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Causal Modeling and Thermodynamics: Towards a new convergence of the two fields

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ABSTRACT

In 1824, Nicolas Léonard Sadi Carnot paid for the publication of his first book. Unfortunately it sparked little interest, and the young engineer never published another. In quick succession, Carnot served in the military, suffered from scarlet fever, mania, and cholera, and passed away in obscurity at age 36. Two centuries have elapsed since Carnot published his only book. Recognition came later. In particular, Carnot's reasoning inspired scientists to formulate the first and second laws of Thermodynamics. The new science that has emerged around these physical laws is nothing short of breathtaking. Yet, with success and growth, critical attention and skepticism have followed. In 1924, Louis de Broglie lauded the first law of Thermodynamics, while remaining more reserved to wards the second. The first law builds on a long history rooted in Causal Modeling, while the second does less so. Today, physicists such as Adrian Bejan continue praising Thermodynamics but contend that some formulations of the second law may have attracted broken science. The present article revisits this history in an attempt to cut through some of the fog. As an outcome of this re-evaluation, the article outlines a new convergence of Thermodynamics and Causal Modeling.

1. Uneven histories

Early in the 1920s, about a century ago, Louis de Broglie began his doctoral dissertation with a brief history of Physics. He covered multiple topics of historical relevance: the origin of modern Physics in the Italian Renaissance, the achievement of exact measurements in Astrophysics, and the development of Statics, Dynamics, and Mechanics (de Broglie, 1924).

De Broglie went on to praise Mechanics for its broad applicability. With applications across Astrophysics, Hydrodynamics, Optics, Acoustics, and other fields of study, "Mechanics reigned over all physical phenomena." Indeed, Mechanics extended even beyond Physics. According to de Broglie, "multiple generations of scientists refined Mechanics to such an extent that it nearly lost its character as Physics."

While these broad applications demonstrated unparalleled success, Mechanics fascinated de Broglie even more through its mathematical elegance. In all of its varied applications, Mechanics relied on one core principle—the "principle of least action". Reuniting all branches of Mechanics, this principle stated that nature operated in the most efficient way possible.

In the context of the principle of least action, de Broglie mentioned the formulations of Pierre Louis Maupertuis and William Rowan Hamilton in his thesis. He did not trace the principle's ancient history, but he was almost certainly aware of it. The principle of least action not only unites various scientific fields but also serves as a cohesive force threading through the history of philosophy across different historical epochs.

Already ancient philosophers posited that nothing happened without a cause. This statement was highlighted as a first principle, for example, in Lucretius's didactic poem *On the Nature of Things* (Lucretius). In beautiful verses, Lucretius explained that illusions are best dispelled not by blinding light, but through careful observation and reasoning. In this context, the first principle of reasoning that Lucretius formulated was that nothing appeared without a cause. Lucretius's poem was rediscovered in the late Middle Ages, and today scholars believe it played an important role in initiating the Italian Renaissance (Greenblatt, 2011). Historically, this is where de Broglie began his review of Statics, Dynamics, Mechanics, and the principle of least action.

The philosophical tenet that nothing happens without a cause has been reformulated in multiple ways. On one hand, it supported causal thinking in fields such as Statics, Dynamics, and Mechanics. In these fields, the tenet that nothing happens without a cause paved the path for the formulation of the principle of least action.

Additionally, the same ancient principle has been reformulated in Thermodynamics. Notably, energy cannot appear or disappear without a cause. In other words, energy is transformed and conserved. In Thermodynamics, this idea eventually became a first principle, also known as the "law of conservation of energy", encapsulated in the "first law of

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Thermodynamics".

Thus, the first law of Thermodynamics and the principle of least action share their historical origins, and they beautifully match up. De Broglie noticed this in his review of the history of Physics (de Broglie, 1924). However, where there is one first principle, it can invite reasoning about additional principles.

In this respect, Mechanics and Thermodynamics diverged. While Mechanics remained centered around only one first principle, Thermodynamics did not. When Thermodynamics emerged as a discipline, the first law was complemented by the second law.

From the beginning, the second law was unlike the first. The first law had been formulated based on two or more millennia of experience and philosophy. By contrast, the second law was formulated primarily in response to the first law. Yet, how did the second law historically emerge? What did it initially say?

To answer these questions, Sadi Carnot's considerations require some attention. He is regarded as one of the founding figures of Thermodynamics, and some of the thermodynamic language that he initially utilized is still in place today. Two centuries after Carnot, we still say that heat "flows", and that thermodynamic systems can be in "equilibrium". In Carnot's time, these terms were taken quite literally.

Heat was conceived as some kind of fluid—the "caloric"—which flowed from bodies of more highly elevated temperature to those of lower temperature. If two connected bodies had different temperatures, it meant that the caloric could "fall" until, eventually, "equilibrium" was reached. Here is a sample of Carnot's own language:

> "The production of motion in steam-engines is always accompanied by a circumstance that is important to consider. This circumstance is the re-establishment of equilibrium in the caloric; that is, its passage from a body in which the temperature is more highly elevated, to another in which it is lower. […] The production of motive power in steam-engines is not due to a consumption of caloric, but *to its transfer from a warm to a cold body,* that is, to the re-establishment of equilibrium—an equilibrium considered as destroyed by any cause whatever, including chemical action such as combustion or any other cause. […] The fall of caloric produces […] motive power. […] The motive power of a waterfall depends on its height and on the quantity of falling liquid. Similarly, the motive power of heat depends also on the quantity of caloric used, and on what we shall call, the height of its fall, that is to say, the difference of temperature of the bodies between which the exchange of caloric is made." (Carnot, 1824)

The caloric theory of heat was replaced by molecular models in the 20th century. Nevertheless, some of Carnot's theories remained unchallenged. Most notably, Carnot suggested that a heat engine could be run in reverse. Meanwhile, such a reverse heat engine has become known as a heat pump or refrigerator. The second law of Thermodynamics is directly linked to Carnot's reasoning around this type of device.

Initially, Carnot's question was how efficient heat engines could ever get. The answer was evident to him. It was dictated by the law of conservation of energy (Carnot, 1824).

Imagine using a heat pump, applying mechanical work to create a temperature difference between two bodies. A heat engine can then extract work from this difference, but even at maximum efficiency, the output work that can be extracted must always be less than the input work initially applied. Otherwise, one would be able to build a perpetual motion machine.

The second law of Thermodynamics came as an afterthought of such considerations. Heat engines and heat pumps never reach maximum efficiency. They cannot be coupled to create perpetual motion machines. Without an external source of power, they eventually stop working. Thus, thermal equilibrium is always reached, and all potential to extract work disappears. From such considerations, the idea emerged that, to run a heat engine, one had to consider both the first law of Thermodynamics and the circumstance that there were inefficiencies (Thomson, 1849). Nature operated the most efficient way possible, but in heat engines, inefficiencies could not be avoided. Heat flowed naturally from hot to cold bodies, even if the bodies were thermally isolated, and this could only result in inefficiencies.

As Thermodynamics grew, new revelations broadened the scientific understanding of the second law. Notably, researchers discovered that heat flowed from hot to cold bodies not only through conduction and convection, but also through radiation (Fourier, 1822; Planck, 1901).

Every object emits blackbody radiation. This radiation can be bundled with lenses, but it cannot be used to heat up a point to a temperature hotter than the body from which the radiation originated. Thus, heat naturally flows from hot to cold bodies through radiation, as well as through conduction and convection. This observation broadened the spectrum of applicability of Thermodynamics.

With such and similar revelations making their way into scientific publications, Thermodynamics became increasingly broadly applicable. It linked engineers of heat engines with physicists who studied radiation. Among these latter scientists was Max Planck, a pivotal figure in the field. He redefined thermodynamic theory with his unique perspective on the second law. In his textbook, *Lectures in Thermodynamics*, Planck stated,

> "Every process occurring in nature proceeds in the sense in which the sum of the entropies of all bodies taking part in the process is increased. In the limit, i.e. for reversible processes, the sum of the entropies remains unchanged." (Planck, 1897/1901)

Planck's statement marked a pivotal moment in the history of the second law. Initially, the second law was concerned with heat transfer. However, in Planck's statement, the term "heat transfer" is entirely absent, replaced by the idea of "increasing entropy".

Clearly, the second law of Thermodynamics became a branch of science that grew rapidly, inviting new perspectives. Yet, this also meant that heat engines were no longer a research focus for all scientists contributing to this growth. At some point, some scientists came to view entropy as a measure of disorder or randomness, and the idea emerged that these system properties tended to increase in natural systems (Lotka, 1945).

The newly emerging ideas did not mean that earlier ideas and language were always replaced. Heat was still "transferred", "flowing" from "higher" to "lower" temperatures until thermal "equilibrium" was established. Scientists who held that disorder and randomness increased in the universe did not contradict this. Thus, heterogeneous conceptualizations coexisted. Even Planck himself provided multiple formulations of the second law.

Such success and growth attracted enthusiasm as well as critical attention from other fields. Eventually, Thermodynamics and Mechanics had to clash. The question emerged whether the two fields were compatible with each other. In 1924, Louis de Broglie brought into focus that the second law of Thermodynamics did not immediately find a matching counterpart in Mechanics.

> "Although one of the main fundamental principles of Thermodynamics, namely conservation of energy, can easily be interpreted in terms of Mechanics, the other, that entropy either remains constant or increases, has no mechanical clarification. The work of CLAUSIUS and BOLTZMANN, which is currently quite topical, shows that there is an analogy between certain quantities relevant to periodic motions and thermodynamic quantities, but has not yet revealed fundamental connections. Only the imposing theory of gases by MAXWELL and BOLTZMANN, as well as the general Statistical Mechanics of GIBBS and BOLTZMANN, teach us that, Dynamics complemented with probabilistic notions yields a mechanical understanding of Thermodynamics." (de Broglie, 1924)

De Broglie compared the first and second laws of Thermodynamics on grounds on which they were uneven. On one hand, the first law built on millennia of history, and it had entered both Mechanics and

Thermodynamics. On the other, the second law was comparatively new and had yet to be translated to other sciences.

De Broglie's considerations were not isolated nor did they lack relevance. They were linked to a doctoral thesis that was sent to Albert Einstein after the evaluators did not seem to understand it. Yet Einstein's reaction brought de Broglie closer to earning the Nobel Prize only five years later, in 1929.

