Models in Topology

Berd van der Steeg

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On Transitive Sets of Functions and Continua

Proefschrift

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Introduction

1. A new cardinal invariant

In chapter 1 a cardinal number $\mathfrak{tf}(X)$ is associated with a topological space X. The cardinal number $\mathfrak{tf}(X)$ is defined as the minimal size of a set $\mathcal{F}$ of continuous functions from X into itself with the property that for every two points $x, y \in X$, there exists a function $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$. Any set of functions with these properties is called a transitive set of functions on the space X. In [**15**] Dow showed that for the spaces $\beta\mathbb{N}$ and $\mathbb{N}^*$ the minimal size of a transitive set of functions is at least $\mathfrak{c}^+$, the least cardinal larger than the cardinality of the reals. For some Cook continuum X, $\mathfrak{tf}(X)$ equals $\mathfrak{c}$, as by [**12**] these (compact metric) the only continuous self maps are the identity and the constant mappings.

Van Mill asked us what the cardinal $\mathfrak{tf}(C)$ would be, where C is the Cantor set. It is obvious that $\mathfrak{tf}(C)$ is at most $\mathfrak{c}$ and Van Mill observed that it also must be the case that $\mathfrak{tf}(C)$ is at least $\aleph_1$.

In chapter 1 it is shown that the cardinal $\mathfrak{c}$ is at least equal to $\mathfrak{tf}(C)$ and at most equal to $\mathfrak{tf}(C)^+$. It is also shown that the equality $\mathfrak{tf}(C) = \mathfrak{c}$ is independent of ZFC.

2. Wallman spaces

A family $\mathcal{F}$ of subsets of a space X is called a *lattice of sets* if it is closed under finite unions and finite intersections. A subfamily u of nonempty members of the lattice $\mathcal{F}$ is called a $\mathcal{F}$-filter if

(1) for all U and V in u we have that $U \cap V$ is in u,
(2) for all $U \in u$ and all V, if $U \subset V$ then V is in u.

The $\mathcal{F}$-filter u is called an $\mathcal{F}$-ultrafilter if it is not properly contained in any other $\mathcal{F}$-filter.

The filter concept was introduced in Topology in order to study convergence. Suppose that the lattice $\mathcal{F}$ contains a neighborhood base for every point of the space X. A filter u is said to converge to a point x if it contains all neighborhoods of x, it is said to cluster to a point x if every member of u meets every neighborhood of x, i.e. if x belongs to $\bigcap\{\operatorname{cl} U : U \in u\}$.

In addition to convergence, collections of $\mathcal{F}$-ultrafilters have been used to construct topological spaces. Let $\operatorname{Ult}(\mathcal{F})$ denote the set of all $\mathcal{F}$-ultrafilters over the space X. For $U \in \mathcal{F}$ let U^* be the set defined by $U^* = \{u \in \mathrm{Ult}(\mathcal{F}) : U \in u\}$. Taking $\{U^* : U \in \mathcal{F}\}$ as a base for the closed sets gives us a topology on the set $\mathrm{Ult}(\mathcal{F})$.

Stone introduced this topology with $\mathcal{F}$ the family of all clopen subsets of X. He showed that $\mathrm{Ult}(\mathcal{F})$ is compact and that if X has a base for the open sets of clopen sets then X can be embedded in $\mathrm{Ult}(\mathcal{F})$.

In [**55**] Wallman introduced this topology where $\mathcal{F}$ is the family of all closed subsets of some T_1-space X. He showed that in that case $\mathrm{Ult}(\mathcal{F})$ is compact and contains X as a dense subspace. He also showed that if X is normal then $\mathrm{Ult}(\mathcal{F})$ would be Hausdorff, in this case U^* is the closure of U in the space $\mathrm{Ult}(\mathcal{F})$ and $\mathrm{Ult}(\mathcal{F})$ is the Čech-Stone compactification βX of X. Wallman used this compactification to show that one cannot distinguish between T_1-spaces and compact spaces by means of homology theory.

3. Elementary reflection of topological properties

Several mathematicians have been investigating ways to use model theory in Topology.

The method of Bankston is taking Wallman representations of ultrapowers or ultraproducts of lattices (see for instance [**4**] and [**5**]).

Tall et al. take an arbitrary space X with topology τ, put this in an elementary substructure $\mathfrak{M}$ of some $H(\theta)$ (θ large enough), and investigate what kind of properties the topological space (X, τ) and $X \cap \mathfrak{M}$ with the topology induced by the base $\tau \cap \mathfrak{M}$ have in common (see for instance [**32**] and [**54**]).

Bandlow investigates compact spaces X by embedding them into a Tychonoff cube $\mathbb{I}^\kappa$ for some κ, taking some elementary substructure $\mathfrak{M}$ of some $H(\theta)$ (θ large enough) and see what properties the compacta X and the projection $\pi_{\mathfrak{M}\cap\kappa}[X]$ (see for instance [**2**] and [**3**]).

In this thesis another model theoretic approach to investigate compact (connected) Hausdorff spaces is considered. Given a compact Hausdorff space, consider the lattice of all closed subsets 2^X, the ultrafilter space $\mathrm{Ult}(2^X)$ is homeomorphic to the space X, as all points $x \in X$ are uniquely defined by the closed sets of X containing them and every ultrafilter in $\mathrm{Ult}(\mathcal{F})$ is fixed. Taking an elementary sublattice L of 2^X gives us a lattice whose corresponding Wallman space, denoted by wL is compact Hausdorff and has a base for its closed sets which is isomorphic with the lattice L. Topological properties that must be shared between the spaces X and wL are called *elementarily reflected* properties. Lots of properties of compact spaces are elementarily reflected, these kinds of results are mentioned in 2.

Further investigations on connected compact Hausdorff spaces include the relation between span an chainability of (metric) continua in chapter 3 and a theorem of Maćkowiak and Tymchatyn in chapter 4. In chapter 5 we show how two other known results can be proved more efficiently by model-theoretic means.

4. Span versus chainability

Mappings $f, g : X \to Y$ are called *disjoint* if $f(x) \neq g(x)$ for all $x \in X$. The mappings f and g can be seen as subsets of the product space $X \times Y$ and then disjoint means that these sets have empty intersection. Lelek considered mappings from X to Y as points of the function space ${}^XY$; the presence of a metric structure on ${}^XY$ gave rise to the question of how far apart the distance between two disjoint mappings from X onto Y can be. Lelek introduced the notion of *span* of a space. Roughly speaking, the span of a country Y is a maximal number d such that two persons can pass over the same part of Y, keeping the distance at least d from one another.

If X is a metric space and d is the metric on X, the *span* of X is defined by

$$\inf\{\epsilon : \text{there exists a subcontinuum } Z \text{ of } X \times X \text{ such that } \pi_1[Z] = \pi_2[Z] \text{ and } d(x_1, x_2) \geq \epsilon \text{ for each } (x_1, x_2) \in Z\},$$

where π_1 and π_2 are the standard projections onto the axes. The span is a monotone function, if $X \subset Y$ then the span of X is at most the span of Y. If A is a connected space and $f_1, f_2 : A \to X$ are continuous maps such that $f_1[A] = f_2[A]$ then the span of X is at least equal to

$$\inf_{a \in A} d(f_1(a), f_2(a)).$$

As the projection maps π_1 and π_2 are continuous, we can say that the span of the space X is equal to

$$\sup_{A, f_i} \inf_{a \in A} d(f_1(a), f_2(a)),$$

where A ranges over all the connected spaces and f_i ($i = 1, 2$) over all continuous mappings of A into X. Lelek also showed the following theorem.

THEOREM 0.1 (Lelek [**38**]). *If $f : X \to Y$ is a continuous map between metric spaces and X is compact, then there are points x and x' such that*

$$d_X(x, x') \geq \text{ the span of } X \text{ and } d_Y(f(x), f(x')) \leq \text{ the span of } Y.$$

Thus, if the span of Y is zero, there is a point $y \in Y$ such that the span of X is at most the diameter of $f^{-1}(y)$.

This theorem implies that the span of chainable continua is zero, as on these spaces there exist real continuous functions with arbitrary small point inverses.

In [**39**] Lelek posed the following question

QUESTION 0.2 (Lelek [**39**]). *If a metric continuum has span zero, is it also chainable?*

This problem has become a classic in the theory of continua. A positive answer would complete the classification of homogeneous plane continua (see [**52**] and also [**13**]) It is known that continua of span zero are tree-like (see [**50**]).

Weaker notions of span have been introduced to find an answer to question 0.2 like

(1) *surjective span*, where the subcontinuum Z in the definition of span of the space X is projected onto X by both projections,
(2) *surjective semi span*, where Z projects onto X by at least one of the projection maps,
(3) *symmetric span*, where Z is symmetric, this means that if $(x, y) \in Z$ then also $(y, x) \in Z$.

Another result on span states that the continua of surjective span zero are also tree-like (see [**33**]).

There are more questions on the different kinds of span of a metric continuum, but we will only be interested in question 0.2.

We broaden our horizon by considering arbitrary continua. Extending the notions of span and chainability to include non-metric continua, we have that chainability still implies span zero for (arbitrary) continua. Without the metric we can only distinguish between span zero and span nonzero.

In chapter 3 we translate the properties of chainability and having span zero or nonzero for continua in the lattice language. We will investigate if these are properties that are elementary reflected. If they are, any non-chainable continuum of span zero would gives us, by the use of Wallman's representation theorem and taking a countable elementary sublattice, a counterexample to Lelek's question 0.2.

After extending the notions of span and chainability to arbitrary compact spaces, we are interested in the span and chainability of the continua $\mathbb{H}^*$ and $\mathbb{I}_u$. As these spaces are connected with the unit interval $\mathbb{I}$ (see [**24**] and [**46**]), which is a chainable metric compact space hence of span zero, we wondered if this would give us continua which would be non-chainable but of span nonzero. The results of this investigation are in chapter 3.

5. The Maćkowiak-Tymchatyn theorem

A continuum is said to be *decomposable* if it has two proper subcontinua that together cover the whole continuum, if there are not two proper subcontinua with this property we say that the continuum is *indecomposable.* A hereditarily indecomposable continuum is a continuum for which every subcontinuum is indecomposable.

In [**43**] Maćkowiak and Tymchatyn proved the follwoing theorem.

THEOREM 0.3 ([**43**]). *Every metric continuum is a weakly confluent image of some hereditarily indecomposable metric curve.*

The proof runs along the following lines. By a result from [**23**] every metric continuum can be embedded in a compactification $\gamma[0,\infty)$ of the half line such that each continuous mapping from some metric continuum onto $\gamma[0,\infty)$ is weakly confluent. Fix some hereditarily indecomposable metric continuum M, with $\dim(M) = 3$ and some weakly confluent onto mapping $f : M \to \mathbb{I}^3$ (these exist by [**11**], [**30**] and [**47**]). Next fix some curve $K \subset \mathbb{I}^3$ and a monotone mapping g from K onto the Hilbert cube Q, which, without loss of generality contains $\gamma[0,\infty)$. Now $g^{-1}[\gamma[0,\infty)] \subset K$ is a continuum and there exists some subcontinuum N of M such that $f[N] = g^{-1}[\gamma[0,\infty)]$. Now $(h = g \circ f) \restriction N$ is a weakly confluent map. With the result from [**41**] stating that every hereditarily indecomposable metric continuum is an image under an open monotone map of a hereditarily indecomposable metric curve, Theorem 0.3 is proved.

In the same paper Maćkowiak and Tymchatyn asked the question whether their result was true for every Hausdorff continuum. In [**25**] Hart, van Mill and Pol extended theorem 0.3.

THEOREM 0.4 ([**25**]). *Every continuum is a weakly confluent image of some 1-dimensional hereditarily indecomposable continuum of the same weight.*

The proof of this theorem used Wallman's representation theorem for lattices. Considering the lattice of all closed sets of some continuum X, the authors translated the properties of being a 1-dimensional hereditarily indecomposable continuum and the property of being the pre-image of the continuum X by a weakly confluent map into the lattice language. This gave the authors a theory, which, if consistent would provide (by Wallman's representation theorem) some continuum with the desired properties. By the compactness and completeness theorems of model theory it was only necessary to show the consistency of finite subsets of the theory. Roughly said, this was done using theorem 0.3, as any finite subset of the theory does mention only finitely many closed sets and these finite number of closed sets form a finite lattice which corresponds to some metric continuum.

Part 1

Transitive sets of functions, a consistency result

CHAPTER 1

A new cardinal invariant

1. Introduction

In [**15**] Dow gave a proof of the Rudin-Shelah theorem on the existence of $2^{\mathfrak{c}}$ points in $\beta\mathbb{N}$ that are Rudin-Keisler incomparable. The proof actually shows that whenever a family $\mathcal{F}$ of $\mathfrak{c}$ continuous self-maps of $\beta\mathbb{N}$ (or $\mathbb{N}^*$) are given there is a set S of $2^{\mathfrak{c}}$ many $\mathcal{F}$-independent points in $\beta\mathbb{N}$ (or $\mathbb{N}^*$). This suggests that we measure the complexity of a space X by the cardinal number $\mathfrak{tf}(X)$ defined as the minimum cardinality of a set $\mathcal{F}$ of continuous self maps such that for all $x, y \in X$ there is $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$. Let us call such an $\mathcal{F}$ transitive. Thus Dow's proof shows $\mathfrak{tf}(\beta\mathbb{N}), \mathfrak{tf}(\mathbb{N}^*) \geq \mathfrak{c}^+$.

We investigate $\mathfrak{tf}(C)$, where C denotes the Cantor set. Van Mill observed that $\mathfrak{tf}(C) \geq \aleph_1$; a slight extension of his argument shows that MA(countable) implies $\mathfrak{tf}(C) = \mathfrak{c}$. Our main result states that in the Sacks model the continuous functions on the Cantor set that are coded in the ground model form a transitive set. Thus we get the consistency of $\mathfrak{tf}(C) = \aleph_1 < \aleph_2 = \mathfrak{c}$.

The gap between $\mathfrak{tf}(C)$ and $\mathfrak{c}$ cannot be arbitrarily wide, because Hajnal's free set lemma implies that for any space X one has $|X| \leq \mathfrak{tf}(X)^+$.

In [**45**] Miller showed that it is consistent with ZFC that for every set of reals of size continuum there is a continuous map from that set onto the closed unit interval. In fact he showed that the iterated perfect set model of Baumgartner and Laver (see [**7**]) is such a model, and noted that the continuous map can even be coded in the ground model.

Here we will show that in the iterated perfect set model, for every two reals x and y there exists a continuous function with code in the ground model that maps x onto y or y onto x.

Definition 1.1. Let X be a topological space. By a transitive set of functions $\mathcal{F}$ on a space X we mean a set of continuous functions from X to itself such that for every two points x and y of X there exists an element $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$ holds.

Let us also define the cardinal number $\mathfrak{tf}(X)$ by

$$\mathfrak{tf}(X) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a transitive set of functions of } X\}.$$

The chapter is organized as follows: in section 2 we prove some simple facts on $\mathfrak{tf}$, the minimal size of transitive sets of functions. We also state and prove the main theorem of this chapter in section 2, using theorems proved later on in section 3. As a corollary to the main theorem we have the consistency of $\mathfrak{tf} < \mathfrak{c}$ with ZFC. Finally in section 4 we will make a remark on the effect on $\mathfrak{tf}$ when we add κ many Sacks reals side-by-side to a model of ZFC + CH.

2. Some equations concerning $\mathfrak{tf}(C)$

We will now prove some relations concerning the cardinals $\mathfrak{tf}(C)$, $\mathfrak{c}$ and $\aleph_1$. The first is Van Mill's observation alluded to above.

THEOREM 1.2. $\mathfrak{tf}(C) \geq \aleph_1$.

PROOF. Suppose $\mathcal{F}$ is a countable set of functions. Let A_f denote the set $\{x : \mathrm{int}(f^{-1}(x)) \neq \emptyset\}$ for every $f \in \mathcal{F}$. Every A_f is at most countable because 2^ω is separable. So choose an x in $2^\omega \setminus \bigcup_{f\in\mathcal{F}} A_f$, then we know that for every $f \in \mathcal{F}$ the set $f^{-1}(x)$ is nowhere dense in 2^ω. For such an x the set $\{f^{-1}(x) : f \in \mathcal{F}\}$ is countable. Because the set $\{f(x) : f \in \mathcal{F}\}$ is also countable the Baire category theorem tells us that the set $2^\omega \setminus \bigcup_{f\in\mathcal{F}}(\{f(x)\} \cup f^{-1}(x))$ is nonempty, thus showing that $\mathcal{F}$ is not transitive. □

Recall that the cardinal number $\mathfrak{d}$ is the dominating number, the minimal cardinality of a subset D of ${}^\omega\omega$ such that any $f \in {}^\omega\omega$ is eventually dominated by some element d of D, this means that there is an $n < \omega$ such that $f(m) \leq d(m)$ for all $m \geq n$. Geschke, Goldstern and Kojman proved in **[20]** the following relation between the cardinal numbers $\mathfrak{tf}(C)$ and $\mathfrak{d}$.

THEOREM 1.3 (**[20]**). $\mathfrak{tf}(C) \geq \mathfrak{d}$.

This theorem will help in the determination of the cardinals $\mathfrak{tf}(\mathbb{R})$, $\mathfrak{tf}(\omega^\omega)$ and $\mathfrak{tf}([0,1])$ in section 4 (see theorem 1.23). The following theorem shows that the cardinals $\mathfrak{tf}(C)$ and $\mathfrak{c}$ cannot differ very much from each other, in fact if they are different then $\mathfrak{c}$ is $\mathfrak{tf}(C)$'s successor.

THEOREM 1.4. $\mathfrak{tf}(C) \leq \mathfrak{c} \leq \mathfrak{tf}(C)^+$.

REMARK 1.5. The proof of theorem 1.2 shows that $\mathfrak{tf}(C)$ is at least equal to the minimum number of nowhere dense sets needed to cover C. Theorem 1.4 and MA(countable) then imply the equality $\mathfrak{tf}(C) = \mathfrak{c}$.

REMARK 1.6. If one replaces 'nowhere dense' by 'measure zero' in the proof of theorem 1.2 then one finds that $\mathfrak{tf}(C)$ is also at least as big as the minimum number of measure zero sets needed to cover the Cantor set C.

The second inequality in theorem 1.4 is a consequence of the following lemma. The proof of this lemma can be found in [**56**].

For this we need some more notation. Let S be an arbitrary set. By a *set mapping on* S we mean a function f mapping S into the power set of S. The set map is said to be of *order* λ if λ is the least cardinal such that $|f(x)| < \lambda$ for each x in S. A subset S' of S is said to be *free for* f if for every $x \in S'$ we have $f(x) \cap S' \subset \{x\}$.

LEMMA 1.7 (Free set lemma). *Let S be a set with $|S| = \kappa$ and f a set map on S of order λ where $\lambda < \kappa$. Then there is a free set of size κ for f.*

PROOF OF THEOREM 1.4. The proof of the first inequality is easy. We simply have to observe that the set of all constant functions on the reals is a transitive set of functions.

Now for the second inequality. Striving for a contradiction suppose that $\mathfrak{c} \geq \mathfrak{tf}(C)^{++}$. Let $\mathcal{F}$ be a transitive set of functions such that $|\mathcal{F}| = \mathfrak{tf}(C)$. We define a set map F on the reals by $F(x) = \{f(x) : f \in \mathcal{F}\}$ for every $x \in 2^\omega$. Because $|F(x)| \leq \mathfrak{tf}(C)$, this set map F is of order $\mathfrak{tf}(C)^+$, which is less than $\mathfrak{c}$. According to the free set lemma there exists a set $X \subset 2^\omega$ such that $|X| = \mathfrak{c}$ and for every $x \in X$ we have $F(x) \cap X \subset \{x\}$. This is a contradiction, because every two reals in X provide a counterexample of $\mathcal{F}$ being a transitive set. □

This theorem also gives us the following corollary.

COROLLARY 1.8. *If V is some model of* ZFC *such that $\mathfrak{c}$ is a limit cardinal or equal to $\aleph_1$, then V is a model of the equation $\mathfrak{tf}(C) = \mathfrak{c}$.*

3. Consistency of $\mathfrak{tf}(C) < \mathfrak{c}$

In this section we will prove that Baumgartner and Laver's model V_{ω_2} is a model of ZFC where $\mathfrak{tf}(C)$ is smaller than the cardinal $\mathfrak{c}$. We will show this by proving that given any two reals in V_α, the extension of a model V of ZFC after adding iteratively α many Sacks reals, there is a code in the ground model V of a continuous map between two closed nowhere dense disjoint subsets of the Cantor set such that in the extension V_α, this continuous map maps one of the reals onto the other. Then, given that V is a model of CH and V_{ω_2} models that $\mathfrak{c}$ equals $\aleph_2$, we see that V_{ω_2} is a model of $\mathfrak{tf}(C) < \mathfrak{c}$.

3.1. Preliminaries. For the rest of this paper let V be a model of ZFC. We will use the same notations and definitions as Baumgartner and Laver in [**7**], so for any ordinal α we let $\mathbb{P}_\alpha$ denote the poset that iteratively adds α Sacks reals to the model V, using countable support. Let $\mathbb{P}_1 = \mathbb{P}$, where $\mathbb{P}$ denotes the 'normal' Sacks poset for the addition of one Sacks real.

Let G_α be $\mathbb{P}_\alpha$-generic over V, we define V_α by $V_\alpha = V[G_\alpha]$ for every ordinal α. Note that if $\beta < \alpha$ we have that $G_\alpha \restriction \beta$ is a $\mathbb{P}_\beta$-generic subset over V. If we denote the $(\alpha+1)$-th added Sacks real by s_α then we can also write $V_\alpha = V[\langle s_\beta : \beta < \alpha\rangle]$.

Assuming that V is a model of CH, the proof of the following facts can be found in [**7**]:

(1) Forcing with $\mathbb{P}_\alpha$ does not collapse cardinals for $\alpha \leq \omega_2$.
(2) V_{ω_2} is a model of ZFC $+ 2^{\aleph_0} = \aleph_2$.
(3) Let $\dot{\mathbb{P}}_\beta$ denote the result of defining $\mathbb{P}_\beta$ in V_α. Then for any $\alpha, \beta \geq 1$, $\Vdash_\alpha$ "$\mathbb{P}_{\alpha,\alpha+\beta}$ is isomorphic to $\dot{\mathbb{P}}_\beta$".

We note that Sacks reals are added at successor ordinal stages only. Because we force with countable support there are reals added at countable limit stages, which are obviously not Sacks reals, and at uncountable limit stages no reals are added.

We make the following definition. For any $\sigma \in {}^{<\omega}2$ we let $l(\sigma) \in \omega$ denote the *length of* σ. So for every $\sigma \in {}^{<\omega}2$ we have $\sigma \in {}^{l(\sigma)}2$.

To show how we construct our continuous maps we reprove the familiar fact that Sacks reals are minimal, see [**31**].

LEMMA 1.9. *Suppose x is a real in $V[G] \setminus V$, where G is a $\mathbb{P}$-generic filter over V, and that $p \in \mathbb{P}$ is such that $p \Vdash$ "$\dot{x} \notin V$". Then there exists a $q \geq p$ and a homeomorphism f defined in V such that $q \Vdash$ "$f(\dot{s}) = \dot{x}$". Here $\dot{s}$ denotes the name of the added Sacks real.*

PROOF. We will construct a fusion sequence $\{\langle p_i, n_i\rangle : i \in \omega\}$ such that each p_{i+1} will know all the first i splitting nodes of every branch of the perfect tree p_i and $(p_{i+1}, n_{i+1}) > (p_i, n_i)$ for every i.

Because p forces that $\dot{x}$ is a new real, there exists an element $u_\emptyset \in {}^{<\omega}2$ with maximal length $m_\emptyset$, such that $p \Vdash$ "$\dot{x} \restriction m_\emptyset = u_\emptyset$" and p does not decide $\dot{x}(m_\emptyset)$. There exist $p_{\langle 0\rangle}, p_{\langle 1\rangle} \geq p_0$ such that $p_{\langle k\rangle} \Vdash$ "$\dot{x}(m_\emptyset) = k$" for $k \in \{0,1\}$. Without loss of generality the stems of $p_{\langle 0\rangle}$ and $p_{\langle 1\rangle}$ are incompatible. Let $n_0 = \min\{n \in \omega : p_{\langle 0\rangle} \restriction n \neq p_{\langle 1\rangle} \restriction n\}$ and let p_0 denote the element $p_{\langle 0\rangle} \cup p_{\langle 1\rangle}$.

Now assume we have $p_i = \bigcup\{p_\sigma : \sigma \in {}^{i+1}2\}$. Consider $\tau \in {}^{i+1}2$, we have an element $u_\tau \in {}^{<\omega}2$ of maximal length m_τ such that $p_\tau \Vdash$ "$\dot{x} \restriction m_\tau = u_\tau$". There exist $p_{\tau\frown 0}, p_{\tau\frown 1} \geq p_\tau$ such that $p_{\tau\frown k} \Vdash$ "$\dot{x}(m_\tau) = k$" for $k \in \{0,1\}$. Again without loss of generality the stems of $p_{\tau\frown 0}$ and $p_{\tau\frown 1}$ are incompatible. Let n_τ denote the integer $\min\{n \in \omega : p_{\tau\frown 0}\restriction n \neq p_{\tau\frown 0}\restriction n\}$ and $n_{i+1} = \max\{n_\sigma : \sigma \in {}^{i+1}2\}$. We let p_{i+1} denote the element $\bigcup\{p_\sigma : \sigma \in {}^{i+2}2\}$. Now the induction step is completed, because p_{i+1} knows all the first $i+1$ splitting nodes of every branch in p_i and $(p_{i+1}, n_{i+1}) > (p_i, n_i)$ for every $i \in \omega$.

We define the function f by

$$f^{-1}([u_\sigma]) \supset [\text{stem}(p_\sigma)] \text{ for } \sigma \in {}^{<\omega}2.$$

As stem(p_σ) is a finite approximation of the added Sacks real $\dot{s}$, we have by the construction of our p_σ for $\sigma \in {}^{<\omega}2$ and the function f that $p_\sigma \Vdash$ "$f(\dot{s}) \in [u_\sigma]$" for every $\sigma \in {}^{<\omega}2$. And so the fusion q of the sequence $\{\langle p_i, n_i\rangle : i \in \omega\}$ forces that in the extension $V[G]$ the equality $f(s) = x$ holds. This f, being a continuous bijection between two Cantor sets, is (of course) a homeomorphism. □

REMARK 1.10. In the lemma we have also defined a map ϕ from the finite sub trees of the fusion q onto the finite sub trees of $T = \bigcup_{\sigma \in {}^{<\omega}2} u_\sigma$ which induces our homeomorphism. We have $\phi(q) = T$ and

$$\phi([q \restriction \sigma]) = \bigcup \{u_\tau : \sigma \subset \tau \text{ and } \tau \in {}^{<\omega}2\}.$$

We note that $[T]$ is the set of all the possible interpretations of $\dot{x}$ in $V[G]$ and that T depends on ϕ and q only. In theorem 1.16 we will use this interpretation of the previous lemma.

We make the following definitions. For $p \in \mathbb{P}$ and $s \in {}^{<\omega}2$ we let p_s denote the sub-tree $\{t \in p : s \subseteq t \text{ or } t \subseteq s\}$ of p. Of course p_s is a perfect tree if and only if $s \in {}^{<\omega}2 \cap p$. To generalize this to $\mathbb{P}_\alpha$, suppose p is an element of $\mathbb{P}_\alpha$, F is a finite subset of dom(p) and $n \in \omega$, we say that a function $\tau : F \longrightarrow {}^n2$ *is consistent with* p if the following holds for every $\beta \in F$:

$$(p \restriction \tau) \restriction \beta \Vdash_\beta \text{“}\tau(\beta) \in p(\beta)\text{”}.$$

So we have for every $\beta \in F$ that

$$(p \restriction \tau) \restriction \beta \Vdash_\beta \text{“}(p(\beta))_{\tau(\beta)} \text{ is a perfect tree”}.$$

Furthermore let us suppose that F and H are two sets such that $F \subset H$, and n and m are two integers such that $m < n$, if τ is a function mapping F into ${}^m2$ then we say that a function $\sigma : H \longrightarrow {}^n2$ *extends* the function τ if for every $i \in F$ we have $\sigma(i) \restriction m = \tau(i)$.

For later use we will prove the following:

LEMMA 1.11. *Let $p \in \mathbb{P}_\alpha$, $F \in [\text{dom}(p)]^{<\omega}$ and $n \in \omega$. Suppose $\tau : F \longrightarrow {}^n2$ is consistent with p then for every $r \geq p \restriction \tau$ there exists a $q \geq p$ such that $q \restriction \tau = r$ and $q \restriction \beta \Vdash_\beta$ “$(p(\beta))_s = (q(\beta))_s$ for every $s \in {}^n2$ such that $s \neq \tau(\beta)$” for every $\beta \in F$.*

PROOF. Define the element $q \in \mathbb{P}_\alpha$ as follows for $\beta < \alpha$:

$$q \restriction \beta \Vdash_\beta \text{“}q(\beta) = \begin{cases} r(\beta) & \beta \notin F \\ r(\beta) \cup \{(p(\beta))_s : s \in {}^n2 \cap p(\beta) \setminus \{\tau(\beta)\}\} & \beta \in F\text{”} \end{cases}$$

In this way we strengthen the tree $p(\beta)$ above $\tau(\beta)$ keeping the rest of the perfect tree intact (according to F anyway). □

3.2. Continuous functions with code in the ground model and the main theorem. Closed subsets of the Cantor set can be coded by sub trees of ${}^{<\omega}2$, as follows. If A is a closed subset of C, then let $T_A = \{x \restriction n : x \in A, n \in \omega\}$; one can recover A from T_A by observing that

$$A = \{x \in {}^\omega 2 : \forall n \in \omega, x \restriction n \in T_A\}.$$

When we say that a closed set A is *coded in the ground model* we understand that T_A belongs to the ground model.

We shall always construct a continuous function f between closed sets A and B by specifying an order-preserving map ϕ from T'_A to T_B, where T'_A denotes the set of splitting nodes of T_A. Once ϕ is found one defines f by

$$f(x) = \text{"the path through } T_B \text{ determined by } \phi \text{ and } \{x \restriction n : n \in \omega\}\text{"}.$$

We say that f is coded in the ground model if ϕ belongs to V. In what follows we shall denote the map ϕ by f as well.

The remaining part of this section will be devoted to the following theorem.

THEOREM 1.12. *The family of continuous functions on C with code in the ground model is a transitive family in V_α for every ordinal α.*

This theorem and fact 2 on page 12 gives us the following corollary.

COROLLARY 1.13. *V_{ω_2} is a model of* ZFC $+ \mathfrak{tf}(\mathrm{C}) < \mathfrak{c}$.

The proof theorem 1.12 will be as follows. We will show that for any α and any two reals from some V_α there exist some code for a continuous function between two closed subsets A and B of the Cantor set, that in V_α maps one of the reals onto the other. Without loss of generality we can assume that the two reals are different as the identity is obviously a continuous map that is coded in the ground model. Because the reals are different we can make sure that the closed sets A and B of the Cantor set, each of them containing one of the reals in question are disjoint. Furthermore we can make sure that the closed sets A and B are nowhere dense. The following lemma show that this continuous map can be extended to a continuous self map of the Cantor set.

Let $\triangle : {}^\omega 2 \times {}^\omega 2 \to \omega$ be the map given by

$$\triangle(x, y) = \min\{n < \omega : x(n) \neq y(n)\}.$$

Let $\triangleleft$ be the linear order on ${}^\omega 2$ defined by

$$x \triangleleft y \leftrightarrow x(\triangle(x, y)) < y(\triangle(x, y)).$$

The following lemma shows that continuous mappings between closed nowhere dense disjoints subsets of the Cantor set $C = {}^\omega 2$ can be extended to the whole set C.

LEMMA 1.14. *Let $f : A \to B$ be a continuous function, where A and B are disjoint closed nowhere dense subsets of the Cantor set C, then there is a continuous self map F of C that is an extension of f.*

PROOF. Let x be some point in ${}^\omega 2$. If x is not an element of the set A, then as $\triangleleft$ induces the topology on ${}^\omega 2$ there is a minimal $n < \omega$ such that $[x \restriction (n + 1)] \cap A = \emptyset$.

Let $g : C \to A$ be defined by $g(x) = x$ if x is an element of A, and $g(x)$ equals $\lhd$-minimal element a of A such that $\triangle(x,a) = n$, for the minimal $n < \omega$ such that $[x \restriction (n+1)] \cap A = \emptyset$. The map g is easily seen to be a retraction of C. If we define the map $F : C \to C$ by $F = f \circ g$, then we have found our continuous self map of C, extending the map f. □

If the continuous function between A and B was actually a homeomorphism we can do better than just extend it to a continuous self map of C, By the following theorem of Knaster and Reichbach we can extend it to an autohomeomorphism of C.

THEOREM 1.15 ([**35**]). *Given two closed nowhere dense subsets of the Cantor set, any homeomorphism between these sets can be extended to an autohomeomorphism of the Cantor set.*

A proof of the following two theorems will be given in subsection 3.3 and subsection 3.4 respectively.

THEOREM 1.16. *For all x in $V_{\alpha+1}$ there exists a continuous function f on C with code in the ground model such that $f(s_\alpha) = x$ holds in the model $V_{\alpha+1}$. If x is an element of $V_{\alpha+1} \setminus V_\alpha$ then this map f can be chosen to be a homeomorphism.*

THEOREM 1.17. *If $\alpha < \omega_2$ is a limit ordinal of cofinality ω, then for all x in $V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ and all y in V_α there is an f with code in the ground model such that in V_α it holds that $f(x) = y$. If, moreover y is not an element of V_β for all $\beta < \alpha$ then f can be chosen to be a homeomorphism.*

Together with theorem 1.16 the previous theorem gives the following corollary.

COROLLARY 1.18. *For any α and every $x, y \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ there is some autohomeomorphism of C with code in the ground model that maps x onto y in V_α.*

The proof of theorem 1.12 now follows easily using transfinite induction and theorems 1.16 and 1.17

3.3. Successor stages. This subsection is completely devoted to the proof of theorem 1.16. The proof of theorem 1.16 will be given in two lemmas, lemma 1.20 and 1.21.

We need the following lemma to make sure that the maps we will construct in the lemmas 1.20 and 1.21 are well-defined and continuous.

LEMMA 1.19. *Let $p \in \mathbb{P}_{\alpha+1}$. Suppose $F, H \in [\mathrm{dom}(p)]^{<\omega}$ are such that $F \subset H$ and $m, n \in \omega$ are such that $m < n$. If $\tau : F \to {}^m 2$ is consistent with p, N is an integer and T is a finite tree such that*

$$(p \restriction \tau) \restriction \alpha \Vdash \text{“}p(\alpha) \cap {}^{\leq N}2 = T\text{”},$$

then there exist a $(q,j) >_H (p \restriction \tau, n)$ and an $M > N$ such that for every $\sigma : H \to {}^n2$ extending τ, if σ is consistent with q, then there exists T_σ such that $q \restriction \sigma \Vdash$ "$q(\alpha) \cap {}^{\leq M}2 = T_\sigma$". Also $|(T_\sigma)_t \cap {}^M2| \geq 2$ for every $t \in T$ and $[T_\sigma] \cap [T_\varsigma] = \emptyset$ whenever σ and ς are distinct and consistent with q.

PROOF. Let Σ_τ denote the set of all $\sigma : H \to {}^n2$ extending τ. Because $p(\alpha)$ is a perfect tree there exists a $\mathbb{P}_\alpha$-name $\dot{M}$ such that for every $t \in T$ we have

$$(p \restriction \tau) \restriction \alpha \Vdash \text{"}|(p(\alpha))_t \cap {}^{\dot{M}}2| \geq 2|\Sigma_\tau|\text{"}.$$

According to lemma 2.3 of [7] there exists a $(q^\dagger, j^\dagger) >_H ((p \restriction \tau) \restriction \alpha, n)$ such that if $\sigma \in \Sigma_\tau$ is consistent with $q^\dagger$ we have an M_σ such that $q^\dagger \restriction \sigma \Vdash$ "$\dot{M} = M_\sigma$". Put $M = \max\{M_\sigma : \sigma \in \Sigma_\tau \text{ consistent with } q^\dagger\}$. We have $q^\dagger \Vdash$ "$|(p(\alpha))_t \cap {}^M2| \geq 2|\Sigma_\tau|$" for every $t \in T$.

Enumerate $\{\sigma \in \Sigma_\tau : \sigma \text{ consistent with } q^\dagger\}$ as $\{\sigma_k : k < K\}$. Let $r \geq q^\dagger \restriction \sigma_0$ be such that $r \Vdash$ "$p(\alpha) \cap {}^{\leq M}2 = S_{\sigma_0}$", where S_{σ_0} is such that $|(S_{\sigma_0})_t \cap {}^M2| \geq 2|\Sigma_\tau|$ for every $t \in T$. Use lemma 1.11 to find a $q_0 \geq q^\dagger$ such that $q_0 \restriction \sigma_0 = r$.

We continue this procedure with all the $\sigma_k \in \Sigma_\tau$. So if σ_k is consistent with q_{k-1} we find an $r \geq q_{k-1} \restriction \sigma_k$ such that $r \Vdash$ "$p(\alpha) \cap {}^{\leq M}2 = S_{\sigma_k}$", and also that $|(S_{\sigma_k})_t \cap {}^M2| \geq 2|\Sigma_\tau|$ for every $t \in T$. And we use lemma 1.11 to define $q_k \geq q_{k-1}$ such that $q_k \restriction \sigma_k = r$. If σ_k is not consistent with q_{k-1} we choose $q_k = q_{k-1}$.

