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P. A. T. Christopher, D. C. Ae., A. F. I. M. A.

SUMMARY

In Ref. 6 Cesari presented a general method, based on concepts from functional analysis and algebraic topology, for establishing the existence of periodic solutions to nonlinear differential equations. The present study shows that this method can be extended to solutions which are not periodic. In particular it is demonstrated that the equation

$$x'' + hx' + c_1 x + c_3 x^3 = Q(t), \qquad \text{(A)}$$

where Q(t) is a step-function and $x' = dx/dt$, possesses a general solution of the form

$$x = x_s + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ a_{mn} + b_{mn} \operatorname{Sin} n\omega t + c_{mn} \operatorname{Cos} n\omega t \right\}, \qquad \text{(B)}$$

where x_s is a static equilibrium value. Further, this result indicates the validity of using truncated forms of (B) for the purpose of approximation.

CONTENTS

1. Introduction

The vector equation

$$x' = X(x) + Q(t), \tag{1.1}$$

where X is a real, nonlinear, analytic, vector function of x, Q(t) a vector step-function and $x' = dx/dt$, arises in a wide class of problems in science and engineering. Unfortunately for those seeking to apply it, the general solution of this equation is unknown, except for a very limited class of special cases. See Refs. 1 and 2. It is, therefore, a worthwhile objective to seek the form of the solution to equation (1.1) or to special cases of this equation.

In the present study attention will be confined to the scalar equation

$$x'' + hx' + c_1x + c_3x^3 = Q(t), \quad h > 0, \tag{1.2}$$

where

$$t \leqslant 0, \quad Q(t) = 0$$
$$t > 0, \quad Q(t) = Q,$$

and having initial conditions $t(-0)$, $x = 0$, $x' = 0$. Alternatively this equation may be written as the equivalent system

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -hx_2 - c_1x_1 - c_3x_1^3 + Q(t) \end{aligned}\right\} \tag{1.3}$$

For values of $t > 0$ the solution of this system may be expressed as

$$x_1 = x_1(-0) + \int_{-0}^{t>0} x_2 dt = x_1(-0) + \int_{-0}^{+0} x_2 dt + \int_{+0}^{t>0} x_2 dt = \int_{+0}^{t>0} x_2 dt \tag{1.4}$$

and

$$x_2 = x_2(-0) + \int_{-0}^{t>0} x_2' dt$$

$$= x_2(-0) + \int_{-0}^{t>0} \left\{ -hx_2 - c_1x_1 - c_3x_1^3 + Q(t) \right\} dt$$

$$= x_2(-0) - h\,[x_1]_{-0}^{t>0} - \int_{-0}^{t>0} (c_1x_1 + c_3x_1^3)dt + \int_{-0}^{t>0} Q(t)dt$$

$$= x_2(-0) - h\,[x_1]_{-0}^{+0} - h\,[x_1]_{+0}^{t>0} - \int_{-0}^{+0}(c_1x_1 + c_3x_1^3)dt - \int_{+0}^{t>0}(c_1x_1 + c_3x_1^3)dt$$

$$+ \int_{-0}^{+0} Q(t)dt + \int_{+0}^{t>0} Q(t)dt$$

$$= -h\,[x_1]_{+0}^{t>0} - \int_{+0}^{t>0}(c_1x_1 + c_3x_1^3)dt + \int_{+0}^{t>0} Q(t)dt$$

or

$$x_2 = - h\,[x_1]_{+0}^{t>0} - \int_{+0}^{t>0}(c_1x_1 + c_3x_1^3)dt + \int_{+0}^{t>0} Q\,dt \qquad (1.5)$$

Differentiating these expressions with respect to t then gives

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -hx_2 - c_1x_1 - c_3x_1^3 + Q. \end{aligned}\right\} \qquad (1.6)$$

This means that, for values of $t > 0$, the solution of (1.6) with initial values taken at $t(+0)$ is the same as that of (1.3) with initial values taken at $t(-0)$.

The equivalent scalar equation to (1.6) is

$$x'' + hx' + c_1x + c_3x^3 = Q, \qquad (1.7)$$

and of particular interest are those solutions which tend asymptotically, as $t \to \infty$, to finite constant values. These static equilibrium, or steady-state, values are defined by

$$x = x_s, \qquad x' = x'' = 0 \qquad (1.8)$$

Substituting this condition into (1.7) gives

$$c_1x_s + c_3x_s^3 = Q; \qquad (1.9)$$

from which it may be concluded that, for a given value of Q, there are, at most, three values of x_s to which the solution may tend asymptotically as $t \longrightarrow \infty$.

Define the new variable ξ by

$$\xi = x - x_s. \qquad (1.10)$$

This means that if the solution of (1.7) is asymptotically stable with respect to a value x_s, then ξ is the ordinate measured from this final steady-state value. Now

$$x' = \xi', \; x'' = \xi''$$

and upon substitution into (1.7) gives

$$\xi'' + h\xi' + c_1(\xi + x_s) + c_3(\xi + x_s)^3 = Q.$$

Subtracting (1.9) from this equation then yields

$$\xi'' + h\xi' + (c_1 + 3c_3x_s^2)\xi + \theta(\xi, c_3) = 0, \qquad (1.11)$$

where $\theta(\xi, c_3) = c_3(3x_s\xi^2 + \xi^3)$, (1.12)

and with initial values $t = 0$, $\xi = -x_s$, $\xi' = 0$.

As defined, the static equilibrium points x_s are stable spiral or nodal points in the sense of Poincaré, of which there are several, depending on the signs of c_1 and c_3 and on the relative magnitude of h. See Ref. 3, Chapter 15. Initially, in order to simplify the problem, only the case $c_1 > 0$, $c_3 > 0$ will be considered. For this case only one position of static equilibrium exists, i.e. one real root of (1.9), and this is either a stable spiral or stable nodal point. The situation may conveniently be described by means of the diagram shown in Fig. 1. See, for example, Ref. 4. The points A, C and E correspond to typical initial values, $Q = 0$, $x_s = 0$, whilst the points B, D and F are related final values for which $Q \neq 0$, $x_s \neq 0$. The problem is then to determine the form of the solution of (1.11) between the static equilibrium points (they are static equilibrium points prior to the application of the step-function) A, C and E, and the final static equilibrium points B, D and F, respectively. Further, it is required to know whether the form of the solution is dependent on the position of the points A, ..., F on Fig. 1, i.e. how the form of the solution correlates with the nature of the singularities at A, ..., F.

Since equation (1.11) is analytic in ξ, then provided $\xi(t = +0) = -x_s$, and thereby Q, is not too great, an answer to the previous problem can be obtained from Lyapunov's Expansion Theorem. See Ref. 5, pp. 95-106. Writing (1.11) in vector form

$$\xi' = \begin{bmatrix} \xi'_1 \\ \xi'_2 \end{bmatrix} = A\xi + q(\xi, c_3), \tag{1.13}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -(c_1 + 3c_3 x_s^2) & -h \end{bmatrix} \text{ and } q(\xi, c_3) = \begin{bmatrix} 0 \\ -\theta(\xi_1, c_3) \end{bmatrix},$$

then (1.13) is in the form discussed in Ref. 5. Let the characteristic roots, λ_j, of A be distinct with negative real parts and satisfy no relation of the form

$$\lambda_j = m_1\lambda_1 + m_2\lambda_2$$

having integers

$$m_1, m_2 \geqslant 0, \; m_1 + m_2 > 1.$$

Let $\left\{u^1(t), \; u^2(t)\right\}$ be a base for the solution of

$$\xi' = A\xi, \tag{1.14}$$

the general solution of which is

$$u(t) = a_1 u^1(t) + a_2 u^2(t), \tag{1.15}$$

where the a_j are constants. The Expansion Theorem then states that the general solution of (1.13), in a restricted neighbourhood of $\xi = 0$, defined by $\| a \| \leqslant \rho > 0$, is

$$\xi(t) = \sum_{m=1}^{\infty} Z^m(t;a), \tag{1.16}$$

where

$$\left.\begin{aligned} Z^1(t;a) &= u(t), \\ Z^m(t;a) &= X(a) \exp\left\{ t \sum_{j=1}^{2} m_j\lambda_j \right\}, \end{aligned}\right\} \tag{1.17}$$

a is the vector $\left\{a_1, \; a_2\right\}$ and X(a) is a polynomial of degree m in the a_j. Since, for a given Q, only one singularity exists, then it may be argued from the fundamental existence theorem that the solution (1.16) will vary continuously with c_3 and a, or x_s. It may, therefore, reasonably be anticipated that the solution (1.16) can be extended to a neighbourhood whose size is much greater than that envisaged in the theorem. A shortcoming of the Expansion Theorem is that it offers no means of determining the size of the region for which the theorem is valid.

A technique which has been applied successfully to existence proofs related to periodic solutions of nonlinear differential equations is that due to Cesari. See Refs. 6, 7 and 8. It is the purpose of the present paper to explore the application of this method, suitably modified to deal with a class of non-periodic solutions, to the solution existence problem associated with equation (1.13)and hence with (1.2).

2. The Existence Theorem

In formulating the existence theorem it is convenient, for the purpose of simplifying some of the algebraic and trigonometric manipulation, to divide the solutions up into two classes. Those for whom the characteristic roots of A are real and negative, and those for whom the roots are complex and have negative real parts. The components of the solutions to equation (1.13) may then be expressed as

$$\xi_j(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_j a_m + {}_j b_{mn} e^{-nlt} + {}_j c_{mn} e^{nlt} \right\}, \qquad (2.1)$$

in the case of real roots, and

$$\xi_j(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_j a_m + {}_j b_{mn} \operatorname{Sin} n\omega t + {}_j c_{mn} \operatorname{Cos} n\omega t \right\}, \qquad (2.2)$$

in the case of complex roots, where m, n, k, l and ω are real and m, n are positive integers. Since these are only a modified re-statement of (1.16), and the solutions given by the Expansion Theorem are general solutions (See Ref. 5, p. 99), then (2.1) or (2.2) are general solutions. Further, it is intended to discuss only the case of solutions of the form (2.2).

Consider the function space S of real vector functions $\xi(t) = \{ \xi_1(t), \xi_2(t) \}$ whose components have the form (2.2) and having a norm $\nu(\xi)$ defined by

$$\nu(\xi) = \sup_{(0 \leq t \leq \infty)} \sum_{j=1}^{2} |\xi_j| \qquad (2.3)$$

A projection operator P may be defined in S by the relations

$$P\xi = (P_1\xi_1, P_2\xi_2) \qquad (2.4)$$

$$P_j\xi_j = \sum_{n=1}^{n=r} \sum_{m=1}^{m=s} e^{mkt} \left\{ {}_j a_m + {}_j b_{mn} \operatorname{Sin} n\omega t + {}_j c_{mn} \operatorname{Cos} n\omega t \right\}, \qquad (2.5)$$

where r, s > 1. Also, by definition, $P = P^2$.

The subspace $\tilde{S}$ of S is defined by

$$\tilde{S} = \{\xi : \xi \epsilon S,\ P\xi = 0\} \tag{2.6}$$

and, thereby, if $\xi \epsilon \tilde{S}$ then

$$\xi_j(t) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} e^{mkt} \left\{ {}_j a_m + {}_j b_{mn} \text{ Sin } n\omega t + {}_j c_{mn} \text{ Cos } n\omega t \right\} \tag{2.7}$$

An operator H on $\tilde{S}$ may be defined by the relation

$$H\xi_j = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} e^{mkt} \left\{ \frac{{}_j a_m}{mk} + \frac{mk\,{}_j b_{mn} + n\omega\,{}_j c_{mn}}{(mk)^2 + (n\omega)^2} \text{ Sin } n\omega t \right.$$

$$\left. + \frac{mk\,{}_j c_{mn} - n\omega\,{}_j b_{mn}}{(mk)^2 + (n\omega)^2} \text{ Cos } n\omega t \right\}, \tag{2.8}$$

which corresponds to the integration of $\xi_j \epsilon \tilde{S}$ with the constants of integration taken to be zero. Further, consider the operators f, F and T on S defined by

$$f\xi = q\xi - Pq\xi \tag{2.9}$$

$$F\xi = Hf\xi \tag{2.10}$$

and

$$y = T\xi = P\xi + F\xi; \tag{2.11}$$

or, in more detail,

$$f\xi = \begin{bmatrix} \xi_2 - P\xi_2 \\ -(c_1 + 3c_3x_s^2)\xi_1 - \theta(\xi_1) - h\xi_2 + P\{(c_1 + 3c_3x_s^2)\xi_1 + \theta(\xi_1) + h\xi_2\} \end{bmatrix}, \tag{2.12}$$

$$F\xi = \begin{bmatrix} H(\xi_2 - P\xi_2) \\ H\left[-(c_1 + 3c_3x_s^2)\xi_1 - \theta(\xi_1) - h\xi_2 + P\{(c_1 + 3c_3x_s^2)\xi_1 + \theta(\xi_1) + h\xi_2\}\right] \end{bmatrix} \tag{2.13}$$

and

$$y = \begin{bmatrix} P\xi_1 + H(\xi_2 - P\xi_2) \\ P\xi_2 + H\left[-(c_1 + 3c_3x_s^2)\xi_1 - \theta(\xi_1) - h\xi_2 + P\{(c_1 + 3c_3x_s^2)\xi_1 + \theta(\xi_1) + h\xi_2\}\right] \end{bmatrix} \tag{2.14}$$

In Section 3 an approximate solution (3.1) will be developed whose components correspond to $P_j\xi_j$ of (2.5) with r = s = 3. By placing certain bounds on $\nu(\xi)$, $|\xi|$, $\nu(\xi - P\xi)$ and $|\xi - P\xi|$ it is possible to define a subspace S_R^* of S and, provided certain inequalities are satisfied, it will be shown that $T: S_R^* \longrightarrow S_R^*$ and is also a contraction mapping. Because T is a contraction in S_R^*, Banach's fixed point theorem (See Ref. 9, p. 141) may be invoked to conclude that y(t) exists uniquely in S_R^* and is continuously dependent on the approximate solution (3.1). This means that ${}_ja_m$, ${}_jb_{mn}$ and ${}_jc_{mn}$ with m, n = 4, 5, ... are uniquely determined by and continuously dependent on ${}_1a_1, \ldots, {}_2a_3, {}_1b_{11}, \ldots, {}_2b_{33}, {}_1c_{11}, \ldots, {}_2c_{33}$.