2. Negating entropy or developing a self-standing concept

Around the time de Broglie completed his dissertation in Physics, the concept of entropy also entered other fields of study. In Physiology, it attracted additional critical attention from yet another perspective.

Physics and Physiology have much in common. Both fields study nature, which is reflected in their names. "Physics" and "Physiology" are derived from the same ancient Greek word "φύσις" (physis), which means "nature".

In the 19th century, when the separate use of "Physics" and "Physiology" emerged, the term "Physiology" appeared in print more frequently than "Physics". This showcases that Physiology had fascinating discoveries to show for. It had brought everything from the development of cell theory (Schleiden, 1838; Schwann, 1839) to the discovery of neurons (Cajal, 1904; Golgi, 1873), from the germ theory of disease (Pasteur, 1861; Koch, 1876) to the discovery of viruses (Ivanovsky, 1892), and from the development of endocrinology (Addison, 1855) to the concept of homeostatic regulation (Bernard, 1865). By 1900, Physiology had become a field of science of hallmark importance.

Nevertheless, the distinction between Physics and Physiology remained somewhat blurred. Around 1900, another field "Physical Chemistry" grew explosively. It linked Physics, Physiology, and Chemistry in new ways. Physical Chemistry utilized mathematics of the kind used in Mechanics, yet it often applied this mathematics to describe cause-effect relationships specifically in Chemistry and Physiology.

As Physiology and Physics were growing with Physical Chemistry between them, similarities and differences were frequently discussed. Physiology was commonly recognized as the science that studied how life grew. The choice of the name "Physiology" could not be better. The ancient Greek word "φύσις" (physis) is related to the verb "φύω" (phyō), which means "to grow".

Physiology often studied growth: Living beings grow or they die, and entire populations evolve and increase in size or they shrink and become extinct. In Physiology, everything grew and increased in size or number—or else it disappeared. Was this also true in Physics? What, then, increased in Physics? According to Max Planck there was one thing that always increased: the sum of entropies (Planck, 1897).

In this sense, Physics and Physiology differed. While physicists such as Max Planck wrote that the sum of entropies increased in all processes in nature, physiologists primarily studied processes in which entropy did not seem to increase.

However, could entropy actually decrease? Around the time Louis de Broglie wrote his dissertation, the life scientist Ervin Bauer applied Thermodynamics to the study of biological life, publishing his considerations in 1920 (Igamberdiev, 2024; Elek and Müller, 2024). He sought to reconcile Physiology and Physics, while also identifying distinctions.

On the side of Physics and Thermodynamics, Bauer's understanding of the second law was similar to that of de Broglie and Planck. He wrote,

> "The second law of Thermodynamics states in general that only such processes occur in nature, in which a certain magnitude—the entropy—increases." (Bauer 1920).

Rephrasing Planck's entropy-based formulation of the second law, Bauer also posited that the second law predicted that systems evolved towards a state of maximum entropy. In this context, Carnot's idea of equilibrium resurfaces,

> "If the maximum entropy is reached, [...] the system is in a state of equilibrium. The second law also states that every system tries to reach equilibrium, and that every process takes place in the direction of equilibrium." (Bauer 1920).

This equilibrium-based formulation of the second law raised some questions. Equilibrium meant that all forces and energies in the system were balanced, and the system could no longer change state. Yet, Bauer was a life scientist, and he knew that living organisms became ever more active. Furthermore, living organisms grew and evolved, and they often maintained their body temperature. This seemed to work against the accumulation of entropy. He explained,

> "The first thing we have to say about the characterization of the living being is that it represents a system that is not in a state of equilibrium." (Bauer 1920).

Then again, Bauer seems to have glimpsed that the problem ran deeper. He could also think of non-living systems that behaved in this way. He specifically mentioned phenomena of the water cycle, such as waterfalls and whirlpools.

Waterfalls kept falling, and whirlpools kept turning. They never reached "equilibrium" or a "state of maximum entropy".

To improve his thermodynamic definition of life, Bauer felt compelled to add an additional criterion, requiring that one only speak of life if a system not only stays outside equilibrium, but also moves away from it:

> "[Life must] take place *against* the direction of equilibrium." (Bauer 1920).

This additional criterion for defining life can be expressed in simpler terms with the phrase "living beings must persist and prevail". "Persist" then refers to what Bauer called "stay outside equilibrium", and "prevail" refers to what Bauer called "take place against the direction of equilibrium".

As a matter of fact, the phrase "persist and prevail" was used with increasing frequency in 1920 (Google Ngrams). Certainly, during and in the immediate aftermath of WW I, it made sense for people to wish to persist and prevail despite the chaos of war. Bauer turned this wish into life science.

The idea that life required organisms to persist and prevail returned multiple times in Bauer's work. In 1935, Bauer published a book in which he explained his theory even more clearly (Bauer, 1935; Igamberdiev et al., 2024). Unfortunately, he was killed in Stalin's Great Terror in 1938.

During this time, another person who shared Bauer's first name as well as political vulnerability further continued this line of thought. Erwin Schrödinger asked the question "What is Life?" In brief, the answer was: "negative entropy" or "negentropy". In a lecture delivered in 1943 and published a year later, Schrödinger explained,

> "A living organism continually increases its entropy—or, as you may say, produces positive entropy—and thus tends to approach the dangerous state of maximum entropy, which is of death. It can only keep aloof from it, i.e. alive, by continually drawing from its environment negative entropy." (Schrödinger, 1944)

In line with Bauer's earlier considerations, Schrödinger iterated that living beings persisted and prevailed in spite of the accumulation of entropy. Life was not just some kind of zero entropy; it was negative entropy. Suddenly, the term "negative entropy" started to spread, and it was very successful, perhaps because it came from a Nobel laureate of 1933.

Only a few years later, a similar idea surfaced again in the work of the physical chemist Alfred Lotka, also known as author of the Lotka-Volterra equations. The difference was that Lotka called for an entirely new law of Thermodynamics (Lotka, 1945). Rather than negating entropy, he wished for a new law that described life in a more self-standing manner. In 1945, this new law would have been the third law of Thermodynamics.

Both the negation of entropy and the call for a new law returned in the second half of the 20th century (Prigogine and Stengers, 1984). Simultaneously, new scientific disciplines were conquered. For example, "negentropy" was broadened in "non-equilibrium mechanics" and "out-of-equilibrium dynamics", which continued the equilibrium terminology that had originated in the work of Bauer and Carnot. Negentropy and non-equilibrium mechanics were later also mirrored in "antifragility", a term coined by the provocative skeptic Nassim Nicholas Taleb (2012).

Both negentropy and antifragility are terms formed by negation, but while negentropy was initially coined for biological life, antifragility was applied to reasoning about markets and social systems. The question remained the same: How did life persist and prevail in a disordered world? Biological life thrived and grew in a disordered world. Economies did that too. Taleb found particular interest in social systems that were not only resilient but, when faced with disorder and stress, could also learn and improve. These systems not only persisted, but also prevailed.

From today's perspective, looking back at the work of authors such as Bauer, Schrödinger, and Taleb, it is evident that negentropy and antifragility have built on a century of rethinking. This has made them great. Their joint problem is evident as well. Negentropy and antifragility first need to define entropy, disorder, or fragility, before proceeding by negation.

Developing a theory of life based on the negation of "disorder" seems a particularly difficult approach. In times of war, it may make sense to say that most of the world is in a dramatic state of disarray. Yet, is the world always disordered by default? And is "dis-order" not already a term obtained by negation? What does a double negation mean? Wouldn't it be better to start from scratch, asking directly how life persists and prevails, without the detour of negations?

This approach has been taken by Adrian Bejan who formulated the Constructal Law in 1996—during a comparably peaceful decade. The 1990s saw the end of the Cold War and the growth of global cooperation.

The Constructal Law is unlike the theories of negentropy, non-equilibrium mechanics, or antifragility. It is not a theory reached by negation. The Constructal Law does not assume that the world is disordered by default.

Furthermore, Bejan significantly expanded and improved upon earlier theories. Already the question that he asked was broader. The initial question was how living organisms persisted and prevailed. Bejan's question is broader. He asked what the necessary conditions were for any system to persist and prevail. Thus, through the lens of the Constructal Law, we are looking at any system, not just biological organisms.

Once again, the question is how any system persists and prevails. How does any system maintain and further construct itself? Bejan's answer could not have been more candid. To maintain and further construct itself, any system has to gain access to resources that it needs. In plain words, construction requires resources. In his hallmark 1997-article, Bejan formulated the Constructal Law as follows:

> "For a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed (or inherent) currents that flow through it." (Bejan 1997)

This statement holds that to maintain and further construct itself, any system must gain easier access to resources. In the sophisticated thermodynamic language of the day, Bejan refers to these resources as "currents that flow through [the system]".

The requirement of accessing resources leaves two options open. Either a system becomes more efficient at accessing resources that it already taps into, or the system gains access to a broader, more diverse range of resources. Yet, what does this mean, concretely? Are there any examples?

Bejan was initially inspired to formulate the constructal law while addressing a problem in computer hardware engineering. Initially, computers had loud and distractive ventilators, necessary to cool them down. Faced with the question of how to perform the cooling without a ventilator, Bejan designed a tree-shaped cooling structure that facilitated the natural flow of heat (The Franklin Institute, 2018).

The tree-shaped structure efficiently solved the problem of heat transfer. Through a well-designed hierarchy of branches and twigs, Bejan's structure connected the heat source with the cold environment fast and broadly.

Over the ensuing decades, Bejan and his followers found many similar examples of tree-shaped structures. Many of these structures were able to maintain and further construct themselves, tapping into resources with increasing ease. Perhaps the most obvious example was that of actual botanical trees with their trunks, branches, twigs, and tree-shaped leaf structures. Another example was that of vascular systems with arteries, arterioles, capillaries, venules, and veins in Physiology. Yet another example was that of river basins with rivers, rivulets, brooks, and creeks in Geography, or that of transportation systems with traffic arteries, distribution hubs, and spokes in Urbanism. In order to maintain and further construct themselves, all of these tree-shaped structures had to be able to access resources with increasing ease (Bejan, 1998; Bejan, 2000; Bejan, 2016a; Bejan, 2016b; Bejan, 2020; Bejan and Errera, 2016; Bejan and Lorente, 2013; Bejan and Lorente, 2012; Bejan and Merkx, 2007; Bejan and Zane, 2012).

