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Interacting agent modellen voor rijkdom verdeling

(Engelse titel: Interacting agent models for wealth distribution)

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Abstract

In this paper a two agent wealth distribution model for a closed economic system developed in [2] is presented and extended. We first extend the model by randomly distributing the propensity to save of the agents. We derive a closed form of the stationary relative wealth measure of an agent. We also see that if we take both the propensity to save and the redistribution measure to be uniformly distributed, then the stationary wealth distribution of agent 1 cannot be Beta distributed. Furthermore we conjecture that given a uniform redistribution measure and Beta distributed propensity to save, the resulting wealth distribution cannot be Beta distributed either. The absence of Beta distributions in the wealth distribution shows that there cannot be product stationary measures in these cases. We also extend the model by assuming zero propensity and that the stationary product measure of one agent is conditionally Gamma(α, β) distributed, where we condition on α be independently distributed as well. We find that the class of distributions for α defined by $\psi(\alpha) = a^{-k}$, $k \in \mathbb{N}$ always leads to the wealth distribution for agent 1 to be heavy-tailed. We also take steps in showing that there exists a distribution for α that solves for the wealth distribution of agent 1 to be Pareto Lomax distributed.

1. Introduction

The wealth distribution of a given population is a problem that has interested economists, governments, and other parties alike for centuries. Given such a distribution, understanding the mechanics behind the economy of a population would take great leaps forward. However this remains to be quite a challenging task. In recent years, techniques and ideas from mathematical statistical mechanics have entered this area of research in order to explain the wealth distribution in an economy of interacting agents. The term *econophysics* was first used in 1995 to describe this particular field of physics that had to do with the statistics of the economy and finance. However using years of research in statistical mechanics as a mathematical foundation, this section of wealth distribution analysis soon became quite prominent in its own right [9].

The wealth distribution models are inspired by a similar energy redistribution model from physics literature. Heat (or energy) transfer can be modelled with a basis in the individual particle heat transfer; assuming we have a closed system (i.e. the total energy within the system is constant), the interaction of particles through collisions causes energy to be randomly redistributed among these particles. The field of econophysics takes this energy model and applies it to a system of wealth instead of energy. Now instead of particles we have *economic agents*; these are entities in the system that can interact in the form of transfer of wealth as opposed to transfer of energy in the original model. The distribution of wealth of the whole system is then comparable to that of the energy distribution in the physical model. Due to its strong roots in physics it becomes clear why analyzing these models using mathematical phenomena produces relevant results [9].

In this paper we take the model introduced in [2] and investigate some realistic extensions that pertain to an economic agent's propensity to save. In [2] a two agent model is used to represent the interactions of the system as a whole. The two economic agents interact through wealth transactions, where the total wealth in the system stays constant. In each transaction, a certain amount of wealth is saved for each agent and the rest is redistributed among them according to some random mechanism. The saved wealth corresponds to the aforementioned propensity. In [2] the focus lies on the different types of wealth redistribution measures that can be used for this model to give realistic results. The purpose in [2] is to find the conditions under which there are product stationary measures, and how the system evolves in time (for example the expected wealth of an agent). For the latter duality techniques are used as well.

In this paper we first mathematically define the model from [2]. We then look at the influence of randomly distributing the propensity to save of an agent in this model, and focus on the stationary measures. We also look at the influence on the model when randomly distributing the parameters and taking propensity to be equal to zero. We ask the following questions

- For the random propensities independently distributed, are there corresponding wealth distribution models that have stationary product measures?

- We conjecture that with the propensity and the redistribution of wealth per transaction Beta distributed that there are no stationary product measures for the wealth distribution.
- What form does the general stationary wealth distribution have with random propensity to save?
- When randomly distributing the parameters in the Gamma redistribution measure for the zero propensity model, can the resulting wealth distribution be heavy tailed? Specifically can it be (Lomax) Pareto distributed?

2. Two agent model

In this section the two agent the model for wealth distribution from [2] is introduced. We first introduce some definitions and concepts from probability theory that will be helpful later on.

2.1. Markov Processes

The two agent model assumes that the process of redistributing the total wealth in the system is a *Markov Process*. The motivation behind this is as follows: Markov processes have the appealing quality that their distribution of future states only depend on the current state; i.e. conditional on the present, the past is independent of the future. Computationally (as well as mathematically as we shall see) this is more efficient, and it does not produce extremely unrealistic results in the model; on the contrary, the Markov property is one that is often used in other financial models, see [7]. Furthermore we choose a continuous state space Markov process with discrete time.

Mathematically speaking, a discrete time Markov Process can be defined as follows. We have a stochastic process $\{X_t, t \in \mathbb{N}\}$ defined on a given measurable space $(\Omega, \mathscr{F})$. The Markov property described in the previous paragraph is then defined as: for all $t_n \in \mathbb{N}$, $n \in \mathbb{N}$, $0 < t_1 < t_2 \cdots < t_n < t$, and $f : \Omega \to \mathbb{R}$ bounded and measurable, we have

$$\mathbb{E}(f(X_t)|X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = \mathbb{E}(f(X_t)|X_{t_n}) \tag{2.1}$$

This essentially mathematically expresses that conditioned on X_t, X_{t+1} and X_{t-1} are independent. We call 2.1 the Markov property. This can be easily extended to continuous time processes, as shown in [6].

2.2. Transition operator

Having defined the Markov process for the redistribution of wealth between two economic agents, we now continue to define its transition operator. Let $\{X_t, t \geq 0\}$ be a Markov process as before defined on a given measurable space (Ω, F). We define the operator S_t on all bounded, measurable functions $f : \Omega \to \mathbb{R}$ as

$$S_t f(x) = \mathbb{E}_x(f(X_t)) = \mathbb{E}(f(X_t|X_0 = x) \tag{2.2}$$

We notice that this operator (known as the *semi-group*) also has the property [6]:

$$S_{t+s} f(x) = S_t S_s f(x) \tag{2.3}$$

Now we define the *transition operator* S_n as P^t, where

$$P f(x) = \mathbb{E}(f(X)|X_0)$$

2.3. Model definition

We are now ready to mathematically define a general redistribution method of the total wealth in a closed economic system with two economic agents. Recall that this model comes directly from [2] and will therefore be defined in a similar manner.

We denote the wealth configuration as the pair (x, y) within the state space $E = [0, \infty)^2$. Furthermore we have the propensity to save, λ, and redistribution value, ϵ, which take values in $[0, 1]$. The propensity to save is the fraction of wealth that an agent decides to keep and not redistribute. The remaining fraction of wealth is then redistributed according to ϵ. We then define the following map as the redistribution of the wealth configuration after one transaction. For $T_\epsilon^\lambda(x, y) : E \to E$ we have

$$T_\epsilon^\lambda(x, y) = (\lambda x + (1 - \lambda)\epsilon(x + y), \lambda y + (1 - \lambda)(1 - \epsilon)(x + y)) \tag{2.4}$$

It should be noted that the redistribution map $T_\epsilon^\lambda(x, y)$ conserves the total wealth in a closed economy; i.e. if $(w, z) = T_\epsilon^\lambda(x, y)$, then $x + y = w + z = s$ is a constant.

We first observe the transition operator with fixed $\lambda \in [0, 1]$ and redistribution measure $\nu(\epsilon)$:

$$P^\lambda f(x, y) = \int_0^1 (T_\epsilon^\lambda(x, y))\nu(\epsilon)d\epsilon \tag{2.5}$$

This is the model as described in [2]. We note that this can be written in recursive notation as follows: if we have the current state of the wealth distribution as (X_n, Y_n), then the wealth distribution after one transaction (i.e. at time $n + 1$) has the form

$$(X_{n+1}, Y_{n+1}) = (X_n + (1 - \lambda)\epsilon_{n+1}(X_n + Y_n), Y_n + (1 - \lambda)\epsilon_{n+1}(X_n + Y_n)) \tag{2.6}$$

where the ϵ_n are i.i.d. according to $\nu(\epsilon)$.

2.4. Stationary product measures

When inspecting wealth distribution we look at the stationary distributions that could be formed from a given transition operator. The idea behind this is that under a Markov Process, a distribution would be *stationary* if it does not change throughout time. This is usually seen as taking the limit $t \to \infty$ over the process. The resulting process is of great interest, if it exists at all. We want the change in time to be irrelevant, meaning that the change from one state to the next after an exponential waiting time should not influence the wealth distribution; transactions can keep occurring without the wealth being distributed any differently. In terms of our process, this means that after a transaction there is no change. Thus for our transition operator this means that for the wealth distribution $\mu(x, y)$:

$$\int P^\lambda f(x, y)\mu(x, y)dxdy = \int f(x, y)\mu(x, y)dxdy$$

for all bounded and continuous functions f. Perhaps a more intuitive form of stationary measures is obtained using the recursive notation of the process. For a stationary process, if we have $(X_0, Y_0) = \mu$, then for all $i \in \mathbb{N}$, $(X_i, Y_i) = \mu$. This once again states that through time the distribution of the process stays the same. Another name for stationary measures is *invariant* measures.

A specific type of (stationary) measure that we are interested in is a *product* measure. These are measures of the form $\mu(x, y) = \mu(x)\mu(y)$. These are the type of distributions we look for in the wealth distribution model. The motivation behind this is as follows; while a two agent model is relatively simple and thus easy to interpret, the real life applications are limited. Expansions to a model with N agents is possible, however the distribution if such a model is not necessarily comparable to that of two agents. However if we have product measures, we see that as $\mu(x, y) = \mu(x)\mu(y)$, we can easily extend this to, for example, $\mu(x, y, z) = \mu(x)\mu(y)\mu(z)$. Thus for an N agent model, the distribution of a two agent model would be directly applicable. Because of this we focus primarily on finding product stationary measures for the wealth distribution.