If $y(t) \in S_R^*$ is a fixed element of T in S_R^*, then

$$y = Py + Fy$$

or

$$y - Py = \begin{bmatrix} H(y_2 - Py_2) \\ H\left[-(c_1 + 3c_3x_s^2)y_1 - \theta(y_1) - hy_2 + P\{(c_1 + 3c_3x_s^2)y_1 + \theta(y_1) + hy_2\}\right] \end{bmatrix}$$

Differentiating this expression with respect to t gives

$$y_1' = y_2 + P(y_1' - y_2)$$

$$y_2' = -(c_1 + 3c_3x_s^2)y_1 - \theta(y_1) - hy_2 + P\{y_2' + (c_1 + 3c_3x_s^2)y_1 + \theta(y_1) + hy_2\}$$

Thus y(t) will satisfy (1.13) provided

$$P(y_1' - y_2) = 0 \qquad (2.15)$$

$$P\{y_2' + hy_2 + (c_1 + 3c_3x_s^2)y_1 + \theta(y_1)\} = 0 \qquad (2.16)$$

If y(t) is the fixed element described, then $y = \xi$ and $y_j(t)$ will be given by (2.2). Consider an approximate form for $y_j(t)$ given by

$$y_j(t) = \sum_{n=1}^{3} \sum_{m=1}^{3} e^{mkt} \left\{ {}_ja_m + {}_jb_{mn} \text{ Sin } n\omega t + {}_jc_{mn} \text{ Cos } n\omega t \right\}. \qquad (2.17)$$

Then by a procedure, closely paralleling that in Section 3, of substitution into (2.15) and (2.16) and equating of coefficients of distinct terms there arise an approximate set of determining equations which are the same as those in the set of equations (3.68) to (3.76). If, instead of the approximation (2.17), the proposed exact solution (2.2) is used in this procedure there will arise an exact set of determining equations

$$
\left.
\begin{aligned}
V_1 &\equiv {}_1a_2(\omega^2 + k^2) + c_3\alpha_1 &&= 0\\
V_2 &\equiv (k^2 - 3\omega^2)\,{}_1b_{22} - 4\omega k\,{}_1c_{22} + c_3\alpha_2 &&= 0\\
V_3 &\equiv 4\omega k\,{}_1b_{22} + (k^2 - 3\omega^2)\,{}_1c_{22} + c_3\alpha_3 &&= 0\\
V_4 &\equiv 4k^2\,{}_1b_{31} - 4\omega k\,{}_1c_{31} + c_3\alpha_4 &&= 0\\
V_5 &\equiv 4\omega k\,{}_1b_{31} + 4k^2\,{}_1c_{31} + c_3\alpha_5 &&= 0\\
V_6 &\equiv 4(k^2 - 2\omega^2)\,{}_1b_{33} - 12\omega k\,{}_1c_{33} + c_3\alpha_6 &&= 0\\
V_7 &\equiv 12\omega k\,{}_1b_{33} + 4(k^2 - 2\omega^2)\,{}_1c_{33} + c_3\alpha_7 &&= 0\\
V_8 &\equiv \xi_1(t = 0) + x_s = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}({}_1a_m \infty {}_1c_{mn}) + x_s &&= 0\\
V_9 &\equiv \xi_2(t = 0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}({}_2a_m + {}_2c_{mn})\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{ mk({}_1a_m + {}_1c_{mn}) + n\omega\,{}_1b_{mn}\right\} &&= 0
\end{aligned}
\right\} \quad (2.18)
$$

Taking the exact values of ${}_1a_2, \ldots, {}_1b_{33}$ and substituting these into the expressions on the left hand side of equations (3.68) to (3.76), respectively, will then give rise to the set of equations

$$
\left.
\begin{aligned}
v_1 &\equiv {}_1a_2(\omega^2 + k^2) + c_3\alpha_{10}\\
v_2 &\equiv (k^2 - 3\omega^2)\,{}_1b_{22} - 4\omega k\,{}_1c_{22} + c_3\alpha_{20}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
v_8 &\equiv {}_1c_{11} + {}_1a_2 + {}_1c_{22} + {}_1c_{31} + {}_1c_{33} + x_s\\
v_9 &\equiv k\left\{{}_1c_{11} + 2({}_1a_2 + {}_1c_{22}) + 3({}_1c_{31} + {}_1c_{33})\right\}\\
&\quad + \omega\left\{{}_1b_{11} + 2\,{}_1b_{22} + {}_1b_{31} + 3\,{}_1b_{33}\right\},
\end{aligned}
\right\} \quad (2.19)
$$

where $\alpha_{10}, \ldots, \alpha_{70}$ are given in Section 3. Thus, if $\alpha_{10}, \ldots, \alpha_{70}$ are good approximations to $\alpha_1, \ldots, \alpha_7$, respectively, and α_8 and α_9 are very small then $v_1, \ldots, v_9$ will be only slightly different from zero.

Denote by Λ the nine-cell defined by $|{}_1b_{11}| \leqslant \mu_1 |x_s|, |{}_1c_{11}| \leqslant \mu_2 |x_s|, \ldots, |{}_1c_{33}| \leqslant \mu_9 |x_s|, \mu_1, \ldots, \mu_9 > 0$, in the Euclidean nine-space of Cartesian co-ordinates ${}_1b_{11}, \ldots, {}_1c_{33}$. Let M and M_o be mappings, described by the sets of equations (2.18) and (2.19), respectively, from the vector space of components $({}_1b_{11}, \ldots, {}_1b_{33})$ to the space of components $(V_1, \ldots, V_9)$ and $(v_1, \ldots, v_9)$, respectively. These mappings are single-valued and continuous. Define C and C_o as the closed eight-cells described by $M\Lambda_B$ and $M_o\Lambda_B$, respectively, where Λ_B is the boundary of Λ. It may be verified directly whether, or not, the origin of the image nine-space lies in C_o, whether C_o has non-zero order, $v(C_o, 0)$, with respect to the origin (see Ref. 9, p. 15 and p. 30), and the distance

$$|(u, v) - 0| = |\{v_1^2 + v_2^2 + \ldots + v_9^2\}^{\frac{1}{2}}| \tag{2.20}$$

of the origin from points on C_o, may be determined. Further, the estimate for $|(U, V) - (u, v)|$, the distance between points in C and points in C_o, given by equation (6.1) may be used. If it can be established that

$$\text{glb}|(U, V) - (u, v)| < \text{lub}|(u, v) - 0|, \tag{2.21}$$

then by Rouché's theorem (see Ref. 10, Vol. 3, p. 103) it follows that

$$v(C, 0) = v(C_o, 0) \neq 0 \tag{2.22}$$

or that

$$\gamma(M, \Lambda, 0) = \gamma(M_o, \Lambda, 0) \neq 0 \tag{2.23}$$

where $\gamma(M, \Lambda, 0)$ is the local topological degree of M at the origin relative to Λ. It then follows from Ref. 9, p. 32, Theorem 6.6 that there is a point in the interior of Λ for which $v_1 = v_2 = \ldots = v_9 = 0$, and another point in the interior of Λ for which $V_1 = V_2 = \ldots = V_9 = 0$. This implies that the exact system of determining equations (2.18) are satisfied for certain values of ${}_1b_{11}, \ldots, {}_1b_{33}$ contained in the cell Λ and, therefore, $y = \xi$, as given by (2.2), is an exact solution of (1.13) and, thereby, (1.11), for certain values ${}_1b_{11}, \ldots, {}_1b_{33}$ contained in Λ.

3. An Approximate Solution

As a first step in proving that (2.2) is a solution of the equation (1.13), it is proposed to show that

$$\xi_j(t) = \sum_{n=1}^{3} \sum_{m=1}^{3} e^{mkt} \left\{ {}_j a_m + {}_j b_{mn} \text{ Sin } n\omega t + {}_j c_{mn} \text{ Cos } n\omega t \right\} \quad (3.1)$$

is an approximate solution to (1.13). Thus ξ_j may be written as

$$\xi_j = A_j e^{kt} + B_j e^{2kt} + C_j e^{3kt}, \quad (3.2)$$

where

$$A_j = {}_j a_1 + {}_j b_{11} \text{ Sin } \omega t + \ldots. + {}_j c_{13} \text{ Cos } 3\omega t, \quad (3.3)$$

$$B_j = {}_j a_2 + {}_j b_{21} \text{ Sin } \omega t + \ldots. + {}_j c_{23} \text{ Cos } 3\omega t \quad (3.4)$$

and

$$C_j = {}_j a_3 + {}_j b_{31} \text{ Sin } \omega t + \ldots. + {}_j c_{33} \text{ Cos } 3\omega t \quad (3.5)$$

Differentiating (3.2) with respect to t gives

$$\xi_j' = (kA_j + A_j')e^{kt} + (2kB_j + B_j')e^{2kt} + (3kC_j + C_j')e^{3kt}, \quad (3.6)$$

where

$$A_j' = \omega({}_j b_{11} \text{ Cos } \omega t - {}_j c_{11} \text{ Sin } \omega t) + 2\omega({}_j b_{12} \text{ Cos } 2\omega t - {}_j c_{12} \text{ Sin } 2\omega t)$$
$$+ 3\omega({}_j b_{13} \text{ Cos } 3\omega t - {}_j c_{13} \text{ Sin}$$

$$B_j' = \omega({}_j b_{21} \text{ Cos } \omega t - {}_j c_{21} \text{ Sin } \omega t) + 2\omega({}_j b_{22} \text{ Cos } 2\omega t - {}_j c_{22} \text{ Sin}$$
$$+ 3\omega({}_j b_{23} \text{ Cos } 3\omega t - {}_j c_{23} \text{ Sin } 3\omega t)$$

and

$$C_j' = \omega({}_j b_{31} \text{ Cos } \omega t - {}_j c_{31} \text{ Sin } \omega t) + 2\omega({}_j b_{32} \text{ Cos } 2\omega t - {}_j c_{32} \text{ Sin } 2\omega t)$$
$$+ 3\omega({}_j b_{33} \text{ Cos } 3\omega t - {}_j c_{33} \text{ Sin } 3\omega t).$$

Substituting for ξ_1' and ξ_2 in the first component equation of (1.13) then gives

$$(kA_1 + A_1')e^{kt} + (2kB_1 + B_1')e^{2kt} + (3kC_1 + C_1')e^{3kt}$$
$$= A_2 e^{kt} + B_2 e^{2kt} + C_2 e^{3kt}, \quad (3.7)$$

which upon equating the coefficients of the distinct terms (Note. Two terms are distinct if they are not linearly dependent. Thus the distinct terms are of the form ae^{mkt}, be^{mkt} Sin $n\omega t$ and ce^{mkt} Cos $n\omega t$, respectively) gives rise to the system of equations

$$\left.\begin{array}{lll}
{}_2a_1 = k_1a_1 & {}_2a_2 = 2k_1a_2 & {}_2a_3 = 3k_1a_3 \\
{}_2b_{11} = k_1b_{11} - \omega_1c_{11} & {}_2b_{21} = 2k_1b_{21} - \omega_1c_{21} & {}_2b_{31} = 3k_1b_{31} - \omega_1c_{31} \\
{}_2c_{11} = k_1c_{11} + \omega_1b_{11} & {}_2c_{21} = 2k_1c_{21} + \omega_1b_{21} & {}_2c_{31} = 3k_1c_{31} + \omega_1b_{31} \\
{}_2b_{12} = k_1b_{12} - 2\omega_1c_{12} & {}_2b_{22} = 2(k_1b_{22} - \omega_1c_{22}) & {}_2b_{32} = 3k_1b_{32} - 2\omega_1c_{32} \\
{}_2c_{12} = k_1c_{12} + 2\omega_1b_{12} & {}_2c_{22} = 2(k_1c_{22} + \omega_1b_{22}) & {}_2c_{32} = 3k_1c_{32} + 2\omega_1b_{32} \\
{}_2b_{13} = k_1b_{13} - 3\omega_1c_{13} & {}_2b_{23} = 2k_1b_{23} - 3\omega_1c_{23} & {}_2b_{33} = 3(k_1b_{33} - \omega_1c_{33}) \\
{}_2c_{13} = k_1c_{13} + 3\omega_1b_{13} & {}_2c_{23} = 2k_1c_{23} + 3\omega_1b_{23} & {}_2c_{33} = 3(k_1c_{33} + \omega_1b_{33})
\end{array}\right\} \tag{3.8}$$

which will be required later.

