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Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities

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Abstract—We consider dynamic games with linear dynamics and quadratic objective functions. We observe that the unconstrained open-loop Nash equilibrium (NE) coincides with a linear-quadratic regulator in an augmented space, thus deriving an explicit expression of the cost-to-go. With such cost-to-go as a terminal cost, we show asymptotic stability for the receding-horizon solution of the finite-horizon, constrained game. Furthermore, we show that the problem is equivalent to a nonsymmetric variational inequality, which does not correspond to any NE problem. For unconstrained closed-loop Nash equilibria, we derive a receding-horizon controller that is equivalent to the infinite-horizon one and ensures asymptotic stability.

Index Terms—Game theory, multi-agent systems, optimal control.

NOMENCLATURE

Notation	Definition
$\text{row}(M_i)_{i \in \mathcal{I}}$	Row stack: $[M_1, \dots, M_N]$.
$\text{col}(M_i)_{i \in \mathcal{I}}$	Column stack: $[M_1^\top, \dots, M_N^\top]^\top$.
$\text{blkdiag}(M_i)_{i \in \mathcal{I}}$	Diagonal stack of $(M_i)_{i \in \mathcal{I}}$.
$\ x\ _M$	Weighted Euclidean norm: $\sqrt{x^\top M x}$.
\mathcal{S}_T^n	Set of T -long sequences in \mathbb{R}^n .
$v[t]$	t th element of $v \in \mathcal{S}_T^n$.
$v[t p]$	For a function $p \mapsto v(p) \in \mathcal{S}_T^n$, denotes the t th element of $v(p)$.
$f(x y)$	For each value of a parameter y , denotes the function $f(\cdot y) : x \mapsto f(x y)$.
δ_i^K	Vector in \mathbb{R}^K with only nonzero element 1 at index i .
I_k	Identity matrix of size k .
N	Number of agents.
n	Dimension of state variable.
m	Dimension of input variable.
\mathcal{I}	Set of all agents: $\{1, \dots, N\}$.
\mathcal{I}_{-i}	Set of all agents except i : $\mathcal{I} \setminus \{i\}$.
T	Length of the control horizon.

\mathcal{T}	Control horizon: $\{0, \dots, T-1\}$.
\mathcal{T}^+	Shifted horizon: $\{1, \dots, T\}$.
x	State variable.
u_i	Input sequence of agent i .
\mathbf{u}	Collective input sequence: $(u_i)_{i \in \mathcal{I}}$.
\mathbf{u}_{-i}	Input sequence for all agents except i : $(u_i)_{i \in \mathcal{I}_{-i}}$.
$\phi(x_0, \mathbf{u})$	State sequence of system (1) with initial state x_0 and input \mathbf{u} .
Q_i	State weight matrix for agent i .
R_i	Input weight matrix for agent i .
C_i	Square root of Q_i : $Q_i = C_i^\top C_i$.
ℓ_i	Stage cost for agent i , see (3).
S_i	$:= B_i R_i^{-1} B_i^\top$.
\mathbb{X}	Set of admissible states.
\mathbb{U}_i	\mathbb{M} to the set of admissible inputs for agent i , see (2b).
$\mathbb{U}_{i,T}$	Maps to the set of T -long admissible input sequences for agent i , see (4).
\mathbb{U}_T	maps to the set of T -long admissible collective input sequences, see (5).
$\text{VI}(\mathcal{F}, \mathcal{C})$	Variational inequality [28]: for \mathcal{C} closed, convex set and operator \mathcal{F} , find $v^* \in \mathcal{C}$ such that $\inf_{v \in \mathcal{C}} \langle \mathcal{F}(v^*), v - v^* \rangle \geq 0$.

I. INTRODUCTION

WE CONSIDER a regulation problem for a constrained, discrete-time linear-quadratic (LQ) dynamic game, which emerges when the agents have interest in reaching and maintaining a known attractor while optimizing an individual objective. Dynamic games (precisely, their continuous-time counterpart, *differential* games) were first studied in the seminal paper [1]. They model a discrete-time dynamical system governed by multiple inputs, each controlled by a decision maker (or agent) with a self-interested objective. Applications include robotics [2], [3], robust control [4], logistics planning [5], and energy markets [6], to cite a few. An input strategy that no agent can improve via unilateral changes is called a Nash equilibrium (NE). Interestingly, depending on the information structure of the problem, dynamic games may admit different types of NE. In particular, an open-loop trajectory, which is for each agent, an optimal response to the trajectories of the other agents, is an open-loop Nash equilibrium (OL-NE); instead, a closed-loop Nash equilibrium (CL-NE) is an optimal feedback policy given the current state and the control policies of the other agents. It is well-known that a linear infinite-horizon CL-NE policy is

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linked to the solution of coupled Algebraic Riccati equations (AREs); see, e.g., [7, Proposition 6.3]. Recently, Monti et al. [8] develop novel sufficient conditions for a similar OL-NE characterization. In [9], the infinite-horizon continuous-time CL-NE problem was related to the problem of finding invariant linear subspaces: this observation has led to a significant development of the field [10] and solution algorithms based on a geometric approach [11]. Other algorithms, based on the iterative solution of Lyapunov or Riccati equations have been presented in [12], [13], and [14]. The recent works [15] and [16] study approximate solutions to differential games, with [17] providing a distributed computation approach. Crucially, none of these works consider the inclusion of constraints, and no established algorithm to compute infinite-horizon NE trajectories exists, with only [14] providing a posteriori, local convergence guarantees for the discrete-time CL-NE setting.

Computational methods for the constrained, finite-horizon case are instead available both for OL-NE [2] and CL-NE [18]. These have sparked the interest for receding-horizon controllers, which compute the NE for a finite-horizon game at each time step and apply the first input of the horizon as a control action. Such approach carries multiple advantages, as the recomputation allows the agents to react to unexpected disturbances, and the inclusion of constraints becomes tractable. Indeed, receding-horizon solutions of dynamic games were successfully employed in autonomous racing [3], autonomous driving [2], logistics planning [5], and electricity market clearance [6] applications. Generally, the stability properties of the closed-loop system are not analyzed, with the exception of [6] and [19], under the restrictions of considering a *potential* game and a *stable* plant, respectively. The game-theoretic receding-horizon setup is also related to distributed, noncooperative model predictive control (MPC). Early attempts at noncooperating MPC formulations resulted in unstable closed-loop dynamics, see [20, Sect. 4], unless additional constraints enforce a cooperation between the agents [21], [22]. In our view, the introduction of these constraints do not fully capture the noncooperative nature of the problem. Compared to the latter references, we do not introduce additional constraints; and compared to [23], [24], and [25], we do not assume that the agents cooperate toward minimizing a common objective function.

Fundamentally, we show that the infinite-horizon unconstrained OL-NE can be interpreted as the input sequence generated by coupled linear-quadratic regulator (LQR) problems in an augmented state space. Considering the problem in this augmented space is a genuinely novel perspective that yields to the first direct connection between the OL-NE and the LQR. Building on this novel insight, we derive a new closed-form expression for the value achieved by the OL-NE. We include this expression as terminal cost for a receding-horizon game-theoretic controller. With this terminal cost, the OL-NE trajectories coincide with the constrained infinite-horizon ones—a result that generalizes the infinite-horizon optimality property of single-agent MPC [26]. Furthermore, we show that the resulting controller is stabilizing: This is the first stability result for a game-theoretic predictive controller that enforces state and input constraints on a nonpotential game. From a computational perspective, we show that the finite-horizon problem can be cast

as a variational inequality (VI) [27]. The extensive literature on VIs allows one to design efficient, decentralized algorithms with convergence guarantees under loose assumptions. Interestingly, we find that the additive terminal cost jeopardizes the structure of the problem, as the resulting VI (despite being solved by the truncation of an infinite-horizon NE) is not associated to the Karush–Kuhn–Tucker (KKT) conditions of any finite-horizon NE problem. In the CL-NE case, the expression of the unconstrained, infinite-horizon cost-to-go is well-known in the literature. We formulate a receding-horizon control problem with such cost-to-go as terminal cost. We show that the receding-horizon solution coincides with the infinite-horizon one in the unconstrained case. Our technical contributions are summarized as follows.

- 1) For the infinite-horizon, unconstrained OL-NE case, we propose a solution algorithm inspired by [14] based on the iterative solution of Stein equations (see Section III-A). We show that the OL-NE is equivalent to a set of LQRs in an augmented space and we derive a novel expression for its cost-to-go (see Section III-B).
- 2) Leveraging the findings in Sections III-A and III-B, we derive a finite-horizon constrained problem whose solutions coincide with the infinite-horizon one, thus generalizing the single-agent MPC infinite-horizon optimality property [26]. We show that the system in closed loop with the receding-horizon solution of the considered finite-horizon problem is asymptotically stable and it can be cast as a VI (see Section III-C).
- 3) By an appropriate choice of the terminal cost and prediction model, we derive a receding-horizon controller whose trajectories are equivalent to the ones of the infinite-horizon CL-NE in the unconstrained case (see Section IV). Compared to the OL-NE case, the CL-NE solutions anticipate and account for the future reactions of the other players [18], and this case admits an infinite-horizon stabilizing solution under less restrictive conditions.

In Section V,¹ we illustrate two application examples via numerical simulations on distributed automatic generation control and vehicle platooning.

II. NOTATION AND PROBLEM STATEMENT

We consider the problem of regulating to the origin the state of the dynamical system

$$x[t+1] = Ax[t] + \sum_{i \in \mathcal{I}} B_i u_i[t] \quad (1)$$

subject to state and coupling input constraints

$$x[t] \in \mathbb{X} \subseteq \mathbb{R}^n \quad \forall t \in \mathcal{T} \quad (2a)$$

$$u_i[t] \in \mathbb{U}_i(\mathbf{u}_{-i}[t]) \subseteq \mathbb{R}^m \quad \forall t \in \mathcal{T}, i \in \mathcal{I}. \quad (2b)$$

Each input is determined by a self-interested agent. Without loss of generality, we consider inputs with equal dimension m for each agent. We denote the state sequence of the system resulting by (1) with initial state x_0 and collective input sequence $\mathbf{u} = (u_i)_{i \in \mathcal{I}}$ as $\phi(x_0, \mathbf{u})$. According to the notation, we denote

¹Code available at github.com/bemilio/Receding-Horizon-GNE.

the state evolution at time t as $\phi[t|x_0, \mathbf{u}]$. We consider control problems over a horizon T , possibly infinite, with quadratic stage costs

$$\forall i \in \mathcal{I} : \ell_i(x, u_i) = \frac{1}{2} \|x\|_{Q_i}^2 + \frac{1}{2} \|u_i\|_{R_i}^2. \quad (3)$$

Define for each i , the feasible input sequences

$$\begin{aligned} \mathcal{U}_{i,T}(x_0, \mathbf{u}_{-i}) &:= \{u_i \in \mathcal{S}_T^m | \\ &u_i[t] \in \mathbb{U}_i(\mathbf{u}_{-i}[t]) \forall t \in \mathcal{T}; \\ &\phi[t|x_0, \mathbf{u}] \in \mathbb{X} \forall t \in \mathcal{T}^+\} \end{aligned} \quad (4)$$

and the collective input sequences

$$\mathcal{U}_T(x_0) := \{\mathbf{u} \in \mathcal{S}_T^{Nm} | u_i \in \mathcal{U}_{i,T}(x_0, \mathbf{u}_{-i}) \forall i \in \mathcal{I}\}. \quad (5)$$

Finally, we assume that the origin is strictly feasible and that the state and input weights are positive semidefinite and definite, respectively.