We now have for every $\sigma \in \Sigma_\tau$ consistent with q_{K-1} a finite tree $S_\sigma \subset {}^{\leq M}2$ extending the tree T such that every branch in T has (at least) $2|\Sigma_\tau|$ different extensions in $S_\sigma \cap {}^M2$ and $q_{K-1} \restriction \sigma \Vdash$ "$p(\alpha) \cap {}^{\leq M}2 = S_\sigma$".

As q_{K-1} forces that, for each $y \in T$ the size of the set $p(\alpha))_t \cap {}^M2$ is at least $2|\Sigma_\tau|$ we can find for $\sigma \in \Sigma_\tau$ consistent with q_{K-1} a sub tree T_σ of S_σ such that $|(T_\sigma)_t \cap {}^M2| \geq 2$ and whenever σ and ς are distinct and consistent with q_{K-1} we have $[T_\sigma] \cap [T_\varsigma] = \emptyset$.

Define $q \in \mathbb{P}_{\alpha+1}$ such that $q \restriction \alpha = q_{K-1}$ and choose $q(\alpha)$ such that for every consistent $\sigma \in \Sigma_\tau$ we have $q \restriction \sigma \Vdash$ "$q(\alpha) = p(\alpha) \cap [T_\sigma]$". If we let j be equal to $\max\{j^\dagger, M\}$ the proof is complete. □

LEMMA 1.20. *Given an ordinal α, a $p \in \mathbb{P}_{\alpha+1}$ and a $\mathbb{P}_{\alpha+1}$-name $\dot{x}$ for a real such that $p \Vdash$ "$\dot{x} \notin V_\alpha$" then there exists a homeomorphism f defined in V and a $q \geq p$ such that $q \Vdash$ "$f(\dot{s_\alpha}) = \dot{x}$".*

PROOF. By remark 1.10 we know that there is an $r \geq p \restriction \alpha$ and there exist $\mathbb{P}_{\alpha+1}$ names $\dot{\phi}$ for a map on the finite sub trees of $p(\alpha)$ and $\dot{T}$ for a perfect tree such that $r \Vdash$ "$\dot{\phi}(p(\alpha)) = \dot{T}$". Without loss of generality we assume that $p \restriction \alpha = r$.

Let us construct a fusion sequence $\{\langle p_i, n_i, F_i\rangle : i \in \omega\}$. Let $p_0 = p_1 = p$, $n_0 = n_1 = 0$, $F_0 = \emptyset$ and choose $F_1 \in [\mathrm{dom}(p)]^{<\omega}$ in such a way that we are building a fusion sequence.

Suppose we have constructed the sequence up to i, let us construct the next element of the fusion sequence. We let $\{\tau_k : k < K\}$ denote all $\tau : F_{i-1} \longrightarrow {}^{n_{i-1}}2$ consistent with p_i. If we choose in lemma 1.19 $\tau = \tau_0$, $F = F_{i-1}$ and $m = n_{i-1}$ we get a $(q_0, m_0) >_{F_i} (p_i \restriction \tau_0, n_i)$ such that for every $\sigma : F_i \longrightarrow {}^{\leq n_i}2$ extending τ_0, consistent with q_0, we have a finite sub tree $T_\sigma \subset {}^{\leq M(\tau_0)}2$ ($M(\tau_0) \in \omega$ follows from lemma 1.19) of $p_i(\alpha) = p(\alpha)$ such that

(1) T_σ is an extension of T_{τ_0}.
(2) For every branch t in T_{τ_0} there exist exactly two different branches of length $M(\tau_0)$ in T_σ extending t.
(3) If σ and ς are two distinct mappings from F_i to ${}^{n_i}2$ extending τ_0 that are consistent with q_0 we have $[T_\sigma] \cap [T_\varsigma] = \emptyset$.

We choose $r_0 \in \mathbb{P}_{\alpha+1}$ with lemma 1.11 such that $r_0 \geq q_0$ and $r_0 \restriction \tau_0 = q_0$.

We iteratively consider all the $\tau : F_{i-1} \longrightarrow {}^{n_{i-1}}2$. In the general case if τ_k is consistent with r_{k-1} then lemma 1.19 gives us a q_k and an $m_k \in \omega$ such that $(q_k, m_k) >_{F_i} (r_{k-1} \restriction \tau_k, n_i)$. We choose r_k in the same way as above, using lemma 1.11 such that $r_k \geq q_k$ and $r_k \restriction \tau_k = q_k$. If τ_k is inconsistent with r_{k-1} then we choose $r_k = r_{k-1}$ and $m_k = m_{k-1}$. After considering all the τ_k's we define $p_{i+1} = r_{K-1}$ and $n_{i+1} = \max\{m_k : k < K\}$. This ends the construction of the next element of the fusion sequence.

For every $i < \omega$ if $\sigma : F_i \longrightarrow {}^{n_i}2$ is consistent with p_{i+1} and extends $\tau : F_{i-1} \longrightarrow {}^{n_{i-1}}2$ then

$$p_{i+1} \restriction \sigma \Vdash \text{“}p(\alpha) \cap {}^{\leq M(\tau)}2 = T_\sigma\text{”}.$$

Considering our function $\dot{\phi}$, let us denote the finite tree $\dot{\phi}(T_\sigma)$ by S_σ. We have

$$p_{i+1} \restriction \sigma \Vdash \text{“}\dot{\phi}(T_\sigma) = S_\sigma\text{”}.$$

We are ready to define the homeomorphism f in V that maps s_α onto x in the extension. Suppose $\tau : F_i \longrightarrow {}^{n_i}2$ and $\sigma : F_{i+1} \longrightarrow {}^{n_{i+1}}2$ such that σ extends τ. Every maximal branch in $(T_x)_\tau$ corresponds to exactly one maximal branch in S_τ. Let f map the splitting point in S_σ above any maximal branch in S_τ onto the splitting point in $(T_x)_\sigma$ above the corresponding maximal branch in $(T_x)_\tau$. The function f thus defined will be a continuous and one-to-one mapping between two Cantor sets, so a homeomorphism. Furthermore the fusion q forces that f maps s_α onto x in the extension.

We could have made sure that $p(\alpha)$ is a nowhere dense subset of the Cantor set, and also that the perfect tree we have constructed by the fusion sequence that will determine $\dot{x}$ is nowhere dense in C. The map function f, whose code we have seen is in the ground model, maps homeomorphically one nowhere dense closed subset of C onto another closed nowhere dense subset of C, as the reals $\dot{x}$ and $\dot{s}_\alpha$ are different and this is forced by p, these subsets of C are also disjoint. By theorem 1.15 the function f can be extended to an autohomeomorphism of the Cantor set. □

LEMMA 1.21. *Given an ordinal α, a $p \in \mathbb{P}_{\alpha+1}$ and a $\mathbb{P}_{\alpha+1}$-name $\dot{x}$ such that $p \Vdash$ "$\dot{x} \in V_\alpha$" then there exists a continuous f, with code in V, and a $q \geq p$ such that $q \Vdash$ "$f(\dot{s}_\alpha) = \dot{x}$".*

PROOF. This proof is similar to the previous one. Apart from the finite sub trees T_σ we are construction we also construct $t_\sigma \in {}^{<\omega}2$ that determine the real $\dot{x}$. We do this by adding the following item on the list in the proof of lemma 1.20 determining the t_σ's

4. t_σ is an extension of t_{τ_0} of length at least $i+2$, and we have $q_0 \upharpoonright \sigma \Vdash$ "$t_\sigma \subset \dot{x}$".

We follow the proof of lemma 1.20. In the end we define a function f in V by $f(b) = t_\sigma$ for every maximal branch b of S_σ for every $\sigma : F_i \to {}^{n_i}2$ consistent with p_i for some $i < \omega$. This function is continuous and maps in V_α the Sacks real s_α onto the real x. Now using lemma 1.14, we can extend this continuous function on a subset of the Cantor set to a continuous function on the whole of C when we make sure that $p(\alpha)$ and $\bigcup\{t_\sigma : \sigma : F_i \to {}^{n_i}2$ consistent with some $p_i, i < \omega\}$ are to disjoint nowhere dense subsets of the Cantor set, which we can. □

3.4. Limit stages of countable cofinality. This subsection is completely devoted to the proof of theorem 1.17.

LEMMA 1.22. *Suppose that α is a limit ordinal of cofinality $\aleph_0$. Let x be a real in V_α such that $x \notin \bigcup_{\beta<\alpha} V_\beta$, and let $p \in \mathbb{P}_\alpha$ be a witness of this. Also let $F, H \in [\mathrm{dom}(p)]^{<\omega}$ such that $F \subset H$ and let n and m be two integers such that $m < n$. If $\tau : F \to {}^m2$ is consistent with p, and $u_\tau \in {}^{<\omega}2$ is such that*

$$p \upharpoonright \tau \Vdash \text{“}u_\tau \subset \dot{x}\text{”},$$

then there exists a $(q, j) >_H (p \upharpoonright \tau, n)$ such that for every $\sigma : H \to {}^n2$ consistent with q, we have a $u_\sigma \in {}^{<\omega}2$ such that $q \upharpoonright \sigma \Vdash$ "$u_\sigma \subset \dot{x}$"; in addition we have $l(u_\sigma) = l(u_\varsigma)$ and $u_\sigma \neq u_\varsigma$ whenever σ and ς are distinct and consistent with q.

Before we prove the lemma we need some more notation. We let $\Vdash^*$ denote forcing in V_δ over $\mathbb{P}_{\delta\alpha}$. Here we use again the same notation as in [7] where for $\delta < \alpha$ $P_{\delta\alpha} = \{p \in \mathbb{P}_\alpha : \mathrm{dom}(p) \subset \{\xi : \delta \leq \xi < \alpha\}\}$, and if $p \in \mathbb{P}_\alpha$ then $p^\delta = p \setminus (p \upharpoonright \delta) \in \mathbb{P}_{\delta\alpha}$. The mapping which carries p into $(p \upharpoonright \delta, p^\delta)$ is an isomorphism of $\mathbb{P}_\alpha$ onto a dense subset of $\mathbb{P}_\delta \times \mathbb{P}_{\delta\alpha}$ (see [7]).

PROOF OF LEMMA 1.22. Choose a δ such that $\max(H) < \delta < \alpha$. Let $\tau : F \to {}^m2$ be consistent with p and let Σ_τ denote all the τ extending functions $\sigma : H \to {}^n2$.

Because p forces that $x \notin V_\delta$, there is an antichain below p^δ of size $|\Sigma_\tau|$ such that all these elements force different interpretations of $\dot{x}$ in the extension. In other words there exist a sequence $\{\dot{f}_\sigma : \sigma \in \Sigma_\tau\}$ of $\mathbb{P}_\delta$ names for elements of $\mathbb{P}_{\delta\alpha}$ and a sequence $\{\dot{u}_\sigma : \sigma \in \Sigma_\tau\}$ of $\mathbb{P}_\delta$ names for elements of ${}^{<\omega}2$ such that for all $\sigma \in \Sigma_\tau$ we have

$$(1) \qquad (p \restriction \tau) \restriction \delta \Vdash \text{“}\dot{f}_\sigma \geq p^\delta \text{ and } \dot{f}_\sigma \Vdash^* \text{“}\dot{u}_\sigma \subset \dot{x}\text{””},$$

and if σ and ς are distinct then

$$(2) \qquad (p \restriction \tau) \restriction \delta \Vdash \text{“}l(\dot{u}_\sigma) = l(\dot{u}_\varsigma) \text{ and } \dot{u}_\sigma \neq \dot{u}_\varsigma\text{”}.$$

Repeatedly using lemma 2.3 of [7] we see that there exist a $(q^\dagger, j) >_H ((p \restriction \tau) \restriction \delta, n)$ and sequences $\{f_\sigma : \sigma \in \Sigma_\tau\}$, $\{u_\sigma : \sigma \in \Sigma_\tau\} \subset {}^i2$ for some integer i such that for every $\sigma \in \Sigma_\tau$ we have

$$(3) \qquad q^\dagger \Vdash_\delta \text{“}\dot{f}_\sigma = f_\sigma \text{ and } \dot{u}_\sigma = u_\sigma\text{”}.$$

Now let q denote the element of $\mathbb{P}_\alpha$ such that $q \restriction \delta = q^\dagger$, and $(q \restriction \sigma) \restriction \delta \Vdash$ “$q^\delta = f_\sigma$” for every $\sigma \in \Sigma_\tau$ consistent with $q^\dagger$. This completes the proof. □

We are now able to give the proof of theorem 1.17.

Proof of Theorem 1.17. For the first part of the theorem suppose that we have $p \in \mathbb{P}_\alpha$ such that $p \Vdash$ “$\dot{x} \notin \bigcup_{\beta<\alpha} V_\beta$ and $\dot{y} \in \bigcup_{\beta<\alpha} V_\beta$”. We will construct a fusion sequence below p and define a continuous function f in V such that the fusion of the sequence forces that $f(x) = y$ holds in V_α.

Let $p_0 = p_1 = p$, $n_0 = n_1 = 0$, $F_0 = \emptyset$, and choose $F_1 \in [\mathrm{dom}(p)]^{<\omega}$ in such a way that we are building a fusion sequence. Suppose we have constructed the sequence up to i, we will construct the next element of the fusion sequence. Let $\{\tau_k : k < K\}$ denote an enumeration of all maps from F_{i-1} into ${}^{n_{i-1}}2$ consistent with p_i.

According to lemma 1.22 there exists a $(q_0, j_0) >_{F_i} (p_i \restriction \tau_0, n_i)$ such that for every $\sigma : F_i \to {}^{n_i}2$ consistent with q_0 we have distinct u_σ's in ${}^{m(\tau_0)}2$ (where $m(\tau_0)$ follows from lemma 1.22), such that $q_0 \restriction \sigma \Vdash$ “$u_\sigma \subset \dot{x}$”. Now use lemma 1.11 to construct $r_0 \in \mathbb{P}_\alpha$ such that $r_0 \geq q_0$ and $r_0 \restriction \tau_0 = q_0$.

We now iteratively consider all the τ_k. In the general case if τ_k is not consistent with r_{k-1} then we make sure that $r_k = r_{k-1}$ and $j_k = j_{k-1}$. If τ_k is consistent with r_{k-1} we find by lemma 1.22 a $(q_k, j_k) >_{F_i} (r_{k-1} \restriction \tau_k, n_i)$ such that for every $\sigma : F_i \to {}^{n_i}2$ consistent with q_k we have distinct u_σ's in ${}^{m(\tau_k)}2$ such that $q_k \restriction \sigma \Vdash$ “$u_\sigma \subset \dot{x}$”. Now use lemma 1.11 to construct $r_k \in \mathbb{P}_\alpha$ such that $r_k \geq r_{k-1}$ and $r_k \restriction \tau_k = q_k$. After considering all τ_k we define $p_{i+1} = r_{K-1}$ and $n_{i+1} = \max\{j_k : k < K\}$.

If we take a closer look at lemma 1.22 we can also let the fusion sequence that we just constructed determine $\dot{y}$. Because if we have $p \restriction \tau \Vdash$ “$t_\tau \subset \dot{y}$”, following the proof of lemma 1.22 we can make sure that (by some strengthening of $q^\dagger$ or the f_σ's, if necessary) there exist t_σ's in ${}^{<\omega}2$, not necessarily distinct, extending t_τ such that for

$\sigma : H \to {}^n2$ consistent with q we also have $q \restriction \sigma \Vdash$ "$t_\sigma \subset \dot{y}$". So assume we have done this. We have for every $\sigma : F_i \to {}^{n_i}2$ consistent with p_{i+1}

$$(4) \qquad p_{i+1} \restriction \sigma \Vdash \text{“}u_\sigma \subset \dot{x} \text{ and } t_\sigma \subset \dot{y}\text{”}.$$

Now we are ready to define our function f which will map x in V_α continuously onto y. Let $f([u_\sigma]) \subset [t_\sigma]$ for all $\sigma : F_i \to {}^{n_i}2$ and all $i \in \omega$. Then $p_i \restriction \sigma \Vdash$ "$f(\dot{x}) \in [t_\sigma]$" for $\sigma : F_i \to {}^{n_i}2$ consistent with p_i and $i \in \omega$. It follows that the fusion q forces that in V_α we have $f(x) = y$. Moreover f is a continuous function, this follows from lemma 1.22.

For the second part of the theorem suppose that $p \Vdash$ "$\dot{x}, \dot{y} \notin \bigcup_{\beta<\alpha} V_\beta$". Just as in lemma 1.22 we can choose not only the u_σ's in equation 4 distinct but also the t_σ's for $\sigma \in \Sigma_\tau$ and $\tau : F_i \to {}^{n_i}2$ for some $i \in \omega$. With this, the constructed continuous function f is actually a homeomorphism. □

4. Concluding remarks

4.1. The cardinal $\mathfrak{tf}$ for other spaces. It is not the case that the $\mathfrak{tf}$ - number is the same for all compact metric spaces, e.g., every Cook continuum X has $\mathfrak{tf}(X) = \mathfrak{c}$ (it only has the identity and constant mappings as self-maps, see [**12**]).

Note also that if the cardinal number cov(*nowhere dense*) equals $\mathfrak{c}$ for the unit interval $\mathbb{I}$, then remark 1.5 shows that $\mathfrak{tf}(\mathbb{I}) = \mathfrak{c}$. Suppose that cov(*nowhere dense*) $= \kappa < \mathfrak{c}$, for $\mathbb{I}$, then we can cover $\mathbb{I}$ by κ many Cantor sets $\{C_\alpha\}_{\alpha<\kappa}$ in such a way that for every two reals x and y there exists an α such that $x, y \in C_\alpha$. For every α we have a transitive family of continuous functions $\mathcal{F}_\alpha$ on C_α such that $|\mathcal{F}_\alpha| = \mathfrak{tf}(C)$. We can extend every $f \in \mathcal{F}_\alpha$ to a continuous self map $\tilde{f}$ of $\mathbb{I}$. So $\mathcal{F} = \{\tilde{f} : f \in \mathcal{F}_\alpha, \alpha < \kappa\}$ is a transitive set of continuous functions on $\mathbb{I}$, and its cardinality is less than or equal to $\kappa \times \mathfrak{tf}(C) = \mathfrak{tf}(C)$. So if we can cover the unit interval with fewer than $\mathfrak{c}$ many nowhere dense sets we have $\mathfrak{tf}(\mathbb{I}) \leq \mathfrak{tf}(C)$.

Theorem 1.3 from section 2 gives the following theorem which is stated in [**20**].

THEOREM 1.23 ([**20**]). *The cardinals* $\mathfrak{tf}(C)$, $\mathfrak{tf}(\mathbb{R})$ *and* $\mathfrak{tf}(\omega^\omega)$ *are one and the same.*

As the proof is not that long we give it here for completeness sake. The proof uses the following lemma.

LEMMA 1.24 ([**20**]). *Let K be a compact subset of ω^ω and $f : K \to \omega^\omega$ be continuous. Then f can be continuously extended to the whole of ω^ω.*

PROOF. Consider K as a subset of $(\omega+1)^\omega$. The latter space is homeomorphic to 2^ω. As $f[K]$ is bounded and thus a subset of ω^ω

there is a copy of 2^ω including $f[K]$. The lemma now follows from the well-known fact that every continuous mapping from a closed subset of a Boolean space to 2^ω can be continuously extended to the whole space (which follows from 2^ω being the Stone space of a free Boolean algebra). □

PROOF OF THEOREM 1.23. We first show $\mathfrak{tf}(C)$ $(= \mathfrak{tf}(2^\omega))$ is less than or equal to $\mathfrak{tf}(\omega^\omega)$. Let $f : \omega^\omega \to \omega^\omega$ be continuous. Then $f^{-1}[2^\omega]$ is closed and thus $A = f^{-1}[2^\omega] \cap 2^\omega$ is a closed subset of 2^ω. We can extend $f \restriction A$ to a continuous function $\overline{f} : 2^\omega \to 2^\omega$ as in the proof of lemma 1.24. This shows that a transitive family of ω^ω gives rise to a transitive family of 2^ω of no greater size. The same argument goes through for $\mathbb{R}$ instead of ω^ω using the Tietze-Urysohn extension theorem.

Observe that ω^ω can be covered by $\mathfrak{d}$ copies of 2^ω, since $\mathfrak{d}$ is the covering number of the ideal of bounded subsets of ω^ω. Let $\mathcal{D}$ be a collection of size $\mathfrak{d}$ copies of 2^ω covering ω^ω.

To each pair $(D, E) \in \mathcal{D} \times \mathcal{D}$ assign a family $\mathcal{F}_{D,E}$ of size $\mathfrak{tf}(2^\omega)$ of continuous functions on ω^ω such that

$$D \times E \subset \bigcup\{\text{ran}(f) \cup \text{dom}(f) : f \in \mathcal{F}_{D,E}\}$$

This is possible by lemma 1.24. Let $\mathcal{F}$ be the union of all $\mathcal{F}_{D,E}$ where D and E range over $\mathcal{D}$. Then $\mathcal{F}$ is a transitive family of ω^ω and the size of $\mathcal{F}$ is $\mathfrak{d} + \mathfrak{tf} = \mathfrak{tf}$, by theorem 1.3.

Again the same argument works for $\mathfrak{tf}(\mathbb{R})$ as $\mathbb{R}$ is just ω^ω (the irrationals) together with countably many additional points (the rationals) and can therefore also be covered by $\mathfrak{d}$ many copies of the Cantor set 2^ω. We again use the Tietze-Urysohn extension theorem to extend continuous functions defined on closed subspaces of $\mathbb{R}$ to the whole of $\mathbb{R}$. □

REMARK 1.25. Knowing that $\mathfrak{tf}(C)$ is equal to $\mathfrak{tf}(\mathbb{R})$ also shows us that $\mathfrak{tf}(C)$ equals $\mathfrak{tf}([0,1])$ by the same reasons.

4.2. The cardinal $\mathfrak{tf}$ and side-by-side Sacks forcing. In the previous sections we showed that after adding $\aleph_2$ many Sacks reals iteratively to a model of ZFC + CH we end up with a model of $\mathfrak{tf}(C) < \mathfrak{c}$. Now consider $\mathbb{P}(\kappa)$, the poset for adding κ many Sacks reals side-by-side (see [**6**]). We have that $\mathbb{P}(\kappa)$ has the $(2^{\aleph_0})^+$-chain condition and preserves $\aleph_1$. Suppose that $\kappa \geq \aleph_1$ and $\text{cf}(\kappa) \geq \aleph_1$. If V is a model of CH and G is $\mathbb{P}(\kappa)$-generic over V, we have in $V[G]$ that $2^{\aleph_0} = \kappa$ and all cardinals are preserved. If κ happens to be some limit cardinal, then we must have that $\mathfrak{tf}(C)$ equals $\mathfrak{c}$ by the restrictions we found on the minimal size of a transitive family on the Cantor set.

A natural question would be if we get a model of $\mathfrak{tf} < \mathfrak{c}$ when we add $\aleph_2$ many Sacks reals side-by-side to a model of ZFC + CH. The answer to this question is in the negative. We will prove that adding $\aleph_2$

many Sacks reals to a model of ZFC + CH we do not produce a model of $\mathfrak{tf}(C) < \mathfrak{c}$.

Let $\mathbb{P}$ denote the poset for adding side-by-side ω_2 many Sacks reals.

THEOREM 1.26. *Suppose $V \models$ CH and G is a $\mathbb{P}$-generic filter over V then $V[G] \models \mathfrak{tf}(C) = \mathfrak{c}$.*

PROOF. Suppose that $\mathfrak{tf}(C) < \mathfrak{c}$. Then for some $\mathbb{P}$-name $\dot{\mathcal{F}} = \{\dot{f}_\alpha : \alpha < \omega_1\}$ we have

(5) $\quad \Vdash_{\omega_2}$ "$\dot{\mathcal{F}}$ is transitive on the Cantor set and of size $\aleph_1$".

As every one of the functions $\dot{f}_\alpha$ is supported on a set of size $\aleph_1$, there is some $A \subset \omega_2$ of size $\aleph_1$ such that for every $\alpha < \omega_1$ we have that $\dot{f}_\alpha$ is a $\mathbb{P} \restriction A$-name for some function on the Cantor set.

Let G be $\mathbb{P}$-generic over V. Pick some $\alpha \in \omega_2 \setminus A$ and let s_α denote the α-th Sacks real added to the model V. For any $f \in \mathcal{F}$ we have

$$f(s_\alpha) \in V[G \restriction A \cup \{\alpha\}].$$

This would imply that there is no function $f \in \mathcal{F}$ for which $f(s_\alpha) = s_\beta$ or $f(s_\beta) = s_\alpha$ for any $\beta \in \omega_2 \setminus A$ which is not equal to α. This clearly contradicts the assumption on the transitivity of $\mathcal{F}$ in $V[G]$. □

COROLLARY 1.27 (Corollary of Proof of Theorem 1.26). *For any cardinal number $\kappa > \aleph_1$, after adding κ many Sacks reals side-by-side to a model of CH we have a model of $\mathfrak{tf}(C) = \mathfrak{c}$.*

4.3. The cardinal $\mathfrak{hm}$. In [21] Geschke, Kojman, Kubriś and Schipperus investigate the minimal number $\gamma(S)$ of convex subsets of a closed subset S of the plane needed to cover S. They relate this number (for specific closed subsets of the plane) to a cardinal number $\mathfrak{hm}$ that is defined as the minimal cardinality of subsets of 2^ω needed to cover 2^ω, which are c-homogeneous, where $c : [2^\omega]^2 \to \{0, 1\}$ is a certain continuous coloring of the unordered pairs of elements of the Cantor set. They show that $\aleph_1 \leq \mathfrak{hm}$ and also that $\mathfrak{hm}$ and also that $\mathfrak{hm} \leq \mathfrak{c}$ and $\mathfrak{c} \leq \mathfrak{hm}^+$. Also, in Baumgartner and Laver's model the inequality $\mathfrak{hm} < \mathfrak{c}$ is valid, as is shown in [21]. All these properties are shared with the cardinal $\mathfrak{tf}(C)$ and the question arises if these cardinal numbers are the same.

In [20] the authors show that the cardinal number $\mathfrak{hm}$ is equal to the minimal cardinality of a set $\mathcal{F}$ of Lipschitz functions $f : 2^\omega \to 2^\omega$ such that for every x and y in 2^ω there is a $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$.

In this paper the authors also show that it is consistent that we need strictly more Lipschitz than continuous functions on 2^ω to make sure that for any pair of elements $\{x, y\}$ of 2^ω there is some function f that maps either x onto y or y onto x. To be more precise.

THEOREM 1.28 ([20]). $\mathfrak{tf}(C) < \mathfrak{hm}$ *is consistent with* ZFC.

The proof of the theorem uses a countable support forcing iteration of length ω_2 of forcing notions satisfying Axiom A of size $\aleph_1$ over a model of CH. So no cardinals are collapsed and in the extension the continuum $\mathfrak{c}$ equals $\aleph_2$. In this model we then have that $\mathfrak{hm}$ is equal to $\mathfrak{c}$ which in turn is equal to the successor cardinal of $\mathfrak{tf}(C)$.

In [**20**] it is also shown that the inequality $\mathfrak{d} < \mathfrak{tf}(C)$ is consistent with ZFC. In particular, after forcing with the measure algebra over $2^{\aleph_2}$ over a model of CH, one obtains a model (the Solovay model) in which $\mathfrak{d} = \aleph_1$, as the ground model elements of ${}^{\omega}\omega$ dominate the new elements, and $\mathfrak{tf}(C) = \aleph_2$. This last equality follows from the fact that if x and y are generic reals over some model V of ZFC such that $x \notin V[y]$ and $y \notin V[x]$, then no continuous function $f \in V$ on 2^{ω} can map x onto y. We used this line of thought to show in subsection 4.2 that $\mathfrak{tf}(C)$ equals $\mathfrak{c}$ when we add Sacks reals side-by-side.

Part 2

A model theoretic approach to continuum theory

CHAPTER 2

Model-theoretic continuum theory

1. Introduction

In [**55**] Wallman formulated a generalization of Stone's representation theorem for Boolean algebras to distributive lattices with zero and unit. This theorem shows that a compact T_1-space is determined by any base for its closed sets that is closed under finite unions and finite intersections. Note that such a base is a lattice.

Given a topological space X, any base for its closed subsets which is closed under finite unions and finite intersections is a lattice, we will call such a base a *lattice base for* X, or a lattice base for short, if no confusion can arise about the topological space X.

A *compact* space will be considered a space for which every family of closed sets with the finite intersection property has nonempty intersection and which is also a Hausdorff space (unless specifically stated otherwise).

If X is a compact space, $\mathcal{C}$ a lattice base for its closed sets and L a lattice that is elementarily equivalent to $\mathcal{C}$ then its Wallman representation wL is also compact and Hausdorff. We will see that wL shares many geometrical properties with X.

Typical ways of getting elementarily equivalent structures of a given structure are by elementary submodels e.g., via the Löwenheim-Skolem theorem and by elementary extension e.g., by taking ultrapowers.

In [**8**] and [**9**] Bankston investigates geometrical properties of Wallman representations of ultrapowers of lattice bases. He introduces a construction of a compact Hausdorff space given a set of compact Hausdorff spaces $\{X_i : i \in I\}$ and an ultrafilter u on I, called the *ultracoproduct*, $\Sigma_u X_i$. This ultracoproduct is defined as an inverse limit of coproducts of compact Hausdorff spaces and can be seen to be the Wallman representation of the ultraproduct lattice $\Pi_u A_i$, where for all $i \in I$ the lattice A_i is a base for the closed sets of X_i. He then shows that certain properties of compact spaces are reflected to the ultracoproduct $\Sigma_u X_i$ if ultrafilter u many of the compact spaces X_i have the property.

As usual if all the compact spaces X_i are the same space X the ultracoproduct is called the *ultracopower* of X. We will encounter an ultracopower in chapter 3. In that chapter we will be investigating the span and chainability of the continuum $\mathbb{I}_u$ (the ultracoproduct of the unit interval $\mathbb{I}$) and the related continuum $\mathbb{H}^*$, the Čech-Stone remainder of the half line $\mathbb{H} = [0, \infty)$. In that chapter we will try to shed some light on the conjecture of Lelek, which states that every metric continuum which has zero span is chainable.

In the present chapter, however, we will restrict ourselves to elementary submodels. It appears that the Wallman representation of an elementary sublattice L of the lattice of all closed sets (even a base for the closed sets closed under finite union and finite intersection) of some continuum X is itself a *continuum* (a connected compact space, a space is said to be *connected* if it cannot be written as a union of two closed disjoint, nonempty sets). And so continua are determined by any of its lattice bases. We will show that many, but not all properties of X are shared by the Wallman representation of L.

2. Preliminaries

2.1. Wallman's representation theorem for lattices.

2.1.1. *Lattices.* A *lattice* is a structure in the language $\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}$ that models the universal closures of the following formulas.

$$\begin{aligned} a \sqcap b &= b \sqcap a \\ a \sqcup b &= b \sqcup a \end{aligned} \quad \text{(Commutativity)} \tag{6}$$

$$\begin{aligned} a \sqcap (b \sqcap c) &= (a \sqcap b) \sqcap c \\ a \sqcup (b \sqcup c) &= (a \sqcup b) \sqcup c \end{aligned} \quad \text{(Associativity)} \tag{7}$$

$$\begin{aligned} a \sqcap a &= a \\ a \sqcup a &= a \end{aligned} \quad \text{(Idempotence)} \tag{8}$$

$$\begin{aligned} a \sqcap \mathbf{0} = \mathbf{0} &\text{ and } a \sqcup \mathbf{0} = a \\ a \sqcap \mathbf{1} = a &\text{ and } a \sqcup \mathbf{1} = \mathbf{1} \end{aligned} \quad \text{(Zero and Unit)} \tag{9}$$

We can introduce an order in a lattice L by defining $a \leq b$ if $a \sqcap b = a$ or equivalently, if $a \sqcup b = b$. With this order we see that the for any lattice L, the (representation of the) constants $\mathbf{0}$ and $\mathbf{1}$ are the smallest respectively the largest elements of L.

EXAMPLE 2.1. If X is any topological space, we let 2^X denote the set of all its closed subsets. This set 2^X, with the operations $\cup$ and $\cap$ and constants $\emptyset$ and X is an example of a lattice. The order in this lattice is then given by set inclusion.

This lattice, for certain topological spaces X, will be at the base of our investigations in this chapter and the chapters following.

A lattice is called *distributive* if it also models the universal closure of the following formulas.

$$\begin{aligned} a \sqcap (b \sqcup c) &= (a \sqcap b) \sqcup (a \sqcap c) \\ a \sqcup (b \sqcap c) &= (a \sqcup b) \sqcap (a \sqcup c) \end{aligned}$$

A lattice is called *disjunctive* or separative if it models the sentence

$$\forall ab \exists c [a \not\leq b \rightarrow c \neq \mathbf{0} \wedge c \leq a \wedge c \sqcap b = \mathbf{0}]$$

A lattice is called *normal* if it models the sentence

$$\forall ab \exists cd[a \sqcap b = \mathbf{0} \rightarrow a \sqcap d = \mathbf{0} \wedge b \sqcap c = \mathbf{0} \wedge c \sqcup d = \mathbf{1}]$$

Note that this definition of normality of a lattice is similar to the definition of normality in a topological space using closed sets only.

2.1.2. *Wallman's representation theorem.* The Stone representation theorem tells us that every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a compact totally disconnected space. Wallman extended this theorem to distributive lattices.

THEOREM 2.2 ([55]). *If L is a distributive lattice then there is a compact T_1-space X with a base for its closed sets that is a homomorphic image of the lattice L. If L is also a disjunctive lattice then X has a base for its closed sets that is an isomorphic image of the lattice L. Furthermore, the space X is Hausdorff if and only if L is a normal lattice.*

REMARK 2.3. We will denote the compact T_1-space X from the theorem by wL to emphasize the fact that it is related to the lattice L.

The compact T_1-space wL is the space of ultrafilters of L where the topology of wL is generated by the subbase for the closed sets $\{A^* : A \in L\}$, where A^* is the set of ultrafilters on L that contain the element A of L. It is then easily seen that wL being Hausdorff and L being normal have the same truth value, and that L being a disjunctive lattice gives us that the (onto) homomorphism between L and $\{A^* : A \in L\}$, mapping A to A^* is one-to-one, and hence an isomorphism.

REMARK 2.4. If X is a compact Hausdorff space, then 2^X is a normal distributive and disjunctive lattice. Furthermore, the space $w(2^X)$ is homeomorphic to X. This easily follows from the fact that every ultrafilter on 2^X, for X compact Hausdorff, is fixed.

As mentioned in the introduction we will use Wallman's representation theorem to investigate compact Hausdorff spaces (and in particular continua). By taking an elementary sublattice L of 2^X we have that wL is a compact Hausdorff space which has a base for its closed sets which is isomorphic to the lattice L. That this holds follows from the fact that the properties of being normal, disjunctive and distributive are first order properties of lattice and hence reflected when we take elementary sublattices.

REMARK 2.5. Almost all of the time we will be dealing with a normal disjunctive distributive lattice L, mostly it will be an elementary sublattice of 2^X, where X is some compactum. By Wallman's representation theorem we know that the lattice L is isomorphic to a lattice base for the compactum wL. We will often abuse notation and say that L actually is a base for the topology on wL.

When we look at L as a lattice we will use the operations $\sqcup$, $\sqcap$ and the induced order $\leq$, when we see L as a base for the closed sets of wL we will use the operations $\cup$, $\cap$ and the order relation of set inclusion $\subset$, so no confusion can arise.

We shall frequently take countable elementary sublattices of 2^X; the following result, which follows from Urysohn's metrization theorem, shows that this will give us metrizable compact spaces.

THEOREM 2.6. *If L is a distributive normal and disjunctive lattice and also countable then wL is a compact and metrizable space.*

PROOF. This follows easily from the fact that wL is compact Hausdorff hence normal and the fact that L is (isomorphic to) a countable base for the closed sets of wL. □

2.2. Elementary reflection of properties of compacta. In the previous section we saw that a compact Hausdorff space is homeomorphic to the Wallman representation of the lattice 2^X, consisting of all closed subsets of X. Now consider an elementary sublattice L of 2^X; the Wallman representation wL of the lattice L is a compact space and will, in general, be a less complicated space then X. For instance, with the aid of theorem 2.6 we see that if L is countable then wL is metrizable.

We will be interested in what kind of properties the compact spaces X and wL must have in common. Consider the following definition.

DEFINITION 2.7. We will say that a property $\mathcal{P}$ of a compact space is *elementarily reflected* if whenever some compact space X has the property $\mathcal{P}$ then the Wallman representation wL of any elementary sublattice L of 2^X also has property $\mathcal{P}$.

We can extend this definition to *countably elementary reflection* if we consider countable elementary sublattices. We can also take an elementary sublattice in a special way, when we take a θ large enough and consider the model $H(\theta)$ of ZFC$^-$; when we take an elementary submodel $\mathfrak{M}$ of $H(\theta)$ containing the lattice 2^X as an element then $2^X \cap \mathfrak{M}$ is an elementary sublattice of the lattice 2^X. In this way the Wallman representation of the elementary sublattice $2^X \cap \mathfrak{M}$ will have more properties in common with the compact space X, as it is a subset of a submodel of ZFC$^-$, and therefore has more structure which can be seen from outside of the lattice (inside the elementary submodel $\mathfrak{M}$).

DEFINITION 2.8. We will say that a property $\mathcal{P}$ of continua is *elementarily reflected by submodels* if the property $\mathcal{P}$ is elementarily reflected when we restrict the elementary sublattices L of 2^X to the lattices of the form $2^X \cap \mathfrak{M}$, where $\mathfrak{M}$ is an elementary submodel of some $H(\theta)$ (θ large enough) which contains 2^X as one of its elements.

The goal of this chapter is to show that certain properties are elementarily reflected in some way.