2.5. Relevant theorems

Now that we have fully laid a mathematical foundation regarding the wealth distribution of two agents as done in [2] we can continue with some results from this paper. Naturally not all results are mentioned here, but there are a few that pertain to our own goals of analyzing the more complicated model of taking random propensity, as well as a class of functions derived in [2]. Showing these results here allows us to use them later on.

We define a new variable $r = \frac{x}{x+y} = \frac{x}{s}$ and call it the relative wealth of agent 1. Note that as we have used the same information as the pair (x, y) in this change of variables, (r, s) follows the same process as the wealth configuration (x, y) as described in previous sections. Thus seeing as we take s to be constant, $(r_n, s_n) = (r_n)$ then defines a Markov process. In terms of the transition operator we then get for all bounded and continuous f:

$$Pf(r) = \int f(\lambda r + (1 - \lambda)\epsilon)\nu(\epsilon)d\epsilon \tag{2.7}$$

The recursive form is given by

$$r_{n+1} = \lambda r_n + (1 - \lambda)\epsilon_n$$

This recursion represents the transaction after an exponential waiting time; r_{n+1} is the relative wealth of agent 1 after the transaction , r_n is the relative wealth of agent 1 before the transaction, and ϵ_n is the redistributed according to $\nu(\epsilon)$. Solving for this recursion gives the following stationary measure

$$\epsilon_\infty^\lambda = \sum_{n=0}^{\infty} \lambda^n (1 - \lambda)\epsilon_n \tag{2.8}$$

A result from [2] is then that a random variable of the form 2.8 has stationary measures of the form $(\epsilon_\infty^\lambda S, (1 - \epsilon_\infty^\lambda)S)$, for some non-negative random variable S. Furthermore, given $\lambda > 0$ there are no product stationary measures for this distribution.

It should be noted that in this paper we work with an agent independent propensity, meaning that we take the propensity per transaction to be the same for both agents. Other more realistic models could include that there are two propensities, λ_1 and λ_2, one for each agent per transaction. However this would complicate computations significantly and thus we will not be pursuing such methods. For results concerning such models, refer to [2].

One particular class of stationary distributions that was deduced in [2] is where the propensity is taken to be 0. If we take the redistribution measure $\nu(\epsilon)$ to be Beta(α, β) distributed, i.e.

$$\nu(\epsilon) = \frac{\epsilon^{\alpha-1}(1-\epsilon)^{\beta-1}}{B(\alpha, \beta)}$$

then we necessarily have a product stationary wealth distribution $\mu(x, y) = \mu(x) \times \mu(y)$ of Gamma$(\zeta_1, \omega)\times$ Gamma(ζ_2, ω), i.e.

$$\mu(x) = \frac{\omega^\zeta}{\Gamma(\zeta)} x^{\zeta-1} e^{-\omega x}$$

Furthermore, this is the only stationary product measure for an s-independent redistribution measure: $\nu(\epsilon, s) = \nu(\epsilon)$. When looking at heavy tailed distributions later on, this is the set of wealth measures that will be analyzed. For more information regarding these particular types of redistribution measures and their connection to the corresponding stationary product measures, refer to [2].

3. Random propensity

We now have a strong mathematical definition for the wealth distribution between two agents. In the previous section we were restricted to a constant propensity to save of both agents. This section focuses on the expansion of this model to the propensity to save to be random in each transaction. This means that we let λ be distributed in a certain way, where each transaction will have an i.i.d. instance of this distribution for λ.

3.1. New transition operator

Taking random λ does not have an extremely large influence on the model itself, however mathematically we do need to make some changes. Previously the redistribution process was defined by the transition operator 2.5 and had the following expansion

$$\begin{aligned} P^{\lambda} f(x,y) &= \int_0^1 f(T_\epsilon^\lambda(x,y))\nu(\epsilon)d\epsilon \\ &= \int_0^1 f((\lambda x + (1-\lambda)\epsilon(x+y), \lambda y + (1-\lambda)(1-\epsilon)(x+y)))\nu(\epsilon)d\epsilon \end{aligned}$$

In the expanded version above, we see that we integrate over ϵ which has a wealth redistribution measure $\nu(\epsilon)$. However when considering random λ, this alone does not suffice; λ now has a distribution of its own which requires integration as well.

For the propensity to save λ we assume the distribution $\phi(\lambda)$ so as to keep things general. Naturally since we cannot have a negative propensity we assume this distribution takes only non-negative values. We can then define the new transition operator as follows

$$Kf(x,y) = \int_0^1 \int_0^1 f(T_\epsilon^\lambda(x,y))\nu(s,\epsilon)\phi(\lambda)d\epsilon d\lambda \tag{3.1}$$

Furthermore it can be noted the requirement for stationary measures does not change, however the resulting equation would require integrating over 3 measures, which is a considerably more difficult task.

We now have a two agent model for wealth distribution with the propensity to save of both agents being randomly distributed according to $\phi(\lambda)$. We are interested in finding out whether stationary measures for this redistribution process exist, and what these may entail. Due to the complicated nature of using two arbitrary distributions (i.e. the distribution of ϵ and the distribution of λ) we will focus on certain specific examples to draw results. In particular, using common distributions such as uniform or beta distributions leads to comparable results as derived in [2].

3.2. Stationary measure of the relative wealth distribution

We take the same approach as done in [2] with the relative wealth variables (r, s). Naturally we assume that ϵ_n and λ_n are instances of their respective distributions, with both ϵ_n and λ_n being i.i.d. for all $n \in \mathbb{N}$. Furthermore we assume that each ϵ_n and λ_n is independent for all $n \in \mathbb{N}$. This gives a helpful result in terms of indexing a sequence of i.i.d. instances between the distributions: if we have two initial sequences $(\lambda_1, \lambda_2, \dots)$ and $(\epsilon_1, \epsilon_2, \dots)$, then for indexing sequences $(\sigma(1), \sigma(2), \dots)$ and $(\rho(1), \rho(2), \dots)$ we have the following equivalence

$$(\lambda_1, \lambda_2, \dots, \epsilon_1, \epsilon_2, \dots) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \epsilon_1, \epsilon_2, \dots) \tag{3.2}$$

$$= (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \epsilon_{\rho(1)}, \epsilon_{\rho(2)}, \dots) \tag{3.3}$$

Given the transition operator defined in 3.1, we then have the following recursion for the relative wealth of agent 1, r:

$$r_{n+1} = \lambda_{n+1} r_n + (1 - \lambda_{n+1})\epsilon_{n+1} \tag{3.4}$$

This recursion equation represents the same transaction qualities as that in described in the previous section. We state the following theorem regarding this recursion:

Theorem 1. *The stationary distribution generated by the recursion 3.4 is given by*

$$r_\infty = \sum_{n=0}^{\infty} (1 - \lambda_n) \bigg(\prod_{j=0}^{n-1} \lambda_j \bigg) \epsilon_n \tag{3.5}$$

Proof. Iterating backwards from 3.4 we get

$$\begin{aligned}
r_{n+1} &= \lambda_{n+1} r_n + (1 - \lambda_{n+1})\epsilon_{n+1} \\
&= \lambda_{n+1}(\lambda_n r_{n-1} + (1 - \lambda_n)\epsilon_n) + (1 - \lambda_{n+1})\epsilon_{n+1} \\
&= \lambda_{n+1}(\lambda_n(\lambda_{n-1} r_{n-2} + (1 - \lambda_{n-1})\epsilon_{n-1}) + (1 - \lambda_n)\epsilon_n) + (1 - \lambda_{n+1})\epsilon_{n+1} \\
&= (1 - \lambda_{n+1})\epsilon_{n+1} + \lambda_{n+1}(1 - \lambda_n)\epsilon_n + \lambda_{n+1}\lambda_n(1 - \lambda_{n-1})\epsilon_{n-1} \\
&+ \lambda_{n+1}\lambda_n\lambda_{n-1}(1 - \lambda_{n-2})\epsilon_{n-2} \dots + (\lambda_{n+1} \dots \lambda_2)(1 - \lambda_1)\epsilon_1 + (\lambda_{n+1} \dots \lambda_1) r_1
\end{aligned}$$

We apply the following indexing re-sequencing to λ_n and ϵ_n: $(1, 2, \dots, n+1) \to (n, n-1, \dots, 0)$. We then see from 3.2 and 3.3 that the following is well defined

$$\begin{aligned}
r_{n+1} &= (1 - \lambda_0)\epsilon_{n+1} + \lambda_0(1 - \lambda_1)\epsilon_n \lambda_0\lambda_1(1 - \lambda_2)\epsilon_{n-1} \\
&+ \lambda_0\lambda_1\lambda_2(1 - \lambda_3)\epsilon_{n-2} \dots + (\lambda_0 \dots \lambda_{n-1})(1 - \lambda_n)\epsilon_1 + (\lambda_0 \dots \lambda_n) r_1 \\
&= (1 - \lambda_0)\epsilon_0 + \lambda_0(1 - \lambda_1)\epsilon_1 \lambda_0\lambda_1(1 - \lambda_2)\epsilon_2 \\
&+ \lambda_0\lambda_1\lambda_2(1 - \lambda_3)\epsilon_3 \dots + (\lambda_0 \dots \lambda_{n-1})(1 - \lambda_n)\epsilon_n + (\lambda_0 \dots \lambda_n) r_1
\end{aligned}$$

In this form we can take the limit $\lim_{n\to\infty} r_{n+1}$. We notice that since $\lambda_n \in (0, 1)$ for all $n \in \mathbb{N}$, the last term in sequence above tends to 0 as $n \to \infty$; we conveniently do not have to worry about the r_1 factor. Thus we can conclude that with λ_n, ϵ_n being i.i.d. with their respective distributions $\phi(\lambda)$, $\nu(\epsilon)$ and independent from each other for all $n \in \mathbb{N}$, the stationary distribution of the recursive formula in 3.4 is given by 3.5 □

The underlying wealth distribution of a model with redistribution measure $\nu(\epsilon)$ and propensity to save with a measure of $\phi(\lambda)$ converges to the stationary form of the one described in 3.5. Note that at this point very little is known about these distributions; their underlying assumptions mentioned above and that they take non-negative values. Furthermore we can see that if we take λ to be constant in 3.5 it turns into 2.8, thereby further solidifying our argument.