Assumption 1:

- i) $0 \in \text{int}(\mathbb{X}) \forall i \in \mathcal{I}, 0 \in \text{int}(\mathbb{U}_i(0))$.
- ii) $Q_i = C_i^\top C_i \succcurlyeq 0, R_i = R_i^\top \succ 0 \forall i \in \mathcal{I}$.

III. OPEN-LOOP NASH TRAJECTORIES

Depending on the information structure of the problem assumed for the agents, dynamic games can have different solutions concepts [29, Ch. 6]. In this section, we consider the infinite-horizon OL-NE trajectories, where each agent assumes that the opponents only observe the initial state, and subsequently, commit to the sequence of inputs computed using such observations. Define the objective

$$\forall i : J_i^\infty(u_i|x_0, \mathbf{u}_{-i}) := \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0, \mathbf{u}], u_i[t]). \quad (6)$$

The OL-NE trajectory is defined as follows.

Definition 1 [29, Def. 6.2]: Let $x_0 \in \mathbb{X}$. The sequences $\mathbf{u} \in \mathcal{U}_\infty(x_0)$ are an open-loop Nash trajectory at x_0 if for all $i \in \mathcal{I}$,

$$J_i^\infty(u_i|x_0, \mathbf{u}_{-i}) \leq \inf_{v_i \in \mathcal{U}_{i,\infty}(x_0, \mathbf{u}_{-i})} J_i^\infty(v_i|x_0, \mathbf{u}_{-i}). \quad (7)$$

A. Unconstrained Infinite-Horizon Case

Let us first consider the unconstrained OL-NE problem, which is related to the Riccati equations

$$\forall i \in \mathcal{I} : \begin{cases} P_i^{\text{OL}} = Q_i + A^\top P_i^{\text{OL}} \bar{A}^{\text{OL}} & (8a) \\ K_i^{\text{OL}} = -R_i^{-1} B_i^\top P_i^{\text{OL}} \bar{A}^{\text{OL}} & (8b) \end{cases}$$

where

$$\bar{A}^{\text{OL}} := A + \sum_{i \in \mathcal{I}} B_i K_i^{\text{OL}}. \quad (9)$$

By leveraging [8, Thm. 4.10], a solution to (8) defines an OL-NE trajectory, as we report next.

Assumption 2 [8, Assumptions 4.6 and 4.7]:

- i) The matrix A is invertible.
- ii) For all $i \in \mathcal{I}$, the pairs (A, B_i) and (A, C_i) introduced in (1) and Assumption 1 are, respectively, stabilizable and detectable.

Algorithm 1: OL-NE Solution via Stein Recursion.

```

1: Initialization:  $(K_i^{(0)})_{i \in \mathcal{I}}$  such that
    $\bar{A}^{(0)} = A + \sum_{i \in \mathcal{I}} B_i K_i^{(0)}$  is Schur;
2: for  $k \in \mathbb{N}$  do :
3:   for  $i \in \mathcal{I}$  do :
4:     Solve  $P_i^{(k+1)} = Q_i + A^\top P_i^{(k+1)} \bar{A}^{(k)}$ ;
5:   end for
6:    $\bar{A}^{(k+1)} \leftarrow (I + \sum_{j \in \mathcal{I}} S_j P_j^{(k+1)})^{-1} A$ 
7:   for  $i \in \mathcal{I}$  do :
8:      $K_i^{(k+1)} \leftarrow -R_i^{-1} B_i^\top P_i^{(k+1)} \bar{A}^{(k+1)}$ .
9:   end for
10: end for

```

Assumption 3 [8, Assumption 4.9]: The matrix

$$H := \begin{bmatrix} A + \sum_{j \in \mathcal{I}} (S_j A^{-\top} Q_j) & \text{row}(-S_j A^{-\top})_{j \in \mathcal{I}} \\ \text{col}(-A^{-\top} Q_j)_{j \in \mathcal{I}} & I_N \otimes A^{-\top} \end{bmatrix} \quad (10)$$

possesses exactly n eigenvalues with modulus smaller than 1. Moreover, an n -dimensional stable invariant subspace of H is complementary to

$$\text{Im} \left(\begin{bmatrix} \mathbf{0}_{n \times Nn} \\ I_{Nn} \end{bmatrix} \right).$$

Proposition 1: Let Assumptions 1, 2, and 3 hold true and let $(K_i^{\text{OL}}, P_i^{\text{OL}})_{i \in \mathcal{I}}$ satisfy (8). Let \bar{A}^{OL} be defined as in (9). For any $x_0 \in \mathbb{R}^n$, let $\mathbf{u}^{\text{OL}}(x_0) \in \mathcal{S}_\infty^{Nm}$, $x^{\text{OL}}(x_0) \in \mathcal{S}_\infty^n$ be defined as

$$\forall t \in \mathbb{N}_0 : x^{\text{OL}}[t|x_0] := (\bar{A}^{\text{OL}})^t x_0 \quad (11a)$$

$$\forall i \in \mathcal{I}, t \in \mathbb{N}_0 : u_i^{\text{OL}}[t|x_0] := K_i^{\text{OL}} x^{\text{OL}}[t|x_0]. \quad (11b)$$

Then, $\mathbf{u}^{\text{OL}}(x_0)$ is an OL-NE trajectory at the initial state x_0 and \bar{A}^{OL} is Schur.

Proof: See Appendix A. ■

If we disregard the dependence of \bar{A}^{OL} on $(P_j^{\text{OL}})_{j \in \mathcal{I}}$, (8a) is a set of N Stein equations, for which off-the-shelf solvers exist [30]. One can then iteratively find $(P_i^{\text{OL}})_{i \in \mathcal{I}}$ that solves (8a) with \bar{A}^{OL} fixed, and then, update \bar{A}^{OL} according to

$$\bar{A}^{\text{OL}} = \left(I + \sum_{j \in \mathcal{I}} S_j P_j^{\text{OL}} \right)^{-1} A. \quad (12)$$

The equality in (12) is proven in Fact 1 (see Appendix A) when A is not singular. This approach is formalized in Algorithm 1.

B. OL-NE as an LQR

In this section, we show that the matrices $(K_i^{\text{OL}})_{i \in \mathcal{I}}$ in (11b) that characterize the unconstrained OL-NE are related to N LQR controllers defined in a higher dimensional space. This result offers a novel perspective on OL-NE and it is instrumental for our main results in Section III-C. For each agent i , let us denote with P_i^{LQR} and K_i^{LQR} , the solution to the ARE that solves the standard LQR problem for the linear time-invariant (LTI) system (A, B_i) , namely

$$P_i^{\text{LQR}} = Q_i + A^\top P_i^{\text{LQR}} \bar{A}_i^{\text{LQR}} \quad (13a)$$

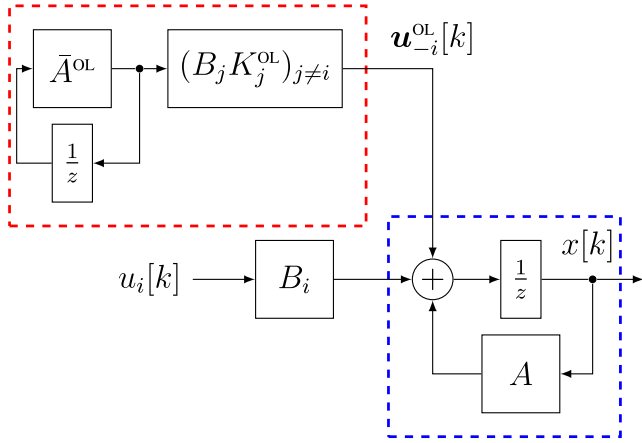


Fig. 1. Block diagram of the augmented system in (14a) for the agent i : The red dashed line highlights an uncontrollable mode that generates $\mathbf{u}_{-i}^{\text{OL}}$. Following (21), u_i^{OL} is the optimal control for this system if both the subsystems highlighted in red and in blue have the same initial state.

$$K_i^{\text{LQR}} = -R_i^{-1} B_i^\top P_i^{\text{LQR}} \bar{A}_i^{\text{LQR}} \quad (13b)$$

$$\bar{A}_i^{\text{LQR}} := A + B_i K_i^{\text{LQR}}. \quad (13c)$$

Monti et al. [8] note that, for each agent i , the OL-NE is the optimal control for the system (A, B_i) perturbed by the actions of the other agents. We observe that, along the trajectory defined by the control laws $(K_i^{\text{OL}})_{i \in \mathcal{I}}$, such perturbation is fully determined by the initial state and by the dynamics of the autonomous system \bar{A}^{OL} . Remarkably, the problem can then be cast as an augmented regulator problem by considering a system with $2n$ states that incorporates the dynamics of the perturbation.

Lemma 1 (NE as augmented LQR solution): Let $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i \in \mathcal{I}}$ solve (8) and let \bar{A}^{OL} as in (9). Let Assumptions 1(ii), 2, and 3 hold true. For all $i \in \mathcal{I}$, define

$$\hat{A}_i := \begin{bmatrix} A & \sum_{j \neq i} B_j K_j^{\text{OL}} \\ 0 & \bar{A}^{\text{OL}} \end{bmatrix}, \quad \hat{B}_i := \begin{bmatrix} B_i \\ 0 \end{bmatrix} \quad (14a)$$

$$\hat{Q}_i := \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{R}_i := R_i. \quad (14b)$$

Then, the ARE

$$\hat{P}_i = \hat{Q}_i + \hat{A}_i^\top \hat{P}_i (\hat{A}_i + \hat{B}_i \hat{K}_i) \quad (15a)$$

$$\hat{K}_i = -(\hat{R}_i + \hat{B}_i^\top \hat{P}_i \hat{B}_i)^{-1} \hat{B}_i^\top \hat{P}_i \hat{A}_i \quad (15b)$$

admits a unique positive semidefinite solution

$$\hat{P}_i = \begin{bmatrix} P_i^{\text{LQR}} & \tilde{P}_i \\ \tilde{P}_i^\top & * \end{bmatrix}; \quad \hat{K}_i = \begin{bmatrix} K_i^{\text{LQR}} & \tilde{K}_i \end{bmatrix} \quad (16)$$

where $\tilde{P}_i = P_i^{\text{OL}} - P_i^{\text{LQR}}$ and $\tilde{K}_i = K_i^{\text{OL}} - K_i^{\text{LQR}}$. Furthermore, for all $i \in \mathcal{I}$, $\hat{A}_i + \hat{B}_i \hat{K}_i$ is Schur. \square

Proof: See Appendix B. \blacksquare

A schematic representation of the system in (14a) is provided in Fig. 1. Let us consider the lifted system (\hat{A}_i, \hat{B}_i) in (14a) for some $i \in \mathcal{I}$ controlled by \hat{K}_i in (16) with initial state $\text{col}(x_0, y_0)$.