Now from (3.2)

$$\xi_1^2 = (A_1e^{kt} + B_1e^{2kt} + C_1e^{3kt})^2$$

$$= A_1^2e^{2kt} + 2A_1B_1e^{3kt} + (B_1^2 + 2A_1C_1)e^{4kt} + 2B_1C_1e^{5kt} + C_1^2e^{6kt}$$

and

$$\xi_1^3 = A_1^3e^{3kt} + 3A_1^2B_1e^{4kt} + 3(A_1^2C_1 + A_1B_1^2)e^{5kt} + (B_1^3 + 6A_1B_1C_1)e^{6kt}$$

$$+ 3(A_1C_1^2 + B_1^2C_1)e^{7kt} + 3B_1C_1^2e^{8kt} + C_1^3e^{9kt},$$

thus

$$(c_1 + 3c_3x_s^2)\xi_1 + \theta(\xi_1, c_3)$$

$$= A_1(c_1 + 3c_3x_s^2)e^{kt} + \left\{B_1(c_1 + 3c_3x_s^2) + 3c_3x_s(P_1A_1^2)\right\}e^{2kt}$$

$$+ \left\{C_1(c_1 + 3c_3x_s^2) + 6c_3x_s(P_1A_1B_1) + c_3(P_1A_1^3)\right\}e^{3kt}, \tag{3.9}$$

where, to be consistent with the degree of approximation made in equation (3.2), only terms up to e^{3kt} have been retained and the projection operator P_1 now refers specifically to the conditions $r = s = 3$. Substituting from

(3.2), (3.6) and (3.9) into the second component equation of (1.13) then gives

$$(kA_2 + A_2')e^{kt} + (2kB_2 + B_2')e^{2kt} + (3kC_2 + C_2')e^{3kt}$$

$$+ h(A_2e^{kt} + B_2e^{2kt} + C_2e^{3kt}) + A_1(c_1 + 3c_3x_s^2)e^{kt}$$

$$+ \left\{B_1(c_1 + 3c_3x_s^2) + 3c_3x_s(P_1A_1^2)\right\} e^{2kt}$$

$$+ \left\{C_1(c_1 + 3c_3x_s^2) + 6c_3x_s(P_1A_1B_1) + c_3(P_1A_1^3)\right\} e^{3kt} = 0$$

or, upon re-arranging,

$$\left\{(k + h)A_2 + A_2' + A_1(c_1 + 3c_3x_s^2)\right\} e^{kt}$$

$$+ \left\{(2k + h)B_2 + B_2' + B_1(c_1 + 3c_3x_s^2) + 3c_3x_s(P_1A_1^2)\right\} e^{2kt}$$

$$+ \left\{(3k + h)C_2 + C_2' + C_1(c_1 + 3c_3x_s^2) + 6c_3x_s(P_1A_1B_1) + c_3(P_1A_1^3)\right\}e^{3kt} = 0 \quad (3.10)$$

Excluding the trivial case $k \rightarrow \infty$, then in order to satisfy equation (3.10) each of the coefficients of e^{kt}, e^{2kt} and e^{3kt}, respectively, must be zero for all t. This means that the constant term and the coefficients of Sin ωt, Cos ωt,..... .., Cos $3\omega t$ in each of the $\{\}$ brackets must be zero. Thus for each $\{\}$ bracketed expression there is established a set of seven equations. Taking the three sets of seven equations arising from (3.10), together with two more equations arising from the initial conditions at t(+0) and twenty-one equations in (3.8), there are a total of fourty-four equations whose simultaneous solution defines the fourty-two coefficients ${}_1a_1$, ${}_1b_{11}$,...., ${}_2b_{33}$, ${}_2c_{33}$ in A_1,....., C_2 and the values of k and ω.

The terms A_1^2, A_1B_1 and A_1^3 are, obviously, going to give rise to considerable labour in their expansion. Considerable reduction in the amount of work may be achieved by first considering the set of seven equations associated with the coefficient of e^{kt} in (3.10). These are

$$(k + h)_2a_1 + (c_1 + 3c_3x_s^2)_1a_1 = 0 \quad (3.11)$$

$$(k + h)_2b_{11} - \omega_2c_{11} + (c_1 + 3c_3x_s^2)_1b_{11} = 0 \quad (3.12)$$

$$(k + h)_2c_{11} + \omega_2b_{11} + (c_1 + 3c_3x_s^2)_1c_{11} = 0 \quad (3.13)$$

$$(k + h)\,{}_2b_{12} - 2\omega\,{}_2c_{12} + (c_1 + 3c_3x_s^2)\,{}_1b_{12} = 0 \qquad (3.14)$$

$$(k + h)\,{}_2c_{12} + 2\omega\,{}_2b_{12} + (c_1 + 3c_3x_s^2)\,{}_1c_{12} = 0 \qquad (3.15)$$

$$(k + h)\,{}_2b_{13} - 3\omega\,{}_2c_{13} + (c_1 + 3c_3x_s^2)\,{}_1b_{13} = 0 \qquad (3.16)$$

$$(k + h)\,{}_2c_{13} + 3\omega\,{}_2b_{13} + (c_1 + 3c_3x_s^2)\,{}_1c_{13} = 0 \qquad (3.17)$$

Substituting from the first seven equations of (3.8) into equations (3.11) to (3.17) then gives

$$E\,{}_1a_1 = 0 \qquad (3.18)$$

$$(E - \omega^2)\,{}_1b_{11} - \omega J\,{}_1c_{11} = 0 \qquad (3.19)$$

$$(E - \omega^2)\,{}_1c_{11} + \omega J\,{}_1b_{11} = 0 \qquad (3.20)$$

$$(E - 4\omega^2)\,{}_1b_{12} - 2\omega J\,{}_1c_{12} = 0 \qquad (3.21)$$

$$(E - 4\omega^2)\,{}_1c_{12} + 2\omega J\,{}_1b_{12} = 0 \qquad (3.22)$$

$$(E - 9\omega^2)\,{}_1b_{13} - 3\omega J\,{}_1c_{13} = 0 \qquad (3.23)$$

$$(E - 9\omega^2)\,{}_1c_{13} + 3\omega J\,{}_1b_{13} = 0, \qquad (3.24)$$

where

$$E = k^2 + hk + c_1 + 3c_3x_s^2 \qquad (3.25)$$

and

$$J = 2k + h. \qquad (3.26)$$

In addition there arise two equations from the initial conditions on ξ_1 and ξ_2 at t(+0). When $t = +0$, $\xi_1 = -x_s$, or, from (3.2) to (3.5)

$$\sum_{m=1}^{3} ({}_1a_m + {}_1c_{m1} + {}_1c_{m2} + {}_1c_{m3}) + x_s = 0. \qquad (3.27)$$

Also when $t = +0$, $\xi_1' = \xi_2 = 0$ or, from (3.2) to (3.6)

$$\sum_{m=1}^{3}\left\{mk({}_1a_m + {}_1c_{m1} + {}_1c_{m2} + {}_1c_{m3}) + \omega({}_1b_{m1} + 2\,{}_1b_{m2} + 3\,{}_1b_{m3})\right\} = 0 \qquad (3.28)$$

In order to determine the various unknowns in the set of equations (3.18) to (3.28) it is convenient to first consider the solution of the linear equation obtained by putting $c_3 \equiv 0$ in (1.13). The characteristic roots associated with this equation are

$$\lambda_{1,2} = \frac{1}{2}\left\{-h \pm (h^2 - 4c_1)^{\frac{1}{2}}\right\}. \qquad (3.29)$$

Three cases exist: (a) roots real and unequal, (b) roots complex, and (c) roots real and equal; the case of purely imaginary roots does not occur unless $h = 0$. Only in case (b) does the solution of this linear equation have the form of (3.1). Attention, therefore, will be restricted to this case, where $4c_1 > h^2$. This being so, then the solution to the linear problem will contain only the coefficients ${}_1b_{11}$ and ${}_1c_{11}$, whilst all the other coefficients are zero. The set of equations (3.18) to (3.28) thereby reduce to

$$(E - \omega^2)\,{}_1b_{11} - \omega J\,{}_1c_{11} = 0 \qquad (3.30)$$

$$(E - \omega^2)\,{}_1c_{11} + \omega J\,{}_1b_{11} = 0 \qquad (3.31)$$

$${}_1c_{11} + x_s = 0 \qquad (3.32)$$

$$k\,{}_1c_{11} + \omega\,{}_1b_{11} = 0 \qquad (3.33)$$

together with (3.25), in which c_3 is taken to be zero, and (3.26). From (3.32) and (3.33)

$${}_1b_{11} = kx_s/\omega \qquad \text{and} \qquad {}_1c_{11} = -x_s$$

which upon substitution into (3.30) and (3.31) gives

$$(E - \omega^2)x_s k/\omega + \omega J x_s = 0$$

$$-(E - \omega^2)x_s + kJx_s = 0.$$

Since the case $x_s = 0$ is trivial, then it may be assumed that in general $x_s \neq 0$ and these equations may be divided through by x_s to give

$$k(E - \omega^2) + \omega^2 J = 0 \tag{3.34}$$

$$(E - \omega^2) - kJ = 0 \tag{3.35}$$

Multiplying (3.35) by k and subtracting from (3.34) gives

$$(k^2 + \omega^2)J = 0.$$

Now k and ω are real and, therefore, $k^2 + \omega^2 \neq 0$ unless $k = 0$, $\omega = 0$. This would be a trivial result, therefore it follows that $J = 0$. Thus, from (3.26),

$$k = -\frac{1}{2}h. \tag{3.36}$$

Putting $J = 0$ in (3.35) gives

$$E - \omega^2 = 0 = k^2 + hk + c_1 - \omega^2, \text{ from (3.25)}$$

$$= \frac{1}{4}h^2 - \frac{1}{2}h^2 + c_1 - \omega^2, \text{ from (3.36)}$$

or

$$\omega^2 = c_1 - \frac{1}{4}h^2 = c_1 - k^2 \tag{3.37}$$

Returning to the problem of solving (3.18) to (3.28) simultaneously when $c_3 \neq 0$, it will be observed that (1.13) is analytic in ξ and c_3 and it follows from Ref. 3, p. 36, Theorem 8.4 that the solution of (1.13) is continuous in c_3 for t in some open interval I_1 and c_3 in some open interval I_2. The proposed solution (2.2) is to be valid for $(+0) \leq t \leq \infty$, and it follows from the formulation of the above theorem, that if (2.2) is the solution described in the theorem then I_1: $(+0) \leq t \leq \infty$ and I_2: $0 \leq c_3 \leq \mu$, where $\mu > 0$. Thus the coefficients ${}_ja_m$, ${}_jb_{mn}$ and ${}_jc_{mn}$ in (2.2) must be continuous in c_3 for c_3 in I_2. Now the coefficients ${}_1b_{11}$, ${}_1c_{11} \neq 0$ for $c_3 = 0$, $x_s \neq 0$, and since these coefficients are continuous in c_3 then ${}_1b_{11}$, ${}_1c_{11} \not\equiv 0$ for $c_3 \neq 0$, $x_s \neq 0$.

Now upon squaring (3.19) and (3.20) and adding there arises the equation

$$\left\{(E - \omega^2)^2 + \omega^2 J^2\right\}({}_1b_{11}^2 + {}_1c_{11}^2) = 0$$

Since ${}_1b_{11}$, ${}_1c_{11} \not\equiv 0$ for $c_3 \neq 0$, $x_s \neq 0$, then it follows that

$$(E - \omega^2)^2 + \omega^2 J^2 = 0 \tag{3.38}$$

From (3.18) either $E = 0$ or ${}_1a_1 = 0$. If $E = 0$ then upon substitution into (3.38)

$$\omega^2(\omega^2 + J^2) = 0$$

Since J is real this equation can only be satisfied by the trivial result $\omega = 0$. It follows that

$$ {}_1a_1 = 0 \tag{3.39}$$

Subtracting ${}_1c_{11} \times$ (3.19) from ${}_1b_{11} \times$ (3.20) gives

$$\omega J({}_1b_{11}^2 + {}_1c_{11}^2) = 0,$$

which implies that

$$\omega J = 0 \tag{3.40}$$

For ω to be zero would be trivial, so $J = 0$. Thus from (3.26)

$$k = -\frac{1}{2}h \tag{3.41}$$

Substituting $J = 0$ into (3.38) gives

$$\omega^2 = E = k^2 + hk + c_1 + 3c_3x_s^2$$

$$= c_1 + 3c_3x_s^2 - \frac{1}{4}h^2 \text{, from (3.41)} \tag{3.42}$$

It is worthy of note that the values of k and ω, as given by (3.41) and (3.42), respectively, will be the same regardless of the degree of the approximation used in (3.1).