3.3. Uniform ϵ and λ

While the closed form of the stationary relative wealth distribution 3.5 is helpful, it does not give a lot of information on the stationary distribution of the actual wealth distribution (x, y), or in particular whether there are product stationary distributions. In this section we look at the simplest case of our model: we take both the redistribution measure and the propensity measure to be uniform. We manipulate these assumptions into ruling out certain common distributions that 3.5 could not uphold.

We first have an important theorem regarding the stationary recursion from the previous section

Theorem 2. *The invariant distribution generated by 3.5, where the propensity to save λ has distribution $\phi(\lambda)$, has for its moment generating function $M_\infty(t)$ the following identity*

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda)\phi(\lambda)\frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)}d\lambda \tag{3.6}$$

Proof. We begin with taking only the redistribution measure to be uniform, i.e. $\nu(\epsilon) = 1$. For now we assume that λ still has an arbitrary distribution $\phi(\lambda)$. Then for the moment generating function we have

$$M_{r_{n+1}}(t) = M_{n+1}(t) = \mathbb{E}(e^{tr_{n+1}}) \tag{3.7}$$

This becomes

$$\begin{aligned} M_{n+1}(t) &= \int_0^1 \int_0^1 \mathbb{E}(e^{t(\lambda r_n + (1-\lambda)\epsilon})\nu(\epsilon)\phi(\lambda)d\epsilon d\lambda \\ &= \int_0^1 \int_0^1 \mathbb{E}(e^{t(\lambda r_n + (1-\lambda)\epsilon})\phi(\lambda)d\epsilon d\lambda \\ &= \int_0^1 \int_0^1 \mathbb{E}(e^{t\lambda r_n})e^{t(1-\lambda)\epsilon}\phi(\lambda)d\epsilon d\lambda \end{aligned}$$

The decomposition of the expected value over two different measures is justified as we have taken them to be independent. We see that $\mathbb{E}(e^{t\lambda R_n}) = M_n(t\lambda)$ is contained within the integral, meaning we can define a type of recursive equation

$$M_{n+1}(t) = \int_0^1 \int_0^1 M_n(t\lambda)e^{t(1-\lambda)\epsilon}\phi(\lambda)d\epsilon d\lambda \tag{3.8}$$

Now we need to integrate over λ and ϵ; ϵ is only apparent in a minor part of the integrand so we shall start there. We use

$$\int_0^1 e^{t(1-x)y}dy = \frac{e^{(1-x)t} - 1}{t(1-x)} \tag{3.9}$$

As $M_n(t\lambda)\phi(\lambda)$ is not dependent on ϵ, we can apply 3.9 directly to 3.8 to obtain

$$M_{n+1}(t) = \int_0^1 M_n(t\lambda)\phi(\lambda)\frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)}d\lambda \tag{3.10}$$

In 3.10 we now have a recursive relation of the moment-generating function of r_{n+1}. Seeing that we know from 3.5 that r_∞ exists and is defined under the current conditions, we are justified in taking the limit as $n \to \infty$ in $M_{n+1}(t)$. The required identity follows. □

Applying the assumption to 3.6 that λ is uniformly distributed as well, meaning that $\phi(\lambda) = 1$, produces the following identity

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda) \frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)} d\lambda \tag{3.11}$$

This is the relation that the moment generating function of the stationary distribution of the relative wealth must satisfy in the case that ϵ and λ are uniformly distributed.

3.4. Numerical approximation

We now have the general framework for the type of stationary measures one would be looking for in our model. In this subsection we show some numerical methods that lead to graphical representation of these distributions. While exact results cannot be obtained this way, it does give the general direction that these distributions would be taking. From here we know what attributes these distributions might have and can thus start looking for analytic answers in those areas. All the simulations run and approximations made in this section were made using MATLAB. Further information as to how these were produced can be found in Appendix A.

3.4.1. Probability density functions

We are interested in the resulting wealth distribution after a large number of transactions. In this simulation we let 10000 transactions take place according to transaction operator 3.1, where ϵ and λ are distributed in several ways.

We first look at both measure being uniformly distributed. Using a kernel density estimator, the following graph was produced.

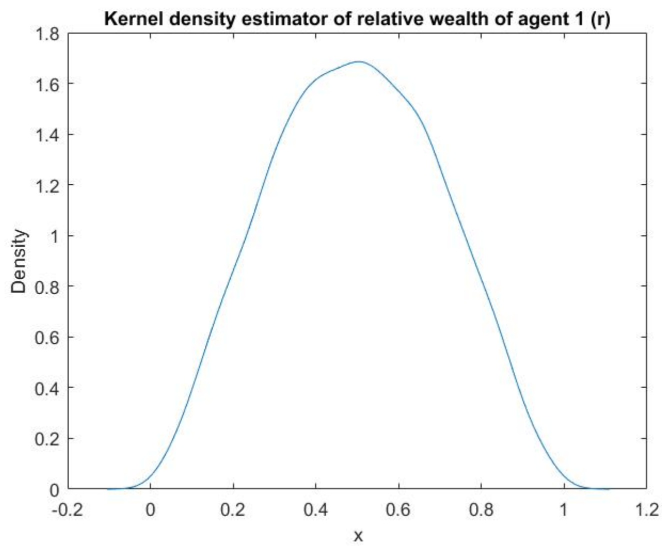

Figure 3.1.: Kernel density estimator with ϵ and λ uniformly distributed

Based on the simulated transaction data we also let MATLAB attempt to fit a Beta distributed density. This gives an idea as to whether uniform λ and ϵ are likely to lead to a wealth measure with a type of Beta distribution. The Beta distribution of best fit had parameters (2.44383, 2.48190), whose probability density function is graphed below

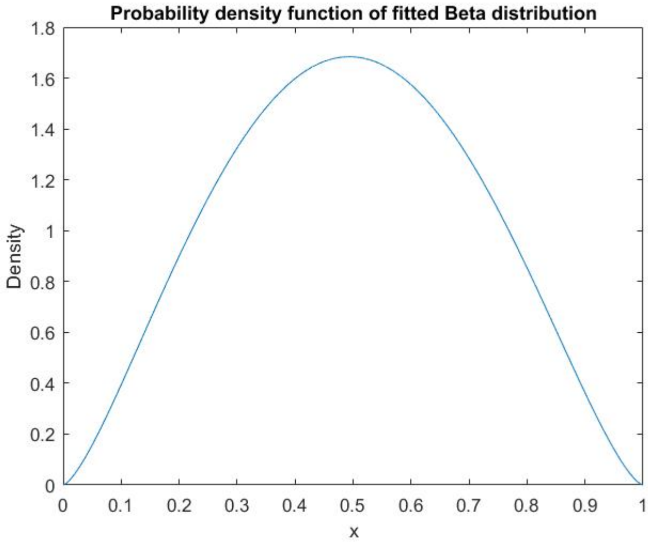

Figure 3.2.: Probability density function of a Beta(2.44383, 2.48190) distributed random variable

While the kernel density estimator and the Beta fitted probability density function look significantly different, when taking a goodness of fit using the **chi2gof** function in MATLAB the null hypothesis that the data fits this Beta(2.44383, 2.48190) distribution is not rejected at significance level 0.01.

This numerical result shows that it seems unlikely that the wealth distribution would have a Beta distribution given the fact that ϵ and λ were uniformly distributed. We try a different numerical approach through the moment generating functions of the wealth distribution. For more information regarding this numerical integration and the graphs produced, refer to the Appendix.