Denote by $\text{col}(x^*, y^*)$ and u_i^* , the resulting state and input sequences, that is,

$$u_i^*[t] = K_i^{\text{LQR}} x^*[t] + \tilde{K}_i y^*[t] \quad (17a)$$

$$x^*[t+1] = Ax^*[t] + B_i u_i^*[t] + \sum_{j \neq i} B_j K_j^{\text{OL}} y^*[t] \quad (17b)$$

$$y^*[t+1] = \bar{A}^{\text{OL}} y^*[t]. \quad (17c)$$

Note that (17c) implies $y^*[t] = (\bar{A}^{\text{OL}})^t y_0$. Furthermore, from (11), $u_j^{\text{OL}}[t|y_0] = K_j^{\text{OL}} (\bar{A}^{\text{OL}})^t y_0$ for all $j \in \mathcal{I}$. Thus, substituting the latter into (17b)

$$x^*[t+1] = Ax^*[t] + B_i u_i^*[t] + \sum_{j \neq i} B_j u_j^{\text{OL}}[t|y_0].$$

In other words, x^* is the sequence of states of the (nonlifted) dynamics in (1) controlled by u_i^* and $\mathbf{u}_{-i}^{\text{OL}}(y_0)$. We write this compactly as

$$x^*[t] = \phi[t|x_0, u_i^*, \mathbf{u}_{-i}^{\text{OL}}(y_0)]. \quad (18)$$

Consider the function

$$V_i(x_0, y_0) := \frac{1}{2} \left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_{\hat{P}_i}^2 \quad (19)$$

where \hat{P}_i solves (15). By applying a known result in the LQR literature [31, Thm. 21.1],

$$\frac{1}{2} \left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_{\hat{P}_i}^2 = \frac{1}{2} \sum_{t=0}^{\infty} \left\| \begin{bmatrix} x^*[t] \\ y^*[t] \end{bmatrix} \right\|_{\hat{Q}_i}^2 + \|u_i^*[t]\|_{\hat{R}_i}^2.$$

By substituting (14b) into the latter to eliminate \hat{Q}_i and \hat{R}_i , we find

$$\begin{aligned} V_i(x_0, y_0) &= \frac{1}{2} \sum_{t=0}^{\infty} \|x^*[t]\|_{\hat{Q}_i}^2 + \|u_i^*[t]\|_{\hat{R}_i}^2 \\ &\stackrel{(18),(3)}{=} \frac{1}{2} \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0, u_i^*, \mathbf{u}_{-i}^{\text{OL}}(y_0)], u_i^*[t]) \\ &\stackrel{(6)}{=} J_i^\infty(u_i^*|x_0, \mathbf{u}_{-i}^{\text{OL}}(y_0)) \\ &\leq J_i^\infty(u_i|x_0, \mathbf{u}_{-i}^{\text{OL}}(y_0)) \quad \forall u_i \in \mathcal{S}_\infty^m \end{aligned} \quad (20)$$

where the latter inequality follows from the optimality of u_i^* . In particular, for $x_0 = y_0$, we have

$$V_i(x_0, x_0) \leq J_i^\infty(u_i|x_0, \mathbf{u}_{-i}^{\text{OL}}(x_0)) \quad \forall u_i \in \mathcal{S}_\infty^m. \quad (21)$$

Note that, from Definition 1 for all $u_i \in \mathcal{S}_\infty^m$ and \mathbf{u}^{OL} defined as in (11)

$$J_i^\infty(u_i^{\text{OL}}(x_0)|x_0, \mathbf{u}_{-i}^{\text{OL}}(x_0)) \leq J_i^\infty(u_i|x_0, \mathbf{u}_{-i}^{\text{OL}}(x_0)). \quad (22)$$

Thus, comparing the latter with (21), we conclude that $V_i(x_0, x_0)$ is the infinite-horizon cost-to-go of the OL-NE trajectory \mathbf{u}^{OL} from state x_0

$$V_i(x_0, x_0) = J_i^\infty(u_i^{\text{OL}}(x_0)|x_0, \mathbf{u}_{-i}^{\text{OL}}(x_0)). \quad (23)$$

Finally, we observe from the Bellman optimality principle applied to the system (\hat{A}_i, \hat{B}_i) that

$$V_i(x_0, y_0) = \min_{u_i \in \mathbb{R}^m} \frac{1}{2} (\|x_0\|_{Q_i}^2 + \|u_i\|_{R_i}^2) + V_i \left(Ax_0 + B_i u_i + \sum_{j \neq i} B_j K_j^{\text{OL}} y_0, \bar{A}^{\text{OL}} y_0 \right) \quad (24)$$

and the latter optimization problem has solution $K_i^{\text{LQR}} x_0 + \bar{K}_i y_0$.

C. Receding-Horizon Open-Loop Nash Equilibria

In this section, we construct a finite-horizon problem whose solution is a truncation of the constrained infinite-horizon OL-NE. This result, shown in Theorem 1, generalizes the infinite-horizon optimality property of single-agent MPC [26]. Intuitively, the finite-horizon problem in (25a) is defined by including the infinite-horizon cost-to-go of the OL-NE V_i in (19) as a terminal cost: However, as V_i is defined in an augmented space, special care is needed. We show in Theorem 2 that, with this design choice, the receding-horizon control law derived by the finite-horizon problem makes the origin asymptotically stable. Finally, in Proposition 2, we cast the finite-horizon problem as a VI. Consider the problem, parametrized in $x_0 \in \mathbb{X}$, of finding $\mathbf{u}^{\text{FH}}(x_0)$ such that

$$\forall i \in \mathcal{I} : u_i^{\text{FH}}(x_0) \in \underset{u_i \in \mathcal{U}_{T,i}(x_0, \mathbf{u}^{\text{FH}})}{\text{argmin}} J_i(u_i | x_0, \mathbf{u}^{\text{FH}}) \quad (25a)$$

$$J_i(u_i | x_0, \mathbf{u}^{\text{FH}}) := \sum_{t \in T} (\ell_i(\phi[t|x_0, u_i, \mathbf{u}_{-i}^{\text{FH}}], u_i[t])) + V_i(\phi[T|x_0, u_i, \mathbf{u}_{-i}^{\text{FH}}], \phi[T|x_0, \mathbf{u}^{\text{FH}}]) \quad (25b)$$

where V_i is as in (19). Note that, for each agent i , the optimization problem in (25a) is parametric in the decision variables of all agents \mathbf{u}^{FH} . Interestingly, this structure differs from the one of a static NE problem, where the parametrization is on the decision variables of the *remaining* agents; see, e.g., [32, eq. (1)]. Let \mathbb{X}_f^{OL} be a constraints-admissible forward invariant set for the dynamics $x[t+1] = \bar{A}^{\text{OL}} x[t]$. Techniques for computing \mathbb{X}_f^{OL} can be found in [33] and references therein. Define the set

$$\mathcal{X} := \{x_0 \in \mathbb{X} | \exists \mathbf{u}^{\text{FH}} \text{ that solves (25a); } \phi[T|x_0, \mathbf{u}^{\text{FH}}] \in \mathbb{X}_f^{\text{OL}}\}. \quad (26)$$

We show next that the infinite-horizon constrained OL-NE trajectory can be recovered by the solutions to (25a).

Theorem 1: Let Assumptions 1–3 hold true. Let $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i \in \mathcal{I}}$ solve (8) and let $\mathbf{u}^{\text{OL}}(x) \in \mathcal{S}_\infty^{Nm}$ be the unconstrained OL-NE sequence for any $x \in \mathbb{X}$, as defined in (11). Let $\mathbf{u}^{\text{FH}} \in \mathcal{S}_T^{Nm}$ solve (25a) for some $x_0 \in \mathcal{X}$, with associated state sequence $x^{\text{FH}} := \phi(x_0, \mathbf{u}^{\text{FH}})$, and define the *extended input sequence* $\mathbf{u}^{\text{EX}} \in \mathcal{S}_\infty^{Nm}$ as

$$\forall i : u_i^{\text{EX}}[t] := \begin{cases} u_i^{\text{FH}}[t], & \text{if } t < T \\ u_i^{\text{OL}}[t - T | x^{\text{FH}}[T]], & \text{if } t \geq T. \end{cases} \quad (27)$$

Then, \mathbf{u}^{EX} is an infinite-horizon OL-NE trajectory for the system in (1) with state and input constraint sets $\mathbb{X}, (\mathbb{U}_i)_{i \in \mathcal{I}}$, respectively, and initial state x_0 .

Proof: See Appendix C. ■

In light of Proposition 1, by solving (25a) at subsequent time instants, one expects the agents not to deviate from the previously found solution. This is because they will recover shifted truncations of the same infinite-horizon OL-NE trajectory. Indeed, in Lemma 2, we show that a trajectory solving the problem in (25a) when shifted by one time step still solves the problem in (25a) for the subsequent state. This is crucial because the control action remains the same between subsequent computations of the solution, keeping the evaluated objective constant—except for the first stage cost, which does not appear in the summation. The cumulative objective of the agents decreases then at each time step, and it can be used as a Lyapunov function to show the stability of the origin (modulo some technicalities, due to the fact that Q_i is only positive semidefinite for all i). Let us formalize this next.

Lemma 2: Let Assumptions 1, 2, and 3 hold. Let $x_0 \in \mathcal{X}$, with \mathcal{X} defined in (26). Define the *shifted input sequence* $\mathbf{u}^{\text{SH}} \in \mathcal{S}_T^{Nm}$ as follows:

$$\forall i \in \mathcal{I} : u_i^{\text{SH}}[t] = \begin{cases} u_i^{\text{FH}}[t+1], & \text{if } t < T-1 \\ K_i^{\text{OL}} x^{\text{FH}}[T], & \text{if } t = T-1 \end{cases} \quad (28)$$

where \mathbf{u}^{FH} solves (25a) with initial state x_0 , $x^{\text{FH}} := \phi(x_0, \mathbf{u}^{\text{FH}})$ and $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i \in \mathcal{I}}$ solve (8). Then, \mathbf{u}^{SH} is a solution for the problem in (25a) with initial state $x^{\text{FH}}[1]$.

Proof: See Appendix C. ■

In view of Lemma 2 and given that the problem in (25a) might admit multiple solutions, we assume that the shifted solution in (28) is actually employed when the agents solve subsequent instances of the problem in (25a). Assumption 4 is practically reasonable as, when implementing a solution algorithm for (25a), one can warm-start the algorithm to the shifted sequence \mathbf{u}^{SH} defined in (28).

Assumption 4: For any $x_0 \in \mathbb{X}$, if the problem in (25a) at the initial state x_0 admits a solution \mathbf{u}^{FH} and if the shifted sequence \mathbf{u}^{SH} defined in (28) is a solution of (25a) with initial state $\phi[1|x_0, \mathbf{u}^{\text{FH}}]$, then \mathbf{u}^{SH} is selected by all agents when solving (25a) with initial state $\phi[1|x_0, \mathbf{u}^{\text{FH}}]$.

We are now ready to conclude on asymptotic stability of the system controlled in receding horizon.

Theorem 2: Let Assumptions 1–4 hold true. Consider the system in (1) with feedback control $x \mapsto \mathbf{u}^{\text{FH}}[0|x]$, where $\mathbf{u}^{\text{FH}}(x)$ solves (25a) for the initial state x . The origin is asymptotically stable for the closed-loop system with region of attraction \mathcal{X} , defined in (26).

Proof: See Appendix C. ■

D. OL-NE as a VI

Theorem 1 bridges the infinite-horizon constrained OL-NE trajectory with the solution to a finite-horizon equilibrium problem, whose solutions can be computed algorithmically. In fact,

we recast the problem in (25a) as a VI, for which a plethora of efficient solution algorithms exist under some standard monotonicity and convexity assumptions [28].