2.2.1. *Atoms form a dense subset.* Suppose that X is a compact space and L an elementary sublattice of 2^X. Let $\text{atom}(L)$ denote the elements a of L for which we have

$$L \models \forall x[x \leq a \rightarrow x = \mathbf{0} \vee x = a]$$

By elementarity every a in $\text{atom}(L)$ is a point of the space X.

LEMMA 2.9. *If A is an element of L then* $\text{atom}(L) \cap A$ *is a dense set of A in the subspace topology.*

PROOF. Note that the lattice 2^X, and by elementarity also L models the following sentence

$$\forall ab \exists x[b \sqcap a \neq a \rightarrow \text{atom}(x) \wedge x \leq a \wedge x \sqcap b = \mathbf{0}]$$

As a is an element of L, this shows that for every open set U of wL which has nonempty intersection with a, we have $\text{atom}(L) \cap (a \cap U) \neq \emptyset$. This follows from the fact that if $U \cap a \neq \emptyset$ then there is a basic closed set b with $X \setminus U \subset b$ and $a \setminus b \neq \emptyset$. □

COROLLARY 2.10. *As X is certainly an element of the lattice L we have that the set* $\text{atom}(L)$ *is a dense subset of wL.*

REMARK 2.11. If X is a topological space and L is an elementary sublattice of 2^X then

$$\text{atom}(L) = L \cap \text{atom}(2^X).$$

2.2.2. *Components.* A set C is said to be a *component* of some topological space X if it is connected and if the inclusion $C \subset C^*$ implies $C = C^*$ for all connected subsets C^* of X. It is easily seen that the components of a topological space are closed and pairwise disjoint sets, and that every connected set is contained in one and only one component of the space.

For any lattice L and any $x \in L$ Let $\text{conn}(x)$ be shorthand for the following formula

$$\forall ab[a \sqcap b = \mathbf{0} \wedge a \sqcup b = x \rightarrow a \sqcap x = \mathbf{0} \vee b \sqcap x = \mathbf{0}]$$

So, for a compact space X the statement $2^X \models \text{conn}(\mathbf{1})$ actually says that X is connected, hence a continuum.

Suppose that X is a continuum and L an elementary sublattice of 2^X. Let x and y be elements of L such that $L \models x \leq y$ and $L \models \text{conn}(x)$. By elementarity this implies that x also is a connected subset of y in X. The component of x in y according to X is a closed subset C of y. The set C is uniquely defined by the equation

$$(10) \quad x \leq C \leq y \wedge \text{conn}(C) \wedge \forall D[C \leq D \leq y \wedge \text{conn}(D) \rightarrow C = D]$$

As for every such x and y there is some C as above in 2^X, by elementarity there is some element $\hat{C}$ in L such that L is a model of equation 10 when C is replaced by $\hat{C}$. By elementarity again $\hat{C}$ also satisfies equation 10 in 2^X, so $\hat{C} = C$. In general, elements of 2^X that are defined uniquely from elements of L must belong to L as well, hence $C \in L$. So L contains the component in y of any connected subset x, when x and y are elements of the lattice L.

3. A continuous image

The following theorem from [**17**] shows under what conditions between (certain) lattice bases for compact spaces X and Y, there exist a continuous map between X and Y. We give the proof of the theorem here as well, as it is one of the central theorems of this chapter and the ones following.

THEOREM 2.12 ([**17**]). *Let X and Y be compact Hausdorff spaces and let $\mathcal{C}$ be a lattice base for X. Then X is a continuous image of Y if and only if there is a map $\phi : \mathcal{C} \to 2^Y$ such that*

(1) *$\phi(\emptyset) = \emptyset$, and if $F \neq \emptyset$ then $\phi(F) \neq \emptyset$,*
(2) *if $F \cup G = X$ then $\phi(F) \cup \phi(G) = X$, and*
(3) *if $F_0 \cap \ldots \cap F_n = \emptyset$ then $\phi(F_0) \cap \cdots \cap \phi(F_n) = \emptyset$.*

PROOF. Necessity is easy: given a continuous onto map $f : Y \to X$, let $\phi(F) = f^{-1}[F]$. Then ϕ is even a lattice embedding.

To prove sufficiency, let $\phi : \mathcal{C} \to 2^Y$ be given and consider for every $y \in Y$ the family $\mathcal{F}_y = \{F \in \mathcal{C} : y \in \phi(F)\}$. The intersection $\bigcap \mathcal{F}_y$ consists of exactly one point. Indeed, by condition 3 the family $\mathcal{F}_y$ has the finite intersection property, so the intersection is non-empty. Take $x_1 \neq x_2$ in X; there exist, as X is normal, F and G in $\mathcal{C}$ such that $x_1 \notin F$, $x_2 \notin G$ and $F \cup G = X$. Then by condition 2 we have that either $y \in \phi(F)$ so $F \in \mathcal{F}_y$ and thus $x_1 \notin \bigcap \mathcal{F}_y$ or $y \in \phi(G)$, $G \in \mathcal{F}_y$ and thus $x_2 \notin \bigcap \mathcal{F}_y$.

We define $f(y)$ to be the unique point in $\bigcap \mathcal{F}_y$.

To show that f is continuous and onto we will show that for every closed subset F of X we have

$$f^{-1}[F] = \bigcap \{\phi(G) : G \in \mathcal{C} \text{ and } F \subset \operatorname{int} G\}. \tag{11}$$

We will first show that the set on the right-hand side has the finite intersection property. Even though F and the complement K of $\bigcap_i G_i$ need not belong to $\mathcal{C}$ we can still find G and H in $\mathcal{C}$ such that $G \cap K = H \cap F = \emptyset$ and $G \cup H = X$. Indeed, apply compactness and the fact that $\mathcal{C}$ is closed under finite intersections and finite unions to find a C in $\mathcal{C}$ such that $F \subset C \subset \bigcap_i \operatorname{int} G$ and then $D \in \mathcal{C}$ with $K \subset D$ and $C \cap D = \emptyset$. Apply normality of $\mathcal{C}$ to C and D to find the G and H in $\mathcal{C}$ we need. Once we have these G and H we see that for each i we also have $H \cup G_i = X$, and so $\phi(H) \cup \phi(G_i) = Y$. Combined with $\phi(G) \cap \phi(H) = \emptyset$ we get $\phi(F) \subset \bigcap_i \phi(G_i)$.

To verify equation 11, first let $y \in Y \setminus f^{-1}[F]$. As above we find G and H in $\mathcal{C}$ such that $f(y) \notin G$, $G \cup H = X$, and $H \cap F = \emptyset$. The first property gives us that $x \notin \phi(G)$, the other two imply that $F \subset \operatorname{int} G_i$.

Second, if $F \subset \operatorname{int} G_i$, then we can find $H \in \mathcal{C}$ such that $G \cup H = Y$ and $F \cap H = \emptyset$. It follows that if $x \notin \phi(G)$ then $x \in \phi(G)$. Hence $f(x) \in H$ and so $f(x) \notin F$. □

COROLLARY 2.13. *If X is a compact space, and L an elementary sublattice of 2^X then the map sending $x \in X$ to the ultrafilter $\{A \in L : x \in A\}$ of L is a continuous onto map from X onto wL. Because X is compact and wL Hausdorff this map is closed.*

By the last corollary, a lot of properties of compact spaces are elementarily reflected.

THEOREM 2.14. *Being connected is a property of compact spaces that is elementarily reflected.*

So this theorem gives us a way to investigate a continuum by one of its lattice bases, as the Wallman representation of any lattice base of a continuum is homeomorphic to this continuum.

THEOREM 2.15. *Being locally connected is a property of compact spaces that is elementarily reflected.*

Being metrizable is a property of compact spaces that is also elementarily reflected, as the following theorem shows.

THEOREM 2.16. *If X is a metrizable compact space and L an elementary sublattice of 2^X, then wL is also metrizable.*

PROOF. By theorem 2.12 and its corollary there exist a closed onto map $f : X \to wL$. By compactness of wL it suffices to show that wL is second countable.

As the space X is compact and metrizable, its weight is countable. As closed maps preserve weight and wL is a continuous image of X under a closed map, it also has countable weight. □

Under some restrictions on X, we have that the map from theorem 2.12 has a certain nice form.

THEOREM 2.17. *If L is a elementary sublattice of 2^X, where X is some locally connected continuum, then the map from theorem 2.12 is monotone.*

PROOF. We know that wL is a locally connected continuum. We want to show that for every x in wL the fiber $f^{-1}(x)$ is connected.

As wL is compact Hausdorff every one of its points is equal to the intersection of the closures of its neighborhoods. Because L is (isomorphic to) a lattice base of wL we have for every $F \in L$ such that $x \notin F$ an element G of L such that $F \subset G$ and $wL \setminus G$ is a connected subset of wL. By elementarity this gives us that $X \setminus G$ is a connected open subset of X, and so, in wL is x the intersection of the subcontinua of wL containing it. We see that $f^{-1}(x)$ also is an intersection of a filter of continua, and hence a nonempty continuum itself, as f maps onto wL. □

The last theorem will provide us with a number of properties that are elementarily reflected.

4. Base-free properties

Following Bankston (see for instance [**4**], [**5**]) we call a $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$-sentence ϕ *base-free* if, for any compactum X and any lattice base $\mathcal{C}$ of X we have

$$\mathcal{C} \models \phi \text{ if and only if } 2^X \models \phi.$$

These base-free properties are properties that are elementarily reflected, because by elementarity if 2^X models such a base-free property ϕ, a sentence in the first-order language over $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$ then any elementary sublattice L of 2^X models such a sentence, hence the compact space wL shares this base-free property with X.

4.1. Being connected. By theorem 2.14 we know that connectedness is a property of compact spaces that is elementarily reflected. We will give here another proof using elementarity of L, the proof will be illustrative for the way we prove theorems in the rest of this chapter.

LEMMA 2.18. *If $\mathcal{C}$ is a lattice base for X, a compact space, then $\mathcal{C}$ contains all the clopen subsets of X.*

PROOF. Let A be a clopen subset of X then A, being closed can be seen as an intersection of elements from the base $\mathcal{C}$. As A also is open, by compactness of X there exist a finite number of elements $A_1, \ldots, A_n$ of $\mathcal{C}$ such that $A \subset \bigcap_i A_i \subset A$. As $\mathcal{C}$ is closed under finite intersections we have that $A = \bigcap_i A_i \in \mathcal{C}$. □

Now suppose that X is a continuum, L an elementary sublattice of 2^X and wL is not connected. By the previous lemma there are closed disjoint nonempty elements A and B of L such that $wL = A \cup B$. So the lattice L models the following sentence

$$\exists ab[a \sqcup b = \mathbf{1} \wedge a \sqcap b = \mathbf{0} \wedge a \neq \mathbf{0} \wedge b \neq \mathbf{0}]$$

As L is an elementary sublattice of 2^X, it then also holds that 2^X models this same sentence. But this implies that X is not connected in contradiction with the assumptions we made on X. This ends the alternative proof of theorem 2.14.

What we in fact proved here is the following theorem.

THEOREM 2.19. *The lattice sentence* conn($\mathbf{1}$), *which expresses connectedness, is base-free.*

4.2. Indecomposability. A continuum X is *indecomposable* if it cannot be written as the union of two proper subcontinua. In [**5**] the following lattice sentence is shown to be a base-free lattice sentence indicating exactly when a compact space is indecomposable.

$$\begin{aligned}\forall ab \exists xy \quad & [a \sqcup b = \mathbf{1} \wedge a \neq \mathbf{1} \wedge b \neq \mathbf{1} \rightarrow \\ & x \sqcap y = \mathbf{0} \wedge x \sqcup y = b \wedge x \not\leq a \wedge y \not\leq a].\end{aligned}$$

So the following theorem follows instantly.

THEOREM 2.20. *Indecomposability is a property of compact spaces that is elementarily reflected.*

4.3. Hereditary indecomposability. A *hereditarily indecomposable* continuum is a continuum for which each of its subcontinua is indecomposable. This is equivalent to saying that for every two subcontinua of X that meet, one is contained in the other.

This property also makes sense for arbitrary compact Hausdorff spaces, so we can extend the notion of hereditary indecomposability to compact Hausdorff spaces by saying that a compact Hausdorff space X is hereditary indecomposable if and only if for every two subcontinua of X that meet one must be contained in the other.

We shall use a characterization of hereditary indecomposability that can be gleaned from [**36**]. For this we will introduce some terminology.

Let X be a compact Hausdorff space and let A and B be disjoint closed subsets of X; we say that (X, A, B) is *crooked* between the neighborhoods U of A and V of B if we can write $X = X_0 \cup X_1 \cup X_2$, where each X_i is closed, and, moreover, $A \subset X_0$, $X_0 \cap X_1 \subset V$, $X_0 \cap X_2 = \emptyset$, $X_1 \cap X_2 \subset U$ and $B \subset X_2$. We say that X is crooked between A and B if (X, A, B) is crooked between any pair of neighborhoods of A and B; no generality is lost if we consider pairs of disjoint neighborhoods only, as crookedness between small neighborhoods implies crookedness between large neighborhoods.

The characterization of hereditary indecomposability we will use is the following.

THEOREM 2.21 (Krasinkiewicz and Minc [**36**]). *A compact space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed (nonempty) subsets.*

We can translate this characterization of hereditary indecomposability in terms of closed sets only. We get the following formulation.

THEOREM 2.22. *A compact Hausdorff space X is hereditarily indecomposable if and only if whenever four closed sets C, D, F and G in X are given such that $C \cap D = C \cap F = D \cap G = \emptyset$ one can write X as the union of three closed sets X_0, X_1 and X_2 such that $C \subset X_0$, $D \subset X_2$, $G \cap X_0 \cap X_1 = \emptyset$, $X_0 \cap X_2 = \emptyset$, and $F \cap X_1 \cap X_2 = \emptyset$.*

In [**17**] it is shown that this characterization of hereditarily disconnectedness of a compact Hausdorff space depends only on a base for the closed sets of X closed under finite intersections.

So if a normal distributive and disjunctive lattice L models the sentence

$$\begin{aligned}&\forall abcd\exists xyz\,[a\sqcap b=\mathbf{0}\wedge a\sqcap c=\mathbf{0}\wedge b\sqcap d=\mathbf{0}\rightarrow\\ (12)\qquad &a\sqcap(y\sqcup z)=\mathbf{0}\wedge b\sqcap(x\sqcup y)=\mathbf{0}\wedge x\sqcap z=\mathbf{0}\wedge\\ &x\sqcap y\sqcap d=\mathbf{0}\wedge y\sqcap z\sqcap c=\mathbf{0}\wedge x\sqcup y\sqcup z=\mathbf{1}].\end{aligned}$$

then its Wallman representation wL is hereditarily indecomposable compact space. So hereditary indecomposability is a base-free property, which gives us immediately the following reflection theorem.

THEOREM 2.23. *Hereditary indecomposability is a property of compact spaces that is elementarily reflected.*

4.4. Covering dimension. The following theorem is well known, a proof of it can be found for instance in [**19**].

THEOREM 2.24. *A normal space X has covering dimension* $\dim X \leq n$ *if and only for if every $(n+2)$-element family of $\{B_m\}_{m=1}^{n+2}$ of closed subsets of the space X satisfying $\bigcap_{m=1}^{n+2} B_m = \emptyset$ there exists a closed cover $\{F_m\}_{m=1}^{n+2}$ of the space X such that $\bigcap_{m=1}^{n+2} F_m = \emptyset$ and $B_m \subset F_m$ for every m.*

If X is compact and C is a base for the closed sets of X closed under finite intersections then we can sharpen the theorem by stating that all the closed sets mentioned in the theorem come from the base C.

Recall that a *swelling* of a family $\{A_s\}_{s\in S}$ of subsets of a space X is a family $\{B_s\}_{s\in S}$ of subsets of X such that $A_s \subset B_s$ for every s and for every finite set $\{s_1,\ldots,s_n\}\subset S$ we have

$$A_{s_1}\cap\cdots\cap A_{s_n}=\emptyset \text{ if and only if } B_{s_1}\cap\cdots\cap B_{s_n}=\emptyset.$$

A well known fact is that every finite family $\{F_i\}_{i=1}^k$ of closed subsets of some compact space X has a swelling $\{U_i\}_{i=1}^k$ consisting of closed sets. If also a base for the closed sets of X is given, these U_i can be chosen in such a way that they are intersections of a finite number of base elements. If, moreover, a family $\{V_i\}_{i=1}^k$ of open subsets of X satisfying $F_i \subset V_i$ for all i is given then the swelling $\{U_i\}_{i=1}^k$ can be chosen such that $U_i \subset V_i$ for all i (for a proof see for instance [**19**]).

This brings us to the following theorem on the covering dimension of the Wallman representation certain lattices.

THEOREM 2.25. *If L is a normal distributive and disjunctive lattice with zero and a unit that models the sentence $\phi_{\leq n}$, which is given by*

$$\begin{aligned}&\forall x_1\cdots x_{n+2}\exists y_1\cdots y_{n+2}[x_1\sqcap\cdots\sqcap x_{n+2}=\mathbf{0}\rightarrow\\ (13)\quad &y_1\sqcap\cdots\sqcap y_{n+2}=\mathbf{0}\wedge y_1\sqcup\cdots\sqcup y_{n+2}=\mathbf{1}\wedge\bigwedge_{i=1}^{n+2}(x_i\leq y_i)],\end{aligned}$$

then its Wallman representation wL is a compact Hausdorff space with covering dimension $\leq n$.

The lattice sentence $\phi_{\leq n}$ is base-free, so having covering dimension $\leq n$ is elementarily reflected for compact spaces. This also holds for the other direction; if X is a compact space and L is some elementary sublattice of 2^X such that $\dim wL > n$ then $L \models \neg\phi_{\leq n}$ and, by elementarity so does 2^X. We have the following theorem.

THEOREM 2.26. *The lattice sentence $\phi_{\leq n} \wedge \neg\phi_{\leq n-1}$ is base free.*

COROLLARY 2.27. *Having covering dimension equal to n ($n \leq \infty$) is a property of compact spaces that is elementarily reflected.*

REMARK 2.28. Using the reflection result from the previous section, corollary 2.27, we can easily show that the value of the large inductive dimension is, in general, not elementarily reflected. There exist compact spaces for which its covering dimension is strictly smaller than its large inductive dimension (see, for instance [**42**]). Let X be such a compact space and L a (any) countable elementary sublattice of 2^X. By theorem 2.6 the Wallman representation wL of L is a compact metrizable space. The large inductive dimension and the covering dimension of such spaces are equal to each other. As $\dim X = \dim wL$ this shows that $\operatorname{Ind} X \neq \operatorname{Ind} wL$.

4.5. Strong infinite (covering) dimension. As we have already seen, the covering dimension of a continuum is elementarily reflected. So, if a continuum X is infinite - dimensional and L an elementary sublattice of 2^X, then the (covering) dimension of wL also is infinite. But there are different kinds of infinite - dimensionality. We will look at one of its strongest forms.

We call a family $\{(A_i, B_i) : i \in I\}$ (I can be finite or infinite) of pairs of closed disjoint subsets *essential* if, whenever we take for all i L_i's that separate the sets A_i and B_i, the intersection $\bigcap L_i$ is nonempty.

If X is a normal space then $\dim X \geq n$ if and only if there exists an essential family of size n. We write $\dim X = \infty$ if $\dim X \geq n$ for all n. So if $\dim X = \infty$ then X has arbitrarily large finite essential families. If X has an infinite essential family we say that X is *strongly infinite - dimensional.*

EXAMPLE 2.29. Consider a Tychonoff cube $\mathbb{I}^\kappa$. For every $\alpha < \kappa$ let $A_\alpha = \{x \in \mathbb{I}^\kappa : x(\alpha) = 0\}$ and $B_\alpha = \{x \in \mathbb{I}^\kappa : x(\alpha) = 1\}$. Then the family $\{(A_\alpha, B_\alpha) : \alpha < \kappa\}$ is an essential family of $\mathbb{I}^\kappa$.

For finite κ this follows from the Brouwer fixed point theorem (for a proof see [**19**], 7.3.19). If κ is infinite we put, for every finite subset a of κ, $F_a = \pi_a^{-1}[\bigcap_{\alpha\in a} \pi_a[L_\alpha]]$, where π_a denotes the projection onto the sub cube $\mathbb{I}^a$. By the finite case, each F_a is nonempty and $a \subset b$ implies $F_a \supset F_b$, so $\bigcap_a F_a \neq \emptyset$. Also $\bigcap_a F_a = \bigcap_\alpha L_\alpha$ as the sets A_α, B_α and

L_α can be separated by canonical closed sets, which depend only on a finite number of coordinates.

This gives us that the Hilbert cube $Q = \mathbb{I}^\infty$ is strongly infinite - dimensional.

Following the example above we call a continuous map $f : X \to \mathbb{I}^\kappa$ *essential* if the family $\{(f^{-1}[A_\alpha], f^{-1}[B_\alpha]) : \alpha < \kappa\}$ is essential. Using Urysohn's lemma it is easy to see that X has an essential family of size κ if and only if X admits an essential map onto $\mathbb{I}^\kappa$.

The question arises if the property of being a strongly infinite - dimensional compactum is elementarily reflected. We have the following partial result.

THEOREM 2.30. *If L is an elementary sublattice of 2^Q, the lattice of all closed subsets of the Hilbert cube, then wL is strongly infinite - dimensional.*

Let us translate some properties related to strong infinite - dimensionality into the lattice language.

Let ψ_n be the lattice sentence which is the universal closure of the conjunction of the following two formulas

(1) $\bigwedge_{i\leq n} a_i \sqcap b_i = \mathbf{0}$.
(2) $\bigwedge_{i\leq n}(a_i \sqcap c_i = \mathbf{0} \wedge b_i \sqcap d_i = \mathbf{0} \wedge c_i \sqcup d_i = \mathbf{1}) \to c_1 \sqcap d_1 \sqcap \cdots \sqcap c_n \sqcap d_n \neq \mathbf{0}$,

whose free variables are exactly the ones from $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. If X is a compact space and $\mathcal{C}$ a lattice base for X, if

$$\mathcal{C} \models \exists a_1 \cdots a_n b_1 \cdots b_n \psi_n(a_1, \ldots, a_n, b_1, \ldots, b_n),$$

then X has an essential family of size n.

Let ϕ_n denote the universal closure of the following lattice formula

$$\text{(14)} \quad \exists xy \psi_n(a_1, \ldots, a_n; b_1, \ldots, b_n) \to \psi_{n+1}(a_1, \ldots, a_n, x; b_1, \ldots, b_n, y),$$

Now, if the lattice base $\mathcal{C}$ of X above also models the sentences $\{\phi_n : n < \omega\}$ then we can say that $w\mathcal{C}$ is strong infinite - dimensional as this space has some essential family of size nonzero and every essential family can be extended to a (larger) essential family. We will show that this is the case for elementary sublattices of a lattice base of the Q.

LEMMA 2.31. *Every finite essential family of the Hilbert cube can be extended to a larger essential family.*

PROOF. Suppose that $\{(A_i, B_i) : i \leq n\}$ is an essential family of the Hilbert cube Q. Consider the base $\mathcal{B}$ of all closed sets of Q that has as subbase the family $\mathcal{P}$ consisting of closed sets of the form $\pi_n^{-1}[[p, q]]$, where n ranges over ω and $p < q$ are rational numbers from the interval $\mathbb{I}$. We can find elements C_i and D_i of $\mathcal{B}$ that contain the sets A_i and B_i respectively, such that the family $\{(C_i, D_i) : i \leq n\}$ also is an essential family. The sets C_i and D_i depend only on a finite number of coordinates, so we can find a coordinate $m < \omega$ such that all m-th coordinate of the closed sets C_i and D_i (so also A_i and B_i) equal $\mathbb{I}_m$. This implies that we can extend the essential family to an essential family of size $n+1$ when we add the pair $(\{x \in Q : x(m) = 0\}, \{x \in Q : x(m) = 1\})$. □

We are now able to give the proof of theorem 2.30.

PROOF. (theorem 2.30) Let L be an elementary sublattice of 2^Q, the lattice of all closed subsets of the Hilbert cube Q. And suppose that $\{(A_i, B_i); i \leq n\}$ is an essential family in wL. We can find elements a_i and b_i of L such that $A_i \subset a_i$ and $B_i \subset b_i$ for all i and $\{(a_i, b_i) : i \leq n\}$ is an essential family in wL. So we have

$$L \models \psi_n[a_1, \ldots, a_n; b_1, \ldots, b_n].$$

As $\mathcal{C}$ models the universal closure of formula 14, and L is an elementary sublattice of $\mathcal{C}$, there exists members x and y of L such that

$$L \models \psi_{n+1}[a_1, \ldots, a_n, x; b_1, \ldots, b_n, y].$$

Which means that any finite essential family in wL can be extended to a larger essential family in wL, so wL is strongly infinite - dimensional. □

The following questions remain.

QUESTION 2.32. *Is having strong infinite (covering) dimension elementarily reflected, and is having not strong infinite (covering) dimension elementarily reflected.*

5. Other elementarily reflected properties

5.1. Connectedness im Kleinen. A continuum X is said to be *connected im Kleinen at x* (cik at x) if for every open neighborhood U of x there exists a subcontinuum K of X and an open subset V of X such that

$$x \in V \subset K \subset U.$$

A continuum is called *connected im Kleinen* (cik) if it is cik at x for all $x \in X$.

THEOREM 2.33. *A continuum X is cik if and only if X is locally connected.*

For a proof see for instance [**48**].

As being locally connected is elementarily reflected for continua, we also have that being cik is elementarily reflected for continua.

THEOREM 2.34. *If X is cik at x and L is an elementary sublattice of 2^X containing $\{x\}$, then wL is cik at x.*

PROOF. The lattice 2^X models the following lattice sentence with respect to the point x

$$\forall F \exists KG \quad [x \sqcap F = \mathbf{0} \rightarrow \\ \text{conn}(K) \wedge K \sqcup G = \mathbf{1} \wedge x \sqcap G = \mathbf{0} \wedge K \sqcap F = \mathbf{0}].$$

Let $U \subset wL$ be an open neighborhood of x. There exists an $F \in L$ such that $x \notin F$ and $F \cup U = X$. By elementarity there exist $K, G \in L$ such that $K \subset wL$ is a subcontinuum containing x, $x \notin G$ and $K \cup G = wL$. This shows that wL is cik at x. □

5.2. Aposyndesis. A space X is said to be *aposyndetic at p with respect to q*, provided that there is a subcontinuum A of $X \setminus \{q\}$ such that p is an interior point of A (in X).

$$2^X \models \exists AF[\text{conn}(A) \wedge A \sqcap q = \mathbf{0} \wedge p \sqcap F = \mathbf{0} \wedge A \sqcup F = \mathbf{1}]$$

A continuum X is said to be *aposyndetic at p* provided it is aposyndetic at p with respect to each point $q \in X \setminus \{p\}$.

$$2^X \models \forall q \exists AF\, [\text{atom}(q) \wedge q \neq p \rightarrow \\ \text{conn}(A) \wedge A \sqcap q = \mathbf{0} \wedge p \sqcap F = \mathbf{0} \wedge A \sqcup F = \mathbf{1}].$$

A continuum X is said to be *aposyndetic* provided it is aposyndetic at each of its points.

$$2^X \models \forall pq \exists AF\, [\text{atom}(p) \wedge \text{atom}(q) \wedge p \neq q \rightarrow \\ \text{conn}(A) \wedge A \sqcap q = \mathbf{0} \wedge p \sqcap F = \mathbf{0} \wedge A \sqcup F = \mathbf{1}].$$

It is obvious that the following theorem holds.

THEOREM 2.35. *Let X be a continuum which is aposyndetic at p with respect to q then for any elementary sublattice L of 2^X containing p and q its Wallman extension wL is aposyndetic at p with respect to q.*

On the other hand, it is not straightforward that if X is an aposyndetic continuum and $L \prec 2^X$ that wL is also an aposyndetic continuum. To see if wL is aposyndetic at p with respect to q we then only have to consider the cases where at least one of the points p and q is not an atom of L.

Consider the case we want to show that wL is aposyndetic at p with respect to q, where $p \in \textit{atom}(L)$ and $q \in wL \setminus \text{atom}(L)$. For this to be true we need 2^X to contain for every closed set F not containing p some subcontinuum A that contains p in its interior and misses F, which would mean that X is connected im Kleinen at p.

THEOREM 2.36. *If X is cik at p and L is an elementary sublattice of 2^X such that $p \in$ atom(L) then wL is aposyndetic at p.*

In [**48**] the following theorem is mentioned as an exercise.

THEOREM 2.37. *A continuum is aposyndetic if and only if it is semi - locally connected.*

We say that a continuum X is semi - locally connected at p if for every open neighborhood U of p contains a neighborhood V of p such that $X \setminus V$ has finitely many components.

This definition of semi-locally connectedness is not first - order, but it does show the following reflection result.

THEOREM 2.38. *Aposyndesis is a property of compacta that is elementary reflected by submodels.*

The following question remains.

QUESTION 2.39. *Is aposyndesis a property of compact spaces that is elementarily reflected?*

5.3. Unicoherence. We say that a nonempty space X is *unicoherent* if it is connected and if for each pair $\{A, B\}$ of closed connected subsets of X, we have that $A \cup B = X$ implies that $A \cap B$ is connected.

Translating the property unicoherence into lattice formulas, a topological space X is unicoherent if and only if 2^X models the lattice sentence

$$\forall AB[\text{conn}(A) \wedge \text{conn}(B) \wedge A \cup B = \mathbf{1} \rightarrow \text{conn}(A \cap B)] \tag{15}$$

THEOREM 2.40. *Unicoherence is elementarily reflected.*

PROOF. Suppose this is not the case and there is some unicoherent continuum X and an elementary sublattice L of 2^X such that wL is not unicoherent. Suppose that A and B are closed connected subsets of wL that have a disconnected intersection. Without loss of generality we can assume that there are closed disjoint nonempty subsets C and D of wL such that $A \cap B = C \cup D$. As L is a lattice base for wL there are x and y in L such that $C \subset x$, $D \subset y$, $x \cap y = \emptyset$, and $A \setminus (x \cup y)$ and $B \setminus (x \cup y)$ are both nonempty.

CLAIM 2.41. $(\text{atom}(L) \cap B) \setminus (x \cup y)$ *is nonempty.*

PROOF. Note that, without loss of generality, by compactness of wL we can assume that $C \subset \text{int}(x)$ and $D \subset \text{int}(y)$.

By compactness of wL and the fact that L is a lattice base for wL, there some a in L such that $A \subset a \subset A \cup (\text{int}(x) \cup \text{int}(y))$. This implies that $a \cap B \subset x \cup y$. As $B \setminus (x \cup y) \neq \emptyset$ we have that $B \setminus (a \cup x \cup y)$ is an open nonempty subset of wL and therefore contains elements of L that are atoms. □

Let $z_a \in A$ be an atom of L that is not contained in B, and $z_b \in B$ and atom of L not contained in A.

There exist Z_a and Z_b in L such that $A \subset Z_a \cup x \cup y$, $B \subset Z_b \cup x \cup y$ and $Z_a \cap Z_b \subset x \cup y$. Let C_a denote the component of z_a in $Z_a \cup x \cup y$ and C_b be the component of z_b in $Z_b \cup x \cup y$. By elementarity $C_a, C_b \in L$, see section 2.2. As A and B are connected we have that $A \subset C_a$ and $B \subset C_b$. As we also have that $Z_a \cap Z_b \subset x \cup y$ we have $C_a \cap C_b \subset x \cup y$, but this is in contradiction with equation 15 and the fact that L is an elementary sublattice of 2^X. □

5.4. Discoherence. A space X is called *discoherent* if it holds that for every closed subsets A and B of X such that $A \cup B = X$ and $A \neq X \neq B$ we also have that $A \cap B$ is not connected. In lattice terms we have

$$(16) \qquad 2^X \models \forall ab[a \neq \mathbf{0} \wedge b \neq \mathbf{0} \wedge a \sqcup b = \mathbf{1} \rightarrow \neg\,\mathrm{conn}(a \sqcap b)]$$

EXAMPLE 2.42. The circle S is a locally connected discoherent continuum.

THEOREM 2.43 ([**34**]). *The properties of being unicoherent and discoherent are invariant under continuous monotone mappings.*

PROOF. Let $f : X \rightarrow Y$ be continuous monotone and onto. Let A and B be two closed connected sets such that $A \cup B = f(X) = Y$. Hence $f^{-1}(A) \cup f^{-1}(B) = X$ and the sets $f^{-1}(A)$ and $f^{-1}(B)$ are closed and connected. Therefore, if the space X is unicoherent the set

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

is connected and so is the set

$$A \cap B = f[f^{-1}(A \cap B)].$$

hence the space $f(X) = Y$ is unicoherent.

If X is discoherent, the set $f^{-1}(A \cap B)$ is not connected. If $A \cap B$ were connected, by monotonicity we would have that $f^{-1}(A \cap B)$ is connected, so we can conclude that the set $A \cap B$ is not connected. Hence the space $f(X) = Y$ is discoherent. □

Together with theorem 2.17, this implies the following reflection theorem for compact spaces.

THEOREM 2.44. *Being locally connected and discoherent is a property of compact spaces that is elementarily reflected.*

QUESTION 2.45. *Is being not discoherent a property of compact spaces that is elementarily reflected?*

5.5. Hereditary unicoherence. A continuum is said to be *hereditary unicoherent* if every subcontinuum is unicoherent. Equivalently, if every two subcontinua have connected intersection.

A continuum X is hereditary unicoherent if the lattice 2^X models the following lattice sentence.

$$(17) \qquad \forall ab[\mathrm{conn}(a) \wedge \mathrm{conn}(b) \rightarrow \mathrm{conn}(a \sqcap b)].$$

THEOREM 2.46. *Being a hereditarily unicoherent continuum is a property of continua that is elementary reflected.*

Proof. Suppose that X is a hereditarily unicoherent continuum. Let L be some elementary sublattice of 2^X and suppose that wL is not hereditary unicoherent. Let A and B be closed connected subsets of wL such that $A \cap B$ is not connected.

There exist nonempty disjoint C and D in L such that $A \cap B \subset \operatorname{int}(C \cup D)$ and $C \cap (A \cap B)$ and $D \cap (A \cap B)$ are both nonempty.

There exist F and G in L such that $A \subset F$ and $F \cap B \setminus \operatorname{int}(C \cup D) = \emptyset$ and also $B \subset G$ and $G \cap A \setminus \operatorname{int}(C \cup D) = \emptyset$.

If A^+ and B^+ denote the components of A in F respectively of B in G, then we have that the lattice 2^X models the lattice formula which states that there exist elements x and y of the lattice, components of F and G respectively, such that $x \sqcap y \leq C \cup D$, $x \sqcap y \sqcap C \neq \mathbf{0}$ and $x \sqcap y \sqcap D \neq \mathbf{0}$. This is in contradiction with the assumption of hereditary unicoherence of X. □

Remark 2.47. Note also that hereditary unicoherence is invariant under monotone mappings. Suppose X is hereditary unicoherent and $f : X \to Y$ is a monotone onto map. Let A and B be two nonempty subcontinua of Y. As $f^{-1}[A]$ and $f^{-1}[B]$ are both subcontinua of X and X is hereditary unicoherent, we have that $f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B]$ is also a subcontinuum of X. This implies that $ff^{-1}[A \cap B] = A \cap B$ is a subcontinuum of Y.

5.6. Triodicity. A continuum is called *triodic* if it contains three proper subcontinua whose common intersection is nonempty and for which every union of two of these subcontinua is a proper subset of the whole continuum.

Translated into a lattice formula we have that a continuum X is triodic if the lattice 2^X is a model for

$$\begin{aligned} \exists xyz \quad & [x \neq y \wedge x \neq z \wedge y \neq z \wedge \operatorname{conn}(x) \wedge \operatorname{conn}(y) \wedge \\ & \operatorname{conn}(z) \wedge x \sqcup y \sqcup y = \mathbf{1} \wedge x \sqcap y \sqcap z \neq \mathbf{0} \wedge \\ & x \sqcup y \neq \mathbf{1} \wedge x \sqcup z \neq \mathbf{1} \wedge y \sqcup z \neq \mathbf{1}]. \end{aligned} \tag{18}$$

It is easily seen that an elementary sublattice L of 2^X contains three elements that are witnesses of the fact that L models the sentence in equation 18. This triple shows then that the Wallman representation of L is triodic.

5.7. Decomposability. A continuum is *decomposable* if it contains two proper subcontinua such that the union covers the whole space.

Translated into a lattice formula we have that a continuum X is decomposable if the lattice 2^X is a model for

$$\exists xy[\operatorname{conn}(x) \wedge \operatorname{conn}(y) \wedge x \neq \mathbf{1} \wedge y \neq \mathbf{1} \wedge x \sqcup y = \mathbf{1}]. \tag{19}$$

As triodicity, it is easy to see that being a decomposable continuum is elementarily reflected.

5.8. Irreducible. A continuum is said to be *irreducible between p and q* if the only subcontinuum of X containing both points p and q is X itself. A continuum is called *irreducible* if there exist points p and q such that the continuum is irreducible between p and q.

Let X be an irreducible continuum and let L be some elementary sublattice of 2^X. As X is irreducible, the lattice 2^X models the following lattice sentence.

$$\exists xy\forall A[\text{atom}(x) \wedge \text{atom}(y) \wedge x \neq y \wedge \text{conn}(A) \wedge x \sqcup y \leq A \rightarrow A = \mathbf{1}].$$

As L is an elementary sublattice of 2^X, it also models this sentence. Let $p, q \in L$ be two different atoms that fulfill sentence above for L.

