced by $\sum_{n=1}^{\infty} \varepsilon_n x_n$.

Proof. The implication ‘(ii) \Rightarrow (iii)’ is trivial and ‘(iii) \Rightarrow (i)’ is classic due to Markov’s inequality. Thus, we only have to check ‘(i) \Rightarrow (ii)’. But this can be proven by mimicking the proof of the corresponding result for Banach spaces; see, for instance, the proof of [2, Proposition 6.4.5] and the subsequent example. An alternative strategy relying on the Paley–Zygmund inequality can be found, e.g., in [45, Corollaries 6.4.2 and 6.4.4]. ■

We close this section with a simple criterion for convergence of bounded random sums, which we use to prove Proposition 3.14 later on.

As for Banach spaces, an r -Banach space is said to have (*Rademacher*) *cotype* $q \in [2, \infty]$ if there exists a constant $C_q \in (0, \infty)$ such that for all finite sequences $x_1, \dots, x_N \in E$ we have

$$\left(\sum_{n=1}^N \|x_n\|_E^q \right)^{1/q} \leq C_q \left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_E^2 \right)^{1/2},$$

with the usual modification of $\max_{n=1, \dots, N} \|x_n\|_E$ replacing the sum on the left hand side for $q = \infty$ (see also, e.g., [5, 61] as well as [2, 59]). Similarly, one can define *type* p for $p \in (0, 2]$, but this notion is less useful for r -Banach spaces with $0 < r < 1$ (see [50]). The following function spaces are classical examples of r -Banach spaces with finite cotype:

- For $p \in (0, \infty)$, the space L^p has cotype $p \vee 2$ (see [61]). This result remains true for non-commutative L^p -spaces (see [78, Corollary 5.8]).
- One easily checks that if X has cotype $q \in [2, \infty)$, $p \in (0, \infty)$, and (S, \mathcal{A}, μ) is a σ -finite measure space, then $L^p(S; X)$ has cotype $p \vee q$. Consequently, for $p, q \in (0, \infty)$ and $s \in \mathbb{R}$, the Besov space $B_{p,q}^s$ has cotype $p \vee q \vee 2$.

PROPOSITION 2.18. *The following assertions are equivalent:*

- (i) *For all sequences $(X_n)_{n \in \mathbb{N}}$ of independent symmetric E -valued random variables satisfying $\mathbb{P}(\sup_{n \in \mathbb{N}} \|\sum_{k=1}^n X_k\|_E < \infty) = 1$ the series $\sum_{k=1}^{\infty} X_k$ converges \mathbb{P} -almost surely.*
- (ii) *For all sequences $(x_n)_{n \in \mathbb{N}} \subseteq E$ satisfying $\mathbb{P}(\sup_{n \in \mathbb{N}} \|\sum_{k=1}^n \varepsilon_k x_k\|_E < \infty) = 1$ one has $\lim_{n \rightarrow \infty} x_n = 0$.*

In particular, both assertions hold if E has finite cotype.

Proof. The equivalence of (i) and (ii) can be proven as in the Banach space case; see e.g. [69, Theorem 4.2]. To see that finite cotype implies (ii) let $q \in [2, \infty)$ and $C_q(E) \in [0, \infty)$ be such that E has cotype q with cotype constant $C_q(E)$. Assume $(\sum_{k=1}^n \varepsilon_k x_k)_{n \in \mathbb{N}}$ is almost surely bounded and hence bounded in probability. Choose $\lambda > 0$ such that for all $n \geq 1$,

$$\mathbb{P}\left(\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|_E > \lambda\right) \leq \frac{1}{4C_{2q,q,r}^2},$$

where $C_{2q,2,r}$ is the constant in the Kahane–Khinchin inequality (Theorem 2.16). By [45, Corollary 6.2.9] (which remains valid in the r -Banach space setting), we obtain

$$\sup_{n \in \mathbb{N}} \left\|\sum_{k=1}^n \varepsilon_k x_k\right\|_{L^q(\Omega; E)} \leq 2^{1/q} \lambda.$$

By definition of cotype we find that

$$\sum_{n \in \mathbb{N}} \|x_n\|_E^q = \sup_{N \in \mathbb{N}} \sum_{n=1}^N \|x_n\|_E^q \leq [C_q(E)]^q \sup_{N \in \mathbb{N}} \mathbb{E} \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_E^q < \infty.$$

Therefore, $\lim_{n \rightarrow \infty} \|x_n\|_E^q = 0$, and this yields (ii). ■

2.4. Gaussian random variables. In this section E denotes a separable r -Banach space for some $r \in (0, 1]$. If $r = 1$, i.e., if E is a Banach space, an E -valued random variable X is usually called Gaussian if $\langle X, x^* \rangle$ is Gaussian for every $x^* \in E^*$. This makes sense, as the points of normed spaces can be separated by their duals due to the Hahn–Banach theorem. However, as already mentioned, r -Banach spaces do not necessarily have a rich dual; it can even be trivial. Nevertheless, we can introduce the notion of ‘Gaussian random variable’ by using what is known as Bernstein’s characterization of Gaussian measures.

DEFINITION 2.19. An E -valued random variable X is *Gaussian* if the random variables $X + Y$ and $X - Y$ are independent for any independent copy Y of X .

REMARK 2.20. The fact that if $E = \mathbb{R}$ this definition coincides with what is usually known to be a real-valued Gaussian random variable has been proven by Bernstein [7] and independently by Kac [48] (see [6, 10, 34] for historical comments and proofs). An analogous definition has been also suggested for introducing Gaussian measures on more general structures like abelian groups; see, e.g., [15, 33].

REMARK 2.21. In the construction of the stochastic integral we present below, all Gaussian random variables are symmetric. Therefore, throughout the paper, if not explicitly stated otherwise, when we say ‘Gaussian’ we

mean ‘symmetric Gaussian’. Note that, as in the Banach space setting, if X is a Gaussian random variable (not necessarily symmetric) then there exists a symmetric Gaussian random variable Y and a point $x_0 \in E$ such that $X = Y + x_0$; see, for instance, [16, Corollary 3.2].

Due to Lévy’s theorem on the equivalence of convergence in distribution and convergence of the characteristic functions for real-valued random variables, it is easy to prove that the convergence in distribution preserves Gaussianity. This is also the case for r -Banach-space-valued Gaussian variables.

LEMMA 2.22. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of E -valued Gaussian random variables converging in distribution to an E -valued random variable X . Then X is Gaussian.*

Proof. Let Y be an independent copy of X and for all $n \in \mathbb{N}$ let Y_n be an independent copy of X_n . Then $(X_n + Y_n, X_n - Y_n)$ converges in distribution to $(X + Y, X - Y)$. Moreover, for all $n \in \mathbb{N}$, $X_n + Y_n$ and $X_n - Y_n$ are independent, whence by Lemma 2.8 it follows that $X + Y$ and $X - Y$ are independent. ■

The following generalization of Fernique’s theorem holds for Gaussian random variables on r -Banach spaces; see, for instance, [19, 47, 97].

THEOREM 2.23. *Let X be an E -valued Gaussian random variable. Then there exists an $\alpha > 0$ such that*

$$\mathbb{E}[\exp(\alpha \|X\|_E^2)] < \infty.$$

2.5. Gaussian random variables in r -Banach spaces with separating dual. One can say more about Gaussian random variables if E has a separating dual. We start with the fact that, in this case, Definition 2.19 extends the usual definition of Gaussian random variables via duality; see also [9] for the Banach space analogue. The results in this subsection will not be used in the rest of the paper.

PROPOSITION 2.24. *Assume that E^* separates the points of E . An E -valued random variable X is Gaussian if and only if $\langle X, x^* \rangle$ is Gaussian for all $x^* \in E^*$.*

Proof. The ‘only if’ part is an easy consequence of Bernstein’s theorem; see [7]. The ‘if’ part follows by the fact that if E^* separates the points of E , then the Fourier transform is unique as $\mathcal{B}(E) = \sigma(x^* : x^* \in E^*)$; see Proposition 2.6 and Example 2.7. ■

In the Banach space setting, Karhunen–Loève expansions can be used to extend many results for Gaussian sums, like, for instance, Theorem 2.16 and Proposition 2.17, to Gaussian random variables. Unfortunately, such

an expansion is not available for general r -Banach spaces. However, if E^* separates the points of E , the following result follows from [17, Theorem 1].

PROPOSITION 2.25. *Assume that E^* separates the points of E . Let $X: \Omega \rightarrow E$ be a Gaussian random variable. Then there exists a Gaussian sequence $(\gamma_n)_{n \in \mathbb{N}} \subseteq L^2(\Omega)$ and a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ such that $\sum_{n=1}^{\infty} \gamma_n x_n$ converges \mathbb{P} -almost surely to a Gaussian random variable Y which has the same distribution as X .*

Proof. Using the notation from [17], it is easy to see that if E^* separates the points of E , then there exists a sequence of elements of $E^2(\mathbb{P}_X)$ separating points of $E \pmod{\mathbb{P}_X}$, as $E^* \subseteq E^2(\mathbb{P}_X)$; see also the proof of Proposition 2.15. Moreover, for every orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of $E^2(\mathbb{P}_X)$, the random variables $(h_n(X))_{n \in \mathbb{N}}$ form a Gaussian sequence. Thus, the assertion is just a very particular instance of [17, Theorem 1]. ■

REMARK 2.26. It is not immediately clear from [17] whether in the proof of Proposition 2.25 one may choose the orthonormal basis $(h_n)_{n \in \mathbb{N}}$ out of E^* . It is also unclear whether it is possible to choose $(h_n)_{n \in \mathbb{N}}$ in such a way that $\sum_{n=1}^{\infty} h_n(X) x_n = X$ \mathbb{P} -a.s.

Now we explain how to use Proposition 2.25 to prove the generalizations of Theorem 2.16 and Proposition 2.17 to Gaussian random variables. Note, however, that these results are not needed in any other proof in this paper.

PROPOSITION 2.27. *Assume that E^* separates the points of E . Then, for every Gaussian random variable $X: \Omega \rightarrow E$ and arbitrary $0 < p, q < \infty$,*

$$(2.7) \quad (\mathbb{E} \|X\|_E^p)^{1/p} \leq C_{p,q,r} (\mathbb{E} \|X\|_E^q)^{1/q}$$

with $C_{p,q,r}$ as in Theorem 2.16.

Proof. This is an immediate consequence of Proposition 2.25, Theorem 2.16 and Proposition 2.17. ■

PROPOSITION 2.28. *Assume that E^* separates the points of E . Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of E -valued Gaussian random variables. Then the following are equivalent:*

- (i) *The sequence $(X_n)_{n \in \mathbb{N}}$ converges in probability.*
- (ii) *The sequence $(X_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega; E)$ for all $0 < p < \infty$.*
- (iii) *The sequence $(X_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega; E)$ for some $0 < p < \infty$.*

Proof. Since Proposition 2.27 holds one can extend the argument in the Banach space setting of [83, Lemma 2.1]. ■

3. γ -norms and square functions. In this section we first introduce and prove fundamental properties of so-called γ -radonifying operators $R: H \rightarrow E$, where H is a separable Hilbert space and E is a separable

r -Banach space for some fixed $r \in (0, 1]$. These operators play a key role in the development of stochastic Itô integrals in the (quasi-)Banach space setting. For details and historical remarks on this class of operators in the Banach space setting, we refer the reader to the survey paper [69]. In the final part of this section we consider the special case of r -Banach function spaces $E = E(S)$ over a measure space (S, \mathcal{A}, μ) . In addition, we replace H by $L^2(\tilde{S}; H)$, where $(\tilde{S}, \tilde{\mathcal{A}}, \tilde{\mu})$ is a measure space such that $L^2(\tilde{S}; H)$ is separable, and connect γ -radonifying operators $R: L^2(\tilde{S}; H) \rightarrow E(S)$ to the theory of (generalized) square functions (see [55]). Our presentation mostly follows [45, Chapter 9].

NOTATION. Throughout this section, H denotes a separable Hilbert space, E denotes a separable r -Banach space, $r \in (0, 1]$ is fixed, and $(\gamma_n)_{n \in \mathbb{N}}$ denotes a Gaussian sequence.

3.1. γ -summing and γ -radonifying operators. For a linear operator $R: H \rightarrow E$ set

$$\|R\|_{\gamma_\infty(H, E)} := \sup_h \left(\mathbb{E} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_E^2 \right)^{1/2} \in [0, \infty],$$

where the supremum is taken over all finite orthonormal systems $h = \{h_1, \dots, h_N\}$ in H . The operator R is called γ -*summing* if $\|R\|_{\gamma_\infty(H, E)} < \infty$. We write $\gamma_\infty(H, E)$ for the set of γ -summing operators from H to E . By considering singletons $\{h\}$ we see that every γ -summing operator is bounded and $\|R\|_{\mathcal{L}(H, E)} \leq \|R\|_{\gamma_\infty(H, E)}$. This can be used to show that the space $\gamma_\infty(H, E)$, endowed with the r -norm $\|\cdot\|_{\gamma_\infty(H, E)}$, is an r -Banach space.

REMARK 3.1. Let $0 < p < \infty$ and write $\gamma_\infty^p(H, E)$ for the set of all linear operators $R: H \rightarrow E$ such that

$$\|R\|_{\gamma_\infty^p(H, E)} := \sup_h \left(\mathbb{E} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_E^p \right)^{1/p} < \infty,$$

the supremum being taken over all finite orthonormal systems $h = \{h_1, \dots, h_N\}$ in H . Then, by the Kahane–Khinchin inequality for Gaussian sums (see Theorem 2.16), for all $p \in (0, \infty)$ we have $\gamma_\infty^p(H, E) = \gamma_\infty(H, E)$ and

$$(3.1) \quad C_{2,p,r}^{-1} \|R\|_{\gamma_\infty(H, E)} \leq \|R\|_{\gamma_\infty^p(H, E)} \leq C_{p,2,r} \|R\|_{\gamma_\infty(H, E)}$$

for all $R \in \gamma_\infty(H, E)$.

For $h \in H$ and $x \in E$ we denote by $h \otimes x$ the rank one operator in $\mathcal{L}(H, E)$ defined by

$$(h \otimes x)g := \langle g, h \rangle_H x, \quad g \in H.$$

The linear span of these operators is usually denoted by $H \otimes E$ and can be viewed as the subspace of finite rank operators in $\mathcal{L}(H, E)$. Every such operator can be represented in the form

$$(3.2) \quad R = \sum_{n=1}^N h_n \otimes x_n$$

with $N \in \mathbb{N}$, $(h_n)_{n=1}^N$ orthonormal in H and $(x_n)_{n=1}^N$ a finite sequence in E . It is known that, if E is a Banach space (i.e., if $r = 1$), then the γ -summing norm $\|R\|_{\gamma_\infty(H, E)}$ of a finite rank operator of this form is equal to the $L_2(\Omega; E)$ -norm of $\sum_{n=1}^N \gamma_n x_n$; see, e.g., [45, Proposition 9.1.3]. For general r -Banach spaces with $r \in (0, 1]$ we obtain the following generalization.

PROPOSITION 3.2. *If $R = \sum_{n=1}^N h_n \otimes x_n$ is a finite rank operator with $N \in \mathbb{N}$, $\{h_1, \dots, h_N\}$ orthonormal in H and $x_1, \dots, x_N \in E$, then $R \in \gamma_\infty(H, E)$ and for all $0 < p < \infty$ we have*

$$(3.3) \quad \left\| \sum_{n=1}^N \gamma_n x_n \right\|_{L^p(\Omega; E)} \leq \left\| \sum_{n=1}^N h_n \otimes x_n \right\|_{\gamma_\infty^p(H, E)} \\ \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \left\| \sum_{n=1}^N \gamma_n x_n \right\|_{L^p(\Omega; E)}.$$

For the proof of this two-sided estimate, we first prove the following auxiliary result. It turns out to be very useful when it comes to handling γ_∞ -norms of finite rank operators. Recall that, for arbitrary $n \in \mathbb{N}$, we write ℓ_n^2 for the space \mathbb{R}^n endowed with the Euclidean norm.

LEMMA 3.3. *Let $0 < p < \infty$ and fix $m, n \in \mathbb{N}$. For all $m \times n$ matrices $A = (a_{ij})_{i,j=1}^{m,n}$ and all finite sequences $(x_j)_{j=1}^n$ in E ,*

$$(3.4) \quad \left\| \sum_{i=1}^m \gamma_i \sum_{j=1}^n a_{ij} x_j \right\|_{L^p(\Omega; E)} \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \|A\|_{\mathcal{L}(\ell_n^2, \ell_m^2)} \left\| \sum_{j=1}^n \gamma_j x_j \right\|_{L^p(\Omega; E)}.$$

In the Banach space setting, i.e., if $r = 1$, Lemma 3.3 has been proven in several ways. Most proofs rely either on a duality argument using covariance operators or on the fact that every matrix A with $\|A\| \leq 1$ is a convex combination of unitary operators. Clearly both methods fail in the r -Banach space setting for $0 < r < 1$. However, as shown in [45, Proposition 6.1.23], Lemma 3.3 can also be proven from basic principles which turn out to remain valid for $0 < r < 1$ and $0 < p < 1$. For the convenience of the reader, we provide the proof with minor modifications.

Proof of Lemma 3.3. (See also [45, proof of Proposition 6.1.23].) We may assume that $\|A\|_{\mathcal{L}(\ell_n^2, \ell_m^2)} = 1$. By adding zero terms to the matrix $A =$

$(a_{ij})_{i,j=1}^{m,n}$, we may also assume that $m = n$. Let B be the $2n \times 2n$ -matrix given by

$$B = \begin{pmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{pmatrix}.$$

Using the simple fact that $A(I - A^*A)^{1/2} = (I - AA^*)^{1/2}A$ we find that $B^*B = I$. Writing $B = (b_{ij})_{i,j=1}^{2n}$ we have $a_{ij} = b_{ij}$ for $1 \leq i, j \leq n$. By Lemma 2.9,

$$\begin{aligned} \mathbb{E} \left\| \sum_{i=1}^n \gamma_i \sum_{j=1}^n a_{ij} x_j \right\|^p &\leq 2^{\frac{(1-(r \wedge p))p}{r \wedge p}} \mathbb{E} \left\| \sum_{i=1}^{2n} \gamma_i \sum_{j=1}^n b_{ij} x_j \right\|^p \\ &= 2^{\frac{(1-(r \wedge p))p}{r \wedge p}} \mathbb{E} \left\| \sum_{j=1}^n G_j x_j \right\|^p, \end{aligned}$$

where $G_j = \sum_{i=1}^{2n} \gamma_i b_{ij}$, for $1 \leq j \leq n$. The result now follows from the fact that $(G_j)_{j=1}^n$ is a sequence of independent real-valued standard Gaussian random variables. ■

REMARK 3.4. In contrast to the Banach space setting, the constant $C_{r,p} = 2^{\frac{1-(r \wedge p)}{r \wedge p}}$ in inequality (3.4) cannot be replaced by $C = 1$: Indeed, setting $(E, \|\cdot\|) = (\ell_2^{1/2}, \|\cdot\|_{\ell_2^{1/2}})$, $n = 2$, $m = 1$, $e_1 = (1, 0)$, $e_2 = (0, 1)$, $x_1 = \frac{1}{4}(e_1 + e_2)$, $x_2 = \frac{1}{16}(e_1 - e_2)$, and $A = (1 \ 0)$, we recover the Gaussian case in Example 2.10.

Lemma 3.3 allows for the following short proof of Proposition 3.2.

Proof of Proposition 3.2. Let $\{u_1, \dots, u_M\}$ be an arbitrary orthonormal system in H . Let A be the $M \times N$ matrix given by $a_{mn} = \langle u_m, h_n \rangle$. It is easily checked that $\|A\| \leq 1$ and therefore, by Lemma 3.3,

$$\begin{aligned} \left\| \sum_{m=1}^M \gamma_m R u_m \right\|_{L^p(\Omega; E)} &= \left\| \sum_{m=1}^M \gamma_m \sum_{n=1}^N a_{mn} x_n \right\|_{L^p(\Omega; E)} \\ &\leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \left\| \sum_{n=1}^N \gamma_n x_n \right\|_{L^p(\Omega; E)}. \end{aligned}$$

This shows that $R \in \gamma_\infty(H, E)$ and that the second estimate in (3.3) holds. To prove the first estimate in (3.3) it suffices to use the orthonormal system $\{h_1, \dots, h_N\}$. ■

In Proposition 3.2 we have seen that every finite rank operator from H to E belongs to $\gamma_\infty(H, E)$. It is demonstrated in e.g. [71, 74] that the closure of the space of finite rank operators in $\gamma_\infty(H, E)$ is the right space to study vector-valued stochastic Itô integrals; see also Sections 4 and 5 below.

DEFINITION 3.5. The space $\gamma(H, E)$ is defined as the closure of the space of finite rank operators in $(\gamma_\infty(H, E), \|\cdot\|_{\gamma_\infty(H, E)})$. The operators in $\gamma(H, E)$ are called γ -radonifying.