Putting $J = 0$ in equations (3.21) to (3.24) gives

$$(E - 4\omega^2)_1b_{12} = (E - 4\omega^2)_1c_{12} = (E - 9\omega^2)_1b_{13} = (E - 9\omega^2)_1c_{13} = 0,$$

which upon substituting $E = \omega^2$, from (3.42), gives

$$-3\omega^2{}_1b_{12} = -3\omega^2{}_1c_{12} = -8\omega^2{}_1b_{13} = -8\omega^2{}_1c_{13} = 0.$$

Since $\omega = 0$ would be trivial, then

$$ {}_1b_{12} = {}_1c_{12} = {}_1b_{13} = {}_1c_{13} = 0 \tag{3.43}$$

Substituting from (3.39) and (3.43) into (3.3) gives

$$A_1 = {}_1b_{11}\ \text{Sin}\ \omega t + {}_1c_{11}\ \text{Cos}\ \omega t, \tag{3.44}$$

from which

$$\begin{aligned} A_1^2 = P_1A_1^2 &= {}_1b_{11}^2\ \text{Sin}^2\omega t + {}_1c_{11}^2\ \text{Cos}^2\omega t + 2\,{}_1b_{11}\,{}_1c_{11}\ \text{Sin}\ \omega t\ \text{Cos}\ \omega t \\ &= \frac{1}{2}\,{}_1b_{11}^2(1 - \text{Cos}\ 2\omega t) + \frac{1}{2}\,{}_1c_{11}^2(1 + \text{Cos}\ 2\omega t) + {}_1b_{11}\,{}_1c_{11}\ \text{Sin}\ 2\omega t \\ &= \frac{1}{2}({}_1b_{11}^2 + {}_1c_{11}^2) + {}_1b_{11}\,{}_1c_{11}\ \text{Sin}\ 2\omega t - \frac{1}{2}({}_1b_{11}^2 - {}_1c_{11}^2)\ \text{Cos}\ 2\omega t \end{aligned} \tag{3.45}$$

The set of equations associated with the coefficient of e^{2kt} in (3.10) now become

$${}_1a_2(c_1 + 3c_3x_s^2) + \frac{3}{2}c_3x_s({}_1b_{11}^2 + {}_1c_{11}^2) = 0 \tag{3.46}$$

$$-\omega_2c_{21} + (c_1 + 3c_3x_s^2)\,{}_1b_{21} = 0 \tag{3.47}$$

$$\omega_2b_{21} + (c_1 + 3c_3x_s^2)\,{}_1c_{21} = 0 \tag{3.48}$$

$$-2\omega_2c_{22} + (c_1 + 3c_3x_s^2)\,{}_1b_{22} + 3c_3x_s\,{}_1b_{11}\,{}_1c_{11} = 0 \tag{3.49}$$

$$2\omega_2b_{22} + (c_1 + 3c_3x_s^2)\,{}_1c_{22} - \frac{3}{2}c_3x_s({}_1b_{11}^2 - {}_1c_{11}^2) = 0 \tag{3.50}$$

$$-3\omega_2c_{23} + (c_1 + 3c_3x_s^2)\,{}_1b_{23} = 0 \tag{3.51}$$

$$3\omega_2b_{23} + (c_1 + 3c_3x_s^2)\,{}_1c_{23} = 0 \tag{3.52}$$

Substituting for ${}_2c_{21}$ from (3.8) into (3.47) gives

$$-\omega(2k\,{}_1c_{21} + \omega_1b_{21}) + (c_1 + 3c_3x_s^2)\,{}_1b_{21} = 0$$

or

$$-2k\omega\,{}_1c_{21} + (c_1 + 3c_3x_s^2 - \omega^2)\,{}_1b_{21} = 0$$

or, from (3.42)

$$-2k\omega_1c_{21} + k^2\,{}_1b_{21} = 0 \tag{3.53}$$

Similarly, from (3.48)

$$2k\omega\,{}_1b_{21} + k^2\,{}_1c_{21} = 0 \qquad (3.54)$$

Squaring (3.53) and (3.54) and adding the squares gives

$$k^2(k^2 + 4\omega^2)({}_1b_{21}^2 + {}_1c_{21}^2) = 0.$$

Since k = 0 and ω = 0 is trivial, then

$${}_1b_{21} = {}_1c_{21} = 0 \qquad (3.55)$$

Substituting for ${}_2c_{23}$ in (3.51) and ${}_2b_{23}$ in (3.52) then gives the pair of equations

$$-6\omega k\,{}_1c_{23} + (c_1 + 3c_3x_s^2 - 9\omega^2)\,{}_1b_{23} = 0$$

$$6\omega k\,{}_1b_{23} + (c_1 + 3c_3x_s^2 - 9\omega^2)\,{}_1c_{23} = 0,$$

which upon squaring and adding these equations yields

$$\left\{36\omega^2k^2 + (c_1 + 3c_3x_s^2 - 9\omega^2)^2\right\}({}_1b_{23}^2 + {}_1c_{23}^2) = 0,$$

or, from (3.42),

$$\left\{(6\omega k)^2 + (k^2 - 8\omega^2)^2\right\}({}_1b_{23}^2 + {}_1c_{23}^2) = 0.$$

The expression in the $\{\}$ brackets can only be zero if both ω and k are zero, and since this would be trivial then

$${}_1b_{23} = {}_1c_{23} = 0 \qquad (3.56)$$

Substituting from (3.55) and (3.56) into (3.4) gives

$$B_1 = {}_1a_2 + {}_1b_{22}\,\text{Sin}\,2\omega t + {}_1c_{22}\,\text{Cos}\,2\omega t,$$

and, therefore,

$$A_1B_1 = P_1A_1B_1 =$$

$$= ({}_1b_{11}\,\text{Sin}\,\omega t + {}_1c_{11}\,\text{Cos}\,\omega t)({}_1a_2 + {}_1b_{22}\,\text{Sin}\,2\omega t + {}_1c_{22}\,\text{Cos}\,2\omega t)$$

$$= {}_1a_2\,{}_1b_{11}\,\text{Sin}\,\omega t + {}_1a_2\,{}_1c_{11}\,\text{Cos}\,\omega t + {}_1b_{11}\,{}_1b_{22}\,\text{Sin}\,\omega t\,\text{Sin}\,2\omega t$$

$$+ {}_1c_{11}\, {}_1b_{22} \text{ Cos } \omega t \text{ Sin } 2\omega t + {}_1b_{11}\, {}_1c_{22} \text{ Sin } \omega t \text{ Cos } 2\omega t$$

$$+ {}_1c_{11}\, {}_1c_{22} \text{ Cos } \omega t \text{ Cos } 2\omega t$$

$$= ({}_1a_2\, {}_1b_{11} + \frac{1}{2}\, {}_1c_{11}\, {}_1b_{22} - \frac{1}{2}\, {}_1b_{11}\, {}_1c_{22}) \text{ Sin } \omega t$$

$$+ ({}_1a_2\, {}_1c_{11} + \frac{1}{2}\, {}_1b_{11}\, {}_1b_{22} + \frac{1}{2}\, {}_1c_{11}\, {}_1c_{22}) \text{ Cos } \omega t$$

$$+ \frac{1}{2} ({}_1b_{11}\, {}_1c_{22} + {}_1c_{11}\, {}_1b_{22}) \text{ Sin } 3\omega t - \frac{1}{2} ({}_1b_{11}\, {}_1b_{22} - {}_1c_{11}\, {}_1c_{22}) \text{ Cos } 3\omega t \tag{3.57}$$

Also

$$A_1^3 = P_1 A_1^3 = ({}_1b_{11} \text{ Sin } \omega t + {}_1c_{11} \text{ Cos } \omega t)^3$$

$$= \frac{3}{4}\, {}_1b_{11}({}_1b_{11}^2 + {}_1c_{11}^2) \text{ Sin } \omega t + \frac{3}{4}\, {}_1c_{11}({}_1b_{11}^2 + {}_1c_{11}^2) \text{ Cos } \omega t$$

$$+ \frac{1}{4}\, {}_1b_{11}(3\, {}_1c_{11}^2 - {}_1b_{11}^2) \text{ Sin } 3\omega t - \frac{1}{4}\, {}_1c_{11}(3\, {}_1b_{11}^2 - {}_1c_{11}^2) \text{ Cos } 3\omega t \tag{3.58}$$

The set of equations associated with the coefficient of e^{3kt} in (3.10) are now

$$k_2 a_3 + (c_1 + 3c_3 x_s^2)\, {}_1a_3 = 0 \tag{3.59}$$

$$k_2 b_{31} - \omega_2 c_{31} + (c_1 + 3c_3 x_s^2)\, {}_1b_{31}$$

$$+ 6c_3 x_s ({}_1a_2\, {}_1b_{11} + \frac{1}{2}\, {}_1c_{11}\, {}_1b_{22} - \frac{1}{2}\, {}_1b_{11}\, {}_1c_{22})$$

$$+ \frac{3}{4} c_3\, {}_1b_{11}({}_1b_{11}^2 + {}_1c_{11}^2) = 0 \tag{3.60}$$

$$k_2 c_{31} + \omega_2 b_{31} + (c_1 + 3c_3 x_s^2)\, {}_1c_{31}$$

$$+ 6c_3 x_s ({}_1a_2\, {}_1c_{11} + \frac{1}{2}\, {}_1b_{11}\, {}_1b_{22} + \frac{1}{2}\, {}_1c_{11}\, {}_1c_{22}$$

$$+ \frac{3}{4} c_3\, {}_1c_{11}({}_1b_{11}^2 + {}_1c_{11}^2) = 0 \tag{3.61}$$

$$k_2 b_{32} - 2\omega_2 c_{32} + (c_1 + 3c_3 x_s^2)_1 b_{32} = 0 \qquad (3.62)$$

$$k_2 c_{32} + 2\omega_2 b_{32} + (c_1 + 3c_3 x_s^2)_1 c_{32} = 0 \qquad (3.63)$$

$$k_2 b_{33} - 3\omega_2 c_{33} + (c_1 + 3c_3 x_s^2)_1 b_{33}$$

$$+ 3c_3 x_s ({}_1b_{11}\, {}_1c_{22} + {}_1c_{11}\, {}_1b_{22}) + \frac{1}{4} c_3\, {}_1b_{11} (3\, {}_1c_{11}^2 - {}_1b_{11}^2) = 0 \qquad (3.64)$$

$$k_2 c_{33} + 3\omega_2 b_{33} + (c_1 + 3c_3 x_s^2)_1 c_{33}$$

$$-3c_3 x_s ({}_1b_{11}\, {}_1b_{22} - {}_1c_{11}\, {}_1c_{22}) - \frac{1}{4} c_3\, {}_1c_{11} (3\, {}_1b_{11}^2 - {}_1c_{11}^2) = 0 \qquad (3.65)$$

Substituting for ${}_2a_3$ from (3.8) into (3.59) gives

$$(\omega^2 + 4k^2)_1 a_3 = 0,$$

from which it follows that

$${}_1a_3 = 0. \qquad (3.66)$$

Substituting for ${}_2b_{32}$ and ${}_2c_{32}$ from (3.8) into (3.62) and (3.63) gives the pair of equations

$$k(3k_1 b_{32} - 2\omega_1 c_{32}) - 2\omega(3k_1 c_{32} + 2\omega_1 b_{32}) + (c_1 + 3c_3 x_s^2)_1 b_{32} = 0$$

$$k(3k_1 c_{32} + 2\omega_1 b_{32}) + 2\omega(3k_1 b_{32} - 2\omega_1 c_{32}) + (c_1 + 3c_3 x_s^2)_1 c_{32} = 0,$$

which upon substitution from (3.42) and re-arrangement gives

$$(4k^2 - 3\omega^2)_1 b_{32} - 8\omega k_1 c_{32} = 0$$

$$8\omega k_1 b_{32} + (4k^2 - 3\omega^2)_1 c_{32} = 0$$

Squaring these equations and then adding the squares gives

$$\left\{(4k^2 - 3\omega^2)^2 + (8\omega k)^2\right\} ({}_1b_{32}^2 + {}_1c_{32}^2) = 0.$$