Over time, the question was asked what made tree-shaped structures so successful, supporting them to easily access resources. The answer was clear. Geometrically, trees have a marked hierarchy in their trunks, branches, and twigs. This hierarchy allows for efficient access to resources. In parallel, broad tree crowns support access that is not only efficient but also widely distributed.

Thus, in order to persist and prevail, living systems construct tree-shaped structures. Through the lens of the Constructal Law, life is described without a detour over entropy, disorder, or fragility. It is described as a tree of life, with a hierarchy of trunk, branches, and twigs, which arrange themselves into a lush crown. Over the course of the tree's life time, the crown must grow ever broader, in order to access useful resources.

Now that thermodynamics has been expanded with the Constructal Law, it may be time to broaden and reassess the question de Broglie posed a century ago. How does Thermodynamics match up with Mechanics? Clearly, to answer this question, one also has to revisit Mechanics, which also includes Social Mechanics and Humanities Mechanics. Perhaps all of Mechanics, relying on cause-effect reasoning, is best collectively referred to as Causal Modeling. Will Causal Modeling support Thermodynamics? Are there any trees of life in Causal Modeling, too?

3. What is Causal Modeling?

The basic assumption of Causal Modeling can be stated as follows: a given set of causes, transformed by a given causal mechanism, always results in the same set of effects (Baciu 2023). For example, if certain quantities of oxygen and hydrogen are mixed, they react, forming a certain quantity of water. In Causal Modeling, the exact same conditions always lead to the same outcomes.

Causal models can be formulated in many different ways. For example, a cause-effect relation can be expressed in words. Philosophers developed verbal causal explanations already in antiquity. However, causal thinking is not limited to spoken languages. Words can be translated into flow diagrams. An example of causal flow diagrams are Sewall Wright's path analyses that use arrows to link causes and effects (1921). In turn, the graphical language of the flow diagrams can be translated into mathematics. In mathematics, the arrows are translated into functions that have the role of linking causes and effects the same way arrows do it graphically. For example, Wright also featured equations next to his flow diagrams to provide detail to his causal models. Another, even more famous example of causal explanation models formulated as equations are those of Isaac Newton, which refined the mathematical notion of functions in the 17th century. Today, such equations can be further translated into computer logic. Computers do not have numbers. They have binary code that is passed through logic gates. This makes the translation of mathematics into computer logic necessary towards computer-processing. In turn, the computer logic can be translated back into natural language. One can always describe in plain words what the computer is doing. Long story short, causal models can be expressed in many human and machine languages.

Causal models are most useful when they can be applied to describe a broad array of phenomena. If a causal model describes a one-time event that happened only once in the past and will never happen again, it is as good as useless. Important questions are those that are faced repeatedly: What happens when clouds pass over mountains? What happens with the water after it rains? How do biological organisms use the water? How do they reproduce, adapt, and evolve? How do people use the water? What happens when they are creative or playful in their interactions? All of these and many more questions can be formulated and answered with causal explanation models.

As just mentioned, causal explanations are most useful when they describe recurrent events. This should make it evident that the models are mere simplifications of a more complex reality. In reality, no event occurs twice and no two objects in the universe are exactly the same object. Yet causal models tell of events that occur once, twice, and many more times. In addition, causal models—especially if formulated with mathematical statements—make ample use of numbers. In causal models, one can refer to two atoms, two molecules, two viruses, two organisms, two humans, and two ideas. However, the ideas, humans, organisms, virions, molecules, or atoms that are described in causal models are rarely identical. Their differences are simply neglected.

There are two reasons why it is important to understand that Causal Modeling is just a simplified way to describe a more complex world: First, one should be guarded against orthodoxies, believing that only one causal model is correct, while all others are flawed. Second, one should be guarded against the idealist stance that the causal model *is* the reality. If it were the reality, it wouldn't be a "model".

A comparison with Art clearly renders these points. Artists never mistake pictures of subjects to be the subjects themselves. Only the legendary Pygmalion fell in love with the woman he sculpted in stone. Knowing this legend, one should not fall to the illusion that a causal model really *is* the subject that it describes. It's only a description.

The same comparison with Art can also guard scientists against orthodoxies. Obviously, one can draw the same face in many different styles. The styles are arbitrary choices that are up to the individual artist. Just the same way, causal descriptions may be fitting, but they are never perfect. All causal descriptions have margins of error. Errors emerge because differences are disregarded from the start. On the other hand, if one were to take all differences into account, causal models would need to have distinct statements for every atom, every electron, every proton, every photon, etc. In the end, if one were to take all of these details into account, the behavior of a grain of salt in a cup of water could not be modeled. Similarly, a portrait that draws every single hair on a person's head is not necessarily a good portrait, and it is sometimes impossible to draw something like it on a single sheet of paper.

A last comparison with Art further illustrates one of the most important aspects of Causal Modeling. On a white sheet of paper, one can draw anything from a face to the shape of a cloud, or from the path of a river in a delta to the paths of galactic clusters in the universe. One can draw all of these using only lines and curves. In addition, one can change subject while drawing. One can begin by drawing the paths of galactic clusters, or the outlines of a cloud, and then change subject, reusing the same lines and curves to represent a new subject. For example adding eyes and a mouth can make almost anything look like a face. Similarly, through Causal Modeling, one can provide explanations about processes that take place in the universe as well as those that take place in human culture. Furthermore, one can reuse verbal descriptions, flow diagrams, or mathematical equations when changing subjects. And no matter what is being modeled, it's done with just linear and nonlinear models. Is it astonishing that one can describe so much using only linear and nonlinear models? When comparing Causal Modeling to Art, everything makes sense: One can draw every subject on a white sheet of paper, utilizing just lines and curves. Just the same way, one can develop causal models in any scientific discipline, using just linear and nonlinear descriptions.

4. Examples of causal models

All causal models begin with a set of causes. As soon as the causes appear, a causal mechanism must act on them. Through this action, a set of effects appears.

All such cause-effect relationships formulated in words can also be translated into mathematics. This is mostly achieved with variables and functions. To refer to causes, let us use the variables $x_1, x_2, x_3 \ldots$ To refer to effects, we can use $\dot{x}_1, \dot{x}_2, \dot{x}_3 \ldots$ We have not yet specified which exact causal mechanism is at work. Let us therefore use an unknown function φ to represent a causal mechanism that has yet to be defined. Utilizing this mathematical notation, we can write $\dot{x}_1, \dot{x}_2, \dot{x}_3 \ldots = \varphi\ (x_1, x_2, x_3 \ldots)$. Literally, this statement translates to: $\dot{x}_1, \dot{x}_2, \dot{x}_3 \ldots$ represent the changes that appear when the unknown causal mechanism φ acts on the causes $x_1, x_2, x_3 \ldots$.

Causal mechanisms always operate in one direction. The reverse direction that goes from effects back to causes requires a reverse causal mechanism, which may or may not be available. Mathematically, functions reflect this property of causal mechanisms by default. A function always operates in one direction. The opposite direction is the inverse function, and not every function has an inverse function.

While functions are deterministic, causal models can be embellished with probabilistic refinements. These refinements reflect uncertainties about the causes involved, the effects observed, or the causal mechanism at work. Graphically, probabilistic refinements can be represented as blurred or dotted lines or curves.

In the previous mathematical notation, the choice of x for causes and $\dot{x}$ for effects reflects that causal mechanisms are at work constantly. They transform causes into effects, as soon as the causes appear. This results in long causal chains in which effects immediately become causes for new effects.

Causal models can focus on individual causal chains, or they can link multiple chains into more complex causal systems. This can be emphasized through vector notation, for example.

The previous equation $\dot{x}_1, \dot{x}_2, \dot{x}_3 \ldots = \varphi\ (x_1, x_2, x_3 \ldots)$ can also be written in vector notation. The causes $x_1, x_2, x_3 \ldots$ can then be denoted with only one symbol as a vector $\boldsymbol{x}$. Technically, the vector stores the total amount of each cause before the causal mechanism acts on it. The changes effectuated through the causal mechanism are then stored in the vector $\dot{\boldsymbol{x}}$. The function φ now takes in a vector of causes and returns another vector of effects of the same length. The equation becomes $\dot{\boldsymbol{x}} = \varphi(\boldsymbol{x})$.

With this basic setup in mind, let us now proceed to a couple of examples.

In the simplest form of this equation, we can focus on one single item x. Let x denote a certain amount of money on a bank account. Our task is now to explain what happens to the money if the bank offers a constant interest rate. To describe this, we can set the causal mechanism φ to be a constant interest rate q. We thus write $\dot{x} = qx$, with $\dot{x}$ being the interest that the bank constantly pays.

And so, what happens to the money? The amount grows. To be specific, it grows exponentially.

Historically, this model is one of the oldest known in human societies. It is uncertain when it first appeared, perhaps as soon as money could be lent, and as soon as money lenders charged interest.

The consequence of charging interest at constant interest rates was that the debt of the borrower grew exponentially. Ancient societies already understood this, and they often found it troubling. In consequence, some societies forbid compounding interest to avoid the effect. This demonstrates not only that they understood the consequences of compound interest, but they also acted in response to that understanding.

The same old growth model was refined in the 18th century (Fig. 1.), and it became infamous around Sadi Carnot's time. Between 1798 and 1826, Thomas Robert Malthus used the model to provocatively describe population growth (Malthus 1826).

Malthus did not believe that populations could indeed grow exponentially, concluding instead that growth was curbed by causes such as famine, which were not considered in the exponential growth model. Ironically, population growth remained exponential long after Malthus's death. Today, population numbers have been reached that most people in Malthus's time would have believed to be astronomical. Indeed, if population growth is modeled for the entire globe from antiquity to recent decades, an exponential growth curve can provide an accurate description (United Nations, Department of Economic and Social Affairs, Population Division, 2019).