REMARK 3.6. By definition, the space $\gamma(H, E)$ is an r -Banach space with respect to the r -norm inherited from $\gamma_\infty(H, E)$. For simplicity, we write

$$\|R\|_{\gamma(H, E)} := \|R\|_{\gamma_\infty(H, E)}, \quad R \in \gamma(H, E),$$

and, more generally,

$$\|R\|_{\gamma^p(H, E)} := \|R\|_{\gamma_\infty^p(H, E)}, \quad R \in \gamma(H, E),$$

for $0 < p < \infty$.

For the case that E is a Banach space it is well-known that the spaces $\gamma_\infty(H, E)$ and $\gamma(H, E)$ are operator ideals in $L(H, E)$. This property plays an important role in the construction of stochastic Itô integrals in Banach spaces (see [71, 74]). By using Lemma 3.3, we can use verbatim the same proof as presented, e.g., in [45, Chapter 9], to verify that the ideal property also holds in the case that E is an r -Banach space with $0 < r < 1$. Indeed, we obtain the following.

THEOREM 3.7 (Ideal property). *Let $R \in \gamma_\infty(H, E)$. Furthermore, let F be a separable r_0 -Banach space for some $r_0 \in (0, 1]$ and let G be a separable Hilbert space. Then for all $V \in \mathcal{L}(G, H)$ and $U \in \mathcal{L}(E, F)$ we have $URV \in \gamma_\infty(G, F)$, and for all $0 < p < \infty$,*

$$(3.5) \quad \|URV\|_{\gamma_\infty^p(G, F)} \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \|U\|_{\mathcal{L}(E, F)} \|R\|_{\gamma_\infty^p(H, E)} \|V\|_{\mathcal{L}(G, H)}.$$

If, moreover, $R \in \gamma(H, E)$, then $URV \in \gamma(G, F)$.

Proof. Mimic the proof of [45, Theorem 9.1.10] and use Lemma 3.3 where needed. ■

REMARK 3.8. Close inspection of the proof of Theorem 3.7 reveals that the constant $2^{\frac{1-(r \wedge p)}{r \wedge p}}$ in (3.5) is only due to the right multiplication, i.e., in the setting of Theorem 3.7, if $H = G$ and $V = \text{Id}$, then

$$\|UR\|_{\gamma_\infty^p(H, F)} \leq \|U\|_{\mathcal{L}(E, F)} \|R\|_{\gamma_\infty^p(H, E)}.$$

3.2. Convergence in $\gamma(H, E)$. We have seen that the γ -radonifying property of an operator is preserved under left and right multiplication. Our next result considers the situation where we multiply with convergent sequences of operators.

THEOREM 3.9. *Let $R \in \gamma(H, E)$. Furthermore, let F be a separable r_0 -Banach space for some $r_0 \in (0, 1]$ and let G be a separable Hilbert space. Let $V, V_1, V_2, \dots \in \mathcal{L}(G, H)$ and $U, U_1, U_2, \dots \in \mathcal{L}(E, F)$ be such that*

- (i) $\lim_{n \rightarrow \infty} V_n^* h = V^* h$ for all $h \in H$,

(ii) $\lim_{n \rightarrow \infty} U_n x = Ux$ for all $x \in E$.

Then $\lim_{n \rightarrow \infty} U_n R V_n = URV$ in $\gamma(G, F)$.

Proof. This result can be obtained by the same strategy as in the proof of [45, Theorem 9.1.14], i.e., by using the ideal property (Theorem 3.7) together with the uniform boundedness principle (which holds on arbitrary r -Banach spaces; see, e.g., [54]). ■

The following two sample applications of this result are frequently used in forthcoming proofs.

EXAMPLE 3.10 (Approximation). Let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis for H and for all $n \in \mathbb{N}$ let P_n denote the orthogonal projection onto $\text{span}(\{h_1, \dots, h_n\})$. Then for all $R \in \gamma(H, E)$ we have $\lim_{n \rightarrow \infty} R P_n = R$ in $\gamma(H, E)$.

EXAMPLE 3.11 (Strong measurability). Let (S, \mathcal{A}, μ) be a measure space. For a function $\Phi: S \rightarrow \gamma(H, E)$ and $h \in H$ define $\Phi h: S \rightarrow E$ by $(\Phi h)(\xi) := \Phi(\xi)h$, $\xi \in S$. Then by employing the same arguments as in the proof of [45, Example 9.1.16] it can be shown that the following assertions are equivalent:

- (i) Φ is strongly measurable.
- (ii) Φh is strongly measurable for all $h \in H$.

We proceed with a characterization of γ -summing and γ -radonifying operators in terms of orthonormal bases.

THEOREM 3.12 (Testing against an orthonormal basis). *Let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis for H .*

- (i) *An operator $R \in \mathcal{L}(H, E)$ belongs to $\gamma_\infty(H, E)$ if and only if for some (equivalently, for all) $0 < p < \infty$,*

$$(3.6) \quad \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_{L^p(\Omega; E)} < \infty.$$

In this case,

$$(3.7) \quad \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_{L^p(\Omega; E)} \leq \|R\|_{\gamma_\infty(H, E)} \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|_{L^p(\Omega; E)}.$$

- (ii) *An operator $R \in \mathcal{L}(H, E)$ belongs to $\gamma(H, E)$ if and only if for some (equivalently, for all) $0 < p < \infty$,*

$$\sum_{n=1}^{\infty} \gamma_n R h_n \text{ converges in } L^p(\Omega; E).$$

In this case, the sum converges almost surely and

$$(3.8) \quad \left\| \sum_{n=1}^{\infty} \gamma_n R h_n \right\|_{L^p(\Omega; E)} \leq \|R\|_{\gamma^p(H, E)} \leq 2^{\frac{1-(r \wedge p)}{r \wedge p}} \left\| \sum_{n=1}^{\infty} \gamma_n R h_n \right\|_{L^p(\Omega; E)}.$$

Proof. We have proven all the ingredients we need in order to obtain this result by mimicking the proof of its Banach space analogue presented in [45, Theorem 9.1.17]: The Kahane–Khinchin inequality for Gaussian sums proven in Theorem 2.16 guarantees that once we have the results for one p , we also have it for arbitrary $0 < p < \infty$. Part (i) follows from a simple application of Lemma 3.3 and Fatou’s lemma. The equivalence of the γ -radonifying norm of a finite rank operator with the $L^p(\Omega; E)$ -norm of the corresponding Gaussian sum provided by Proposition 3.2 in combination with Example 3.10 are the ingredients needed to prove (ii). The almost sure convergence follows from the extension of Hoffmann-Jørgensen’s theorem presented in Proposition 2.14. ■

Note that Theorem 3.12 provides a simple proof of the main result in [18] for the special (and more simple) case that the metric space E is an r -Banach space.

COROLLARY 3.13. *Let $R \in \mathcal{L}(H, E)$. If for some orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of H , the series $\sum_{n=1}^{\infty} \gamma_n R h_n$ converges in probability, then for every orthonormal basis $(h'_n)_{n \in \mathbb{N}}$ of H , the series $\sum_{n=1}^{\infty} \gamma_n R h'_n$ converges in $L^p(\Omega; E)$ for all $0 < p < \infty$ and \mathbb{P} -almost surely. Moreover, the distribution of the limit random variable does not depend on the particular basis.*

Proof. The assumption and Proposition 2.17 imply that $\sum_{n=1}^{\infty} \gamma_n R h_n$ converges in $L^p(\Omega; E)$ for all $0 < p < \infty$. Therefore, $R \in \gamma(H, E)$ by Theorem 3.12. Applying Theorem 3.12 again with an arbitrary orthonormal basis $(h'_n)_{n \in \mathbb{N}}$ we find that $\sum_{n=1}^{\infty} \gamma_n R h'_n$ converges in $L^p(\Omega; E)$ for all $0 < p < \infty$ and \mathbb{P} -almost surely. The independence of the distribution of the series on the orthonormal basis can be easily verified for finite rank operators and then extended to arbitrary $R \in \gamma(H, E)$ by exploiting Theorem 3.12(ii). ■

In general, $\gamma(H, E)$ is a proper closed subspace of $\gamma_{\infty}(H, E)$. In the case that E is a Banach space, it was proven by Hoffmann-Jørgensen and Kwapien [42, 56] that $\gamma(H, E) = \gamma_{\infty}(H, E)$ if E fails to contain a copy of c_0 . More precisely, Hoffmann-Jørgensen and Kwapien proved that for a Banach space E , the space c_0 fails to embed continuously into E if and only if for certain E -valued random sums, almost sure boundedness of the partial sums implies almost sure convergence. The proof uses a characterization of Bessaga and Pełczyński, which seems to be unavailable for r -Banach spaces. Fortunately Proposition 2.18 in combination with Proposition 2.17 implies the following quasi-Banach space analogue of the result by Hoffmann-Jørgensen

and Kwapien (one may copy verbatim the relevant parts of the proof provided in [69, Theorem 4.3]).

PROPOSITION 3.14. *If E has finite cotype, then $\gamma(H, E) = \gamma_\infty(H, E)$.*

3.3. γ -norms in r -Banach function spaces and square functions.

It is well-known that, if $E = E(S)$ is a Banach function space with finite cotype over a σ -finite measure space (S, \mathcal{A}, μ) , then $\|\cdot\|_{\gamma(H, E)}$ has an equivalent square function interpretation; see, e.g., [45, Theorem 9.3.6]. In particular, $\gamma(H, E(S)) \approx E(S; H)$, where $E(S; H)$ is the vector space of all (equivalence classes of) strongly measurable functions $f: S \rightarrow H$ such that $\xi \mapsto \|f(\xi)\|_H$ is an element of $E(S)$ (naturally normed by $\|f\|_{E(S; H)} := \|(\xi \mapsto \|f(\xi)\|_H)\|_{E(S)}$, $f \in E(S; H)$). In this section, we prove that this result extends verbatim to r -Banach function spaces with finite cotype (see Theorem 3.16 below).

We begin with a definition of an r -Banach function spaces analogue to the definition of a Banach function space as considered in e.g. [45, Definition F.3.1]. For details in the case $r = 1$ we refer the reader to [60, Definition 1.b.17], [96, Chapter 15]. In the case $0 < r < 1$ these spaces have been studied in [52, 53].

DEFINITION 3.15. Let (S, \mathcal{A}, μ) be a σ -finite measure space and let $r \in (0, 1]$. We say that a vector space $E(S) \subseteq L^0(S)$ is an r -Banach function space if it is complete under some r -norm $\|\cdot\|_{E(S)}: E(S) \rightarrow [0, \infty)$ and moreover for all $f \in L^0(S)$, $g \in E(S)$ such that $f \leq g$ one has $f \in E(S)$ and $\|f\|_{E(S)} \leq \|g\|_{E(S)}$.

Before we turn to our next result we recall some standard results for r -Banach function spaces. For details on r -convexity and q -concavity we refer to [53, Section 3], or, e.g., to [60]). If $E(S)$ has finite cotype, say cotype t , it satisfies a lower t -estimate. Therefore, it follows from [53, Theorem 3.2] that X is s -convex for some $s \in (0, 1]$. It follows from [61, Theorem 4.4] that $E(S)$ is q -concave for any $q > t$. Therefore, as in [60, Theorem 1.d.6] (using s -convexity to prove the \gtrsim part) one can prove the Maurey–Khinchin inequalities: for all $0 < p < \infty$, $n \in \mathbb{N}$, and $x_1, \dots, x_n \in E(S)$,

$$(3.9) \quad \left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_{L^p(\Omega; E(S))} \widetilde{\approx}_{q, s, E(S)} \left\| \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2} \right\|_{E(S)}.$$

Moreover, in (3.9) one can replace the Rademacher sum by a Gaussian sum as well. Indeed, this follows from [52, Proposition 2.5]. For completeness we mention that finite cotype is actually equivalent to q -concavity for some $q < \infty$. Indeed, the remaining implication follows from [60, Proposition 1.f.3].

The theorem below can be found in [45, Chapter 9] for the case that $r = 1$. The proof extends to the quasi-Banach space case by taking into

account that the Maurey–Khinchin inequalities remain valid in this setting, as explained above. For an r -Banach function space $E(S) \subseteq L^0(S)$ we set

$$E(S; H) := \{f \in L^0(S; H) : s \mapsto \|f(s)\|_H \in E(S)\},$$

which is an r -Banach space when endowed with the r -norm $\|f\|_{E(S; H)} = \|\|f\|_H\|_{E(S)}$, $f \in E(S; H)$.

THEOREM 3.16. *Let $0 < r \leq 1$. Let $E(S)$ be a separable r -Banach function space with finite cotype over a σ -finite measure space (S, \mathcal{A}, μ) . Then the mapping $U : E(S; H) \rightarrow \mathcal{L}(H, E(S))$ defined by*

$$(Uf)h := \langle h, f(\cdot) \rangle_H, \quad h \in H,$$

defines an isomorphism U of Banach spaces

$$E(S; H) \approx \gamma(H, E(S)).$$

Furthermore, for an operator $R \in \mathcal{L}(H, E(S))$ the following assertions are equivalent:

- (i) $R \in \gamma(H, E(S))$.
- (ii) There exists a function $0 \leq g \in E(S)$ such that for all finite orthonormal systems $(h_n)_{n=1}^N \subseteq H$ we have

$$\left(\sum_{n=1}^N |Rh_n|^2 \right)^{1/2} \leq g \quad \mu\text{-a.e.}$$

- (iii) There exists a function $0 \leq g \in E(S)$ such that for all $h \in H$ we have

$$|Rh| \leq \|h\|_H \cdot g \quad \mu\text{-a.e.}$$

- (iv) There exists a function $k \in E(S; H)$ such that for all $h \in H$ we have

$$Rh = \langle h, k(\cdot) \rangle_H \quad \mu\text{-a.e.}$$

- (v) The function $(\sum_{n \in \mathbb{N}} |Rh_n|^2)^{1/2}$ belongs to $E(S)$.

In this situation, in (iii) we may take $g = (\sum_{n=1}^{\infty} |Rh_n|^2)^{1/2}$ and we have

$$(3.10) \quad \|R\|_{\gamma(H, E(S))} \approx_E \|k\|_{E(S; H)} = \left\| \left(\sum_{n=1}^{\infty} |Rh_n|^2 \right)^{1/2} \right\|_{E(S)}.$$

In the setting of stochastic integrals the case that H in Theorem 3.16 is an L^2 -space is of particular relevance (see Example 4.10). In this case we obtain the following corollary.

COROLLARY 3.17. *Let $0 < r \leq 1$. Let $E(S)$ be a separable r -Banach function space with finite cotype over a σ -finite measure space (S, \mathcal{A}, μ) . Furthermore, let $(\tilde{S}, \tilde{\mathcal{A}}, \tilde{\mu})$ be a σ -finite measure space such that $L^2(\tilde{S}; H)$ is separable.*

(i) If $\varphi: \tilde{S} \times S \rightarrow H$ is strongly $\tilde{\mathcal{A}} \otimes \mathcal{A}$ -measurable and such that

$$(3.11) \quad \left\| \left(\int_{\tilde{S}} \|\varphi(\tilde{\xi}, \cdot)\|_H^2 d\tilde{\mu}(\tilde{\xi}) \right)^{1/2} \right\|_{E(S)} < \infty,$$

then $R: L^2(\tilde{S}; H) \rightarrow E(S)$ given by

$$(3.12) \quad Rf(\xi) = \int_{\tilde{S}} \langle \varphi(\tilde{\xi}, \xi), f(\tilde{\xi}) \rangle_H d\tilde{\mu}(\tilde{\xi}), \quad \xi \in S,$$

for $f \in L^2(\tilde{S}; H)$, is in $\gamma(L^2(\tilde{S}; H), E(S))$, and

$$(3.13) \quad \|R\|_{\gamma(L^2(\tilde{S}; H), E(S))} \approx_{E(S)} \left\| \left(\int_{\tilde{S}} \|\varphi(\tilde{\xi}, \cdot)\|_H^2 d\tilde{\mu}(\tilde{\xi}) \right)^{1/2} \right\|_{E(S)}.$$

(ii) Conversely, if $R \in \gamma(L^2(\tilde{S}; H), E(S))$, then we can find a $\tilde{\mu} \otimes \mu$ -almost everywhere unique φ such that (3.12) holds.

Moreover, if $E(S)$ has cotype 2, then given φ such that (3.11) holds we can define $\Phi \in L^2(\tilde{S}; \gamma(H, E(S)))$ by $\Phi(\tilde{\xi})h = \langle \varphi(\tilde{\xi}, \cdot), h \rangle_H$, $\tilde{\xi} \in \tilde{S}$, $h \in H$.

Proof. Part (i) is an immediate consequence of Theorem 3.16 with H replaced by $L^2(\tilde{S}; H)$. To prove part (ii), we choose a sequence $(R_n)_{n \in \mathbb{N}} \subseteq L^2(\tilde{S}; H) \otimes E(S)$ of finite rank operators such that $R_n \rightarrow R$ in the space $\gamma(L^2(\tilde{S}; H), E(S))$. For every $n \in \mathbb{N}$, since R_n is of finite rank, we can obviously find a strongly measurable $\varphi_n: S \times \tilde{S} \rightarrow H$ such that R_n is given by (3.12) with φ replaced by φ_n . From (3.13) it follows that (φ_n) is a Cauchy sequence and hence convergent to some φ in $E(S; L^2(\tilde{S}; H))$. Thus,

$$Rf = \lim_{n \rightarrow \infty} R_n f = \lim_{n \rightarrow \infty} \int_{\tilde{S}} \langle \varphi_n(\tilde{\xi}, \cdot), f(\tilde{\xi}) \rangle_H d\mu(\tilde{\xi}) = \int_{\tilde{S}} \langle \varphi(\tilde{\xi}, \cdot), f(\tilde{\xi}) \rangle_H d\mu(\tilde{\xi}),$$

where the convergence is in $E(S)$; the last step follows from the fact that evaluation against $f \in L^2(\tilde{S}; H)$ is a bounded operator from $E(S; L^2(\tilde{S}; H))$ into $E(S)$ by the Cauchy-Schwarz inequality. The fact that we can choose φ to be strongly $\tilde{\mathcal{A}} \otimes \mathcal{A}$ -measurable can be obtained by mimicking the proof of [45, Lemma 9.3.7]. We conclude that (3.12) holds. The $\tilde{\mu} \otimes \mu$ -uniqueness follows by a standard Fubini argument.

For the final assertion note that cotype 2 implies 2-concavity by (3.9). Therefore, the result follows from the continuous version of 2-concavity which is characterized by (see [92, proof of Theorem 3.9(2)])

$$\|\Phi\|_{L^2(\tilde{S}; \gamma(H, E(S)))} \approx_{E(S)} \|\varphi\|_{L^2(\tilde{S}; E(S; H))} \lesssim_{E(S)} \left\| \left(\int_{\tilde{S}} \|\varphi\|_H^2 d\tilde{\mu} \right)^{1/2} \right\|_{E(S)}. \quad \blacksquare$$

REMARK 3.18. If E is a quasi-Banach space isomorphic to a subspace of an r -Banach function space, then one can deduce a similar characterization. Indeed, for a characterization of $\gamma(L^2(\tilde{S}; H); E)$ in the case of $E = B_{p,q}^\sigma(\mathbb{R}^d)$

(i.e., a Besov space which is not a Banach function space) and $\tilde{S} = [0, T]$ we refer to Proposition A.1.

REMARK 3.19. Note that, in particular, if $\tilde{\mu}(\tilde{S}) < \infty$, Corollary 3.17 provides an example of a class of functions $\Phi: \tilde{S} \rightarrow \mathcal{L}(H, E(S))$ for which an integral $\int_{\tilde{S}} \Phi d\tilde{\mu} := R\mathbb{1}_{\tilde{S}}$ exists in $\mathcal{L}(H, E(S))$. Recall from the introduction that some integration theory in r -Banach spaces is available (see [95]), but that the results in [1] show that the integral does not obey the usual properties of Bochner integrals.

4. Stochastic integration I: deterministic integrands. The theory developed in the previous sections enables us to extend the notion of a stochastic integral from Banach spaces to quasi-Banach spaces by closely following the ideas developed in [71, 74]; see also [73] for a survey. In this section we assume that the integrand is deterministic, i.e., we develop what is sometimes called a Wiener integral. We first construct an abstract stochastic integral for γ -radonifying operators on a separable Hilbert space \mathcal{H} with values in a separable quasi-Banach space E (Subsection 4.1). We then proceed in Subsection 4.2 to consider the case $\mathcal{H} = L^2(0, T; H) := L^2([0, T]; H)$, where $T \in (0, \infty)$ and H is a separable Hilbert space—in this case the abstract stochastic integral introduced in Subsection 4.1 can be interpreted as a continuous, E -valued stochastic process. Examples 4.10 and 4.11 provide characterizations of functions on $[0, T]$ for which an E -valued stochastic integral is well-defined in the case that E is an r -Banach function space or a Besov space.

If E has a separating dual, then explicit characterizations of stochastically integrable $\mathcal{L}(H, E)$ -valued functions can be given. This is the topic of Section 4.3. More specifically, we show that the notion of γ -radonifying operators as defined above coincides with the classical notion of γ -radonifying operators which goes back to Gel'fand [39], Segal [86], and Gross [40, 41]—see [69] for more historical remarks. As a consequence, we can define a weak (Pettis type) stochastic Itô integral for functions $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ as in the Banach space setting.