The expression in the $\{\}$ brackets can only be zero if both ω and k are zero, and it follows that

$$ {}_1b_{32} = {}_1c_{32} = 0 \qquad (3.67) $$

Substituting for ${}_2b_{22}$, ${}_2c_{22}$, ${}_2b_{31}$, ${}_2c_{31}$, ${}_2b_{33}$ and ${}_2c_{33}$, as appropriate, in equations (3.46), (3.49), (3.50), (3.60), (3.61), (3.64), (3.65), (3.27) and (3.28), due regard being taken of the results given in (3.39), (3.41), (3.42), (3.43), (3.55), (3.56), (3.66) and (3.67), gives the following set of nine equations, in the nine unknowns ${}_1b_{11}$, ${}_1c_{11}$, ${}_1a_2$, ${}_1b_{22}$, ${}_1c_{22}$, ${}_1b_{31}$, ${}_1c_{31}$, ${}_1b_{33}$ and ${}_1c_{33}$, to be solved simultaneously:

$$ {}_1a_2(\omega^2 + k^2) + c_3\alpha_{10} = 0 \qquad (3.68) $$

$$ (k^2 - 3\omega^2)\,{}_1b_{22} - 4\omega k\,{}_1c_{22} + c_3\alpha_{20} = 0 \qquad (3.69) $$

$$ 4\omega k\,{}_1b_{22} + (k^2 - 3\omega^2)\,{}_1c_{22} + c_3\alpha_{30} = 0 \qquad (3.70) $$

$$ 4k^2\,{}_1b_{31} - 4\omega k\,{}_1c_{31} + c_3\alpha_{40} = 0 \qquad (3.71) $$

$$ 4\omega k\,{}_1b_{31} + 4k^2\,{}_1c_{31} + c_3\alpha_{50} = 0 \qquad (3.72) $$

$$ 4(k^2 - 2\omega^2)\,{}_1b_{33} - 12\omega k\,{}_1c_{33} + c_3\alpha_{60} = 0 \qquad (3.73) $$

$$ 12\omega k\,{}_1b_{33} + 4(k^2 - 2\omega^2)\,{}_1c_{33} + c_3\alpha_{70} = 0 \qquad (3.74) $$

$$ {}_1c_{11} + {}_1a_2 + {}_1c_{22} + {}_1c_{31} + {}_1c_{33} + x_s + c_3\alpha_{80} = 0 \qquad (3.75) $$

$$ k\{{}_1c_{11} + 2({}_1a_2 + {}_1c_{22}) + 3({}_1c_{31} + {}_1c_{33})\} $$

$$ + \omega\{{}_1b_{11} + 2\,{}_1b_{22} + {}_1b_{31} + 3\,{}_1b_{33}\} + c_3\alpha_{90} = 0, \qquad (3.76) $$

where

$$ \alpha_{10} = \frac{3}{2} x_s({}_1b_{11}^2 + {}_1c_{11}^2) $$

$$ \alpha_{20} = 3x_s\,{}_1b_{11}\,{}_1c_{11} $$

$$ \alpha_{30} = -\frac{3}{2} x_s({}_1b_{11}^2 - {}_1c_{11}^2) $$

$$ \alpha_{40} = 6x_s({}_1a_2\,{}_1b_{11} + \frac{1}{2}\,{}_1c_{11}\,{}_1b_{22} - \frac{1}{2}\,{}_1b_{11}\,{}_1c_{22}) + \frac{3}{4}\,{}_1b_{11}({}_1b_{11}^2 + {}_1c_{11}^2) $$

$$\alpha_{50} = 6x_s({}_1a_2\,{}_1c_{11} + \frac{1}{2}\,{}_1b_{11}\,{}_1b_{22} + \frac{1}{2}\,{}_1c_{11}\,{}_1c_{22}) + \frac{3}{4}\,{}_1c_{11}({}_1b_{11}^2 + {}_1c_{11}^2)$$

$$\alpha_{60} = 3x_s({}_1b_{11}\,{}_1c_{22} + {}_1c_{11}\,{}_1b_{22}) + \frac{1}{4}\,{}_1b_{11}(3\,{}_1c_{11}^2 - {}_1b_{11}^2)$$

$$\alpha_{70} = -3x_s({}_1b_{11}\,{}_1b_{22} - {}_1c_{11}\,{}_1c_{22}) - \frac{1}{4}\,{}_1c_{11}(3\,{}_1b_{11}^2 - {}_1c_{11}^2)$$

$$\alpha_{80} = 0$$

$$\alpha_{90} = 0.$$

The coefficients ${}_1b_{11}, \ldots, {}_1c_{33}$ may be expressed in terms of x_s in the following way:

$${}_1b_{11} = (k/\omega + \epsilon_1)x_s,\quad {}_1c_{11} = (-1 + \epsilon_2)x_s,\quad {}_1a_2 = \epsilon_3 x_s,\quad {}_1b_{22} = \epsilon_4 x_s,$$

$${}_1c_{22} = \epsilon_5 x_s,\quad {}_1b_{31} = \epsilon_6 x_s,\quad {}_1c_{31} = \epsilon_7 x_s,\quad {}_1b_{33} = \epsilon_8 x_s,\quad {}_1c_{33} = \epsilon_9 x_s,$$

whereupon equations (3.68) to (3.76) become

$$3c_3\frac{k}{\omega}x_s^3\epsilon_1 - 3c_3x_s^3\epsilon_2 + x_s(\omega^2 + k^2)\epsilon_3 + \frac{3}{2}c_3x_s^3[1 + (\frac{k}{\omega})^2] + G_1(\ ) = 0 \qquad (3.77)$$

$$-3c_3x_s^3\epsilon_1 + 3c_3\frac{k}{\omega}x_s^3\epsilon_2 + x_s(k^2 - 3\omega^2)\epsilon_4 - 4\omega k x_s\epsilon_5 - 3c_3x_s^3\frac{k}{\omega} + G_2(\ ) = 0 \qquad (3.78)$$

$$-3c_3\frac{k}{\omega}x_s^3\epsilon_1 - 3c_3x_s^3\epsilon_2 + 4\omega k x_s\epsilon_4 + (k^2 - 3\omega^2)x_s\epsilon_5 + \frac{3}{2}c_3x_s^3[1 - (\frac{k}{\omega})^2] + G_3() = 0$$
$$(3.79)$$

$$\frac{3}{4}c_3x_s^3\left[1 + 3(\frac{k}{\omega})^2\right]\epsilon_1 - \frac{3}{2}c_3x_s^3\frac{k}{\omega}\epsilon_2 + 6c_3x_s^3\frac{k}{\omega}\epsilon_3 - 3c_3x_s^3\epsilon_4 - 3c_3x_s^3\frac{k}{\omega}\epsilon_5$$
$$+ 4k^2x_s\epsilon_6 - 4\omega k x_s\epsilon_7 + \frac{3}{4}c_3x_s^3\frac{k}{\omega}[1 + (\frac{k}{\omega})^2] + G_4(\ ) = 0 \qquad (3.80)$$

$$-\frac{3}{2}c_3x_s^3\frac{k}{\omega}\epsilon_1 + \frac{3}{4}c_3x_s^3[3 + (\frac{k}{\omega})^2]\epsilon_2 - 6c_3x_s^3\epsilon_3 + 3c_3x_s^3\frac{k}{\omega}\epsilon_4 - 3c_3x_s^3\epsilon_5 + 4\omega k x_s\epsilon_6$$
$$+ 4k^2x_s\epsilon_7 - \frac{3}{4}c_3x_s^3[1 + (\frac{k}{\omega})^2] + G_5(\ ) = 0 \qquad (3.81)$$

$$\frac{3}{4}c_3x_s^2[1 - (\frac{k}{\omega})^2]\epsilon_1 - \frac{3}{2}c_3x_s^3\frac{k}{\omega}\epsilon_2 - 3c_3x_s^3\epsilon_4 + 3c_3x_s^3\frac{k}{\omega}\epsilon_5 + 4(k^2 - 2\omega^2)x_s\epsilon_8$$
$$- 12\omega k x_s\epsilon_9 + \frac{1}{4}c_3x_s^3[3 - (\frac{k}{\omega})^2]\frac{k}{\omega} + G_6(\ ) = 0 \qquad (3.82)$$

$$\frac{3}{2} c_3 x_s^3 \frac{k}{\omega} \epsilon_1 - \frac{3}{4} c_3 x_s^3 [(\frac{k}{\omega})^2 - 1]\epsilon_2 - 3c_3 x_s^3 \frac{k}{\omega} \epsilon_4 - 3c_3 x_s^3 \epsilon_5 + 12\omega k x_s \epsilon_8$$

$$+ 4(k^2 - 2\omega^2) x_s \epsilon_9 + \frac{1}{4} c_3 x_s^3 [3 (\frac{k}{\omega})^2 - 1] + G_7(\ ) = 0 \qquad (3.83)$$

$$(\epsilon_2 + \epsilon_3 + \epsilon_5 + \epsilon_7 + \epsilon_9) x_s = 0 \qquad (3.84)$$

$$k x_s \{\epsilon_2 + 2(\epsilon_3 + \epsilon_5) + 3(\epsilon_7 + \epsilon_9)\} + \omega x_s \{\epsilon_1 + 2\epsilon_4 + \epsilon_6 + 3\epsilon_8\} = 0, \qquad (3.85)$$

where

$$G_1 = \frac{3}{2} c_3 x_s^3 (\epsilon_1^2 + \epsilon_2^2) \qquad (3.86)$$

$$G_2 = 3c_3 x_s^3 \epsilon_1 \epsilon_2 \qquad (3.87)$$

$$G_3 = -\frac{3}{2} c_3 x_s^3 (\epsilon_1^2 - \epsilon_2^2) \qquad (3.88)$$

$$G_4 = c_3 x_s^3 \{6\epsilon_1\epsilon_3 + 3\epsilon_2\epsilon_4 - 3\epsilon_1\epsilon_5 + \frac{3}{4}\frac{k}{\omega}(3\epsilon_1^2 + \epsilon_2^2) - \frac{3}{2}\epsilon_1\epsilon_2 + \frac{3}{4}\epsilon_1(\epsilon_1^2 + \epsilon_2^2)\} \qquad (3.89)$$

$$G_5 = c_3 x_s^3 \{6\epsilon_2\epsilon_3 + 3\epsilon_1\epsilon_4 + 3\epsilon_2\epsilon_5 - \frac{3}{4}(\epsilon_1^2 + 3\epsilon_2^2) + \frac{3}{2}\frac{k}{\omega}\epsilon_1\epsilon_2 + \frac{3}{4}\epsilon_2(\epsilon_1^2 + \epsilon_2^2)\} \qquad (3.90)$$

$$G_6 = c_3 x_s^3 \{3\epsilon_1\epsilon_5 + 3\epsilon_2\epsilon_4 - \frac{3}{2}\epsilon_1\epsilon_2 - \frac{1}{2}\frac{k}{\omega}\epsilon_1^2 + \frac{1}{4}(\epsilon_1 + \frac{k}{\omega})(3\epsilon_2^2 - \epsilon_1^2)\} \qquad (3.91)$$

$$G_7 = c_3 x_s^3 \{-3\epsilon_1\epsilon_4 + 3\epsilon_2\epsilon_5 - \frac{3}{2}\frac{k}{\omega}\epsilon_1\epsilon_2 - \frac{1}{2}\epsilon_2^2 - \frac{1}{4}(1 - \epsilon_2)(3\epsilon_1^2 - \epsilon_2^2)\} \qquad (3.92)$$

$$G_8 = 0 \qquad (3.93)$$

$$G_9 = 0 \qquad (3.94)$$

The task of determining $\epsilon_1, \ldots, \epsilon_9$ in terms of c_3 from these nine simultaneous cubic equations would be formidable, this, however, is not required. The reason for deriving this system of equations is firstly that they define, in detail, the mapping M_o, and secondly they provide a guide to the choice of the nine-cell Λ. Provided $\epsilon_1, \ldots, \epsilon_9$ are of the first order of small quantities compared with unity, then $G_1, \ldots, G_7$ contain terms of the second and higher orders of small quantities only. Under these conditions the equations may be approximated by the linear set defined by (3.77) to (3.85) with $G_1, \ldots, G_9$ all zero. Since the terms not containing $\epsilon_1, \ldots, \epsilon_9$ all possess $c_3 x_s^3$ as a multiplier then it follows that the solutions to the degenerate linear set may be written

$$\epsilon_i = c_3 x_s^3 \, \Phi_i \, (c_3, \; x_s, \; k, \; \omega), \tag{3.95}$$

where $\epsilon_i \longrightarrow 0$ as $c_3 \longrightarrow 0$.

4. Some Norms

From (2.2)

$$\xi_1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_1a_m + {}_1b_{mn} \text{ Sin } n\omega t + {}_1c_{mn} \text{ Cos } n\omega t \right\}.$$

Differentiating this expression with respect to t gives

$$\xi_2 = \xi_1' = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ mk \, {}_1a_m + (mk \, {}_1b_{mn} - n\omega \, {}_1c_{mn}) \text{ Sin } n\omega t + (mk \, {}_1c_{mn} + n\omega \, {}_1b_{mn}) \text{ Cos } n\omega t \right\}.$$

But from (2.2)

$$\xi_2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_2a_{mn} + {}_2b_{mn} \text{ Sin } n\omega t + {}_2c_{mn} \text{ Cos } n\omega t \right\},$$

therefore it must follow that

$$\left. \begin{aligned} {}_2a_m &= mk \, {}_1a_m \\ {}_2b_{mn} &= mk \, {}_1b_{mn} - n\omega \, {}_1c_{mn} \\ {}_2c_{mn} &= mk \, {}_1c_{mn} + n\omega \, {}_1b_{mn} \end{aligned} \right\} \tag{4.1}$$

This is, of course, a generalization of the results set out in (3.8).