CLAIM 2.48. *wL is irreducible between p and q.*

PROOF. Suppose not, then for some proper subcontinuum $A \subset wL$ we have $p, q \in A$. By elementarity A cannot be an element of L, but it is an intersection of elements of L. Let $\mathcal{A}$ be a finite subset of L such that its intersection contains A and is not equal to wL. As $\bigcap \mathcal{A} \in L$, let $A^* \in L$ be the component of $\bigcap \mathcal{A}$ containing the point p (hence the point q). This implies that L models the following sentence

$$p \sqcup q \leq A^* \wedge A^* \neq \mathbf{1} \wedge \text{conn}(A^*),$$

which contradicts the fact that L is an elementary sublattice of 2^X. □

THEOREM 2.49. *Being irreducible is a property of continua that is elementarily reflected.*

5.9. (Locally) arc/path-wise connectedness. A space X is *pathwise connected* if for every pair x, y of points of X there exists a continuous mapping $f : \mathbb{I} \rightarrow X$ of the closed unit interval $\mathbb{I}$ to the space X satisfying $f(0) = x$ and $f(1) = y$. The space X is called *arc-wise connected* if the map f above may be chosen to be one-to-one. It is easily seen that pathwise connectedness is an invariant of continuous mappings.

THEOREM 2.50 ([**19**]). *A Hausdorff space is pathwise connected if and only if it is arcwise connected.*

This theorem and theorem 2.12 give us the following elementary reflection result for compact spaces.

THEOREM 2.51. *The property of being path/arc-wise connected is elementarily reflected for compact spaces.*

Something similar can be said for locally pathwise connectedness and locally arcwise connectedness. A space X is *locally pathwise (arc-wise) connected* if for every $x \in X$ and every open neighborhood U of x there exists an open neighborhood V of x such that for every $y \in V$ there exists a continuous (one-to-one) mapping $f : \mathbb{I} \rightarrow U$ satisfying $f(0) = x$ and $f(1) = y$.

THEOREM 2.52 ([**19**]). *Local pathwise connectedness is an invariant of quotient mappings.*

As Hausdorff spaces are locally arcwise connected if and only if they are locally pathwise connected, and the continuous map from theorem 2.12 is closed and hence quotient we get the following reflection theorem.

THEOREM 2.53. *Local pathwise connectedness (hence arcwise connectedness) is a property of compact spaces that is elementarily reflected.*

5.10. Peano continua. Locally connected metric continua are also known as *Peano continua*. Theorem 2.12 and theorem 2.16 give us immediately the following elementary reflection result on Peano continua.

THEOREM 2.54. *Being a Peano continuum is a property that is elementarily reflected.*

We see that the Wallman representation of a countable elementary sublattice of 2^X, for some locally connected continuum X, is actually a Peano continuum.

6. The spaces $\mathbb{I}$, $\mathbb{I}^2$, S and S^2

6.1. The unit interval $\mathbb{I}$. A continuum is said to be an *arc* if it is homeomorphic to the unit interval $\mathbb{I}$. The unit interval contains exactly two points that do not separate it, and so does every arc. These two points are called the *endpoints* of the arc.

THEOREM 2.55 ([**53**]). *If a metric continuum contains two points a and b such that to every point x correspond two closed sets A and B satisfying the conditions*

$$X = A \cup B,\ a \in A,\ b \in B \text{ and } A \cap B = \{x\},$$

then X is an arc.

THEOREM 2.56. *Being an arc is a property of compact Hausdorff spaces that is elementarily reflected.*

PROOF. Let X be an arc and L an elementary sublattice of 2^X. The lattice 2^X models the following sentence

$$\exists ab[a \neq b \wedge \text{atom}(a) \wedge \text{atom}(b) \wedge \text{conn}^*(a \sqcup b) \wedge \\ \forall x(\text{atom}(x) \wedge x \neq a \wedge x \neq b \rightarrow \neg\,\text{conn}^*(x))].$$

Where $\text{conn}^*(x)$ is shorthand for the following lattice formula

$$\forall ab[a \sqcap b \leq x \wedge a \sqcup b \sqcup x = \mathbf{1} \rightarrow a \leq x \vee a \sqcup x = \mathbf{1}],$$

which expresses that the complement of the set x is connected.

The lattice L, being an elementary submodel of 2^X also models the sentence 20 and contains the endpoints of X; these are the only atoms of L that do not separate the continuum wL. But not all points of wL are atoms of L. We want to show that wL has the properties stated in theorem 2.55.

By theorem 2.16 wL is a metrizable continuum. Let y be any point in $wL \setminus \{a, b\}$. We want to find closed subsets A and B of wL such that $wL = A \cup B$ and $A \cap B = \{y\}$ and A contains the endpoint a and B contains the endpoint b of X.

For any atom p of L there exist two connected closed sets A_p and B_p in X such that $a \in A_p$, $b \in B_p$ and $A_p \cap B_p = \{p\}$. Being uniquely defined in 2^X, L also contains these connected closed sets for every $p \in \text{atom}(L)$. If y is any element in wL then we can form sets A_y and B_y by

$$A_y = \bigcap\{A_p : p \in \text{atom}(L) \text{ and } y \in A_p\}$$
$$B_y = \bigcap\{B_p : p \in \text{atom}(L) \text{ and } y \in B_p\}.$$

As for any $p, q \in \text{atom}(L)$ we either have $A_p \subset A_q$ or $A_q \subset A_p$, and likewise for the connected sets $\{B_p : p \in \text{atom}(L)\}$, we have that A_y and B_y are connected closed sets, being intersections of a directed family of connected closed sets, containing a and b respectively. We also have that $A_y \cap B_y = \{y\}$, as the atoms of L form a dense subset of wL. □

6.2. The simple closed curve S. A space homeomorphic to the circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is said to be a *simple closed curve*.

THEOREM 2.57 (R. L. Moore). *If every pair of points separates the metric continuum X, then X is a simple closed curve.*

The following theorem is an immediate consequence of the previous theorem.

THEOREM 2.58. *If to every pair of points a and b of a metric continuum X correspond two closed sets A and B such that*

$$X = A \cup B,\ A \cap B = \{a, b\} \textit{ and } A \neq X \neq B$$

then X is a simple closed curve.

For a proof see, for instance [**37**].

THEOREM 2.59. *Being the simple closed curve S is a property of compact spaces that is elementarily reflected.*

PROOF. Let L be an elementary sublattice of 2^S.

CLAIM 2.60. *For every point x of wL and every $F \in L$ such that $x \notin F$ we can find a G in L that is an arc in wL, contains x and misses F.*

PROOF. If x is an atom of L this is obvious, so suppose that $x \notin \text{atom}(L)$. Let a and b be two distinct atoms of L. By theorem 2.58 and elementarity there exist $A, B \in L$ such that

$$L \models A \sqcap B = a \sqcup b \wedge A \sqcup B = \mathbf{1} \wedge A \neq 1 \wedge B \neq \mathbf{1}.$$

Without loss of generality we can assume that $x \in A$. The set A is an arc by theorem 2.56 and elementarity, so for some element $G \in L$ we have $x \in G$, G is connected, $G \leq A$ and misses F. □

So every point in wL is the intersection of elements of L that are arcs. Note that for every $F, G \in L$ such that

$$L \models \text{conn}(F) \wedge \text{conn}(G) \wedge F \sqcap G = \mathbf{0},$$

by elementarity there exist elements C, D of L such that the lattice L models the sentence

$$\text{conn}(C) \wedge \text{conn}(D) \wedge c \sqcup D = \mathbf{1} \wedge C \sqcap D = F \sqcup G,$$

Consider for every two points $a, b \in wL$ the family $\mathcal{F}_{ab}$ consisting of pairs of elements (C, D) of L such that C and D are connected subsets of wL, $C \cup D = wL$ and $a, b \in C \cap D$. Partially order this family by

$$(C, D) \leq (C', D') \text{ if and only if } C \subset C' \text{ and } D \subset D'.$$

With the claim made above, the intersection of any maximal chain in $\mathcal{F}_{ab}$ will give us a pair of closed connected subsets (A, B) of wL such that $A \cup B = wL$ and $A \cap B = \{a, b\}$. As this holds for all points a and b of wL, by theorem 2.58 we have that wL is a simple closed curve. □

6.3. The sphere S^2. A *constituent* of a point p in a space X is the union of all subcontinua of X containing the point p. A subset C of X is said to be a *cut between the points a and b* of X if a and b have different constituents in $X \setminus C$.

A subset $C \subset X$ is a *cut of the space X* if $X \setminus C$ has two points such that any subcontinuum of X containing these points must intersect with C.

DEFINITION 2.61. A space X is called a *Janiszewski space* if it is a locally connected metric continuum with the property that for any continua C and D whose intersection $C \cap D$ is not connected, their union $C \cup D$ is a cut of the space X.

THEOREM 2.62. *The sphere S^2 is a Janiszewski space.*

Janiszewski proved that among all locally connected metric continua without cut points, the property mentioned in definition 2.61 characterizes the 2-sphere.

THEOREM 2.63 ([**34**]). *The concept of a Janiszewski space is invariant under monotone continuous transformations.*

This gives us the following reflection theorem.

THEOREM 2.64. *Being a Janiszewski space is a property of compact spaces that is elementarily reflected.*

Consider the following theorem of Moore.

THEOREM 2.65 (R. L. Moore [45]). *If f is a continuous mapping of S^2 such that, for each y, $f^{-1}(y)$ is a continuum which does not cut the space, then $f[S^2]$ and S^2 are homeomorphic.*

We will use this theorem to prove the following theorem.

THEOREM 2.66. *Being the 2-sphere, S^2, is a property of compact spaces that is elementarily reflected.*

Suppose that L is an elementary sublattice of 2^{S^2}. By theorem 2.12 there exists a continuous map $f : S^2 \to wL$ and as S^2 is locally connected, so is wL. By theorem 2.17 the continuous map f is actually monotone. The theorem is proved once we have proved the following lemma.

LEMMA 2.67. *No inverse image $f^{-1}(x)$ of some point $x \in wL$ cuts the sphere S^2.*

PROOF. Suppose that the fiber $f^{-1}(x)$ cuts the sphere S^2 for some $x \in wL$. Let y and z be two atoms of L that correspond to two points in different components of $wL \setminus \{x\}$. So $f^{-1}(x)$ cuts the sphere between y and z.

In 2^{S^2} we can find F_y and G_y such that both sets are connected subsets of S^2, y is contained in the interior of F_y, $f^{-1}(x)$ in the interior of G_y, $F_y \cup G_y = S^2$ and $F_y \cap G_y$ is a simple closed curve. Similar sets F_z and G_z can be found for the point z. By elementarity we can assume that these sets are elements of L. There is an element C of L, that contains $f^{-1}(x)$ and is contained in $G_y \cap G_z$.

In S^2 the intersection $G_y \cap G_z$ is a cylinder and cutting S^2 in half, by some simple closed curve containing the points y and z, we see that there is a closed cover $\{E, F\}$ of S^2 consisting of connected sets that do not cut the sphere and moreover $E \cap (G_y \cap G_z)$ and $F \cap (G_y \cap G_z)$ also do not cut the sphere. By elementarity we can assume that E and F are elements of L.

We have reached our contradiction, as x is an ultrafilter of L containing $G_y \cap G_z$, and as it either contains E or F, we have that $f^{-1}(x)$ cannot be a cut of S^2 between y and z. □

6.4. The unit square $\mathbb{I}^2$. Using the fact that the property of being the 2-sphere is elementarily reflected (theorem 2.66), we prove the following reflection theorem.

THEOREM 2.68. *Being the unit square $\mathbb{I}^2$ is a property of compact spaces that is elementarily reflected.*

We know that the unit square is homeomorphic to the closure of any disc A of the sphere S^2, such that $\overline{A} \neq S^2$. Pick such a disc A, and let $\phi : 2^{\mathbb{I}^2} \to 2^{S^2} \upharpoonright \overline{A}$ be an isomorphism between lattices.

Let L be an elementary sublattice of $2^{\mathbb{I}^2}$. Define the subset K of 2^{S^2} by

$$K = \{B \in 2^{S^2} : B \cap \overline{A} \in \phi[L]\}.$$

CLAIM 2.69. *K is a distributive lattice.*

PROOF. This is easily seen, as $\phi[L]$ is an elementary sublattice of the distributive lattice $2^{S^2} \upharpoonright \overline{A}$, and for $a, b \in K$ we have

$$(a \cup b) \cap \overline{A} = (a \cap \overline{A}) \cup (b \cap \overline{A}) \text{ and } (a \cap b) \cap \overline{A} = (a \cap \overline{A}) \cap (b \cap \overline{A}).$$

□

CLAIM 2.70. *K is a normal lattice.*

PROOF. Let $a, b \in K$ such that $a \cap b = \emptyset$.

In the case that $b \cap \overline{A} = \emptyset$ then $a \cup \overline{A}$ and b are two closed sets of S^2 that are disjoint, hence there exist closed sets C and D of S^2 such that $C \cap (a \cup \overline{A}) = \emptyset$, $D \cap b = \emptyset$ and $C \cup D = S^2$. As $\overline{A}$ is a subset of D we know that C and D are elements of K.

Now, for the case that $a \cap \overline{A} \neq \emptyset$ and $b \cap \overline{A} \neq \emptyset$. As the lattice L is normal, we can find elements C^* and D^* in $\phi[L]$ such that $(a \cap \overline{A}) \cap D^* = \emptyset$, $(b \cap \overline{A}) \cap C^* = \emptyset$ and $C^* \cup D^* = \overline{A}$. The space $S^2 \setminus A$ being normal, we can find closed sets C^{**} and D^{**} such that $((a \cup C^*) \setminus A) \cap D^{**} = \emptyset$, $((b \cup D^*) \setminus A) \cap C^{**} = \emptyset$ and $C^{**} \cup D^{**} = S^2 \setminus A$. The closed sets $C = C^* \cup C^{**}$ and $D = D^* \cup D^{**}$ are elements of K that show that K is normal between the sets a and b. □

CLAIM 2.71. *The lattice K is disjunctive.*

PROOF. Given $a, b \in K$ such that $K \models a \not\leq b$, we have that $a \cap \overline{A} \not\subset b \cap \overline{A}$ or $a \setminus A \not\subset b \setminus A$. In the first case, the fact that L is an elementary sublattice of $2^{\mathbb{I}^2}$ and $\phi^{-1}[a \cap \overline{A}]$ and $\phi^{-1}[b \cap \overline{A}]$ are elements of L give us a $c \in K$ (actually in $\phi[L]$) such that $c \subset a \cap \overline{A}$ and $c \cap (a \cap \overline{A}) = \emptyset$. In the second case, by normality of S^2, we can find a $c \in K$ such that $c \subset a$ and $c \cap (\overline{A} \cup b) = \emptyset$. □

CLAIM 2.72. *The compact space wK is an image of the 2-sphere S^2 by a monotone map.*

PROOF. It is easy to see that the map $\phi : K \to 2^{S^2}$ which maps every $a \in K$ onto itself, fulfills all requirements of theorem 2.12, so wK is a continuous image of S^2, hence locally connected. We have that the map $f : S^2 \to wK$ is actually monotone. □

Let $x \in wK$. the fiber $f^{-1}(x)$ of x does not cut the sphere S^2 by the fact that K is an elementary sublattice of 2^{S^2} and lemma 2.67.

By the previous theorem and the claims of this section we see that the compact space wK is homeomorphic to the sphere S^2. The set $\overline{A} \setminus A$ is an element of K. In S^2 this is a simple closed curve, hence by theorem 2.66 also in wK. Shrinking the set $wK \setminus A$ to a point we end up with a homeomorphic copy of the 2-sphere, so $\overline{A}$ is homeomorphic to a disc of S^2. So by the following theorem we have proved the reflection theorem, theorem 2.68.

THEOREM 2.73. *If D is a disc, then its closure is homeomorphic to $\mathbb{I}^2$.*

For a proof of this last theorem, see [**37**].

6.5. $\mathbb{I}^n$ and elementary reflection w.r.t. submodels. This section will be devoted to the proof of the following theorem.

THEOREM 2.74. *Being the unit n-cube $\mathbb{I}^n$ is a property that is elementarily reflected by submodels.*

Let $\mathcal{B}$ be an open subbase for the space X. The closed subbase $\mathcal{B}$ is called a *binary (closed) subbase* if every subfamily of $\mathcal{B}$ with an empty intersection contains two disjoint base elements. The closed subbase $\mathcal{B}$ is called *comparable* if whenever $B_0 \cap B_1 = \emptyset$ and $B_0 \cap B_2 = \emptyset$ then either $B_1 \subset B_2$ or $B_2 \subset B_1$.

In [**22**] De Groot gave the following characterization of the n-dimensional cubes $\mathbb{I}^n$ and the Hilbert cube $\mathbb{I}^\infty$.

THEOREM 2.75. *A topological space X is homeomorphic to $\mathbb{I}^n$ if and only if X has the following properties:*

(1) *X is T_1,*
(2) *X is connected,*
(3) *X has dimension n,*
(4) *X has a countable comparable binary subbase.*

A characterization of the Hilbert cube is obtained if condition 3 is replaced by

3*. *X is infinite dimensional.*

EXAMPLE 2.76. The (countable) base

$$\mathcal{B} = \{[0, q] : q \in [0, 1) \cap \mathbb{Q}\} \cup \{[q, 1] : q \in (0, 1] \cap \mathbb{Q}\}$$

of $\mathbb{I}$ is easily seen to have the properties of being binary and comparable. This subbase can be used to get a countable binary and comparable subbase for the space $\mathbb{I}^n$. Simply consider the subbase $\{\pi_i^{-1}[A] : A \in \mathcal{B}$ and $i \leq n\}$. For the Hilbert cube consider the subbase consisting of closed subsets of Q of the form $\pi_i^{-1}[A]$, where $i < \omega$, π_i the projection of Q onto its i-th coordinate and A an element of $\mathcal{B}$.

PROOF OF THEOREM 2.74. We will use De Groot's characterization to prove the theorem. We will do this by showing that, if $\mathfrak{M}$ is an elementary submodel of some $H(\theta)$ and $2^{\mathbb{I}^n}$ is an element of $\mathfrak{M}$ then there exists a countable subset $\mathcal{B}$ of $2^{\mathbb{I}^n} \cap \mathfrak{M}$ that is a binary and comparable subbase of the space $w(2^{\mathbb{I}^n} \cap \mathfrak{M})$. We show this of course by induction. By theorem 2.56 we know this is true for $n = 1$. Now suppose that we have proven the theorem for all $m < n$, and consider some elementary submodel $\mathfrak{M}$ of $H(\theta)$, which contains $2^{\mathbb{I}^n}$ as one of its elements.

The model $\mathfrak{M}$ contains the elements $\mathbb{I}^m$ and $2^{\mathbb{I}^m}$ for $m = 0, 1, \ldots, \infty$, as they have a first order definition. So, as $2^{\mathbb{I}}$ is an element of $\mathfrak{M}$, we also have that $\mathfrak{M}$ contains an element $\mathcal{B}$ that is a countable subbase of $\mathbb{I}$ which is binary and comparable.

If $B \in \mathcal{B}$ then we have that $B \times \mathbb{I}^{n-1}$, $\mathbb{I} \times B \times \mathbb{I}^{n-2}$, ..., $\mathbb{I}^{n-1} \times B$ are all elements of $\mathfrak{M} \cap 2^{\mathbb{I}^n}$. It is also not hard to show that all these closed sets combined form a subbase $\mathcal{B}^*$ for the closed sets of $\mathbb{I}^n$. As $\mathcal{B}$ is countable, binary and comparable, so is $\mathcal{B}^*$. We see that $w(2^{\mathbb{I}^n} \cap \mathfrak{M})$ is an n-dimensional connected space which has a countable binary and comparable subbase for its closed sets and thus by de Groot's characterization of $\mathbb{I}^n$ homeomorphic to the space $\mathbb{I}^n$. This ends the proof of theorem 2.74. □

REMARK 2.77. Another approach to finding out if the property of being $\mathbb{I}^n$ for compact spaces is elementarily reflected is by defining a new predicate B for the binary comparable subbase. Comparability would then be given by the universal closure of the formula

$$B(x) \wedge B(y) \wedge B(z) \wedge x \sqcup y = \mathbf{1} \wedge x \sqcup z = \mathbf{1} \rightarrow y \leq z \vee z \leq y$$

And binarity for instance with the countable many sentences

$$\forall x_1 \cdots x_n \exists yz \, [\textstyle\bigwedge_i B(x_i) \wedge x_1 \sqcap \cdots \sqcap x_n = \mathbf{0} \rightarrow \\ B(y) \wedge B(z) \wedge y \sqcap z = \mathbf{0} \wedge \textstyle\bigvee_i x_i = y \wedge \bigvee_i x_i = z].$$

But now we run into trouble, as being a subbase is second-order. So this approach will not help us very much.

So the following questions remains open.

QUESTION 2.78. *Is being the unit n-cube or the n-sphere a property of compact spaces that is elementarily reflected?*

7. Concluding remarks

We have seen that there are a lot of properties of compact spaces that are elementarily reflected. In chapter 3 we will see if this can shed some light on a conjecture of Lelek in metric continuum theory. There we will investigate if the properties mentioned in the conjecture are elementarily reflected. This is of importance, as if this is the case we can extend our horizon when trying to look for a counterexample to the conjecture, i.e. we are then no longer restricted to metric counterexamples.

In chapter 5 we will give an alternative proof of some theorems in continuum theory. These proofs will be model theoretic and rely heavily on Wallman's representation theorem on lattices. These proofs show that sometimes we can extend theorems on metric continua to arbitrary continua. In the case of the model theoretic proof of a theorem of Maćkowiak and Tymchatyn we actually found another topological proof of the theorem, which easily extends to a proof in the arbitrary continuum setting (this extension was already shown to hold by Hart, Van Mill and Pol in [**25**]).

CHAPTER 3

Span and chainability

1. Introduction

The notion span of a metric continuum was introduced by Lelek in [**38**] to capture the spirit that an arc-like continuum must be long and thin. He also proved that every chainable metric continuum has span zero. As chain covers of certain continua can be rather complicated, it can be difficult to show that a metric continuum is chainable. For example it is known that there exists embeddings of arc-like continua in the plane such that the continuum can not be covered by a chain of connected sets of arbitrary small diameter (see [**10**]).

The notion of span and the fact that chainability of a metric continuum implies that it has span zero has been useful to prove that certain metric continua are not chainable. The converse of this theorem is one of the main open problems in continuum theory today.

CONJECTURE 3.1 (Lelek). *If X is a metric continuum with span zero, then X is chainable.*

The notion of span is easily transferred to non-metric spaces, be it that we can only distinguish the cases of span zero and nonzero span. Likewise we can set the notion of chainability in a more general setting including the non-metric continua. We will show that chainable continua have span zero.

In section 3 we will investigate if the notions of span zero and chainability can be translated into lattice properties as we have done in chapter 2. We want to reflect the notions of span zero and non-chainability, as then we will not be restricted to metric continua when looking for a counterexample to Lelek's conjecture. Any counterexample X to the conjecture will then give us a metric counterexample when we take an elementary sublattice of the lattice of all closed subsets 2^X of the continuum X. Unfortunately, we will see that these notions are not captured in this way, as stronger forms of logic are needed to write down the sentences that describe these notions.

In subsection 4 we will investigate the span and chainability of the continua $\mathbb{H}^*$, the remainder of the Čech-Stone extension of the half line $\mathbb{H} = [0, \infty)$ and its standard subcontinua, in the form of $\mathbb{I}_u$, for some $u \in \omega^*$. We will show that these continua have span nonzero, surjective (semi) span nonzero and symmetric span nonzero and are also non-chainable.

In section 7 we will use a different approach using elementary submodels, to take elementary sublattices of 2^X of some continuum. These elementary sublattices are more powerful than in section 3 and we can show that in this case chainability is reflected. Unfortunately we do not know if span zero is reflected for arbitrary continua in this way.

2. Preliminaries

In this section we will extend the definitions for span and chainability to non-metric continua, and try to find sentences in the lattice language that capture the notion span respectively chainability.

2.1. Span. In [**38**], Lelek defined the *span* of a metric compact space (X, d) as the supremum of all real numbers $\epsilon > 0$ such that there exists a connected subset Z of the square $X \times X$ which has the following two properties:

(1) Z projects onto the same sets on both axes.
(2) For all $(x, y) \in Z$ we have $d(x, y) \geq \epsilon$.

Note that, if we discard of the metric d in the definition above we can distinguish only the cases span zero and span nonzero.

We make the following definition.

DEFINITION 3.2. A continuum X has *span* zero if every subcontinuum Z of its square $X \times X$, which projects onto the same set on both axes, has nonempty intersection with the diagonal $\Delta_X = \{(x, x) : x \in X\}$ of X; otherwise it is said to have span nonzero.

After Lelek introduced the notion of span and posed his conjecture, quite a number of articles by various authors have been produced on the subject (see for instance [**40**], [**49**], [**50**], [**51**] and [**34**]). In [**40**], Lelek introduced weaker forms of span, by changing some of the properties of the subcontinuum Z of $X \times X$ in the definition of span. He called these notions *surjective span* and *surjective semi span*.

DEFINITION 3.3. A continuum X has *surjective (semi) span* zero if every subcontinuum Z of $X \times X$ which projects onto X in both coordinates (in at least one coordinate) has nonempty intersection with the diagonal Δ_X.

In [**40**], Lelek showed that the span and the semi span of metric spaces need not be equal. That these forms of span are equal for compact metric spaces was proved by Davis in [**13**]. For his proof he introduces another version of span, the symmetric span.

DEFINITION 3.4. A continuum X has *symmetric span* zero if every subcontinuum Z of $X \times X$ for which $Z = Z^{-1}$, where $Z^{-1} = \{(y, x) : (x, y) \in Z\}$, has nonempty intersection with the diagonal Δ_X.

The following relations between the different kinds of span of some continuum X follow directly from the definition.

(1) If the surjective span of A is zero for all connected nonempty $A \subset X$ then X has span zero,
(2) If X has span zero then X has surjective and symmetric span zero.
(3) If X has surjective semi span zero then X has surjective span zero.
(4) If X has surjective span zero then X has symmetric span zero.

2.2. Chainability. A cover $\mathcal{C} = \{C_1, \ldots, C_m\}$ of a space is a *chain cover* if $C_i \cap C_j$ is nonempty if and only if $|i - j| \leq 1$. The elements C_i of $\mathcal{C}$ are called the *links* of the chain $\mathcal{C}$. A continuum is *chainable* if every open cover has a refinement which is a chain cover.

This definition is in accordance with the definition of chainable in the metric case which states that a metric continuum is chainable if for every $\epsilon > 0$ there exists a chain cover for which the diameter of the links are less than ϵ. In [**38**] Lelek showed that every chainable metric continuum has span zero. The same statement is true for non-metric chainable continua.

THEOREM 3.5. *Every chainable continuum has span zero.*

PROOF. Let X be a chainable continuum and suppose that X has surjective span nonzero. Let $Z \subset X \times X$ be a subcontinuum which has empty intersection with the diagonal $\triangle_X$, and which maps onto X by both projections π_1 and π_2. For every x in X let U_x be an open neighborhood of x such that $U_x \times U_x$ has empty intersection with the subcontinuum Z. As $\{U_x : x \in X\}$ is a open cover of X it has a chain cover refinement $\{V_1, \ldots, V_n\}$. Let the open sets O_1 and O_2 of $X \times X$ be defined by

$$O_1 = \bigcup\{V_i \times V_j : i < j\}$$
$$O_2 = \bigcup\{V_i \times V_j : i > j\}.$$

We have $Z \subset O_1 \cup O_2$. Also $Z \cap O_1 \cap O_2$ is empty; if not, Z would intersect some $V_i \times V_i$, which would contradict the fact that V_i is contained in some U_x. So we can write Z as the disjoint union of the open sets $Z \cap O_1$ and $Z \cap O_2$. As Z is connected, one of these must be empty, say $Z \cap O_2$. Now we have a contradiction, as $\pi_2[Z \cap O_1] \subset \bigcup_{1<m\leq n} V_m$ and $V_1 \setminus V_2$ is nonempty. $\square$

Note that chainability of a space X is a property of X we can read of from any of its bases.

THEOREM 3.6. *A continuum X is chainable if and only if every cover of base elements of some base (closed under finite unions and intersections) of X has a chain cover refinement consisting entirely of base elements.*

To prove this we need the well - known theorem on swellings mentioned in section 4.4.

PROOF OF THEOREM 3.6. Let X be a continuum and $\mathcal{B}$ a base closed under finite unions and finite intersections.

Suppose X is chainable and $\mathcal{U} \subset \mathcal{B}$ is a cover of X. There is a chain cover refinement $\mathcal{C} = \{C_1, \ldots, C_n\}$ of $\mathcal{U}$. Consider the family $\{X \setminus C_i\}_{i=1}^n$ of closed subsets of X. As $\{X \setminus B : B \in \mathcal{B}\}$ is a base for the closed sets of X closed under finite unions and finite intersections. We can find a swelling $\{F_i\}_{i=1}^n$ of the family $\{X \setminus C_i\}_{i=1}^n$ consisting of elements from $\{X \setminus B : B \in \mathcal{B}\}$. The family $\{X \setminus F_i\}_{i=1}^n$ is a chain cover and a refinement of $\mathcal{C}$ consisting entirely of elements of the base $\mathcal{B}$. □

3. Chainability and span in the lattice language

3.1. Chainability. We can formulate chainability in the lattice language if we are allowed to take a conjunction over the natural numbers; this leads to the logic $\mathcal{L}_{\omega_1\omega}$.

Let $\psi_m(u_1, \ldots, u_m)$ be the following formula in the lattice language (in the logic $\mathcal{L}_{\omega_1\omega}$).

$$u_0 \sqcap \cdots \sqcap u_m = \mathbf{0} \rightarrow \bigvee_n \exists v_1 \cdots v_n \theta(u_1, \ldots, u_m; v_1, \ldots, v_n). \tag{20}$$

Here $\theta(u_1, \ldots, u_m; v_1, \ldots, v_n)$ is the conjunction of the formulas

(1) $v_1 \sqcap \cdots \sqcap v_n = \mathbf{0}$

(2) $\bigwedge_i \bigvee_j v_i \geq u_j$

(3) $\bigwedge_{i=1}^{n-1} (v_{i-1} \sqcup v_i \neq \mathbf{1} \wedge v_i \sqcup v_{i+1} \neq \mathbf{1}) \wedge \bigwedge_{i=1}^{n-2} \bigwedge_{j=i+2}^{n} v_i \sqcup v_j = \mathbf{1}$.

In equation 20 the symbol $\bigvee_n$ denotes the conjunction over all $n \in \omega$. The first formula in the list shows that the family of complements of the closed sets v_i in wL will be an open cover of wL. The second formula of the list then shows that the complement of any v_i is contained in a complement of some u_j. So if $\{wL \setminus u_j\}_j$ and $\{wL \setminus v_i\}_i$ are both (open) covers of wL then $\{wL \setminus v_i\}_i$ refines $\{wL \setminus u_j\}_j$. And the last formula shows that $\{wL \setminus v_i\}_i$ will be a chain cover.

This gives us the following lemma.

LEMMA 3.7. *If L is any normal distributive disjunctive lattice then its Wallman representation wL is chainable if and only if*

$$L \models \forall u_1 \cdots u_m \psi_m(u_1, \ldots, u_m) \text{ for all } m < \omega.$$

3.2. Span. We know that if a continuum X has span zero then every subcontinuum Z of the square $X \times X$, for which $\pi_1[Z] = \pi_2[Z]$ has nonempty intersection with the diagonal $\triangle_X$. We want to capture the notion of span in a lattice formula. Below we will give a formula in the lattice language such that all the continua whose lattice of closed sets fulfill this property must have span zero. But we also show that this property is much too restrictive as the unit interval $\mathbb{I}$ is a continuum which does not satisfy this property.

Let $G = (V, E)$ be a connected graph and $V = \{1, 2, \dots, n\}$. Let $\phi_{E,n}$ denote the following formula in the lattice language

$$\bigwedge_{\{i,j\}\in E} (u_i \sqcap u_j \neq \mathbf{0} \wedge v_i \sqcap v_j \neq \mathbf{0}) \wedge \bigsqcup_i u_i = \bigsqcup_i v_i \rightarrow \bigvee_i u_i \sqcap v_i \neq \mathbf{0}$$

If X is a continuum such that for every connected graph $G = (\{1, \dots n\}, E)$ we have $2^X \models \phi_{E,n}$ then it must hold that X has span zero. Because if X has span nonzero we would have a subcontinuum Z of $X \times X$ for which $\pi_1[Z] = \pi_2[Z]$ and $Z \cap \triangle_X = \emptyset$. For every point (x, y) in Z there are open sets $F_x \subset \pi_1[Z]$ and $G_y \subset \pi_2[Z](= \pi_1[Z])$ such that

$$(x, y) \in F_x \times G_y \subset \overline{F_x} \times \overline{G_y} \subset X \setminus \triangle_X.$$

As Z is a continuum there are a finite number of points $\{(x_i, y_i)\}_{i=1}^n$ such that $Z \subset \bigcup\{F_{x_i} \times G_{y_i} : 1 \leq i \leq n\}$. Let Y denote the closed set $\bigcup_i \overline{F_{x_i}} \cap \bigcup_i \overline{G_{x_i}}$ and define the graph $(\{1, \dots, n\}, E)$ by

$$\{i, j\} \in E \leftrightarrow \overline{F_{x_i}} \cap \overline{F_{x_j}} \cap Y \neq \emptyset \text{ and } \overline{G_{y_i}} \cap \overline{G_{y_j}} \cap Y \neq \emptyset.$$

This graph is connected, because Z is and so, as $2^X \models \phi_{n,E}$ we must have some i such that the closed set $\overline{F_{x_i}} \times \overline{G_{y_i}}$ intersects the diagonal $\triangle_X$; in contradiction with the construction of these sets.

Unfortunately this sentence $\phi_{E,n}$ does not capture the notion span zero as it is too strong a requirement as the following example will show.

EXAMPLE 3.8. Consider the unit interval $\mathbb{I}$. Let the open subsets V_1, V_2, V_3 and V_4, and U_1, U_2, U_3 and U_4 of $\mathbb{I}$ be given by

$$V_1 = U_3 = (\frac{1}{12}, \frac{5}{12}), \ V_2 = U_4 = (\frac{4}{12}, \frac{8}{12}),$$
$$V_3 = U_1 = (\frac{7}{12}, \frac{11}{12}), \ V_4 = U_2 = (\frac{10}{12}, 1] \cup [0, \frac{2}{12}).$$

The intersection $U_i \times V_i \cap U_j \times V_j$ is nonempty if and only if $|i - j| \leq 1$ or if $i = 1$ and $j = 4$. The sets $\{U_1, U_2, U_3, U_4\}$ and $\{V_1, V_2, V_3, V_4\}$ are both open covers of the interval $\mathbb{I}$. The intersection graph $G = (V, E)$ is given by $V = \{1, 2, 3, 4\}$ and $E = \{\{i, j\} : |i - j| \leq 1\} \cup \{\{1, 4\}\}$ so this is a connected graph. However none of the $U_i \times V_i$ has an nonempty intersection with the diagonal $\triangle_{\mathbb{I}}$.

So $2^{\mathbb{I}} \not\models \phi_{n,E}$ and as $\mathbb{I}$ is obviously chainable we also have that $\mathbb{I}$ has span zero.

Note that the covers $\{U_i\}_{i=1}^4$ and $\{V_i\}_{i=1}^4$ do not consist entirely of connected sets. So maybe if we add the requirement that the u_i and v_i are connected to the sentence $\phi_{E,n}$ we get a weaker notion that might say more than the strong notion $\phi_{E,n}$.

Let us define the formula $\phi^*_{E,n}$ by

$$\bigwedge_i \mathrm{conn}(u_i) \wedge \bigwedge_i \mathrm{conn}(v_i) \wedge \bigwedge_{\{i,j\}\in E}(u_i \sqcap u_j \neq \mathbf{0} \wedge v_i \sqcap v_j \neq \mathbf{0}) \wedge \\ \bigsqcup_i u_i = \bigsqcup_i v_i \rightarrow \bigvee_i u_i \sqcap v_i \neq \mathbf{0}.$$

We now look at the smaller class of connected spaces, namely the class of locally connected spaces.

Suppose we have a locally connected continuum X such that there is a connected graph $(\{1,\dots,n\},E)$ and there are closed sets $U1,\dots,U_n$ and $V_1,\dots,V_n$ such that

$$2^X \models \neg\phi^*_{E,n}[U_1,\dots,U_n;V_1,\dots,V_n]. \tag{21}$$

This means that the U_i's and V_i's are subcontinua of X and the intersection graph of the subcontinua $U_i \times V_i$ of $X \times X$ is given by the graph $(\{1,\dots,n\},E)$, so as this graph is connected we have that $\bigcup_i U_i \times V_i$ is a subcontinuum of $X \times X$. The projections on the axes are equal as the U_i's and the V_i's are witnesses of $\bigsqcup_i u_i = \bigsqcup_i v_i$. Also it has empty intersection with the diagonal by equation 21. So the space X must have span non-zero by the existence of such a subcontinuum of its square.

Suppose now that the locally connected continuum X has non-zero span. So there exists a subcontinuum Z of $X \times X$ such that its projections on the axes are equal and it does not intersect the diagonal $\triangle_X$. For every $(x,y) \in Z$ we can find open connected subsets A_x and B_y of x and y respectively in the space X such that $A_x \times B_y$ misses the diagonal $\triangle_X$. By compactness of Z there exist $(x_1,y_1),\dots,$ (x_n,y_n) in Z such that $Z \subset \bigcup_i A_{x_i} \times B_{y_i}$. Let Y denote the closed set $\bigcup_i \overline{A_{x_i}} \cap \bigcup_i \overline{B_{y_i}}$ of X. It must be that $Z \subset Y \times Y$. Let $(\{1,\dots,n\},E)$ denote the intersection graph of the closed sets $(\overline{A_{x_i}} \cap Y) \times (\overline{B_{y_i}} \cap Y)$ for $i = 1 \dots n$. This graph is connected as Z is connected. And we also have

$$\bigcup_i(\overline{A_{x_i}} \cap Y) = Y \cap \bigcup_i \overline{A_{x_i}} = Y \cap \bigcup_i \overline{B_{y_i}} = \bigcup_i(\overline{B_{y_i}} \cap Y),$$

so we found closed sets $\{\overline{A_{x_i}} \cap Y\}_{i=1}^n$ and $\{\overline{B_{y_i}} \cap Y\}_{i=1}^n$ of X and a connected graph $(\{1,\dots,n\},E)$ such that

$$2^X \models \neg\phi^*_{E,n}[\overline{A_{x_1}} \cap Y,\dots,\overline{A_{x_n}} \cap Y; \overline{B_{y_1}} \cap Y,\dots,\overline{B_{y_n}} \cap Y].$$

This gives us the following lemma.