Proposition 2: Assume $\mathcal{U}_T(x_0)$ nonempty, closed, and convex for all $x_0 \in \mathbb{X}$. For some $T \in \mathbb{N}$, define for all $i \in \mathcal{I}$

$$\begin{aligned} \Gamma_i &:= \begin{bmatrix} B_i & 0 & \dots & 0 \\ AB_i & B_i & \dots & 0 \\ \vdots & & \ddots & \\ A^{T-1}B_i & A^{T-2}B_i & \dots & B_i \end{bmatrix} \\ \Theta &:= \text{col}(A^k)_{k \in T+} \\ \bar{R}_i &:= I_T \otimes R_i \\ \bar{Q}_i &:= \begin{bmatrix} I_{T-1} \otimes Q_i & 0 \\ 0 & P_i^{\text{OL}} \end{bmatrix} \end{aligned} \quad (29)$$

and define $F(\cdot|x_0) : \mathbb{R}^{TNm} \rightarrow \mathbb{R}^{TNm}$, parametric in $x_0 \in \mathbb{R}^n$, as

$$\begin{aligned} F(\mathbf{u}|x_0) &:= \text{blkdiag}(\bar{R}_i)_{i \in \mathcal{I}} \mathbf{u} \\ &+ \begin{bmatrix} \Gamma_1^\top \bar{Q}_1 \\ \vdots \\ \Gamma_N^\top \bar{Q}_N \end{bmatrix} \begin{bmatrix} \Gamma_1 & \dots & \Gamma_N \end{bmatrix} \mathbf{u} + \begin{bmatrix} \Gamma_1^\top \bar{Q}_1 \\ \vdots \\ \Gamma_N^\top \bar{Q}_N \end{bmatrix} \Theta x_0. \end{aligned} \quad (30)$$

Then, any solution of $\text{VI}(F(\cdot|x_0), \mathcal{U}_T(x_0))$ is a solution to (25a).

Proof: See Appendix C. \blacksquare

It can be shown via the generalized Gerschgorin disk theorem [34] that the VI in Proposition 2 is strongly monotone [28, Def. 2.3.1] if for all i , $R_i = r_i I$, with $r_i > 0$ large enough. A strongly monotone VI admits a unique solution [28, Def. 2.3.3], thus this design choice makes Assumption 4 redundant. The proposed control algorithm is summarized in Algorithm 2. We refer to [35] for a runtime benchmark and implementation of different solution algorithms available to perform step 3.

Remark 1: It is well-known that a static convex game can be cast as a VI defined via the stacked partial gradients of the agents cost functions [27]. We note, however, that the matrix multiplying \mathbf{u} in (30) has nonsymmetric diagonal blocks, since $(P_i^{\text{OL}})_{i \in \mathcal{I}}$ are in general nonsymmetric matrices. This implies that there do not exist N cost functions (one for each block) such that F corresponds to their stacked gradients. Therefore, there is no static game whose NE corresponds to the solution of (25a). This confirms the observation made in Section III-C that the structure of (25a) differs from the one of a static NE problem.

IV. CLOSED-LOOP NASH EQUILIBRIA

We now turn our attention to the CL-NE solution concept, where each agent assumes that the opponents can observe the state at each time step and recompute their input sequence accordingly. The game is defined over the feedback control functions $\sigma_i : \mathbb{N} \times \mathbb{R}^n \rightarrow \mathbb{R}^m$. Denote $\sigma = (\sigma_i)_{i \in \mathcal{I}}$. We overload the notation for the state sequence of the system in (1) controlled