Now

$$|\xi_1| = \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_1a_m + {}_1b_{mn} \text{ Sin } n\omega t + {}_1c_{mn} \text{ Cos } n\omega t \right\} \right|,$$

and since, from (3.41), k < 0, then

$$|\xi_1| \leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ |{}_1a_m| + |{}_1b_{mn}| + |{}_1c_{mn}| \right\} .$$

Thus

$$\nu(\xi_1) = \sup_{0 \leqslant t \leqslant \infty} |\xi_1| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ |{}_1a_m| + |{}_1b_{mn}| + |{}_1c_{mn}| \right\} \tag{4.2}$$

Also

$$|\xi_2| = \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt} \left\{ {}_2a_m + {}_2b_{mn} \text{ Sin } n\omega t + {}_2c_{mn} \text{ Cos } n\omega t \right\} \right|$$

$$\leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ |{}_2a_{mn}| + |{}_2b_{mn}| + |{}_2c_{mn}| \right\}, \text{ from (3.41)},$$

$$\leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ |mk\,{}_1a_m| + |mk\,{}_1b_{mn} - n\omega\,{}_1c_{mn}| + |mk\,{}_1c_{mn} + n\omega\,{}_1b_{mn}| \right\},$$

from (4.1),

$$\leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ m|k|\,|{}_1a_m| + (m|k| + n\omega)(|{}_1b_{mn}| + |{}_1c_{mn}|) \right\},$$

thus

$$\nu(\xi_2) = \sup_{0 \leqslant t \leqslant \infty} |\xi_2| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ m|k|\,|{}_1a_m| + (m|k| + n\omega)(|{}_1b_{mn}| + |{}_1c_{mn}|) \right\} \tag{4.3}$$

From (2.3)

$$\nu(\xi) = \sup_{0 \leqslant t \leqslant \infty} \sum_{j=1}^{2} |\xi_j| = \sup (|\xi_1| + |\xi_2|) = \sup |\xi_1| + \sup |\xi_2|$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ (1 + m|k|)\,|{}_1a_m| + (1 + m|k| + n\omega)(|{}_1b_{mn}| + |{}_1c_{mn}|) \right\} \tag{4.4}$$

From (2.5)

$$P_j\xi_j = \sum_{n=1}^{n=r} \sum_{m=1}^{m=s} e^{mkt} \left\{ {}_ja_m + {}_jb_{mn} \text{ Sin } n\omega t + {}_jc_{mn} \text{ Cos } n\omega t \right\}, \; r, \; s > 1,$$

and, thereby,

$$\xi_j - P\xi_j = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} e^{mkt}\left\{ {}_ja_m + {}_jb_{mn} \text{ Sin } n\omega t + {}_jc_{mn} \text{ Cos } n\omega t \right\} \quad (4.5)$$

Thus

$$\nu(\xi - P\xi) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ (1 + m|k|)|{}_1a_m| + (1 + m|k| + n\omega)(|{}_1b_{mn}| + |{}_1c_{mn}|) \right\} \quad (4.6)$$

It follows that

$$\left.\begin{array}{l} \nu(P\xi) \leqslant \nu(\xi) \\ \text{and} \\ \nu(\xi - P\xi) \leqslant \nu(\xi) \end{array}\right\} \quad (4.7)$$

for all ξ in S.

Now from (2.8)

$$H(\xi_j - P\xi_j) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} e^{mkt}\left\{ \frac{{}_ja_m}{mk} + \frac{mk\,{}_jb_{mn} + n\omega\,{}_jc_{mn}}{(mk)^2 + (n\omega)^2} \text{ Sin } n\omega t \right.$$

$$\left. + \frac{mk\,{}_jc_{mn} - n\omega\,{}_jb_{mn}}{(mk)^2 + (n\omega)^2} \text{ Cos } n\omega t \right\};$$

thus

$$H(\xi_1 - P\xi_1) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} e^{mkt}\left\{ \frac{{}_1a_m}{mk} + \frac{mk\,{}_1b_{mn} + n\omega\,{}_1c_{mn}}{(mk)^2 + (n\omega)^2} \text{ Sin } n\omega t \right.$$

$$\left. + \frac{mk\,{}_1c_{mn} - n\omega\,{}_1b_{mn}}{(mk)^2 + (n\omega)^2} \text{ Cos } n\omega t \right\},$$

whilst

$$H(\xi_2 - P\xi_2) = H\xi_2 - HP\xi_2 = \xi_1 - PH\xi_2 = \xi_1 - P\xi_1.$$

The norm of $H(\xi - P\xi)$ is given by

$$\nu H(\xi - P\xi) = \sup \sum_{j=1}^{2} |H(\xi_j - P\xi_j)|$$

$$= \sup|H(\xi_1 - P\xi_1)| + \sup|H(\xi_2 - P\xi_2)|$$

$$= \sup \left| H(\xi_1 - P\xi_1) \right| + \sup \left| \xi_1 - P\xi_1 \right| .$$

Now

$$\sup_{0 \leqslant t \leqslant \infty} \left| \xi_1 - P\xi_1 \right| = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ \left| {}_1a_m \right| + \left| {}_1b_{mn} \right| + \left| {}_1c_{mn} \right| \right\}$$

and

$$\sup_{0 \leqslant t \leqslant \infty} \left| H(\xi_1 - P\xi_1) \right| = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ \frac{\left| {}_1a_m \right|}{m|k|} + \frac{m|k| + n\omega}{(mk)^2 + (n\omega)^2} \left(\left| {}_1b_{mn} \right| + \left| {}_1c_{mn} \right| \right) \right\},$$

thus

$$\nu H(\xi - P\xi) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ \left(1 + \frac{1}{m|k|}\right) \left| {}_1a_m \right| \right.$$

$$\left. + \left(1 + \frac{m|k| + n\omega}{(mk)^2 + (n\omega)^2}\right) \left(\left| {}_1b_{mn} \right| + \left| {}_1c_{mn} \right| \right) \right\} \qquad (4.8)$$

If the values of m and k are such that $m|k| \geqslant 1$, then the following inequality

$$(m|k|)^{-1}(1 + m|k| + n\omega) > 1 + \left\{ (mk)^2 + (n\omega)^2 \right\}^{-1} (m|k| + n\omega) \qquad (4.9)$$

holds for all values of $\omega > 0$. This may be shown by the following obvious steps

$$(m|k|)^{-1}(1 + m|k| + n\omega) > \left\{ (mk)^2 + (n\omega)^2 \right\}^{-1} \left\{ (mk)^2 + (n\omega)^2 + m|k| + n\omega \right\}$$

$$\left\{ (mk)^2 + (n\omega)^2 \right\} (1 + m|k| + n\omega) > m|k| \left\{ (mk)^2 + (n\omega)^2 + m|k| + n\omega \right\}$$

$$n\omega \left\{ (mk)^2 + (n\omega)^2 + n\omega \right\} > m|k| \, n\omega$$

$$(mk)^2 + (n\omega)^2 + n\omega > m|k|, \text{ for } m|k| \geqslant 1.$$

Using this result in (4.8) gives

$$\nu H(\xi - P\xi) \leqslant \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} (m|k|)^{-1} \left\{ (1 \quad m|k|) \left| {}_1a_m \right| \right.$$

$$\left. + (1 + m|k| + \text{ı} \quad \left| b_{mn} \right| + \left| {}_1c_{mn} \right|) \right\}$$

$$\leqslant \frac{1}{(s + 1)|k|} \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ (1 + m|\text{ı} \;) \left| {}_1a_m \right| \right.$$

$$\left. + (1 + n \; k| + n\omega) \left(\left| {}_1b_{mn} \right| + \left| {}_1c_{mn} \right| \right) \right\}$$

or

$$\nu H(\xi - P\xi) \leqslant g^{-1}\nu(\xi - P\xi) \leqslant g^{-1}\nu(\xi), \tag{4.10}$$

where $g = (s + 1)|k|$.

If h is a real analytic operator in S, then it is true that

$$\nu H[h\xi - P(h\xi)] \leqslant g^{-1}\nu[h\xi - P(h\xi)] \leqslant g^{-1}\nu(h\xi) \tag{4.11}$$

It will also be observed that

$$\sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} (1 + m|k|)(|_1a_m| + |_1b_{mn}| + |_1c_{mn}|) \leqslant (\xi - P\xi)$$

or $(1 + g)\nu(\xi_1 - P\xi_1) \leqslant \nu(\xi - P\xi)$

or

$$\nu(\xi_1 - P\xi_1) \leqslant (1 + g)^{-1}\nu(\xi - P\xi); \tag{4.12}$$

and that

$$\sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} m|k|(|_1a_{mn}| + |_1b_{mn}| + |_1c_{mn}|) \leqslant \nu(\xi_2 - P\xi_2)$$

or

$$g\nu(\xi_1 - P\xi_1) \leqslant \nu(\xi_2 - P\xi_2)$$

or

$$\nu(\xi_1 - P\xi_1) \leqslant g^{-1}\nu(\xi_2 - P\xi_2) \tag{4.13}$$

From (4.12) and (4.13)

$$\max \nu(\xi_1 - P\xi_1) = g^{-1}\nu(\xi_2 - P\xi_2) = (1 + g)^{-1} \nu(\xi - P\xi)$$

or

$$\nu(\xi_2 - P\xi_2) = \frac{g}{1 + g} \nu(\xi - P\xi) \tag{4.14}$$

Also

$$\nu H(\xi_1 - P\xi_1) = \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ (m|k|)^{-1}|_1a_m| + \frac{m|k| + n\omega}{(mk)^2 + (n\omega)^2} (|_1b_{mn}| + |_1c_{mn}|)\right\}.$$

But

$$(mk)^{-2}(m|k| + n\omega) > \{(mk)^2 + (n\omega)^2\}^{-1} (m|k| + n\omega), \quad \omega > 0, \; m|k| \geqslant 1,$$

therefore

$$\nu H(\xi_1 - P\xi_1) \leqslant \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} (mk)^{-2}\{m\,k\,|_1a_m| + (m|k| + n\omega)(|_1b_{mn}| + |_1c_{mn}|)\}$$

$$\leqslant g^{-2} \sum_{n=r+1}^{\infty} \sum_{m=s+1}^{\infty} \left\{ m|k| \, |{}_1a_m| + (m|k| + n\omega)(|{}_1b_{mn}| + |{}_1c_{mn}|) \right\}$$

$$\leqslant g^{-2} \, \nu(\xi_2 - P\xi_2)$$

$$\leqslant g^{-1} (1+g)^{-1} \, \nu(\xi - P\xi) \qquad (4.15)$$

5. Conditions for T to be a Contraction Mapping in S_R^*

Consider the nine cell Λ defined in Section 2. Then ξ^* is defined as the vector having components ξ_1^*, ξ_2^*, where

$$\xi_j^* = \sum_{n=1}^{3} \sum_{m=1}^{3} e^{mkt} \left\{ {}_ja_m + {}_jb_{mn} \text{ Sin } n\omega t + {}_jc_{mn} \text{ Cos } n\omega t \right\}, \quad r,\ s > 1, \qquad (5.1)$$

with ${}_1a_2$, ${}_1b_{11}$, ${}_1c_{11}$, ${}_1b_{22}$, ${}_1c_{22}$, ${}_1b_{31}$, ${}_1c_{31}$, ${}_1b_{33}$ and ${}_1c_{33}$ contained in Λ; the remaining coefficients ${}_1a_1$, etc. being zero. Thus

$$\xi_j^* = A_j e^{kt} + B_j e^{2kt} + C_j e^{3kt} \qquad (5.2)$$

where

$$\left.\begin{aligned} A_1 &= {}_1b_{11} \text{ Sin } \omega t + {}_1c_{11} \text{ Cos } \omega t \\ B_1 &= {}_1a_2 + {}_1b_{22} \text{ Sin } 2\omega t + {}_1c_{22} \text{ Cos } 2\omega t \\ C_1 &= {}_1b_{31} \text{ Sin } \omega t + {}_1c_{31} \text{ Cos } \omega t + {}_1b_{33} \text{ Sin } 3\omega t + {}_1c_{33} \text{ Cos } 3\omega t \end{aligned}\right\} \qquad (5.3)$$

and

$$\left.\begin{aligned} A_2 &= (k_1b_{11} - \omega_1c_{11}) \text{ Sin } \omega t + (k_1c_{11} + \omega_1b_{11}) \text{ Cos } \omega t \\ B_2 &= 2k_1a_2 + 2(k_1b_{22} - \omega_1c_{22}) \text{ Sin } 2\omega t + 2(k_1c_{22} + \omega_1b_{22}) \text{ Cos } 2\omega t \\ C_2 &= (3k_1b_{31} - \omega_1c_{31}) \text{ Sin } \omega t + (3k_1c_{31} + \omega_1b_{31}) \text{ Cos } \omega t \\ &\quad + 3(k_1b_{33} - \omega_1c_{33}) \text{ Sin } 3\omega t + 3(k_1c_{33} + \omega_1b_{33}) \text{ Cos } 3\omega t \end{aligned}\right\} \qquad (5.4)$$

It follows from (4.2), (4.3) and (4.4) that

$$\nu(\xi_1^*) = |{}_1b_{11}| + |{}_1c_{11}| + |{}_1a_2| + |{}_1b_{22}| + |{}_1c_{22}|$$
$$+ |{}_1b_{31}| + |{}_1c_{31}| + |{}_1b_{33}| + |{}_1c_{33}|$$
$$\leqslant \tau_1(\mu), \tag{5.5}$$

$$\nu(\xi_2^*) = (|k| + \omega)(|{}_1b_{11}| + |{}_1c_{11}|) + 2|k||{}_1a_2| + 2(|k| + \omega)(|{}_1b_{22}| + |{}_1c_{22}|)$$
$$+ (3|k| + \omega)(|{}_1b_{31}| + |{}_1c_{31}|) + 3(|k| + \omega)(|{}_1b_{33}| + |{}_1c_{33}|)$$
$$\leqslant \tau_2(\mu), \tag{5.6}$$

$$\nu(\xi^*) \leqslant c(\mu), \tag{5.7}$$

where

$$\tau_1(\mu) = \sum_{j=1}^{9} \mu_j \tag{5.8}$$

$$\tau_2(\mu) = |k|\{\mu_2 + \mu_3 + 2(\mu_1 + \mu_4 + \mu_5) + 3(\mu_6 + \mu_7 + \mu_8 + \mu_9)\}$$
$$+ \omega\{\mu_2 + \mu_3 + \mu_6 + \mu_7 + 2(\mu_4 + \mu_5) + 3(\mu_8 + \mu_9)\} \tag{5.9}$$

and

$$c(\mu) = \tau_1(\mu) + \tau_2(\mu) \tag{5.10}$$

Define S_R^* to be the set of all $\xi(t)$, as given by (2.2), which satisfy the conditions

$$\left.\begin{aligned} P\xi &= \xi^* \\ \nu(\xi) &\leqslant d \\ \nu(\xi - P\xi) &\leqslant \delta \end{aligned}\right\} \tag{5.11}$$

where d and δ will be chosen below. Then

$$\nu P\xi = \nu(\xi^*) \leqslant c$$

for every ξ in S_R^*.