3.4.2. Moment generating functions

In 3.11 we saw a requirement for the moment generating function of the stationary wealth distribution. While this involves several complicated integrals, we can show that given the moment generating functions of the Uniform and Beta distributions, numerically the left and right hand side to not approach each other. We begin with assuming that wealth distribution is uniformly distributed, meaning that

$$M_\infty(t) = \frac{e^t - 1}{t} \tag{3.12}$$

Then from 3.11 we numerically approximate the following integral

$$\int_0^1 M_\infty(t\lambda)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda = \int_0^1 \frac{e^{t\lambda}-1}{t\lambda}\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda \tag{3.13}$$

From here $M_\infty(t)$ as in 3.12 was computed for various t. The error between these two is graphed below, first for values for $t \in [1, 10]$ and second for $t \in [1, 100]$

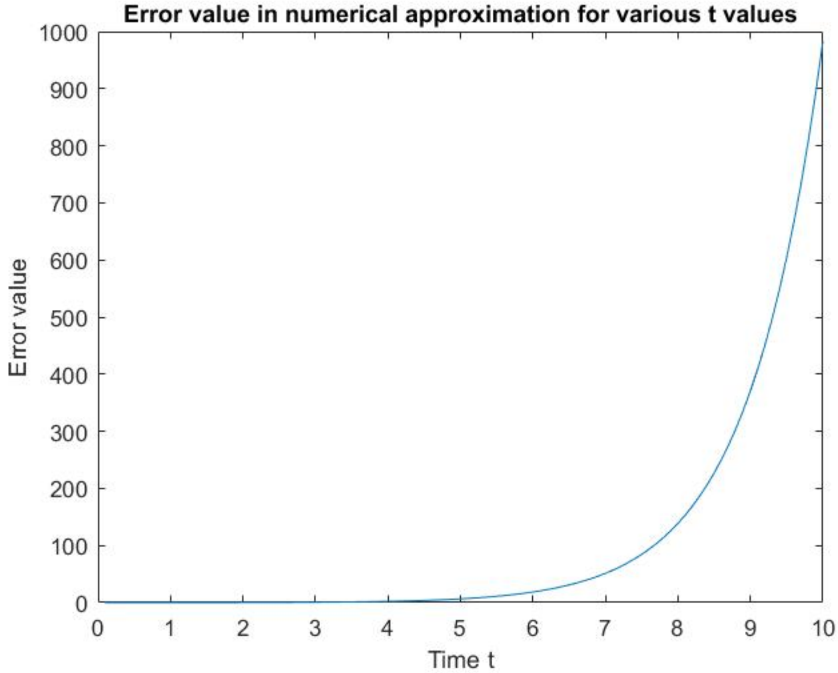

Figure 3.3.: Error value of 3.11 for $t \in [1, 10]$

As can be seen the error seems to be minimal for the first few t values, however starts to climb exponentially afterwards. This can be observed in the graph below

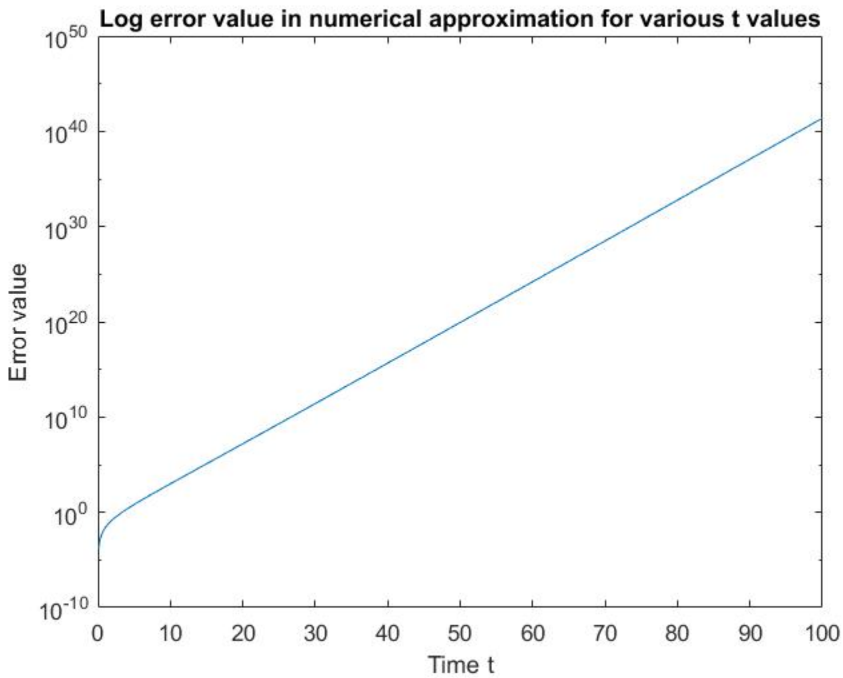

Figure 3.4.: Log error value of 3.11 for $t \in [1, 100]$

A logarithmic scale was chosen in order to more accurately show the error. Therefore it seems unlike the wealth distribution is uniformly distributed in this case.

We now look at the possibility that the wealth distribution is Beta(α, α) distributed given ϵ, λ uniform. The method is as in the uniform case: were the wealth distribution Beta distributed, its probability density functions would be of the form

$$f_X(x) = \frac{(x(1-x))^{\alpha-1}}{B(\alpha,\alpha)} \tag{3.14}$$

Here we have restricted ourselves to taking $\beta = \alpha$. From the definition of the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$ we therefore see that we must numerically validate the following equality

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda$$
$$\int_0^1 e^{t\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}}{B(\alpha,\alpha)}d\lambda = \int_0^1\int_0^1 e^{t\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}}{B(\alpha,\alpha)}d\lambda\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda$$

We notice that on the right hand side we have a double integral in the same measure, meaning that this becomes

$$\int_0^1 e^{t\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}}{B(\alpha,\alpha)}d\lambda = \left(\int_0^1 e^{t\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}}{B(\alpha,\alpha)}d\lambda\right)\left(\int_0^1 \frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda\right) \tag{3.15}$$

We numerically integrate the integrals in 3.15 for various values of t as well as α. We know that the moment generating function should apply for all $t \in R$. Therefore we have taken the α with the lowest error value for all respective t values, instead of making a 2D plot with all the values. These error values were then graphed and provide a lower boundary on the error for the numerical approximation, which is graphed below.

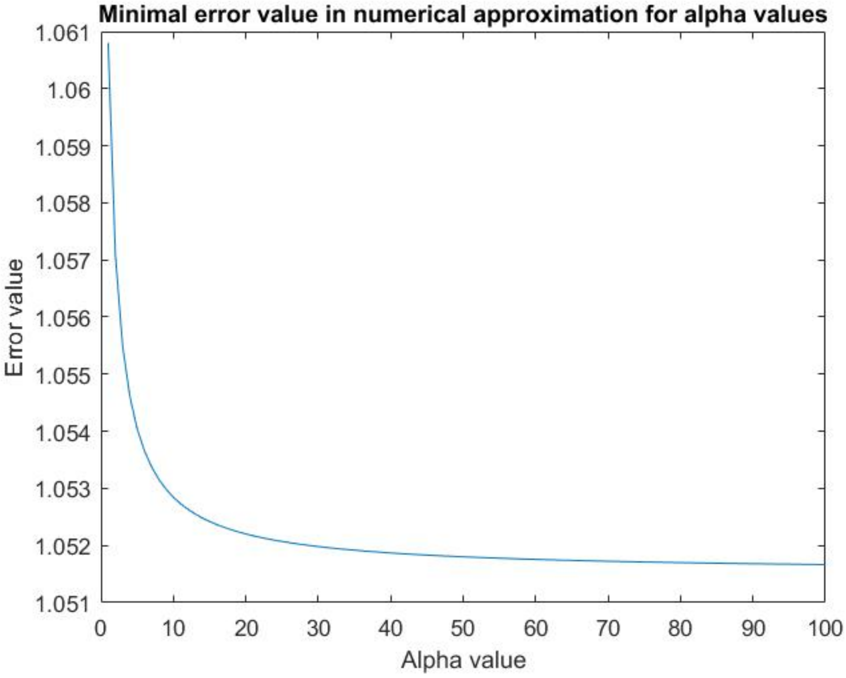

Figure 3.5.: Error value of 3.15 for $t \in [1, 10]$, $\alpha \in [1, 100]$

We see that while the error boundary has gone down through the values for α, it appears to be reaching some sort of limit that does not equal to 0. This again suggests that given a uniformly distributed ϵ and λ, the stationary wealth distribution cannot be of the form $\text{Beta}(\alpha, \alpha)$. For more information regarding this numerical integration and the graphs produced, refer to the Appendix.

3.5. Uniform and Beta(α, β) distributions

From the approximations in the previous section we suspect that given uniform redistribution law ϵ and propensity to save λ, we cannot have the wealth distribution be either uniformly distributed or $\text{Beta}(\alpha, \beta)$ distributed. We show in this section that this is indeed the case by evaluating the requirement 3.11 for the moment generating function. Seeing that uniqueness of distributions is preserved in these functions, showing that the right hand side of 3.11 is unequal to the known moment generating functions of these distributions shows that the stationary wealth distribution is not of this form.

3.5.1. Uniform wealth distribution

In this subsection we state and prove the following theorem which confirms the conjecture formed by the numerical analysis.