NOTATION. Throughout this section, E is a separable r -Banach space for some $0 < r \leq 1$, \mathcal{H} and H are separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Moreover, $\mathcal{W}_{\mathcal{H}}: \mathcal{H} \rightarrow L^2(\Omega)$ is an \mathcal{H} -isonormal process (see e.g. [73, Definition 2.1]). If $\mathcal{H} = L^2(0, T; H)$, we write W_H instead of $\mathcal{W}_{L^2(0, T; H)}$.

4.1. An abstract definition of the stochastic integral. The results obtained in Section 3 allow us to introduce the notion of stochastic integration with respect to an \mathcal{H} -isonormal process $\mathcal{W}_{\mathcal{H}}$ for γ -radonifying operators. We start with finite rank operators.

DEFINITION 4.1. Let $R \in \mathcal{H} \otimes E$, i.e., $R = \sum_{n=1}^N h_n \otimes x_n$ with $\{h_1, \dots, h_N\}$ orthonormal in \mathcal{H} and $x_1, \dots, x_N \in E$ for some $N \in \mathbb{N}$. Then

$$R \cdot \mathcal{W}_{\mathcal{H}} := \sum_{n=1}^N \mathcal{W}_{\mathcal{H}}(h_n)x_n \in L^2(\Omega; E)$$

is called *the stochastic integral of R with respect to $\mathcal{W}_{\mathcal{H}}$* .

REMARK 4.2. (i) The stochastic integral of a finite rank operator $R \in \mathcal{H} \otimes E$ with respect to an \mathcal{H} -isonormal Wiener process is well-defined, i.e., if

$$R = \sum_{n=1}^N h_n \otimes x_n = \sum_{m=1}^M h'_m \otimes x'_m$$

with different orthonormal systems $\{h_1, \dots, h_N\}$ and $\{h'_1, \dots, h'_M\}$ in \mathcal{H} and $x_1, \dots, x_N, x'_1, \dots, x'_M \in E$ for some $N, M \in \mathbb{N}$, then

$$\sum_{n=1}^N \mathcal{W}_{\mathcal{H}}(h_n)x_n = \sum_{m=1}^M \mathcal{W}_{\mathcal{H}}(h'_m)x'_m \quad (\text{in } L^2(\Omega; E)).$$

This can be seen by using the linearity of the \mathcal{H} -isonormal process $\mathcal{W}_{\mathcal{H}}$.

(ii) Standard calculations show that the mapping $\mathcal{W}_{\mathcal{H}}: \mathcal{H} \otimes E \rightarrow L^2(\Omega; E)$, $R \mapsto R \cdot \mathcal{W}_{\mathcal{H}}$, is linear. We also call this mapping the *stochastic integral*.

(iii) For any real-valued standard Gaussian random variable γ and any $x \in E$, the E -valued random variable $\gamma(\cdot)x$ is Gaussian. Also, the sum of finitely many independent E -valued Gaussian random variables is Gaussian. Consequently, $R \cdot \mathcal{W}_{\mathcal{H}}$ is Gaussian for any $R \in \mathcal{H} \otimes E$.

(iv) Due to Proposition 3.2,

$$(4.1) \quad 2^{-\frac{1-(r \wedge p)}{(r \wedge p)}} \|R\|_{\gamma^p(\mathcal{H}, E)} \leq \|R \cdot \mathcal{W}_{\mathcal{H}}\|_{L^p(\Omega; E)} \leq \|R\|_{\gamma^p(\mathcal{H}, E)}, \quad R \in \mathcal{H} \otimes E,$$

for arbitrary $0 < p < \infty$.

Since the finite rank operators form a dense subset of the space $\gamma(\mathcal{H}, E)$ of γ -radonifying operators and (4.1) holds, we can extend the notion of the stochastic integral $\mathcal{W}_{\mathcal{H}}: \mathcal{H} \otimes E \rightarrow L^2(\Omega; E)$ uniquely to a bounded linear operator $\mathcal{W}_{\mathcal{H}}: \gamma(\mathcal{H}, E) \rightarrow L^2(\Omega; E)$.

DEFINITION 4.3. Let $R \in \gamma(\mathcal{H}, E)$ and fix a sequence $(R_n)_{n \in \mathbb{N}} \subset \mathcal{H} \otimes E$ of finite rank operators such that

$$(4.2) \quad \lim_{n \rightarrow \infty} \|R_n - R\|_{\gamma(\mathcal{H}, E)} = 0.$$

Then the E -valued random variable

$$(4.3) \quad R \cdot \mathcal{W}_{\mathcal{H}} := \lim_{n \rightarrow \infty} R_n \cdot \mathcal{W}_{\mathcal{H}} \quad (\text{in } L^2(\Omega; E))$$

is called *the stochastic integral of R with respect to $\mathcal{W}_{\mathcal{H}}$* .

REMARK 4.4. (i) The stochastic integral of a γ -radonifying operator $R \in \gamma(\mathcal{H}, E)$ with respect to an \mathcal{H} -isonormal process $\mathcal{W}_{\mathcal{H}}$ is well-defined, i.e., $R \cdot \mathcal{W}_{\mathcal{H}}$ in (4.3) exists as a limit in $L^2(\Omega; E)$ and does not depend on the approximating sequence $(R_n)_{n \in \mathbb{N}} \subset \mathcal{H} \otimes E$ fulfilling (4.2). This is an immediate consequence of the completeness of $L^2(\Omega; E)$ and the estimates in (4.1).

(ii) Standard calculations show that the mapping $\mathcal{W}_{\mathcal{H}}: \gamma(\mathcal{H}, E) \rightarrow L^2(\Omega; E)$, $R \mapsto R \cdot \mathcal{W}_{\mathcal{H}}$, is linear. We also call this mapping the *stochastic integral*.

(iii) $R \cdot \mathcal{W}_{\mathcal{H}}$ is Gaussian for any $R \in \gamma(\mathcal{H}, E)$. This follows immediately from the definition due to Lemma 2.22 and Remark 4.2(iii).

(iv) Due to Remark 4.2(iv), for arbitrary $0 < p < \infty$,

$$(4.4) \quad 2^{-\frac{1-(r \wedge p)}{(r \wedge p)}} \|R\|_{\gamma^p(\mathcal{H}, E)} \leq \|R \cdot \mathcal{W}_{\mathcal{H}}\|_{L^p(\Omega; E)} \leq \|R\|_{\gamma^p(\mathcal{H}, E)},$$

for all $R \in \gamma(\mathcal{H}, E)$.

4.2. Stochastic integrals as stochastic processes. In this subsection we assume that H is a separable Hilbert space and investigate the stochastic integral introduced in Definition 4.3 with $\mathcal{H} = L^2(0, T; H)$.

In this particular case, for every $t \in [0, T]$, if $R \in \gamma(\mathcal{H}, E)$, then the operator $R|_{[0, t]}: \mathcal{H} \rightarrow E$, $f \mapsto R(\mathbb{1}_{[0, t]}f)$, is γ -radonifying as well, due to the right ideal property proven in Theorem 3.7. Thus, we can define the stochastic process $R \cdot W_H(t) := R|_{[0, t]} \cdot W_H$, $t \in [0, T]$, which we also refer to as the *stochastic integral* or the *stochastic integral process of R with respect to W_H* —recall that $W_H := \mathcal{W}_{L^2(0, T; H)}$. Our next goal is to prove that this process has a version with continuous paths and that it satisfies appropriate estimates.

In the situation of normed spaces the continuity proof usually relies on Doob's maximal inequality. By lack of convexity properties of $\|\cdot\|_E^p$, a different argument is needed for r -Banach spaces.

LEMMA 4.5. *Let $(X_t)_{t \in [0, T]}$ be a continuous E -valued stochastic process such that $X_0 \equiv 0$ and for all $0 \leq s \leq t \leq T$, $X_t - X_s$ is symmetric and independent of $\sigma(\{X_u : u \in [0, s]\})$. Then for all $0 < p < \infty$,*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|X_t\|_E^p \right] \leq 2^{1+p/r-p} \mathbb{E} \|X_T\|_E^p.$$

Proof. For $n \in \mathbb{N}$ set $D_n = \{j2^{-n}T : j \in \{0, 1, \dots, 2^n\}\}$. The result follows from the continuity of $(X_t)_{t \in [0, T]}$, Fatou's lemma and Lemma 2.12:

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} \|X_t\|_E^p \right] &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in D_n} \|X_t\|_E^p \right] \\
&= \liminf_{n \rightarrow \infty} \mathbb{E} \left[\sup_{k \in \{1, \dots, 2^n\}} \left\| \sum_{j=1}^k X_{j2^{-n}T} - X_{(j-1)2^{-n}T} \right\|_E^p \right] \\
&\leq 2^{1+p/r-p} \mathbb{E} \|X_T\|_E^p. \quad \blacksquare
\end{aligned}$$

DEFINITION 4.6. An operator-valued function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ of the form

$$\Phi(t) = \sum_{j=1}^J \mathbb{1}_{(t_{j-1}, t_j]}(t) \sum_{k=1}^K h_k \otimes x_{j,k}, \quad t \in [0, T],$$

with $0 = t_0 < t_1 < \dots < t_J = T$, $\{h_1, \dots, h_K\} \subseteq H$ orthonormal, and $x_{j,k} \in E$, $1 \leq j \leq J$, $1 \leq k \leq K$, for some $J, K \in \mathbb{N}$, is called a *finite rank step function*.

Note that every finite rank step function can be canonically identified with an element of $\gamma(L^2(0, T; H), E)$. Indeed, if $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ is as in Definition 4.6, then the operator $R_\Phi: L^2(0, T; H) \rightarrow E$ defined by

$$(4.5) \quad R_\Phi f := \int_0^T \Phi(t) f(t) dt, \quad f \in L^2(0, T; H),$$

is γ -radonifying. We say that Φ *represents* R_Φ and sometimes write Φ instead of R_Φ (note that the mapping $\Phi \mapsto R_\Phi$ is one-to-one). The following lemma shows that the class of operators represented by finite rank step functions is actually dense in $\gamma(L^2(0, T; H), E)$.

LEMMA 4.7. *Let $R \in \gamma(L^2(0, T; H), E)$. Then there exists a sequence of finite rank step functions $(\Phi_n)_{n \in \mathbb{N}}$ such that*

$$\lim_{n \rightarrow \infty} \|R_{\Phi_n} - R\|_{\gamma(L^2(0, T; H), E)} = 0.$$

Proof. Let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis for H , let $(f_n)_{n \in \mathbb{N}}$ be the Haar basis for $L^2(0, T)$. For $n \in \mathbb{N}$ let $Q_n \in \mathcal{L}(H)$ be the orthogonal projection on $\text{span}(\{h_1, \dots, h_n\})$ and let $P_n \in \mathcal{L}(L^2(0, T))$ be the orthogonal projection on the subspace $\text{span}(\{f_1, \dots, f_n\})$. Then, for every $n \in \mathbb{N}$, $R(P_n \otimes Q_n)$ is represented by a finite rank step function, and as $\{f_j \cdot h_k : (j, k) \in \mathbb{N} \times \mathbb{N}\}$ is an orthonormal basis of $L^2(0, T; H)$, $R(P_n \otimes Q_n) \rightarrow R$ in $\gamma(L^2(0, T; H), E)$; see also Example 3.10. \blacksquare

PROPOSITION 4.8. *Let $R \in \gamma(L^2(0, T; H), E)$. Then the stochastic integral process $R \cdot W_H$ has a version with continuous paths and for every $0 < p < \infty$,*

$$(4.6) \quad 2^{-\frac{1-(r \wedge p)}{r \wedge p}} \|R\|_{\gamma^p(L^2(0,T;H),E)} \leq \left(\mathbb{E} \left[\sup_{t \in [0,T]} \|R \cdot W_H(t)\|_E^p \right] \right)^{1/p} \\ \leq 2^{\frac{1}{p} + \frac{1}{r} - 1} \|R\|_{\gamma^p(L^2(0,T;H),E)}.$$

Proof. First assume R is represented by a finite rank step function. In this case continuity of $R \cdot W_H$ follows from classical Itô theory, observing that if $(h_k)_{k=1}^n$ is an orthonormal sequence in H , then $(W_H(1_{[0,t]} \cdot h_k))_{t \in [0,T], k=1}^n$ is a sequence of independent standard Brownian motions. The first inequality in (4.6) follows by Remark 4.2(iv), and the second from Lemma 4.5 and Remark 4.2(iv).

For general $R \in \gamma(L^2(0,T;H),E)$, the result now follows from Lemma 4.7 and the fact that (4.6) holds for operators represented by finite rank step functions. ■

As in the Banach space setting, we have the following series expansion of the stochastic integral. For $R \in \gamma(L^2(0,T;H),E)$ and $h \in H$ we write $R(\cdot \otimes h)$ for the γ -radonifying operator $R(\cdot \otimes h): L^2(0,T) \rightarrow E$, $f \mapsto R(f \otimes h)$. Moreover, we write $W_H(\cdot \otimes h)$ for the $L^2(0,T)$ -isonormal process $W_H(\cdot \otimes h): L^2([0,T]) \rightarrow L^2(\Omega)$, $f \mapsto W_H(f \otimes h)$, and $W_H h$ for the Brownian motion $W_H h(t) := W_H(1_{(0,t]} \otimes h)$, $t \in [0,T]$.

COROLLARY 4.9. *Let $R \in \gamma(L^2(0,T;H),E)$. Then, for every orthonormal basis $(h_n)_{n \in \mathbb{N}}$ in H ,*

$$(4.7) \quad R \cdot W_H = \sum_{n \in \mathbb{N}} R(\cdot \otimes h_n) \cdot W_H(\cdot \otimes h_n),$$

where the series converges in $L^p(\Omega; \mathcal{C}([0,T]; E))$, $0 < p < \infty$. In particular, for every $x^* \in E^*$,

$$(4.8) \quad \langle R \cdot W_H, x^* \rangle = \sum_{n \in \mathbb{N}} \int_0^{\cdot} \langle R^* x^*(t), h_n \rangle_H dW_H h_n(t),$$

where the series converges in $L^p(\Omega; \mathcal{C}([0,T]))$, $0 < p < \infty$.

Proof. For $n \in \mathbb{N}$ let $Q_n \in \mathcal{L}(H)$ be the orthogonal projection on the subspace $\text{span}(\{h_1, \dots, h_n\})$. Moreover, let $(f_n)_{n \in \mathbb{N}}$ be the Haar basis for $L^2(0,T)$, and for every $n \in \mathbb{N}$ write $P_n \in \mathcal{L}(L^2(0,T))$ for the orthogonal projection on $\text{span}(\{f_1, \dots, f_n\})$. The representation (4.7) of $R \cdot W_H$ holds due to Example 3.10, Proposition 4.8 and the fact that it obviously holds with R replaced by $R(P_M \otimes Q_N)$ for arbitrary $M, N \in \mathbb{N}$. To obtain (4.8), we only have to check that for arbitrary $x^* \in E^*$ and $n \in \mathbb{N}$,

$$\langle R(\cdot \otimes h_n) \cdot W_H(\cdot \otimes h_n), x^* \rangle = \int_0^{\cdot} \langle R^* x^*, h_n \rangle_H dW_H h_n.$$

But this is clear due to the construction of the stochastic integral. ■

In the Banach space setting the dual space can be used to relate operator-valued functions to operators $R \in \mathcal{L}(L^2(0, T; H), E)$; see e.g. [74, Theorem 2.5]. Recall however that the dual of a quasi-Banach space may be trivial. In Subsection 4.3 we provide an analogue of [74, Theorems 2.3 and 2.5] in the case that E^* separates the points in E . The following two examples show that even if E^* does not separate points, it can still be possible to relate functions and operators.

EXAMPLE 4.10. Let (S, \mathcal{A}, μ) be a σ -finite measure space and let $E(S)$ be a separable r -Banach function space, $0 < r \leq 1$, with finite cotype. It follows from Corollary 3.17 that for every strongly measurable $\varphi: [0, T] \times S \rightarrow H$ such that $\|(\int_0^T \|\varphi(t, \cdot)\|_H^2 dt)^{1/2}\|_E < \infty$ we have $R_\varphi \in \gamma(L^2(0, T; H), E(S))$, where

$$(R_\varphi f)(s) = \int_0^T \langle \varphi(t, s), f(t) \rangle_H dt, \quad f \in L^2(0, T; H).$$

Consequently, $R_\varphi \cdot W_H$ is well-defined (see Definition 4.3) and moreover Corollary 3.17 and Proposition 4.8 guarantee that for all $p \in (0, \infty)$ there exist constants $c_{p,E}, C_{p,E} \in (0, \infty)$ (independent of the choice of φ) such that

$$\left\| \left(\int_0^T \|\varphi(t, \cdot)\|_H^2 dt \right)^{1/2} \right\|_E \widetilde{\sim}_{p,E} \|R_\varphi \cdot W_H\|_{L^p(\Omega; \mathcal{C}([0, T]; E))}.$$

More specifically, if $E(S) = L^q(S)$ for some $q \in (0, \infty)$, then for all $p \in (0, \infty)$,

$$\int_S \left(\int_0^T \|\varphi(t, s)\|_H^2 dt \right)^{q/2} ds \widetilde{\sim}_{p,q} \|R_\varphi \cdot W_H\|_{L^p(\Omega; \mathcal{C}([0, T]; L^q(S)))}^q.$$

EXAMPLE 4.11. Let $d \in \mathbb{N}$, $0 < p, q < \infty$, and $\sigma \in \mathbb{R}$. Moreover, let $\varphi \in B_{p,q}^\sigma(\mathbb{R}^d; L^2([0, T]; H))$. By Proposition A.1,

$$R_\varphi \in \gamma(L^2(0, T; H), B_{p,q}^\sigma(\mathbb{R}^d)),$$

where $(R_\varphi f)(s) = \int_0^T \langle \varphi(s)(t), f(t) \rangle_H dt$. Consequently, $R_\varphi \cdot W_H$ is well-defined (see Definition 4.3) and moreover Proposition A.1 and Proposition 4.8 guarantee that for all $\tau \in (0, \infty)$ there exist constants $c_{\tau, B_{p,q}^\sigma}, C_{\tau, B_{p,q}^\sigma} \in (0, \infty)$ (independent of the choice of φ) such that

$$\|\varphi\|_{B_{p,q}^\sigma(\mathbb{R}^d; L^2(0, T; H))} \widetilde{\sim}_{p,q,\sigma,\tau} \|R_\varphi \cdot W_H\|_{L^\tau(\Omega; \mathcal{C}([0, T]; B_{p,q}^\sigma(\mathbb{R}^d)))}.$$

4.3. Stochastic integration in r -Banach spaces with separating dual. Recall that although the dual of a quasi-Banach space may be empty, there exist quasi-Banach spaces with separating dual. An important example in the context of nonlinear approximation is the scale of Besov spaces $B_{\tau,\tau}^\alpha(\mathcal{O})$

satisfying $1/\tau = \alpha/d + 1/p$ for a fixed $p \in (1, \infty)$. As announced above Example 4.10, the aim of this section is to prove Proposition 4.13, i.e., an analogue of [74, Theorem 2.5]. We begin by proving the following key result in a similar fashion as if E were a Banach space; see, e.g., [74, Theorem 2.3].

THEOREM 4.12. *Assume that E^* separates the points of E and let $R \in \mathcal{L}(\mathcal{H}, E)$. Then the following are equivalent:*

- (i) $R \in \gamma(\mathcal{H}, E)$.
- (ii) *There exists a random variable $X: \Omega \rightarrow E$ such that*

$$(4.9) \quad \forall x^* \in E^*: \mathcal{W}_{\mathcal{H}}(R^*x^*) = \langle X, x^* \rangle \quad \mathbb{P}\text{-a.s.}$$

In this case $X = R \cdot \mathcal{W}_{\mathcal{H}}$ \mathbb{P} -a.s.

Proof. ‘(i) \Rightarrow (ii)’: This implication holds even without assuming that E^* separates the points of E : The equality $\mathcal{W}_{\mathcal{H}}(R^*x^*) = \langle R \cdot \mathcal{W}_{\mathcal{H}}, x^* \rangle$ for all $x^* \in E^*$ is easily verified if R is a finite rank operator and can be extended to arbitrary $R \in \gamma(\mathcal{H}, E)$ by using the continuity of $\mathcal{W}_{\mathcal{H}}$ and the fact that, if we choose an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ in \mathcal{H} , then, due to Theorem 3.12(ii), the series

$$X := \sum_{n=1}^{\infty} \mathcal{W}_{\mathcal{H}}(h_n) R h_n = R \cdot \mathcal{W}_{\mathcal{H}}$$

converges in $L^2(\Omega; E)$.