The same mathematical model can be applied not only to money and population dynamics but also to the expansion of the universe. In this context, distances in the universe are represented with the universe's scale factor a, while the expansion rate of the universe is the Hubble constant H. If the universe's expansion is dominated by the effect of dark energy, one speaks of a "de Sitter universe" (de Sitter, 1917). This particular expansion can be described simply as $\dot{a} = Ha$ (Peebles and Ratra, 2003).

Evidently, this is the same equation as before. The only thing that changes is how the mathematical symbols are named. Before, the constant was named q; now it is named H, yet it's still a constant. The variable was named x; now it is named $\dot{a}$, yet it's still a variable. The operations remain the same. The old and new formulas are mathematically equivalent.

The revelation that the universe indeed expands exponentially came in the late 20th century. Dark energy drives the expansion, creating more space. Yet, the newly created space contains additional dark energy, so to say, by default. Thus, space gives birth to more space. The same way, people give birth to more people.

The universe does not always expand exponentially. Many cosmological models are more complex. Similarly, the model of exponential population growth is not the only growth model used in population dynamics. In the 1830s (a decade after Malthus), two Belgian scientists—Quetelet and Verhulst—proposed to introduce a nonlinear element into the equation.

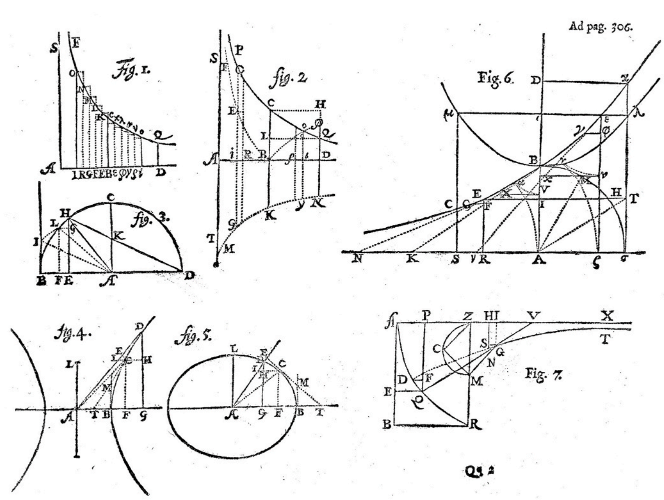

Fig. 1. Compound interest in the 18th century. Excerpts from Jacob Bernoulli's Ars Conjectandi, Basel 1713, in which the mathematician developed important concepts for calculating interest.

Like Malthus, Adolphe Quetelet did not believe in exponential growth over extended periods of time, suggesting that population growth had to be S-shaped, not exponential. Initially, populations did grow exponentially, but then, the exponential growth had to ebb out and stagnate (Quetelet 1835). Evidently, describing such a curve required a nonlinear component. The linear equation of population growth cannot produce S-curves.

Pierre François Verhulst agreed with Quetelet. He empirically treated the subject, working with population growth data available to him. Verhulst specifically introduced a negative nonlinear component into the Malthusian growth model. Mathematically, this could be written as $\dot{x} = qx - \varphi(x)$, where $- \varphi(x)$ was an unknown nonlinear function (Verhulst 1845).

Verhulst then evaluated multiple options for $- \varphi(x)$, finally selecting $- q_2 x^2$ as the best fit (where q_2 is just another constant). This left him with $\dot{x} = q_1 x - q_2 x^2$. Thus, Verhulst obtained a nonlinear, S-shaped population growth equation, fitted to empirical data (Verhulst 1845).

Verhulst and his followers went on to provide verbal translations for this equation. The initial source of inspiration for the S-curve was Mechanics. Yet, the model had been empirically developed. It wasn't clear what the component $- q_2 x^2$ actually meant, translated from mathematics into a natural language such as French or English.

Unfortunately, the verbal translations of the model were not perfect. When discussing the equation, researchers sometimes came to speak of limitations in the 'carrying capacity' of the environment. It should be evident that the equation has no specific parameter for the environment. If the environment is anywhere in the equation, it is somewhere hidden in the constants q_1 and q_2. This may be a nuance, but can a verbal translation be found that—like the model—does not explicitly mention the environment?

Formulating such a translation of the mathematics can be obtained by imagining that the population x is split into two groups x_1 and x_2. The first variable, x_1, represents friends and positive connections around each individual. x_2 represents foes who can alienate the positive connections. As the population grows, the encounters between friends (which are a fraction of x) and foes (which are also a fraction of x) can be modeled as a term that grows proportional to the probability that the two groups meet. Thus, we must begin by calculating x times x, or simply x^2 —which is the term explicitly found in the equation.

These considerations suggest that the equation represents a case of self-controlled population growth. Interactions between friends and foes have a negative effect on population dynamics. As the population grows, these negative effects appear with rapidly increasing frequency, which, in turn, controls the population growth from within. (Conversely, in a population in which encounters between pairs of groups have no negative effect whatever, the coefficient of x^2 must be zero, and the population growth curve remains exponential. This case may be found in human populations up into the early 21st century, as humans have learned to form larger and larger networks, befriending and cooperating with anyone to more easily gain access to resources together.)

Verhulst's formula of population growth can be repurposed for other applications. It also leads seamlessly into the basic model of Ecology (Lotka-Volterra equations; Lotka 1925), the basic model of Epidemiology (SIR model; Kermack and McKendrick, 1927), the basic model of Game Theory (generalized Lotka-Volterra or Replicator equation; Hofbauer and Sigmund, 1998), transformer layers in Artificial Neural Networks, and diversity indices (Simpson's diversity index; Simpson, 1949).

Together, these mathematical formulas are some of the greatest products of applied mathematics in the last century. They have in common that they make use of the same type of nonlinear modeling (Baciu et al., 2022; Baciu, 2023). Notably, x_1 and x_2 reappear in these models to represent diverse system components such as prey and predators in the case of ecosystems or mainstream and opposition in the case of human cultures.

Now that x_1 and x_2 are different values, we can concatenate them into a vector and return to our initial equation $\dot{\boldsymbol{x}} = \varphi(\boldsymbol{x})$. Lotka-Volterra equations are a nonlinear special case of this formula. They can be stated as $\dot{x}_1 = q_1 x_1 - q_2 x_1 x_2$, $\dot{x}_2 = q_3 x_1 - q_4 x_2$. Here, the multiplication $x_1 x_2$ replaces Verhulst's x^2.

The Lotka-Volterra model can serve as a causal description of human culture, modeling the effects of a spreading new idea and opposition to it. In this model, a given idea x_1 spreads if the growth rate q_1 is greater than 1. This is represented by $\dot{x}_1 = q_1 x_1$. In response to the spreading idea, opposition x_2 also spreads. It does so at rate q_3 in response to the spreading idea x_1. This is represented as $\dot{x}_2 = q_3 x_1$. As both idea and opposition spread, they can bump into each other, which has a negative effect on the spread of the initial idea. This statement is represented by the term $- q_2 x_1 x_2$. (This is the term that finds a matching counterpart in Verhulst's $- q_2 x^2$.) The last component of the model says that opposition does not spread on its own, in the absence of the idea that it opposes. Rather opposition disappears at rate q_4 if it is left alone. This is represented as $- q_4 x_2$.

The resulting behavior of this system resembles multiple S-curves in close succession. Verbally, one can describe this behavior as "oscillations" or "waves". Empirical examples for such recurrent oscillations are the spread of ideas and opposition, returning fashions, or other, similar phenomena that involve both positive action and negative reaction.

The model was first formulated by Lotka in Physical Chemistry, initially inspired by child growth curves, yet it resurfaced in other disciplines, as mentioned. Notably, Lotka's model is related to Simpson's diversity index. Simpson's index estimates the probabilities that any two system components meet. These probabilities are represented as all pairwise multiplications $x_i \, x_j$. We have once again the nonlinear component historically inherited from Verhulst. This suggests an important insight. Nonlinear models can be used to describe diversity and interaction between diverse system components (Baciu 2023).

Our initial model $\dot{\boldsymbol{x}} = \varphi(\boldsymbol{x})$ can be turned into nonlinear equations, as just described. However, it can also be turned into linear equations with multiple variables but without multiplications between variables. For example, we can write $\dot{x}_1 = q_1 x_1 + q_2 x_2$, $\dot{x}_2 = q_3 x_1 + q_4 x_2$. Thus, we have avoided the multiplication $x_1 x_2$.

While the previous, nonlinear model was used to describe interplay between mainstream and opposition, this new model can serve as a description of creativity in human culture (Baciu 2023). In this context, the variables x_1 and x_2 are used to refer to two interrelated ideas. Each idea spreads by itself. Thus, we can write $\dot{x}_1 = q_1 x_1$ and $\dot{x}_2 = q_4 x_2$. In addition, each idea inspires thinking about the other idea. We have $\dot{x}_2 = q_3 x_1$ and $\dot{x}_1 = q_2 x_2$. Because x_1 and x_2 often create each other, they can be considered to be each other's offspring, and therefore related. They form a group of interrelated ideas.

The prediction that can be extracted from this type of model is that groups of interrelated ideas are hierarchical. Evolution within groups gives rise to pyramids of influence. In the concrete case of culture, some cultural content is extremely influential, while most cultural material almost entirely lacks influence (Baciu 2023).

Mathematically, the pyramid is the result of different growth rates and creativity. If one idea spreads slightly faster than all others, this leads to exponential growth, and one can observe competitive exclusion. Only the fastest-growing idea survives by itself, while all the others that survive do so because they are directly or indirectly related to the fastest-growing idea. Typically, this most-successful idea has a few immediate connections and a much larger number of indirect ones. These indirect connections make up the bulk of the many variants that are numerous together, although each of them is rare. They thrive indirectly, so to say, only through connections of connections.

This pyramid of influence was first empirically observed by Vilfredo Pareto who studied income data and discovered a "pyramid of income" (Pareto 1896).

Pareto did not provide a convincing causal explanation model for the distribution he observed. The causal explanation model appeared in life science literature only much later. At first, it appeared in Physical Chemistry in the 1970s, when it became known as the Quasispecies equation. Later, the same model reappeared in Virus Dynamics and in Evolutionary Dynamics, where it is known as a variation-selection process. Regardless of the field, variation-selection-models support uneven distributions. Such distributions are seen between rich and poor, influential and less influential individuals, widespread and rare beliefs, frequent and rare genetic setups, etc. (Baciu 2020). This model behavior is linear in the sense that hierarchies are established and maintained, rather than reversed over time.

