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Publication date
2020

Document Version
Final published version

Published in
Adaptive Optics

New dimension for the phase retrieval problem

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Abstract: We consider the extension of the traditional projection-based phase retrieval algorithms by increasing the problem dimensionality and introducing novel projection operators. The approach is demonstrated on an example of phase retrieval for the high-NA case.

1. Phase retrieval in optics

In optics, the phase retrieval (PR) problem is a general problem of getting the wavefront aberration information from the point-spread functions (PSF) of an optical system. For low values of numerical aperture (NA), the PSF $I(\mathbf{u})$ can be related to the wavefront aberration $\varphi(\mathbf{x})$ as

$$I(\mathbf{u}) = \left| \mathscr{F}\left(a(\mathbf{x})\mathrm{e}^{\mathrm{i}\varphi(\mathbf{x})}\right)\right|^2, \tag{1}$$

where $\mathbf{x}, \mathbf{u} \in \mathbb{R}^2$ are the coordinates in the pupil and focal planes respectively, $I(\mathbf{u})$ is the intensity of the optical field in the focal plane, $a(\mathbf{x}), \varphi(\mathbf{x})$ are the amplitude and phase of the collimated beam in the pupil plane, and $\mathscr{F}$ is the (two-dimensional continuous) Fourier transform.

The problem has been addressed in a large volume of publications and several algorithms have been proposed to solve it (see, for instance, [1–3] for the review).

From mathematical point of view, for low NA case, the PR problem can be formulated as getting the phase of a 2D signal (of dimension $I \times J$, say), when only the amplitudes of the signal in the spatial and Fourier domain are known:

$$\text{find } x, X \in \mathbb{C}^{I\times J}: \quad \begin{cases} X &= \mathscr{F}_2 x \\ |x| &= a \\ |X| &= b \end{cases}, \tag{2}$$

for some $a, b \in \mathbb{R}^{I\times J}$, and $\mathscr{F}_2$ denoting the 2-dimensional discrete Fourier Transform (DFT). This can be shown to be a feasibility problem [1], *i.e.* the problem of finding a point x belonging to the intersection of two sets $A \subset \mathbb{C}^{I\times J}$ and $B \subset \mathbb{C}^{I\times J}$ defined as

$$A = \{x : |x| = a\}, \quad \text{and} \quad B = \{x = \mathscr{F}_2^{-1} X : |X| = b\}, \tag{3}$$

if such point exists, or to find a point x in one set closest in some sense to the second set, if the sets do not intersect.

The problem can be solved, for instance, by alternating projections (AP) on sets A and B (also referred to as alternating minimisation, error reduction) or by Fienup variant (hybrid input-output). For example, as sets A and B are both the Cartesian products of circles, the alternating operation of "resetting the amplitude and keeping the phase" of the Gerchberg-Saxton algorithm can be seen as alternating projections on sets A and B:

$$x^{n+1} = \mathscr{P}_A \mathscr{P}_B x^n, \tag{4}$$

where x^n is n-th approximation to the problem solution. The convergence of the sequence was analysed in the literature [1,4], and the proof actively uses the properties of the two-dimensional DFT and projection operators.

Obviously, as the alternative projection framework does not depends on the problem dimensionality, the phase retrieval invites for the extension to the cases of higher dimensions. Here we consider two possible ways to do it.

2. Extending the dimensionality of the problem

As it was noted already more than 50 years ago [5], Eq. (1) can be extended to a three-dimensional case by considering $\mathbf{u}, \mathbf{x} \in \mathbb{R}^3$ being 3D coordinates in the areas near the exit pupil and focal plane, with 3D Fourier transform $\mathscr{F}_3$ relating the 3D spectrum of the optical field to the 3D PSF (see Fig. 1). While this approach provides insight about physics of the problem, can be used for the simulation of the vector light [6], and has the advantage of direct application of the developed 2D PR techniques, it would be very inefficient in practice, as it would require the information about the intensity at every voxel of the 3D cube near the focal plane, which does increase the computational costs significantly. However, using additional constraint on the support of the optical spectrum for the monochromatic light, the algorithm can be simplified to the use of the intensity values in only two "slices" around the focal plane (as it is formulated in the classical case of the phase-diverse PR problem).