‘(ii) \Rightarrow (i)’: Given a random variable $X: \Omega \rightarrow E$ such that (4.9) is fulfilled, we set

$$G := \overline{\text{span}(\{\langle X, x^* \rangle : x^* \in E^*\})}^{\|\cdot\|_{L^2(\Omega)}} = \overline{\{\langle X, x^* \rangle : x^* \in E^*\}}^{\|\cdot\|_{L^2(\Omega)}}.$$

By a Gram–Schmidt argument, we can choose a sequence $(x_i^*)_{i \in I} \subset E^*$ such that $\{\langle X, x_i^* \rangle : i \in I\}$ is an orthonormal basis of $(G, \langle \cdot, \cdot \rangle_{L^2(\Omega)})$, where $I \subseteq \mathbb{N}$ (note that G is separable since X is strongly measurable). We assume that $I = \mathbb{N}$, the other case can be treated analogously. Since $\mathcal{W}_{\mathcal{H}}$ is an \mathcal{H} -isonormal process and (4.9) holds, $(\langle X, x_i^* \rangle)_{i \in \mathbb{N}}$ is a sequence of independent real-valued standard Gaussian random variables, i.e., a Gaussian sequence. Put $h_i := R^*x_i^*$, $i \in \mathbb{N}$. Then $(h_i)_{i \in \mathbb{N}}$ is an orthonormal basis of the closure in \mathcal{H} of the space spanned by $\{R^*x^* : x^* \in E^*\}$, i.e., of

$$\mathcal{H}_0 := \overline{\text{span}(\{R^*x^* : x^* \in E^*\})}^{\|\cdot\|_{\mathcal{H}}} = \overline{\{R^*x^* : x^* \in E^*\}}^{\|\cdot\|_{\mathcal{H}}},$$

endowed with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ inherited from \mathcal{H} . This can be seen as follows. Since $\mathcal{W}_{\mathcal{H}}$ is an \mathcal{H} -isonormal process and (4.9) holds,

$$\begin{aligned} \langle h_i, h_j \rangle_{\mathcal{H}} &= \langle R^*x_i^*, R^*x_j^* \rangle_{\mathcal{H}} = \mathbb{E}[\mathcal{W}_{\mathcal{H}}(R^*x_i^*)\mathcal{W}_{\mathcal{H}}(R^*x_j^*)] \\ &= \mathbb{E}[\langle X, x_i^* \rangle \langle X, x_j^* \rangle] = \delta_{ij}, \quad i, j \in \mathbb{N}. \end{aligned}$$

Thus, $(h_i)_{i \in \mathbb{N}}$ is an orthonormal system in \mathcal{H}_0 . In order to prove the maximality of $(h_i)_{i \in \mathbb{N}}$, fix an arbitrary $h \in \mathcal{H}_0$ with $\langle h, R^*x_i^* \rangle_{\mathcal{H}} = 0$ for all $i \in \mathbb{N}$.

Then, since $\mathcal{W}_{\mathcal{H}}$ is an \mathcal{H} -isonormal process and due to (4.9),

$$(4.10) \quad \mathbb{E}[\mathcal{W}_{\mathcal{H}}(h)\langle X, x_i^* \rangle] = \mathbb{E}[\mathcal{W}_{\mathcal{H}}(h)\mathcal{W}_{\mathcal{H}}(R^*x_i^*)] = \langle h, R^*x_i^* \rangle_{\mathcal{H}} = 0$$

for all $i \in \mathbb{N}$. Also, since $h \in \mathcal{H}_0$, the very definitions of the spaces \mathcal{H}_0 and G , together with the continuity of $\mathcal{W}_{\mathcal{H}}: \mathcal{H} \rightarrow L^2(\Omega)$, yield $\mathcal{W}_{\mathcal{H}}(h) \in G$. Thus, since $(\langle X, x_i^* \rangle)_{i \in \mathbb{N}}$ is an orthonormal basis of G , Eq. (4.10) yields $\mathcal{W}_{\mathcal{H}}(h) = 0$. Using again the properties of $\mathcal{W}_{\mathcal{H}}$, we obtain $\|h\|_{\mathcal{H}}^2 = \mathbb{E}|\mathcal{W}_{\mathcal{H}}(h)|^2 = 0$ and therefore $h = 0$. Consequently, $(h_i)_{i \in \mathbb{N}}$ is an orthonormal basis of \mathcal{H}_0 .

Fix $x^* \in E^*$. Recalling that $(\langle X, x_i^* \rangle)_{i \in \mathbb{N}}$ is an orthonormal basis for G we have

$$\langle X, x^* \rangle = \sum_{i \in \mathbb{N}} c_i \langle X, x_i^* \rangle \quad \text{in } L^2(\Omega)$$

with

$$c_i := \mathbb{E}[\langle X, x^* \rangle \langle X, x_i^* \rangle] = \langle R^*x^*, R^*x_i^* \rangle_{\mathcal{H}} = \langle Rh_i, x^* \rangle, \quad i \in \mathbb{N}.$$

Therefore,

$$\langle X, x^* \rangle = \sum_{i \in \mathbb{N}} \langle X, x_i^* \rangle \langle x^*, Rh_i \rangle \quad \text{in } L^2(\Omega).$$

Therefore, by Proposition 2.15, the sequence

$$S_n := \sum_{i=1}^n \langle X, x_i^* \rangle Rh_i, \quad n \in \mathbb{N},$$

of partial sums converges \mathbb{P} -a.s. to X . It follows from (4.9), Theorem 2.23, and Proposition 2.24 that $X \in L^2(\Omega; E)$, whence Proposition 2.14 yields

$$\sum_{i=1}^{\infty} \langle X, x_i^* \rangle Rh_i = X \quad \text{in } L^2(\Omega; E).$$

Thus, $R \in \gamma(\mathcal{H}_0, E)$ due to Theorem 3.12(ii). The assertion that $R \in \gamma(\mathcal{H}, E)$ follows now if we prove that R vanishes on the orthogonal complement \mathcal{H}_0^{\perp} of \mathcal{H}_0 in \mathcal{H} . This is indeed the case since for a fixed $h \in \mathcal{H}_0^{\perp}$, $\langle Rh, x^* \rangle = \langle h, R^*x^* \rangle_{\mathcal{H}} = 0$ for all $x^* \in E^*$. Since E^* separates the points of E , $Rh = 0$ for all $h \in \mathcal{H}_0^{\perp}$.

In order to prove that $X = R \cdot \mathcal{W}_{\mathcal{H}}$ \mathbb{P} -a.s., first recall from the proof of Proposition 2.15 that there exists a countable set in E^* that separates the points of E . Hence every random variable fulfilling (4.9) is just a version of $R \cdot \mathcal{W}_{\mathcal{H}}$. Next note that we have seen in the proof of '(i) \Rightarrow (ii)' that (4.9) holds for $X = R \cdot \mathcal{W}_{\mathcal{H}}$. ■

We have now all ingredients we need to prove a characterization of the stochastic integral introduced above as a Pettis type stochastic Itô integral (see Proposition 4.13 below); for $r = 1$ this result can be found, e.g., in [71, Proposition 3.2]; see also [74]. Before we state this result, let us generalize some notions, which are part of the folklore in Banach space theory: We

say a function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ is *H-measurable* if for every $h \in H$ the mapping $t \mapsto \Phi(t)h$ is strongly measurable. Moreover, an *H-measurable* function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ *belongs to* $L^2(0, T; H)$ *scalarly* if for every $x^* \in E^*$ the mapping $\Phi^*x^*: [0, T] \rightarrow H$, $t \mapsto \Phi(t)^*x^*$, is square integrable. If E^* separates the points of E , then we say that a function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ that belongs to $L^2(0, T; H)$ (*scalarly*) *represents* an operator $R \in \mathcal{L}(L^2(0, T; H), E)$ if $R^*x^* = \Phi^*x^*$ in $L^2(0, T; H)$ for all $x^* \in E^*$, i.e., for all $x^* \in E^*$,

$$\langle Rf, x^* \rangle = \int_0^T \langle \Phi(t)f(t), x^* \rangle dt, \quad f \in L^2(0, T; H).$$

Recall from (4.5) that every finite rank step function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ represents an operator $R \in \mathcal{L}(L^2(0, T; H), E)$.

PROPOSITION 4.13. *Assume that E^* separates the points of E . For a function $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ belonging to $L^2(0, T; H)$ scalarly, the following are equivalent:*

- (i) *There exists a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of finite rank step functions, such that*
 - $\lim_{n \rightarrow \infty} \Phi_n^*x^* = \Phi^*x^*$ in $L^2(0, T; H)$ for all $x^* \in E^*$;
 - $(R_{\Phi_n} \cdot W_H)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^0(\Omega; E)$ (i.e., Cauchy in probability).
- (ii) *There exists a random variable $X: \Omega \rightarrow E$ such that for all $x^* \in E^*$ we have*

$$\langle X, x^* \rangle = \int_0^T \Phi^*(t)x^* dW_H(t) \quad \mathbb{P}\text{-a.s.}$$

- (iii) Φ represents an operator $R \in \gamma(L^2(0, T; H), E)$.

In this case, $R \cdot W_H = X$ in $L^p(\Omega; E)$ for all $p \in (0, \infty)$, and

$$(4.11) \quad \|R\|_{\gamma^p(L^2(0, T; H); E)} \leq \|X\|_{L^p(\Omega; E)} \leq 2^{\frac{1-r\wedge p}{r\wedge p}} \|R\|_{\gamma^p(L^2(0, T; H); E)}.$$

We call a function Φ satisfying these equivalent conditions stochastically integrable with respect to W_H . The random variable X is called the stochastic integral of Φ with respect to W_H and is denoted by

$$\int_0^T \Phi(t) dW_H(t) := X.$$

Proof. ‘(i) \Rightarrow (iii)’: We claim that $(R_{\Phi_n} \cdot W_H)$ converges in $L^q(\Omega; E)$ for any $q \in (0, \infty)$. In order to show this let $\varepsilon > 0$ be arbitrary. Note that $R_{\Phi_n} \cdot W_H - R_{\Phi_m} \cdot W_H = R_{\Phi_n - \Phi_m} \cdot W_H$ is a Gaussian sum, so that Theorem 2.16 applies and we will use the constant $C_{p,q,r}$ introduced there. Let $\delta = \varepsilon 2^{-1/q}$.

Choose N such that for all $m, n \geq N$,

$$\mathbb{P}(\|R_{\Phi_n} \cdot W_H - R_{\Phi_m} \cdot W_H\| > \delta) \leq \frac{1}{4C_{2q,q,r}^2}.$$

Then by [45, Corollary 6.2.9] (which remains valid in the r -Banach space setting), we find that for all $n, m \geq N$,

$$\|R_{\Phi_n} \cdot W_H - R_{\Phi_m} \cdot W_H\|_{L^q(\Omega; E)} \leq 2^{1/q} \delta = \varepsilon.$$

This proves the claim. In particular, $(R_{\Phi_n} \cdot W_H)$ converges in $L^2(\Omega; E)$. This, in turn, implies the existence of an operator $R \in \gamma(L^2(0, T; H), E)$ such that $\lim_{n \rightarrow \infty} \|R_{\Phi_n} - R\|_{\gamma(L^2(0, T; H), E)} = 0$ due to inequality (4.1) and the completeness of $\gamma(L^2(0, T; H), E)$. Obviously, for all $x^* \in E^*$, we have $R_{\Phi_n}^* x^* = \Phi_n^* x^*$, and, as a consequence, $R^* x^* = \Phi^* x^*$.

‘(iii) \Rightarrow (i)’: Since $R \in \gamma(L^2(0, T; H), E)$, Lemma 4.7 implies that there exists a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of finite rank step functions such that

$$\lim_{n \rightarrow \infty} \|R_{\Phi_n} - R\|_{\gamma(L^2(0, T; H), E)} = 0.$$

Thus, due to inequality (4.1), $(R_{\Phi_n} \cdot W_H)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega; E)$, and hence in $L^0(\Omega; E)$. Moreover, for every $x^* \in E^*$, we have

$$\|R_{\Phi_n}^* x^* - R^* x^*\|_{L^2(0, T; H)} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Thus, $\Phi_n^* x^* \rightarrow \Phi^* x^*$ in $L^2(0, T; H)$.

The equivalence of (ii) and (iii) as well as the final statement are immediate consequences of Theorem 4.12. ■

5. Stochastic integration II: random integrands. In this section we extend the stochastic integral introduced above by allowing for random integrands. More specifically, we define the stochastic integral for $R \in L^0(\Omega; \gamma(L^2(0, T; H), E))$, where $T > 0$, H is a Hilbert space, E is a suitable type of r -Banach space, and R satisfies an adaptedness condition. The approach is in the spirit of [71] (see also the survey [73]): We first define the stochastic integral for stochastic processes with a very simple structure, the so-called adapted elementary processes. Then we use a decoupling inequality to obtain a Burkholder–Davis–Gundy type estimate. Finally, using this estimate, we extend the stochastic integral to a wider class of random integrands: first in L^p and then in L^0 by localization.

The main difference to the Banach space theory is that we do not assume that E has the UMD property. Instead, we consider a (weaker) one-sided decoupling property. This property was investigated in the quasi-Banach space setting in [24]. It turns out to be sufficient for the one-sided estimate required for the extension of the stochastic integral. In fact, as Proposition 5.1

below shows, the UMD property fails to make sense in the quasi-Banach space setting.

NOTATION. Throughout this section, E is a separable r -Banach space for some $0 < r \leq 1$, H is a separable Hilbert space, and $0 < T < \infty$. In addition, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space endowed with a normal filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, and W_H is an $\mathbb{F}/L^2(0, T; H)$ -isonormal process, i.e., an $L^2(0, T; H)$ -isonormal process such that for all $0 \leq s < t \leq T$ and all $h \in L^2(0, T; H)$ such that $\text{supp } h \subseteq [s, t]$, $W_H(h)$ is \mathcal{F}_t -measurable and independent of \mathcal{F}_s .

5.1. Intermezzo: The UMD property in quasi-Banach spaces.

In order to extend the class of admissible integrands and to develop a stochastic calculus in the Banach space setting one typically assumes the Banach space has the UMD (*unconditional martingale differences*) property; see, e.g., the survey [73]. Although the concept of an E -valued martingale, with E an r -Banach space and $r \in (0, 1)$, does not make sense in general (for lack of a Bochner integral), it does make sense to consider E -valued martingales with respect to a filtration of finitely generated σ -algebras. Moreover, it is known from [65] (see also e.g. [44, Theorem 4.2.5]) that in the Banach space setting the UMD property is equivalent to the dyadic UMD property (i.e., where one considers only martingales with respect to a dyadic filtration). It is also known (see [23, Section 5]) that the decoupling inequalities needed to develop the stochastic integral of random integrands can be derived directly from the dyadic UMD property. Unfortunately, Proposition 5.1 below states that if an r -Banach space satisfies the dyadic UMD property, then it is isomorphic to a Banach space.

Moreover, one can combine Proposition 5.1 with [36, Theorem 2] (the proof carries over to the r -Banach space setting *mutatis mutandis*) to prove that if E is an r -Banach space such that a two-sided version of inequality (5.2) below holds (or, equivalently, a two-sided version of inequality (5.3)), then E is isomorphic to a Banach space.

A sequence $(d_n)_{n \in \mathbb{N}}$ of E -valued random variables is called a *Paley–Walsh martingale* (or *dyadic martingale*) if there exists a Rademacher sequence $(r_k)_{k \in \mathbb{N}}$, a $v_0 \in E$, and mappings $v_n: \{-1, 1\}^n \rightarrow E$, $n \in \mathbb{N}$, such that

$$d_1 = r_1 v_0 \quad \text{and} \quad d_n = r_1 v_0 + \sum_{k=2}^n r_k v_{k-1}(r_1, \dots, r_{k-1}), \quad n \in \{2, 3, \dots\}.$$

If E is a Banach space, this is equivalent to saying that $(d_n)_{n \in \mathbb{N}}$ is a martingale with respect to a dyadic filtration.

PROPOSITION 5.1. *Let $0 < r < 1$, $1 < p < \infty$, and let E be a separable r -Banach space that satisfies the dyadic UMD_p -property, i.e., there exists a*

constant $\beta \in (0, \infty)$ such that

$$\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n d_n \right\|_E^p \leq \beta^p \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|_E^p \quad \text{for all } N \in \mathbb{N},$$

for all Paley–Walsh martingales $(d_n)_{n \in \mathbb{N}}$ and all $(\varepsilon_n)_{n \in \mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$. Then E is isomorphic to a Banach space.

Proof. Let $\beta_p(\ell_{2^n}^\infty)$, $n \in \mathbb{N}$, denote the dyadic UMD_p constant of $\ell_{2^n}^\infty$. It is well-known (see e.g. [44, Proposition 4.2.19]) that there exists a constant $c_p \in (0, \infty)$ such that $\beta_p(\ell_{2^n}^\infty) \geq c_p \sqrt{n}$ for all $n \in \mathbb{N}$. Note that if, for some $\lambda \in (0, \infty)$ and $n \in \mathbb{N}$, E contains a λ -isomorphic copy of $\ell_{2^n}^\infty$ in the sense of [45, Definition 7.1.8] then $\beta_p(\ell_{2^n}^\infty) \leq \lambda\beta$. It follows that c_0 is not ε -finitely represented in E for any $\varepsilon \in (0, \infty)$, in the sense of [2, Definition 11.1.1]. Thus [5, Theorem 1] implies that there exists a $q \in [2, \infty)$ such that E has cotype q .

Minor modifications of the Burkholder stopping time proof (see [14, Theorem 1.1, ‘(1.2) \Rightarrow (1.3)’], and see [24, Lemma 3.2] for the minor modifications) shows that if E satisfies the dyadic UMD_s -property for some $s \in (1, \infty)$, then it satisfies the dyadic UMD_s -property for all $s \in (1, \infty)$. It follows immediately (see e.g. [73, Proposition 5.3]) that E has Paley–Walsh martingale cotype q . Now the proof of [77, Theorem 3.1(a)] (which also works for E a quasi-Banach space) provides a norm $|\cdot|_E: E \rightarrow \mathbb{R}$ and constants $c, C \in (0, \infty)$ such that $c\|x\|_E \leq |x|_E \leq C\|x\|_E$ for all $x \in E$. ■

5.2. Stochastic integrals of adapted elementary processes and a one-sided Burkholder–Davis–Gundy inequality. As usual, our construction starts with the following class of integrands.

DEFINITION 5.2. We say that $\Phi: [0, T] \times \Omega \rightarrow \mathcal{L}(H, E)$ is an *adapted elementary process* if there are $0 = t_0 < t_1 < \dots < t_J = T$, $\{h_1, \dots, h_K\} \subseteq H$ orthonormal and $\mathcal{F}_{t_{j-1}}$ -measurable simple random variables $X_{j,k}: \Omega \rightarrow E$, $j \in \{1, \dots, J\}$, $k \in \{1, \dots, K\}$, for some $J, K \in \mathbb{N}$, such that

$$\Phi(t, \omega) = \sum_{j=1}^J 1_{(t_{j-1}, t_j]}(t) \sum_{k=1}^K h_k \otimes X_{j,k}(\omega).$$

Moreover, we say a random variable $R \in L^0(\Omega; \gamma(L^2(0, T; H), E))$ is *represented* by an adapted elementary process Φ if

$$(5.1) \quad Rf = R_\Phi f := \int_0^T \Phi(t, \cdot) f(t) dt, \quad f \in L^2(0, T; H).$$

For this class of processes/operator-valued random variables we define the stochastic integral the usual way as a pathwise Lebesgue–Stieltjes type integral.

DEFINITION 5.3. Let $R \in L^0(\Omega; \gamma(L^2(0, T; H), E))$ be represented by an adapted elementary process $\Phi = \sum_{j=1}^J \sum_{k=1}^K \mathbb{1}_{(t_{j-1}, t_j]} \otimes h_k \otimes X_{j,k}$ with t_j , h_k and $X_{j,k}$ as in Definition 5.2. Then the *stochastic integral of R with respect to W_H* is denoted by $R \cdot W_H$ and defined by

$$R \cdot W_H := \sum_{j=1}^J \sum_{k=1}^K W_H(\mathbb{1}_{(t_{j-1}, t_j]} \otimes h_k) X_{j,k}.$$

Moreover, the *stochastic integral process $(R \cdot W_H(t))_{t \in [0, T]}$* is defined by

$$R \cdot W_H(t) := \sum_{j=1}^J \sum_{k=1}^K W_H(\mathbb{1}_{(t_{j-1}, t_j] \cap [0, t]} \otimes h_k) X_{j,k}, \quad t \in [0, T].$$

REMARK 5.4. Let Φ be an adapted elementary process.

(i) With slight abuse of notation we use $R_\Phi \cdot W_H$ to denote both the stochastic integral and the stochastic integral process. We may also write $\Phi \cdot W_H$ instead of $R_\Phi \cdot W_H$.

(ii) The stochastic integral process $\Phi \cdot W_H$ is an E -valued \mathbb{F} -adapted process, and there exists a version of $\Phi \cdot W_H$ that has continuous paths.

(iii) Recall that the Bochner integral (and hence the conditional expectation) fails to extend to the quasi-Banach space setting. However, clearly $\langle \Phi \cdot W_H, x^* \rangle$ is a martingale for all $x^* \in E^*$.