Theorem 3. *Given that the redistribution measure ϵ and propensity to save λ are both uniformly distributed, the stationary relative wealth distribution generated by 3.5 cannot be uniformly distributed.*

Proof. We begin with assuming that the wealth distribution is uniformly distributed. We then find a contradiction in 3.11. Given a uniform wealth distribution the moment generating function would be of the form 3.12. We see that if we take the infinite series representation, we get the following for real c

$$\frac{e^c - 1}{c} = \sum_{k=1}^{\infty} \frac{c^{k-1}}{k!} \tag{3.16}$$

Then using 3.12 and 3.16 on the right hand side of 3.11, we obtain the following representation

$$\begin{aligned}
M_\infty(t) &= \int_0^1 M_\infty(t\lambda) \frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)} d\lambda \\
&= \int_0^1 \frac{e^{t\lambda} - 1}{t\lambda} \frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)} d\lambda \\
&= \int_0^1 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(t\lambda)^{k-1}}{k!} \frac{(t(1-\lambda))^{n-1}}{n!} d\lambda \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k-1}}{k!} \frac{t^{n-1}}{n!} \int_0^1 \lambda^{k-1} (1-\lambda)^{n-1} d\lambda
\end{aligned}$$

The integral at the end of the summation above looks familiar, and for good reason seeing that for a Beta(α, β) distributed random variable, we have have probability density function 3.14 where $B(\alpha, \beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$, the Beta function. We know that integrating this function over $(0, 1)$ gives 1, thus we simplify

$$\begin{aligned}
M_\infty(t) &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k-1}}{k!} \frac{t^{n-1}}{n!} B(k, n) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k-1}}{k!} \frac{t^{n-1}}{n!} \frac{(k-1)!(n-1)!}{(k+n-1)!} \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k+n-2}}{kn(k+n-1)!}
\end{aligned}$$

In order to rid ourselves of the double infinite summation we apply a change of variables described below

$$\begin{aligned}
k + n &= v \\
k &= v - n \\
n &= v - k
\end{aligned}$$

We then get an equivalent representation using a single infinite sum of finite sums

$$M_\infty(t) = \sum_{v=2}^{\infty} \frac{t^{v-2}}{(v-1)!} \sum_{k=1}^{v-1} \frac{1}{k(v-k)} \tag{3.17}$$

In 3.17 we now have a single infinite sum representation for the moment generating function $M_\infty(t)$. Recall that in 3.12 we have another infinite sum representation for this function, meaning

that these sums should have equal terms.
1st term
We look at the first term of 3.12 and 3.17 and see that they are equal

$$\frac{t^{1-1}}{1!} = 1 = \frac{t^{2-2}}{(2-1)!}\left(\frac{1}{1(2-1)}\right)$$

2nd term
At the second term we see that for 3.12 we have

$$\frac{t^{2-1}}{2!} = \frac{t}{2},$$

and for 3.17 we have

$$\frac{t^{3-2}}{(3-1)!}\left(\frac{1}{1(3-1)} + \frac{1}{2(3-2)}\right) = \frac{t}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{t}{2}$$

3rd term
Here we see a discrepancy. For 3.12 we have

$$\frac{t^{3-1}}{3!} = \frac{t^2}{6},$$

and for 3.17 we have

$$\frac{t^{4-2}}{(4-1)!}\left(\frac{1}{1(4-1)} + \frac{1}{2(4-2)} + \frac{1}{3(4-3)}\right) = \frac{t^2}{6}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{3}\right) = \left(\frac{11}{12}\right)\frac{t^2}{6}$$

Thus from the terms of the infinite series in 3.12 and 3.17 being unequal, we see that 3.11 does not hold. Therefore because the moment generating function of the uniform distribution does not follow the moment generating function of the wealth distribution, we can conclude that they are distributed differently.

□

3.5.2. Beta(α, β) wealth distribution

We now look at the case where we assume the wealth distribution is Beta(α, β) distributed. While the uniform distribution in the previous section is a special case of this ($\alpha = 1$, $\beta = 1$), there might still be other parameters α, β for which 3.11 is valid. The following theorem shows that there are no such α, β.

Theorem 4. *Given that the redistribution measure ϵ and propensity to save λ are both uniformly distributed, the stationary relative wealth distribution generated by 3.5 cannot be Beta(α, β) distributed for any $\alpha, \beta \in \mathbb{R}$.*

Proof. We start by first assuming that for certain $a_n \in \mathbb{R}$ with $n = 0, 1, 2 \ldots$, the moment generating function has the form

$$M_\infty(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \tag{3.18}$$

We then apply 3.11 and 3.16 and produce the following equation

$$
\begin{aligned}
M_\infty(t) &= \int_0^1 M_\infty(t\lambda)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda \\
&= \int_0^1 \left(\sum_{k=0}^{\infty} a_k \frac{(t\lambda)^k}{k!}\right)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda \\
&= \int_0^1 \left(\sum_{k=0}^{\infty} a_k \frac{(t\lambda)^k}{k!}\right)\left(\sum_{n=1}^{\infty} \frac{(t(1-\lambda))^{n-1}}{n!}\right)d\lambda
\end{aligned}
$$

We can now apply a similar method to that used in the previous section with the uniform wealth distribution. Rearranging gives

$$
\begin{aligned}
M_\infty(t) &= \int_0^1 \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} a_k \frac{(t\lambda)^k}{k!}\frac{(t(1-\lambda))^{n-1}}{n!}d\lambda \\
&= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} a_k \frac{t^{k+n-1}}{k!n!}\int_0^1 \lambda^k(1-\lambda)^{n-1}d\lambda \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k \frac{t^{k+n}}{k!(n+1)!}\int_0^1 \lambda^k(1-\lambda)^{n}d\lambda
\end{aligned}
$$

We once again use the fact that integrating 3.14 over [0,1] is equal to 1 and that for the Beta function we have $B(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$, giving

$$
\begin{aligned}
M_\infty(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k \frac{t^{k+n}}{k!(n+1)!}\beta(k+1,n+1) \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k \frac{t^{k+n}}{k!(n+1)!}\frac{k!n!}{(k+n+1)!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k \frac{t^{k+n}}{(k+n+1)!(n+1)}
\end{aligned}
$$

The same change in notation with the following substitution is applied in order to get rid of the double infinite summation

$$
\begin{aligned}
k+n &= v \\
k &= v-n \\
n &= v-k
\end{aligned}
$$

This then gives following infinite sum for the moment generating function of the wealth equation

$$
M_\infty(t) = \sum_{v=0}^{\infty}\frac{t^v}{(v+1)!}\sum_{k=0}^{v}\frac{1}{v-k+1}a_k \tag{3.19}
$$

Recall we assumed that the wealth distribution is Beta(α,β) distributed and thus has moment generating function of the form

$$
M_\infty(t) = 1 + \sum_{k=0}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}
$$

We make a change in notation in order to work more comfortably in the field of infinite series. The product operator $\prod_{n=0}^{k} x_n$ is usually only defined for all $k \in \mathbb{Z}^+$. We now add a definition for the term at $k = -1$ as simply an empty product; i.e. it is equal to 1. Due to this change in notation and the fact that $\frac{t^0}{0!} = 1$ we rewrite

$$M_\infty(t) = \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r} \right) \frac{t^k}{k!} \tag{3.20}$$

This means that the coefficients a_n need to be of the form

$$a_k = \prod_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r}$$

We now compare the first few terms of the infinite series representation of 3.19 and 3.20 to see that there are no parameters α, β for which they are equal. This comes down to the following equality's in v

$$\prod_{k=0}^{v-1} \frac{\alpha + k}{\alpha + \beta + k} = \frac{1}{v+1} \sum_{k=0}^{v} \frac{1}{v+1-k} \prod_{s=0}^{v-1} \frac{\alpha + s}{\alpha + \beta + s}$$

or equivalently

$$\frac{1}{v+1} \sum_{k=0}^{v} \frac{1}{v+1-k} \prod_{s=k}^{v-1} \frac{\alpha + \beta + s}{\alpha + s} = 1$$

We compare these for the first 3 terms in v

$\underline{v{=}0}$

$$\frac{1}{1}\frac{1}{1}(1) = 1$$

$\underline{v{=}1}$

$$\frac{1}{2}\left(\frac{1}{2}\frac{\alpha+\beta}{\alpha} + \frac{1}{1}\right) = 1$$

This is only the case when $\alpha = \beta$. We continue with this requirement to the third term.

$\underline{v{=}2}$

$$\frac{1}{3}\left(\frac{1}{3}\frac{\alpha+\beta}{\alpha}\frac{\alpha+\beta+1}{\alpha+1} + \frac{1}{2}\frac{\alpha+\beta+1}{\alpha+1} + \frac{1}{1}\right) = 1$$

Taking the requirement from the previous term $\alpha = \beta$, we get an answer for $\alpha = 3$ where this is equal.

v=3

$$\frac{1}{4}\left(\frac{1}{4}\frac{\alpha+\beta}{\alpha}\frac{\alpha+\beta+1}{\alpha+1}\frac{\alpha+\beta+2}{\alpha+2}+\frac{1}{3}\frac{\alpha+\beta+1}{\alpha+1}\frac{\alpha+\beta+2}{\alpha+2}+\frac{1}{2}\frac{\alpha+\beta+2}{\alpha+2}+\frac{1}{1}\right)=\frac{31}{30}\neq=1$$

We see that for $\alpha = \beta = 3$ the term does not satisfy the requirement. From the term by term expansion of infinite series we can thus see that the moment generating function of the Beta distribution does not satisfy 3.11. Finally we can conclude that given a uniformly distributed propensity λ and redistribution ϵ, the wealth distribution cannot be Beta distributed. □

Corollary 1. *When the redistribution ϵ and propensity to save are both uniformly distributed there cannot be stationary product measure for the two agent model*

Proof. The result follows directly from the result of [2] as described in previous sections. □

It should be noted that the term comparison done in this section is different from that in the numerical approximations done in previous sections. Here we have looked at the infinite expansion of the moment generating function and determined term by term that these cannot be the same. However in our numerical approximation we numerically estimated what the difference in 3.11 would be for various values of $t \in \mathbb{R}$. There was no infinite series expansion and was thus a completely different computation whose purpose was to examine how the difference in 3.11 would act.

3.6. Uniform ϵ and Beta λ

Having investigated the stationary wealth distribution with s-independence for ϵ and λ being distributed uniformly, we now turn our attention to a different example; how would the wealth distribution fare when we once again take a uniform redistribution measure ϵ, but now with a Beta(z_1, z_2) distributed propensity to save λ.