LEMMA 3.9. *If L is any normal distributive disjunctive lattice such that its Wallman representation wL is locally connected then wL has span zero if and only if*

$$L \models \phi^*_{E,n} \textit{ for all } n < \omega \textit{ and all connected graphs } E \textit{ on } n.$$

REMARK 3.10. Note however that we do not have a first-order formula that describes span zero, as the property of being locally connected is not first order (or base free).

So the following questions still remain open.

QUESTION 3.11. *Is the property of having span zero a first-order property?*

QUESTION 3.12. *Is the property of having span zero a base-free property?*

QUESTION 3.13. *Is chainability a property of compact spaces that is elementary reflected?*

4. The spaces $\mathbb{I}_u$ and $\mathbb{H}^*$

In this section we will state some preliminary results on the spaces $\mathbb{I}_u$ and $\mathbb{H}^*$ that we will need in our investigation of the span of the spaces $\mathbb{I}_u$ and $\mathbb{H}^*$. The survey [**24**] will provide the proofs for the statements only mentioned here about these spaces.

The space $\mathbb{H}^*$ is the Čech-Stone remainder of the half line $\mathbb{H} = [0,\infty)$. As $\mathbb{H}$ is connected so is $\beta\mathbb{H}$. It is not hard to see that the remainder $\mathbb{H}^*$ is connected as well, as $\mathbb{H}$ is locally connected, $\mathbb{H}^*$ is also closed, so $\mathbb{H}^*$ is a continuum.

Following the survey article of Hart [**24**] and Mioduszewski's paper [**45**] we will investigate the space $\mathbb{H}^*$ by studying the space $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I}$ denotes the unit interval $[0,1]$. We shall write $\mathbb{I}_n$ for the space $\{n\} \times \mathbb{I}$.

The map $\pi : \mathbb{M} \to \omega$ given by $\pi(n,x) = n$ is perfect and monotone, and so the Čech-Stone extension $\pi : \beta\mathbb{M} \to \beta\omega$ is also monotone (for a proof see [**24**]). We denote the inverse image of the ultrafilter u of ω^* by $\mathbb{I}_u$. In the hyperspace of $\beta\mathbb{H}$ this is the u-limit of the sequence $\{\mathbb{I}_n\}_{n<\omega}$, where $\mathbb{I}_n$ denotes the subspace $\{n\} \times \mathbb{I}$ of $\mathbb{M}$. We can write $\mathbb{I}_u$ as

$$\mathbb{I}_u = \bigcap_{U\in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n\in U} \mathbb{I}_n.$$

REMARK 3.14. The space $\mathbb{I}_u$ is, in the notation of Bankston (see for instance [**4**] or [**9**]), the ultra co-power of the continuum $\mathbb{I}$ with respect to the ultrafilter u. In this section we will use Mioduszewski's approach to the space $\mathbb{I}_u$ from the paper [**45**], instead of the ultra co-power approach of Bankston as it is much easier to understand.

Let $\{x_n\}_{n<\omega}$ be a sequence in the unit interval $\mathbb{I}$. Given any $u \in \omega^*$, the point $x_u \in \mathbb{I}_u$ is the unique point x in $\mathbb{I}_u$ such that for every $\beta\mathbb{M}$-neighborhood O of x, the set $\{n : (n,x_n) \in O\}$ is an element of the ultrafilter u. We will denote this by $x_u = u - \lim\{x_n\}$. The point $u - \lim\{0\}$ will be denoted by 0_u and the point $u - \lim\{1\}$ by 1_u.

Let $\mathbb{P}_u$ denote all the points of $\mathbb{I}_u$ that we can get in this way, so

$$\mathbb{P}_u = \{x_u : \{x_n\}_{n<\omega} \subset \mathbb{I} \text{ and } x_u = u - \lim\{x_n\}\}.$$

Here are a couple of properties of the set $\mathbb{P}_u$.

PROPOSITION 3.15 ([**24**]). *The set $\mathbb{P}_u \setminus \{0_u, 1_u\}$ is a dense set of cut points of $\mathbb{I}_u$ and its subspace topology is the same as the order topology induced by $<_u$.*

For $a_u <_u b_u$ points of $\mathbb{P}_u$ we let the *interval from a_u to b_u* denoted by $[a_u, b_u]$, be the set of points of $\mathbb{I}_u$ that are in the closure of $\bigcup_{n<\omega}\{n\} \times [a_n, b_n]$.

If x is any point of the space $\mathbb{I}_u$ then we define the *layer L_x* of x in $\mathbb{I}_u$ as the intersection of all subintervals of $\mathbb{I}_u$ that contain x.

The following proposition shows the existence of non-trivial layers of $\mathbb{I}_u$.

PROPOSITION 3.16 ([**24**]). *Suppose that $\langle a_n \rangle_n$ is a strictly increasing sequence in $\mathbb{P}_u$, let $B = \{b \in \mathbb{P}_u : a_n <_u b$ for all $n < \omega\}$ then*

$$L = \bigcap\{[a_n, b] : n < \omega, b \in B\},$$

is a non-trivial layer of $\mathbb{I}_u$.

PROOF. As $\mathbb{P}_u$ is dense in $\mathbb{I}_u$ we know that L is a layer of $\mathbb{I}_u$. As the set $A = \{a_n : n < \omega\}$ is relatively discrete its closure is homeomorphic to $\beta\omega$, but $\overline{A} \setminus A \subset L$. □

Such a non-trivial layer will be used to show that the span and the surjective (semi)span of $\mathbb{I}_u$ are nonzero.

We state some properties of layers and intervals

PROPOSITION 3.17 ([**24**]). *Let a_u and b_u be points of $\mathbb{P}_u$ with $a_u <_u b_u$ and let x be a point of $\mathbb{I}_u$.*

(1) *The interval $[a_u, b_u]$ is homeomorphic with $\mathbb{I}_u$.*
(2) *The interval $[a_u, b_u]$ is irreducible between a_u and b_u.*
(3) *$[a_u, b_u] = [0_u, b_u] \cap [a_u, 1_u]$, so L_x, the layer of x is the intersection of intervals of all intervals of the form $[0_u, b_u]$ and $[a_u, 1_u]$ that contain it.*
(4) *The layers of the points in $\mathbb{P}_u$ are one-point sets.*

We can extend the order $<_u$ to the whole set of layers in the following way: $L_x <_u L_y$ if and only if there is some $a \in \mathbb{P}_u$ such that $x \in [0_u, a]$ and $y \in [a, 1]$. This extension of the order gives the following continuity property in $\mathbb{I}_u$.

PROPOSITION 3.18 ([**24**]). *For $x \in \mathbb{I}_u \setminus \{0_u, 1_u\}$, the closure of the half open intervals $[0_u, L_x)$ and $(L_x, 1_u]$, equal $[0_u, L_x]$ and $[L_x, 1_u]$ respectively.*

The shift σ on ω^* is the Čech-Stone extension of the shift σ on ω defined by $\sigma(n) = n + 1$ for all n. For a point $u \in \omega^*$ its image under σ is the ultrafilter $\{\{n+1 : n \in U\} : U \in u\}$. We call a subset F of ω^* *σ-invariant* if it holds that $\sigma[F] \cup \sigma^{-1}[F] \subset F$.

We are interested in the continua $\mathbb{I}_u$, as they provide us with a lot of subcontinua of the continuum $\mathbb{H}^*$.

Let $q : \mathbb{M} \to \mathbb{H}$ be such that $q((n,x)) = n + x$, then q is a perfect map and its Čech-Stone extension $q : \beta\mathbb{M} \to \beta\mathbb{H}$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$.

If $\{[a_n, b_n] : n < \omega\}$ is a sequence of closed subsets of $(0,1)$ then q maps $\bigcup_n \{n\} \times [a_n, b_n]$ homeomorphically onto the subset $\bigcup_n [n + a_n, n + b_n]$ of $\mathbb{H}$. Furthermore, the map q is exactly two-to-one on the set $\omega \times \{0,1\}$ with the exception of the point $(0,0)$. This implies that the map q is one-to-one on every $\mathbb{I}_u$ and the only identifications that are made by q on $\mathbb{M}^*$ are those of 1_u and $0_{\sigma(u)}$ for every $u \in \omega^*$.

Using any sequence $\mathbf{a} = \{a_n : n < \omega\}$ in $\mathbb{H}$ increasing to infinity, we can define a perfect map $q_{\mathbf{a}} : \mathbb{M} \to \mathbb{H}$ like q by putting $q_{\mathbf{a}}(n,x) = a_n + x(a_{n+1} - a_n)$ for every $(n,x) \in \mathbb{M}$. The map $q_{\mathbf{a}}$ will be called a *parameterization of* $\mathbb{H}^*$ *determined by* $\mathbf{a}$. The map $q_{\mathbf{a}}$ is, like the map q, one-to-one on every $\mathbb{I}_u$ and identifies the points 1_u and $0_{\sigma(u)}$ for every $u \in \omega^*$. The map q from the previous paragraph will be called the *standard parameterization*.

If $q_{\mathbf{a}}$ is a parameterization of $\mathbb{H}^*$ then as it is one-to-one on the continua $\mathbb{I}_u$, $q_{\mathbf{a}}[\mathbb{I}_u]$ and $\mathbb{I}_u$ are homeomorphic. The subcontinua of $\mathbb{H}^*$ of the form $q_{\mathbf{a}}[\mathbb{I}_u]$ are called *standard subcontinua of* $\mathbb{H}^*$. These standard continua are of a lot of interest to us as the following theorems show.

THEOREM 3.19 ([**24**]). *Every proper subcontinuum of* $\mathbb{H}^*$ *is the intersection of all standard continua containing it.*

So the standard subcontinua determine virtually the whole structure of the subcontinua of $\mathbb{H}^*$.

LEMMA 3.20 ([**24**]). *Every subcontinuum of* $\mathbb{H}^*$ *contains a standard subcontinuum and hence no subcontinuum of* $\mathbb{H}^*$ *is hereditarily indecomposable.*

LEMMA 3.21 ([**24**]). *A subcontinuum of* $\mathbb{H}^*$ *is decomposable if and only if it is a nondegenerate interval of some standard continuum.*

PROPOSITION 3.22 ([**24**]). *All layers of* $\mathbb{I}_u$ *are indecomposable.*

THEOREM 3.23 ([**24**]). *If K and L are subcontinua of* $\mathbb{H}^*$ *that intersect and one of K and L is indecomposable then $K \subset L$ or $L \subset K$.*

Let L be a layer of $\mathbb{I}_u$. If A_L denotes the set of all $\langle a_n \rangle_{n<\omega} \subset \mathbb{I}$ such that $a_u <_u L$, and B_L the set of all $\langle b_n \rangle_{n<\omega} \subset \mathbb{I}$ such that $L <_u b_u$ then we have

$$L = \bigcap \{[a_u, b_u] : \langle a_n \rangle_{n<\omega} \in A_L \text{ and } \langle b_n \rangle_{n<\omega} \in B_L\}.$$

It follows that if L is non-trivial then $\langle A_L, B_L \rangle$ determines a gap in $\mathbb{P}_u$, and conversely. We say that $\langle A_L, B_L \rangle$ determines a gap in $\mathbb{P}_u$ when there is no $\langle c_n \rangle_{n<\omega}$ in $\mathbb{I}$ such that $a_u <_u c_u <_u b_u$ for all $\langle a_n \rangle_{n<\omega}$ in A_L and $\langle b_n \rangle_{n<\omega}$ in B_L.

The continuum hypothesis (CH) implies there is a non - trivial layer L of $\mathbb{I}_u$ such that A_L has cofinality ω, and B_L has coinitiality ω_1 in the order $<_u$. This layer will be used in subsection 6.4 to find a subcontinuum of the square $\mathbb{I}_u \times \mathbb{I}_u$ that shows that the symmetric span of the continuum $\mathbb{I}_u$ is nonzero.

In [**16**] Dow and Hart gave a representation of indecomposable subcontinua of $\mathbb{H}^*$. They showed that the closed and σ-invariant subsets parameterize the indecomposable subcontinua of $\mathbb{H}^*$ by the following theorem.

THEOREM 3.24 ([**16**]). *If L is a nontrivial indecomposable subcontinuum of $\mathbb{H}^*$ then there are a σ-invariant subset F of ω^* and a parameterization $q_{\mathbf{a}}$ of $\mathbb{H}^*$ such that $L = q_{\mathbf{a}}[\bigcup_{u \in F} \mathbb{I}_u]$.*

5. Chainability of $\mathbb{H}^*$ and $\mathbb{I}_u$

If X is a completely regular space and V is an open subset of X, then $\operatorname{Ex} V$ is the commonly used notation for the largest open subset of βX, the Čech-Stone compactification of X, that, intersected with X is equal to V:

$$\operatorname{Ex} V = \beta X \setminus \operatorname{cl}_{\beta X}(X \setminus V).$$

A chain cover is called *minimal* if without one of its links it is no longer a (chain) cover.

LEMMA 3.25. *Suppose that $\mathcal{O} = \{O_i : i < n\}$ is a minimal finite chain cover of some space X and x and y are two points of X such that none of the links of $\mathcal{O}$ contains both the elements x and y, then there exists a unique minimal sub chain $\mathcal{O}^*$ of $\mathcal{O}$ such that $x, y \in \bigcup \mathcal{O}^*$.*

PROOF. Let i_1 be the minimal number $i < n$ such that

$$O_i \cap \{x, y\} \neq \emptyset \text{ and } O_{i+1} \cap \{x, y\} \neq O_i \cap \{x, y\}.$$

Let i_2 be the minimal $i < n$ such that

$$\{x, y\} \subset \bigcup_{i_1 \leq m \leq i} O_m.$$

By assumption we have $i_1 < i_2$. The sub chain $\mathcal{O}^* = \{O_i : i_1 \leq i \leq i_2\}$ is as required, as it is obviously a minimal sub chain of $\mathcal{O}$ containing the points x and y in its union and all other sub chains containing x and y in its union must contain it. $\square$

Consider the open subsets W_i of $\mathbb{H}$ for $i < 4$ given by

$$W_i = \bigcup_{n<\omega} (8n + 2i, 8n + 2i + 3) \qquad (i < 4).$$

As these sets cover the whole of $\mathbb{H}$ save the compact subset $\{0\}$, the open cover $\mathcal{U} = \{U_i\}_{i \leq 3}$ given by

$$U_i = \operatorname{Ex} W_i \cap \mathbb{H}^* \text{ for all } i \leq 3,$$

is an open cover of the space $\mathbb{H}^*$. We will see that there exists no finite chain cover of $\mathbb{H}^*$ that refines the cover $\mathcal{U}$, showing that $\mathbb{H}^*$ is non-chainable.

Striving for a contradiction, suppose that $\mathcal{V} = \{V_i : i < n\}$ is a chain cover of $\mathbb{H}^*$ that refines the cover $\mathcal{U}$.

We may assume that $\mathcal{V}$ is minimal and that each V_i is of the form $\mathrm{Ex}(O_i) \cap \mathbb{H}^*$, where

$$O_i = \bigcup_{n<\omega} (a_n^i, b_n^i) \text{ for every } i < n,$$

for two sequences $\{a_n^i\}_{n<\omega}$ and $\{b_n^i\}_{n<\omega}$ of reals increasing to infinity, such that $a_n^i < b_n^i < a_{n+1}^i$ for every $i < n$ and $n < \omega$.

As $\mathcal{V}$ is a cover of $\mathbb{H}^*$, for some $m < \omega$ we know that $[m, \infty) \subset \bigcup_{i<n} O_i$.

By construction of the cover $\mathcal{U}$ we have

$$U_i \setminus \bigcup_{j \neq i} U_j \neq \emptyset$$

for all $i < 4$. Let $n_i < n$ be such that the following holds for all $i < 4$.

$$V_{n_i} \cap U_i \setminus \bigcup_{j \neq i} U_j \neq \emptyset.$$

By (possibly) permuting the indexes of the open sets of the cover $\mathcal{U}$ we may assume that $n_0 < n_1 < n_2 < n_3$. This implies that the sub chain $\mathcal{V}^* = \{V_{n_0}, V_{n_0+1}, \ldots, V_{n_2}\}$ of $\mathcal{V}$ connects points from $U_0 \setminus \bigcup_{j\neq 0} U_j$ and $U_2 \setminus \bigcup_{j \neq 2} U_j$, and also the other way around. Hence the minimal sub chain of $\mathcal{V}$ with this property is a sub chain of $\mathcal{V}^*$. The order of $\mathbb{H}$ then gives us that $\mathcal{V}^*$ covers all of $\mathbb{H}^*$, which can obviously not be true as it misses the set $U_3 \setminus \bigcup_{j\neq 3} U_j$ completely; a contradiction.

We have proven the following theorem.

THEOREM 3.26. *$\mathbb{H}^*$ is non-chainable.*

The proof above can be slightly altered to give a proof of the following theorem.

THEOREM 3.27. *$\mathbb{I}_u$ is non-chainable.*

SKETCH OF PROOF. Note that it suffices to show that $[0_u, L]$ is non-chainable for some layer L of $\mathbb{I}_u \setminus \{0_u\}$.

Let $\{x_n; n < \omega\}$ be any increasing sequence in $\mathbb{I}$ and consider the open subsets $\{V_i : i < 4\}$ of $\mathbb{M}$ given by

$$V_i = \bigcup_{m<\omega} \{m\} \times (x_{8m+2i}, x_{8m+2i+3}) \text{ for } i < 4.$$

Let L be the limit of the sequence $\{x_{n,u} : n < \omega\}$ in $\mathbb{I}_u$, then $L <_u x_u$, and L is a non-trivial layer of $\mathbb{I}_u$.

Let $\mathcal{S} = \{S_i\}_{i<4}$ be the open cover of $\mathbb{I}_u$ defined by $S_0 = \{0_u\} \cup \operatorname{Ex} V_0 \cap \mathbb{I}_u$ and $S_i = \operatorname{Ex} V_i \cap \mathbb{I}_u$ for $1 \leq i < 4$.

Assuming that this open cover $\mathcal{S}$ has a finite chain refinement, ends in a contradiction similar to the one in the proof above, this time using the order of $\mathbb{I}$. □

As $\mathbb{I}$ is chainable we have the following corollary.

COROLLARY 3.28. *Chainability is not a first-order property in the language of lattices.*

6. Different kinds of span of $\mathbb{I}_u$ and $\mathbb{H}^*$

In this section we investigate the different kinds of span of the continua $\mathbb{I}_u$ and $\mathbb{H}^*$.

6.1. The span and the surjective (semi)span of $\mathbb{H}^*$.

LEMMA 3.29. *There exists a fixed point free onto self map of* $\mathbb{H}^*$.

PROOF. Let $f : \mathbb{H} \to \mathbb{H}$ be the map which assigns to every point x in $\mathbb{H}$ the point $x + 2$ then βf maps $\mathbb{H}^*$ onto $\mathbb{H}^*$. We will show that βf is a fixed point free self map of $\mathbb{H}^*$.

Let x be a point in $\mathbb{H}^*$. Consider the closed sets F_i $(i = 0, \ldots, 3)$ of $\mathbb{H}$ defined by

$$F_i = \bigcup_{n<\omega} [4n + i, 4n + i + 1].$$

The closures of these closed sets form a closed cover of $\beta\mathbb{H}$. There is an F_i which contains the point x in its closure. As f maps F_i onto $F_{(i+2) \bmod 4}$ and F_i and $F_{(i+2) \bmod 4}$ are disjoint, $\beta f(x) \neq x$. □

The graph of the continuous map of the previous lemma gives us the following corollary.

COROLLARY 3.30. *The surjective (semi)span of* $\mathbb{H}^*$ *is nonzero.*

6.2. The span of $\mathbb{I}_u$. The proof of the following theorem will be given in the rest of this subsection in a number of claims.

THEOREM 3.31. *The continuum* $\mathbb{I}_u$ *has span nonzero for any ultrafilter* $u \in \omega^*$.

Let $\{x_n\}_{n<\omega}$ be some increasing sequence in $\mathbb{I}$ which converges to 1. Without loss of generality assume that $x_0 = 0$. Let $x_{n,u}$ denote the point of $\mathbb{I}_u$ that corresponds to the point $\{(m, x_n) : m < \omega\}$ of $\mathbb{M}$. As $\{x_n\}_{n<\omega}$ is strictly increasing in $\mathbb{I}$ we have that $\{x_{n,u}\}_{n<\omega}$ is a strictly increasing sequence in $\mathbb{I}_u$, let L denote the limit of this sequence. By proposition 3.16 L is a non-trivial layer of $\mathbb{I}_u$. And as $x_u = u\text{–}\lim_n\{x_n\}$ is a point of $\mathbb{P}_u$ for which we have that $x_{n,u} <_u x_u$ we also have that $L <_u x_u$.

The proof of the theorem will now be as follows. We will define a continuous map f on $\mathbb{I}_u$ that maps L onto itself without a fixed point. The graph of the map $f \restriction L$ will then be a subcontinuum of $\mathbb{I}_u \times \mathbb{I}_u$ that projects onto L in both axes and which has an empty intersection with the diagonal. This implies that the span of the continuum $\mathbb{I}_u$ is non-zero.

We define the map $f : \mathbb{I}_u \to \mathbb{I}_u$ by defining it on $\mathbb{M}$, taking its Čech-Stone extension and restricting it to the space $\mathbb{I}_u$.

(1) Let $f \restriction \mathbb{I}_0$ be equal to the identity.
(2) For all $n \geq 1$ let $f \restriction \mathbb{I}_n$ be given by the following restrictions
    (a) $f \restriction \mathbb{I}_n$ maps the point (n, x_k) onto the point (n, x_{k+1}) for all $k < n$ and the point $(n, 1)$ onto the point $(n, 1)$.
    (b) $f \restriction \mathbb{I}_n$ is linear on all the intervals $\{n\} \times [x_k, x_{k+1}]$ for all $k < n$ and on the interval $\{n\} \times [x_n, 1]$.

CLAIM 3.32. *The Čech-Stone extension of the map f maps $[0_u, L]$ continuously onto $[x_{1,u}, L]$.*

PROOF. It is not hard to see that βf maps the interval $[x_{k,u}, x_{k+1,u}]$ of $\mathbb{I}$ onto $[x_{k+1,u}, x_{k+2,u}]$ (in a one-to-one way) for all $k < \omega$. So βf maps $[0_u, L)$ continuously onto $[x_{1,u}, L)$. Furthermore, $\beta f \restriction \mathbb{I}_u : \mathbb{I}_u \to \mathbb{I}_u$ is a closed map. As by proposition 3.18 the closure of $[0_u, L)$ in $\mathbb{I}_u$ is $[0_u, L]$ we have

$$\begin{aligned}\beta f[0_u, L] &= \beta f(\mathrm{cl}_{\mathbb{I}_u}([0_u, L))) = \mathrm{cl}_{\mathbb{I}_u}(\beta f([0_u, L))) \\ &= \mathrm{cl}_{\mathbb{I}_u}([x_{1,u}, L)) = [x_{1,u}, L].\end{aligned}$$

□

CLAIM 3.33. *The restriction $(\beta f) \restriction L$ maps L onto L.*

PROOF. The construction of f gives us that $\beta f[0_u, L) = [x_{1,u}, L)$, and as we have seen above this gives us $L \subset \beta f[L]$.

Again by construction of f we have for every $n < \omega$ that βf maps $[x_{n,u}, L)$ onto $[x_{n+1,u}, L)$ in one-to-one way. So we have, as $\beta f[L] \subset \beta f[x_{n,u}, L] = [x_{n+1,u}, L]$ for all $n < \omega$ that $\beta f[L] \subset \bigcap_n [x_{n,u}, L] = L$. □

Now we have defined the self map $\beta f \restriction L$ of L onto L, we only need to show that it is fixed point free, to prove theorem 3.31.

CLAIM 3.34. *For all x in L we have $\beta f(x) \neq x$, in other words $\beta f \restriction L$ is a fixed point free self map of L onto L.*

PROOF. Consider a point x in the layer L. For every n let a_n be the middle point of the interval (x_n, x_{n+1}).

Note that the map f maps (n, a_k) onto the point (n, a_{k+1}) for every $k < n$. Define the following closed subsets F_i for $i = 0, \dots, 3$:

$$F_0 = \bigcup_n \{\{n\} \times \bigcup_{k<n} [x_k, a_k], \quad F_2 = \bigcup_n \{\{n\} \times \bigcup_{k<n-1} [x_{k+1}, a_{k+1}],$$

$$F_1 = \bigcup_n \{\{n\} \times \bigcup_{k<n} [a_k, x_{k+1}] \text{ and } F_3 = \bigcup_n \{\{n\} \times \bigcup_{k<n-1} [a_{k+1}, x_{k+2}].$$

Note that the closure in $\beta\mathbb{M}$ of the union of the F_i's contains the interval $[0_u, L]$ of $\mathbb{I}_u$. Also note that the closed set F_i is mapped onto the the closed set $F_{(i+2) \bmod 4}$, so $f[F_i] \cap F_i = \emptyset$. As x is contained in one of the sets $\mathrm{cl}_{\beta\mathbb{M}} F_i$, $i = 0, \dots, 3$, we see that $\beta f(x) \neq x$. □

As $\mathbb{I}$ is a continuum which has span zero and $\mathbb{I}_u$ is a continuum with span nonzero we have the following corollary.

COROLLARY 3.35. *Having span zero is not preserved by ultracopowers.*

6.3. The surjective (semi) span of $\mathbb{I}_u$. In section 6 we showed that the surjective (semi) span of $\mathbb{H}^*$ is non-zero, by constructing a fixed point free self map of $\mathbb{H}^*$ onto $\mathbb{H}^*$. In this section we will see that the continuum $\mathbb{I}_u$ has (at least under CH) also non-zero surjective (semi) span. We show this by constructing a continuous self map f of $\mathbb{I}_u$ whose graph has empty intersection with the diagonal $\Delta_{\mathbb{I}_u}$.

As the graph of every continuous self map of $\mathbb{I}_u$ which does not hit the diagonal contains points of the form (x, y) with $x <_u y$ and points (s, t) of the form $t <_u s$ somewhere we have to "cross" the diagonal. As this is obviously impossible in a cut-point of $\mathbb{I}_u$, it must happen in some non-trivial layer L.

Consider an increasing sequence $\{x_n : n < \omega\}$ in $\mathbb{I}$, and the non-trivial layer L, the limit of the sequence $\{x_{n,u} : n < \omega\}$ in $\mathbb{I}_u$.

In section 6 we constructed a mapping $f : \mathbb{M} \to \mathbb{M}$ such that the restriction of its Čech-Stone extension βf to $[0_u, L]$ is a continuous mapping of $[0_u, L]$ onto $[x_{1,u}, L]$ which does not have a fixed point. This map βf and the retraction we get from the following theorem will show that the surjective (semi) span of $\mathbb{I}_u$ is non-zero.

6.3.1. *Preliminaries.* Let $\{a_n\}_{n<\omega}$ and $\{b_\alpha\}_{\alpha<\omega_1}$ be sequences in $\mathbb{P}_u$ that form an $\langle\omega, \omega_1\rangle$-gap in $\mathbb{P}_u$ (exists under CH) and let L be the corresponding layer of $\mathbb{I}_u$.

$$L = \bigcap \{[a_n, b_\alpha] : n < \omega \text{ and } \alpha < \omega_1\}.$$

Let $\{a_n(k)\}_{k<\omega}$ and $\{b_\alpha(k)\}_{k<\omega}$ be sequences in $\mathbb{I}$ such that $a_n = u-\lim_k \{a_n(k)\}$ for every $n < \omega$ and $b_\alpha = u-\lim_k \{b_\alpha(k)\}$ for every $\alpha < \omega_1$.

THEOREM 3.36. *L is a retract of $[L, 1_u]$.*

To prove the theorem we will construct a map ϕ between lattice bases $\mathcal{B}$ and $\mathcal{C}$ of L and $[L, 1_u]$ respectively having certain properties, such that the continuous map we get from theorem 2.12 will be a retraction.

Let $\mathcal{R}$ be the family of finite unions of closed subsets of $\mathbb{I}$ with rational endpoints. For every $f \in {}^\omega\mathcal{R}$ let us define the closed subset A_f of $\mathbb{M}$ by

$$A_f = \bigcup_{n<\omega} \{n\} \times f(n).$$

These sets form a lattice base for $\mathbb{M}$, and by normality of $\mathbb{M}$, the closures of these sets in $\beta\mathbb{M}$ will be a lattice base for $\beta\mathbb{M}$. The lattice base $\mathcal{B}$ of L consists of intersections of the form $\overline{A_f} \cap L$ and the lattice base $\mathcal{C}$ of $[L, 1_u]$ consists of intersections of the form $\overline{A_f} \cap [L, 1_u]$.

For $f, g \in {}^\omega\mathcal{R}$ let $f \sim g$ denote the existence of an $n < \omega$ and an $\alpha < \omega_1$ such that

$$\{k < \omega : A_f \cap \{k\} \times [a_n(k), b_\alpha(k)] = A_g \cap \{k\} \times [a_n(k), b_\alpha(k)]\} \in u.$$

It is easily seen that this relation $\sim$ on ${}^\omega\mathcal{R}$ is an equivalence relation.

LEMMA 3.37. *For all $f, g \in {}^\omega\mathcal{R}$, $f \sim g$ if and only if for some $n < \omega$ and some $\alpha < \omega_1$ we have $\overline{A_f} \cap [a_n, b_\alpha] = \overline{A_g} \cap [a_n, b_\alpha]$.*

PROOF. One implication follows directly from the definition of the equivalence relation. For the reverse implication, note that for every $h \in {}^\omega\mathcal{R}$ the restriction of $\overline{A_h}$ to $\mathbb{I}_k$ is equal to $\{k\} \times h(k)$. □

LEMMA 3.38. *For all $f, g \in {}^\omega\mathcal{R}$, $f \sim g$ if and only if $\overline{A_f} \cap L = \overline{A_g} \cap L$.*

PROOF. Again, one implication follows directly from the definition of the equivalence relation.

For the other implication suppose that $f \not\sim g$. Without loss of generality we can assume that $(\overline{A_f} \setminus \overline{A_g}) \cap [a_n, b_\alpha] \neq \emptyset$ for all $n < \omega$ and $\alpha < \omega_1$. Let F be the closed subset of A_f consisting of all center points of the intervals in $\{k\} \times (f(k) \setminus g(k))$ for every $k < \omega$. The closure of this set F in $\beta\mathbb{M}$ contains elements of L, is contained in $\overline{A_f}$ and is disjoint with $\overline{A_g}$ implying that $\overline{A_f} \cap L \neq \overline{A_g} \cap L$. □

LEMMA 3.39. *For any $h \in {}^\omega\mathcal{R}$ if $L \subset \overline{A_h}$ then there are $n < \omega$ and $\alpha < \omega_1$ such that $[a_n, b_\alpha] \subset \overline{A_h}$.*

PROOF. Let D be a closed discrete subset of $\mathbb{M}$ we get by picking one point in every complementary interval of $\{k\} \times h(k)$ for every $k < \omega$.

Suppose now that for every $n < \omega$ and any $\alpha < \omega_1$ we have $[a_n, b_\alpha] \not\subset \overline{A_h}$.

Let $n < \omega$ and $\alpha < \omega_1$ the set

$$V = \{k < \omega : \{k\} \times [a_n(k), b_\alpha(k)] \cap D \neq \emptyset\}$$

is an element of the ultrafilter u, for if not, $\omega \setminus V$ would be an element of u which would imply that

$$[a_n, b_\alpha] \subset \overline{\bigcup_{k \in \omega \setminus V} \{k\} \times [a_n(k), b_\alpha(k)]} \subset \overline{A_h},$$

in contradiction with the assumptions made on $\overline{A_h}$.

This implies that $[a_n, b_\alpha] \cap \overline{D} \neq \emptyset$, for all $n < \omega$ and $\alpha < \omega_1$, hence $L \cap \overline{D} \neq \emptyset$. This is in contradiction with the fact that D and A_h are closed and disjoint subsets of $\mathbb{M}$ and the assumption that L is contained in $\overline{A_h}$. □

LEMMA 3.40. *For any $h \in {}^\omega\mathcal{R}$ if $L \cap \overline{A_h} = \emptyset$ then there are $n < \omega$ and $\alpha < \omega_1$ such that $[a_n, b_\alpha] \cap \overline{A_h} = \emptyset$.*

PROOF. As the layer L is the intersection of the closed intervals $[a_n, b_\alpha]$. As $\overline{A_h} \cap \mathbb{I}_u$ is a closed set which misses L, by compactness of $\mathbb{I}_u$ there exist $n_0, \ldots, n_{m-1} < \omega$ and $\alpha_0, \ldots, \alpha_{m-1} < \omega_1$ such that

$$\overline{A_h} \cap \bigcap_{i<m} [a_{n_i}, b_{\alpha_i}] = \emptyset.$$

For $n = \max\{n_i : i < m\}$ and $\alpha = \max\{\alpha_i : i < m\}$ we have $[a_n, b_\alpha] \cap \overline{A_h} = \emptyset$. □

6.3.2. *Constructing the retraction.* For the construction of the retraction we will make use of theorem 2.12. We will construct a map $\phi : \mathcal{B} \to \mathcal{C}$ such that

(1) $\phi(\emptyset) = \emptyset$ and for all nonempty $B \in \mathcal{B}$ the set $\phi(B)$ is nonempty;
(2) if $F \cup G = L$ for $F, G \in \mathcal{B}$ then $\phi(F) \cup \phi(G) = [L, 1_u]$;
(3) if $\mathcal{F}$ is a finite subset of $\mathcal{B}$ such that $\bigcap \mathcal{F} = \emptyset$ then we have $\bigcap_{F \in \mathcal{F}} \phi(F) = \emptyset$;
(4) $\phi(F) \cap L = F$ for all $F \in \mathcal{B}$.

Together with theorem 2.12 this implies that L is a retract of $[L, 1_u]$. Let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathcal{B}$ (note that we assumed CH). By an ω_1-recursion we will define $\phi(A_\alpha)$ in $\mathcal{C}$. Consider first the following situation.

Suppose $\{f_n\}_{n \leq \omega}$ is a subset of ${}^\omega\mathcal{R}$ such that

$$\overline{A_{f_m}} \cap L \neq \overline{A_{f_n}} \cap L \text{ for all } m < n \leq \omega.$$

Let us define the following sets B_n, C_n and D_n for $n \leq \omega$

$$B_n = \{m < n : \overline{A_{f_n}} \cup \overline{A_{f_m}} \supset L\}$$

$$C_n = \{F \subset n : |F| < \omega \text{ and } \overline{A_{f_n}} \cap \bigcap_{m \in F} \overline{A_{f_m}} = \emptyset\}$$

$$D_n = \{G \subset n : |G| < \omega \text{ and } \bigcap_{m \in G} \overline{A_{f_m}} \subset \overline{A_{f_n}}\}.$$

Suppose we also have a subset $\{g_n\}_{n<\omega}$ of ${}^\omega\mathcal{R}$ such that for every $n < \omega$ we have an $\alpha_n < \omega_1$ such that

$$\overline{A_{f_n}} \cap [L, b_{\alpha_n}] = \overline{A_{g_n}} \cap [L, b_{\alpha_n}],$$

and the following properties are also satisfied

(1) if $m \in B_n$ then $(\overline{A_{g_m}} \cup \overline{A_{f_n}}) \cap [L, b_{\alpha_n}] = [L, b_{\alpha_n}]$ and for ultrafilter many $k < \omega$ we have $[b_{\alpha_n}(k), 1] \subset g_n(k) \cup g_m(k)$,

(2) if $F \in C_n$ then $\overline{A_{f_n}} \cap \bigcap_{m\in F} \overline{A_{g_m}} \cap [L, b_{\alpha_n}] = \emptyset$ and for ultrafilter many $k < \omega$ we have $g_n(k) \cap \bigcap_{m\in F} g_m(k) = \emptyset$,

(3) if $G \in D_n$ then $\bigcap_{m\in F} \overline{A_{g_m}} \cap [L, b_{\alpha_n}] \subset \overline{A_{f_n}}$ and for ultrafilter many $k < \omega$ we have $\bigcap_{m\in F} g_m(k) \subset g_n(k)$,

CLAIM 3.41. *There is a $g_\omega \in {}^\omega\mathcal{R}$ that has all these properties related to the map f_ω, the sets B_ω, C_ω and D_ω and some $\alpha_\omega < \omega_1$.*

PROOF. As there are only countably many restrictions on g_ω, by the lemmas 3.39 and 3.40 there exists an $\alpha < \omega_1$ such that on the interval $[L, b_\alpha]$ the set $\overline{A_{f_\omega}}$ already fulfills all requirements on g_ω, so we only need to define $g_\omega(k)$ on the interval $[b_\alpha(k), 1]$, for ultrafilter many $k < \omega$.

We will do this in an ω-recursion, constructing at step $n < \omega$ a $g^n_\omega \in {}^\omega\mathcal{R}$ and a $U_n \in u$ such that $\{U_n\}_{n<\omega}$ is strictly descending and g^n_ω has all properties that g_ω must have that follow from $B_\omega \cap n$, $C_\omega \cap \mathcal{P}(n)$ and $D_\omega \cap \mathcal{P}(n)$.