The distinction between such linear behavior that maintains hierarchies and the nonlinear behavior described earlier in this article can be summarized as follows: In human culture, linear behavior is linked to creativity, progress, the establishment and maintenance of pyramidal hierarchies, the dominance of the most successful groups of ideas, and competitive exclusion. By contrast, nonlinear behavior reflects the playful nature of culture. Play, as Johan Huizinga (1938) already emphasized, isn't about competition and the survival of the fittest; it's about surprises, chaotic behavior, and opposition. Most games end when there is no more opposition to play with. Play turns hierarchies upside down, supporting independent perspectives and diversity (Baciu 2023).

The question whether a model behaves in a linear or nonlinear manner is most accurately answered with matrix algebra. The initial equation $\dot{\boldsymbol{x}} = \varphi(\boldsymbol{x})$ can be rewritten in matrix notation as $\dot{\boldsymbol{x}} = \boldsymbol{Q}\boldsymbol{x}$. As before, $\boldsymbol{x}$ is a vector containing all causes that are considered. $\dot{\boldsymbol{x}}$ represents all changes that are being effectuated. The previous function φ is now written a square matrix $\boldsymbol{Q}$ containing all operations performed on each cause. To study how the system changes over time, the matrix can now be multiplied with the vector.

The matrix representation of causal modeling makes it evident how a model behaves. It indicates when a model behaves in linear or nonlinear ways, and clarifies how these two types of behavior differ. Furthermore, this representation underscores what are the most commonly expected outcomes of Causal Modeling.

Mostly, the square matrix $\boldsymbol{Q}$ will have pairs of eigenvectors and eigenvalues. If the largest eigenvalue is a real number greater than 1, multiplying matrix and vector will result in exponential growth in the direction of the eigenvector associated with this largest eigenvalue (Gloub and Van Loan, 2013; Axler, 2015; Brezis, 2010; Nagle, 2017; Petersen and Petersen, 2012). Because eigenvectors are mostly composed of different numbers, not just values of 1, the hierarchy of the numbers in the eigenvector will eventually become visible, and it will be maintained (Strang, 2016; Lay et al., 2016; Horn and Johnson, 2012; Trefethen and Bau, 1997; Meyer, 2000). This hierarchy of numbers can be visualized as the silhouette of a pyramid of influence.

Eigenvalues can be real, but they can also be imaginary numbers. With imaginary eigenvalues growth does not move linearly in the direction of the eigenvector, rather, it rotates around the vector's tip. If the magnitude of the imaginary part of the eigenvalue is greater than one, the growth will spiral out. Otherwise, it will spiral inward. This is the other most common result obtained in matrix algebra (Demmel, 1997; Lancaster and Tismenetsky, 1985; Shilov 1977; Lutkepohl, 1997; Kreyszig, 2010). Compared to the previous stretching of the space in the direction of the eigenvector, rotation reflects changes in hierarchy, which may be experienced as 'cultural revolution' and 'diversity'. Mathematically, the phenomenon is described through rotation that spirals in or out.

While much of the present section is accompanied by mathematical equations, the phenomena that have been discussed do not necessarily rely on mathematics to be described. Mathematical operations can be translated into logical operations (Knuth, 1997; Cormen et al., 2009; Turing, 1936; Epp, 2011; Rosen, 2018; Lipson, 2009). Logic does not have numbers, it only has truth tables with true or false answers. These can be represented by the binary numbers 0 and 1. If one performs additions and multiplications that rely solely on these two numbers, the additions can be performed with OR operations ($0 + 0 = 0$ OR $0 = 0$; $0 + 1 = 0$ OR $1 = 1$), while the multiplications are performed with AND operations ($0 \times 0 = 0$ AND $0 = 0$; $0 \times 1 = 0$ AND $1 = 0$; $1 \times 1 = 1$ AND $1 = 1$). This means that one can roughly express linearity with OR operations between variables, while nonlinearity is better expressed with AND operations between them.

Even intuitively, such a translation does make sense. Think of OR and AND operations in Venn diagrams. Two circles linked with an OR operation become a larger circle, and there is unity. Once again, linearity is useful for modeling relatedness and unity. By contrast, two circles linked with an AND operation are separated into three different areas—one for each circle and one joint area. Thus, with AND operations, there is diversity, which we have already associated with nonlinearity.

Of course, translations into logic can be much more complex. All the math mentioned earlier is solved on a computer without math, being first translated into computer logic to be solved by the computer entirely through logical operations. Even so, the phenomena are once again linear and nonlinear.

To summarize, the most commonly observed results that we can expect to see in Causal Modeling is that systems grow, replacing systems that shrink. If the growth can be described with linear equations, this mostly results in pyramidal hierarchies. However, with nonlinear descriptions, the hierarchies that emerge can change, being turned upside down. This rotational dynamic can reverse steep hierarchies and foster diversity.

These types of behavior are the ones that should be most commonly observed. The question now is what concrete examples of applications exist, and whether they align with Thermodynamics.

The remaining parts of this article will provide examples and attempt to make matches, searching for new convergences between Thermodynamics and Causal Modeling. In particular the following questions are asked: Does the concept of entropy find new causal clarifications? How does the Constructal Law with its living, tree-shaped systems match up with Causal Modeling?

5. Entropic dead end?

In the late 1940s, Claude Shannon devised a model of human communication. His model was inspired by problems in telecommunication. Specifically, Shannon was faced with the problem that noise accumulated when messages were repeatedly transmitted through telephone lines (Shannon, 1948; Shannon and Weaver, 1949).

Shannon's model of communication maps directly onto Causal Modeling as follows.

1) Causes: A) an acoustic message and B) noise.
2) Causal mechanism: overwriting part of the message with noise during transmission.
3) Effect: a received message in which some useful information is replaced with noise.

With repeated transmission, the received message becomes increasingly noisy, until it is lost. This effect is obtained because the causal mechanism that is at work gradually overwrites the message with noise. If this noise is equated with entropy, it is evident that entropy must increase in the process. The question remains: does this transmission process offer a good context to translate the second law of Thermodynamics into communication theory?

Claude Shannon also developed an index to quantify what he called 'information entropy'. Such information entropy can indeed increase during communication. The case is easy to make. Here are the instructions.

1) Begin with a regular message that ranks low in information entropy
2) In multiple transmissions gradually overwrite this message with some kind of noise that ranks high in information entropy.
3) Evidently, the result of this overwriting-process will be an overwritten message, which will rank high in information entropy.

Just like the accumulation of thermodynamic entropy described by Max Planck, the process of accumulating information entropy only goes one way, though this is also true for any other causal process. The reverse process is always a different causal mechanism. Sometimes, this mechanism exists; other times, it doesn't. How about the present case? Are there also cases in which information entropy decreases?

This question can be answered in the affirmative. Namely, if the noise that is used to overwrite the initial message has lower information entropy than the message itself, then although noise levels increase, the overall information entropy of the system decreases. Thus.

1) Begin with a regular message that ranks high in information entropy (e.g., a whisper)
2) In multiple transmissions gradually overwrite this message with some kind of noise that ranks low in information entropy (e.g., a disturbing whistling).
3) Now the result is the opposite. Information entropy decreases over transmission.

Under these new circumstances, information entropy decreases, although noise levels still increase during transmission. Has the concept of information entropy run into a problem? Actually, the problem runs deeper.

In a 2018-article, I showed that Shannon's model of communication is a special case of communication. The model assumes that there is only one line of communication. If there are multiple intertwined lines of communication, the message that is transmitted can actually become clearer—it can even end up providing more useful information. Thus, the postulated increase in noise during communication appears to be a special case of a causal model with a single line of communication (Baciu, 2018).

With multiple lines of communication, a variation-selection process can emerge. The effects could seem paradoxical: even if noise makes messages noisier, in most instances, overall noise levels can nevertheless decrease. This effect is obtained because of selective success of messages that are meaningful. The few meaningful messages are shared widely, having a significant competitive edge over the many noisy messages. Thus, in populations that are selective, good things become influential, and they can sometimes emerge from noise or misunderstandings. Over time, noise can naturally be transformed into meaning. Shannon and my generalized causal models are shown in Fig. 2.

Given that information entropy more generally decreases during human communication, does it still make sense to speak of 'entropy'? In an interview, Shannon explained how he chose the term. According to him, the name was suggested by the mathematician John von Neumann, who said, "You should call it entropy, […] nobody knows what entropy really is, so in a debate you will always have the advantage" (Eigen, 2013). Today "information entropy" is often referred to as "Shannon index" or "Shannon diversity". Perhaps the choice of a new term illustrates that many researchers really were at a loss in debates.

The story about Shannon choosing the term "entropy" based on Neumann's whimsical suggestion found many interested ears. In a book published by Oxford University Press in 2013, Nobel laureate Manfred Eigen still recalled the interview. Initially, Eigen was fascinated with Shannon's ideas about information. In an early article, he once asked whether Shannon's theory of communication could become useful in genetics, which was then Eigen's own field of research.