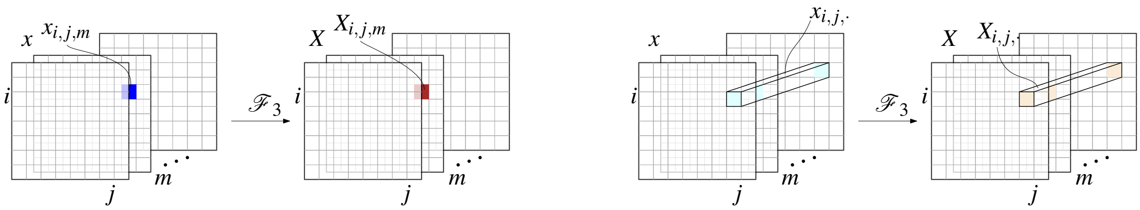

Fig. 1. Coherent (left) and incoherent (right) 3D formulation of the phase retrieval problem. Complex 3D arrays x and X are related by 3D DFT, $X = \mathscr{F}_3 x$. Amplitude of x is known in every point, and is given by a real 3D array a, $|x| = a$. For the coherent case, the amplitude of X should be known in every point too, $|X| = b$; in the incoherent case it is less strictly constrained, and only Euclidean length of its fibre along index m is known $\|X_{i,j,\cdot}\| = \sqrt{I_{i,j}}$. This is compensated by an additional constraint on the constant phase of fibre of x along index m: $\arg x_{i,j,1} = \ldots = \arg x_{i,j,M}$.

Here, we propose another way to exploit the 3D formulation of the PR problem, which is based on the incoherent sum of several 2D PSFs. For instance, for the diffraction taking into account the vectorial nature of light, like a PSF model for a high-NA lens [7]:

$$I(\mathbf{u}) = \sum_{m=1}^{6} \left| \mathscr{F}_2 \left(a(\mathbf{x}) E_m(\mathbf{x}) \mathrm{e}^{\mathrm{i}\varphi(\mathbf{x})} \right) \right|^2, \tag{5}$$

where $E_m(\mathbf{x})$ are known polarisation-dependent aperture modulations; or when a significant level of approximately constant, but unknown background illumination b is present in the PSF measurements:

$$I(\mathbf{u}) = \left| \mathscr{F}_2 \left(a(\mathbf{x}) \mathrm{e}^{\mathrm{i}\varphi(\mathbf{x})} \right) \right|^2 + b = \left| \mathscr{F}_2 \left(a(\mathbf{x}) \mathrm{e}^{\mathrm{i}\varphi(\mathbf{x})} \right) \right|^2 + \left| \mathscr{F}_2 \left(\sqrt{b}\, \delta(\mathbf{x}) \mathrm{e}^{\mathrm{i}\varphi(0)} \right) \right|^2. \tag{6}$$

Here, using the Parseval theorem, the sum along the third dimension of the squares of the components of the 2F Fourier transform can be presented as sum of the squares of the 3D transform (see Fig. 1 for details), and the incoherent sum of PSFs can be presented in a similar to Eq. (2) way using the 3D DFT and other constraints. By adjusting the corresponding projection operator (see [7] for the technical details), *any PR algorithm using AP framework* can be used now for solving Eq. (5) and Eq. (6). Moreover, both described generalisations can be combined in one 4D problem in a natural way, requiring almost no further adjustment, and resulting in the whole class of phase-diverse PR algorithms for high NA values.

3. Conclusion

We have shown two ways of generalising of the 2D PR problem to higher dimensions. The novel incoherent 3D formulation brings the plethora of well-developed tools for other application scenarios, like vectorial PR problem.

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