Algorithm 2: Receding Horizon OL-NE Control.

```

1: for  $t \in \mathbb{N}$  do
2:   Measure  $x[t]$ 
3:   Find  $\mathbf{u}^{\text{FH}}$  that solves  $\text{VI}(F(\cdot|x[t]), \mathcal{U}_T(x[t]))$ 
4:   for  $i \in \mathcal{I}$  do
5:     Apply the input  $u_i^{\text{FH}}[0]$ 
6:   end for
7: end for

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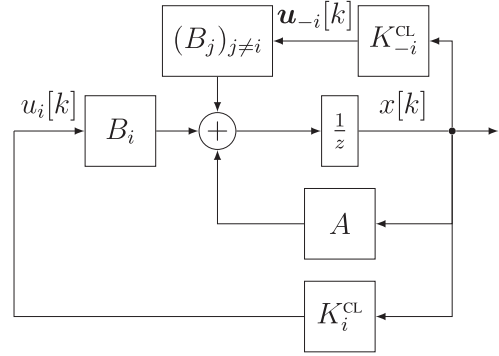


Fig. 2. Block diagram of the CL-NE control problem for the agent i . The linear feedback K_i^{CL} is optimal for the depicted system.

by the feedback law σ as $\phi(x_0, \sigma)$. Furthermore, we denote as $u_i(x_0, \sigma)$ the sequence of inputs for agent $i \in \mathcal{I}$ resulting from σ : that is, for all $t \in \mathcal{T}$

$$u_i[t|x_0, \sigma] = \sigma_i(t, \phi[t|x_0, \sigma]) \quad \forall i \in \mathcal{I}$$

$$\phi[t+1|x_0, \sigma] = A\phi[t|x_0, \sigma] + \sum_{i \in \mathcal{I}} B_i u_i[t|x_0, \sigma].$$

We also overload the notation for the objective in (6) as

$$\begin{aligned} \forall i \in \mathcal{I} : J_i^\infty(\sigma_i|x_0, \sigma_{-i}) \\ = \sum_{t=0}^{\infty} \ell_i(\phi[t|x_0, \sigma], u_i[t|x_0, \sigma]). \end{aligned} \quad (31)$$

We consider the unconstrained case with $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U}_i = \mathbb{R}^m$ for all $i \in \mathcal{I}$.

Definition 2 [29, Def. 6.3]: The feedback strategies $\sigma^* : \mathbb{R}^n \rightarrow \mathbb{R}^{Nm}$ are a CL-NE if for all $x_0 \in \mathbb{R}^n$, for all $i \in \mathcal{I}$

$$J_i^\infty(\sigma_i^*|x_0, \sigma_{-i}^*) \leq \inf_{\sigma_i : \mathbb{R}^n \rightarrow \mathbb{R}^m} J_i^\infty(\sigma_i|x_0, \sigma_{-i}^*). \quad (32)$$

The unconstrained CL-NE problem admits a linear feedback solution. We report a known characterization and stability result for the CL-NE in terms of coupled AREs, which we then employ to design a stabilizing receding-horizon controller in Section IV.

Assumption 5: $(A, \text{row}(B_i)_{i \in \mathcal{I}})$ is stabilizable and $(A, \sum_{i \in \mathcal{I}} Q_i)$ is detectable.

Lemma 3 [8, Cor. 3.3]: Let Assumptions 1(ii) and 5 hold true. Let $(P_i^{\text{CL}}, K_i^{\text{CL}})_{i \in \mathcal{I}}$ solve

$$\forall i \in \mathcal{I} : \begin{cases} P_i^{\text{CL}} = Q_i + (\bar{A}_{-i}^{\text{CL}})^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}} & (33a) \\ K_i^{\text{CL}} = -R_i^{-1} B_i^\top P_i^{\text{CL}} \bar{A}^{\text{CL}} & (33b) \end{cases}$$

with $P_i^{\text{CL}} = (P_i^{\text{CL}})^\top \succ 0$,

$$\begin{aligned}\bar{A}^{\text{CL}} &:= A + \sum_{j \in \mathcal{I}} B_j K_j^{\text{CL}} \\ \bar{A}_{-i}^{\text{CL}} &:= A + \sum_{j \neq i} B_j K_j^{\text{CL}}.\end{aligned}\quad (34)$$

Then, the linear feedback control $\sigma = (K_i^{\text{CL}})_{i \in \mathcal{I}}$ is a CL-NE, the resulting closed-loop dynamics is asymptotically stable and $J_i(\sigma_i | x_0, \sigma_{-i}) = \frac{1}{2} \|x_0\|_{P_i^{\text{CL}}}^2$ for all $i \in \mathcal{I}$ and $x_0 \in \mathbb{R}^n$.

Proof: It follows directly via algebraic calculations from [8, Cor. 3.3]. \blacksquare

If one ignores the dependence of \bar{A}_{-i}^{CL} on K_i^{CL} [which emerges via (33b)], (33a) is a standard Riccati equation that solves the LQR problem with state evolution matrix \bar{A}_{-i}^{CL} . This is schematically represented in Fig. 2. Therefore, one can expect its solution to be symmetric. For this reason, (33) is sometimes referred to in the literature as a symmetric coupled ARE, as opposed to (8), whose solutions are in general not symmetric. As proposed in [14], the CL-NE can either be computed by iteratively fixing $(K_j^{\text{CL}})_{j \neq i}$ for each i and solving (33a) with a Riccati equation solver, or by rewriting (33a) as

$$\begin{aligned}P_i^{\text{CL}} &= Q_i + (\bar{A}^{\text{CL}})^\top P_i^{\text{CL}} \bar{A}^{\text{CL}} - (K_i^{\text{CL}})^\top B_i^\top P_i^{\text{CL}} \bar{A}^{\text{CL}} \\ &\stackrel{(33b)}{=} \underbrace{Q_i + (P_i^{\text{CL}} \bar{A}^{\text{CL}})^\top S_i P_i^{\text{CL}} \bar{A}^{\text{CL}}}_{\tilde{Q}_i} + (\bar{A}^{\text{CL}})^\top P_i^{\text{CL}} \bar{A}^{\text{CL}}\end{aligned}$$

and by solving the latter via a Lyapunov equation solver, considering \tilde{Q}_i fixed at each iteration.

A. Receding-Horizon Closed-Loop Nash Equilibria

In this section, we study the stability of the origin for the system (1) in closed loop with the receding-horizon solution of a finite-horizon CL-NE problem. As a CL-NE is a feedback law valid on the whole state space, one needs not recomputing it at each iteration, and thus, the concept of a receding-horizon CL-NE is counterintuitive. However, one must consider that the state-of-the-art solution algorithm for finite-horizon CL-NE problems [18], which handles state and input constraints, only computes a single trajectory resulting from a CL-NE given an initial state. By receding-horizon CL-NE, we then mean computing at each time step a trajectory resulting from a finite-horizon CL-NE policy, and applying the first input of the sequence. For stability purposes, a natural choice for the terminal cost is $\|\cdot\|_{P_i^{\text{CL}}}^2$, where $(K_i^{\text{CL}}, P_i^{\text{CL}})_{i \in \mathcal{I}}$ solve (33). The finite-horizon CL-NE problem can be written as an NE problem with nested equilibrium constraints parametrized in the initial state [18, Thm. 2.2]

$$\begin{aligned}\forall i : \sigma_{T,i}^*(x[0]) &\in \\ \arg_{u_i[0]} \min_{u_i \in S_T^m} &\frac{1}{2} \|x[T]\|_{P_i^{\text{CL}}}^2 + \sum_{t=0}^{T-1} \ell_i(x[t], u_i[t]) \quad (35a) \\ \text{s.t. } x[t+1] &= Ax[t] + B_i u_i[t] + \sum_{j \in \mathcal{I}_{-i}} B_j \sigma_{T-t,j}^*(x[t]), \quad (35b) \\ &\forall t \in \mathcal{T}.\end{aligned}$$

In the latter, the notation $\arg_{u_i[0]} \min_{u_i \in S_T^m}$ denotes that the minimum is taken over u_i , while only the first element $u_i[0]$ is returned. The equilibrium constraints emerge in (35b), as σ_{T-t}^* is defined by an instance of the problem in (35) over the shortened horizon $T-t$. The equilibrium constraints cannot be expressed explicitly as a constraint set, and thus (35) cannot be directly written as a VI. We then consider a surrogate problem, obtained by substituting $\sigma_{T-t,j}^*$ with the linear feedback K_j^{CL} for all $t \in \mathcal{T}, j \neq i$. The resulting problem only presents standard affine coupling constraints in the game decision variables, which allows for a VI reformulation.

$$\forall i : u_i^*(x[0]) \in \quad (36a)$$

$$\arg \min_{u_i \in S_T^m} \frac{1}{2} \|x[T]\|_{P_i^{\text{CL}}}^2 + \sum_{t=0}^{T-1} \ell_i(x[t], u_i[t]) \quad (36b)$$

$$\text{s.t. } x[1] = Ax[0] + B_i u_i[0] + \sum_{j \neq i} B_j u_j[0] \quad (36c)$$

$$x[t+1] = \bar{A}_{-i}^{\text{CL}} x[t] + B_i u_i[t] \quad \forall t \neq 0. \quad (36d)$$

We remark that, given $x[0]$, (36) is a static NE problem (without equilibrium constraints) and it thus admits a VI reformulation via standard results [32]. Intuitively, this substitution modifies the prediction model of each agent: instead of expecting the remaining agents to apply a finite-horizon CL-NE with shrinking horizon, it is predicted that they will apply the infinite-horizon CL-NE. Next, we show that the choice of prediction model and terminal cost ensures that $(K_i^{\text{CL}} x_0)_{i \in \mathcal{I}}$ is a solution to (36) from any initial state x_0 .

Theorem 3: Let Assumptions 1(ii) and 5 hold. For all $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$, \mathbf{u}^{CL} is a solution to (36) for the initial state x_0 , defined as

$$\begin{aligned}x^{\text{CL}}[t] &:= (\bar{A}^{\text{CL}})^t x_0 \quad \forall t \in \mathcal{T} \\ u_i^{\text{CL}}[t] &:= K_i^{\text{CL}} x^{\text{CL}}[t] \quad \forall t \in \mathcal{T} \forall i \in \mathcal{I}\end{aligned}\quad (37)$$

where $(K_i^{\text{CL}})_{i \in \mathcal{I}}$ solve (33) and \bar{A}^{CL} is in (34).

Proof: See Appendix D. \blacksquare

The stability of the origin under the receding-horizon controller follows then from the one of $(K_i^{\text{CL}})_{i \in \mathcal{I}}$. Note that the results of Theorem 3 trivially hold also when input and state constraints are included, if the initial state is in a constraints-admissible forward invariant set for the autonomous system $x[t+1] = \bar{A}^{\text{CL}} x[t]$, thus still ensuring local asymptotic stability of the origin. This is because, from Theorem (3), the input u_i^* defined in (37) is the minimizer for the i th unconstrained problem in (36) when the other agents apply the input u_{-i}^* . Since u_i^* is constraint-admissible, it is also a minimizer of the constrained problem: note that the introduction of the constraints does not modify the model of the dynamics assumed by the agent i . The same considerations do not hold for the case in (35), because the equilibrium constraints in (35b) are affected by the introduction of state and input constraints, thus \mathbf{u}^* might not satisfy the equation in (35b) even when it is constraint-admissible for the state and input constraints. Intuitively speaking, the feedback $(K_i)_{i \in \mathcal{I}}$ might not be a CL-NE even when state-and-input constraint admissible, as one agent could “take advantage” of

the opponents' constraints by driving the state outside of the constraint-admissible region.

V. APPLICATION EXAMPLES

A. Vehicle Platooning

We consider the vehicle platooning scenario in [36]. The leading vehicle, indexed by 1, aims at reaching a reference speed v^{ref} , while the remaining agents $i \in \{2, \dots, N\}$ aim at matching the speed of the preceding vehicle, while maintaining a desired distance d_i , plus an additional speed-dependent term $h_i v_i$, where v_i is the speed of agent i and h_i is a design parameter. For all $i \neq 1$, the local state is

$$x_i = \begin{bmatrix} p_{i-1} - p_i - d_i - h_i v_i \\ v_{i-1} - v_i \end{bmatrix} \quad (38)$$