Unfortunately, Eigen later discovered that genetic evolution was unlike telecommunication at least in one important aspect: In telecommunication, noise mostly increased in the line. However, in genetic evolution, the quality of the genetic information present in a system

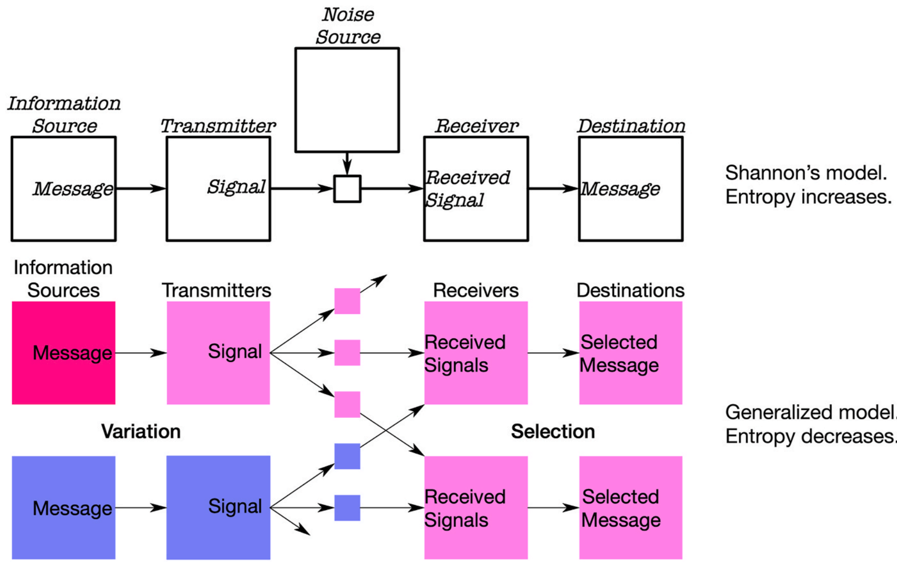

Fig. 2. Increasing and decreasing entropy. Above: Shannon's model is a special case where selection is absent; therefore entropy increases. Below: The general case includes selection; therefore entropy decreases. Source: Baciu (2018).

evolved for the better. Organisms adapted, and useful genetic information persisted and prevailed.

Finally, in his 2013-book, Eigen compared Shannon's communication theory to Darwinian evolution. His conclusion was that the two theories were antithetical. The distance between them could be compared to that from Shannon Airport in Ireland to Darwin Airport in Australia. Additionally, Eigen imagined that a crow had to fly the distance (Eigen, 2013).

In this context, it is important to appreciate both the differences and similarities between Shannon and Eigen's work. Both theories were mathematical theories. Even more specifically, both were causal explanation models. However, while noise increased in Shannon's case, it decreased in Eigen's model.

It appears that Eigen never picked up the phone to call Shannon and explain how genetic information evolved. Had he done so, Shannon could have countered that this did not matter in telecommunication engineering. On the other hand, Eigen would have had the last word. He could have made it clear that Shannon described telecommunication without taking into account the most important element of it: the people who communicated. Take people into account, and noise decreases!

Technically, Eigen's mathematical model—the Quasispecies equation—was a variation-selection-model. It is equivalent to the generalized model of communication of Fig. 2, and it can be used to explain the pyramidal distributions discovered by Pareto (Baciu, 2020). Had Eigen picked up the phone to call Shannon, he could have told him that information entropy did increase in certain aspects of telecommunication, not considering evolution. However, evolution was the more general case.

Faced with such challenging words, Shannon would have had one last resort. He could have argued that, evolution wasn't everything. Evolution explained how species evolved, which helped them become ever more efficient at what they were doing. This resulted in adaptation, competition for resources, and competitive exclusion. Yet next to evolution, there was also ecology. Compared to evolution, ecology was not about competitive exclusion and the survival of the fittest, it was about co-existence and ecosystem diversity.

More generally, evolution could be brought down to variation-selection-models, which could be represented with matrices that had real eigenvalues and demonstrated linear behavior. However, ecological diversity required more advanced nonlinear modeling—and the Shannon index was eventually referred to as "Shannon diversity index". Perhaps the name "entropy" had been a capricious choice, but the Shannon index started to be reused in a context where it provided important insights about living systems and their diversity.

6. Convergence of causal modeling and constructal law

Successful ecosystems evolve and diversify in ways that invite both linear and nonlinear descriptions. This dual nature of description—-linear and nonlinear—is also applicable to the study of cities. The social sciences have brought forth two antithetical models of the city: the concentric "Chicago model" (Burgess, 1925) and the polycentric "LA model" (Harris and Ullman, 1945).

The Chicago model can be expressed with linear equations, whereas the LA model requires nonlinear components. Thus, the two antithetical models of the city are yet another echo of the widespread distinction between linear and nonlinear behavior in Causal Modeling.

The Chicago model can be expressed with linear equations because it really needs only one variable: the city. Everything depends on it—except only for the environment. However, the environment can be modeled as a constant, without requiring another variable.

The result is a linear system of equations. The boundary between city and its environment can be blurred, but the city has only one center and it evolves to be home for only one most productive industry. This city of the Chicago model is all about unity. One can subdivide the city into multiple circles, but these circles all share the same center. They are concentric. In essence, this presents a concentric city model (Park and Burgess, 1925; Batty, 2013).

The concentric model of the city supports progress, steep hierarchies, and pyramids of influence, productivity, and density. The center of the city is the dense and productive tip of the pyramid. Density tends to decrease towards the periphery, following a Pareto distribution. The system of equations that describe the Chicago model can be pressed into a matrix with real eigenvalues. The uneven distribution that emerges is often reflected in the height of buildings. For example, Chicago's center long had the world's tallest skyscraper, while buildings in the periphery barely rose over the flat horizon of the Great Plains.

Most cities do have linear, concentric aspects, but they also exhibit nonlinear aspects. The San Francisco Bay Area and Southern California are clear examples of cities where polycentricity is strongly pronounced (Scott and Soja, 1996; Gordon and Richardson, 1996). To describe the dynamics of these polycentric urban areas, one requires models with multiple variables that are multiplied with each other (as in the term x_1x_2 in the Causal Modeling section). These variables interact in the way shown in a Venn diagram with an AND operation.

In this Venn diagram, the circles are clearly not concentric. They are independent of each other. The environment can also be interpreted as a variable or as a set of multiple variables. If one now goes ahead and builds a model with multiple variables and multiplications between them, the behavior of this new model becomes very different. It is nonlinear. The nonlinear city has multiple centers, industries, and perhaps also a greater cultural diversity (Baciu, 2023; Baciu et al., 2022). Fig. 3 visualizes the difference between concentric and polycentric cities.

The concentric and polycentric models of the city clearly illustrate the distinction between linear and nonlinear phenomena that is also observed across evolution and ecology. In the Chicago model, winners take all, establishing hierarchies and disparities, while in the LA model, hierarchies are often turned upside-down. The same distinction returns between evolution and ecology, with evolution supporting the survival of the fittest, while ecology allows for diversification. More broadly, this observed distinction is reflected in the difference between real and imaginary numbers. A matrix with eigenvectors with real eigenvalues stretches a space apart in the direction of the eigenvector, which establishes hierarchies and disparity, while a matrix with eigenvectors with imaginary eigenvalues rotates a space around the vector's tip, turning hierarchies upside-down.

At this point, it may be important to remember that causal models are merely descriptions of a more complex reality. The distinction between linear and nonlinear model behavior is not necessarily a distinction present in the underlying reality. Instead, it is often made simply to facilitate the development causal models. Furthermore, once the models become more complex, they often unite both linear and nonlinear behavior. Indeed, most matrices have many eigenvectors, some of which have real while others have imaginary eigenvalues.

A more complex mathematical model that behaves sometimes in linear and other times in nonlinear ways is shown in Fig. 4. The model involves multiple nested layers of cultural groups and subgroups that interact with each other in linear and nonlinear ways. The switch from linear to nonlinear behavior happens at certain threshold values.

The good news is that even in this more complex model, the most general types of linear and nonlinear behavior return. They result in the emergence of observable distribution patters that we have already discussed. In particular:

- Emergence of pyramidal hierarchies of influence within each group,
- Polarization between groups,
- Revolutions that turn hierarchies upside down.

As already discussed, these distribution-patterns can be observed in cities, and they can also be observed in ecosystems. In addition, they are also observed on the largest scale of the universe. Galaxies and galactic clusters also have centers of density separated by voids.

The stars are best seen at night. Let us, therefore, compare them with images of Earth taken at night, as well. While telescopes are pointed at the night sky to observe the stars, there are also satellites that point their cameras in the opposite direction, taking images of urban night lights (NASA Earth Observatory, 2020; Imhoff and Lawrence, 2003). In these nighttime images of stars and Earth, the stars and the urbanized areas look remarkably similar. In addition, if the urban areas are analyzed even further, cultural groups are found in them, and these groups are also hierarchical and separated by polarization. This is shown in Fig. 5.

The distribution patterns that are observed are momentary snapshots. Causal Modeling helps us describe how the snapshots change over time. Thus, we obtain causal descriptions of the history of both stars and cities. These causal descriptions reunite all sciences once again. The most commonly observed processes are linear and nonlinear, and they lead to the following two types of behavior.

- Linear behavior reflects the emergence of lineages.
- Nonlinear behavior reflects interplay and diversification between lineages.

When the evolution and diversification of cities, ecosystems, star clusters, etc. are described with linear and nonlinear causal models, the resulting representation mostly resembles a tree.

Historically, many cities started off with just one industry, which has gradually diversified. Cities that were initially concentric often gained