We note that the deductions produced in the previous section are a specific case of the current example (where we take $(z_1, z_2) = (1, 1)$), so we take a similar approach and develop a requirement for which the s-independent wealth distribution is Beta(α, β) distributed. We start by once again stating the requirement for the stationary wealth distribution shown in 3.6 with an unknown λ distribution $\phi(\lambda)$:

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda)\phi(\lambda)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda$$

We have taken λ to be Beta(α, β) distributed, meaning that the measure for λ is of the form

$$\phi(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}$$

Thus the requirement for the moment generating function of the stationary wealth distribution in this specific case becomes

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda)\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda \tag{3.21}$$

We now apply the same infinite series expansion techniques shown in the previous section to give a requirement for the the wealth distribution to be Beta(z_1, z_2) distributed. We know that the moment generating function of such a distribution would have the form

$$M_\infty(t) = \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{t^k}{k!}$$

Here we have once again defined the empty product in $k = -1$ to be equal to 1. Thus 3.21 turns into

$$\begin{aligned}
M_\infty(t) &= \int_0^1 M_\infty(t\lambda) \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} \frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)} d\lambda \\
&= \int_0^1 \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{(t\lambda)^k}{k!} \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} \frac{e^{(1-\lambda)t} - 1}{t(1-\lambda)} d\lambda \\
&= \int_0^1 \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{(t\lambda)^k}{k!} \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} \left(\sum_{n=1}^{\infty} \frac{(t(1-\lambda))^{n-1}}{n!} \right) d\lambda \\
&= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{t^k}{k!} \frac{1}{B(\alpha,\beta)} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{t^{n-1}}{n!} \int_0^1 \lambda^{k+\alpha-1}(1-\lambda)^{n+\beta-2} d\lambda \\
&= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k+n-1}}{n!k!} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{B(k+\alpha, \beta+n-1)}{B(\alpha,\beta)}
\end{aligned}$$

To once again use work in the space of a single infinite series we apply the transformation

$$\begin{aligned}
k + n &= v \\
k &= v - n \\
n &= v - k
\end{aligned}$$

This gives the following identity for the wealth distribution

$$M_\infty(t) = \sum_{v=1}^{\infty} t^{v-1} \sum_{k=0}^{v} \frac{1}{(v-k)!k!} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{B(k+\alpha, \beta+v-k-1)}{B(\alpha,\beta)} \tag{3.22}$$

The question then becomes, can the coefficients of the moment generating function of the Beta distribution solve for this identity, or are there z_1, z_2, α, and β such that for all $k \in \mathbb{N}$

$$\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \frac{1}{k!} = \sum_{k=0}^{v} \frac{1}{(v-k)!k!} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{B(k+\alpha, \beta+v-k-1)}{B(\alpha,\beta)} \tag{3.23}$$

This is a analytically complex identity to solve for parameters z_1, z_2, α, and β; however it seems unlikely that parameters of this form exist that would satisfy 3.22. We end the section with the following conjecture regarding this matter.

Conjecture 1. *Given that the redistribution measure ϵ is uniformly distributed and propensity to save λ is Beta(α, β) distributed, the stationary relative wealth distribution generated by 3.5 cannot be Beta(z_1, z_2) distributed for any $\alpha, \beta, z_1, z_2 \in \mathbb{R}$.*

4. Heavy-tailed distributions

In this section we take a step back from the propensity to save of the economic agents and instead look at ways in which the classical energy redistribution model described in [2] can be expanded. This can be done by letting the redistribution model have random (and thus distributed) parameters. Given the right distribution this leads to heavy-tailed distributions, the likes of which have been empirically observed in finance and economics; while the previous norm was to take all distributions to be normal, recent analysis and advances in computing power show that heavy tailed distributions are realistic. For example, [1] states that financial asset returns are now considered to be heavy-tailed. With this assumption, risk in such financial assets is increased as heavy-tailed distributions associated a much higher volatility than those of the normal distribution. Even in an agent based model volatility phenomena associated with heavy-tailed distribution can be observed, as shown in [3]. Therefore it is logical to attempt to expand the agent based model for a closed economy to produce heavy-tailed distributions as well; this is then exactly what will be discussed in this chapter. We look at the case where $\lambda = 0$ and attempt to find stationary product measures with heavy tails.

4.1. The zero-propensity model

In [2] forming a model for the distribution of wealth between two agents was done by observing the existing model for the redistribution of energy on the particle scale in a closed physical system. From this point one introduced the notion of economic agents and saw that these two systems could be modelled in the same way. The adding of propensity was then an extension only applicable to the wealth model. To add the extension of distributed parameters, we first mathematically describe the model.

We now explain the energy distribution model at the hand of the existing wealth distribution model, however one should note that originally this was derived in the opposite manner. The irrelevance in direction shows that the this energy model is also a type of wealth distribution model, given the right assumptions. We thus continue with the notation of economic agents used in our previous model; we have agents with wealth (x, y) respectfully, where $x + y = s$ remains constant. We define the following map for use in the generator: for $T_\epsilon(x, y) : E \to E$ we have

$$T_\epsilon(x, y) = (\epsilon(x + y), (1 - \epsilon)(x + y)) \tag{4.1}$$

Note that $T_\epsilon(x, y) = T_\epsilon^0(x, y)$ as defined in previous chapters. Now for a given distribution measure $v(\epsilon)$ the transition operator for the zero-propensity distribution model becomes, for all bounded continuous function $f(x, y)$

$$Pf(x, y) = \int_0^1 f(T_\epsilon(x, y))\nu(s, \epsilon)d\epsilon \tag{4.2}$$

4.2. Characteristics of agents

From [2] it is known that given the wealth distribution model with zero propensity to save ($\lambda = 0$) and redistribution measure Beta(α_1, α_2), that the resulting stationary wealth measure will be product. Furthermore this is the only case in which stationary product measures occur, with the stationary distribution being Gamma distributed, i.e. $\mu(x, y) = \Gamma(\alpha_1, \beta) \times \Gamma(\alpha_2, \beta)$ for scale parameters β. Here a $\Gamma(\alpha_1, \beta)$ distribution has the following probability density function:

$$\mu(x) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} \tag{4.3}$$

We therefore take interest in this α parameter that seems to be conserved in these product stationary measures. We can see α as a 'characteristic' of the economic agent. With this view it is reasonable to assume that α_1, α_2 are independent for different agents; this inevitably leads to the question of whether they can be seen as randomly distributed and of the form $\psi(\alpha_1\psi(\alpha_2)$. This is we investigate in this section; we ask whether this random distributing of the the agents quality can lead to heavy tailed product measures for the stationary wealth distribution.

We now look at how the model evolves when one takes a randomly distributed shape parameter α. For generality we say that α has distribution $\psi(\alpha)$ with support on $[0, \infty)$. We note that because we have another random variable, the wealth distribution of agent 1 no longer looks like 4.3, but now has the form

$$\mu(x) = \int_0^\infty \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} \psi(\alpha) d\alpha \tag{4.4}$$

Having the general wealth distribution for agent 1 given s-independence and no propensity to save, we show some interesting properties given certain distributions for α.

What we are particularly interested in when it comes to wealth distributions of the kind described in 4.4 is how their moments function. Particularly interesting is the class of distributions that have infinite moments passed a certain point; these are called *heavy-tailed* distributions. One such type of distribution is the Pareto distribution which has been found empirically in the world of finance and economics [9]. Therefore we first take a look at what could possibly produce such heavy tailed distributions.

The q'th moment of a continuous non-negative random variable X with distribution $f_X(x)$ is defined as follows

$$\mathbb{E}(X^q) = \int_0^\infty x^q f_X(x) dx$$

Seeing that the wealth distributions are based on continuous non-negative random variables this is the definition we will use. We then define a heavy tailed distribution as a distribution who for some $q_c \in \mathbb{R}$ has the following quality

$$\int_0^\infty x^q f_X(x) dx = \infty \qquad q > q_c$$

This means that after a certain point the moments of the distribution are infinite as mentioned before.

We apply this to the general wealth distribution for agent 1. For simplicity we take $\beta = 1$ for the scale parameter. We then get the following expression for the q'th moment of agent 1, X_1

$$\begin{aligned}\mathbb{E}(X_1^q) &= \int_0^\infty \int_0^\infty x^q \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \psi(\alpha) d\alpha dx \\ &= \int_0^\infty \int_0^\infty \frac{x^{q+\alpha-1}}{\Gamma(\alpha)} e^{-x} \psi(\alpha) d\alpha dx\end{aligned}$$

The following theorem then defines a class of distributions for α that lead to heavy tailed wealth distributions.

Theorem 5. *The class of distributions for shape parameter α given by*

$$\psi(\alpha) = \alpha^{-k}, \qquad k \in \mathbb{N} \tag{4.5}$$

lead to heavy tailed product invariant measures, given that they follow 4.4.

Proof. We use an alternative representation of the Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ for $z \in \mathbb{R}^+$. Thus integrating over only x the above double integral simplifies to

$$\begin{aligned}\mathbb{E}(X_1^q) &= \int_0^\infty \frac{\Gamma(q+\alpha)}{\Gamma(\alpha)} \psi(\alpha) d\alpha \\ &= \int_0^\infty (q+\alpha-1)(q+\alpha-2)\dots(\alpha)\psi(\alpha)d\alpha\end{aligned}$$

Here we have used the original definition of the Gamma function $\Gamma(z) = (z-1)!$.