where p_i denotes the position of the agent i with respect to the one of agent 1. As the position of the agent 1 with respect to itself is 0, we define $x_1 = [0, v^{\text{ref}} - v_1]^\top$. The dynamics is that of a single integrator and $N - 1$ double integrators sampled with rate $\tau_s = 0.1$ s. With algebraic calculations, we obtain

$$\begin{aligned} A &= \text{blkdiag} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I_{N-1} \otimes \begin{bmatrix} 1 & \tau_s \\ 0 & 1 \end{bmatrix} \right) \\ B_1 &= \delta_2^N \otimes \begin{bmatrix} \tau_s^2/2 \\ \tau_s \end{bmatrix} - \delta_1^N \otimes \begin{bmatrix} 0 \\ \tau_s \end{bmatrix} \\ B_N &= -\delta_N^N \otimes \begin{bmatrix} h_i \tau_s + \tau_s^2/2 \\ \tau_s \end{bmatrix} \\ &\forall i \in \{2, \dots, N-1\} : \\ B_i &= \delta_{i+1}^N \otimes \begin{bmatrix} \tau_s^2/2 \\ \tau_s \end{bmatrix} - \delta_i^N \otimes \begin{bmatrix} h_i \tau_s + \tau_s^2/2 \\ \tau_s \end{bmatrix} \end{aligned} \quad (39)$$

where $\delta_i^n \in \mathbb{R}^n$ is a vector with only nonzero element 1 at index i . We impose the following safety distance, speed and input constraints:

$$\begin{aligned} p_{i-1} &\geq d_i^{\min} + p_i \\ v_i^{\min} &\leq v_i \leq v_i^{\max} \\ u_i^{\min} &\leq u_i \leq u_i^{\max}. \end{aligned} \quad (40)$$

As the system in (39) does not satisfy the stabilizability Assumption 2(ii), we apply to each agent i a prestabilizing local controller

$$K_i^{\text{stab}} = (\delta_i^N)^\top \otimes [-1, -1].$$

We then apply the OL-NE receding-horizon control with state and input weights $Q_i = I$, $R_i = 1$, and horizon $T = 10$. The VI defined in Proposition 2 is solved via the forward-backward splitting method [37, Sect. 12.4.2], see also [38, Algorithm 1] for an implementation in the context of NE problems. This method employs dualization of the constraints that couple the decision variables of each agents (i.e., state constraints and coupling input constraints), while the local input constraints are handled via a

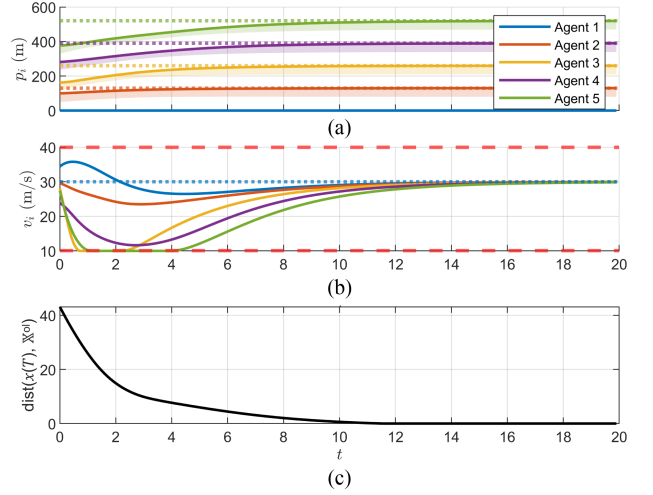


Fig. 3. Vehicle platooning test case. (a) Position with respect to the leading agent. The shaded areas in the first plot represent the distance constraints for each agent. The dotted lines represent the desired values at steady state. (b) Velocity. The dotted lines represent the desired values at steady state. The dashed red lines represent the constraints. (c) Distance of the terminal state to \mathbb{X}_f^{OL} .

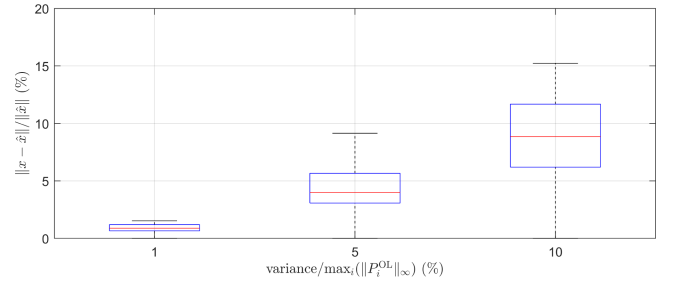


Fig. 4. Deviation between the nominal state sequence \hat{x} and the state sequence x resulting from the receding-horizon solution of (26) when the matrices $(P_i^{\text{OL}})_{i \in \mathcal{I}}$ are perturbed by an additive error sampled from a normal distribution.

projection step. A sample trajectory is shown in Fig. 3, where we observe that the vehicles achieve the desired equilibrium state while satisfying all the constraints. We compute a suitable set \mathbb{X}_f^{OL} by inscribing a level set of a quadratic Lyapunov function of the autonomous system with dynamics \bar{A}^{OL} in the polyhedron defined by (40). As shown in Fig. 3(c), the state enters the set \mathcal{X} defined in (26) after $t = 11.7$ s: At this instant, all conditions of Theorem 2 are satisfied and convergence to the origin is guaranteed. Furthermore, we verify numerically that, for each subsequent time step τ , the input sequences u^{SH} computed at time τ as in (28) are a solution for the game at time step $\tau + 1$, which is to be expected due to Lemma 2. We report that we are currently unable to compute an accurate a priori estimate of the region of attraction, and this test shows that it is larger than the set \mathcal{X} . In addition, we test the robustness of the method to errors in the terminal cost function. We perturb the matrices $(P_i^{\text{OL}})_{i \in \mathcal{I}}$ with an additive error matrix, generated by sampling a normal distribution. We perform 100 state realizations for each tested value of the variance. In Fig. 4, we plot the difference between the resulting state trajectory and the nominal state trajectory \hat{x} . We observe that the problem is robust to small perturbations of the terminal cost function, as the state sequence

deviates by approximately 2% when the variance is 1% of the maximum element of $(P_i^{\text{OL}})_{i \in \mathcal{I}}$. The error increases significantly for increasing error values. We observe that the system remains asymptotically stable in all the tests.

B. Distributed Control of Interconnected Generators

We test the inclusion of the proposed terminal costs on the automatic generation control problem for the power system application considered in [20]. The power system under consideration is composed of four generators interconnected via tie lines arranged in a line graph. The models for the dynamics of the generators and of the tie lines linearized around a steady-state reference are the ones in [20, eq. (17)], with the model parameters specified in [39, Sect. A.1]. This application example is chosen as it is observed in [20] that the distributed MPC control architecture (equivalent to the receding-horizon OL-NE without terminal cost in this article) leads to the system being unstable and thus it is a challenging distributed control problem. The model considered has three states for each generator (namely, the angular velocity of the rotating element, the mechanical power applied to the rotating element, and the position of the steam valve) and one for each tie line (namely, the power flow). Each agent has control authority over the reference point of their respective governor. The control objective for each agent is to regulate the deviation from the reference angular speed of the generator rotating part and power flow at the tie-line they are connected to, with the exception of agent 1 that does not control the tie line. In numbers, we have

$$\begin{aligned} R_i &= 1 \quad \forall i \in \{1, 2, 3, 4\}; \\ Q_1 &= \text{blkdiag}(5, \mathbf{0}_{14 \times 14}) \\ &\quad \forall i \in \{2, 3, 4\}: \\ Q_i &= \begin{bmatrix} \delta_i^4 (\delta_i^4)^\top \otimes \text{diag}(5, 0, 0) & \mathbf{0}_{12 \times 3} \\ \mathbf{0}_{3 \times 12} & 5\delta_{i+1}^3 (\delta_{i+1}^3)^\top \end{bmatrix}. \end{aligned} \quad (41)$$

We observe that the system does not satisfy Assumption 3, thus we could not test the performance of the OL-NE receding-horizon controller. Conversely and as expected from Lemma 3, we find a stabilizing infinite-horizon CL-NE using the iterations in [14, eq. (16)]. We test the receding-horizon CL-NE controller which solves the problem in (36) from a randomly generated initial state, and we compare its stabilizing property with respect to the controller which implements (36) without a terminal cost in Fig. 5. We estimate a constrained admissible forward invariant set of the form

$$\mathbb{X}_f^{\text{CL}} = \{x \in \mathbb{X} \mid \|x\|_P^2 < r\} \quad (42)$$

where P defines a quadratic Lyapunov function for the autonomous system \bar{A}^{CL} defined as in (34) and r is determined numerically such that the controller $(K_i^{\text{CL}})_{i \in \mathcal{I}}$ is feasible. We observe that the system is asymptotically stable when the initial state is in \mathbb{X}_f^{CL} . In general, the inclusion of the terminal cost is beneficial for the asymptotic convergence of the closed-loop system.

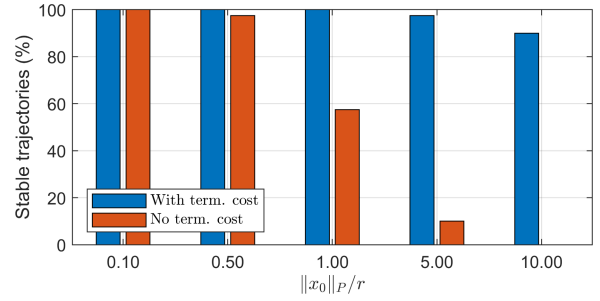


Fig. 5. Percentage of asymptotically stable trajectories for the four-zone power systems in [20] controlled by the receding-horizon CL-NE for different initial states x_0 . P and r define a constraint-admissible forward invariant set as in (42).

VI. CONCLUSION

The constrained infinite-horizon OL-NE can be computed by solving a finite-horizon control problem, if the infinite-horizon cost-to-go over an augmented state-space is used as terminal cost function. The receding-horizon solution is asymptotically stable and the problem can be cast as a VI. Instead, the CL-NE case requires solving a game with nested equilibrium constraints. With an appropriate relaxation, we find a finite-horizon problem whose solution coincides with the infinite-horizon unconstrained CL-NE. Open problems left for future work include developing algorithms with convergence guarantees for computing infinite-horizon unconstrained Nash equilibria, studying the constrained receding-horizon CL-NE case, and determining a characterization of the monotonicity of the finite-horizon variational inequalities with respect to the problem data.

APPENDIX

A. Additional Results and Proofs to Section III-A

Proof of Proposition 1: The proof is based on showing that (8) implies [8, eq. (4.32)], and then, applying [8, Thm. 4.10]. We left-multiply (8a) by $S_i A^{-\top}$ and sum over \mathcal{I} to obtain

$$\begin{aligned} \sum_{j \in \mathcal{I}} S_j A^{-\top} (Q_j - P_j^{\text{OL}}) &= \sum_{j \in \mathcal{I}} -S_j P_j^{\text{OL}} \bar{A}^{\text{OL}} \\ &\stackrel{(8b)}{=} \sum_{j \in \mathcal{I}} B_j K_j^{\text{OL}}. \end{aligned} \quad (43)$$

Thus

$$\bar{A}^{\text{OL}} \stackrel{(9),(43)}{=} A + \sum_{j \in \mathcal{I}} S_j A^{-\top} (Q_j - P_j^{\text{OL}}). \quad (44)$$

By substituting (44) into (8a), we find

$$\begin{aligned} A^{-\top} (Q_i - P_i^{\text{OL}}) &= -P_i^{\text{OL}} (A + \sum_{j \in \mathcal{I}} S_j A^{-\top} (Q_j - P_j^{\text{OL}})) \end{aligned} \quad (45a)$$

$$K_i^{\text{OL}} = -R_i^{-1} B_i^\top P_i^{\text{OL}} (A + \sum_{j \in \mathcal{I}} S_j A^{-\top} (Q_j - P_j^{\text{OL}})). \quad (45b)$$

The latter reads as [8, eq. (4.32)]. From [8, Thm. 4.10], \mathbf{u}^{OL} defined in (11) is an OL-NE and $\lim_{t \rightarrow \infty} x[t|x_0, \mathbf{u}^{\text{OL}}(x_0)] = 0$ for all x_0 , thus implying that \bar{A}^{OL} is Schur. ■

Fact 1: Let $P_i^{\text{OL}}, K_i^{\text{OL}}$ solve (8). Let A be invertible. Then, (12) holds true.

Proof:

$$\bar{A}^{\text{OL}} \stackrel{(9)}{=} A + \sum_{i \in \mathcal{I}} B_i K_i^{\text{OL}} \quad (46a)$$

$$\stackrel{(8b)}{=} A - \sum_{i \in \mathcal{I}} S_i P_i^{\text{OL}} \bar{A}^{\text{OL}} \quad (46b)$$