In the Banach space setting the UMD property is not strictly necessary to extend the definition of the stochastic integral to a suitable class of operator-valued random variables $R \in L^0(\Omega; \gamma(L^2(0, T; H), E))$. Indeed, as shown in [24, Theorem 5.4(ii)], to obtain an extension together with a p -independent one-sided (‘upper’) Burkholder–Davis–Gundy inequality, it is enough to assume that E satisfies a decoupling property for tangent sequences as introduced by Kwapien and Woyczyński [57]; see also [58] for details on this topic. In this section we show that in the quasi-Banach space setting we can extend the class of integrands using a similar strategy.

The precise definition of the decoupling property is technical and understanding the definition is not essential for the comprehension of this section; one may take Proposition 5.8 as a starting point. However, for the readers’ convenience we recall here the definition of decoupling that we have in mind (see also [24, Definition 1.2 and Theorem 4.1]).

DEFINITION 5.5. Let E be a quasi-Banach space, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{G}_n)_{n \in \mathbb{N}_0})$ a filtered probability space, and let $(d_n)_{n \in \mathbb{N}}, (e_n)_{n \in \mathbb{N}}$ be $L^p(\Omega; E)$ -sequences such that $(d_n)_{n \in \mathbb{N}}$ is $(\mathcal{G}_n)_{n \in \mathbb{N}}$ -adapted. We say that $(e_n)_{n \in \mathbb{N}}$ is a $\mathcal{G}_\infty := \sigma(\mathcal{G}_n : n \in \mathbb{N})$ -decoupled tangent sequence of $(d_n)_{n \in \mathbb{N}}$ if $(e_n)_{n \in \mathbb{N}}$ is a \mathcal{G}_∞ -conditionally independent sequence and for all $B \in \mathcal{B}(E)$ and $n \in \mathbb{N}$,

$$\mathbb{P}(d_n \in B \mid \mathcal{G}_{n-1}) = \mathbb{P}(e_n \in B \mid \mathcal{G}_\infty).$$

By passing to a larger probability space, one can always construct a \mathcal{G}_∞ -decoupled tangent sequence to a given $(\mathcal{G}_n)_{n \in \mathbb{N}}$ -adapted $L^p(\Omega; E)$ -sequence $(d_n)_{n \in \mathbb{N}}$ (see e.g. [58, Section 4.3]).

DEFINITION 5.6. We say that a quasi-Banach space E satisfies the decoupling inequality if for all $p \in (0, \infty)$ there exists a constant $D_p \in (0, \infty)$ such that for all filtered probability spaces $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{G}_n)_{n \in \mathbb{N}_0})$ and all $L^p(\Omega; X)$ -sequences $(d_n)_{n \in \mathbb{N}}$ and $(e_n)_{n \in \mathbb{N}}$ such that $(d_n)_{n \in \mathbb{N}}$ is $(\mathcal{G}_n)_{n \in \mathbb{N}}$ -adapted and $(e_n)_{n \in \mathbb{N}}$ is a \mathcal{G}_∞ -decoupled tangent sequence of $(d_n)_{n \in \mathbb{N}}$ one has

$$\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|_E^p \leq D_p^p \mathbb{E} \left\| \sum_{n=1}^N e_n \right\|_E^p \quad \text{for all } N \in \mathbb{N}.$$

In contrast to the UMD property, the decoupling property is satisfied by many prominent examples of quasi-Banach spaces. The following examples are particularly important for our purposes.

EXAMPLE 5.7. Every Hilbert space satisfies the decoupling property; see e.g. [24, Corollary 4.9]. Moreover, let F be an r -Banach space satisfying the decoupling property, $0 < r \leq 1$, let (S, \mathcal{A}, μ) be a σ -finite measure space, and let $0 < p < \infty$, then the $r \wedge p$ -Banach space $L^p(S; F)$ satisfies the decoupling property; see [24, Corollary 4.6]. Finally, it is immediate from the definition that any closed subspace of an r -Banach space satisfying the decoupling property again satisfies the decoupling property. In particular, $B_{p,q}^\alpha(\mathbb{R}^d)$ satisfies the decoupling property for all $\alpha \in \mathbb{R}$ and all $p, q \in (0, \infty)$.

Proposition 5.8 below provides the crucial property ensuring that quasi-Banach spaces that satisfy the decoupling property allow for a well-defined stochastic integral; see [24, Example 1.1, Definition 1.2, and Theorem 4.1] for a proof.

PROPOSITION 5.8. *Let E satisfy the decoupling property. Then for all $p \in (0, \infty)$ there exists a constant $C_{E,p} \in (0, \infty)$ such that for every filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G} = (\mathcal{G}_n)_{n \in \mathbb{N}_0})$, all Gaussian sequences $(\gamma_n)_{n \in \mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that γ_n is \mathcal{G}_n -measurable and independent of \mathcal{G}_{n-1} , and all \mathbb{G} -adapted E -valued stochastic processes $(v_n)_{n \in \mathbb{N}_0}$,*

$$(5.2) \quad \mathbb{E} \left[\sup_{1 \leq M \leq N} \left\| \sum_{n=1}^M \gamma_n v_{n-1} \right\|_E^p \right] \leq C_{E,p}^p \mathbb{E} \left\| \sum_{n=1}^N \gamma'_n v_{n-1} \right\|_E^p,$$

where $(\gamma'_n)_{n \in \mathbb{N}}$ is a copy of $(\gamma_n)_{n \in \mathbb{N}}$ independent of $\mathcal{G}_\infty := \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{G}_n)$. Moreover, without loss of generality one may assume $\limsup_{p \rightarrow \infty} C_{E,p}/p < \infty$.

The proposition above allows us to obtain a one-sided Burkholder–Davis–Gundy inequality for the stochastic integral of an adapted elementary process.

PROPOSITION 5.9. *Let E satisfy the decoupling property. Then for all $p \in (0, \infty)$ and all adapted elementary processes $\Phi: [0, T] \times \Omega \rightarrow \mathcal{L}(H, E)$,*

$$(5.3) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\Phi \cdot W_H(t)\|_E^p \right] \leq C_{E,p}^p \mathbb{E} \|R_\Phi\|_{\gamma^p(L^2(0, T; H), E)}^p,$$

where $C_{E,p}$ is the constant introduced in Proposition 5.8 and R_Φ is defined by (5.1).

Proof. The proof is entirely analogous to the proof of [24, Theorem 5.4(2)]. More precisely, we follow the argument in [24, pp. 371–372], but replace the equivalence labeled $\stackrel{(i)}{\sim}$ in [24, p. 371] by an upper estimate which follows from Proposition 5.8. ■

REMARK 5.10. For the purpose of obtaining the one-sided Burkholder–Davis–Gundy inequality (5.3) one merely needs the Gaussian decoupling property, i.e., one needs that (5.2) holds. The reason we do not directly consider quasi-Banach spaces with this property is that it does not seem possible to prove p -independence of this property directly. Note however that results in [23, Section 5] and [91] prove that—at least in the Banach space setting—Gaussian decoupling is equivalent to Rademacher decoupling, for which the p -independence is well-established. We take [24] as a starting point because this is already in the quasi-Banach space setting.

5.3. L^p -stochastic integration. With the Burkholder–Davis–Gundy type inequality from Proposition 5.9 at hand, we can extend the stochastic integral to the closure in $L^p(\Omega; \gamma(L^2(0, T; H), E))$ of the space of operators that represent elementary adapted processes. Before we do so, we want to describe this completion. Recall that if F is an r -Banach space, then $L^0(\Omega; F)$ denotes the space of F -valued random variables. When endowed with the topology of convergence in probability, $L^0(\Omega; F)$ becomes a complete metric space under the metric $d(f, g) = \mathbb{E}(\|f - g\|_F^r \wedge 1)$. In order to state and prove Lemma 5.12 without having to deal with the case $p = 0$ separately, in this section we adopt the convention

$$(5.4) \quad \|f\|_{L^0(\Omega; F)} := \mathbb{E}[\|f\|_F^r \wedge 1].$$

DEFINITION 5.11. Let $0 \leq p \leq \infty$ and let $R \in L^p(\Omega; \gamma(L^2(0, T; H), E))$. We say that R is \mathbb{F} -adapted, and write $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$, if for all $t \in [0, T]$ and all $f \in L^2(0, T; H)$ satisfying $\text{supp } f \subseteq [0, t]$ the function $Rf \in L^p(\Omega; E)$ is (strongly) \mathcal{F}_t -measurable.

LEMMA 5.12. *Let $0 \leq p < \infty$ and let $R \in L^p(\Omega; \gamma(L^2(0, T; H), E))$. Then the following are equivalent:*

- (i) $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$.
 (ii) *There exists a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of adapted elementary processes such that $\lim_{n \rightarrow \infty} \|R - R_{\Phi_n}\|_{L^p(\Omega; \gamma(L^2(0, T; H), E))} = 0$.*

Proof. ‘(ii) \Rightarrow (i)’: The desired implication follows immediately by observing that for all $t \in [0, T]$ and $f \in L^2(0, T; H)$ satisfying $\text{supp } f \subseteq [0, t]$ the function $R_{\Phi_n} f$ is an E -valued, \mathcal{F}_t -measurable simple function and

$$\lim_{n \rightarrow \infty} \|Rf - R_{\Phi_n} f\|_{L^p(\Omega; E)} = 0.$$

‘(i) \Rightarrow (ii)’: By rescaling we may assume $T = 1$. Let $\varepsilon > 0$ and $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, 1; H), E))$ be given.

For $\eta \in [0, 1]$ let $S_\eta \in \mathcal{L}(L^2(0, 1; H))$ be the left shift, i.e., for $f \in L^2(0, 1; H)$ we define

$$S_\eta f(t) = \begin{cases} f(t + \eta), & t \in [0, 1 - \eta], \\ 0, & t \in (1 - \eta, 1]. \end{cases}$$

Note that $S_\eta^* f \rightarrow f$ in $L^2(0, 1; H)$ as $\eta \downarrow 0$ for all $f \in L^2(0, 1; H)$, and that $\sup_{\eta \in [0, 1]} \|S_\eta\|_{\mathcal{L}(L^2(0, 1; H))} = 1$. If $p = 0$ then Theorem 3.9 implies that $\lim_{\eta \downarrow 0} \|R - RS_\eta\|_{L^p(\Omega; \gamma(L^2(0, 1; H), E))} = 0$. The same conclusion holds for $p \in (0, \infty)$ by invoking Theorem 3.9, Theorem 3.7, and the dominated convergence theorem. Hence we can pick $K \in \mathbb{N}$ such that

$$(5.5) \quad \|R - RS_{2^{-K}}\|_{L^p(\Omega; \gamma(L^2(0, 1; H), E))} < \varepsilon.$$

Now let $(h_k)_{k \in \mathbb{N}}$ be an orthonormal basis for H , and for all $k \in \mathbb{N}$ let $Q_k \in \mathcal{L}(H)$ be the orthogonal projection on $\text{span}(\{h_1, \dots, h_k\})$. Let $(f_k)_{k \in \mathbb{N}}$ be the Haar basis for $L^2(0, 1)$, and for $k \in \mathbb{N}$ let $P_k \in \mathcal{L}(L^2(0, 1))$ be the orthogonal projection on $\text{span}(\{f_1, \dots, f_k\})$. Note that

$$(5.6) \quad P_{2^k} f = \sum_{j=1}^{2^k} 2^k \langle \mathbb{1}_{((j-1)2^{-k}, j2^{-k}]}, f \rangle_{L^2(0, 1)} \mathbb{1}_{((j-1)2^{-k}, j2^{-k}]} \quad \text{for all } f \in L^2(0, 1).$$

By Example 3.10 we have

$$\lim_{L \rightarrow \infty} \|RS_{2^{-K}}(P_L \otimes Q_L) - RS_{2^{-K}}\|_{\gamma(L^2(0, 1; H), E)} = 0 \quad \mathbb{P}\text{-a.s.}$$

It follows from the above that for $p = 0$ there exists an $L \geq K$ such that

$$(5.7) \quad \|RS_{2^{-K}} - RS_{2^{-K}}(P_{2L} \otimes Q_{2L})\|_{L^p(\Omega; \gamma(L^2(0, 1; H), E))} < \varepsilon.$$

The same conclusion is obtained for $p \in (0, \infty)$ by invoking Theorem 3.7 and the dominated convergence theorem. Now define $R_{K, L} = RS_{2^{-K}}(P_{2L} \otimes Q_{2L})$ and observe that $R_{K, L}$ is represented by $\Phi_{K, L}: [0, 1] \times \Omega \rightarrow \mathcal{L}(H, E)$ in the

sense of Definition 5.1, where $\Phi_{K,L}$ is given by

$$\Phi_{K,L} = \sum_{j=1}^{2^L} \sum_{\ell=1}^{2^L} \mathbb{1}_{((j-1)2^{-L}, j2^{-L}]} \otimes h_\ell \otimes X_{j,\ell}^{(K,L)},$$

$$\begin{aligned} \text{with } X_{j,\ell}^{(K,L)} &= 2^L R S_{2^{-K}}(\mathbb{1}_{((j-1)2^{-L}, j2^{-L}]} \otimes h_\ell) \\ &= 2^L R(\mathbb{1}_{((j-1)2^{-L-2^{-K}}, j2^{-L-2^{-K}}] \cap [0,1]} \otimes h_\ell). \end{aligned}$$

Therefore, as $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, 1; H), E))$ and $L \geq K$ we have $X_{j,\ell}^{(K,L)} \in L^p(\Omega, \mathcal{F}_{(j-1)2^{-L}}; E)$.

As a final step, note that if $Y_{j,\ell} \in L^p(\Omega, \mathcal{F}_{(j-1)2^{-L}}; \mathbb{P}; E)$, $j, \ell \in \{1, \dots, 2^L\}$, and if $R_{K,L,Y} \in L^p(\Omega; \gamma(L^2(0, 1; H), E))$ is represented by

$$(5.8) \quad \Phi_{K,L,Y} = \sum_{k=1}^{2^L} \sum_{\ell=1}^{2^L} \mathbb{1}_{((j-1)2^{-L}, j2^{-L}]} \otimes h_\ell \otimes Y_{j,\ell},$$

then by estimate (3.3) in Proposition 3.2 we obtain

$$\|R_{K,L} - R_{\Phi_{K,L,Y}}\|_{L^p(\Omega; \gamma(L^2(0,1;H),E))} \sim_{p,r} \left\| \sum_{j=1}^{2^L} \sum_{\ell=1}^{2^L} 2^{-L/2} \gamma_{j,\ell}(X_{j,\ell} - Y_{j,\ell}) \right\|_{L^p(\Omega; E)},$$

where $(\gamma_{j,\ell})_{j,\ell \in \{1, \dots, 2^L\}}$ is a Gaussian sequence on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of $\mathcal{F}_\infty := \sigma(\bigcup_{t \in [0, \infty)} \mathcal{F}_t)$. It follows that we can pick $\mathcal{F}_{(j-1)2^{-L}}$ -measurable simple random variables $Y_{j,\ell}$, $k, \ell \in \{1, \dots, 2^L\}$, such that

$$(5.9) \quad \|R_{K,L} - R_{K,L,Y}\|_{L^p(\Omega; \gamma(L^2(0,1;H),E))} < \varepsilon.$$

Note that $\Phi_{K,L,Y}$ is an adapted elementary process in the sense of Definition 5.2. Estimates (5.5), (5.7), and (5.9) yield

$$\|R - R_{K,L,Y} S\|_{L^0(\Omega; \gamma(L^2(0,1;H),E))} < 3\varepsilon,$$

and $\|R - R_{K,L,Y} S\|_{L^p(\Omega; \gamma(L^2(0,1;H),E))} < 3^{1/(r \wedge p)} \varepsilon$ for all $p \in (0, \infty)$. ■

Now all ingredients are available to extend the stochastic integral to integrands in $L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$ for $0 < p < \infty$. The following theorem is an immediate consequence of Proposition 5.9 and Lemma 5.12.

THEOREM 5.13. *Let $0 < p < \infty$, let E satisfy the decoupling property, and let $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Then there exists a unique continuous E -valued \mathbb{F} -adapted stochastic process $(\Psi(t))_{t \in [0, T]}$ such that for every sequence of adapted elementary processes $(\Phi_n)_{n \in \mathbb{N}}$ satisfying*

$$\lim_{n \rightarrow \infty} \mathbb{E} \|R_{\Phi_n} - R\|_{\gamma(L^2(0, T; H), E)}^p = 0$$

we have

$$\lim_{n \rightarrow \infty} \|\Phi_n \cdot W_H - \Psi\|_{L^p(\Omega; \mathcal{C}([0, T]; E))} = 0.$$

We use the notation $R \cdot W_H = (R \cdot W_H(t))_{t \in [0, T]}$ for Ψ and call this process the stochastic integral of R with respect to W_H . We have

$$(5.10) \quad \mathbb{E} \|R \cdot W_H\|_{\mathcal{C}([0, T]; E)}^p \leq C_{E, p}^p \mathbb{E} \|R\|_{\gamma(L^2(0, T; H), E)}^p,$$

where $C_{E, p}$ is as in Proposition 5.8.

5.4. Localization. We now turn to the localized case. To this end, we need to introduce ‘stopped’ elements of $L_{\mathbb{F}}^0(\Omega; \gamma(L^2(0, T; H), E))$. Hence we define, for $t \in [0, T]$, the operator $P_t \in \mathcal{L}(L^2(0, T; H))$ by $P_t(f) = f \mathbb{1}_{[0, t]}$, $f \in L^2(0, T; H)$.

LEMMA 5.14. *Let $0 \leq p < \infty$, let E satisfy the decoupling property, let $\tau: \Omega \rightarrow [0, T]$ be an \mathbb{F} -stopping time, and let $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Then $RP_{\tau} \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Moreover, if $p > 0$ then*

$$(5.11) \quad R \cdot W_H(\tau) = RP_{\tau} \cdot W_H(T) \quad \text{in } L^p(\Omega; E).$$

Proof. If R is represented by an adapted elementary process, then it follows trivially from the fact that τ is a stopping time, Theorem 3.7, and the fact that $\sup_{t \in [0, T]} \|P_t\|_{\mathcal{L}(L^2(0, T; H))} = 1$ that $RP_{\tau} \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Moreover, identity (5.11) follows from ‘classical’ Itô integration theory.

The general case is obtained by considering an approximating sequence $(\Phi_n)_{n \in \mathbb{N}}$ of adapted elementary processes (see Lemma 5.12) and noting that if

$$\lim_{n \rightarrow \infty} \|R - R_{\Phi_n}\|_{L^p(\Omega; \gamma(L^2(0, T; H), E))} = 0,$$

then by Theorem 3.7 also

$$\lim_{n \rightarrow \infty} \|RP_{\tau} - R_{\Phi_n} P_{\tau}\|_{L^p(\Omega; \gamma(L^2(0, T; H), E))} = 0.$$

Finally, if $p > 0$, then by Theorem 5.13,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \|R \cdot W_H - R_{\Phi_n} \cdot W_H\|_{\mathcal{C}([0, T]; E)}^p \\ = \lim_{n \rightarrow \infty} \mathbb{E} \|RP_{\tau} \cdot W_H - R_{\Phi_n} P_{\tau} \cdot W_H\|_{\mathcal{C}([0, T]; E)}^p = 0. \end{aligned}$$

Recalling that identity (5.11) holds for R_{Φ_n} , we conclude that it also holds for R . ■

The following lemma is the key ingredient that enables us to define a localized version of the stochastic integral. For $r = 1$ it can be found in [71, Lemma 5.4].

LEMMA 5.15. *Let $0 < p < \infty$, let E satisfy the decoupling property, and let $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Then for all $\varepsilon, \delta \in (0, \infty)$,*

$$\mathbb{P}(\|R \cdot W_H\|_{\mathcal{C}([0, T]; E)} > \varepsilon) \leq \frac{C_{E, p}^p \varepsilon^p}{\delta^p} + \mathbb{P}(\|R\|_{L^p(\Omega; \gamma(L^2(0, T; H), E))} > \delta),$$

where $C_{E, p}$ is the constant in Theorem 5.13.