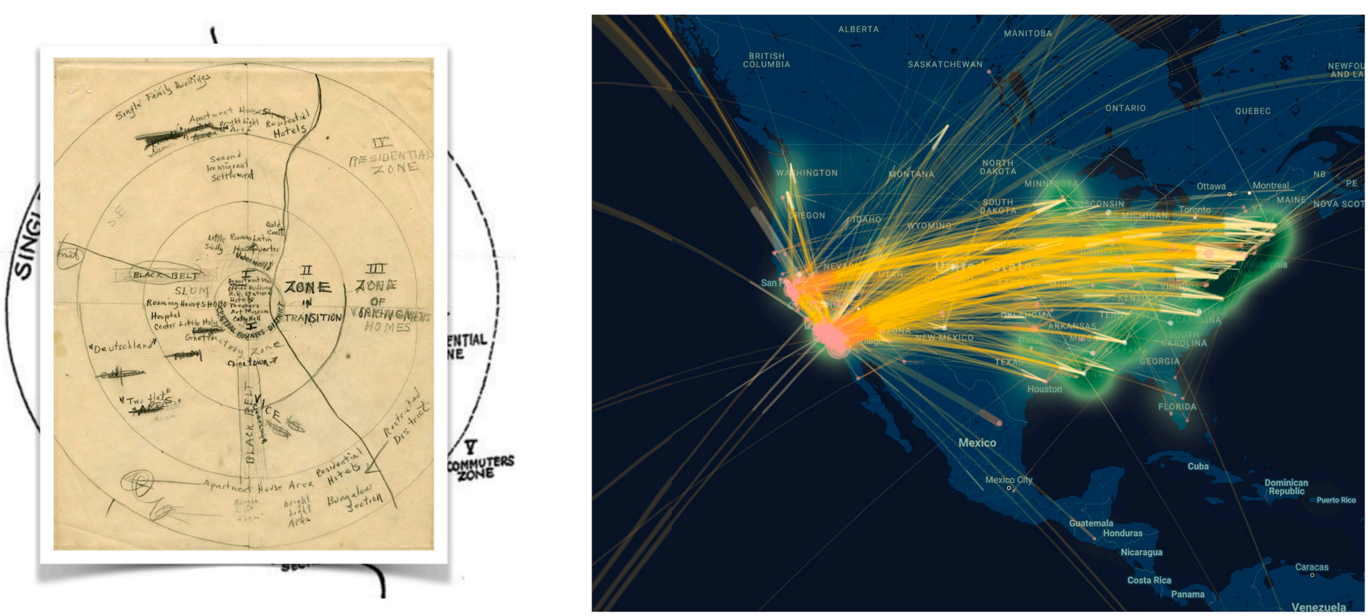

Fig. 3. Linear and nonlinear models in urban space. The social sciences have brought forth two antithetical models of the city, the concentric "Chicago" model (left) and the polycentric "LA" model. These two models represent the most common types of urban growth. Cities become ever more hierarchical with dense centers, and they become ever more diverse, with multiple independent industries and centers. This is supported empirically, but is also the most general outcome of Causal Modeling, with the case on the left being a result of eigenvectors with real eigenvalues, whereas the case on the right can be described as a result of eigenvectors with imaginary numbers as eigenvalues.

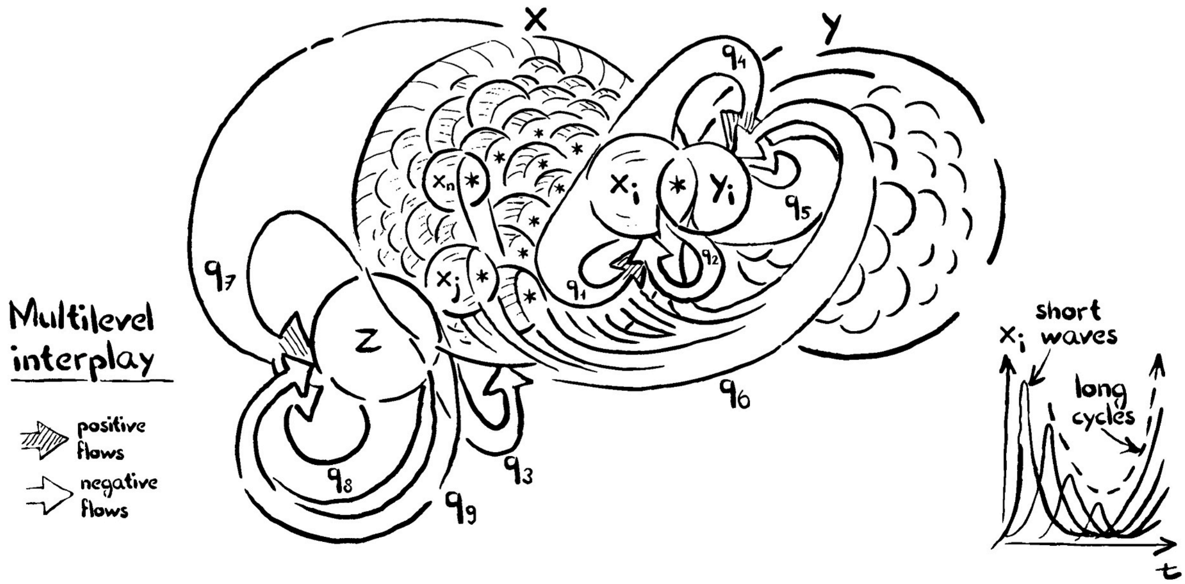

Fig. 4. A multi-level causal model with groups and subgroups. This causal model stands out for producing different types of behavior. Each arrow points from cause to effect. Each effect becomes a cause for new effects, as time passes. Causes can directly create an effect (here with causal flows *q1, 4, 5, 7, 8*), or multiple causes can interact in interplay, which, in turn, creates an effect (here with *q2, 3, 6, 9*). Graphically, each cause is visualized as a circle. Interplay is visualized as intersection between circles, as in an AND Venn diagram. Causes can also act in unison, which is visualized by placing multiple small circles into a larger circle (here, variants of x are in x, variants of y are in y). Each causal flow q has a coefficient that weighs the importance of each cause in creating the effect. The coefficient can be positive (rendered with a hash), or negative (rendered in plain white). In this particular causal model, each cause in y is in interplay with all causes in x. However, to simplify, only one instance of such interplay is drawn explicitly as an example. The equations are found in Baciu (2023), which is also the source of this visualization and contains additional material. As an empirical example, the model is used to explain the interaction between cultural groups and subgroups, for example, Science overall and individual scientific domains (Baciu 2020).

new centers, providing space for different urban activities. Beyond Urbansim, many cultures start off small and homogenous, only to grow and become ever more diverse. Even entire languages can spread while also diversifying into different dialects. Latin languages may serve as a concrete example. What was initially just Latin has become Italian, Spanish, Portuguese, French, Rumanch, Romanian, and multiple other languages. This growth and diversification are mirrored in ecosystems that grow, while phylogenetic trees document diversification.

Likewise, the universe has evolved from a state of equal density to one where dense galactic clusters are separated by increasingly vast voids (Siegel, 2017; Tully et al., 2014; Smolin, 1997; Penrose, 2010). Examples of trees in ecology and cosmology are shown in Fig. 6.

Evidently, this tree-shaped growth is also what the Constructal Law predicts. The Constructal Law says that systems construct themselves, evolving and branching out in order to gain access to resources more efficiently and more broadly. Efficient access is obtained by lineages that evolve. Broad access is obtained through diversification. Thus, it can be concluded that there is an excellent match between Causal Modeling and the Constructal Law. The linear and nonlinear aspects of Causal Modeling are found in the branches and bifurcation of constructal trees and in the idea that systems can only persist if they become more efficient and more diverse, gaining easier access to resources.

7. A new name

The present article reveals that there is a new convergence between Causal Modeling and Thermodynamics. Living systems evolve and diversify. They become ever more efficient at their tasks, and they learn to perform new tasks. Such evolution and diversification can be represented with trees. These trees of evolution and diversification link the Constructal Law and Causal Modeling seamlessly.

Nevertheless, Constructal Law and Causal Modeling provide different perspectives on the trees of evolution and diversification. The Constructal Law takes the perspective of system-analysis. It states that trees of evolution and diversification grow in order to provide easy access to resources. Causal Modeling takes a cause-effect perspective. It describes how exactly causes interact to give rise to evolution and diversification.

Given that the processes that are described are the same—even though seen from different perspectives—the question emerges whether there could be joint vocabulary. The present article has discussed various terms that are in circulation. However, it appears that each term has advantages and disadvantages.

Many terms are obtained through negation. Examples include negentropy, out-of-equilibrium dynamics, and antifragility. However, the phenomena described are of a rather general nature. It would seem more adequate to have a positive term to describe them.

Terms that are not negative often describe only part of the process. For example, the term "evolution" is often used in the narrow sense of a variation-selection process. However, it is also used for both evolution and diversification together. This can lead to confusion and neglect the importance of diversification and nonlinear, playful behavior. Additionally, such usage can make scientists forget that, by default, evolution in the absence of ecological interplay comes with competitive exclusion.

The term "constructal law" in Thermodynamics is perhaps one of the more appealing options, especially for those working in the field. Over time, it has been complemented by terms such as "constructal tree", "constructal design", "constructal process", and the like. It would be useful to incorporate verbs in this terminology. In this article, I have opted for "to construct oneself" as in "the tree-structure constructs itself". Perhaps a regular, non-reflexive verb would feel lighter.

All researchers whose work is discussed in this article have made a direct connection between constructal processes and life. The presence of evolution and diversification is often believed to represent vitality and to be characteristic for living systems. Nevertheless, as the science of such processes grew and diversified, it became evident that living systems are studied not only in Physiology but also in Physics. This sometimes leads to misunderstandings and even partisanship, with life being defined in different ways across the two fields.