$$\Leftrightarrow \left(I + \sum_{i \in \mathcal{I}} S_i P_i^{\text{OL}} \right) \bar{A}^{\text{OL}} = A \quad (46c)$$

$$\Leftrightarrow \left(I + \sum_{i \in \mathcal{I}} S_i P_i^{\text{OL}} \right) \bar{A}^{\text{OL}} A^{-1} = I \quad (46d)$$

thus $(I + \sum_{i \in \mathcal{I}} S_i P_i^{\text{OL}})$ is invertible. Left-multiplying (46c) by $(I + \sum_{i \in \mathcal{I}} S_i P_i^{\text{OL}})^{-1}$ shows (12). ■

B. Additional Results and Proofs of Section III-B

Let us present a preliminary result to the proof of Lemma 1.

Lemma 4: Let $(P_i^{\text{OL}}, K_i^{\text{OL}})_{i \in \mathcal{I}}$ solve (8) and let Assumptions 1(ii) and 2 hold true. Then, for each $i \in \mathcal{I}$, $P_i^{\text{OL}} = P_i^{\text{LQR}} + \tilde{P}_i$ where \tilde{P}_i satisfies

$$\tilde{P}_i = (\bar{A}_i^{\text{LQR}})^\top \tilde{P}_i \bar{A}^{\text{OL}} + (\bar{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} \left(\sum_{j \neq i} B_j K_j^{\text{OL}} \right) \quad (47)$$

and $K_i^{\text{OL}} = K_i^{\text{LQR}} + \tilde{K}_i$, where

$$\begin{aligned} \tilde{K}_i &= - (R_i + B_i P_i^{\text{LQR}} B_i)^{-1} \\ &\quad B_i^\top (\tilde{P}_i \bar{A}^{\text{OL}} + P_i^{\text{LQR}} (\sum_{j \neq i} B_j K_j^{\text{OL}})). \end{aligned} \quad (48)$$

Proof: P_i^{LQR} is symmetric [31, Thm. 22.2], thus (13a) can be written as

$$P_i^{\text{LQR}} = Q_i + (\bar{A}_i^{\text{LQR}})^\top P_i^{\text{LQR}} A. \quad (49)$$

Now note

$$\begin{aligned} \bar{A}_i^{\text{LQR}} P_i^{\text{LQR}} B_i K_i^{\text{OL}} &\stackrel{(8b)}{=} -\bar{A}_i^{\text{LQR}} P_i^{\text{LQR}} B_i R_i^{-1} B_i^\top P_i^{\text{OL}} \bar{A}^{\text{OL}} \\ &\stackrel{(13b)}{=} (B_i K_i^{\text{LQR}})^\top P_i^{\text{OL}} \bar{A}^{\text{OL}}. \end{aligned} \quad (50)$$

Substituting $P_i^{\text{OL}} = P_i^{\text{LQR}} + \tilde{P}_i$ into (8a)

$$\begin{aligned} P_i^{\text{LQR}} + \tilde{P}_i &= Q_i + A^\top (P_i^{\text{LQR}} + \tilde{P}_i) \bar{A}^{\text{OL}} \\ &\stackrel{(13c)}{=} Q_i + (\bar{A}_i^{\text{LQR}})^\top (P_i^{\text{LQR}} + \tilde{P}_i) \bar{A}^{\text{OL}} \\ &\quad - (B_i K_i^{\text{LQR}})^\top P_i^{\text{OL}} \bar{A}^{\text{OL}} \\ &\stackrel{(50)}{=} Q_i + (\bar{A}_i^{\text{LQR}})^\top (P_i^{\text{LQR}} + \tilde{P}_i) \bar{A}^{\text{OL}} \\ &\quad - \bar{A}_i^{\text{LQR}} P_i^{\text{LQR}} B_i K_i^{\text{OL}}. \end{aligned}$$

Subtract (49) from the latter to obtain (47). Let us rewrite (8b) and (13b), respectively, as

$$(R_i + B_i^\top P_i^{\text{OL}} B_i) K_i^{\text{OL}} = -B_i^\top P_i^{\text{OL}} (A + \sum_{j \neq i} B_j K_j^{\text{OL}}); \quad (51)$$

$$(R_i + B_i^\top P_i^{\text{LQR}} B_i) K_i^{\text{LQR}} = -B_i^\top P_i^{\text{LQR}} A. \quad (52)$$

Substitute $P_i^{\text{OL}} = P_i^{\text{LQR}} + \tilde{P}_i$ into (51)

$$\begin{aligned} (R_i + B_i^\top P_i^{\text{LQR}} B_i) K_i^{\text{OL}} \\ = -B_i^\top (P_i^{\text{OL}} (A + \sum_{j \neq i} B_j K_j^{\text{OL}}) + \tilde{P}_i B_i K_i^{\text{OL}}) \end{aligned}$$

which reads as (48) after subtracting (52). ■

Proof of Lemma 1: By the Schur property of \bar{A}^{OL} , it follows that any control action that stabilizes (A, B_i) is stabilizing for (\hat{A}_i, \hat{B}_i) for all i , thus the system in (14a) is stabilizable. Let

$$\hat{C}_i := \begin{bmatrix} C_i & 0 \end{bmatrix}$$

where C_i is as in Assumption 2. Clearly, $\hat{Q}_i = \hat{C}_i^\top \hat{C}_i$. Let $\hat{A}_i z = \lambda z$ for some $\lambda \geq 1$ and $\hat{C}_i z = 0$, that is, let z an unstable unobservable mode of \hat{A}_i . From the stability of \bar{A}^{OL} , it has to be $z = [x^\top, 0^\top]^\top$ for some $x \in \mathbb{R}^n$. Then, $\hat{A}_i z = Ax$ and it has to be $x = 0$ by the detectability of (A, C_i) . Consequently, (\hat{A}_i, \hat{C}_i) is detectable. Following [40, Cor. 13.8], (15a) admits a unique positive semidefinite solution and the corresponding controller is stabilizing. Consider the partition

$$\hat{P}_i = \begin{bmatrix} \hat{P}_i^{1,1} & \hat{P}_i^{1,2} \\ \hat{P}_i^{2,1} & \hat{P}_i^{2,2} \end{bmatrix}; \quad \hat{K}_i = [\hat{K}_i^1, \hat{K}_i^2].$$

By expanding (15) for the blocks $\hat{P}_i^{1,1}, \hat{P}_i^{2,1}$, and \hat{K}_i^1 , one obtains via straightforward calculations

$$\hat{P}_i^{1,1} = Q_i + A^\top \hat{P}_i^{1,1} (A + B_i \hat{K}_i^1) \quad (53a)$$

$$\hat{K}_i^1 = - (R_i + B_i^\top \hat{P}_i^{1,1} B_i)^{-1} B_i^\top \hat{P}_i^{1,1} A \quad (53b)$$

$$\begin{aligned} \hat{P}_i^{2,1} &= \left(\sum_{j \neq i} B_j K_j^{\text{OL}} \right)^\top \hat{P}_i^{1,1} (A + B_i \hat{K}_i^1) \\ &\quad + (\bar{A}^{\text{OL}})^\top \hat{P}_i^{2,1} (A + B_i \hat{K}_i^1). \end{aligned} \quad (53c)$$

We note that (53a) and (53b) have the same expression as (13a) and (13b), respectively. From $\hat{P}_i \succcurlyeq 0$, $\hat{P}_i^{1,1} \succcurlyeq 0$. Thus, $\hat{P}_i^{1,1}$ is the unique positive semidefinite solution of (13a) and $\hat{P}_i^{1,1} = P_i^{\text{LQR}}, \hat{K}_i^1 = K_i^{\text{LQR}}$. Thus, by substituting P_i^{LQR} and K_i^{LQR} into (53c), one obtains that $(\hat{P}_i^{2,1})^\top$ must satisfy (47). As (47) is a Stein equation in \tilde{P}_i , its solution is unique following the Schur property of \bar{A}_i^{LQR} and \bar{A}^{OL} [41, Lemma 2.1]. Thus, $(\hat{P}_i^{2,1})^\top = \tilde{P}_i$. ■

C. Proofs of Sections III-C and III-D

Proof of Theorem 1: By the invariance of $\mathbb{X}_f^{\text{OL}}, \mathbf{u}^{\text{EX}}$ is feasible. To prove it is an OL-NE trajectory, we proceed by

contradiction. Assume there exists $v_i \in \mathcal{U}_{\infty,i}(x_0, \mathbf{u}_{-i}^{\text{EX}})$ such that

$$J_i^\infty(v_i|x_0, \mathbf{u}_{-i}^{\text{EX}}) < J_i^\infty(u_i^{\text{EX}}|x_0, \mathbf{u}_{-i}^{\text{EX}}) \quad (54)$$

where J_i^∞ is defined in (6). Let us substitute the definition (27) of \mathbf{u}^{EX} into (6)

$$\begin{aligned} J_i^\infty(u_i^{\text{EX}}|x_0, \mathbf{u}_{-i}^{\text{EX}}) &= J_i^\infty(u_i^{\text{OL}}(x^{\text{FH}}[T])|x^{\text{FH}}[T], \mathbf{u}_{-i}^{\text{OL}}(x^{\text{FH}}[T])) \\ &\quad + \sum_{t=0}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\stackrel{(23)}{=} V_i(x^{\text{FH}}[T], x^{\text{FH}}[T]) + \sum_{t=0}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\stackrel{(25b)}{=} J_i(u_i^{\text{FH}}|x_0, \mathbf{u}^{\text{FH}}). \end{aligned}$$

Denote the state sequence $x_v = \phi(x_0, v_i, \mathbf{u}_{-i}^{\text{EX}})$: from the definition of (6)

$$\begin{aligned} J_i^\infty(v_i|x_0, \mathbf{u}_{-i}^{\text{EX}}) &= J_i^\infty(v_i|x_v[T], \mathbf{u}_{-i}^{\text{OL}}(x^{\text{FH}}[T])) \\ &\quad + \sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\stackrel{(20)}{\geq} V_i(x_v[T], x^{\text{FH}}[T]) + \sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\stackrel{(25b)}{=} J_i(v_i|x_0, \mathbf{u}^{\text{FH}}). \end{aligned}$$

Thus, (54) contradicts the definition of \mathbf{u}^{FH} in (25a). It follows that \mathbf{u}^{EX} is an OL-NE for all $x_0 \in \mathcal{X}$. ■

Proof of Lemma 2: We first note that, for any x_0 , by evaluating (24) at the optimal input $u_i = (K_i^{\text{LQR}} + \tilde{K}_i)x_0$ for a generic $i \in \mathcal{I}$ and substituting $K_i^{\text{LQR}} + \tilde{K}_i = K_i^{\text{OL}}$ (Lemma 4), we obtain

$$\begin{aligned} V_i(x_0, x_0) &= \ell_i(x_0, K_i^{\text{OL}}x_0) \\ &\quad + V_i(Ax_0 + \sum_{j \in \mathcal{I}} B_j K_j^{\text{OL}}x_0, \bar{A}^{\text{OL}}x_0) \\ &= \ell_i(x_0, K_i^{\text{OL}}x_0) + V_i(\bar{A}^{\text{OL}}x_0, \bar{A}^{\text{OL}}x_0). \end{aligned} \quad (55)$$

We proceed by contradiction and thus assume for some $v_i \in \mathcal{U}_{i,T}(x^{\text{FH}}[1], \mathbf{u}_{-i}^{\text{SH}})$

$$J_i(v_i|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}) < J_i(u_i^{\text{SH}}|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}) \quad (56)$$

where J_i is defined in (25b). Denote $x_v = \phi(x^{\text{FH}}[1], v_i, \mathbf{u}_{-i}^{\text{SH}})$. Define the auxiliary sequence

$$\hat{v}_i[t] := \begin{cases} u_i^{\text{FH}}[t], & \text{if } t = 0 \\ v_i[t-1], & \text{if } t \in \{1, \dots, T-1\}. \end{cases} \quad (57)$$

One can easily verify that

$$\phi[t+1|x_0, \hat{v}_i, \mathbf{u}_{-i}^{\text{FH}}] = x_v[t] \forall t \in \mathcal{T}. \quad (58)$$

As $v_i \in \mathcal{U}_{i,T}(x^{\text{FH}}[1], \mathbf{u}_{-i}^{\text{SH}})$, clearly \hat{v}_i is also feasible, that is, $\hat{v}_i \in \mathcal{U}_{i,T}(x_0, \mathbf{u}_{-i}^{\text{FH}})$. By expanding (25b)

$$\begin{aligned} &J_i(v_i|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}) \\ &= V_i(x_v[T], \phi[T|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}]) + \sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\stackrel{(28)}{=} V_i(x_v[T], \bar{A}^{\text{OL}}x^{\text{FH}}[T]) + \sum_{t=0}^{T-1} \ell_i(x_v[t], v_i[t]) \\ &\stackrel{(24)}{\geq} V_i(x_v[T-1], x^{\text{FH}}[T]) + \sum_{t=0}^{T-2} \ell_i(x_v[t], v_i[t]) \\ &\stackrel{(58)}{=} V_i(\phi[T|x_0, \hat{v}_i, \mathbf{u}_{-i}^{\text{FH}}], x^{\text{FH}}[T]) \\ &\quad + \sum_{t=1}^{T-1} \ell_i(\phi[t|x_0, \hat{v}_i, \mathbf{u}_{-i}^{\text{FH}}], \hat{v}_i[t]) \\ &\stackrel{(25b)}{=} J_i(\hat{v}_i|x_0, \mathbf{u}^{\text{FH}}) - \ell_i(x_0, u_i^{\text{FH}}[0]). \end{aligned} \quad (59)$$

From (25b) and the definition of \mathbf{u}^{SH} in (28)

$$\begin{aligned} J_i(u_i^{\text{SH}}|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}) &= \sum_{t=1}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\quad + \ell_i(x^{\text{FH}}[T], K_i^{\text{OL}}x^{\text{FH}}[T]) \\ &\quad + V_i(\bar{A}^{\text{OL}}x^{\text{FH}}[T], \bar{A}^{\text{OL}}x^{\text{FH}}[T]) \\ &\stackrel{(55)}{=} V_i(x^{\text{FH}}[T], x^{\text{FH}}[T]) + \sum_{t=1}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \\ &\stackrel{(25b)}{=} J_i(u_i^{\text{FH}}|x_0, \mathbf{u}^{\text{FH}}) - \ell_i(x_0, u_i^{\text{FH}}[0]). \end{aligned} \quad (60)$$

By substituting (59) and (60) into (56), one obtains

$$J_i(\hat{v}_i|x_0, \mathbf{u}^{\text{FH}}) < J_i(u_i^{\text{FH}}|x_0, \mathbf{u}^{\text{FH}})$$

which contradicts \mathbf{u}^{FH} being a solution of (25a). ■