Proof. Define the stochastic process $\psi: [0, T] \times \Omega \rightarrow \mathbb{R}$ by setting $\psi(t) = \|RP_t\|_{\gamma(L^2(0, T; H), E)}$. Note that $\lim_{t \rightarrow T} \|P_t f - f\|_{L^2(0, T; H)} = 0$ for all $f \in L^2(0, T; H)$, whence it follows from Theorem 3.9 that ψ is a continuous \mathbb{F} -adapted process. Moreover, by the definition of the γ -radonifying norm (see Definition 3.5), ψ is increasing and $\psi(T) = \|R\|_{\gamma(L^2(0, T; H), E)}$. Finally, it follows from Theorem 5.13 and Lemma 5.14 that for every stopping time $\tau: \Omega \rightarrow [0, T]$,

$$(5.12) \quad \mathbb{E} \|R \cdot W(\tau)\|_{\mathcal{C}([0, T]; E)}^p \leq C_{E, p} \psi(\tau).$$

The result now follows from an argument based on Chebyshev's inequality; see [71, Lemma 5.1]. ■

THEOREM 5.16. *Let E satisfy the decoupling property. Then, for every $R \in L_{\mathbb{F}}^0(\Omega; \gamma(L^2(0, T; H), E))$, there exists a unique continuous E -valued \mathbb{F} -adapted stochastic process $(\Psi(t))_{t \in [0, T]}$ such that for every sequence of adapted elementary processes $(\Phi_n)_{n \in \mathbb{N}}$ satisfying*

$$\lim_{n \rightarrow \infty} \mathbb{E} [\|R\Phi_n - R\|_{\gamma(L^2(0, T; H), E)} \wedge 1] = 0$$

we have

$$\lim_{n \rightarrow \infty} \mathbb{E} [\|\Phi_n \cdot W_H - \Psi\|_{\mathcal{C}([0, T]; E)} \wedge 1] = 0.$$

We use the notation $R \cdot W_H = (R \cdot W_H(t))_{t \in [0, T]}$ for Ψ and call this process the stochastic integral of R with respect to W_H .

Proof. It follows from Lemma 5.15 that the stochastic integral mapping for adapted elementary processes (see Definition 5.3)

$$\Phi \mapsto \Phi \cdot W_H \in L^0(\Omega; \mathcal{C}([0, T]; E))$$

extends continuously to $L_{\mathbb{F}}^0(\Omega; \gamma(L^2(0, T; H), E))$ with respect to the metric of convergence in probability—here we also use the fact that the adapted elementary processes are dense in this space by Lemma 5.12. ■

As for deterministic integrands (see Corollary 4.9), we have the following series expansion of the stochastic integral.

COROLLARY 5.17. *Let E satisfy the decoupling inequality, let $0 \leq p < \infty$, and let $R \in L_{\mathbb{F}}^p(\Omega; \gamma(L^2(0, T; H), E))$. Then, for every orthonormal basis $(h_n)_{n \in \mathbb{N}}$ in H ,*

$$R \cdot W_H = \sum_{n \in \mathbb{N}} R(\cdot \otimes h_n) \cdot W_H(\cdot \otimes h_n),$$

where the series converges in $L^p(\Omega; \mathcal{C}([0, T]; E))$. In particular, for every $x^* \in E^*$,

$$\langle R \cdot W_H, x^* \rangle = \sum_{n \in I} \int_0^{\cdot} \langle R^* x^*(t), h_n \rangle_H dW_H h_n(t),$$

where the series converges in $L^p(\Omega; \mathcal{C}([0, T]))$.

Proof. The proof is entirely analogous to the proof of Corollary 4.9, with the understanding that we apply Theorems 5.13 and 5.16 instead of Proposition 4.8. ■

5.5. Stochastic integration in r -Banach spaces with separating dual. If E^* separates the points in E , then we have the following consequence of Theorems 5.13 and 5.16. See also [71, Theorems 3.6 and 5.9] for the (elementary) proof. Note that [71, Theorems 3.6 and 5.9] also provide the reverse statement. However, in ‘Step 2’ of the proof of the reverse statement a two-sided decoupling inequality is used, which is not available in the r -Banach space setting when $r < 1$.

COROLLARY 5.18. *Let E satisfy the decoupling property, let E^* separate the points of E and let $0 \leq p < \infty$. Let $\Phi: [0, T] \rightarrow \mathcal{L}(H, E)$ be such that $\Phi^*x^* \in L^0(\Omega; L^2(0, T; H))$ for all $x^* \in E^*$, and Φh is strongly measurable and adapted for all $h \in H$. If $R \in L^p(\Omega; \gamma(L^2(0, T; H), E))$ is such that*

$$\langle Rf \otimes h, x^* \rangle = \int_0^T \langle \Phi(t, \cdot) h f(t), x^* \rangle dt \quad \text{for all } f \in L^2(0, T), h \in H, x^* \in E^*,$$

then there exists a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of adapted elementary processes such that

- (1) for all $h \in H$ and $x^* \in E^*$ we have $\lim_{n \rightarrow \infty} \langle \Phi_n h, x^* \rangle = \langle \Phi h, x^* \rangle$ in $L^0(\Omega; L^2(0, T))$;
- (2) there exists a process $\Psi \in L^p(\Omega; \mathcal{C}([0, T]; E))$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}(\|\Phi_n \cdot W_H - \Psi\|_{\mathcal{C}([0, T]; E)} \wedge 1) = 0$$

and, if $p > 0$, $\lim_{n \rightarrow \infty} \mathbb{E} \|\Phi_n \cdot W_H - \Psi\|_{L^p(\Omega; \mathcal{C}([0, T]; E))} = 0$.

6. The stochastic heat equation. Let $d \in \mathbb{N}$, $p, q \in (0, 1)$, and $\sigma \in \mathbb{R}$. Our aim is to prove Hölder continuity in time and spatial Besov regularity, i.e., with respect to $B_{p,q}^\sigma(\mathbb{R}^d)$, for the stochastic heat equation

$$(6.1) \quad \begin{cases} du(t, x) = [\Delta u(t, x) + f(t, x)] dt \\ \quad + \sum_{n \in \mathbb{N}} g_n(t, x) dW_n(t), & x \in \mathbb{R}^d, t \in [0, \infty), \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Here $(W_n)_{n=1}^\infty$ is a sequence of independent \mathbb{F} -standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)})$, $u_0: \Omega \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is \mathcal{F}_0 -measurable, f is an $\mathcal{S}'(\mathbb{R}^d)$ -valued process such that $\langle f, \varphi \rangle$ is progressively measurable for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and $(g_n)_{n \in \mathbb{N}}$ is an $\mathcal{S}'(\mathbb{R}^d; \ell^2)$ -valued process such that $\langle g_n, \varphi \rangle$ is progressively measurable for all $n \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We shall work with the following concept of a solution to (6.1).

DEFINITION 6.1. Assume that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the function $s \mapsto \langle f(s), \varphi \rangle$ is in $L^1(0, T)$ a.s., and $(\langle g_n(t), \varphi \rangle)_{n \in \mathbb{N}}$ is in $L^2(0, T; \ell^2)$ a.s. An adapted process $U: \Omega \times [0, T] \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is called a *solution to (6.1)* if for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $t \in [0, T]$ we have $\langle U, \Delta\varphi \rangle \in L^1(0, T)$ and

$$\begin{aligned} \langle U(t), \varphi \rangle - \langle u_0, \varphi \rangle &= \int_0^t \langle U(s), \Delta\varphi \rangle ds + \int_0^t \langle f(s), \varphi \rangle ds \\ &\quad + \sum_{n \in \mathbb{N}} \int_0^t \langle g_n(s), \varphi \rangle dW_n(s). \end{aligned}$$

Uniqueness of a solution to (6.1) follows from the uniqueness of a solution to the heat equation in the space of tempered distributions. Theorem 6.2 below states existence of a solution to (6.1) assuming certain (spatial) Besov regularity conditions on f and g .

The main challenge in proving well-posedness of (6.1) is that we are dealing with stochastic processes taking values in $B_{p,q}^\sigma(\mathbb{R}^d)$, where possibly p or q is less than 1, i.e., one is in the $p \wedge q \wedge 1$ -Banach space setting. More specifically, we need to work in the space $\gamma(L^2(0, T; H), B_{p,q}^\sigma(\mathbb{R}^d))$. To this end we observe that given $G \in B_{p,q}^\sigma(\mathbb{R}^d; L^2(0, T; H))$, we can associate an operator $R: L^2(0, T; H) \rightarrow B_{p,q}^\sigma(\mathbb{R}^d)$ by setting

$$Rf = \int_0^T \langle G(t), f(t) \rangle_H dt$$

By Proposition A.1 this operator R is γ -radonifying and we know that $\|R\|_{\gamma(L^2(0,T;H), B_{p,q}^\sigma(\mathbb{R}^d))} \sim_{p,q} \|G\|_{B_{p,q}^\sigma(\mathbb{R}^d; L^2(0,T;H))}$ (note that this implies in particular that G is uniquely determined by R). The proof of Theorem 6.2 relies on this characterization of $\gamma(L^2(0, T; H), B_{p,q}^\sigma(\mathbb{R}^d))$. Although it is not used in this section, we wish to stress that because $B_{p,q}^\sigma(\mathbb{R}^d)$ has a separating dual, the tools developed in Section 4.3 could be employed to deal with stochastic integrals of deterministic integrands.

The results and techniques of this section can be extended to the case that f and g depend on u in a suitable way. Another possible extension is to consider more general differential operators (instead of Δ). We prefer to keep the presentation as simple as possible whilst still demonstrating the power of stochastic calculus in r -Banach spaces such as Besov spaces.

We refer to Appendix A for some basic results on vector-valued Besov spaces needed below. In order to phrase our result we introduce, for $r, \alpha, t \in (0, \infty)$ and E a Banach space, the weighted L^r -norm $\|\cdot\|_{L_\alpha^r(0,t;E)}: L^r(0, t; E) \rightarrow [0, \infty]$ defined by $\|f\|_{L_\alpha^r(0,t;E)}^r := \int_0^t (t-s)^{-\alpha r} \|f(s)\|_E^r ds$. Observe that for

$\alpha \in (0, 1/r)$, $L^\infty(0, t; E) \subseteq L_\alpha^r(0, t; E)$ with

$$\|f\|_{L_\alpha^r(0, t; E)} \leq C_{\alpha, r} t^{1/r - \alpha} \|f\|_{L^\infty(0, t; E)}.$$

Our main existence and uniqueness result for the stochastic heat equation (6.1) in arbitrary Besov spaces reads as follows.

THEOREM 6.2. *Let $\sigma \in \mathbb{R}$, $p, q, r, T \in (0, \infty)$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ a filtration on \mathcal{F} , and let W_{ℓ^2} be an $\mathbb{F}/L^2(0, T; \ell^2)$ -isonormal process. Let $u_0 \in L^r(\mathcal{F}_0; B_{p, q}^\sigma(\mathbb{R}^d))$. Let $f: [0, T] \times \Omega \rightarrow B_{p, q}^{\sigma - 2\alpha}(\mathbb{R}^d)$ be progressively measurable, and assume*

$$(6.2) \quad C_f^{(6.2)} := \sup_{t \in [0, T]} \|f|_{\Omega \times [0, t]}\|_{L^r(\Omega; B_{p, q}^{\sigma - 2\alpha}(\mathbb{R}^d; L_\alpha^1(0, t)))} < \infty.$$

Let $g: [0, T] \times \Omega \rightarrow B_{p, q}^{\sigma - 2\beta}(\mathbb{R}^d; \ell^2)$ be progressively measurable, and assume

$$(6.3) \quad C_g^{(6.3)} := \sup_{t \in [0, T]} \|g|_{\Omega \times [0, t]}\|_{L^r(\Omega; B_{p, q}^{\sigma - 2\beta}(\mathbb{R}^d; L_\beta^2(0, t; \ell^2)))} < \infty.$$

Finally, let $K_t \in \mathcal{S}(\mathbb{R}^d)$, $t \in (0, \infty)$, be the standard heat kernel, i.e.,

$$(6.4) \quad K_t(x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad t \in (0, \infty), x \in \mathbb{R}^d.$$

For $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ set $K_0 * \varphi := \varphi$.

Then $K_{t-} * g|_{[0, t]} \in L_{\mathbb{F}}^r(\Omega; \gamma(L^2(0, t; \ell^2), B_{p, q}^\sigma(\mathbb{R}^d)))$ for all $t \in (0, \infty)$. Consequently, the stochastic integral $[(K_{t-} * g)\mathbb{1}_{[0, t]}] \cdot W_{\ell^2}$ is well-defined for all $t \in (0, T]$ by Example 5.7 and Theorem 5.13. Define, for all $t \in [0, T]$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(6.5) \quad \begin{aligned} \langle U(t), \varphi \rangle &= \langle K_t * u_0, \varphi \rangle + \int_0^t \langle K_{t-s} * f(s), \varphi \rangle ds \\ &\quad + [\langle K_{t-} * g, \varphi \rangle \mathbb{1}_{[0, t]}] \cdot W_{\ell^2} \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Then U is an \mathbb{F} -adapted, $B_{p, q}^\sigma(\mathbb{R}^d)$ -valued stochastic process that is a solution to (6.1) in the sense of Definition 6.1.

Moreover, for all $\lambda \in (0, \alpha \wedge \beta]$ there exists a constant $C_{d, p, \alpha, \beta, \lambda}^{(6.6)}$ (independent of u_0 , f , and g) such that

$$(6.6) \quad \begin{aligned} \|U\|_{C^\lambda([0, T]; L^r(\Omega; B_{p, q}^{\sigma - 2\lambda}(\mathbb{R}^d)))} \\ \leq C_{d, p, q, \alpha, \lambda}^{(6.6)} e^{6\pi^2 T} (\|u_0\|_{L^r(\Omega; B_{p, q}^\sigma(\mathbb{R}^d))} + C_f^{(6.2)} + C_g^{(6.3)}), \end{aligned}$$

and if $(\alpha \wedge \beta) - 1/r > 0$, then $U \in L^r(\Omega; C^\lambda([0, T]; B_{p, q}^{\sigma - 2(\lambda + 1/r + \varepsilon)}(\mathbb{R}^d)))$ for all $\lambda, \varepsilon \in (0, \alpha \wedge \beta)$ satisfying $\lambda + \varepsilon \leq (\alpha \wedge \beta) - 1/r$.

Proof. It is an immediate consequence of Corollary B.3 (with $s = \lambda = 0$) that

$$(6.7) \quad \sup_{t \in [0, T]} \|K_t * u_0\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))} \leq (1 + C_{d,p,0}^{(B.15)} e^{6\pi^2 T}) \|u_0\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))}.$$

It also follows from Corollary B.3 that for all $\lambda \in (0, \alpha \wedge \beta]$ and all $0 \leq s < t \leq T$ one has

$$(6.8) \quad \|(K_t - K_s) * u_0\|_{L^r(\Omega; B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d))} \leq C_{d,p,\lambda}^{(B.15)} (t-s)^\lambda e^{6\pi^2 T} \|u_0\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))}.$$

Proposition A.1, estimate (B.17) in Corollary B.4 with $s = \lambda = 0$, and (6.3) yield

$$\begin{aligned} & \sup_{t \in (0, T]} \|K_{t-} * g|_{[0,t]}\|_{L^r(\Omega; \gamma(L^2(0,t;\ell^2), B_{p,q}^\sigma(\mathbb{R}^d)))} \\ & \lesssim_{p,q} \sup_{t \in (0, T]} \|K_{t-} * g|_{[0,t]}\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d; L^2(0,t;\ell^2)))} \leq C_{d,p,\beta,0}^{(B.17)} C_g^{(6.3)} e^{5\pi^2 T}, \end{aligned}$$

whence by Theorem 5.13 and Example 5.7 there exists a constant $C_{d,p,q,\beta}^{(6.9)}$ (independent of g) such that

$$(6.9) \quad \sup_{t \in (0, T]} \|[K_{t-} * g\mathbb{1}_{[0,t]}] \cdot W_{\ell^2}\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))} \leq C_{d,p,q,\beta}^{(6.9)} C_g^{(6.3)} e^{5\pi^2 T}.$$

Similarly, by Theorem 5.13, Example 5.7, Proposition A.1, and the estimates (B.16) and (B.17) in Corollary B.4, for all $\lambda \in (0, \beta]$ there exists a constant $C_{d,p,q,\beta,\lambda}^{(6.9)}$ such that for all $0 \leq s < t \leq T$:

$$(6.10) \quad \begin{aligned} & \|[K_{t-} * g\mathbb{1}_{[0,t]}] \cdot W_{\ell^2} - [K_{s-} * g\mathbb{1}_{[0,s]}] \cdot W_{\ell^2}\|_{L^r(\Omega; B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d))} \\ & \lesssim_{p,q} \left(\|(K_{t-} - K_{s-}) * g\mathbb{1}_{[0,s]}\|_{L^r(\Omega; B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d; L^2(0,s;\ell^2)))} \right. \\ & \quad \left. + \|K_{t-} * g\mathbb{1}_{[s,t]}\|_{L^r(\Omega; B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d; L^2(s,t;\ell^2)))} \right) \\ & \leq (C_{d,p,\alpha,\lambda}^{(B.16)} + C_{d,p,\alpha,\lambda}^{(B.17)}) C_g^{(6.3)} (t-s)^\lambda e^{6\pi^2 T}. \end{aligned}$$

As for the deterministic integral, we have, by estimate (B.19) in Corollary B.5 with $\lambda = s = 0$,

$$(6.11) \quad \sup_{t \in (0, T]} \left\| \int_0^t K_{t-s} * g(s) ds \right\|_{L^r(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))} \leq C_{d,p,\alpha,0}^{(B.19)} C_f^{(6.2)} e^{5\pi^2 T}.$$

Moreover, by estimates (B.18) and (B.19) in Corollary B.5 it follows that for all $\lambda \in (0, \alpha]$ and all $0 \leq s < t \leq T$,

$$(6.12) \quad \begin{aligned} & \left\| \int_0^t K_{t-u} * g(u) ds - \int_0^s K_{t-u} * g(u) du \right\|_{L^r(\Omega; B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d))} \\ & \leq (C_{d,p,\alpha,\lambda}^{(B.18)} + C_{d,p,\alpha,\lambda}^{(B.19)}) C_f^{(6.2)} (t-s)^\lambda e^{6\pi^2 T}. \end{aligned}$$

Estimates (6.7), (6.9), and (6.11) guarantee that U is well-defined as an \mathbb{F} -adapted, $B_{p,q}^\sigma(\mathbb{R}^d)$ -valued process. It follows from estimates (6.8), (6.10), and (6.12) that estimate (6.6) holds for all $\lambda \in (0, \alpha \wedge \beta]$.

It follows from estimate (6.6) and the Kolmogorov–Chentsov criterion—see e.g. [49, Theorem 14.9]—that there exists a version $\bar{U}: [0, T] \times \Omega \rightarrow B_{p,q}^\sigma(\mathbb{R}^d)$ such that

$$\mathbb{E} \|\bar{U}\|_{C^\lambda([0,T]; B_{p,q}^{\sigma-2(\lambda+1/r+\varepsilon)}(\mathbb{R}^d))}^r < \infty.$$

Note that here we applied the Kolmogorov–Chentsov criterion with the metric

$$d(f, g) = \|f - g\|_{B_{p,q}^{\sigma-2(\lambda+1/r+\varepsilon)}(\mathbb{R}^d)}^{p \wedge q \wedge 1}.$$

Next, recall that for all $\lambda_1, \lambda_2 \in (0, 1)$ satisfying $\lambda_2 > \lambda_1$ the ‘small’ Hölder space $c^{\lambda_1}([0, T])$ is separable and $C^{\lambda_2}([0, T]) \hookrightarrow c^{\lambda_1}([0, T])$. Hence we can conclude that \bar{U} is strongly measurable as a mapping taking values in $C^\lambda([0, T]; B_{p,q}^{\sigma-2(\lambda+1/r+\varepsilon)}(\mathbb{R}^d))$, and thus the claim that $U \in L^p(\Omega; C^\lambda([0, T]; B_{p,q}^{\sigma-2(\lambda+1/r+\varepsilon)}(\mathbb{R}^d)))$ is verified.

It remains to show that U satisfies Definition 6.1. This can be proved by repeating the classical argument for mild and weak solutions (see [25, Theorem 5.4]). This completes the proof of the theorem. ■

REMARK 6.3. Given that $(K_t)_{t \in (0, \infty)}$ is the heat kernel, one may think of the process U in Theorem 6.2 as a mild solution to the stochastic heat equation (6.1).

REMARK 6.4. In view of Proposition A.1, the weighted norm considered in (6.3) may be seen as a weighted γ -norm as introduced in [94] in the Banach space setting; it is a useful tool in stochastic evolution equations in Banach spaces (see [72]). Note that if $\alpha \geq 1$ or $\beta \geq 1/2$, then (6.2), respectively (6.3), is only satisfied for $f \equiv 0$, respectively $g \equiv 0$.

REMARK 6.5. If $p = q = r$ in Theorem 6.2, then it is possible to obtain the assertions of this theorem by considering $\langle U(\varphi_m), e_n \rangle_{\ell^2}$ pointwise in $\xi \in \mathbb{R}^d$ (e_n being the n th unit vector in ℓ^2 and φ_m being as in Appendix A.1). Otherwise, however, the theory developed in Sections 1–5 seems necessary to obtain this result. In addition to the theory developed in Sections 1–5, a key element for the proof concerns a technical result on pointwise boundedness of convolutions by maximal functions (see Lemma B.1).

Appendix A. Vector-valued Besov spaces

A.1. Notation. In this article we consider vector-valued Besov spaces defined by means of a Littlewood–Paley type decomposition. We recall some basic definitions and results below, and refer to [3, 67, 79, 85] for more

details on vector-valued Besov spaces defined via this approach (these references ‘only’ deal with the Banach-space parameter range, i.e., the spaces $B_{p,q}^\sigma(\mathbb{R}^d; E)$ with $p, q \in [1, \infty]$ and $\sigma \in \mathbb{R}$). We refer to [87, 88] for a general treatment of the real-valued case (including the quasi-Banach space parameter range $p \in (0, 1)$ or $q \in (0, 1)$).