Consider the mapping $T : S_R^* \to S$ defined by (2.11), then

$$Py = PT\xi = P(P\xi + F\xi) = PP\xi + PF\xi.$$

But $PP\xi$

and $PF\xi = PHf\xi = 0$,

thus

$$Py = P\xi = \xi^*.$$

Also

$$y_1 - Py_1 = H(y_2 - Py_2) = \xi_1 - P\xi_1$$

and

$$y_2 - Py_2 = H[-(c_1 + 3c_3x_s^2)\xi_1 - \theta(\xi_1) - h\xi_2 + P\{(c_1 + 3c_3x_s^2)\xi_1 + \theta(\xi_1) + h\xi_2\}]$$

or, from (3.42),

$$y_2 - Py_2 = H[-(\omega^2 + k^2)\xi_1 - \theta(\xi_1) - h\xi_2 + P\{(\omega^2 + k^2)\xi_1 + \theta(\xi_1) + h\xi_2\}]. \quad (5.12)$$

It is required now to obtain conditions for $T : S_R^* \longrightarrow S_R^*$. For this purpose $\nu(y - Py)$ will be evaluated by means of inequalities involving $\mu_1, \ldots, \mu_9$, τ_1 and δ. Now

$$\nu(y - Py) = \nu(y_1 - Py_1) + \nu(y_2 - Py_2)$$

which from (4.13) gives

$$\nu(y - Py) \leqslant g^{-1}(1 + g)\, \nu(y_2 - Py_2) \quad (5.13)$$

Writing

$$\xi_1 = P\xi_1 + (\xi_1 - P\xi_1),$$

then

$$\theta(\xi_1) = c_3(3x_s\xi_1^2 + \xi_1^3)$$

$$= c_3[3x_s\{(P\xi_1)^2 + 2(P\xi_1)(\xi_1 - P\xi_1) + (\xi_1 - P\xi_1)^2\}$$

$$+ (P\xi_1)^3 + 3(P\xi_1)^2(\xi_1 - P\xi_1) + 3P\xi_1(\xi_1 - P\xi_1)^2 + (\xi_1 - P\xi_1)^3]$$

Thus, from (5.12),

$$\nu(y_2 - Py_2) = \nu H\{(\omega^2 + k^2)(\xi_1 - P\xi_1) + h(\xi_2 - P\xi_2) + [\theta(\xi_1) - P\theta(\xi_1)]\}$$

$$= \nu H\{(\omega^2 + k^2)(\xi_1 - P\xi_1) + h(\xi_2 - P\xi_2)$$

$$+ 3c_3x_s\left[(P\xi_1)^2 - P(P\xi_1)^2\right] + 6c_3x_s\left[(P\xi_1)(\xi_1 - P\xi_1) - P(P\xi_1)(\xi_1 - P\xi_1)\right]$$

$$+ 3c_3x_s\left[(\xi_1 - P\xi_1)^2 - P(\xi_1 - P\xi_1)^2\right] + c_3\left[(P\xi_1)^3 - P(P\xi_1)^3\right]$$

$$+ 3c_3\left[(P\xi_1)^2(\xi_1 - P\xi) - P(P\xi_1)^2(\xi_1 - P\xi_1)\right]$$

$$+ 3c_3\left[(P\xi_1)(\xi_1 - P\xi_1)^2 - P(P\xi_1)(\xi_1 - P\xi_1)^2\right]$$

$$+ c_3\left[(\xi_1 - P\xi_1)^3 - P(\xi_1 - P\xi_1)^3\right]\}$$

or

$$\nu(y_2 - Py_2) \leqslant (\omega^2 + k^2)\nu H(\xi_1 - P\xi_1) + |h|\ \nu H(\xi_2 - P\xi_2)$$

$$+ 3|c_3||x_s|\nu H\left[(P\xi_1)^2 - P(P\xi_1)^2\right]$$

$$+ 6|c_3||x_s|\nu H\left[(P\xi_1)(\xi_1 - P\xi_1) - P(P\xi_1)(\xi_1 - P\xi_1)\right]$$

$$+ 3|c_3||x_s|\nu H\left[(\xi_1 - P\xi_1)^2 - P(\xi_1 - P\xi_1)^2\right] + |c_3|\nu H\left[(P\xi_1)^3 - P(P\xi_1)^3\right]$$

$$+ 3|c_3|\nu H\left[(P\xi_1)^2(\xi_1 - P\xi_1) - P(P\xi_1)^2(\xi_1 - P\xi_1)\right]$$

$$+ 3|c_3|\nu H\left[(P\xi_1)(\xi_1 - P\xi_1)^2 - P(P\xi_1)(\xi_1 - P\xi_1)^2\right]$$

$$+|c_3|\nu H\left[(\xi_1 - P\xi_1)^3 - P(\xi_1 - P\xi_1)^3\right] \qquad (5.14)$$

Consider the terms in this inequality in order, with $\xi \in S^*_R$

Then

$$\nu H(\xi_1 - P\xi_1) \leqslant g^{-1}(1 + g)^{-1} \nu(\xi - P\xi) \leqslant g^{-1}(1 + g)^{-1}\delta \tag{5.15}$$

and

$$\nu H(\xi_2 - P\xi_2) = \nu(\xi_1 - P\xi_1) \leqslant (1 + g)^{-1} \nu(\xi - P\xi)$$

$$\leqslant (1 + g)^{-1}\delta \tag{5.16}$$

Now

$$P\xi_1 = \xi_1^* = A_1 e^{kt} + B_1 e^{2kt} + C_1 e^{3kt},$$

$$(P\xi_1)^2 = A_1^2 e^{2kt} + 2A_1B_1 e^{3kt} + (B_1^2 + 2A_1C_1)e^{4kt} + 2B_1C_1 e^{5kt} + C_1^2 e^{6kt},$$

$$P(P\xi_1)^2 = A_1^2 e^{2kt} + 2A_1B_1 e^{3kt},$$

and, therefore,

$$(P\xi_1)^2 - P(P\xi_1)^2 = (B_1^2 + 2A_1C_1)e^{4kt} + 2B_1C_1 e^{5kt} + C_1^2 e^{6kt}. \tag{5.17}$$

Also

$$|(B_1^2 + 2A_1C_1)e^{4kt} + 2B_1C_1 e^{5kt} + C_1^2 e^{6kt}|$$

$$\leqslant |B_1^2 + 2A_1C_1 + 2B_1C_1 + C_1^2|$$

$$\leqslant |\gamma_{00} + \beta_{01} \operatorname{Sin} \omega t + \gamma_{01} \operatorname{Cos} \omega t + \dots + \gamma_{06} \operatorname{Cos} 6\omega t|$$

$$\leqslant |\gamma_{00}| + |\beta_{01}| + |\gamma_{01}| + \dots + |\gamma_{06}|$$

where $\gamma_{00}, \beta_{01}, \dots, \gamma_{06}$ are quadratic relations in ${}_1b_{11}, {}_1c_{11}, \dots, {}_1c_{33}$.

Thus

$$\nu[(P\xi_1)^2 - P(P\xi_1)^2] = |\gamma_{00}| + |\beta_{01}| + |\gamma_{01}| + \dots + |\gamma_{06}| = \phi_1, \tag{5.18}$$

and from (4.11)

$$\nu H[(P\xi_1)^2 - P(P\xi_1)^2] \leqslant g^{-1} \phi_1 \tag{5.19}$$

$$\nu H[(P\xi_1)(\xi_1 - P\xi_1) - P(P\xi_1)(\xi_1 - P\xi_1)]$$

$$\leqslant g^{-1}\,\nu[(P\xi_1)(\xi_1 - P\xi_1) - P(P\xi_1)(\xi_1 - P\xi_1)], \text{ from (4.11)},$$

$$\leqslant g^{-1}\,\nu[(P\xi_1)(\xi_1 - P\xi_1)]$$

$$\leqslant g^{-1}\,|P\xi_1|\,\nu(\xi_1 - P\xi_1)$$

$$\leqslant g^{-1}\,\tau_1(\mu)\nu(\xi_1 - P\xi_1), \text{ from (5.5)}$$

$$\leqslant g^{-1}(1+g)^{-1}\,\tau_1(\mu)\nu(\xi - P\xi)$$

$$\leqslant g^{-1}(1+g)^{-1}\,\tau_1(\mu)\,\delta \qquad (5.20)$$

$$\nu H[(\xi_1 - P\xi_1)^2 - P(\xi_1 - P\xi_1)^2]$$

$$\leqslant g^{-1}\,\nu[(\xi_1 - P\xi_1)^2 - P(\xi_1 - P\xi_1)^2]$$

$$\leqslant g^{-1}\,\nu(\xi_1 - P\xi_1)^2$$

$$\leqslant g^{-1}[\nu(\xi_1 - P\xi_1)]^2$$

$$\leqslant g^{-1}(1+g)^{-2}[\nu(\xi - P\xi)]^2$$

$$\leqslant g^{-1}(1+g)^{-2}\delta^2 \qquad (5.21)$$

Now

$$(P\xi_1)^3 = A_1^3e^{3kt} + 3A_1^2B_1e^{4kt} + 3(A_1^2C_1 + A_1B_1^2)e^{5kt}$$
$$+ (B_1^3 + 6A_1B_1C_1)e^{6kt} + 3(A_1C_1^2 + B_1^2C_1)e^{7kt}$$
$$+ 3B_1C_1^2e^{8kt} + C_1^3e^{9kt},$$

$$P(P\xi_1)^3 = A_1^3e^{3kt},$$

$$(P\xi_1)^3 - P(P\xi_1)^3 = 3A_1^2B_1e^{4kt} + 3(A_1^2C_1 + A_1B_1^2)e^{5kt}$$
$$+ (B_1^3 + 6A_1B_1C_1)e^{6kt} + 3(A_1C_1^2 + B_1^2C_1)e^{7kt}$$
$$+ 3B_1C_1^2e^{8kt} + C_1^3e^{9kt}.$$

Therefore

$$|(P\xi_1)^3 - P(P\xi_1)^3| \leqslant |3A_1^2B_1 + 3(A_1^2C_1 + A_1B_1^2) + (B_1^3 + 6A_1B_1C_1)$$
$$+ 3(A_1C_1^2 + B_1^2C_1) + 3B_1C_1^2 + C_1^3|,$$
$$\leqslant |\sigma_{00} + \rho_{01} \text{ Sin } \omega t + \sigma_{01} \text{ Cos } \omega t + \ldots . + \sigma_{09} \text{ Cos } 9\omega t|$$
$$\leqslant |\sigma_{00}| + |\rho_{01}| + |\sigma_{01}| + \ldots . + |\sigma_{09}|,$$

where $\sigma_{00}, \ldots, \sigma_{09}$ are cubic relations in ${}_1b_{11}$, ${}_1c_{11}, \ldots, {}_1c_{33}$.

Thus

$$\nu[(P\xi_1)^3 - P(P\xi_1)^3] = |\sigma_{00}| + |\rho_{01}| + |\sigma_{01}| + \ldots . + |\sigma_{09}| = \phi_2 \quad (5.22)$$

and, from (4.11),

$$\nu H[(P\xi_1)^3 - P(P\xi_1)^3] \leqslant g^{-1}\phi_2. \quad (5.23)$$

Also

$$\nu H[(P\xi_1)^2(\xi_1 - P\xi_1) - P(P\xi_1)^2(\xi_1 - P\xi_1)]$$
$$\leqslant g^{-1}\nu[(P\xi_1)^2(\xi_1 - P\xi_1) - P(P\xi_1)^2(\xi_1 - P\xi_1)]$$
$$\leqslant g^{-1}\nu[(P\xi_1)^2(\xi_1 - P\xi_1)]$$
$$\leqslant g^{-1}|P\xi_1|^2 \nu(\xi_1 - P\xi_1)$$
$$\leqslant g^{-1}(1 + g)^{-1}\tau_1^2\delta \quad (5.24)$$

$$\nu H[(P\xi_1)(\xi_1 - P\xi_1)^2 - P(P\xi_1)(\xi_1 - P\xi_1)^2]$$

$$\leqslant g^{-1}\nu[(P\xi_1)(\xi_1 - P\xi_1)^2 - P(P\xi_1)(\xi_1 - P\xi_1)^2]$$

$$\leqslant g^{-1}\nu[(P\xi_1)(\xi_1 - P\xi_1)^2]$$

$$\leqslant g^{-1}|P\xi_1|[\nu(\xi_1 - P\xi_1)]^2$$

$$\leqslant g^{-1}(1 + g)^{-2}\tau_1\delta^2 \qquad (5.25)$$

and

$$\nu H[(\xi_1 - P\xi_1)^3 - P(\xi_1 - P\xi_1)^3]$$

$$\leqslant g^{-1}\nu[(\xi_1 - P\xi_1)^3 - P(\xi_1 - P\xi_1)^3]$$

$$\leqslant g^{-1}\nu[(\xi_1 - P\xi_1)^3]$$

$$\leqslant g^{-1}[\nu(\xi_1 - P\xi_1)]^3$$

$$\leqslant g^{-1}(1 + g)^{-3}\delta^3 \qquad (5.26)$$

Substituting the appropriate inequalities into (5.13) and (5.14) gives

$$\nu(y - Py) \leqslant (1 + g^{-1})\nu(y_2 - Py_2) = \left(\frac{1+g}{g}\right)\nu(y_2 - Py_2) = N \qquad (5.27)$$

where

$$N = \left(\frac{1+g}{g}\right)\{(\omega^2 + k^2)g^{-2}\delta + |h|(1 + g)^{-1}\delta + 3|c_3||x_s|g^{-1}\phi_1$$
$$+ 6|c_3||x_s|g^{-1}(1 + g)^{-1}\tau_1\delta + 3|c_3||x_s|g^{-1}(1 + g)^{-2}\delta^2$$
$$+|c_3|g^{-1}\phi_2 + 3|c_3|g^{-1}(1 + g)^{-1}\tau_1^2\delta + 3|c_3|g^{-1}(1 + g)^{-2}\tau_1\delta^2$$
$$+|c_3|g^{-1}(1 + g)^{-3}\delta^3\} \qquad (5.28)$$