Proof of Theorem 2: Let $\bar{C} := \text{col}(C_i)_{i \in \mathcal{I}}$, where C_i is defined in Assumption 2. Clearly, (A, \bar{C}) is detectable by the Hautus lemma and Assumption 2(ii). (A, \bar{C}) is then uniformly input/output-to-state-stable [42, Def. 2.22], [43, Prop. 3.3]. Furthermore, there exists L such that $A + L\bar{C}$ is Schur. Then, there exists P_L that solves the Lyapunov equation

$$P_L - (A + L\bar{C})^\top P_L (A + LC) = I.$$

For some $\gamma_x, \gamma_y, \gamma_u > 0$ and for all x , it holds that

$$\begin{aligned} &\|Ax + \sum_{i \in \mathcal{I}} B_i u_i\|_{P_L}^2 - \|x\|_{P_L}^2 \\ &\leq -\gamma_x \|x\|^2 + \gamma_y \|\bar{C}x\|^2 + \gamma_u \|\mathbf{u}\|^2. \end{aligned} \quad (61)$$

The proof of the latter inequality can be found in [43, Sect. 3.2] and it is thus omitted. From (61)

$$\begin{aligned} & \|Ax + \sum_{i \in \mathcal{I}} B_i u_i\|_{P_L}^2 - \|x\|_{P_L}^2 \\ & \leq -\gamma_x \|x\|^2 + \sum_{i \in \mathcal{I}} \gamma_y (\|x\|_{Q_i}^2 + \gamma_u \|u_i\|^2) \end{aligned} \quad (62a)$$

$$\leq -\gamma_x \|x\|^2 + \bar{\gamma} \sum_{i \in \mathcal{I}} (\|x\|_{Q_i}^2 + \|u_i\|_{R_i}^2) \quad (62b)$$

where (62a) follows from $\|\bar{C}x\|^2 = \sum_i \|x\|_{Q_i}^2$ and (62b) follows from the equivalence of norms and by taking $\bar{\gamma}$ as the maximum multiplicative constant. From the direct application of [42, Thm. B.53], there exists a continuous Λ , continuous, increasing, unbounded α_1, α_2 , and a positive definite ρ such that

$$\alpha_1(\|x\|) \leq \Lambda(x) \leq \alpha_2(\|x\|) \quad (63a)$$

$$\begin{aligned} & \Lambda(Ax + \sum_{i \in \mathcal{I}} B_i u_i) - \Lambda(x) \\ & \leq -\rho(\|x\|) + \sum_{i \in \mathcal{I}} (\|x\|_{Q_i}^2 + \|u_i\|_{R_i}^2). \end{aligned} \quad (63b)$$

Consider the candidate Lyapunov function

$$V(x) = \Lambda(x) + \sum_{i \in \mathcal{I}} J_i(u_i^{\text{FH}}(x)|x, \mathbf{u}^{\text{FH}}(x)) \quad (64)$$

where $\mathbf{u}^{\text{FH}}(x)$ solves (25a) for the state x . Let $x^{\text{FH}} = \phi(x, \mathbf{u}^{\text{FH}})$. Denote the *shifted sequence* \mathbf{u}^{SH} as in (28), and recall that \mathbf{u}^{SH} solves (25a) for the state $x^{\text{FH}}[1]$ following Lemma 2 and Assumption 4. Furthermore, note that

$$\phi[t|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}] = x^{\text{FH}}[t+1] \quad \forall t \in \mathcal{T}. \quad (65)$$

Then

$$\begin{aligned} & V(x^{\text{FH}}[1]) - \Lambda(x^{\text{FH}}[1]) \stackrel{(64)}{=} \sum_{i \in \mathcal{I}} J_i(u_i^{\text{SH}}|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}) \\ & \stackrel{(25b), (28)}{=} \sum_{i \in \mathcal{I}} \left(V_i(\bar{A}^{\text{OL}} x^{\text{FH}}[T], \bar{A}^{\text{OL}} x^{\text{FH}}[T]) \right. \\ & \quad + \ell_i(x^{\text{FH}}[T], K_i^{\text{OL}} x^{\text{FH}}[T]) \\ & \quad \left. + \sum_{t=0}^{T-2} \ell_i(\phi[t|x^{\text{FH}}[1], \mathbf{u}^{\text{SH}}], u_i^{\text{FH}}[t+1]) \right) \\ & \stackrel{(55), (65)}{=} \sum_{i \in \mathcal{I}} \left(V_i(x^{\text{FH}}[T], x^{\text{FH}}[T]) + \sum_{t=1}^{T-1} \ell_i(x^{\text{FH}}[t], u_i^{\text{FH}}[t]) \right) \\ & \stackrel{(25b)}{=} \sum_{i \in \mathcal{I}} (J_i(u_i^{\text{FH}}|x, \mathbf{u}^{\text{FH}}) - \ell_i(x, u_i^{\text{FH}}[0])) \\ & \stackrel{(64)}{=} V(x) - \Lambda(x) - \sum_{i \in \mathcal{I}} (\ell_i(x, u_i^{\text{FH}}[0])). \end{aligned} \quad (66)$$

We rearrange (66) and substitute (3) to obtain

$$\begin{aligned} & V(x^{\text{FH}}[1]) - V(x) \\ & = \Lambda(x^{\text{FH}}[1]) - \Lambda(x) - \sum_{i \in \mathcal{I}} \|x\|_{Q_i}^2 + \|u_i^{\text{FH}}[0]\|_{R_i}^2 \\ & \stackrel{(63b)}{\leq} -\rho(\|x\|). \end{aligned} \quad (67)$$

From the invariance of \mathbb{X}_f^{OL} , $\bar{A}^{\text{OL}} x \in \mathbb{X}_f^{\text{OL}}$. As $\bar{A}^{\text{OL}} x$ is the terminal state for \mathbf{u}^{SH} with initial state $x^{\text{FH}}[1]$, we obtain from the definition of \mathcal{X} that \mathcal{X} is an invariant set for the closed-loop system. We conclude that the closed-loop system is asymptotically stable with region of attraction \mathcal{X} [42, Thm. B.13] ■

Proof of Proposition 2: For this proof, we treat finite sequences as column vectors, that is, $u_i = \text{col}(u_i[t])_{t \in \mathcal{T}}$. Let \mathbf{u}^* solve $\text{VI}(F(\cdot|x_0), \mathbf{U}_T(x_0))$. Consider the matrix Γ_i , defined in (29): We can rewrite the state evolution as

$$\begin{bmatrix} \phi[1|x_0, u_i, \mathbf{u}_{-i}^*] \\ \vdots \\ \phi[T|x_0, u_i, \mathbf{u}_{-i}^*] \end{bmatrix} = \Theta x_0 + \Gamma_i u_i + \sum_{j \neq i} \Gamma_j u_j^*. \quad (68)$$

Denote the partitions

$$\Gamma_i = \begin{bmatrix} \bar{\Gamma}_i \\ \underline{\Gamma}_i \end{bmatrix}, \quad \Theta = \begin{bmatrix} \bar{\Theta} \\ A^T \end{bmatrix}$$

where $\underline{\Gamma}_i$ is the last block row, that is,

$$\underline{\Gamma}_i := \begin{bmatrix} A^{T-1} B_i & A^{T-2} B_i & \dots & B_i \end{bmatrix}.$$

Using (68), we find the following expression for the terminal states appearing in (25b):

$$\begin{aligned} & \phi[T|x_0, \mathbf{u}^*] = A^T x_0 + \underline{\Gamma}_i u_i^* + \sum_{j \neq i} \underline{\Gamma}_j u_j^* \\ & \phi[T|x_0, u_i, \mathbf{u}_{-i}^*] = A^T x_0 + \underline{\Gamma}_i u_i + \sum_{j \neq i} \underline{\Gamma}_j u_j^*. \end{aligned} \quad (69)$$

Substituting (68) and (69) into (25b), one obtains with straightforward calculations

$$\begin{aligned} J_i(u_i|x_0, \mathbf{u}^*) & = u_i^\top \underline{\Gamma}_i^\top \left(\frac{1}{2} P_i^{\text{LQR}} \underline{\Gamma}_i u_i + \tilde{P}_i \underline{\Gamma}_i u_i^* \right) \\ & \quad + u_i^\top \underline{\Gamma}_i^\top P_i^{\text{OL}} (A^T x_0 + \sum_{j \neq i} \underline{\Gamma}_j u_j^*) \\ & \quad + \frac{1}{2} u_i^\top (\bar{R}_i + \bar{\Gamma}_i^\top (I_{T-1} \otimes Q_i) \bar{\Gamma}_i) u_i \\ & \quad + u_i^\top \bar{\Gamma}_i^\top (I_{T-1} \otimes Q_i) (\bar{\Theta} x_0 + \sum_{j \neq i} \bar{\Gamma}_j u_j^*) \\ & \quad + f(x_0, \mathbf{u}^*) \end{aligned} \quad (70)$$

where f contains the terms that do not depend on u_i . By deriving the latter with respect to u_i

$$\begin{aligned} \nabla J_i(u_i|x_0, \mathbf{u}^*) & = \underline{\Gamma}_i^\top (P_i^{\text{LQR}} \underline{\Gamma}_i u_i + \tilde{P}_i \underline{\Gamma}_i u_i^*) \\ & \quad + \underline{\Gamma}_i^\top P_i^{\text{OL}} (A^T x_0 + \sum_{j \neq i} \underline{\Gamma}_j u_j^*) \\ & \quad + (\bar{R}_i + \bar{\Gamma}_i^\top (I_{T-1} \otimes Q_i) \bar{\Gamma}_i) u_i \\ & \quad + \bar{\Gamma}_i^\top (I_{T-1} \otimes Q_i) (\bar{\Theta} x_0 + \sum_{j \neq i} \bar{\Gamma}_j u_j^*). \end{aligned} \quad (71)$$

By evaluating the latter at u_i^* and by substituting $P_i^{\text{LQR}} + \tilde{P}_i = P_i^{\text{OL}}$ (Lemma 1), it can be verified that

$$F(\mathbf{u}^*|x_0) = \text{col}(\nabla J_i(u_i^*|x_0, \mathbf{u}^*))_{i \in \mathcal{I}}.$$

Furthermore, $J_i(\cdot|x_0, \mathbf{u}^*)$ is convex for every x_0 , because $P_i^{\text{LQR}} \succcurlyeq 0$, $Q_i \succcurlyeq 0$, $R_i \succ 0$. Thus, by the definition of VI solution, for any $u_i \in \mathcal{U}_{i,T}(x_0, \mathbf{u}_{-i}^*)$,

$$\begin{aligned} 0 &\leq F(\mathbf{u}^*|x_0)^\top (\text{col}(u_i, \mathbf{u}_{-i}^*) - \mathbf{u}^*) \\ &= \nabla J_i(u_i^*|x_0, \mathbf{u}^*)^\top (u_i - u_i^*) \\ &\leq J_i(u_i|x_0, \mathbf{u}^*) - J_i(u_i^*|x_0, \mathbf{u}^*) \end{aligned}$$

that is, \mathbf{u}^* solves the problem in (25a). \blacksquare

D. Additional Results and Proofs of Section IV

Before proving Theorem 3, let us show that the CL-NE satisfies an optimality principle.

Lemma 5: Let $(P_i^{\text{CL}}, K_i^{\text{CL}})_{i \in \mathcal{I}}$ satisfy (33). Then, for all $i \in \mathcal{I}$

$$\frac{1}{2} \|x\|_{P_i^{\text{CL}}}^2 = \min_{u_i \in \mathbb{R}^m} \ell_i(x, u_i) + \frac{1}{2} \|\bar{A}_{-i}^{\text{CL}} x + B_i u_i\|_{P_i^{\text{CL}}}^2 \quad (72)$$

and the minimum is achieved by $K_i^{\text{CL}} x$.

Proof: Let us rewrite (33b) as

$$K_i^{\text{CL}} = -(R_i + B_i^\top P_i^{\text{CL}} B_i)^{-1} B_i^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}}.$$

By setting the gradient of (72) to 0 and comparing the resulting equation with the latter, one can see that the minimum is achieved by $K_i^{\text{CL}} x$. By substituting $u_i = K_i^{\text{CL}} x$ into (72), the minimum of (72) is

$$\begin{aligned} \|x\|_{Q_i}^2 + (K_i^{\text{CL}} x)^\top R_i K_i^{\text{CL}} x + (\bar{A}_{-i}^{\text{CL}} x)^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}} x &\stackrel{(33b)}{=} \\ \|x\|_{Q_i}^2 - (K_i^{\text{CL}} x)^\top B_i^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}} x + (\bar{A}_{-i}^{\text{CL}} x)^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}} x &= \\ \|x\|_{Q_i}^2 + x^\top (\bar{A}_{-i}^{\text{CL}} - B_i K_i^{\text{CL}})^\top P_i^{\text{CL}} \bar{A}_{-i}^{\text{CL}} x &\stackrel{(33a)}{=} \|x\|_{P_i^{\text{CL}}}^2. \end{aligned}$$

Proof of Theorem 3: We rewrite the minimization in (36a) as

$$\begin{aligned} &\min_{u_i \in \mathcal{S}_{T-1}^m} \frac{1}{2} \left\{ \sum_{\tau=0}^{T-2} \|x[\tau]\|_{Q_i}^2 + \|u_i[\tau]\|_{R_i}^2 \right. \\ &\left. + \min_{u_i[T-1] \in \mathbb{R}^m} \left(\|x[T-1]\|_{Q_i}^2 + \|u_i[T-1]\|_{R_i}^2 + \|x[T]\|_{P_i^{\text{CL}}}^2 \right) \right\}. \end{aligned} \quad (73)$$

By substituting the constraint (36c) into (73) and by applying Lemma 5, the inner minimization in (73) is solved by $u_i^*[T-1] = K_i^{\text{CL}} x[T-1]$. Substituting (72) into the latter, we obtain

$$\min_{u_i \in \mathcal{S}_{T-1}^m} \frac{1}{2} \left(\|x[T-1]\|_{P_i^{\text{CL}}}^2 + \sum_{\tau=0}^{T-2} \|x[\tau]\|_{Q_i}^2 + \|u_i[\tau]\|_{R_i}^2 \right).$$

By repeating the reasoning backwards in time and substituting the constraint (36b), (36a) becomes

$$\min_{u_i[0] \in \mathbb{R}^m} \frac{1}{2} \left(\|x[0]\|_{Q_i}^2 + \|u_i[0]\|_{R_i}^2 + \left\| Ax[0] + \sum_{j \in \mathcal{I}} B_j u_j[0] \right\|_{P_i^{\text{CL}}}^2 \right). \quad (74)$$

If $u_j[0] = K_j^{\text{CL}} x[0]$ for all $j \in \mathcal{I}_{-i}$, the minimum is obtained by $u_i^*[0] = K_i^{\text{CL}} x[0]$ following Lemma 5. Thus, \mathbf{u}^* is an NE. \blacksquare

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