The map g_ω defined by

$$g_\omega(k) = g^n_\omega(k) \text{ for all } k \in U_n \setminus U_{n+1},$$

will satisfy all the requirements that follow from B_ω, C_ω and D_ω.

Suppose we already have constructed $U_m \in u$ and $g^m_\omega \in {}^\omega\mathcal{R}$ for $m < n$. To find suitable U_n and g^n_ω consider $m \in B_\omega \cap n$, $F \in C_\omega \cap \mathcal{P}(n)$ and $G \in D_\omega \cap \mathcal{P}(n)$.

By construction, for ultrafilter many $k < \omega$ we have

$$\bigcap_{i\in F} g_i(k) \subset \operatorname{int}(g_m(k))$$

$$\Big(\bigcap_{j\in G} g_j(k)\Big) \cap \Big(\bigcap_{i\in F} g_i(k)\Big) = \emptyset.$$

As $B_\omega \cap n$, $C_\omega \cap \mathcal{P}(n)$ and $D_\omega \cap \mathcal{P}(n)$ are all finite, we can find a $h \in {}^\omega\omega$ such that, for ultrafilter many $k < \omega$ and all $m \in B_\omega \cap n$, $F \in C_\omega \cap \mathcal{P}(n)$ and $G \in D_\omega \cap \mathcal{P}(n)$

$$B_{\frac{1}{h(k)}}\Big(\bigcap_{i\in F} g_i(k)\Big) \cap \Big(\bigcap_{j\in G} g_j(k)\Big) = \emptyset,$$

$$B_{\frac{1}{h(k)}}\Big(\bigcap_{i\in F} g_i(k)\Big) \subset g_m(k).$$

When we define the map $g^n_\omega \in {}^\omega\mathcal{R}$ by

$$\begin{aligned} g^n_\omega(k) \cap [0, b_\alpha(k)] &= f_\omega(k) \cap [0, b_\alpha(k)] \text{ for all } k < \omega, \\ g^n_\omega(k) \cap [b_\alpha(k), 1] &= [b_\alpha, 1] \setminus \bigcup_{F\in C_\omega \cap \mathcal{P}(n)} B_{\frac{1}{h(k)}}\Big(\bigcap_{i\in F} g_i(k)\Big) \text{ for all } k < \omega, \end{aligned}$$

we have made sure that all properties following from $B_\omega \cap n$, $C_\omega \cap \mathcal{P}(n)$ and $D_\omega \cap \mathcal{P}(n)$ are satisfied on ultrafilter many coordinates. Choose $U_n \in u$ such that $U_n \subset U_{n-1}$ and $\min U_{n-1} < \min U_n$. □

Without loss of generality we can assume in the enumeration $\{A_\alpha : \alpha < \omega_1\}$ of $\mathcal{B}$ that $A_0 = \emptyset$ and $A_1 = L$. We can pick representing maps from the equivalence classes of $f_\alpha \in {}^\omega\mathcal{R}/_\sim$ such that

$$A_\alpha = \overline{A_{f_\alpha}} \cap L \text{ for every } \alpha < \omega_1.$$

Let the maps g_0, g_1 and g_2 in ${}^\omega\mathcal{R}$ be maps defined by $g_0(k) = \emptyset$, $g_1(k) = [0,1]$ and $g_2(k) = f_2(k)$ for all $k < \omega$. When we need to define $\phi(A_\alpha)$, note that α is countable and we can use the claim in the previous section to find a $g_\alpha \in {}^\omega\mathcal{R}$ such that $\phi(A_\alpha) = \overline{A_{g_\alpha}} \cap [L, 1_u]$ is as required.

Note that if α is finite then the construction in the previous section will still give us a suitable g_α.

We can now prove the following theorem.

THEOREM 3.42 (CH). *The surjective (semi)span of* $\mathbb{I}_u$ *is nonzero.*

PROOF. Let βf denote the Čech-Stone extension of the map f from section 6 that maps $[0_u, L]$ onto $[x_{1,u}, L]$ without fixed points, and g the retraction of $[L, 1_u]$ onto L. Let $h : \mathbb{I}_u \to \mathbb{I}_u$ be the combination $h_1 \bigtriangledown h_2$ of the compatible maps $h_1 : [0_u, L] \to \mathbb{I}_u$ and $h_2 : [L, 1_u] \to \mathbb{I}_u$ defined by

$$\begin{aligned} h_1 &= (\beta f) \restriction [0_u, L] \\ h_2 &= ((\beta f) \circ g) \restriction [L, 1_u]. \end{aligned}$$

Then h is a continuous self map of $\mathbb{I}_u$ without fixed points. The graph of h is a subcontinuum of the square $\mathbb{I}_u \times \mathbb{I}_u$ which maps onto $\mathbb{I}_u \times \{0_u\}$ in one way, so this gives us that the surjective semi span of $\mathbb{I}_u$ is non-zero.

If we join the graph of h with $\{0_u\} \times [x_{1,u}, 1_u]$ and $\{1_u\} \times [0_u, L]$ we have a subcontinuum of the square of $\mathbb{I}_u$ which does not intersect the diagonal $\triangle_{\mathbb{I}_u}$; as both of its projections map onto $\mathbb{I}_u$ we see that the surjective span of $\mathbb{I}_u$ must also be non-zero. □

COROLLARY 3.43. *Having surjective span zero or surjective semi span zero is a property that is not preserved by ultracopowers.*

The continuous self map $h : \mathbb{I}_u \to \mathbb{I}_u$ without fixed points gives us the following interesting corollary.

COROLLARY 3.44. *The fixed-point property is not preserved by ultracopowers.*

6.4. The symmetric span of $\mathbb{I}_u$ and $\mathbb{H}^*$. In this section we will show that under CH the symmetric span of the continuum $\mathbb{I}_u$ is non-zero. We will accomplish this by finding a non-trivial layer L of $\mathbb{I}_u$ and show that

(1) There exists a continuous mapping f from L onto L without a fixed point.
(2) There exists a retraction g of $[0_u, 1_u]$ onto L.

Fix a non-trivial layer L of $\mathbb{I}_u$ which determines an (ω_1, ω_1)-gap in $\mathbb{P}_u$. Choose sequences $\{f_\alpha\}_{\alpha<\omega_1}$ and $\{g_\alpha\}_{\alpha<\omega_1}$ in ${}^\omega\mathbb{I}$ such that if we let $f_{\alpha,u}$ and $g_{\alpha,u}$ denote the points $u - \lim\{f_\alpha(n) : n < \omega\}$ and $u - \lim\{g_\alpha(n) : n < \omega\}$ respectively for all α less than ω_1 then we have

$$f_{\alpha,u} <_u f_{\beta,u} <_u L <_u g_{\beta,u} <_u g_{\alpha,u}, \text{ for all } \alpha < \beta < \omega_1.$$

As L is a (ω_1, ω_1)-gap in $\mathbb{I}^\omega/_u$, we get the retraction g above by applying theorem 3.36 twice. Once to get a retraction from $[L, 1_u]$ onto L and once for the other part of the interval, to get a retraction form $[0_u, L]$ onto L.

To construct a continuous onto self map of L we will use the following theorem from [**16**], theorem 3.24, which describes the structure of the layer L completely.

Fix $\{\{n_i\} \times [a_i, b_i] : i < \omega\}$, a sequence of intervals in $\mathbb{M}$ such that

(1) $\lim_i n_i = \infty$ and $n_i \le n_{i+1} \le n_i + 1$ for all $i < \omega$,
(2) if $n_i = n_{i+1}$ then $a_{i+1} = b_i$, and
(3) if $n_{i+1} = n_i + 1$ then $b_i = 1$ and $a_i = 0$;

let $v \in \omega^*$ such that $[a_v, b_v] \subset L$.

For any $\alpha < \omega_1$ and any $U \in u$ we have that

$$\{i < \omega : \{n_i\} \times [a_i, b_i] \subset \bigcup_{n \in U} [f_\alpha(n), g_\alpha(n)]\},$$

is an element of the ultrafilter v. Without loss of generality we may assume that $b_i \ne 1$ for every i and n, as we would only ignore in every $\mathbb{I}_n$ at most one interval $\{n_i\} \times [a_i, b_i]$.

Let us define the continuous map $h : \mathbb{M} \to \mathbb{M}$ by defining its restrictions to the subsets $\mathbb{I}_n$ of $\mathbb{M}$. The map $h \restriction \mathbb{I}_n$ is equal to the identity if there is no i for which $n_i = n$. If there is some $i < \omega$ such that $n_i = n$, then the map $h \restriction \mathbb{I}_n$ is defined by

(1) $h \restriction \mathbb{I}_n(n, 0)$ is equal to (n_i, a_i) with the smallest index i for which $\{n_i\} \times [a_i, b_i] \subset \mathbb{I}_n$,
(2) $h \restriction \mathbb{I}_n(n_i, a_i)$ is equal to (n_i, b_i) for every i such that $\{n_i\} \times [a_i, b_i] \subset \mathbb{I}_n$,
(3) $h \restriction \mathbb{I}_n(n_i, b_i)$ is equal to (n_{i+1}, a_{i+1}) for every i for which $\{n_i\} \times [a_i, b_i] \cup \{n_{i+1}\} \times [a_{i+1}, b_{i+1}]$ is a subset of $\mathbb{I}_n$,
(4) the restriction $h \restriction \mathbb{I}_n$ is equal to $(n, 1)$ on the interval $\{n_i\} \times [b_i, 1]$ where i is the largest index for which $\{n_i\} \times [a_i, b_i] \subset \mathbb{I}_n$ and
(5) the map $h \restriction \mathbb{I}_n$ is linear between these points just mentioned.

This map h is a continuous map on the space $\mathbb{M}$ whose fixed points are points of the form $(n, 1)$ for some n.

Take some element $V \in v$, then with $\alpha < \omega$ and $U \in u$ consider the sequence $\{m_n : n \in U\}$, defined by

$$m_n = |\{i \in V : \{n_i\} \times [a_i, b_i] \subset [f_\alpha(n), g_\alpha(n)]\}|.$$

As L is a non-trivial layer we have that the sequence $\{m_n : n \in U\}$ must be unbounded. We see that the sequence $\{|\{i \in V : h[\{n_i\} \times [a_i, b_i]] \subset [f_\alpha(n), g_\alpha(n)]\}| : n \in U\}$ is also unbounded as each entry differs at most one from the corresponding number m_n.

For every $\alpha < \omega_1$ and every $U \in u$ we have that the intersection of $[f_{\alpha,u}; g_{\alpha,u}]$ with the set

$$\mathrm{cl}_{\beta\mathbb{M}} \bigcup \{\{n_i\} \times [a_i, b_i] : \{n_i\} \times [a_i, b_i] \subset [f_\alpha(n), g_\alpha(n)], n \in U\},$$

is non-empty. So the intersection of all these sets for $\alpha < \omega_1$ gives us a closed non-empty set in L, namely the interval $[a_v, b_v]$.

Likewise we find that for every $\alpha < \omega_1$ the intersection of $[f_{\alpha,u}; g_{\alpha,u}]$ with the set

$$\mathrm{cl}_{\beta\mathbb{M}} \bigcup \{h[\{n_i\} \times [a_i, b_i]] : \{n_i\} \times [a_i, b_i] \subset [f_\alpha(n), g_\alpha(n)], n \in U\},$$

is non-empty. This gives us also a closed subset of $\mathbb{I}_u$. As $h((n_i, a_i)) = (n_i, b_i)$ for ultrafilter v many i's we have that $\beta h(a_v) = b_v$ and so the standard subcontinua $[a_v, b_v]$ and $\beta h[[a_v, b_v]] = [b_v, a_{\sigma(v)})]$ intersect. So the layer L and $h[[a_v, b_v]]$ intersect, as this last set cannot contain the layer L it must be a subset of L by theorem 3.23.

CLAIM 3.45. *The restriction of the Čech-Stone extension βh of the map h to the layer L is a fixed-point free self map.*

PROOF. By theorem 3.24 we know that if we let $F \subset \omega^*$ be the set of all ultrafilters w on ω such that $[a_w, b_w] \subset L$ then L can be written as $L = \bigcup_{w \in F} [a_w, b_w]$.

We have just seen that every $[a_v, b_v]$ is mapped onto the subinterval $[b_v, a_{\sigma(v)}]$ of L for every $v \in F$.

Every $x \in L$ is contained in some standard subcontinuum $[a_v, b_v]$ of L for some $v \in F$, so the restriction of βh to L is fixed point free. □

Let C be the subcontinuum of the square $\mathbb{I}_u \times \mathbb{I}_u$ that is the graph of the map $h \circ g$. By construction, the intersection of C with the diagonal $\triangle_{\mathbb{I}_u}$ is empty. Define the closed subset $D \subset \mathbb{I}_u \times \mathbb{I}_u$ by

$$D = \{(x, y) \in \mathbb{I}_u \times \mathbb{I}_u : (y, x) \in C\}.$$

CLAIM 3.46. *The intersection of C and D is empty.*

PROOF. Suppose that $(x, y) \in C \cap D$, then $\beta h(x) = y$ and $\beta h(y) = x$. As $x \in L$ there exists a $v \in F$ such that $x \in [a_v, b_v]$, we then also know that $y = h(x) \in [b_v, a_{\sigma(v)}]$ and $x = \beta h(y) \in [a_{\sigma(v)}, b_{\sigma(v)}]$. But this would give that $[a_v, b_v] \cap [a_{\sigma(v)}, b_{\sigma(v)}] \neq \emptyset$, a contradiction. □

If we let X be the union

$$\bigcup \{C, D, \{0_u\} \times [L, 1_u], \{1_u\} \times [0_u, L], [L, 1_u] \times \{0_u\}, [0_u, L] \times \{1_u\}\},$$

then X is a symmetric subset of the square $\mathbb{I}_u \times \mathbb{I}_u$ which has empty intersection with the diagonal. As X is also a subcontinuum of the square, we have the following theorem.

THEOREM 3.47 (CH). *The symmetric span of the continuum* $\mathbb{I}_u$ *is nonzero.*

COROLLARY 3.48 (CH). *Having symmetric span zero is not preserved by ultracopowers.*

7. Another model theoretic approach

The lattice sentence which expresses that the Wallman representation of a distributive lattice that models this sentence is chainable is, as we have seen in section 3, not a first - order sentence. In this section we will take an elementary sublattice of some lattice 2^X in another way, such that we have a lot more structure to work with. To be more precise we will be taking elementary substructures of our lattices with respect to submodels (see chapter 2).

Let for the remainder of this section θ be a large enough cardinal such that $H(\theta)$ reflects all the theorems of ZFC we will need in our reasonings below.

7.1. Preliminaries. Let X be a continuum, and $\mathfrak{M}$ be an elementary substructure of $H(\theta)$ such that $2^X \in \mathfrak{M}$.

The following objects are uniquely defined in terms of elements of $\mathfrak{M}$ and are therefore elements of $\mathfrak{M}$.

(1) X, the union of 2^X and $X \times X$, the product of this union.
(2) The diagonal $\triangle_X$, defined as the set of all ordered pairs of elements from X whose coordinates are equal and which contains, for every element x of X the ordered pair (x, x).
(3) The lattice $2^{X\times X}$ of all closed sets of $X \times X$.

Let L be the elementary sublattice $2^X \cap \mathfrak{M}$ of 2^X, and K the elementary sublattice $2^{X\times X} \cap \mathfrak{M}$ of $2^{X\times X}$.

LEMMA 3.49. $wL \times wL = wK$.

PROOF. As L and K are elementary sublattices of 2^X and $2^{X\times X}$ respectively, by theorem 2.12 there are continuous mappings $f : X \to wL$ and $g : X \times X \to wK$. Consider the mapping $h : wK \to wL \times wL$ defined by

$$h(p) = (\{\pi_1[P] : P \in p\}, \{\pi_2[P] : P \in p\}).$$

CLAIM 3.50. *For* $p \in wK$ *is* $h(p)$ *an element of* $wL \times wL$.

PROOF. Note that by elementarity of $\mathfrak{M}$ we have $\{A \times B : A, B \in L\} \subset K$. Also by elementarity, for every $P \in K$ we have that its projections $\pi_1(P)$ and $\pi_2(P)$ on the axes of $X \times X$ are elements of L.

Suppose that there is some D in L such that $D \cap A \neq \emptyset$ for all $A \in \{A : A \times X \in p\} = \{\pi_1[C] : C \in p\}$. Then we would have that $D \times X \cap P \neq \emptyset$ for all $P \in p$ and so, as p is an ultrafilter, we have $D \times X \in p$ and $D \in \{\pi_1[P] : P \in p\}$. □

CLAIM 3.51. *$h : wK \to wL \times wL$ is continuous and one to one.*

PROOF. As inverse images under h of the closed base element $A \times B$ for any $A, B \in L$ of $wL \times wL$ is equal to the intersection of the closed sets $A \times X$ and $X \times B$ in wK, the map h is continuous.

Suppose that $p \neq q$ are elements of wK, then there are $P \in p$ and $Q \in q$ such that $P \cap Q = \emptyset$. By elementarity we have open disjoint subsets U and V, with disjoint closures of P and Q in $\mathfrak{M}$. For each point in the closed set P we can find an open set that is a product of open sets of X, such that the closure of this product is contained in the closure of U. The set C being compact there are finitely many of these products of open subsets of X needed to cover it. By elementarity there are a finite number of pairs of elements of L whose product cover P and and a finite number of pairs of elements of L whose product cover Q, such that there unions are disjoint. p and q being ultrafilters, and K containing all products of elements of L, p and q contain a product of elements of L that have empty intersection. This implies that $h(p) \neq h(q)$. □

CLAIM 3.52. *$h[wK] = wL \times wL$.*

PROOF. For this we will show that for every $a, b \in wL$ we have $p \circ g \circ (f \times f)^{-1}(a, b) = ((a, b))$. For this we bring the proof of theorem 2.12 again to mind.

Let (x, y) be an element of $X \times X$. Suppose that $f(x) = a$, $f(y) = b$ and $g((x, y)) = p$. We have the following equations $a = \bigcap\{A \in L : x \in A\}$, $b = \bigcap\{B \in L : y \in B\}$ and $p = \bigcap\{P \in K : (x, y) \in P\}$.

Obviously, if $x \in A \in L$ and $y \in B \in L$ then $(x, y) \in A \times B \in K$ by elementarity. Aiming for a contradiction, suppose that $h(p) = (e, f) \neq (a, b)$. Without loss of generality we may assume that $e \neq a$. So there are elements A and E of L such that $A \in a$, $E \in e$ and $A \cap E = \emptyset$. Now $h(p) = (e, f)$ gives us that $(x, y) \in E \times X$, which means that $x \in E$, and thus $a \in E$ which is impossible. □

By the previous three claims we have that h is a one-to-one continuous map of wK onto $wL \times wL$ so, as these spaces are compact the map h is in fact a homeomorphism. □

7.2. Reflecting chainability. Suppose that X is a chainable continuum, and $\mathfrak{M}$ is an elementary submodel of $H(\theta)$ containing 2^X as one of its elements. Let L and K denote the lattices $2^X \cap \mathfrak{M}$ and $2^{X \times X} \cap \mathfrak{M}$ respectively. We will show that the continuum wL is chainable as well.

Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite open cover of the continuum wL.

THEOREM 3.53 ([**19**]). *Every finite open cover $\{U_i\}_{i=1}^k$ of a normal space X has shrinkings $\{F_i\}_{i=1}^k$ and $\{W_i\}_{i=1}^k$, functionally closed and functionally open respectively, such that $F_i \subset W_i \subset \overline{W_i} \subset U_i$ for $i = 1, \dots, k$.*

As L is isomorphic to a lattice base for wL, and $\mathcal{U}$ is finite, we can find an open cover $\mathcal{V} = \{V_1, \ldots, V_n\}$ of wL such that $V_i \subset U_i$ for all i and every V_i is the complement of some base element A from L.

By elementarity we then find a chain cover refinement of $\mathcal{V}$, which obviously is a chain cover refinement of $\mathcal{U}$.

7.3. Reflecting non-chainability. Let X be some non-chainable continuum and suppose that $\mathfrak{M}$ is an elementary submodel of $H(\theta)$ (where θ large enough) such that $2^X \in \mathfrak{M}$. Let L denote the elementary sublattice of 2^X given by $L = 2^X \cap \mathfrak{M}$.

The continuum X being non-chainable, there exists a (finite) open cover of X, every finite refinement of which is not a chain cover of X. By elementarity there exist elements $x_0, \ldots, x_{n-1} \in \mathfrak{M}$, for some $n < \omega$ such that

$$\mathfrak{M} \models \{x_i : i < n\} \subset 2^X \wedge \bigcap_{i<n} x_i = \emptyset.$$

Corresponding to $\{x_i : i < n\}$ is an open cover of which every refinement is not a chain cover. Translated into closed set terminology, for every $m < \omega$ and every $y_0, \ldots, y_{m-1} \in 2^X \cap \mathfrak{M}$, if $\mathfrak{M}$ models the sentence

$$\bigcap_{i<m} y_i = \emptyset \wedge \bigwedge_{i<m} \bigvee_{j<n} x_j \subset y_i$$

then $\mathfrak{M}$ also models the sentence

$$\bigvee \{y_i \cup y_j \neq X : |i-j| \geq 2\} \vee \bigvee \{y_i \cup y_{i+1} = X : i < m\}.$$

Suppose now that wL is chainable, then there exists a finite set of closed sets $\{y_i : i < m\}$ for some $m < \omega$ such that its complements form a chain cover of wL that refine the cover $\{wL \backslash x_i : i < n\}$. We will show that this cannot happen by finding, in $\mathfrak{M}$ a subset of elements of 2^X that correspond to a chain refinement of $\{X \setminus x_i; i < n\}$. This will be a contradiction with the fact that $\mathfrak{M}$ is an elementary submodel of $H(\theta)$, hence a proof that wL is non-chainable.

Without loss of generality we can assume that $m > 2$.

As $y_{i-1} \cup y_i \neq wL$ and $y_i \subset wL \setminus \bigcap_{j \neq i} y_j$ and by the compactness of wL we can pick for any $i < m$ a nonempty F_i in L such that

$$y_i \subset F_i \subset wL \setminus (\bigcup_{j<i} F_j \cup \bigcup_{i<j} y_j).$$

Consider the family $\mathcal{F} = \{F_i : i < m\}$ of elements of L. The F_i's are chosen in such a way that their intersection is empty, hence they correspond to a finite open cover of wL. Also for every $i < m$ we have $y_i \subset F_i$, so $\mathcal{F}$ corresponds to a refinement of $\{wL \backslash y_i : i < m\}$ and this also implies that

$$F_i \cup F_j = wL \text{ whenever } |i-j| \geq 2.$$

Finally, because wL is a continuum and we have chosen the F_i's in such a way that $F_i \cap F_{i+1} = \emptyset$ we also have that

$$F_i \cup F_{i+1} \neq wL \text{ for every } i < m.$$

This shows that $\mathfrak{M}$ will model that $\mathcal{F}$ corresponds to a finite chain refinement of $\{X \setminus x_i : i < n\}$.

7.4. Reflecting span nonzero. Suppose that X is a continuum which has span nonzero. This means that there is some subcontinuum of the square $X \times X$ that projects onto the same subset on both coordinates and whose intersection with the diagonal Δ_X is empty. Let $\mathfrak{M}$ be an elementary submodel of $H(\theta)$ that contains 2^X as one of its elements.

LEMMA 3.54. *The continuum $w(2^X \cap \mathfrak{M})$ has span nonzero.*

PROOF. There exists an element Z in the lattice of closed sets of $X \times X$ such that it is connected and it projects onto both coordinates onto the same set, this implies that for all closed subsets F of X we have

$$Z \subset X \times F \text{ if and only if } Z \subset F \times X.$$

By compactness there exist for some $n < \omega$ closed sets $F_1, \dots, F_n$ of X such that

$$Z \subset \bigcap_{i=1}^{n} (F_i \times X \cup X \times F_i) \text{ and } \bigcap_{i=1}^{n} F_i = \emptyset.$$

By elementarity we can find n of such elements of $2^X \cap \mathfrak{M}$, which shows that $w(2^X \cap \mathfrak{M})$ has span nonzero. □

8. Final remarks

The big question is, if span zero is a property that reflects like the properties (non-)chainable and span nonzero. If this is the case then we are not restricted to metric spaces when we want to find a counter example to Lelek's conjecture on the equivalence of span zero and chainability for metric continua. Given a non-metric counterexample to Lelek's conjecture, with the method described above just take a countable elementary submodel $\mathfrak{M}$ of some $H(\theta)$ such that $2^X \in \mathfrak{M}$ and the continuum $w(2^X \cap \mathfrak{M})$ is a metric counter example to Lelek's conjecture.

QUESTION 3.55. *Are span zero, surjective (semi) span zero and symmetric span zero properties that reflect using the method described in the previous section?*

In section 3 we found a countably many sentences that together describe span of lattices L that model the sentence ϕ_{loc}, the lattices whose Wallman representation is a locally connected continuum. We have thus a partial answer to question 3.55, a positive answer for locally connected continua.

Given a positive answer to one of these questions, we are, as mention earlier no longer restricted to the metric continua if we want to find a counterexample to Lelek's conjecture. We saw that the spaces $\mathbb{I}_u$ and $\mathbb{H}^*$ are continua that are non-chainable and have (different kinds of) span nonzero. The following question arises.

QUESTION 3.56. *Is there a continuum that is non-chainable and has span zero or surjective (semi) span zero or symmetric span zero?*

Or, since we know that the answer to question 3.55 is positive for locally connected continua, a positive answer on the following question would give us immediately a counter example to Lelek's conjecture.

QUESTION 3.57. *Does there exist a locally connected non-chainable continuum which has span zero?*

CHAPTER 4

On a theorem of Maćkowiak and Tymchatyn

1. Introduction and Preliminaries

We call a continuous mapping between two continua *weakly confluent* if every subcontinuum in the range is the image of a subcontinuum in the domain. The following theorem of Maćkowiak and Tymchatyn is the topic of this section

THEOREM 4.1 ([**43**]). *Every metric continuum is a weakly confluent image of some one-dimensional hereditarily indecomposable metric continuum.*

In [**25**] this result was extended to general continua.

THEOREM 4.2 ([**25**]). *Every continuum is a weakly confluent image of some one-dimensional hereditarily indecomposable continuum of the same weight.*

The authors of [**25**] gave two proofs for this theorem, one topological and one model-theoretic. Both proofs made essential use of the metric result.

Our initial interest was to construct a model-theoretic proof of theorem 4.1. After we found this proof we realized that it could be combined with any standard proof of the completeness theorem of first-order logic to produce an inverse-limit proof of the general form of the Maćkowiak-Tymchatyn result. We present both proofs. The model-theoretic argument occupies section 4, and the inverse-limit approach appears in section 3.

We want to take this opportunity to point out some connections with work of Bankston ([4], [5]) who dualized the model-theoretic notions of existentially closed structures and existential maps to that of co-existentially closed compacta and co-existential maps. He proves that co-existentially closed continua are one-dimensional and hereditarily indecomposable, that co-existential maps are weakly confluent and that every continuum is the continuous image of a co-existentially closed one. The map can in general not be chosen co-existential, because co-existential maps preserve indecomposability and do not raise dimension.

Remember that we can extend the notion of hereditary indecomposability to arbitrary compact spaces (see section 4.3 for details). Then a compact Hausdorff space is hereditarily indecomposable if for every two subcontinua that meet, one is contained in the other. In this section we will use the characterization of Krasinkiewicz and Minc mentioned in section 4.3 to show that a compact space is hereditary indecomposable when a base for its closed subsets fulfills all requirements of theorem 2.22.

In this section we will also make use of the characterization of the covering dimension dim for normal spaces mentioned in section 4.4 theorem 2.24. If a base for the closed sets of some compact space fulfills all requirements of this theorem, we know that the covering dimension of this space is less than or equal to n.

This section is put together in such a way that readers who are only interested in the topological (model-theoretic) proof can simply ignore section 4 (section 3, respectively) without loss of continuity.

2. Two main lemmas

The two lemmas in this section stand at the basis of the topological as well as the model-theoretic proof in section 3 and section 4 respectively.

LEMMA 4.3. *If X is a continuum and a, b and c are nonempty closed subsets of X with empty intersection then there exist a continuum Y and a monotone closed onto map $\phi : Y \to X$ such that $w(X) = w(Y)$ and Y has a closed cover $\{A, B, C\}$ with the property that $\phi^{-1}[a] \subset A$, $\phi^{-1}[b] \subset B$, $\phi^{-1}[c] \subset C$ and $A \cap B \cap C = \emptyset$.*

PROOF. We apply normality to find a partition of unity $\{\kappa_a, \kappa_b, \kappa_c\}$ subordinate to $\{X \setminus a, X \setminus b, X \setminus c\}$, i.e. the support of κ_a is a subset of $X \setminus a$, etc. Define the function $f : X \to \mathbb{R}^3$ by $f(x) = (\kappa_a(x), \kappa_b(x), \kappa_c(x))$. The function f maps the space X into the triangle $T = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0 \text{ and } t_1 + t_2 + t_3 = 1\}$. The resulting embedding of X into $X \times T$ defined by $x \mapsto (x, f(x))$, will be denoted by g.

Now consider the space $\partial T \times [0, 1]$, where $\partial T = T \setminus \text{int}(T)$ in $\mathbb{R}^3$. Let h be the map from $\partial T \times [0, 1]$ onto T defined by

$$h((x, t)) = x(1 - t) + t(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

The map h restricted to $\partial T \times [0, 1)$ is a homeomorphism between $\partial T \times [0, 1)$ and $T \setminus \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

We define $Y \subset X \times (\partial T \times [0, 1])$ by $Y = (\text{id} \times h)^{-1}[g[X]]$. And let $\phi : Y \to X$ be the (onto) map $g^{-1} \circ (\text{id} \times h)$. As the inverse images of points $(x, (t_1, t_2, t_3))$ under the map $\text{id} \times h$ are points for $(x, (t_1, t_2, t_3))$ in $X \times T$ with $(t_1, t_2, t_3) \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and equal to $\{x\} \times \partial T \times \{1\}$ for those $(x, (t_1, t_2, t_3))$ in $X \times T$ with $(t_1, t_2, t_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we find that the map $\text{id} \times h : X \times (\partial T \times [0, 1]) \to X \times T$ is monotone. Furthermore it is also closed.

Let p be the line segment between $(0,1,0)$ and $(0,0,1)$, q the line segment between $(1,0,0)$ and $(0,0,1)$ and r the line segment between $(1,0,0)$ and $(0,1,0)$. The sets $A = Y \cap (X \times (p \times [0,1]))$, $B = Y \cap (X \times (q \times [0,1]))$ and $C = Y \cap (X \times (r \times [0,1]))$ form a closed closed cover of Y such that $\phi^{-1}[a] \subset A$, $\phi^{-1}[b] \subset B$, $\phi^{-1}[c] \subset C$, and $A \cap B \cap C = \emptyset$. As it is easily seen that Y and X have the same weight, we have proven the lemma. □

LEMMA 4.4. *If X is a continuum and a, b, c and d are nonempty closed subsets of X such that $a \cap b = a \cap d = b \cap c = \emptyset$ then there exist a continuum Y and a weakly confluent onto map $\psi : Y \to X$ such that $w(X) = w(Y)$ and Y has a closed cover $\{U, V, W\}$ with the property that $\psi^{-1}(a) \subset U$, $\psi^{-1}(b) \subset W$ and $U \cap V \cap \psi^{-1}(c) = U \cap W = V \cap W \cap \psi^{-1}(d) = \emptyset$.*

PROOF. We are going to use an idea from [**25**]. Let a, b, c and d be nonempty closed subsets of X with the property stated in the lemma. With the aid of Urysohn's lemma we can find a continuous function $f : X \to [0,1]$ such that $f[a] \subset \{0\}$, $f[b] \subset \{1\}$, $f[c] \subset [0, \frac{1}{2}]$ and $f[d] \subset [\frac{1}{2}, 1]$.

Let P denote the (closed and connected) subset of $[0,1] \times [0,1]$ given by

$$P = \{\frac{1}{4}\} \times [0, \frac{2}{3}] \cup [\frac{1}{4}, \frac{1}{2}] \times \{\frac{2}{3}\} \cup \{\frac{1}{2}\} \times [\frac{1}{3}, \frac{2}{3}] \cup [\frac{1}{2}, \frac{3}{4}] \times \{\frac{1}{3}\} \cup \{\frac{3}{4}\} \times [\frac{1}{3}, 1].$$

Let $Z \subset [0,1] \times X$ denote the pre-image of the set P under the function $\mathrm{id} \times f$:

$$Z = \{(t,x) \in [0,1] \times X : (t, f(x)) \in P\}.$$

As P is closed and $\mathrm{id} \times f$ is continuous the set Z is compact. Define the (continuous) map $\pi : Z \to X$ by $\pi((t,x)) = x$ for every $(t,x) \in Z$.

Let $\mathcal{F}$ be the set of all clopen subsets of Z that are mapped onto X by π.

CLAIM 4.5. *The set $\mathcal{F}$ is a nonempty ultrafilter in the family of clopen subsets of Z.*

PROOF. Suppose we have closed sets F and G such that $Z = F + G$. Define closed subsets A_i, B_i of X, where $i \in \{0,1,2\}$, by

$$\begin{aligned} A_0 &= \{x \in X : (\tfrac{1}{4}, x) \in F\},\ B_0 = \{x \in X : (\tfrac{1}{4}, x) \in G\} \\ A_1 &= \{x \in X : (\tfrac{1}{2}, x) \in F\},\ B_1 = \{x \in X : (\tfrac{1}{2}, x) \in G\} \\ A_2 &= \{x \in X : (\tfrac{3}{4}, x) \in F\},\ B_2 = \{x \in X : (\tfrac{3}{4}, x) \in G\} \end{aligned}$$

It is clear that $A_i \cap B_i = \emptyset$ for every $i \in \{0,1,2\}$.

If $x \in (A_0 \cap B_1) \cup (B_0 \cap A_1)$ then $f(x) < \frac{2}{3}$ as $f(x) = \frac{2}{3}$ is clearly impossible. Similarly we see that $f[(A_1 \cap B_2) \cup (B_1 \cap A_2)] \subset (\frac{1}{3}, 1]$.

Let A^* and B^* be closed sets of X, such that A^* is equal to the following union of closed sets

$$\bigcup\{f^{-1}[0,\frac{1}{3}]\cap A_0, f^{-1}[\frac{2}{3},1]\cap A_2, A_0\cap A_1\cap A_2, A_0\cap B_1\cap B_2, \\ B_0\cap B_1\cap A_2, B_0\cap A_1\cap B_2\},$$

and we get a description of the closed set B^* by interchanging A's and B's in the above equation. The sets A^* and B^* are disjoint closed subsets of X and their union is the whole of X. As X is connected one of these sets must be empty. So without loss of generality we can assume that $B^* = \emptyset$. We see that $\pi[F] = X$ and furthermore, that π maps G, the complement of F into the set $f^{-1}[\frac{1}{3},\frac{2}{3}]$, a proper subset of X.

This argument shows that if $F_1, F_2 \in \mathcal{F}$ then $\pi[X \setminus (F_1 \cap F_2)] \subset f^{-1}[\frac{1}{3},\frac{2}{3}]$, whence $\mathcal{F}$ is seen to be a filter; it also shows that $\mathcal{F}$ is an ultrafilter. □

Let $Y \subset Z$ be given by $Y = \bigcap\mathcal{F}$ and let $\psi : Y \to X$ be the restriction of π to the continuum Y, $\psi = \pi \restriction Y$.

CLAIM 4.6. *$\psi : Y \to X$ is weakly confluent.*

PROOF. Suppose we have $A \subset X$ connected. If we look at the image of A under the function f there are a number of possibilities:

(1) $f[A] \subset [0,\frac{2}{3}]$ and $f[A]\cap[0,\frac{1}{3}) \neq \emptyset$. As $\pi[Z\setminus Y] \subset f^{-1}[\frac{1}{3},\frac{2}{3}]$ we know that $\{\frac{1}{4}\}\times A$ must intersect Y. As $\{\frac{1}{4}\}\times A$ is connected we even have that $\{\frac{1}{4}\}\times A$ is a subset of Y.

(2) $f[A] \subset [\frac{1}{3},\frac{2}{3}]$. The component Y of Z must intersect at least one of the connected subsets $\{\frac{1}{4}\}\times A$, $\{\frac{1}{2}\}\times A$ or $\{\frac{3}{4}\}\times A$ of Z, because Y is mapped onto X. And so Y must contain at least one of these connected sets.

(3) $f[A]\cap[0,\frac{1}{3}) \neq \emptyset \neq f[A]\cap(\frac{2}{3},1]$. As above, assuming that $A^+(=\pi^{-1}[A]) = F + G$, we can construct closed and disjoint subsets A^* and B^* of A which cover it. Again the image under ψ is either all of A or a proper subset of A. The (unique) component of A^+ that maps onto the whole of A must intersect the set Y, and so is contained in it.

This ends the proof of the claim. □

If we let U be the set $\{(t,x) \in Y : t \in [0,\frac{3}{8}]\}$, $V = \{(t,x) \in Y : t \in [\frac{3}{8},\frac{5}{8}]\}$ and $W = \{(t,x) \in Y : t \in [\frac{5}{8},1]\}$, then $\{U,V,W\}$ is a closed cover of the space Y such that $\psi^{-1}[a] \subset U$, $\psi^{-1}[b] \subset W$, $U\cap V\cap\psi^{-1}[c] = \emptyset$, $V\cap W\cap\psi^{-1}[d] = \emptyset$ and $U\cap W = \emptyset$. This ends the proof of the lemma as it is easily seen that X and Y are both of the same weight. □

3. A topological proof of the theorem

Before we start with the proof of the theorem we restate the following well - known lemma on a base for the closed sets of some (transfinite) inverse sequence.