Let E be a \mathbb{C} -Banach space. For every $d \in \mathbb{N}$ let $\mathcal{S}(\mathbb{R}^d; E)$ denote the space of E -valued Schwartz functions, we set $\mathcal{S}(\mathbb{R}^d) := \mathcal{S}(\mathbb{R}^d; \mathbb{C})$. Let $\mathcal{S}'(\mathbb{R}^d; E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^d), E)$ denote the space of tempered distributions; we write $\mathcal{S}'(\mathbb{R}^d) := \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d; E)$ define $\varphi * f \in \mathcal{S}'(\mathbb{R}^d; E)$ by $(\varphi * f)(g) = f(\varphi(\cdot) * g)$, where $(\varphi(\cdot) * g)(\xi) = \int_{\mathbb{R}^d} \varphi(y - \xi)g(y) \, dy$.

In what follows, we use a slightly different convention than in the main text and define the Fourier transform $\mathcal{F}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ on the Schwartz space to be given by $\mathcal{F}(f)(s) = \int_{\mathbb{R}^d} e^{-2\pi i \langle s, x \rangle} f(x) \, dx$ for $f \in \mathcal{S}(\mathbb{R}^d)$ and $s \in \mathbb{R}^d$ (this has the consequence that $\mathcal{F}^{-1}(f)(s) = \int_{\mathbb{R}^d} e^{2\pi i \langle s, x \rangle} f(x) \, dx$ and $\mathcal{F}(\nabla f)(s) = 2\pi i s \mathcal{F}(f)(s)$). We also use \widehat{f} to denote $\mathcal{F}(f)$. For $\lambda \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d)$ set

$$(A.1) \quad (1 - \lambda \Delta)^\alpha f := \mathcal{F}^{-1}((1 + (2\pi)^2 \lambda |\cdot|^2)^\alpha \widehat{f}) \in \mathcal{S}(\mathbb{R}^d);$$

note that for $\lambda = 1$ it corresponds with the Bessel potential operator (and for $\alpha \in \mathbb{N}$ this corresponds with the classical definition of the Laplacian Δ).

In addition, let $\mathcal{F}: \mathcal{S}'(\mathbb{R}^d; E) \rightarrow \mathcal{S}'(\mathbb{R}^d; E)$ denote the Fourier transform on the space of tempered distributions, i.e., for all $f \in \mathcal{S}'(\mathbb{R}^d; E)$ the distribution $\mathcal{F}(f) \in \mathcal{S}'(\mathbb{R}^d; E)$ is defined by $\langle \mathcal{F}(f), g \rangle = \langle f, \mathcal{F}(g) \rangle$ for $g \in \mathcal{S}(\mathbb{R}^d)$. Again we may write \widehat{f} instead of $\mathcal{F}(f)$. Moreover, for $\lambda \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^d; E)$ we define $(1 - \lambda \Delta)^\alpha f \in \mathcal{S}'(\mathbb{R}^d; E)$ by setting $\langle (1 - \lambda \Delta)^\alpha f, g \rangle = \langle f, (1 - \lambda \Delta)^\alpha g \rangle$ for $g \in \mathcal{S}(\mathbb{R}^d)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that $0 \leq \widehat{\varphi}(\xi) \leq 1$ for $\xi \in \mathbb{R}^d$, $\widehat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$, and $\widehat{\varphi}(\xi) = 0$ for $|\xi| \geq \frac{3}{2}$. Let $\widehat{\varphi}_0 = \widehat{\varphi}$, and for $k \in \mathbb{N}$ define $\widehat{\varphi}_k = \widehat{\varphi}(2^{-k}\cdot) - \widehat{\varphi}(2^{-k+1}\cdot)$. Clearly, for all $n \in \mathbb{N}$ we have $\sum_{k=0}^n \widehat{\varphi}_k = \widehat{\varphi}(2^{-n}\cdot)$ and hence $\sum_{k=0}^\infty \widehat{\varphi}_k = 1$ pointwise, and one may check that

$$(A.2) \quad \text{supp } \widehat{\varphi}_k \subseteq \{\xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 3 \cdot 2^{k-1}\}, \quad k \in \mathbb{N}.$$

For all $\sigma \in \mathbb{R}$, $p, q \in (0, \infty]$, we write $\|\cdot\|_{B_{p,q}^\sigma(\mathbb{R}^d; E)}: \mathcal{S}'(\mathbb{R}^d; E) \rightarrow [0, \infty]$ for the Besov $(p \wedge q \wedge 1)$ -norm defined by

$$(A.3) \quad \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; E)} := \|(2^{k\sigma} \varphi_k * f)_{k \in \mathbb{N}}\|_{\ell^q(L^p(\mathbb{R}^d; E))}, \quad f \in \mathcal{S}'(\mathbb{R}^d; E).$$

The vector-valued Besov space $B_{p,q}^\sigma(\mathbb{R}^d; E)$ is given by

$$B_{p,q}^\sigma(\mathbb{R}^d; E) = \{f \in \mathcal{S}'(\mathbb{R}^d; E) : \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; E)} < \infty\}.$$

One easily verifies that (A.3) indeed defines a $(p \wedge q \wedge 1)$ -norm on $B_{p,q}^\sigma(\mathbb{R}^d; E)$, and one may check that a different choice of φ leads to an equivalent norm

(see e.g. [87, Theorem 2.3.3] for a precise formulation of this result in the real-valued case—the proof carries over to the vector-valued case).

Note that $B_{p,q}^\sigma(\mathbb{R}^d; E)$ is a $(p \wedge q \wedge 1)$ -Banach space, and that $\mathcal{S}(\mathbb{R}^d; E^*) \subseteq [B_{p,q}^\sigma(\mathbb{R}^d; E)]^*$. In particular, $[B_{p,q}^\sigma(\mathbb{R}^d; E)]^*$ contains a subspace that separates the points of $B_{p,q}^\sigma(\mathbb{R}^d; E)$. A precise characterization of $[B_{p,q}^\sigma(\mathbb{R}^d)]^*$ can be found in [87, 2.11.2 and 2.11.3].

A.2. γ -Fubini in Besov spaces

PROPOSITION A.1. *Let H be a separable Hilbert space, $p, q \in (0, \infty)$, and $\sigma \in \mathbb{R}$. Then for all finite rank operators $G \in \mathcal{L}(H, B_{p,q}^\sigma(\mathbb{R}^d))$ one has*

$$\|G\|_{\gamma(H, B_{p,q}^\sigma(\mathbb{R}^d))} \widetilde{\sim}_{p,q} \|G\|_{B_{p,q}^\sigma(\mathbb{R}^d; H)}.$$

Moreover, $I: B_{p,q}^\sigma(\mathbb{R}^d; H) \rightarrow \gamma(H, B_{p,q}^\sigma(\mathbb{R}^d))$, where $I(G)h = \langle G, h \rangle_H$ for $G \in B_{p,q}^\sigma(\mathbb{R}^d; H)$, $h \in H$, defines an isomorphism.

The idea of the proof involves a Fubini argument that is classical for characterizing $\gamma(H, L^q(S; E))$ with E a Banach space (see e.g. [73, p. 5]), and has already been used in the proof of Theorem 3.16—note however that $B_{p,q}^\sigma(\mathbb{R}^d)$ is not a quasi-Banach function space, so that Proposition A.1 is not a straightforward consequence of Theorem 3.16.

Proof of Proposition A.1. Let $G \in \gamma(H, B_{p,q}^\sigma(\mathbb{R}^d))$ be a finite rank operator, i.e., $G = \sum_{n=1}^N h_n \otimes g_n$, where $N \in \mathbb{N}$, $(h_n)_{n=1}^N$ is an orthonormal system in H , and $g_n \in B_{p,q}^\sigma(\mathbb{R}^d)$. Let $(\gamma_n)_{n=1}^N$ be a sequence of i.i.d. standard \mathbb{R} -valued Gaussian random variables, and let $(\varphi_k)_{k \in \mathbb{N}}$ be as in Appendix A.1. By Proposition 3.2, the Kahane–Khinchin inequalities (applied thrice) and Fubini’s theorem (applied twice) we have

$$\begin{aligned} \|G\|_{\gamma(H, B_{p,q}^\sigma(\mathbb{R}^d))} &\widetilde{\sim}_{p,q} \left\| \sum_{n=1}^N \gamma_n g_n \right\|_{L^2(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))} \widetilde{\sim}_q \left\| \sum_{n=1}^N \gamma_n g_n \right\|_{L^q(\Omega; B_{p,q}^\sigma(\mathbb{R}^d))} \\ &= \left(\sum_{k \in \mathbb{N}} 2^{k\sigma q} \left\| \sum_{n=1}^N \gamma_n (\varphi_k * g_n) \right\|_{L^q(\Omega; L^p(\mathbb{R}^d))}^q \right)^{1/q} \\ &\widetilde{\sim}_p \left(\sum_{k \in \mathbb{N}} 2^{k\sigma q} \left\| \sum_{n=1}^N \gamma_n (\varphi_k * g_n) \right\|_{L^p(\Omega; L^p(\mathbb{R}^d))}^q \right)^{1/q} \\ &\widetilde{\sim}_p \left(\sum_{k \in \mathbb{N}} 2^{k\sigma q} \left\| \left(\sum_{n=1}^N |\varphi_k * g_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}^q \right)^{1/q} = \|G\|_{B_{p,q}^\sigma(\mathbb{R}^d; H)}. \end{aligned}$$

In order to complete the proof it suffices to prove that the functions represented by finite rank operators are dense in $B_{p,q}^\sigma(\mathbb{R}^d; H)$ (they are dense in $\gamma(H, B_{p,q}^\sigma(\mathbb{R}^d))$ by definition).

Recall that $\mathcal{S}(\mathbb{R}^d; H)$ is dense in $B_{p,q}^\sigma(\mathbb{R}^d; H)$ (see e.g. [87, Theorem 2.3.3], the proof also applies to the vector-valued case). Let $G \in \mathcal{S}(\mathbb{R}^d; H)$. Moreover, let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\text{span}(\text{Range}(G))$ for some $I \subseteq \mathbb{N}$. For $n \in \mathbb{N}$ let $P_n: H \rightarrow H$ be the orthogonal projection onto $\text{span}(\{h_j: j \leq n\})$. Observe that $P_n G$ represents a $B_{p,q}^\sigma(\mathbb{R}^d)$ -valued finite rank operator, and $\lim_{n \rightarrow \infty} \|(\text{Id} - P_n)G\|_{B_{p,q}^\sigma(\mathbb{R}^d; H)} = 0$ by the dominated convergence theorem. In conclusion, every $G \in \mathcal{S}(\mathbb{R}^d; H)$ can be approximated in $B_{p,q}^\sigma(\mathbb{R}^d; H)$ by a sequence of finite rank operators. This completes the proof of the proposition. ■

Appendix B. The Hardy–Littlewood maximal function and a technical lemma. Lemma B.1 below provides a pointwise estimate for the mapping $f \mapsto K_t * f$, where $f \in \mathcal{S}(\mathbb{R}^d)$ is such that the Fourier transform \widehat{f} of f has compact support, and $(K_t)_{t \in (0, \infty)}$ is the standard heat kernel (see (6.4)). As a corollary, we obtain estimates in the Besov quasi-norm for the heat kernel applied to an element of a Besov space; see Corollaries B.3–B.5. The proof is in spirit a Fourier multiplier argument closely related to [87, Section 1.5.2].

Recall the notation from Appendix A.1. For $f: \mathbb{R}^d \rightarrow \mathbb{R}$ measurable let $M(f): \mathbb{R}^d \rightarrow [0, \infty]$ denote the *Hardy–Littlewood maximal function* of f , i.e.,

$$(B.1) \quad M(f)(x) = \sup_{r \in (0, \infty)} \frac{\Gamma(d/2 + 1)}{(\pi r)^d} \int_{|y| \leq r} |f(x + y)| \, dy, \quad x \in \mathbb{R}^d.$$

LEMMA B.1. *Let M and $(K_t)_{t \in (0, \infty)}$ be as in (B.1) and (6.4), respectively, and define*

$$\mathcal{D}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d): \begin{array}{l} \text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d: |\xi| \leq 3/2\} \text{ or} \\ \exists n \in \mathbb{N}: \text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d: 2^{n-1} \leq |\xi| \leq 3 \cdot 2^{n-1}\} \end{array} \right\}.$$

Then for all $\alpha \in \mathbb{R}$, $\lambda \in [0, 1)$, $r \in (0, 1]$ there exist constants $C_{d,r,\alpha}^{(B.2)}, C_{d,r,\alpha,\lambda}^{(B.3)} \in (0, \infty)$ such that for all $f \in \mathcal{D}(\mathbb{R}^d)$, $0 < s < t < \infty$, and $\xi \in \mathbb{R}^d$, we have

$$(B.2) \quad |(1 - (2\pi)^{-2} \Delta)^\alpha (K_t * f)(\xi)| \leq C_{d,r,\alpha}^{(B.2)} t^{-(\alpha \vee 0)} e^{5\pi^2 t} (M(|f|^r)(\xi))^{1/r},$$

and

$$(B.3) \quad \begin{aligned} & |(1 - (2\pi)^{-2} \Delta)^\alpha (K_t * f - K_s * f)(\xi)| \\ & \leq C_{d,r,\alpha,\lambda}^{(B.3)} s^{(-\alpha - \lambda) \wedge 0} (t - s)^\lambda e^{6\pi^2 t} (M(|f|^r)(\xi))^{1/r}. \end{aligned}$$

Moreover, if $\alpha \leq 0$ then (B.2) holds for $t = 0$, and if $\alpha \leq -\lambda$ then (B.3) holds for $s = 0$ (recall that we set $K_0 * \varphi := \varphi$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$).

Proof. Fix $\alpha \in \mathbb{R}$ and $r \in (0, 1]$. We first prove the statement involving estimate (B.2). To this end we introduce $h: (0, \infty) \rightarrow \mathbb{R}$; $h(x) = x^\alpha e^{-4\pi^2 x}$,

and $g_t, m_t: \mathbb{R}^d \rightarrow \mathbb{R}$, $t \in [0, \infty)$, given by $g_t(\xi) = t(1 + |\xi|^2)$ and $m_t(\xi) = h(g_t(\xi))$. Using the Leibniz rule one may verify that for all $\beta, \gamma \in \mathbb{N}_0$ there exists a constant $c_{\alpha, \beta, \gamma}^{(B.4)}$ such that for all $t \in (0, \infty)$,

$$(B.4) \quad \sup_{x \in [t, \infty)} x^\beta h^{(\gamma)}(x) \leq c_{\alpha, \beta, \gamma}^{(B.4)} (1 + t^{(\alpha + \beta - \gamma) \wedge 0}).$$

Let $\frac{\partial}{\partial \xi}$ denote the Fréchet derivative with respect to $\xi \in \mathbb{R}^d$. Using that for $\xi, x^{(1)}, x^{(2)} \in \mathbb{R}^d$ we have $\frac{\partial}{\partial \xi} g_t(\xi)(x^{(1)}) = 2t \langle \xi, x^{(1)} \rangle_{\mathbb{R}^d}$ as well as $\frac{\partial^2}{\partial \xi^2} g_t(\xi)(x^{(1)}, x^{(2)}) = 2t \langle x^{(1)}, x^{(2)} \rangle_{\mathbb{R}^d}$ and $\frac{\partial^k}{\partial \xi^k} g_t \equiv 0$ for all $k \geq 3$, it follows, by the Faà di Bruno formula, that for all $\gamma \in \mathbb{N}_0$, $t \in (0, \infty)$, $\xi, x^{(1)}, \dots, x^{(\gamma)} \in \mathbb{R}^d$ one has

$$(B.5) \quad \begin{aligned} \frac{\partial^\gamma}{\partial \xi^\gamma} m_t(\xi)(x^{(1)}, \dots, x^{(\gamma)}) &= \sum_{k=\lceil \gamma/2 \rceil}^{\gamma} 2^k t^k h^{(k)}(g_t(\xi)) \\ &\times \sum_{\substack{B \subseteq \{1, \dots, \gamma\}; \\ |B|=2k-\gamma}} \prod_{i \in B} \langle \xi, x^{(i)} \rangle_{\mathbb{R}^d} \\ &\times \sum_{\substack{(n_j, m_j)_{j=1}^{\gamma-k} \\ P_2(\{1, \dots, n\} \setminus B)}} \prod_{j=1}^{\gamma-k} \langle x^{(n_j)}, x^{(m_j)} \rangle_{\mathbb{R}^d}, \end{aligned}$$

where for $A \subseteq \{1, \dots, n\}$ a set with an even number of elements, $P_2(A)$ is the set of all pairings of elements of A (i.e., $P_2(A)$ itself contains $\frac{|A|!}{2^{|A|/2} (|A|/2)!}$ elements). Note that $\text{Range } g_t = [t, \infty)$. It thus follows from (B.4) and (B.5) that for all $\gamma \in \mathbb{N}_0$ there exists a constant $c_{\alpha, \gamma}^{(B.6)}$ such that for all $t \in (0, \infty)$, $\xi \in \mathbb{R}^d$,

$$(B.6) \quad \begin{aligned} \left\| \frac{\partial^\gamma}{\partial \xi^\gamma} m_t(\xi) \right\|_{L^{(\gamma)}(\mathbb{R}^d, \mathbb{R})} &\leq \sum_{k=\lceil \gamma/2 \rceil}^{\gamma} \frac{2^{2k-\gamma} \gamma!}{(2k-\gamma)! (\gamma-k)!} t^k h^{(k)}(g_t(\xi)) |\xi|^{2k-\gamma} \\ &\leq \sum_{k=\lceil \gamma/2 \rceil}^{\gamma} \frac{2^{2k-\gamma} \gamma!}{(2k-\gamma)! (\gamma-k)!} t^{\gamma/2} [g_t(\xi)]^{k-\gamma/2} h^{(k)}(g_t(\xi)) \leq c_{\alpha, \gamma}^{(B.6)} (t^{\gamma/2} + t^{\alpha \wedge 0}). \end{aligned}$$

Moreover, by a similar argument there exists a constant $c_{\alpha, \gamma}^{(B.7)}$ such that for all $t \in (0, \infty)$, $\xi \in \mathbb{R}^d$,

$$(B.7) \quad \begin{aligned} \left\| |\xi|^\gamma \frac{\partial^\gamma}{\partial \xi^\gamma} m_t(\xi) \right\|_{L^{(\gamma)}(\mathbb{R}^d, \mathbb{R})} &\leq \sum_{k=\lceil \gamma/2 \rceil}^{\gamma} \frac{2^{2k-\gamma} \gamma!}{(2k-\gamma)! (\gamma-k)!} (g_t(\xi))^k h^{(k)}(g_t(\xi)) \\ &\leq c_{\alpha, \gamma}^{(B.7)} (1 + t^{\alpha \wedge 0}). \end{aligned}$$

Now fix $f \in \mathcal{D}(\mathbb{R}^d)$. In the case that there exists an $n \in \mathbb{N}$ such that $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : 2^{n-1} \leq |\xi| \leq 3 \cdot 2^{n-1}\}$, choose $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\psi}(\xi) = 1$ for $1 \leq |\xi| \leq 3$ and $\widehat{\psi}(\xi) = 0$ if $|\xi| > 4$ or $|\xi| \leq 1/2$. Otherwise we choose ψ such that $\widehat{\psi}(\xi) = 1$ for $|\xi| \leq 3/2$ and $\widehat{\psi}(\xi) = 0$ if $|\xi| \geq 2$ and set $n = 1$ (so we have $n = 1$ if $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : 1 \leq |\xi| \leq 3\}$, but also if $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq 3/2\}$). We have the following straightforward estimate for all $n \in \mathbb{N}$, $\xi \in \text{supp } \widehat{\psi}$:

$$(B.8) \quad 2^{\ell(n-1)} \leq 1 + 2^{\ell n} |\xi|^\ell.$$

Note that $\widehat{K}_t(\xi) = \exp(-4\pi^2 t |\xi|^2)$ and $\widehat{f} = \widehat{\psi}(2^{-n+1} \cdot) \widehat{f}$ (by assumption on $\text{supp } f$), i.e., for all $t \in (0, \infty)$, $\xi \in \mathbb{R}^d$ one has

$$(B.9) \quad e^{-4\pi^2 t^\alpha} |(1 - (2\pi)^{-2} \Delta)^\alpha (K_t * f)| = |\mathcal{F}^{-1}(m_t \widehat{f})| \\ = |\mathcal{F}^{-1}(m_t \widehat{\psi}(2^{-n+1} \cdot) \widehat{f})|.$$