Given these considerations and the shortcomings of earlier

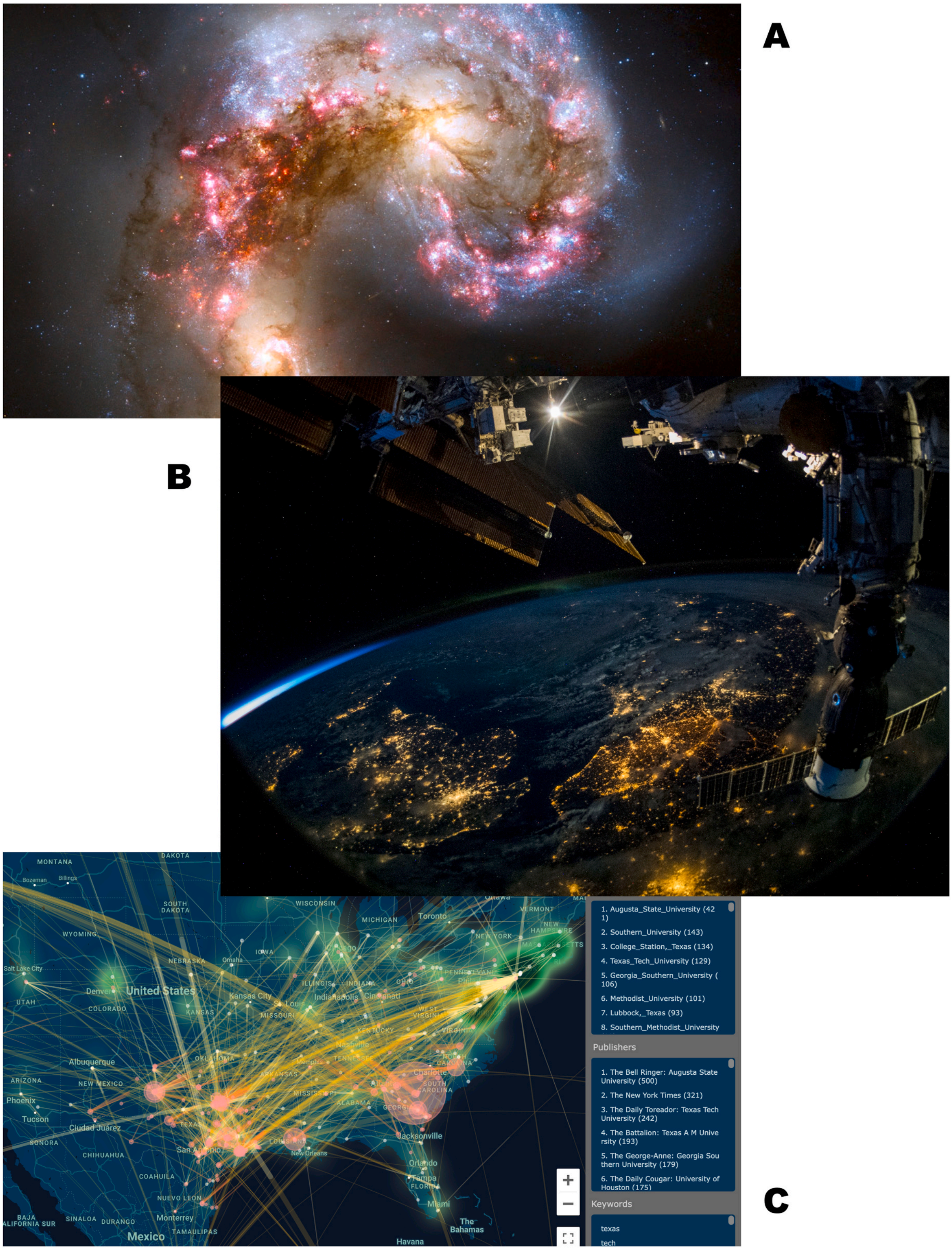

Fig. 5. Galactic clusters, urban clusters, and cultural groups. These are empirical cases showing systems that can evolve and diversify at once. A: Two colliding galaxies (NASA, ESA, and the Hubble Heritage Team STScI/AURA-ESA/Hubble Collaboration). B: Satellite image of Europe with London and Paris (ESA). C: Southern U.S. academic culture based on data extracted from news and university sources (Baciu 2020).

terminology, wouldn't it make sense to coin a new word that signifies both evolution and diversification, a word that can refer to both linear and nonlinear causal mechanisms, a word that celebrates a new convergence between Causal Modeling and Thermodynamics, and one that reunites Physics and Physiology?

Physics and Physiology both stem from the ancient Greek word "φύσις" (physis), which is related to the verb "φύω" (phyō), which means "to grow". Growing can signify that something evolves, but it is also used for things that grow apart and become more diverse. Would it make sense to introduce this root to English to provide a word that reunites the

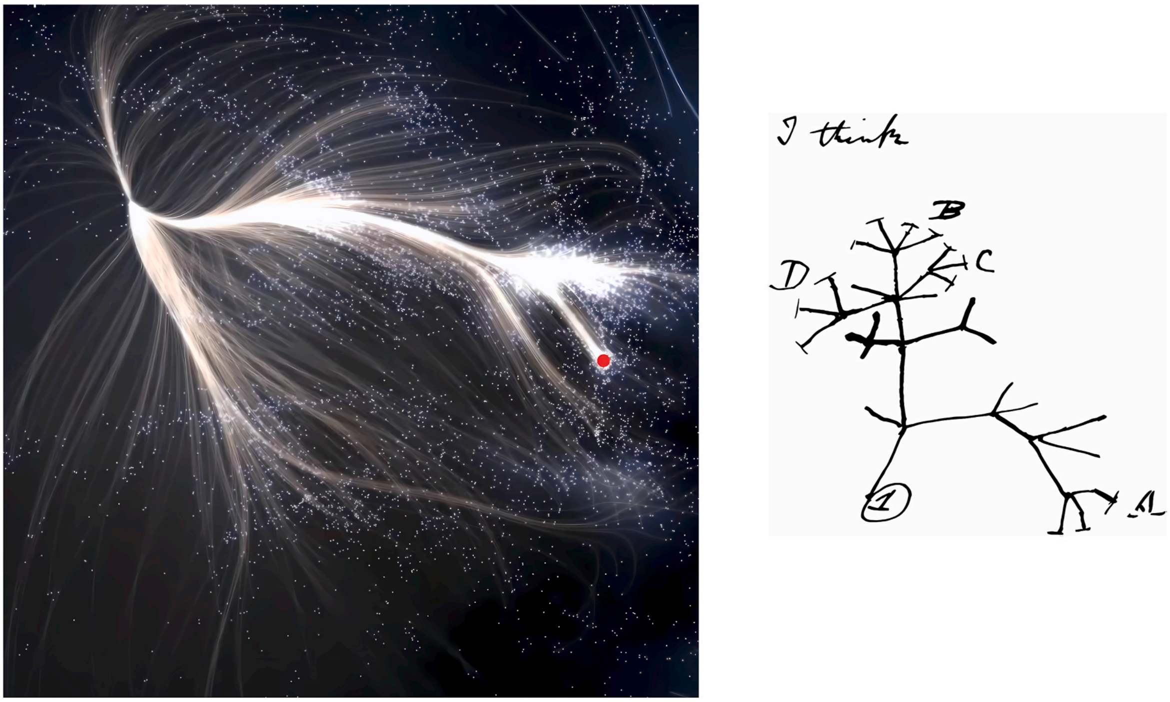

Fig. 6. Tree-shaped histories. The Laniakea supercluster and Darwin's sketch of a phylogenetic tree as examples of tree-shaped histories of evolution and diversification. Similar trees also exist around cultural lineages. Causal Modeling leverages linear and nonlinear components to describe how systems evolve and diversify. The Constructal Law offers an explanation from a systems perspective. These trees persist and prevail as long as they facilitate access to resources necessary for further construction. Left: The supercluster Laniakea according to Tully et al. (2014). Right: Charles Darwin's schematic phylogenetic tree, copied from his personal notebook.

evolutionary and branching aspects of growth?

If so, the verb "to phy" could be used to signify that a system both evolves and diversifies. One could simply say the system "phies", becoming ever more efficient and sophisticated. The term could be applied in any field. An example could be an actual botanical tree that rapidly grows. One might say the tree is rapidly "phying", growing branches in all directions to get as much exposure to sunshine as possible. Another example could be a meme that "phies", circulating and inspiring new memes that depart from the original in different ways, creating new, diverse groups of memes. Another example could be a cancer tumor that "phies", resulting in new metastases that rapidly grow, while also spreading to new types of tissue. A city might also "phy", with industries becoming ever more efficient and diverse, attracting people from different cultural and ethnic backgrounds and from all walks of life.

One could use "phy" as a noun, too. As a noun, "phy" could signify evolution and diversification in one. One could simply say, the "phy" of an ecosystem, referring to evolution and diversification in the ecosystem. Another example could be the "phy" of a city, referring to both the city's rate of growth and parameters linked to the city's diversity.

One could also turn phy into an adjective: "phy", "phyal", "phyable", etc. For example, when speaking about the evolutionary and ecological constraints associated with a system and its environment, one could simply speak of its "phyal constraints", meaning all evolutionary and ecological constraints at once.

Beyond such examples, the word "phy" could refer to the behavior of any thermodynamic system that changes according to the constructal law. Additionally, "phy" could refer to the linear and nonlinear behavior of any causal model expressed with functions—encoded with the greek letter φ. Historically, φ has silently reunited many sciences. The mathematician-astronomer Carl Friedrich Gauss used φ for functions. The refugee, architect, and professor Gottfried Semper used φ for describing creative processes in art. Quetelet and Verhulst used φ in population dynamics. Eigen and Shuster used φ in their Quasispecies model. φ has long "phied", growing powerful branches in all fields at once. Graphically, it unites a line and a curve, and it resembles tree with a trunk and a crown.

8. phy *(noun, verb, adjective)*

8.1. Noun

The process in which any system, entity, or population both evolves and becomes more diverse, especially if the process results into increasingly easy access to useful resources. The term may be used independent of scientific domain across any field of physics, physiology, or social and cultural studies.

- *Example*: "Artistic movements undergo a constant phy, with each generation adding multiple new artistic lineages and fields to the cultural landscape."
- *Example*: "The history of our supergalaxy can be represented as a phy tree, with groups of galaxies branching out as they are pushed apart by dark energy. Phylogenetic trees are another similar category of phy trees."

8.2. Verb

To grow while becoming more efficient and branching further.

- *Example*: "The technology industry continues to phy, with innovations becoming more specialized yet also more complementary."
- *Example*: "Over centuries, languages can gradually phy to create multiple distinct dialects, rich in locally meaningful vocabularies."

8.3. Adjective

Describing a system or entity that exhibits at once increased efficiency and diversity.

- *Example*: "Her approach to education is very phy. It constantly improves as she masters new specialized techniques to make learning both more efficient and diversified."
- *Example*: "This is a truly phy virus. Different new and highly transmittable escape variants are quickly evolving in response to vaccination efforts."

Etymology: Derived from the Greek root "phy" (grow).

8.4. Additional usage examples

The exposed fake news is now phying, inspiring many different memes that invite highly infectious laughter.

The cancer has phied rapidly, resulting in many fast-growing metastases.

Attacked by lions, the gazelles will phy, attempting to escape quickly in multiple directions, though the lions may still catch one.

Uncertain which gazelles are slower, the lions phy, accelerating rapidly in multiple directions.

In deltas, rivers phy, branching out while transporting sediment and pushing away the sea in all directions.

Rivers phy down from the mountains, draining away the water from entire areas through a hierarchy of rivers and rivulets.

Under political pressure, the terror organization has phied, breaking apart in multiple, hidden groups that can become very dangerous at times.

Throughout its history, Thermodynamics has phied, resulting in multiple laws and convincing applications across a wide range of different disciplines.

Airplanes are now employed in many different, highly efficient ways thanks to countless sophisticated innovations and technological phy.

If poplars aren't pollarded, their branches quickly phy to get as much exposure to sunshine as possible.

CRediT authorship contribution statement

Dan Costa Baciu: Writing – review & editing, Writing – original draft, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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