Let $\{X_\alpha; f_\alpha, \kappa\}$ be an (transfinite) inverse sequence with X_κ as its inverse limit space. Let for every $\alpha < \kappa$ the continuous function π_α be defined by $\pi_\alpha = \text{proj}_\alpha \restriction X_\kappa : X_\kappa \to X_\alpha$, where $\text{proj}_\alpha : \Pi_{\alpha<\kappa} X_\alpha \to X_\alpha$ is the projection. The following lemma is well known.

LEMMA 4.7. *The family of all sets of the form $\pi_\alpha^{-1}[F]$, where F is a closed subset of the space X_α and α runs over a subset C cofinal in κ, is a base for the closed sets of X_κ. Moreover, if for every $\alpha < \kappa$ a base $\mathcal{B}_\alpha$ for the closed sets of space X_α is fixed, then the subfamily of those $\pi_\alpha^{-1}[F]$ for which $F \in \mathcal{B}_\alpha$, also is a base for the closed sets of X_κ.*

Let X be a metric continuum. We are going to define a inverse sequence $\{X_n, f_n, \omega\}$,

$$X = X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} X_n \xleftarrow{f_n} \cdots,$$

in such a way that the inverse limit space X_ω is a hereditarily indecomposable one-dimensional continuum of countable weight such that $\pi_0 : X_\omega \to X$ is a weakly confluent map onto X.

For every n we will define a metric continuum X_n, an onto map $f_n : X_n \to X_{n-1}$ and a countable base $\mathcal{B}_n$ for the closed sets of X_n that is closed under finite unions and intersections . Lemma 4.7 tells us that $\mathcal{B} = \bigcup_{n<\omega} \pi_n^{-1}[\mathcal{B}_n]$ will be a countable base for the closed sets of X_ω. If we choose the bases $\mathcal{B}_n$ in such a way that $f_n^{-1}[\mathcal{B}_{n-1}] \subset \mathcal{B}_n$, then we even have that $\mathcal{B}$ is closed under finite unions and intersections.

By theorem 2.22 and corollary 2.27 we know that X_ω is a hereditarily indecomposable continuum, that is one-dimensional and of countable weight if we can make sure that the base $\mathcal{B}$ has the following two properties

(1) For every $a, b, c \in \mathcal{B}$ with empty intersection there are $A, B, C \in \mathcal{B}$ such that $a \subset A$, $b \subset B$ $c \subset C$, $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = X_\omega$.
(2) For every $a, b, c, d \in \mathcal{B}$ such that $a \cap b = a \cap d = b \cap c = \emptyset$ there are $U, V, W \in \mathcal{B}$ such that $a \subset U$, $b \subset W$, $U \cap V \cap c = \emptyset$, $V \cap W \cap d = \emptyset$, $U \cap W = \emptyset$ and $X_\omega = U \cup V \cup W$.

To consider all the triples and quadruples of $\mathcal{B}$ it is more than enough, by the definition of the bases $\mathcal{B}_n$, to consider all the triples and quadruples of every $\mathcal{B}_n$. As there are countably many of those we can find an enumeration σ of these triples and quadruples of length ω in such a way that the n-th element $\sigma(n)$ of this enumeration will be some triple or quadruple of some base $\mathcal{B}_m$ with $m \leq n$.

Furthermore, if all the bonding maps f_n are weakly confluent then the map π_0 will be weakly confluent. This is easily seen: given some subcontinuum A of X we can define an inverse sequence $\{A_n, g_n, \omega\}$, where $A_0 = A$ and, for all n, A_{n+1} is some subcontinuum of X_n such that $g_n[A_{n+1}] = A_n$. Where g_n is the restriction of f_n to the set A_{n+1}. The inverse limit of this sequence is a subcontinuum of X_ω which is mapped onto A by the map π_0.

We will use lemma 4.3 and lemma 4.4 in the construction of the inverse sequence $\{X_n, f_n, \omega\}$. Suppose we have defined all X_m, f_m and $\mathcal{B}_m$ for $m \leq n$. If $\sigma(n)$ is some triple of $\mathcal{B}_m$ then we look at $\{a, b, c\}$, their pre-image under the map f_m^n in X_n. We use lemma 4.3 to find X_{n+1} and f_{n+1}, and we choose a countable base $\mathcal{B}_{n+1}$ for the closed sets of X_n such that it contains $\{A, B, C\}$ and $f_{n+1}^{-1}[\mathcal{B}_n]$, where $\{A, B, C\}$ is the closed cover of X_{n+1} we get from lemma 4.3. When $\sigma(n+1)$ was a quadruple of $\mathcal{B}_m$ then we do something similar as above but this time we use lemma 4.4.

In a similar way we can construct for any continuum X, using lemmas 4.3 and 4.4 some (transfinite) inverse sequence $\{X_\alpha, f_\alpha, w(X)\}$ such that $X_0 = X$ and the inverse limit of this sequence will be a one-dimensional hereditarily indecomposable continuum of weight $w(X)$ that is mapped onto X by the weakly confluent map π_0. This provides an independent proof of a theorem in **[25]** which states just this. The proof in that paper made essential use of the metric case.

4. A model-theoretic proof of the theorem

For the remainder of this section we fix some metric continuum X. In this section we will prove the Maćkowiak-Tymchatyn theorem for this X in two steps. First we show that X is a continuous image of some metric one-dimensional hereditarily indecomposable continuum, and then we show that the map can even be weakly confluent.

We will construct a lattice L such that some lattice base of X is embedded into L, the Wallman representation wL of L is a one-dimensional hereditarily indecomposable continuum and that for every subcontinuum in X there exists a subcontinuum of wL that is mapped onto it.

For this we need to translate things like being hereditarily indecomposable, being of dimension less than or equal to one and being connected in terms of closed sets only.

Using the characterization of hereditary indecomposability as stated in 2.22, we see that a compact Hausdorff space Y is hereditarily indecomposable if the lattice 2^Y models the sentence

$$\begin{aligned} &\forall abcd \exists xyz\, [a \sqcap b = \mathbf{0} \wedge a \sqcap d = \mathbf{0} \wedge b \sqcap c = \mathbf{0} \rightarrow \\ &\quad a \sqcap (y \sqcup z) = \mathbf{0} \wedge b \sqcap (x \sqcup y) = \mathbf{0} \wedge x \sqcap z = \mathbf{0} \wedge \\ &\quad x \sqcap y \sqcap c = \mathbf{0} \wedge y \sqcap z \sqcap d = \mathbf{0} \wedge x \sqcup y \sqcup z = \mathbf{1}]. \end{aligned} \tag{22}$$

Using corollary 2.27, we see that a space Y is of dimension less than or equal to one if the lattice 2^Y models the sentence

$$\begin{aligned} &\forall abc \exists xyz \, [a \sqcap b \sqcap c = \mathbf{0} \rightarrow \\ &\quad a \sqcap x = a \wedge b \sqcap y = b \wedge c \sqcap z = c \wedge \\ &\quad x \sqcap y \sqcap z = \mathbf{0} \wedge x \sqcup y \sqcup z = \mathbf{1}]. \end{aligned} \tag{23}$$

A space Y is connected if the lattice 2^Y models the sentence conn($\mathbf{1}$), where conn(a) is shorthand for the formula $\forall x\, y[((x \sqcap y = \mathbf{0}) \wedge (x \sqcup y = a)) \rightarrow (x = a) \vee (x = \mathbf{0}))]$.

4.1. The space X is a continuous image of some one-dimensional hereditarily indecomposable metric continuum. Using the theorems 2.2 and 2.12, we see that in order to get a hereditarily indecomposable continuum of dimension one and countable weight that maps onto X we must find a countable distributive, disjunctive normal lattice L such that it is a model of the sentences 22, 23 and conn($\mathbf{1}$), and furthermore that some lattice base of X is embedded into this lattice L.

Fix a lattice base $\mathcal{B}$ for X.

For some countable set of constants K we will construct a set of sentences Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. We will make sure that Σ is a consistent set of sentences such that, if we have a model $\mathfrak{A} = (A, \mathcal{I})$ for Σ then

$$L(\mathfrak{A}) = \mathcal{I} \restriction K$$

is the universe of some lattice model in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$ which is normal, distributive and disjunctive and models the sentences 22, 23 and conn($\mathbf{1}$). To make sure that $\mathcal{B}$ is embedded into $L(\mathfrak{A})$ we simply add the diagram of the lattice $\mathcal{B}$ to the set Σ and make sure that there are constants in K representing the elements of $\mathcal{B}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ are given by there interpretations under $\mathcal{I}$ in the model $\mathfrak{A}$.

Let K be the following countable set of constants

$$K = \bigcup_{-1 \leq n < \omega} K_n = \bigcup_{-1 \leq n < \omega} \{k_{n,m} : m < \omega\}.$$

We define sets Σ_n of sentences by an ω-recursion and set $\Sigma = \bigcup_{n<\omega} \Sigma_n$.

To begin we define $K_{-1} = \mathcal{B}$ and $\Sigma_0 = \triangle_{\mathcal{B}}$, the diagram of $\mathcal{B}$.

The sets Σ_n will have the following properties:

(1) The Σ_{5n+1}'s will be sets of sentences that will make sure that the $L(\mathfrak{A})$ is a distributive lattice and that the Wallman space $wL(\mathfrak{A})$ of the lattice $L(\mathfrak{A})$ is connected.
(2) The Σ_{5n+2}'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ is normal.
(3) The Σ_{5n+3}'s will be sets of sentences that will make sure that $L(\mathfrak{A})$ is a disjunctive lattice.
(4) The Σ_{5n+4}'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ will be a model of the sentence 23.

(5) And the $\Sigma_{5(n+1)}$'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ will be a model of the sentence 22.

4.1.1. *Construction of* Σ *in* $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. We now show how to define the sets of sentences of $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \bigcup_{m<5n+4} K_m$ as described in 1 through 5.

We have a natural order $\lhd$ on the set $K = \bigcup_m K_m$ defined by

$$k_{n,m} \lhd k_{r,t} \leftrightarrow [(n < r) \vee ((n = r) \wedge (m < t))].$$

Let $\{p_l\}_{l<\omega}$ be an enumeration of

$$\{p \in [\bigcup_{m \leq 5n} K_m]^2 : p \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}.$$

For every $l < \omega$ write $p_l = \{p_l(0), p_l(1)\}$.

$$\Sigma^0_{5n+1} = \{p_l(0) \sqcap p_l(1) = k_{5n+1,2l} : l < \omega\}$$
$$\Sigma^1_{5n+1} = \{p_l(0) \sqcup p_l(1) = k_{5n+1,2l+1} : l < \omega\}.$$

Furthermore we let Σ^2_{5n+1} be a set of sentences in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \bigcup_{m \leq 5n} K_m$ (without quantifiers) consisting of

(1) sentences stating that we are dealing with a distributive lattice with a $\mathbf{0}$ and a $\mathbf{1}$ according to the constants from $\bigcup_{m \leq 5n} K_m$,
(2) sentences that make sure that no pair of constants from will refute conn($\mathbf{1}$).

Define Σ_{5n+1} by

$$\Sigma_{5n+1} = \Sigma^0_{5n+1} \cup \Sigma^1_{5n+1} \cup \Sigma^2_{5n+1}.$$

This set of sentences will make sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be a distributive lattice with a $\mathbf{0}$ and a $\mathbf{1}$, and also a model of the sentence conn($\mathbf{1}$). Let us denote the following set of sentences by Σ_{5n+2}.

$$\{[(p_l(0) \sqcap p_l(1) = \mathbf{0}) \rightarrow ((p_l(1) \sqcap k_{5n+2,2l} = \mathbf{0}) \wedge (p_l(0) \sqcap k_{5n+2,2l+1} = \mathbf{0}) \wedge (k_{5n+2,2l} \sqcup k_{5n+2,2l+1} = \mathbf{1}))] : l < \omega\}.$$

This set of sentences will make sure that any (lattice) model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be normal.

The following set of sentences makes sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ which is also a lattice is a disjunctive lattice.

$$\Sigma^0_{5n+3} = \{[p_l(0) \sqcap p_l(1) \neq p_l(0) \rightarrow k_{5n+3,2l+1} \sqcap p_l(0) = k_{5n+3,2l+1} \wedge k_{5n+3,2l+1} \sqcap p_l(1) = \mathbf{0})] : l < \omega\}$$
$$\Sigma^1_{5n+3} = \{[p_l(1) \sqcap p_l(0) \neq p_l(1) \rightarrow k_{5n+3,2l} \sqcap p_l(1) = k_{5n+3,2l} \wedge k_{5n+3,2l} \sqcap p_l(0) = \mathbf{0})] : l < \omega\}.$$

And define Σ_{5n+3} by

$$\Sigma_{5n+3} = \Sigma^0_{5n+3} \cup \Sigma^1_{5n+3}.$$

Let ζ denote the following lattice formula

$$\zeta(a,b,c;x,y,z) = [a \sqcap b \sqcap c = \mathbf{0} \rightarrow a \sqcap x = a \wedge b \sqcap y = b \wedge c \sqcap z = c \wedge x \sqcap y \sqcap z = \mathbf{0} \wedge x \sqcup y \sqcup z = \mathbf{1}].$$

Let $\{q_l\}_{l<\omega}$ be an enumeration of the set

$$\{q \in [\bigcup_{m \leq 5n} K_m]^3 : q \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}.$$

For every $l < \omega$ write $q_l = \{q_l(0), q_l(1), q_l(2)\}$.

Now define Σ_{5n+4} by

$$\Sigma_{5n+4} = \{\zeta(q_l(0), q_l(1), q_l(2); k_{5n+4,3l}, k_{5n+4,3l+1}, k_{5n+4,3l+2}) : l < \omega\}.$$

This will make sure that the Wallman space of any lattice model of Σ will be at most one-dimensional.

To make sure that the Wallman space of any model of Σ will be hereditarily indecomposable we introduce the following formulas in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$:

$$\begin{aligned} \phi(a,b,c,d) &= [a \sqcap b = \mathbf{0} \wedge a \sqcap d = \mathbf{0} \wedge b \sqcap c = \mathbf{0}] \\ \psi(a,b,c,d;x,y,z) &= [x \sqcup y \sqcup z = \mathbf{1} \wedge x \sqcap z = \mathbf{0} \wedge \\ &\quad a \sqcap (y \sqcup z) = \mathbf{0} \wedge b \sqcap (x \sqcup y) = \mathbf{0} \wedge \\ &\quad x \sqcap y \sqcap c = \mathbf{0} \wedge y \sqcap z \sqcap d = \mathbf{0}] \\ \text{(24)} \quad \theta(a,b,c,d;x,y,z) &= \phi(a,b,c,d) \rightarrow \psi(a,b,c,d;x,y,z) \end{aligned}$$

Let $\{r_l\}_{l<\omega}$ be an enumeration of the set

$$\{r \in {}^4[\bigcup_{m \leq 5n} K_m] : \mathrm{ran}(r) \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}.$$

Let $\Sigma_{5(n+1)}$ be the following set of sentences :

$$\{\theta(r_l(0), r_l(1), r_l(2), r_l(3); k_{5(n+1),3l}, k_{5(n+1),3l+1}, k_{5(n+1),3l+2}) : l < \omega\}.$$

Here the formula θ is as in equation 24.

4.1.2. *Consistency of* Σ *in* $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. In this section we show that Σ is a consistent set of sentences by finding, for every finite subset Γ of Σ a metric space Y and an interpretation function $\mathcal{I} : K \rightarrow 2^Y$ such that $(2^Y, \mathcal{I})$ is a model for the theory $\Gamma \cup \Delta_{\mathcal{B}}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ will always be $\cap$, $\cup$ (the normal set intersection and union), $\emptyset$ and Y respectively.

For $\Gamma = \emptyset$ we let $Y = X$ and we interpret every constant from K_{-1} as its corresponding base element in $\mathcal{B}$. Extend the interpretation function by assigning the empty set to all constants of $K \setminus K_{-1}$. It is obvious that $(2^Y, \mathcal{I})$ is a model of $\Delta_{\mathcal{B}}$.

REMARK 4.8. As the interpretation of $\sqcap$ and $\sqcup$ in the metric continuum Y will always be the normal set intersection and set union, all the sentences in Σ^2_{5n+1} for some $n < \omega$ are true in the model $(2^Y, \mathcal{I})$.

So we can ignore these sentences and for the remainder of this section concentrate on the remaining sentences of Σ.

We can define a well order $\sqsubset$ on the set $\Sigma \setminus \{\Sigma^2_{5n+1} : n < \omega\}$ by stating that $\phi \sqsubset \psi$ if and only if there are $n < m < \omega$ such that $\phi \in \Sigma_n$ and $\psi \in \Sigma_m$ or there are $k < l < \omega$ and $n < \omega$ such that $\phi, \psi \in \Sigma_n$ and ϕ is a sentence that mentions p_k (q_k or r_k respectively) and ψ is a sentence that mentions p_l (q_l or r_l respectively).

Suppose Γ is a finite subset of Σ such that each of its proper subsets has a model as stated as above. Let θ be the $\sqsubset$-largest sentence in $\Gamma \setminus \{\Sigma^2_{5n+1} : n < \omega\}$ and let Y be a metric continuum and $\mathcal{I} : K \to 2^Y$ be an interpretation function such that $(2^Y, \mathcal{I})$ is a model of the theory $\Gamma \setminus \{\theta\} \cup \triangle_{\mathcal{B}}$.

We will show that there exists a metric space Z and an interpretation function $\mathcal{J} : K \to 2^Z$ such that $(2^Z, \mathcal{J})$ is a model of the theory $\Gamma \cup \triangle_{\mathcal{B}}$. We consider three cases: $\theta \in \Sigma_{5n+1} \cup \Sigma_{5n+2} \cup \Sigma_{5n+3}$, $\theta \in \Sigma_{5n+4}$ and $\theta \in \Sigma_{5(n+1)}$ for some $n < \omega$.

(1) If $\theta \in \Sigma_{5n+1} \cup \Sigma_{5n+2} \cup \Sigma_{5n+3}$, we can simply let $Z = Y$ and either interpret the new constant under $\mathcal{J}$ as the intersection or union of two closed sets in Y if θ is in some Σ_{5n+1} or, if θ is an element of some Σ_{5m+2} or Σ_{5m+3}, using the fact that the space Y is normal find $\mathcal{J}$-interpretations for the newest constants, in an obvious way.

(2) If $\theta \in \Sigma_{5n+4}$, then θ is a sentence of the following form

$$\begin{aligned}\theta = [&a \sqcap b \sqcap c = \mathbf{0} \to a \sqcap x = a \wedge b \sqcap y = b \wedge \\ &c \sqcap z = c \wedge x \sqcap y \sqcap z = \mathbf{0} \wedge x \sqcup y \sqcup z = \mathbf{1}].\end{aligned}$$

Suppose the preamble of θ is true in the model $(2^Y, \mathcal{I})$. If a has a zero interpretation then we can choose $x = \mathbf{0}$, $y = \mathbf{1}$ and $z = \mathbf{1}$, and this interpretation of x, y and z makes sure that θ holds in the model $(2^Y, \mathcal{I})$. So we may assume that a, b and c have non zero interpretations.

By lemma 4.3 there exist a metric continuum Z, a closed, monotone and onto map $f : Z \to Y$ and a closed cover $\{A, B, C\}$ of the space Z, with empty intersection such that $f^{-1}[\mathcal{I}(a)] \subset A$, $f^{-1}[\mathcal{I}(b)] \subset B$ and $f^{-1}[\mathcal{I}(c)] \subset C$.

Define an interpretation function $\mathcal{J} : K \to 2^Z$ by

$$\begin{aligned}&\mathcal{J}(k) = f^{-1}[\mathcal{I}(k)] \text{ for all } k \in K \setminus \{x, y, z\}, \\ &\quad \mathcal{J}(x) = A,\ \mathcal{J}(y) = B \text{ and } \mathcal{J}(z) = C.\end{aligned}$$

With this interpretation function $(2^Z, \mathcal{J})$ is a model for Γ.

(3) If $\theta \in \Sigma_{5(n+1)}$ then it is of the form $\theta(a, b, c, ; x, y, z)$ as in equation 24. Suppose the preamble of θ is true in the model $(2^Y, \mathcal{I})$.

If the interpretation of a is zero we can simply take $x = y = \mathbf{0}$ and $z = \mathbf{1}$ to make $(2^Y, \mathcal{I})$ a model of θ. So we may again assume that the interpretations of a, b, c and d are nonzero.

By lemma 4.4 there exists a metric continuum Z, a weakly confluent onto map $f : Z \to Y$ and a closed cover $\{U, V, W\}$ of Z such that $f^{-1}[\mathcal{I}(a)] \subset V$, $f^{-1}[\mathcal{I}(b)] \subset W$, $U \cap V \cap f^{-1}[\mathcal{I}(c)] = \emptyset$, $U \cap W = \emptyset$, and $V \cap W \cap f^{-1}[\mathcal{I}(d)] = \emptyset$.

Define an interpretation function $\mathcal{J} : K \to 2^Z$ by

$$\mathcal{J}(k) = f^{-1}[\mathcal{I}(k)] \text{ for all } k \in K \setminus \{x, y, z\}$$
$$\mathcal{J}(x) = U,\ \mathcal{J}(y) = V \text{ and } \mathcal{J}(z) = W.$$

The structure $(2^Z, \mathcal{J})$ is a model for Γ.

So the theory Σ is a consistent theory in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$.

4.2. The Maćkowiak-Tymchatyn theorem. Apart from the weakly confluent property of the continuous onto map we have proven the Maćkowiak-Tymchatyn theorem, theorem 4.1.

In this section we will extend the language of the previous section and construct a consistent theory in this extended language that shows that there exists a one-dimensional hereditarily indecomposable continuum Y (of the form $wL(\mathfrak{A})$) that maps onto the continuum X be a weakly confluent map. By this approach the weight of the continuum Y will be greater than the weight of the our space X. We can amend this by taking a countable elementary sublattice of $L(\mathfrak{A})$.

To make sure that the continuous map following from the previous section is weakly confluent, we must consider all the subcontinua of the space X.

We let $\hat{K}$ be the following set

$$\hat{K} = \bigcup_{-2 \leq n < \omega} \hat{K}_n = \bigcup_{-2 \leq n < \omega} \{k_{n,\alpha} : \alpha < |2^X|\}.$$

We will construct some theory $\hat{\Sigma} = \bigcup_{-1 \leq n < \omega} \hat{\Sigma}_n$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ similar as in the previous section such that given any model $\mathfrak{A} = (A, \mathcal{I})$ of $\hat{\Sigma}$, the set $L(\mathfrak{A}) = \mathcal{I} \restriction \hat{K}$ will be the universe of some normal distributive and disjunctive lattice such that it is a model of the sentences 22, 23 and conn($\mathbf{1}$), we can embed the lattice 2^X into $L(\mathfrak{A})$, so there exists a continuous map f from $wL(\mathfrak{A})$ onto X and, for every subcontinuum of X there exists a subcontinuum of $wL(\mathfrak{A})$ that is mapped onto it by f.

4.2.1. *Construction of* $\hat{\Sigma}$ *in* $\{\sqcap, \sup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$. We let $\hat{K}_{-1} = \{k_{-1,\alpha} < |2^X|\}$ correspond to the set $2^X = \{x_\alpha : \alpha < |2^X|\}$ in such a way that the set of all the subcontinua of X corresponds to the set $\{x_{2\alpha} : \alpha < \beta\}$ for some ordinal number $\beta \leq |2^X|$. Let the set of sentences $\hat{\Sigma}_0$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}_{-1}$ correspond to $\triangle_{2^X}$, the diagram of the lattice 2^X.

We want to define a set of sentences $\hat{\Sigma}_{-1}$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}_{-2} \cup \hat{K}_{-1}$ that will make sure that if $\mathfrak{A}$ is a model of $\hat{\Sigma}$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ then we have for every subcontinuum in X a subcontinuum of $wL(\mathfrak{A})$ that will be mapped onto it by the continuous onto map we get by the fact that 2^X is embedded in the lattice $L(\mathfrak{A})$.

$$\begin{aligned}
\hat{\Sigma}^0_{-1} &= \{\mathrm{conn}(k_{-2,\alpha}) \wedge k_{-2,\alpha} \sqcap k_{-1,\alpha} = k_{-2,\alpha} : \alpha < \beta\} \\
\hat{\Sigma}^1_{-1} &= \{\mathrm{conn}(k_{-2,\alpha}) \wedge k_{-2,\alpha} \sqcap k_{-1,\gamma} = k_{-2,\alpha} \rightarrow \\
&\qquad k_{-1,\alpha} \sqcap k_{-1,\gamma} = k_{-1,\alpha} : \alpha < \beta,\ \gamma < |2^X|\} \\
\hat{\Sigma}^2_{-1} &= \{k_{-2,\gamma} = \mathbf{0} : \beta \leq \gamma < |2^X|\}.
\end{aligned}$$

And define the set of sentences $\hat{\Sigma}_{-1}$ as $\hat{\Sigma}_{-1} = \hat{\Sigma}^0_{-1} \cup \hat{\Sigma}^1_{-1} \cup \hat{\Sigma}^2_{-1}$.

Suppose $\mathfrak{A}$ is a model of $\hat{\Sigma}$. The set $\hat{\Sigma}^0_{-1}$ will make sure that for every subcontinuum C of X there is some subcontinuum C' of $wL(\mathfrak{A})$ that is mapped into C by the continuous onto map f we get from theorem 2.12 and the fact that 2^X is embedded into $wL(\mathfrak{A})$. The set $\hat{\Sigma}^1_{-1}$ will then make sure that C' is in fact mapped onto C by the map f.

Let us further construct the sets $\hat{\Sigma}_n$ for $0 < n < \omega$ in the same manner as we have constructed the set Σ_n in the previous section. So that if we have a model $\mathfrak{A}$ of $\hat{\Sigma}$, the lattice $L(\mathfrak{A})$ will be a normal distributive and disjunctive lattice that models the sentences **22**, **23** and $\mathrm{conn}(\mathbf{1})$.

4.2.2. *Consistency of* $\hat{\Sigma}$ *in* $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$. Suppose we go about as in section 4.1.2 and try to prove by that given a model $(2^Y, \mathcal{I})$ for the theory Γ and γ a sentence of $\hat{\Sigma}$ constructed after the sentences from Γ, that there exist a model $(2^Z, \mathcal{J})$ for the theory $\Gamma \cup \{\gamma\}$, either by using lemma 4.3 or 4.4 or the fact that Y and Z are metric continua. A problem may arise if we use lemma 4.4 to find the space Z, as in this case $f : Z \rightarrow Y$ is only weakly confluent so we cannot just take the f-inverse image of the $\mathcal{I}$-interpretation of constants from $\hat{K}_{-2}$ as their $\mathcal{J}$-interpretations, as these might not be connected. We can however always find a connected subset that maps onto the $\mathcal{I}$-interpretation under the map f. These $\mathcal{J}$-interpretations of the c_i's (might) affect all other $\mathcal{J}$-interpretations, and it could happen that some sentence in Γ true in the model $(2^Y, \mathcal{I})$, because its premise was false, has now a true premise in $(2^Z, \mathcal{J})$ and we have to find $\mathcal{J}$-interpretations for the constants introduced by this sentence to make it a true sentence in $(2^Z, \mathcal{J})$. This again could affect the interpretations and the truth value of other sentences in Γ, and so on.

To bypass this problem we will consider every finite set Γ of $\hat{\Sigma}$ separately, and find a model for it.

We fix such a finite set Γ from now on.

Note that there can only be mention of finitely many constants $\{c_1, \ldots, c_k\}$ from the set $\hat{K}_{-2}$. We start by construction a model

$(2^{X^+}, \mathcal{I}^+)$ from X which is not only a model of Δ_{2^X} and all the sentences from $\Gamma \cap \hat{\Sigma}_{-1}$ but also models $c_i \sqcap c_j = \mathbf{0}$ for all $i \neq j$. Denote $\Gamma \setminus \hat{\Sigma}_{-1}$ by $\{\gamma_1, \ldots, \gamma_m\}$ in such a way that the γ in $\Gamma \setminus \hat{\Sigma}_{-1}$ which are in $\hat{\Sigma}_m$ have lower index than those in $\hat{\Sigma}_n$, when $m < n$. We will construct models $(2^{Y_i}, \mathcal{I}_i)$ such that

$$(2^{Y_i}, \mathcal{I}_i) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \Delta_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\} \cup \{\gamma_1, \ldots, \gamma_i\}.$$

All these metric continua are related in the following way

$$X^+ \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xleftarrow{g_3} \cdots \xleftarrow{g_i} Y_i,$$

where the g_i's are either the identity map or come from lemma 4.3 or 4.4.

4.2.3. *Construction of* $(2^{X^+}, \mathcal{I}^+)$. Note that the constants from $\hat{K}_{-2} \cup \hat{K}_{-1}$ correspond with closed sets from the metric continuum X. So c_i corresponds with some subcontinuum C_i of X, and $a \in \hat{K}_{-1}$ corresponds to some closed set A of X.

Let X^+ be the space $X \times [0,1]$ and define the interpretation map $\mathcal{I}^+ : \hat{K} \to 2^{X^+}$ by

$$\mathcal{I}^+(a) = \pi_X^{-1}[A] \text{ for all } a \in \hat{K}_{-1},$$

$$\mathcal{I}^+(c_i) = C_i \times \{\frac{i}{k}\} \text{ for all } i \text{ and}$$

$$\mathcal{I}^+(a) = \emptyset \text{ for all } a \in \hat{K} \setminus (\hat{K}_{-1} \cup \{c_1, \ldots, c_k\}).$$

By construction, we have

$$(2^{X^+}, \mathcal{I}^+) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \Delta_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\}.$$

Suppose now that we have already taken care of the sentences $\{\gamma_1, \ldots, \gamma_{i-1}\}$ of Γ. We will show how to find a model $(2^{Y_i}, \mathcal{I}_i)$ for some γ_i. Let $Y = Y_{i-1}$ and $\mathcal{I} = \mathcal{I}_{i-1}$. We have

$$(2^Y, \mathcal{I}) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \Delta_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\} \cup \{\gamma_j : j < i\}.$$

4.2.4. *The sentence γ_i is in $\hat{\Sigma}_{5n+j}$ for some $j = 1, 2, 3, 4$.* If γ_i is some sentence in one of the sets $\hat{\Sigma}_{5n+1}$, $\hat{\Sigma}_{5n+2}$ or $\hat{\Sigma}_{5n+3}$ for some n then we let $Z = Y$, let the $\mathcal{J}$-interpretation of all constants of $\hat{K}_{-1}$ and those that are mentioned in some γ_j with $j < i$ equal their $\mathcal{I}$-interpretation. We use normal set intersection or union to find interpretations for the constants introduced by γ_i if it is an element of some $\hat{\Sigma}_{5n+1}$ if necessary, and normality of the space Y to find interpretations for the constants introduced by γ_i if it is a sentence in $\hat{\Sigma}_{5n+2}$ or $\hat{\Sigma}_{5n+3}$ for some n.

If γ_i is a sentence in one of the sets $\hat{\Sigma}_{5n+4}$ and its premise is true in the model $(2^Y, \mathcal{I})$ and there is no triple in 2^Y that can make the sentence a true sentence in 2^Y, then we use lemma 4.3 to find a continuum Z and a closed monotone map $f : Z \to Y$ such that if we let the $\mathcal{J}$-interpretation of $a \in \hat{K}$, a not equal to one of the constants introduced by γ_i be defined by

$$\mathcal{J}(a) = f^{-1}[\mathcal{I}(a)],$$

then with the interpretation of the constants introduced by γ_i by the closed sets of Z we get from the lemma, we made, with this interpretation $\mathcal{J}$ the sentence γ_i a true sentence in $(2^Z, \mathcal{J})$. None of the other sentences if affected by this construction as we take pre-images of their $\mathcal{I}$-interpretations as their $\mathcal{J}$-interpretations, and by the fact that f is closed and monotone.

4.2.5. *The sentence γ_i is in $\hat{\Sigma}_{5(n+1)}$.* Without loss of generality we can assume that the premise of $\gamma_i = \gamma_i(a, b, c; x, y, z)$, but not its conclusion is true in the model $(2^Y, \mathcal{I})$. With the aid of lemma 4.4 we find a metric continuum Z and closed sets U, V and W of Z such that

$$\gamma_i(f^{-1}[\mathcal{I}(a)], f^{-1}[\mathcal{I}(b)], f^{-1}[\mathcal{I}(c)]; U, V, W)$$

is a true statement. We will show how to find an interpretation map $\mathcal{J} : \hat{K} \to 2^Z$ such that $(2^Z, \mathcal{J})$ models the sentences from $\{\gamma_j : j \leq i\}$, $\triangle_{2X}$ and $\{c_i \sqcap c_j = \mathbf{0} : i \neq j\}$.

(1) Choose $\mathcal{J}(c_i) \subset f^{-1}[\mathcal{I}(c_i)]$ such that it is a continuum that is mapped onto $\mathcal{I}(c_i)$ by the map f.
(2) Let the $\mathcal{J}$-interpretation of all the constants from $\hat{K}_{-1}$ be equal to the f-inverse of their $\mathcal{I}$-interpretation.

With the interpretation map $\mathcal{J}$ we have so far we already have that $(2^Z, \mathcal{J})$ is a model of the theory $(\Gamma \cap \hat{\Sigma}_{-1}) \cup \triangle_{2X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\}$.

Now we will consider the sentences from $\Gamma \setminus \hat{\Sigma}_{-1} = \{\gamma_j : j < i\}$ one at a time in the order given by their index. These γ_j's will be restrictions on the $\mathcal{J}$-interpretation of constants for which we have so far no $\mathcal{J}$-interpretation in 2^Z. We will find $\mathcal{J}$-interpretation for a constant a introduced by one of the γ_j's inside the f-pre-image of the $\mathcal{I}$-interpretation of a. So for all constants mentioned in some γ_j with $j < i$ we have

$$\mathcal{J}(a) \subset f^{-1}[\mathcal{I}(a)].$$

Note that so far we have that $f[\mathcal{J}(a)] = \mathcal{I}(a)$ for all constants a which $\mathcal{J}$-interpretation we have determined. We will make sure that when we consider the next γ_j in the list and find a $\mathcal{J}$-interpretation of the introduced constant a it has the following two properties

(1) If $x \in \mathcal{I}(a) \cap \mathcal{I}(c_i)$ and $y \in \mathcal{J}(c_i)$ is such that $f(y) = x$, we have $y \in \mathcal{J}(a)$.
(2) If for all i we have $x \notin \mathcal{I}(c_i)$ and $x \in \mathcal{I}(a)$ then we have $f^{-1}(x) \subset \mathcal{J}(a)$.

This will make sure that the premise of the next γ_j to consider (if it has any) has the same truth value in the model $(2^Y, \mathcal{I})$ as it has in the model $(2^Z, \mathcal{J})$ we have constructed so far, as for any finite number of constants $a_1, \dots, a_n$ for which we have defined its $\mathcal{J}$-interpretation we have

$$f[\mathcal{J}(a_1) \cap \cdots \cap \mathcal{J}(a_n)] = \mathcal{I}(a_1) \cap \cdots \cap \mathcal{I}(a_n).$$

(1) Suppose γ_j is of the form $a = b \sqcap c$ or $a = b \sqcup c$. The $\mathcal{J}$-interpretation of a is fully prescribed by $\mathcal{J}(b)$ and $\mathcal{J}(c)$, and it easily seen that $\mathcal{J}(a)$ has the properties 1 and 2 above if $\mathcal{J}(b)$ and $\mathcal{J}(c)$ have it.

(2) Suppose γ_j is of the form $b \not\leq c \to a \leq b \wedge a \sqcap c = \mathbf{0}$. If the premise is false then $\mathcal{I}(a) = \emptyset$ and thus $\mathcal{J}(a) = \emptyset$ will suffice.

If the premise is true $\mathcal{I}(a)$ is a nonempty closed set that witnesses that $\mathcal{I}(b)$ is not a subset of $\mathcal{I}(c)$. We choose the $\mathcal{J}$-interpretation of a by

$$\mathcal{J}(a) = f^{-1}[\mathcal{I}(a)] \cap \mathcal{J}(b).$$

As $\mathcal{J}(b)$ maps onto $\mathcal{I}(b)$ under the map f and as $\mathcal{I}(a)$ is a nonempty subset of $\mathcal{I}(b)$ we see that with this $\mathcal{J}$-interpretation of a, we have a witness for $b \not\leq c$ in $(2^Z, \mathcal{J})$. It is also easily seen that $\mathcal{J}(a)$ has properties 1 and 2 if $\mathcal{J}(b)$ has these properties.

(3) Suppose a is one of the constants introduced by γ_j from some $\hat{\Sigma}_{5n+i}$ where $i = 2, 4, 5$. The $\mathcal{J}$-interpretation of these constants will be given by

$$\mathcal{J}(a) = f^{-1}[\mathcal{I}(a)].$$

This will make the sentence we are considering a true sentence in $(2^Z, \mathcal{J})$. Again $\mathcal{J}(a)$ will have properties 1 and 2.

The closed subsets U, V and W of Z we got from lemma 4.4 will make the sentence γ_i a true sentence in the model $(2^Z, \mathcal{J})$ as the premise of this sentence has the same truth value as in the model $(2^Y, \mathcal{I})$ and the $\mathcal{J}$-interpretation of the constants mentioned in the premise are subsets of the f-inverse of their $\mathcal{I}$-interpretation.

All the constants for which we have not yet given a $\mathcal{J}$-interpretation will have as their $\mathcal{J}$-interpretation the f-inverse image of their $\mathcal{I}$-interpretation, which of course is the empty set.