Looking at $(q+\alpha-1)(q+\alpha-2)\dots(\alpha)$, we see that this can be expanded to form a polynomial of order q in α; i.e. it is of the form $P(\alpha) = c_1\alpha^q + c_2\alpha^{q-1}\dots + c_q$, where c_i are constants in $\mathbb{R}$ for all $i = 1 \dots q$. We are therefore inclined to look at the following for for the distribution of α: $\psi(\alpha) = \alpha^{-k}$ for some $k \in \mathbb{N}$. This can be seen by substituting this form into the expression for the q'th moment:

$$\begin{aligned}\mathbb{E}(X_1^q) &= \int_0^\infty (c_1\alpha^q + c_2\alpha^{q-1}\dots + c_q)\psi(\alpha)d\alpha \\ &= \int_0^\infty \frac{c_1\alpha^q + c_2\alpha^{q-1}\dots + c_q}{\alpha^k} d\alpha \\ &= \int_0^\infty c_1\alpha^{q-k} + c_2\alpha^{q-k-1}\dots + \alpha^{-k}c_q d\alpha\end{aligned}$$

We know that $\int_0^\infty x^k dx = \infty$ if $k > 0$ and $\int_0^\infty x^k dx \neq \infty$ if $k \leq 0$. Thus we have found a class of distributions for which have heavy tails; if the s-independent wealth of agent 1 is Gamma$(\alpha, 1)$ distributed and α has distribution of the form $\psi(\alpha) = \alpha^{-k}$, $k \in \mathbb{N}$, then for moments $q > k$ have a value of infinite. Therefore this class of distributions is heavy tailed. □

4.3. Random scale parameter

In the previous section we looked at the invariant product measure for the s-independent wealth distribution of agent 1 where the shape parameters α was randomly distributed. The probability distribution function then had the form 4.4. We now let the shape parameter α be constant in 4.3 and instead condition the distribution on the scale parameter β, which we give general distribution $\gamma(\beta)$. This produces the following pdf parallel to 4.4

$$\mu(x) = \int_0^\infty \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} \gamma(\beta) d\beta \tag{4.6}$$

As before we make things simpler by this time letting the shape parameter $\alpha = 1$. Instead of viewing the moment generating function of agent 1, we now continue to work with the probability density function. We first off note that in letting $\alpha = 1$ we have created a conditional exponential distribution in the wealth distribution, i.e.

$$\mu(x) = \int_0^\infty \beta e^{-\beta x} \gamma(\beta) d\beta$$

We now assume that β is Gamma distributed with shape and scale parameters θ, k respectively. This means that $\gamma(\beta)$ is of the form 4.3 with $(\alpha, \beta) = (\theta, k)$. We then rewrite 4.6 as

$$\begin{aligned}\mu(x) &= \int_0^\infty \beta e^{-\beta x} \frac{k^\theta \beta^{\theta-1}}{\Gamma(\theta)} e^{-k\beta} d\beta \\ &= \frac{k^\theta}{\Gamma(\theta)} \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta\end{aligned}$$

We know from the probability distribution of Gamma that the following equality holds

$$\int_0^\infty \frac{\beta^\theta}{\Gamma(\theta+1)} (x+k)^{\theta+1} \beta^\theta e^{-\beta(x+k)} d\beta = 1$$

and so

$$\int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta = \frac{\Gamma(\theta+1)}{\beta^\theta (x+k)^{\theta+1}}$$

Thus the probability density becomes

$$\begin{aligned}\mu(x) &= \frac{k^\theta}{\Gamma(\theta)} \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta \\ &= \frac{k^\theta}{\Gamma(\theta)} \frac{\Gamma(\theta+1)}{\beta^\theta (x+k)^{\theta+1}} \\ &= \frac{\theta k^\theta}{(x+k)^{\theta+1}}\end{aligned}$$

This final density is exactly that of the Lomax Pareto distribution with parameters (θ, k) [5]. This distribution is of Pareto Type II which also has heavy tails. Thus we can conclude that given an conditionally exponential distribution s-independent wealth distribution for agent 1, if one conditions the parameter to be Gamma(θ, k), the original distribution in question is distributed to a Lomax Pareto distribution with parameters (θ, k). One can even take this result further if we take the shape parameter β to be Gamma distributed, which leads to a Generalized Pareto distribution as shown in [4].

While these results appear to be positive, they are in fact not applicable to our original model. Recall that we are manipulating the stationary wealth distribution of agent 1 relative to that of the whole economic system with 2 agents. From [2] we recall that this was justified if the redistribution measure $\nu(\epsilon)$ was Beta(α_1, α_2) distributed, as this would produce invariant product measures of the form $u(x,i) \sim \Gamma(\alpha_i, \beta)$, $i = 1, 2$ with some shape parameter β. However if we condition on the shape parameter instead of the scale parameter, these will no longer be the same for both agents; this means that even if we take wealth distribution of agent 1 to be Gamma distributed, the inconsistency in the shape parameter prevents the translation from being justified. Conditioning over the scale parameters $a_{1,2}$ still produces product measure, so the results in the previous section are justified. While it is unfortunate that the result for $\mu(x) = \frac{\theta k^\theta}{(x+k)^{\theta+1}}$ is not applicable in the form that we have presented here, it does give reason to further investigate this type of distribution.

4.4. The Pareto distribution

As previously mentioned, heavy-tailed distributions are common occurrences when viewing real life data and attempting to find a distribution that fits such data. We see that in particular the class of Pareto distribution is of interest, both from a real life perspective and from that of the previous section. Fitting of empirical data to form a wealth distribution has been attempted already, as shown in [9] and [8]. Getting an agent-based justification for the Pareto distribution is thus a logical choice of wealth distribution. In this section we show the mathematical requirements of the chosen parameters in the model to produce such a wealth distribution.

As before we assume that the s-independent stationary wealth distribution of agent 1 is Gamma distributed with parameters (α, β). As we have seen in the previous sections, the only logical choice of conditioning when it comes to these parameters is the scale parameter α. This is due to results found in [2] pertaining to product stationary solutions in the model. We therefore set the shape parameter to 1 for simplicity. Now as we know, conditioning on the parameter α gives the following probability density function for agent 1 as given in 4.4. We then see that the foll lowing must be satisfied

$$\mu(x) = \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \psi(\alpha) d\alpha = \frac{\theta k^\theta}{(x+k)^{\theta+1}} \tag{4.7}$$

This is the probability density function of the Pareto Lomax distribution as seen in the previous section. Here we have chosen for this particular form of Pareto distribution instead of the Pareto Type 1 to have full support on $[0, \infty)$. These two types of distributions are very similar, the Lomax variant essentially being Pareto Type 1 with a shift to include the support on [0,1] that lacks in the Pareto Type 1 distribution.

The identity in 4.7 is once again quite difficult to solve for general distribution $\psi(\alpha)$, so we shall produce a secondary criteria in which both sides of the equation are more comparable. We note the following in the Pareto Lomax probability density function

$$\frac{\theta k^\theta}{(x+k)^{\theta+1}} = \frac{\Gamma(\theta+1) k^\theta}{\Gamma(\theta)(x+k)^{\theta+1}}$$

Manipulating the integral form of the Gamma function we see that

$$\begin{aligned}\Gamma(\theta+1) &= \int_0^\infty (\beta(x+k))^{\theta+1-1} e^{-\beta(x+k)} d\beta \\ &= (x+k)^\theta \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta\end{aligned}$$

Combining these two equations gives

$$\frac{\theta k^\theta}{(x+k)^{\theta+1}} = \frac{k^\theta}{\Gamma(\theta)(x+k)} \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta$$

thus we can conclude that for the probability density function of the wealth distribution of agent 1 to be Pareto Lomax distributed, we must have equivalence in the distribution of α, $\psi(\alpha)$ in the following identity for the probability density function $\mu(x)$:

$$\int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \psi(\alpha) d\alpha = \frac{k^\theta}{\Gamma(\theta)(x+k)} \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta \tag{4.8}$$

If this applies, then the stationary wealth distribution of agent 1 will have a Pareto Lomax distribution with parameters (θ, k). This does not seem like a unlikely scenario, so we end the section with the following conjecture.

Conjecture 2. *There exist a class of distributions $\psi(\alpha)$ for the shape parameter α such that 4.8 is satisfied and thus the stationary product wealth measure of agent 1 is Pareto Lomax distributed.*

5. Conclusion

In this paper we have studied a two agent model for the wealth distribution in a closed economic system introduced in [2] and extended it in two significant ways concerning the parameters given in the model. These expansions were then analyzed to see how the wealth in this closed economic system would be distributed among the agents and given certain assumptions for the model, what distributions we could exclude from consideration.

The first extension added to the model was to introduce a distributed propensity to save. In the original model for wealth distribution, the transactions between the two agents were based on a redistribution measure ϵ and a constant propensity to save λ. In each transaction, a fraction of the total conserved wealth was redistributed among the two agents according to ϵ. When we took the variable λ to be distributed as well, it meant that the amount of wealth that was being redistributed between the agents was decided randomly for each transaction according to the distribution of λ. We then looked at how the wealth of agent 1 would be distributed after a large number of transactions and assuming that the redistribution measure was independent of the total wealth in the system i.e. we looked for stationary distributions for the wealth of agent 1 relative to the total independent wealth in the system. This newer model and its (potential) stationary distributions were first mathematically defined and then analyzed accordingly.