Also note that for all $\varphi \in L_{\text{loc}}^1(\mathbb{R}^d)$, $c \in (0, \infty)$, and $\xi \in \mathbb{R}^d$ we have $M(\varphi(c \cdot))(c^{-1} \xi) = M(\varphi)(\xi)$. It follows from this observation and [87, Theorem 1.3.1, equation (2)] that there exists a constant $c_{d,r}^{(B.10)}$ such that for all $g \in \mathcal{S}(\mathbb{R}^d)$ and $R \in (0, \infty)$ satisfying $\text{supp } \widehat{g} \subseteq B_{4R}(0)$,

$$(B.10) \quad \forall \xi, y \in \mathbb{R}^d: \quad \frac{g(\xi - 2^{-R}y)}{1 + |y|^{d/r}} \leq c_{d,r}^{(B.10)} [M(|g|^r)(\xi)]^{1/r}.$$

We thus find, using the fact that $\mathcal{F}^{-1}(\varphi)(\xi) = c^d \mathcal{F}^{-1}(\varphi(c \cdot))(c\xi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $c \in (0, \infty)$, and $\xi \in \mathbb{R}^d$, and then applying (B.10), that for all $t \in (0, \infty)$, $\xi \in \mathbb{R}^d$ one has

$$(B.11) \quad |\mathcal{F}^{-1}(m_t \widehat{\psi}(2^{-n+1} \cdot) \widehat{f})(\xi)| \\ = |[\mathcal{F}^{-1}(m_t \widehat{\psi}(2^{-n+1} \cdot)) * f](\xi)| \\ = \int_{\mathbb{R}^d} \mathcal{F}^{-1}(m_t \widehat{\psi}(2^{-n+1} \cdot))(y) f(\xi - y) dy \\ = \int_{\mathbb{R}^d} \mathcal{F}^{-1}(m_t(2^{n-1} \cdot) \widehat{\psi})(u) f(\xi - 2^{-n+1}u) du \\ \leq c_{d,r}^{(B.10)} [M(|f|^r)(\xi)]^{1/r} \int_{\mathbb{R}^d} (1 + |u|^{d/r}) \mathcal{F}^{-1}(m_t(2^{n-1} \cdot) \widehat{\psi})(u) du.$$

Thus it remains to estimate the last integral in the expression above. To do so let $j \in \mathbb{N}$ be the smallest even integer such that $j > d + d/r$; then with

$c_{d,r}^{(B.12)} = \|(1 + |\cdot|^2)^{d/2r-j/2}\|_{L^1(\mathbb{R}^d)} < \infty$ for all $t \in (0, \infty)$ we have

$$\begin{aligned}
(B.12) \quad & \|(1 + |\cdot|^{d/r})\mathcal{F}^{-1}(m_t(2^{n-1}\cdot)\widehat{\psi})\|_{L^1(\mathbb{R}^d)} \\
& \leq 2\|(1 + |\cdot|^2)^{d/2r}\mathcal{F}^{-1}(m_t(2^{n-1}\cdot)\widehat{\psi})\|_{L^1(\mathbb{R}^d)} \\
& \leq 2\|(1 + |\cdot|^2)^{d/2r-j/2}\|_{L^1(\mathbb{R}^d)} \\
& \quad \times \|\mathcal{F}^{-1}[(1 - (2\pi)^{-2}\Delta)^{j/2}(m_t(2^{n-1}\cdot)\widehat{\psi})]\|_{L^\infty(\mathbb{R}^d)} \\
& \leq 2c_{d,r}^{(B.12)}\|(1 - (2\pi)^{-2}\Delta)^{j/2}(m_t(2^{n-1}\cdot)\widehat{\psi})\|_{L^1(\mathbb{R}^d)} \\
& \leq \frac{2c_{d,r}^{(B.12)}4^{d\pi d/2}}{\Gamma(d/2+1)} \sup_{\xi \in \mathbb{R}^d} |(1 - (2\pi)^{-2}\Delta)^{j/2}(m_t(2^{n-1}\cdot)\widehat{\psi})(\xi)|,
\end{aligned}$$

where we have used the fact that the inverse Fourier transform is contractive from L^1 into L^∞ and the fact that $\text{supp } \widehat{\psi} = \{\xi \in \mathbb{R}^d : |\xi| \leq 4\}$.

The first inequality below is obtained from the binomial formula, the Leibniz rule and the triangle inequality, while the second inequality follows from (B.8). In the third inequality we use the estimates on the derivative of m_t obtained in (B.6) and (B.7). Altogether we find that there exist constants $c_{d,r}^{(B.13)}$, $c_{d,r,\alpha}^{(B.13)}$ depending only on d and r , respectively d , r , and α , such that for all $t \in (0, \infty)$, $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
(B.13) \quad & |(1 - (2\pi)^{-2}\Delta)^{j/2}(m_t(2^{n-1}\cdot)\widehat{\psi})(\xi)| \\
& \leq c_{d,r}^{(B.13)} \sum_{k=0}^{j/2} \sum_{\ell=0}^{2k} 2^{\ell(n-1)} \left\| \left(\frac{\partial^\ell}{\partial \xi^\ell} m_t \right) (2^{n-1}\xi) \right\|_{L^{(\ell)}(\mathbb{R}^d, \mathbb{R})} \\
& \quad \times \left\| \left(\frac{\partial^{2k-\ell}}{\partial \xi^{2k-\ell}} \widehat{\psi} \right) (\xi) \right\|_{L^{(2k-\ell)}(\mathbb{R}^d, \mathbb{R})} \\
& \leq c_{d,r}^{(B.13)} \sum_{k=0}^{j/2} \sum_{\ell=0}^{2k} (1 + 2^\ell |2^{n-1}\xi|^\ell) \left\| \left(\frac{\partial^\ell}{\partial \xi^\ell} m_t \right) (2^{n-1}\xi) \right\|_{L^{(\ell)}(\mathbb{R}^d, \mathbb{R})} \\
& \quad \times \left\| \left(\frac{\partial^{2k-\ell}}{\partial \xi^{2k-\ell}} \widehat{\psi} \right) (\xi) \right\|_{L^{(2k-\ell)}(\mathbb{R}^d, \mathbb{R})} \\
& \leq c_{d,r,\alpha}^{(B.13)} (1 + t^{\alpha \wedge 0}) e^{\pi^2 t}.
\end{aligned}$$

Combining (B.9), (B.11), (B.12), and (B.13) completes the proof of (B.2).

We now turn to the statement involving estimate (B.3). Fix $f \in \mathcal{D}(\mathbb{R}^d)$, $\lambda \in [0, 1)$, and $0 < s < t < \infty$. Recall that for all $g \in \mathcal{S}(\mathbb{R}^d)$ we have $\frac{\partial}{\partial t}(K_t * g) = \Delta(K_t * g)$. It follows that

$$(1 - (2\pi)^{-2}\Delta)^\alpha (K_t * f - K_s * f) = \int_s^t (1 - (2\pi)^{-2}\Delta)^\alpha \Delta(K_\tau * f) d\tau.$$

Therefore, by (B.2) we can estimate, for all $\xi \in \mathbb{R}^d$,

$$\begin{aligned} & |(1 - (2\pi)^{-2}\Delta)^\alpha (K_t * f - K_s * f)(\xi)| \\ & \leq (2\pi)^2 \int_s^t (|(1 - (2\pi)^{-2}\Delta)^{\alpha+1} (K_\tau * f)(\xi)| + |(1 - (2\pi)^{-2}\Delta)^\alpha (K_\tau * f)(\xi)|) d\tau \\ & \leq (2\pi)^2 e^{5\pi^2 t} \int_s^t (C_{d,r,\alpha+1}^{(B.2)} \tau^{-((\alpha+1)\vee 0)} + C_{d,r,\alpha}^{(B.2)} \tau^{-(\alpha\vee 0)}) d\tau (M(|f|^r)(\xi))^{1/r}. \end{aligned}$$

Estimate (B.3) follows from the above and from the fact that by Hölder's inequality, for all relevant $\beta \in \mathbb{R}$,

$$\begin{aligned} \int_s^t \tau^\beta d\tau & \leq (t-s)^\lambda \left| \frac{1-\lambda}{1-\lambda+\beta} \right|^{1-\lambda} s^{(1-\lambda+\beta)\wedge 0} t^{(1-\lambda+\beta)\vee 0} \\ & \lesssim_{\lambda,\beta} (t-s)^\lambda s^{(1-\lambda+\beta)\wedge 0} e^{\pi^2 t}. \end{aligned}$$

The statement regarding $t = 0$ in (B.2) can be proven e.g. by replacing m_t in the proof above by $m \in \mathcal{S}(\mathbb{R}^d)$, $m(\xi) = (1 + |\xi|^2)^\alpha$. The statement concerning $s = 0$ in (B.3) can then be proven entirely analogously to the above. ■

For convenience we recall the following extension of the Fefferman–Stein theorem [32, Theorem 1]; for a proof see [35, Theorem 2.6].

PROPOSITION B.2. *Let $p, q \in (0, \infty)$, let $r \in (0, p \wedge q) \cap (0, 1]$, and let (S, \mathcal{A}, μ) be a σ -finite measure space. For every measurable $f: \mathbb{R}^d \times S \rightarrow \mathbb{R}$ define $\overline{M}(f): \mathbb{R}^d \times S \rightarrow [0, \infty]$ by $\overline{M}(f)(x, s) = M(f(\cdot, s))(x)$. Then there exists a constant $C_{p/r, q/r}^{(B.14)}$ such that for all $f: \mathbb{R}^d \times S \rightarrow \mathbb{R}$ measurable,*

$$(B.14) \quad \|\overline{M}(|f|^r)\|_{L^p(\mathbb{R}^d; L^q(S))}^{1/r} \leq C_{p/r, q/r}^{(B.14)} \|f\|_{L^p(\mathbb{R}^d; L^q(S))}.$$

Combining Lemma B.1, Proposition B.2, and the lifting property for Besov spaces we obtain the following three corollaries.

COROLLARY B.3. *Let $d \in \mathbb{N}$, $p, q \in (0, \infty)$, $\sigma \in \mathbb{R}$, and let $\lambda \in [0, 1)$. Then there exists a constant $C_{d,p,\lambda}^{(B.15)}$ such that for all $f \in B_{p,q}^\sigma(\mathbb{R}^d)$ and all $0 \leq s < t < \infty$,*

$$(B.15) \quad \|(K_t - K_s) * f\|_{B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d)} \leq C_{d,p,\lambda}^{(B.15)} (t-s)^\lambda e^{6\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d)}.$$

Proof. Let $(\varphi_k)_{k \in \mathbb{N}}$ be as in Section A.1 and let $r \in (0, p) \cap (0, 1]$. It follows from the lifting property (see [87, 2.3.8]), Lemma B.1, and the Fefferman–Stein theorem that for all $f \in B_{p,q}^\sigma(\mathbb{R}^d)$, $0 \leq s < t < \infty$ we have

$$\begin{aligned} \|(K_t - K_s) * f\|_{B_{p,q}^{\sigma-2\lambda}(\mathbb{R}^d)} & \preceq_\lambda \|(1 - (2\pi)^{-2}\Delta)^{-\lambda} (K_t - K_s) * f\|_{B_{p,q}^\sigma(\mathbb{R}^d)} \\ & = \left\| (2^{\sigma k} (1 - (2\pi)^{-2}\Delta)^{-\lambda} (K_t - K_s) * f * \varphi_k)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \end{aligned}$$

$$\begin{aligned}
&\leq C_{d,r,-\lambda,\lambda}^{(B.3)}(t-s)^\lambda e^{6\pi^2 t} \left\| (2^{\sigma k} [M(|f * \varphi_k|^r)]^{1/r})_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&\leq C_{p/r}^{F.-S} C_{d,r,-\lambda,\lambda}^{(B.3)}(t-s)^\lambda e^{5\pi^2 t} \left\| (2^{\sigma k} f * \varphi_k)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&= C_{p/r}^{F.-S} C_{d,r,-\lambda,\lambda}^{(B.3)}(t-s)^\lambda e^{5\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L^r(S))}.
\end{aligned}$$

Taking the infimum over all $r \in (0, p) \cap (0, 1]$ on the right-hand side above we obtain (B.15). ■

COROLLARY B.4. *Let $d \in \mathbb{N}$, $p, q \in (0, \infty)$, $\sigma \in \mathbb{R}$, $\alpha \in [0, \infty)$ and $\lambda \in [0, \alpha] \cap [0, 1/2)$, and recall the definition of $L_\alpha^2(0, t; \ell^2)$ from Section 6. Then there exists a constant $C_{d,p,\alpha,\lambda}^{(B.16)}$ such that for all $0 < s < t < \infty$ and all $f: [0, s] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d; \ell^2)$ measurable,*

$$\begin{aligned}
(B.16) \quad &\|(K_{t-\cdot} - K_{s-\cdot}) * f\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d; L^2(0,s;\ell^2))} \\
&\leq C_{d,p,\alpha,\lambda}^{(B.16)}(t-s)^\lambda e^{6\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^2(0,s;\ell^2))}.
\end{aligned}$$

Moreover, there exists a constant $C_{d,p,\alpha,\lambda}^{(B.17)}$ such that for all $0 \leq s < t < \infty$ and all $f: [s, t] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d; \ell^2)$ measurable,

$$\begin{aligned}
(B.17) \quad &\|K_{t-\cdot} * f\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d; L^2(s,t;\ell^2))} \\
&\leq C_{d,p,\alpha,\lambda}^{(B.17)}(t-s)^\lambda e^{5\pi^2 t} \|f \mathbb{1}_{[s,t]}\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^2(0,t;\ell^2))}.
\end{aligned}$$

Proof. Let $(\varphi_k)_{k \in \mathbb{N}}$ be as in Section A.1 and let $r \in (0, p) \cap (0, 1]$. For brevity we introduce

$$J := 1 - (2\pi)^{-2} \Delta.$$

By the lifting property (see [87, 2.3.8], or [3, Theorem 6.1] for the vector-valued case), Lemma B.1 and Proposition B.2 (with $q = 2$ and $S = [0, T] \times \mathbb{N}$) it follows for all $0 < s < t < \infty$ and all measurable $f: [0, t] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d; \ell^2)$ that

$$\begin{aligned}
&\|(K_{t-\cdot} - K_{s-\cdot}) * f\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d; L^2(0,s;\ell^2))} \\
&\quad \approx_{\alpha-\lambda} \|J^{\alpha-\lambda}(K_{t-\cdot} - K_{s-\cdot}) * f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L^2(0,s;\ell^2))} \\
&= \left\| \left(2^{k\sigma} \left(\int_0^s \|J^{\alpha-\lambda}(K_{t-u} - K_{s-u}) * f(u) * \varphi_k\|_{\ell^2}^2 du \right)^{1/2} \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&\leq C_{d,r,\alpha-\lambda,\lambda}^{(B.3)}(t-s)^\lambda e^{6\pi^2 t} \\
&\quad \times \left\| \left(2^{k\sigma} \left(\int_0^s (s-u)^{-\alpha} \| [M(|f(u) * \varphi_k|^r)]^{1/r} \|_{\ell^2}^2 du \right)^{1/2} \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))}
\end{aligned}$$

$$\begin{aligned}
&\leq C_{p/r,2/r}^{(B.14)} C_{d,r,\alpha-\lambda}^{(B.3)} (t-s)^\lambda e^{6\pi^2 t} \\
&\quad \times \left\| \left(2^{k\sigma} \left(\int_0^s (s-u)^{-\alpha} \|(f(u) * \varphi_k)\|_{\ell^2}^2 du \right)^{1/2} \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&= C_{p/r,2/r}^{(B.14)} C_{d,r,\alpha-\lambda}^{(B.3)} (t-s)^\lambda e^{6\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^2(0,s;\ell^2))}.
\end{aligned}$$

Taking the infimum over all $r \in (0, p) \cap (0, 1]$ on the right-hand side above we obtain (B.16).

Similarly, by the lifting property, Lemma B.1 and Proposition B.2 (with $q = 2$ and $S = [0, T] \times \mathbb{N}$) it follows for all $0 \leq s < t < \infty$ and all measurable $f: [s, t] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d; \ell^2)$ that

$$\begin{aligned}
&\|K_{t-} * f\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d; L^2(s,t;\ell^2))} \\
&\quad \approx_{\alpha-\lambda} \|J^{\alpha-\lambda}(K_{t-} * f)\|_{B_{p,q}^\sigma(\mathbb{R}^d; L^2(s,t;\ell^2))} \\
&\leq C_{d,r,\alpha-\lambda}^{(B.2)} e^{5\pi^2 t} \\
&\quad \times \left\| \left(2^{k\sigma} \left(\int_s^t (t-u)^{-\alpha+\lambda} \|[M(|f(u) * \varphi_k|^r)]^{1/r}\|_{\ell^2}^2 du \right)^{1/2} \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&\leq C_{p/r,2/r}^{(B.14)} C_{d,r,\alpha-\lambda}^{(B.2)} (t-s)^\lambda e^{5\pi^2 t} \\
&\quad \times \left\| \left(2^{k\sigma} \left(\int_s^t (t-u)^{-\alpha} \|(f(u) * \varphi_k)\|_{\ell^2}^2 du \right)^{1/2} \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\
&= C_{p/r,2/r}^{(B.14)} C_{d,r,\alpha-\lambda}^{(B.2)} (t-s)^\lambda e^{5\pi^2 t} \|f \mathbb{1}_{[s,t]}\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^2(0,t;\ell^2))}.
\end{aligned}$$

Again taking the infimum over all $r \in (0, p) \cap [0, 1]$ on the right-hand side above we arrive at (B.17). ■

COROLLARY B.5. *Let $d \in \mathbb{N}$, $p, q \in (0, \infty)$, $\sigma \in \mathbb{R}$, $\alpha \in [0, \infty)$ and $\lambda \in [0, \alpha] \cap [0, 1)$ and recall the definition of $L_\alpha^1(0, t)$ from Section 6. Then there exists a constant $C_{d,p,\alpha,\lambda}^{(B.18)}$ such that for all $0 < s < t < \infty$ and all $f: [0, s] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d)$ measurable,*

$$\begin{aligned}
(B.18) \quad &\left\| \int_0^s (K_{t-u} - K_{s-u}) * f(u) du \right\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d)} \\
&\leq C_{d,p,\alpha,\lambda}^{(B.18)} (t-s)^\lambda e^{6\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^1(0,s))},
\end{aligned}$$

where $\int_0^s K_{r-u} * f du \in \mathcal{S}'(\mathbb{R}^d)$, $r \in \{s, t\}$, is defined by

$$\left\langle \int_0^s K_{r-u} * f du, \varphi \right\rangle = \int_0^s \langle K_{r-u} * f, \varphi \rangle du, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Moreover, there exists a constant $C_{d,p,\alpha,\lambda}^{(B.19)}$ such that for all $0 \leq s < t < \infty$ and all $f: [s, t] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d)$ measurable,

$$(B.19) \quad \left\| \int_s^t K_{t-u} * f(u) \, du \right\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d)} \leq C_{d,p,\alpha,\lambda}^{(B.19)} (t-s)^\lambda e^{5\pi^2 t} \|f \mathbb{1}_{[s,t]}\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^1(0,t))}.$$

Proof. The proof is entirely analogous to the proof of Corollary B.4, hence we only provide the argument for (B.18). Let $(\varphi_k)_{k \in \mathbb{N}}$ be as in Section A.1, $J := (1 - (2\pi)^{-2}\Delta)$, and let $r \in (0, p) \cap (0, 1]$. By the lifting property, Lemma B.1 and Proposition B.2 it follows for all $0 < s < t < \infty$ and all $f: [0, s] \rightarrow B_{p,q}^\sigma(\mathbb{R}^d)$ that

$$(B.20) \quad \begin{aligned} & \left\| \int_0^s (K_{t-u} - K_{s-u}) * f(u) \, du \right\|_{B_{p,q}^{\sigma+2(\alpha-\lambda)}(\mathbb{R}^d)} \\ & \quad \approx_{\alpha-\lambda} \left\| \int_0^s J^{\alpha-\lambda} (K_{t-u} - K_{s-u}) * f(u) \, du \right\|_{B_{p,q}^\sigma(\mathbb{R}^d)} \\ & \leq C_{d,r,\alpha-\lambda,\lambda}^{(B.3)} (t-s)^\lambda e^{6\pi^2 t} \\ & \quad \times \left\| \left(2^{\sigma k} \int_0^s (s-u)^{-\alpha} [M|(f(u) * \varphi_k|^r)]^{1/r} \, du \right)_{k \in \mathbb{N}} \right\|_{\ell^q(L^p(\mathbb{R}^d))} \\ & \leq C_{p/r,1/r}^{(B.14)} C_{d,r,\alpha-\lambda,\lambda}^{(B.3)} (t-s)^\lambda e^{6\pi^2 t} \|f\|_{B_{p,q}^\sigma(\mathbb{R}^d; L_\alpha^1(0,t))}. \end{aligned}$$

Estimate (B.18) follows by taking the infimum over $r \in (0, p) \cap (0, 1]$ on the right-hand side above. ■

Acknowledgements. The first author has been supported by the Deutsche Forschungsgemeinschaft (DFG, grant DA 360/20-1) and partially by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand. The second author is supported by the research program VENI Vernieuwingsimpuls with project number 639.031.549, which is financed by the Netherlands Organization for Scientific Research (NWO). The third named author is supported by the Vidi subsidy 639.032.427 of the Netherlands Organization for Scientific Research (NWO).

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