REMARK 4.9. This consistency proof also shows that there will be a set of disjoint continua in the Wallman representation of the lattice $L(\mathfrak{A})$, where $\mathfrak{A}$ is a model of $\hat{\Sigma}$ that will map onto all the continua in X by the map given by theorem 2.12.

4.2.6. *The Maćkowiak-Tymchatyn theorem.* As $\hat{\Sigma}$ is a consistent theory in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ there is some model $\mathfrak{A}$ for it. This model gives us a normal distributive and disjunctive lattice $L(\mathfrak{A})$ which models the sentences 22, 23 and conn$(\mathbf{1})$. There also exists, using the interpretations of the constants in $\hat{K}_{-1}$, an embedding of 2^X into the $L(\mathfrak{A})$. So the Wallman space $wL(\mathfrak{A})$, is a one-dimensional hereditarily indecomposable continuum which admits a weakly confluent map onto the metric continuum X.

Now we only have to make sure that there exists such a space that is of countable weight to complete the proof of the Maćkowiak-Tymchatyn theorem.

THEOREM 4.10. [**25**] *Let $f : Y \to X$ be a continuous surjection between compact Hausdorff spaces. Then f can be factored as $h \circ g$, where $Y \xrightarrow{g} Z \xrightarrow{h} X$ and Z has the same weight as X and shares many properties with Y (for instance, if Y is one-dimensional so is X or if Y is hereditarily indecomposable, so is X).*

PROOF. Let $\mathcal{B}$ a minimal sized lattice-base for the closed sets of X, and identify it with its copy $\{f^{-1}[B] : B \in \mathcal{B}\}$ in 2^Y. By the Löwenheim-Skolem theorem there is an elementary sublattice of 2^Y, of the same cardinality as $\mathcal{B}$ such that $\mathcal{B} \subset D \prec 2^Y$. The space wD is as required. □

Applying this theorem to the space $wL(\mathfrak{A})$ and the weakly confluent map $f : wL(\mathfrak{A}) \to X$ we get a one-dimensional hereditarily indecomposable continuum wD which admits a weakly confluent map onto the space X and moreover the weight of the space wD equals the weight of the space X. This is exactly what we were looking for.

CHAPTER 5

More old theorems and new proofs

1. On a theorem of Gordh

Bellamy showed in [8] that every metric continuum is homeomorphic to a retract of some metric indecomposable continuum. In [9] Bellamy showed that every irreducible metric continuum is a retract of some indecomposable continuum. Gordh noted in [22] that by construction the indecomposable continuum was also irreducible, and that the metric restriction can be dropped; in the same paper he extended Bellamy's result to the following theorem

THEOREM 5.1. *Every continuum is a retract of some irreducible indecomposable continuum.*

Fix a continuum X for the remainder of this section. We will, using model theory and Bellamy's result for the metric case, construct a continuum Y that is irreducible and indecomposable and that has a retraction homeomorphic to X. This will be a proof of Gordh's result for non-metric continua using the fact that the theorem holds for metric continua.

The continuum Y we want to find will be of the form wL for some lattice L. The properties that we want Y to have can be translated into properties that the lattice L must have.

We will give a set of sentences Σ in some language $\mathcal{L}^+$ such that every model L of this set will be a distributive lattice such that its Wallman interpretation wL has the properties we want. After we find this set Σ, we show that it is consistent, so that there really is a lattice with the wanted properties.

1.1. The set of sentences Σ. Let $\mathcal{L}$ be the lattice language $\{\mathbf{0}, \mathbf{1}, \sqcup, \sqcap\}$. Let LAT be a set of sentences in the language $\mathcal{L}$ such that every model of LAT is a normal distributive and disjunctive lattice (see section 2.1.1).

Let us define the following sentence in the language $\mathcal{L}$

$$\begin{aligned}\theta_{indec} &= \forall xy[\text{conn}(x) \wedge \text{conn}(y) \wedge x \sqcup y = \mathbf{1} \rightarrow x = \mathbf{1} \vee y = \mathbf{1}],\\ \theta_{irr} &= \exists xy \forall a[x \neq y \wedge \text{atom}(x) \wedge \text{atom}(y) \wedge \text{conn}(a) \wedge \\ &\qquad x \sqcup y \leq a \rightarrow a = \mathbf{1}].\end{aligned}$$

The Wallman representation of any normal distributive lattice model in the language $\mathcal{L}$ which models the sentence $\theta_{indec} \wedge \text{conn}(\mathbf{1})$ is an indecomposable Hausdorff continuum and if the lattice models the sentence

$\theta_{irr} \wedge \text{conn}(\mathbf{1})$ its Wallman representation is an irreducible Hausdorff continuum.

Now let $\mathcal{L}^+$ denote the extended language $\mathcal{L} \cup \{\hat{X}, \phi, \psi\} \cup \{c_x : x \in 2^X\}$, where $\hat{X}$ and the c_x's denote constants, and ϕ and ψ denote one-place functions.

The constants $\{c_x : x \in 2^X\}$ will correspond to the lattice 2^X. We let Σ contain Δ_{2^X} the diagram of 2^X, where any constant from $\{c_x : x \in 2^X\}$ will be corresponding to the closed subset of X that is its index. Note that by theorem 2.12, the Wallman representation of any model L of $LAT \cup \Delta_{2^X}$ will map continuously onto the continuum X. This is only part of what we want, because we want X to be homeomorphic to some retract of wL.

We added the constant $\hat{X}$ and the functions ϕ and ψ to the language $\mathcal{L}$ just for this. We will make sure that $\hat{X}$ will be some retract of the Wallman representation of any model of Σ by giving restrictions on ψ, and giving restrictions on ϕ we make sure that X and $\hat{X}$ (seen as a closed subspace of that Wallman representation) are homeomorphic.

1.1.1. *The continuum X and the subspace $\hat{X}$ of wL are homeomorphic.* We want the interpretation of the function variable ϕ to be a mapping between the lattices 2^X, in the form of the interpretations of the constants c_x, and the lattice that is given by the interpretation of the constant $\hat{X}$, $\{x : x \leq \hat{X}\}$, to be of the form as given in theorem 2.12. So we want ϕ to have the following properties.

(1) $\phi(c_\emptyset) = \mathbf{0}$, $\phi(c_X) = \hat{X}$ and for all $x \in 2^X \setminus \{\emptyset\}$ we have $\mathbf{0} < \phi(x) \leq \hat{X}$.
(2) For all x and y in 2^X such that $x \cup y = X$ we have $\phi(c_x) \sqcup \phi(c_y) = \hat{X}$.
(3) For every subset $\{x_1, \ldots, x_n\}$ of 2^X such that $\bigcap_i x_i = \emptyset$ we have
$\phi(c_{x_1}) \sqcap \cdots \sqcap \phi(c_{x_n}) = \mathbf{0}$.

These properties make sure that the Wallman interpretation of the lattice $\{c_x : x \in 2^X\}$ which is homeomorphic to the continuum X, is a continuous image of the Wallman representation of the lattice $\{x : x \leq \hat{X}\}$, the closed subspace $\hat{X}$ of wL.

But what we really want is that the spaces X and $\hat{X}$ are homeomorphic. For this it suffices to show that the continuous mapping is one-to-one, as X and $\hat{X}$ are compact Hausdorff spaces. If we take a look at the proof of theorem 2.12 then we know that the continuous image between $\hat{X}$ and X maps every point x from $\hat{X}$ (seen as a closed subspace of wL) to the point $\bigcap\{c_y : x \in \phi(c_y)\}$. To make sure that this map is one-to-one we want ϕ also to have the following property

4. For all x and y, if $x \sqcup y \leq \hat{X}$, $x \sqcap y = \mathbf{0}$ and $x \neq \mathbf{0}$ then there is a closed set z of X such that $x \leq \phi(c_z)$ and $y \sqcap \phi(c_z) = \mathbf{0}$.

All these restrictions on ϕ we denote by Θ_ϕ.

1.1.2. *$\hat{X}$ as a retract of wL.* We want to define a set of sentences Θ_ψ such that if some (any) lattice L is a model of the sentences from $LAT \cup \Theta_\psi$ then the subspace of wL given by the interpretation of $\hat{X}$ in L is a retract of wL.

(1) $\psi(\mathbf{0}) = \mathbf{0}$ and if $x \leq \hat{X}$ and $x \neq \mathbf{0}$ then $\psi(x) \neq \mathbf{0}$.
(2) If x and y are elements of L such that $x \cup y = \hat{X}$ then $\psi(x) \sqcup \psi(y) = \mathbf{1}$.
(3) If $x_1, \ldots, x_n$ are elements of L such that $x_1 \sqcup \cdots \sqcup x_n \leq \hat{X}$ and $x_1 \cap \cdots \cap x_n = \emptyset$ then we have that $\psi(x_1) \sqcap \cdots \sqcap \psi(x_n) = \mathbf{0}$.
(4) For all x in L if $x \leq \hat{X}$ then $\psi(x) \sqcap x = x$.

All these sentences together form the set Θ_ψ. Suppose L is some (any) model of $LAT \cup \Theta_\psi$. Consider its Wallman interpretation wL. The closed subspace of wL that corresponds to the constant $\hat{X}$ we also denote by $\hat{X}$. As the lattice base $L \cap 2^{\hat{X}}$ is embedded in the lattice L we know that $\hat{X}$ is a continuous image of the space wL. As, by item 4 in the list above this continuous map restricted to the closed set $\hat{X}$ is the identity, we know that $\hat{X}$ is a retract of wL.

1.1.3. *The set of sentences Σ.* Let Σ be the set of sentences in the language $\mathcal{L}^+$ defined by

$$\Sigma = LAT \cup \{\theta_{irr}, \theta_{indec}, \text{conn}(1)\} \cup \triangle_{2^X} \cup \Theta_\phi \cup \Theta_\psi.$$

Any model L of Σ will be a normal distributive and disjunctive lattice such that its Wallman representation wL will be an irreducible indecomposable continuum such that X is an continuous image of wL, X is homeomorphic to the closed subspace $\hat{X}$ of wL and $\hat{X}$ is a retract of the continuum wL.

1.2. The consistency of Σ in the language $\mathcal{L}^+$. By the compactness theorem it suffices to show that any finite subset Σ' of Σ has a model.

Let Σ' be some finite subset of Σ, without loss of generality we can assume that the sentences θ_{irr}, θ_{indec} and conn$(\mathbf{1})$ are contained in Σ'.

As there is only mention of a finite number of constants c_x of $\{c_x : x \in 2^X\}$ there is a countable sublattice $\mathcal{C}$ of 2^X containing the finitely many constants mentioned in Σ'.

Extend the set Σ' to a set $\Sigma^* \subset \Sigma$ such that Σ^* contains the sentences of Σ that mention only the constants from $\mathcal{C} \cup \{\hat{X}, \mathbf{0}, \mathbf{1}\}$. The set Σ^* is countable.

Let $w\mathcal{C}$ be the Wallman representation of the (countable) sublattice $\mathcal{C}$ of 2^X. This $w\mathcal{C}$ is a metric continuum and the theorem of Bellamy says that there exist some space Y that is a irreducible continuum that has a retract R that is homeomorphic to the continuum $w\mathcal{C}$. Using the remark made by Gordh, we may also assume that this continuum is indecomposable.

So, there exist a homeomorphism $f : w\mathcal{C} \to R$ a retraction $r : Y \to R$. With these functions and spaces we are able to give interpretations of the constants and functions of $\mathcal{L}^+ \setminus \mathcal{L}$ and thus get a model for the set Σ^* of sentences in $\mathcal{L}^+$.

The model will be the lattice 2^Y, the interpretation of the constants $\{\mathbf{0}, \mathbf{1}, \hat{X}\}$ and $\{c_x \in \mathcal{C}\}$ will be given by $\emptyset$, Y and R, and the interpretations of c_x in $w\mathcal{C}$ respectively. The functions ϕ and ψ are given by $\phi(c_x) = f[c_x]$ for all c_x in $\mathcal{C}$ (the interpretation of c_x in $w\mathcal{C}$ is denoted by c_x as well), and $\psi(x) = r^{-1}[x]$ for all $x \in 2^{\hat{X}}$ (the interpretation of the x in Y is denoted by x).

So the set Σ has a model, and as we mentioned earlier the Wallman representation of this model (it is of course a distributive lattice with zero and unit) is an irreducible indecomposable continuum which has a retract that is homeomorphic to the continuum X.

2. On a theorem of Van Hartskamp and Vermeer

2.1. Preliminaries. A fixed-point free map $f : X \to X$ is said to be colorable with k colors if there exists a closed cover $\mathcal{C}$ of X of size k (or less) such that $C \cap f[C] = \emptyset$ for all $C \in \mathcal{C}$. The elements C of $\mathcal{C}$ will be called *colors*, and we will say that $\mathcal{C}$ is a *coloring* of the map f.

A theorem of Katětov gives us that every fixed-point free selfmap can be colored with three colors. The theorem of Lusternik and Schnirelmann (see, for instance [**18**]) states that every coloring of the antipodal map $\iota : S^n \to S^n$ on the n-dimensional sphere S^n needs at least $n + 2$ colors. Variations of this theorem are given by Aarts et al. in [**1**].

In this section we will give a shortened proof of the following theorem.

THEOREM 5.2 ([**28**],[**44**]). *Let X be a paracompact Hausdorff space of dimension $\dim X \leq n$. Then any fixed-point free autohomeomorphism of X can be colored with $n + 3$ colors.*

The proof of Van Hartskamp and Vermeer in [**28**] goes along the following lines. The first result used is due to Van Douwen [**14**], who showed that every fixed-point free autohomeomorphism on a finite-dimensional paracompact space can be colored with *finitely* many colors. Then Aarts et al. [**1**], using this theorem of Van Douwen, proved the following theorem.

THEOREM 5.3 ([**1**]). *Let X be a compact metrizable space with $\dim X \leq n$. Every fixed-point free continuous map of X to itself can be colored with $n + 3$ colors.*

Then Van Hartskamp and Vermeer [**28**], again using the theorem of Van Douwen, proved that a fixed-point free autohomeomorphism on an n-dimensional paracompact Hausdorff space is semi-conjugated to a fixed-point free autohomeomorphism on some $\leq n$-dimensional metrizable space. Applying the result on metrizable spaces, and pulling back the obtained coloring, one obtains a coloring of the original homeomorphism.

In [**44**] Van Mill gives another proof of theorem 5.2, using standard facts from dimension theory only.

2.2. Proof of theorem 5.2. Suppose that X is a paracompact space such that $\dim X \leq n$ and that $f : X \to X$ is a fixed-point free autohomeomorphism. Using the following theorem we may assume without loss of generality that X is compact.

THEOREM 5.4 ([**14**]). *Every fixed-point free homeomorphism $f : X \to X$ of a finite-dimensional paracompact Hausdorff space onto itself has a fixed-point free Čech-Stone extension $\beta f : \beta X \to \beta X$.*

Also, as f is fixed-point free, and by assumption X is compact, there exists a finite closed cover $\mathcal{F}$ of X such that $F \cap f[F] = \emptyset$ for all $F \in \mathcal{F}$.

Extend the lattice language $\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}$ by adding two functions θ and θ^* to form $\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}^* = \{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\} \cup \{\theta, \theta^*\}$. Let us define $F, F^* : 2^X \to 2^X$ by

$$F(a) = f[a] \quad \text{for all } a \in 2^X;$$
$$F^*(a) = f^{-1}[a] \quad \text{for all } a \in 2^X,$$

then $(2^X; F, F^*)$ is a model for the language $\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}^*$ (we abuse notation somewhat as 2^X is of course shorthand for $(2^X; \cup, \cap, \emptyset, X)$, a model of $\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}$).

Let $(L; G, G^*)$ be some countable elementary submodel of $(2^X; F, F^*)$ such that $\mathcal{F} \subset L$. From the lattice function G on L we can construct a map $g : wL \to wL$ by

$$g(a) = \bigcap\{G(b) : b \in L \text{ and } b \in a\} \text{ for all } a \in wL.$$

This map is continuous, as $g^{-1}[b] \in L$ for all $b \in L$ (in fact it is not hard to see that g is actually a homeomorphism). It also is fixed-point free by the fact that $\mathcal{F}$ is a closed cover of wL such that $g[F] \cap F = \emptyset$ for all $F \in \mathcal{F}$.

Note that wL is a metric compactum of (covering) dimension $\leq n$, so by theorem 5.3 there is a coloring of g of size $n+3$. We have

$$(L; G, G^*) \models \exists x_1 \cdots x_{n+3}[(\bigwedge_i \theta(x_i) \sqcap x_i = \mathbf{0}) \wedge \bigsqcup_i x_i = \mathbf{1}].$$

So $(2^X; F, F^*)$ also models this sentence, which states that f can be colored with $n+3$ colors.

APPENDIX A

Model Theory

Model theory is the branch of mathematical logic that studies structures from a logical point of view. These structures can be all sorts of mathematical objects like linear orders, groups, fields, Boolean algebras, lattices, etc..

In this appendix I will review some notions and theorems of model theory that are used in the thesis. The goal of this appendix is neither to be complete nor exact, it is just to give the reader, who is not familiar with model theory some idea of what it is about, and how it is used in this thesis.

1. Logic

We shall concentrate on the model theory of first-order predicate logic.

The key notions in model theory are *language* and *theory*.

1.1. Language. Languages consist of two parts. The first is the same for all languages. It is the logical part consisting of the symbols $\exists$, $\forall$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $=$ together with an (countably) infinite number of variables (and the symbols (,), [,] and , used for separation in formulas).

The second part of the language is a set of symbols, specific for the kind of structure we are considering.

EXAMPLE A.1. If we are investigating linearly ordered structures we need a symbol to indicate that, say *a comes before b in the linear order*. Let $<$ be this symbol, then we can write $a < b$ for the above.

EXAMPLE A.2. Similarly, to study groups we use the (additional) symbols $\circ$, for group multiplication and e to denote the identity element of the group.

The symbol $<$ above is called a (2-placed) relation symbol, $\circ$ is called a (2-placed) function symbol and e is a constant symbol. In general a language for mathematical structures consist of the logical symbols and a (possibly infinite) number of relation, function and constant symbols.

EXAMPLE A.3. To study fields we use the language $\{+, \times, 0, 1\}$.

EXAMPLE A.4. To study lattices with zero and unit we use the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$.

1.2. Terms and formulas. With these symbols we can make strings. Some of these will be meaningless, others not. Terms are strings of symbols that label elements of the structure, and formulas are strings of symbols that say something about terms or the structure itself. Formulas without free variables are called sentences. Some of these sentences are used to lay the foundation of the structure in so called theories.

REMARK A.5. We assume that the reader is familiar with first-order mathematical logic. To make formulas more readable, we, as is common in mathematical logic shall assume that $\neg$ is more binding than $\wedge$ and $\vee$, which are itself more binding than $\to$ and $\leftrightarrow$. This allows us to omit some of the parenthesis, which makes the formulas more readable.

1.3. Theories. A theory is simply a set of formulas. Interesting theories should be about something non-trivial and should be consistent, which means that you cannot derive a false statement from it.

Normally one specifies a theory by listing a number of formulas (sentences) as its starting point (its axioms), and we tacitly assume that the consequences of these axioms make up the whole theory.

EXAMPLE A.6. The theory of linearly ordered sets has the following list of axioms (remember that the non-logical part of its language consists of a 2-placed relation symbol $<$).

(1) $\forall x \neg [x < x]$,
(2) $\forall xy [x < y \vee y < x \vee x = y]$,
(3) $\forall xyz [x < y \wedge y < z \to x < z]$.

When we add the sentence

4. $\forall xy \exists z [x < y \to x < z \wedge z < y]$,

to the theory for linear orders we have the theory of dense linear orders. When we add the sentence

5. $\forall x \exists yz [y < x \wedge x < z]$,

to the theory of dense linear orders we have the theory of dense linear orders without endpoints.

EXAMPLE A.7. The theory of groups has the following list of axioms.

(1) $\forall xyz [[x \circ y] \circ z = x \circ [y \circ z]]$,
(2) $\forall x [x \circ e = x]$,
(3) $\forall x \exists y [x \circ y = e]$.

1.4. Models. A *model* for a theory is a structure for the corresponding language where all the formulas of the theory are valid. So a linearly ordered set is a model for the theory of linearly ordered sets and a group is a model for the theory of groups. We already encountered the theory of distributive lattices with zero and unit in chapter 2, and for any topological space X, $(2^X, \cap, \cup, \emptyset, X)$ is a model for this theory.

EXAMPLE A.8. The models $\mathfrak{A} = (\mathbb{N}, <)$, $\mathfrak{B} = (\mathbb{Q} \cap [0, \infty], <)$ and $\mathfrak{C} = (\mathbb{R}, <)$ are models for the theory of linearly ordered sets from example A.6, where $<$ is the usual order on the reals $\mathbb{R}$. The models $\mathfrak{B}$ and $\mathfrak{C}$ are also models for the theory of dense linear orders. The model $\mathfrak{C}$ is the only model of the three that also models the theory for dense linear orders without endpoints.

In the example A.8 above, $\mathbb{N}$ is said to be the *universe* of the model $\mathfrak{A}$.

EXAMPLE A.9. $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}, +, 0)$ are models for the theory of groups from example A.7.

Let $\phi(x_0, \ldots, x_n)$ be some formula in some language L with all its free variables among $\{x_0, \ldots, x_n\}$. Suppose that $\mathfrak{A}$ is some model for the language L and $a_0, \ldots, a_n$ are elements of its universe A. We let

$$\mathfrak{A} \models \phi[a_0, \ldots, a_n]$$

denote that in the model $\mathfrak{A}$, if we substitute all occurrences in ϕ of the free variables x_i with the corresponding a_i we get a true statement.

EXAMPLE A.10. Consider $(\mathbb{Z}; +, 0)$, a model for the theory of groups (in the language $\{\circ, e\}$). Let $\phi(x, y)$ denote the formula $x \circ y = x$ and ψ be the lattice sentence $\forall x[x \circ e = x]$ then

$$(\mathbb{Z}; +, 0) \models \phi[3, 0] \text{ and also } (\mathbb{Z}; +, 0) \models \psi.$$

2. Compactness and completeness

The Compactness theorem and the Extended Completeness Theorem are two of the theorems that play a dominant role in model theory.

THEOREM A.11 (Compactness Theorem). *A set of sentences has a model is and only if every finite subset has a model.*

THEOREM A.12 (Extended Completeness Theorem). *A set of sentences is consistent if and only if it has a model.*

3. Substructures and extensions

If $\mathfrak{D}$ and $\mathfrak{E}$ are models for some language L, then we say that $\mathfrak{D}$ is a submodel of $\mathfrak{E}$, or $\mathfrak{E}$ is an extensions of $\mathfrak{D}$ (notation: $\mathfrak{D} \subset \mathfrak{E}$) if the universe D of $\mathfrak{D}$ is a subset of the universe E of $\mathfrak{E}$, $F_{\mathfrak{D}} = F_{\mathfrak{E}} \restriction D^n$, $R_{\mathfrak{D}} = R_{\mathfrak{E}} \cap D^m$ and $c_{\mathfrak{D}} = c_{\mathfrak{E}}$ for every n-placed function symbol, every m-placed relation symbol and every constant symbol c of L.

EXAMPLE A.13. For the models $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ of the theory of linear orders the following holds

$$\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{C}.$$

3.1. Elementary submodels. To define the notion of elementary submodel we will use the characterization of Tarski and Vaught, which states that given two models $\mathfrak{A}$ and $\mathfrak{B}$ for the first-order language L such that $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ if for every L-formula $\phi(x_0, \ldots, x_n, y)$ and all $a_0, \ldots, a_n$ in A, the universe of $\mathfrak{A}$, if

$$\mathfrak{B} \models \exists y \phi[a_0, \ldots, a_n, y]$$

then there is some $a \in A$ such that

$$\mathfrak{B} \models \phi[a_0, \ldots, a_n, a].$$

The notation we use for $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ is $\mathfrak{A} \prec \mathfrak{B}$, we also say that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

EXAMPLE A.14. Consider the models $\mathfrak{F} = (\mathbb{Q} \cap (0, \infty), <)$, $\mathfrak{G} = (\mathbb{Q}, <)$ and $\mathfrak{H} = (\mathbb{R}, <)$ for the theory of dense linear ordered sets without endpoints. It may come as no surprise that we have the following

$$\mathfrak{F} \prec \mathfrak{G} \prec \mathfrak{H}.$$

Consider also the following models for the theory of dense linear order $\mathfrak{I} = (\mathbb{R} \cap [0, \infty), <)$ and $\mathfrak{J} = (\mathbb{R} \cap [-1, \infty), <)$.

We have $\mathfrak{I} \subset \mathfrak{H}$, but if we consider the sentence $\phi = \exists x \forall y \neg [y < x]$, expressing the existence of a minimal element, then $\mathfrak{I}$ is a model of ϕ and $\mathfrak{H}$ clearly models its negation $\neg\phi$, hence $\mathfrak{I}$ is not an elementary submodel of $\mathfrak{H}$.

Although we have $\mathfrak{I} \subset \mathfrak{J}$, the model $\mathfrak{I}$ is not an elementary submodel of $\mathfrak{J}$. Consider the formula $\phi(x, y) = \exists x [x < y]$, expressing that the free variable y is not the least element of the order. The model $\mathfrak{J}$ models $\phi(x, 0)$, as there are lots of points below 0 in this model, as there are none in the model $\mathfrak{J}$ this shows that $\mathfrak{I}$ cannot be an elementary submodel of $\mathfrak{J}$.

EXAMPLE A.15. If we consider the models $\mathfrak{K} = (\mathbb{Q}, +, \times, 0, 1)$ and $\mathfrak{L} = (\mathbb{R}, +, \times, 0, 1)$ of the theory of fields, although $\mathfrak{K}$ is a submodel of $\mathfrak{L}$, it is not an elementary submodel; just consider the formula $\exists x (x \times x = 2)$, it states that 2 (an element of both models) is a square. This formula holds in $\mathfrak{L}$ but not in $\mathfrak{K}$.

EXAMPLE A.16. A nontrivial result on elementary substructures of fields is the fact that the field of algebraic numbers is an elementary submodel of the field of complex numbers (for a proof see [**29**]).

4. Set Theory

Set theory, loosely stated, says that every mathematical statement can be translated into a statement about sets. In other words, sets are at the basis of mathematics. Over the years a number of theories have been defined for set theory. The one that is commonly used as the basis for set theory is ZFC, Zermelo-Fraenkel set theory and the axiom of Choice.

4.1. The theory ZFC. The language of set theory is, apart from the logical symbols only the symbol $\in$, which stands for membership. The only objects are sets and the axioms give us tools to construct new sets from old ones. the theory ZFC is an infinite list of sentences in the language $\{\in\}$.

Consider again the theory of groups from example A.7. If we add the axiom

4. $\forall xy[x \circ y = y \circ x]$,

then we have the theory of Abelian groups. A model for this sort of group is for instance the additive group of natural numbers $(\mathbb{Z}, +, 0)$. As there are also non-Abelian groups, there is no formal proof from the theory of groups for the following sentence, nor its negation

$$\forall xy[x \circ y = y \circ x]. \tag{25}$$

A model for the non-Abelian groups is for instance the the invertible 2-by-2 matrices, with the group operation of matrix multiplication and with unit the identity matrix.

The sentence from equation 25 is said to be independent of the theory of groups. We say that the theory of groups is not a complete theory, as it does not decide for every sentence if it follows from the theory of groups or not.

Something similar is the case for set theory. A famous statement that cannot be decided in ZFC is the Continuum Hypothesis (CH), the statement that every subset of the reals is either countable (finite or as large as the set of natural numbers) or as large as the set of reals. Gödel showed that there is a model of ZFC where CH holds. By a method known as forcing, Cohen showed that there are also models of ZFC that model $\neg$CH. This implies that CH is independent of ZFC.

In chapter 1 we used the forcing method to show that the statement $\mathfrak{tf}(C) < \mathfrak{c}$ is consistent with ZFC. In the same chapter we also showed that $\neg[\mathfrak{tf}(C) < \mathfrak{c}]$ holds in some model of ZFC (for instance when we have CH) we have thus shown that the statement $\mathfrak{tf}(C) < \mathfrak{c}$ is independent of ZFC.

APPENDIX B

List of problems from the thesis

PROBLEM B.1 (Question 2.32 of this thesis.). Is having strong infinite (covering) dimension elementarily reflected, and is having not strong infinite (covering) dimension elementarily reflected.

PROBLEM B.2 (Question 2.39 of this thesis.). Is aposyndesis a property of compact spaces that is elementarily reflected?

PROBLEM B.3 (Question 2.45 of this thesis.). Is being not discoherent a property of compact spaces that is elementarily reflected?

PROBLEM B.4 (Question 2.78 of this thesis.). Is being the unit n-cube or the n-sphere a property of compact spaces that is elementarily reflected?

PROBLEM B.5 (Question 3.11 of this thesis.). Is the property of having span zero a first-order property?

PROBLEM B.6 (Question 3.12 of this thesis.). Is the property of having span zero a base-free property?

QUESTION B.7 (Question 3.13 of this thesis.). *Is chainability a property of compact spaces that is elementary reflected?*

PROBLEM B.8 (Question 3.55 of this thesis.). Are span zero, surjective (semi) span zero and symmetric span zero properties that reflect using the method described in the previous section?

PROBLEM B.9 (Question 3.56 of this thesis.). Does there exists a continuum that is non-chainable and has span zero, surjective (semi) span zero or symmetric span zero?

PROBLEM B.10 (Question 3.57 of this thesis.). Does there exist a locally connected non-chainable continuum which has span zero?

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Summary

This thesis deals with two unrelated topics. Chapter 1 gives a consistency proof on the minimal cardinality of certain sets of continuous self maps of the Cantor set, the remaining chapters investigate how certain properties of continua (connected compact Hausdorff spaces) relate to their lattices of closed subsets.

In the first chapter we consider, given a topological space X what the minimal cardinality is of a set of continuous self maps that is required to 'connect' any two points of X. More explicit, we look for subsets $\mathcal{F}$ of the self maps of X, such that for every $x, y \in X$ there exists an $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$; we are interested in the minimal cardinality of such a subset $\mathcal{F}$, we denote this minimal cardinal number for X by $\mathfrak{tf}(X)$.

We mainly investigate the cardinal number $\mathfrak{tf}(C)$, where C denotes the Cantor set. Three restrictions on $\mathfrak{tf}(C)$ are easily found:

(1) $\mathfrak{tf}(C)$ is at least $\aleph_1$.
(2) $\mathfrak{tf}(C)$ is less than or equal to $\mathfrak{c}$, the cardinality of the set of all reals.
(3) $\mathfrak{tf}(C)^+$, the successor cardinal of $\mathfrak{tf}(C)$, is at least $\mathfrak{c}$.

Under these restrictions it is not hard to see that when CH holds or when $\mathfrak{c}$ is a limit cardinal, then $\mathfrak{tf}(C)$ and $\mathfrak{c}$ have to be equal to one another. Is this always the case? No; we show that in the Baumgartner-Laver model (a model of ZFC obtained by iteratively adding $\aleph_2$ many Sacks reals to a model of ZFC + CH) the cardinal number $\mathfrak{tf}(C)$ is strictly less than $\mathfrak{c}$. With this we have shown that the statement $\mathfrak{tf}(C) = \mathfrak{c}$ is independent of ZFC.

In the second part, which covers the chapters 2, 3, 4 and 5, we use Wallman's representation theorem on lattices to investigate, by model-theoretic means, how certain properties of compact Hausdorff spaces relate to the lattice of their closed sets. We are particularly interested in the class of continua (connected, compact Hausdorff spaces).

The method in chapter 2 is as follows, starting with the lattice of all closed subsets of some continuum we take an elementary sublattice, by Wallman's representation theorem we know that this lattice again corresponds with some continuum. Some topological properties must be shared by both continua, others may not be; examples of both are mentioned in this chapter.

In chapter 3 we try, using the method described above to find some new insights on an old problem in continuum theory, whether span zero implies chainability for metric continua. One of the results is that, with an extra assumption of local connectivity any counterexample can be transformed into a metric counterexample. Further results in this chapter concern the (extended notions of) span and chainability of the continua $\mathbb{H}^*$ and $\mathbb{I}_u$. It appears that both are non chainable and of span nonzero.

In chapter 4 we give a model theoretic proof of the theorem of Maćowiack and Tymchatyn which shows that every metric continuum is a weakly confluent image of some 1 - dimensional metric continuum. In this chapter there is also a new topological proof of the extended theorem of Maćowiack and Tymchatyn for arbitrary continua.

In Chapter 5 we show how two other known results can be proved efficiently by model-theoretic means.

Samenvatting

Dit proefschrift behandelt twee onderwerpen die los staan van elkaar. Het eerste hoofdstuk geeft een consistentie bewijs over de minimale cardinaliteit van een verzameling continue zelf afbeeldingen, in de resterende hoofdstukken wordt onderzocht hoe somige eigenschappen van een continuum (een samenhangende compacte Hausdorff ruimte) zich verhouden tot het tralie van al zijn gesloten deelverzamelingen.

In het eerste hoofdstuk onderzoeken we, gegeven een topologische ruimte X, hoeveel continue zelfafbeeldingen er nodig zijn om elk tweetal punten van X met elkaar te 'verbinden'. Iets explicieter, we zoeken een deelverzameling $\mathcal{F}$ van de continue zelfafbeeldingen van X, zodanig dat voor alle $x, y \in X$ er een $f \in \mathcal{F}$ bestaat met $f(x) = y$ of $f(y) = x$; we zijn geinteresseerd in de minimale cardinaliteit van zo'n verzameling $\mathcal{F}$, dit cardinaalgetal duiden we aan met $\mathfrak{tf}(X)$.

In hoofdstuk 1 bekijken we het cardinaalgetal $\mathfrak{tf}(C)$, waar C de Cantor verzameling voorstelt. Drie restricties voor $\mathfrak{tf}(C)$ zijn snel gevonden:

(1) $\mathfrak{tf}(C)$ is teninste $\aleph_1$.
(2) $\mathfrak{tf}(C)$ is kleiner of gelijk aan $\mathfrak{c}$, de cardinaliteit van de verzameling der reele getallen.
(3) $\mathfrak{tf}(C)^+$, het kleinste cardinaalgetal groter dan $\mathfrak{tf}(C)$, is een bovengrens voor $\mathfrak{c}$.

Onder deze restricties is het niet moeilijk in te zien dat onder CH of wanneer $\mathfrak{c}$ een limiet cardinaalgetal is, $\mathfrak{tf}(C)$ en $\mathfrak{c}$ aan elkaar gelijk moeten zijn. Maar is dit altijd het geval? Nee; we laten zien dat in het Baumgartner-Laver model (een model van ZFC verkregen door aan een model van ZFC + CH iteratief $\aleph_2$ Sacks-reals toe te voegen) het cardinaalgetal $\mathfrak{tf}(C)$ strikt kleiner is dan $\mathfrak{c}$. Hiermee hebben we laten zien dat de uitspraak $\mathfrak{tf}(C) = \mathfrak{c}$ onafhankelijk is van ZFC.

In het tweede deel, dat de hoofdstukken 2, 3, 4 en 5 beslaat, gebruiken we Wallman's representatie stelling voor traliën om via de modeltheoretische weg te onderzoeken hoe sommige eigenschappen van compacte Hausdorff ruimten zich verhouden tot het tralie van alle gesloten verzamelingen van deze compacta. We zijn in het bizonder geinteresseerd in de deelklasse van de samenhangende compcate Hausdorff ruimten, de zogenaamde continua.

De aanpak in hoofdstuk 2 is als volgt, startend met het tralie van gesloten verzamelingen van een continuum nemen we hiervan een elementaire deelstructuur, dankzij de representatiestelling van Wallman weten we dat er bij dit tralie weer een continuum hoort. Er zijn topologische eigenschappen die beide continua moeten delen, en eigenschappen die niet noodzakelijkerwijs gedeeld hoeven te worden; beide komen in dit hoofdstuk aan de orde.

In hoofdstuk 3 proberen we via de werkwijze zoals hierboven is beschreven nieuw inzicht te krijgen in een oud probleem in de continuumleer, namelijk of span nul ketenbaarheid impliceert voor metrische continua. Een van de resultaten is dat, onder de extra voorwaarde van lokale samenhang een willekeurig tegenvoorbeeld een metrisch tegenvoorbeeld oplevert. Verder wordt in dit hoofdstuk bekeken wat de (uitgebreide vorm van) span en ketenbaarheid is van de continua $\mathbb{H}^*$ en $\mathbb{I}_u$. Het blijkt dat beide niet ketenbaar zijn en een span ongelijk aan nul hebben.

In hoofdtuk 4 wordt er een modeltheoretisch bewijs gegeven van de stelling van Maćowiack en Tymchatyn die zegt dat elk metrisch continuum een zwak confluent beeld is van een 1-dimensionaal metrisch continuum. Tevens wordt in dit hoofdstuk een nieuw topologisch bewijs gegeven voor de uitbreiding van deze stelling Maćowiack en Tymchatyn naar willekeurige continua.

Hoofdstuk 5 geeft van twee bekende stellingen een modeltheoretisch bewijs.

curriculum vitae

Berend Jan van der Steeg was born in Rhoon on September 22 in 1975. In 1993 he finished secondary school at the CSG Jan Arentsz in Alkmaar and started his studies in Technical Mathematics at the Delft University of Technology. His graduation research was concerned with Topology and Set Theory. Under the supervision of dr. K. P. Hart he investigated uniformities, with his thesis 'Cardinaal reflecties van uniforme ruimten' he obtained his M. Sc. degree in 1998.

Since September 1998 he worked on his Ph.D. research in Topology at the Technical University in Delft also under the supervision of dr. K. P. Hart. During this period he attended several conferences and courses in Canada, the United States, the Čech Republic and England.

The book you see before you are the results of his research.

From October 2002 till June 2003, during the final preparation of his thesis he has taught mathematics at the secondary school Comenius College at Capelle aan den IJssel.

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