In the model with random propensity λ, we derived a result concerning the relative wealth distribution of agent 1; through recursion we saw that this had the form

$$r_\infty = \sum_{n=0}^{\infty}(1-\lambda_n)\Big(\prod_{j=0}^{n-1}\lambda_j\Big)\epsilon_n$$

Here the λ_n and ϵ_n are independent instances of the randomly distributed propensity to save λ and redistribution measure ϵ respectfully.

We also examined specific stationary distributions with the assumption that the redistribution measure ϵ was uniformly distributed and the propensity measure λ had density $\phi(\lambda)$. By examining the moment generating function $M_\infty(t)$, we defined a identity that a stationary wealth distribution of this sort has to comply with:

$$M_\infty(t) = \int_0^1 M_\infty(t\lambda)\phi(\lambda)\frac{e^{(1-\lambda)t}-1}{t(1-\lambda)}d\lambda \tag{5.1}$$

After first investigating using numerical methods, we can conclude that given λ uniformly distributed, the stationary wealth distribution of agent 1 cannot be $\mathrm{Beta}(\alpha, \beta)$ distributed for any parameters α, β. As a consequence we saw that for the two agent model no invariant product measure can exist. Furthermore if we take the the the propensity to save to be $\mathrm{Beta}(z_1, z_2)$ distributed, it was shown that the following identity must hold for the moment generating function of the wealth distribution

$$M_\infty(t) = \sum_{v=1}^{\infty} t^{v-1} \sum_{k=0}^{v} \frac{1}{(v-k)!k!} \left(\prod_{r=0}^{k-1} \frac{z_1 + r}{z_1 + z_2 + r} \right) \frac{B(k+\alpha, \beta + v - k - 1)}{B(\alpha, \beta)} \tag{5.2}$$

The second expansion that we considered was where we take the propensity to save to be 0 and focus solely on the case where the redistribution measure ϵ is Beta distributed and once again independent of the total wealth of the system. Based on results from [2], we then know that the stationary wealth distribution of the whole system is comprised of product stationary distributions in the two agents. These product measure are always Gamma(α, β) distributed according to [2]. We then took the parameters (α, β) and distributed them individually and in separate cases (meaning that if α was distributed then β was constant and the other way around). These stationary product measure were mathematically defined and then manipulated to form certain wealth distributions.

In randomly distributing the parameters of the stationary wealth distribution when this is Gamma(α, β), we looked for distributions of these parameters that would lead to heavy-tailed distributions. When taking the shape parameter α to be random, we found a class of distributions that would lead to the wealth distribution of agent 1 to be heavy-tailed; this class was defined as all the distributions of the form $\psi(\alpha) = \alpha^{-k}$ for $k \in \mathbb{N}$. Furthermore it was shown that while taking a random scale parameter β would lead to interesting results, ultimately the restrictions on the model would not allow this for any distribution for β. This did however lead to a requirement for wealth distributions to be Pareto Lomax distributed, given that α is distributed according to $\psi(\alpha)$:

$$\int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \psi(\alpha) d\alpha = \frac{k^\theta}{\Gamma(\theta)(x+k)} \int_0^\infty \beta^\theta e^{-\beta(x+k)} d\beta \tag{5.3}$$

If this applies, then the stationary wealth distribution of agent 1 will have a Pareto Lomax distribution with parameters (θ, k).

The two agent model suggested in [2] is a relatively simple representation of a closed economic system and how the wealth in such a system is distributed. The extensions to this model created in this paper add some variability to the possible parameters by distributing them and show what the effect of this variability to the model as a whole. Furthermore multiple criteria have been proposed such that if the distributed parameters fulfill these criteria, it is clear what type of wealth distribution the model will be able to provide.

6. Discussion

This paper has investigated several methods expanding the model generated in [2] and drawn several conclusions from specific cases of the extended models. However due to time constraints, not all potential results could be pursued in the investigation. This chapter highlights possible extensions of the research and its results.

Based on the identity described in 5.2, we have a requirement for a possible Beta(z_1, z_2) distributed propensity to save λ. It cannot be ruled out that there are not parameters z_1, z_2, α, and β such that the wealth of agent 1 given s-independence is Beta(α, β) distributed. However it does seems unlikely, as posed in conjecture 1. Furthermore any number of moment generating functions of common distributions could be attempted to conclude whether they abide by the identity 5.1; with proper manipulation it may even be possible to construct a class of distributions which always solves this identity.

Using the criteria given in 5.3, one could investigate possible distributions for the shape parameter α in the wealth distribution for agent 1 assuming this is conditionally Gamma(α, β) distributed. In the case that common probability density functions cannot apply, a class of distributions with the appropriate characteristics to satisfy 5.3 could be created based off of the identity alone. Furthermore certain transformations could be investigated in order to solve the integrals given in the identity with success. We thus ask the question: Do distributions exist that satisfy the equations 5.1 and 5.3, and what might these distributions look like?

In this paper we have restricted our queries to the case where we only have two agents in the closed economic system. However in [2] an N-agent model was also introduced. This idea could be extended in the models in this paper, possibly with results concerning duality functions or expected values of the system. For more information regarding such models refer to [2]. Finally many of the methods used in this paper were applied to very specific common distributions in order to produce results. Naturally other common distributions could be used in order to find comparable results; i.e. instead of uniform or Beta distributions as the respective redistribution and propensity to save measures, any number of common distributions could be used. Whether still will provide interesting results is left to the researcher.

A. MATLAB

This appendix contains the MATLAB code used in generating the various graphs shown throughout the report.

Wealth distribution with 2 agents

```
%Wealth distribution model with 2 agents with lambda propensity
clear;
rng(304);
%Parameters
t = 10000; %Amount of transactions
in_wealth_1 = 1000; %initial wealth of first agent
in_wealth_2 = 1000; %initial wealth of second agent
Economy = in_wealth_1 + in_wealth_2; %Total wealth in the economy

agent = zeros(t,2);
agent(1,:) = [in_wealth_1, in_wealth_2];

for i = 1:t
    lambda = betarnd(1,1);  %Amount of propensity
    redist = betarnd(1,1); %Redistribution measure
    agent(i+1,:) = [lambda*agent(i,1)+(1-lambda)*redist*sum(agent(i,:)),
        lambda*agent(i,2)+(1-lambda)*(1-redist)*sum(agent(i,:))];
end

%Estimated probability density function of agent
figure(1);
ksdensity(agent(:,1)/Economy);
title('Kernel density estimator of relative wealth of agent 1 (r)');
xlabel('x');
ylabel('Density');

%Probability density function of fitted Beta distribution
x = 0:1/t:1;
y = fitdist(agent(:,1)/Economy, 'Beta');
figure(2)
plot(x,pdf(y,x))
title('Probability density function of fitted Beta distribution');
xlabel('x');
ylabel('Density');
```

```
%Goodness of fit at significance level 0.01
[h,p]=chi2gof(agent(:,1)/Economy, 'CDF',y, 'Alpha',0.01)
```

Numerical integration for Uniform moment generating function

```
%Numerical integration of M_infinity(t) with R_infinity uniform
%for lambda, epsilon uniform
clear;
dt = 0.1; %timestep
%L = 10; %Largest value of t
L = 100; %Largest value of t

M_uniform = @(t) (exp(t)-1)./t; %Moment generating function of
    unifrom distribution from 0 to 1
M_infinity = @(x,t) (exp(t.*x)-1)./(t.*x).*(exp(t.*(1-x))-1)./(t.*(1-
    x)); %Integrand
difference = zeros(L/dt,1);
int = zeros(L/dt,1);
t = zeros(L/dt,1);
for i = 1:L/dt
    t(i) = i*dt;
    int(i) = integral(@(x)M_infinity(x,t(i)),0,1);
    difference(i) = M_uniform(t(i)) - int(i);
end

%plot(t, difference);
semilogy(t,difference);
xlabel('Time t');
ylabel('Error value');
%title('Error value in numerical approximation for various t values')
    ;
title('Log error value in numerical approximation for various t
    values');
```

Numerical integration for Beta moment generating function

```
%Numerical integration of M_infinity(t) with R_infinity beta(a,a)
    after
%differentiating to x
%for lambda, epsilon uniform
clear;
dt = 0.1; %stepsize t
L = 10; %Largest value of t

dh = 1; %stepsize a
H = 100; %Largest value of a

```

```
M_beta = @(x,t,a) 1./beta(a,a).*exp(t.*x).*(x.*(1-x)).^(a-1); %Integrand for beta(a,a)
M_infinity1 = @(x,t,a) 1./beta(a,a).*exp(t.*x.^2).*(x.*(1-x)).^(a-1); %Integrand 1 for R_infinity
M_infinity2 = @(x,t) (exp(t.*(1-x))-1)./(t.*(1-x)); %Integrand 2 for R_infinity
difference = zeros(L/dh,H/dh);
t = zeros(L/dh);
a = zeros(H/dh);
for i = 1:L/dt
    t(i) = i*dt;
    int_2 = integral(@(x)M_infinity2(x,t(i)),0,1);
    for j = 1:H/dh
        a(j) = j*dh;
        int_beta = integral(@(x)M_beta(x,t(i),a(j)),0,1);
        int_1 = integral(@(x)M_infinity1(x,t(i),a(j)),0,1);
        difference(i,j) = int_1*int_2; - int_beta;
    end
end
min = min(difference); %Shows the lowest difference for all a

plot(min);
xlabel('Alpha value');
ylabel('Error value');
title('Minimal error value in numerical approximation for alpha values');
```

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