The conditions for $T : S_R^* \rightarrow S_R^*$ may now be established. First it is required to ask whether, for ξ in S_R^*,

$$\nu(y - Py) \leqslant \nu(\xi - P\xi) \leqslant \delta$$

or, from (5.27), whether

$$N \leqslant \delta \qquad (5.29)$$

Now for ξ in S_R^* $Py = P\xi = P\xi^* = \xi^*$ and, therefore

$$\nu(\xi) = \nu\{P\xi + (\xi - P\xi)\} \leqslant \nu(P\xi) + \nu(\xi - P\xi)$$

$$\leqslant c + \delta \leqslant d \qquad (5.30)$$

Similarly

$$\nu(y) \leqslant \nu(Py) + \nu(y - Py) \leqslant c + \nu(y - Py) \qquad (5.31)$$

If the inequality (5.29) holds then from (5.31)

$$\nu(y) \leqslant c + \delta \leqslant d$$

and hence y is in S_R^*. Choosing

$$d = c + \delta, \qquad (5.32)$$

with δ satisfying (5.29), then $T : S_R^* \longrightarrow S_R^*$.

Conditions for T to be a contraction mapping in S_R^* may be established in the following way. With ξ, $\bar{\xi}$ in S_R^*

$$\nu(y - \bar{y}) \leqslant (1 + g^{-1})\nu[(y_2 - Py_2) - (\bar{y}_2 - P\bar{y}_2)] \quad \text{from (5.13)}$$

$$\leqslant (1 + g^{-1})\nu(y_2 - \bar{y}_2), \quad \text{since } Py_2 = P\bar{y}_2,$$

$$\leqslant (1 + g^{-1})\nu\{(\omega^2 + k^2)(\xi_1 - \bar{\xi}_1) + h(\xi_2 - \bar{\xi}_2) + \theta(\xi_1) - \theta(\bar{\xi}_1)\}$$

$$\leqslant (1 + g^{-1})\nu\{(\omega^2 + k^2)(\xi_1 - \bar{\xi}_1) + h(\xi_2 - \bar{\xi}_2) + 3c_3x_s(\xi_1^2 - \bar{\xi}_1^2) + c_3(\xi_1^3 - \bar{\xi}_1^3)\}$$

$$\leqslant (1 + g^{-1})\nu\{(\omega^2 + k^2)(\xi_1 - \bar{\xi}_1) + h(\xi_2 - \bar{\xi}_2) + 3c_3x_s(\xi_1 + \bar{\xi}_1)(\xi_1 - \bar{\xi}_1)$$
$$c_3(\xi_1^2 + \xi_1\bar{\xi}_1 + \bar{\xi}_1^2)(\xi_1 - \bar{\xi}_1)\}$$

$$\leqslant (1 + g^{-1})\{(\omega^2 + k^2)\nu(\xi_1 - \bar{\xi}_1) + |h|\nu(\xi_2 - \bar{\xi}_2) + 6|c_3||x_s|d\nu(\xi_1 - \bar{\xi}_1)$$
$$+ 3|c_3|d^2\nu(\xi_1 - \bar{\xi}_1)\}$$

$$\leqslant (1 + g^{-1})\{(\omega^2 + k^2)g^{-1} + |h| + 6|c_3||x_s|dg^{-1} + 3|c_3|d^2g^{-1}\}\nu(\xi_2 - \bar{\xi}_2)$$

$$\leqslant g^{-1}\{(\omega^2 + k^2) + g|h| + 6|c_3||x_s|d + 3|c_3|d^2\}\nu(\xi - \bar{\xi}).$$

It follows that T is a contraction mapping in S_R^* if

$$(\omega^2 + k^2) + g|h| + 6|c_3||x_s|d + 3|c_3|d^2 < g$$

or upon substitution from (5. 32) this becomes

$$(\omega^2 + k^2) + |c_3|\{6|x_s|(c + \delta) + 3(c + \delta)^2\} < g(1 - |h|) \tag{5.33}$$

When the inequalities (5. 29) and (5. 33) hold then T is a contraction in S_R^*, and it may be concluded, on the basis of Banach's fixed point theorem that the fixed element

$$y_j(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{mkt}\{{}_ja_m + {}_jb_{mn} \text{ Sin } n\omega t + {}_jc_{mn} \text{ Cos } n\omega t\}$$

exists, is unique in S_R^* and is continuously dependent on ξ^*. Thus ${}_ja_m$, ${}_jb_{mn}$, ${}_jc_{mn}$ for $m > s$, $n > r$ are uniquely determined by and continuously dependent on ${}_ja_1$, ${}_jb_{11}, \ldots,$ ${}_jc_{sr}$ for these in Λ.

6. An Estimate for glb$|(U, V) - (u, v)|$.

The method used in Ref. 7 for estimating $|\alpha_1 - \alpha_{10}|, \ldots, |\beta_3 - \beta_{30}|$ and hence glb$|(U, V) - (u, v)|$ is only appropriate to the case of periodic solutions. In the present case the cruder, but more general, estimate given by Cesari in Ref. 6, p. 164 will be used. Thus

$$|(U, V) - (u, v)| \leqslant 2^{\frac{1}{2}}Mn\nu(y - Py) \tag{6.1}$$

where M is the Lipschitz constant for (1. 11) and n = 2. The exact value of M is not important, it is sufficient to know that (1. 11) is analytic in ξ and that M will be finite. Also if the inequality (5. 29) is satisfied then (6. 1) becomes

$$|(U, V) - (u, v)| \leqslant 2^{\frac{1}{2}} \times 2M\delta,$$

which means that

$$\text{glb}|(U, V) - (u, v)| \leqslant M_1\delta, \tag{6.2}$$

where M_1 is a finite constant.

7. An Estimate for lub$|(u, v) - 0|$.

Since $\mu_1, \ldots, \mu_9$ define the cell Λ, and hence S_R^*, it is clear that if S_R^* is to contain the proposed exact solution then Λ must contain the point $({}_1b_{11}, \ldots, {}_1c_{33})$ defined by the leading coefficients of the exact solution and $\mu_1, \ldots, \mu_9$ must be chosen so as to make this possible. Now

$$\left.\begin{aligned}
|{}_1b_{11}| &= |(k/\omega + \epsilon_1)x_s| \leqslant (|k/\omega| + |\epsilon_1|)|x_s| \\
|{}_1c_{11}| &= |(-1 + \epsilon_2)x_s| \leqslant (1 + |\epsilon_2|)|x_s| \\
|{}_1a_2| &= |\epsilon_3 x_s| \leqslant |\epsilon_3||x_s| \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
|{}_1c_{33}| &= |\epsilon_9 x_s| \leqslant |\epsilon_9||x_s|
\end{aligned}\right\} \tag{7.1}$$

and if $\mu_1, \ldots, \mu_9$ are chosen so that

$$\left.\begin{aligned}
|k/\omega| + |\epsilon_1| &\leqslant \mu_1 \\
1 + |\epsilon_2| &\leqslant \mu_2 \\
|\epsilon_3| &\leqslant \mu_3 \\
&\ldots\ldots\ldots\ldots \\
|\epsilon_9| &\leqslant \mu_9
\end{aligned}\right\} \tag{7.2}$$

then $|{}_1b_{11}| \leqslant \mu_1|x_s|$, $|{}_1c_{11}| \leqslant \mu_2|x_s|, \ldots, |{}_1c_{33}| \leqslant \mu_9|x_s|$ as required. The nine cell Λ is then defined as the set of points corresponding to all combinations of $\epsilon_1, \ldots, \epsilon_9$ in the intervals

$$k/\omega - \mu_1 \leqslant \epsilon_1 \leqslant \mu_1 - k/\omega, \quad 1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - 1, \quad -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3, \ldots,$$
$$-\mu_9 \leqslant \epsilon_9 \leqslant \mu_9. \qquad (7.3)$$

The set of points defining the boundary eight-cell Λ_B is obtained by taking $\epsilon_1, \ldots, \epsilon_9$, in turn to be their extreme values in the above definition. Thus Λ_B is made up of the following collection of eight-cells

$$\left.\begin{array}{l}
\epsilon_1 = k/\omega - \mu_1, \ 1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - 1, \ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3, \ldots, -\mu_9 \leqslant \epsilon_9 \leqslant \mu_9 \\
\epsilon_1 = \mu_1 - k/\omega, \ 1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - 1, \ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3, \ldots, -\mu_9 \leqslant \epsilon_9 \leqslant \mu_9 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
k/\omega - \mu_1 < \epsilon_1 < \mu_1 - k/\omega, \ 1 - \mu_2 < \epsilon_2 < \mu_2 - 1, \ -\mu_3 < \epsilon_3 < \mu_3, \ldots, \epsilon_9 = \mu_9,
\end{array}\right\} \qquad (7.4)$$

where $\epsilon_1, \ldots, \epsilon_9$ take on all values over their respective intervals.

When the exact values of $\epsilon_1, \ldots, \epsilon_9$ are substituted into (3.77) to (3.85) they give $v_1, \ldots, v_9$ and hence define M_o. Thus C_o is given by $M_o \Lambda_B$ where Λ_B is the eight-cell defined by (7.4). The distance $|(u, v) - 0|$ is then

$$|(u, v) - 0| = \left|\left\{v_1^2 + v_2^2 + \ldots + v_9^2\right\}^{\frac{1}{2}}\right| \qquad (7.5)$$

In order to employ the inequality (2.21) it is required to determine $\text{lub}|(u, v) - 0|$. This may be done by means of the inequality

$$\text{lub}|(u, v) - 0| \geqslant \left|\left\{(\text{lub}v_1)^2 + (\text{lub}v_2)^2 + \ldots + (\text{lub}v_9)^2\right\}^{\frac{1}{2}}\right| \qquad (7.6)$$

which, if required, degenerates to

$$\text{lub}|(u, v) - 0| \geqslant \text{lub} \left\{\begin{array}{l} |v_1| \\ |v_2| \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ |v_9| \end{array}\right. \qquad (7.7)$$

In the present case use will be made of

$$\text{lub}|(u,v) - 0| \geqslant \text{lub}|v_1| \tag{7.8}$$

only.

Consider the evaluation of $\text{lub}v_1$. The first step is to determine whether any minima exist in v_1 as $\epsilon_1, \ldots, \epsilon_9$ vary over Λ_B. Differentiating (3.77) gives

$$\left.\begin{aligned}
\frac{\partial v_1}{\partial \epsilon_1} &= 3c_3 \frac{k}{\omega} x_s^3 + \frac{\partial G_1}{\partial \epsilon_1} = 3c_3 x_s^3\left(\frac{k}{\omega} + \epsilon_1\right) \\
\frac{\partial v_1}{\partial \epsilon_2} &= -3c_3 x_s^3 + \frac{\partial G_1}{\partial \epsilon_2} = 3c_3 x_s^3(-1 + \epsilon_2) \\
\frac{\partial v_1}{\partial \epsilon_3} &= x_s(\omega^2 + k^2) + \frac{\partial G_1}{\partial \epsilon_3} = x_s(\omega^2 + k^2) \\
\frac{\partial v_1}{\partial \epsilon_i} &= \frac{\partial G_1}{\partial \epsilon_i} \text{ for } i = 4, \ldots, 9.
\end{aligned}\right\} \tag{7.9}$$

Now in the subsequent analysis the values of $\mu_1, \ldots, \mu_9$ will be chosen so that the values of $\epsilon_1, \ldots, \epsilon_9$ on Λ are, at most, of the first order of small quantities compared with unity. Thus the derivatives $\partial G_1/\partial\epsilon_1, \ldots, \partial G_1/\partial\epsilon_9$ are, at most, of the first order of small quantities. Excluding the case $x_s = 0$, $c_3 = 0$, then $3c_3x_s^2$ and $x_s(\omega^2 + k^2)$ will never be zero. With $\epsilon_1 = k/\omega - \mu_1$ or $\epsilon_1 = \mu_1 - k/\omega$ on Λ_B, the least condition for a minimum is

$$\frac{\partial v_1}{\partial \epsilon_2} = \frac{\partial v_1}{\partial \epsilon_3} = \ldots. = \frac{\partial v_1}{\partial \epsilon_9} = 0.$$

But $\partial v_1/\partial\epsilon_2 \neq 0$ and $\partial v_1/\partial\epsilon_3 \neq 0$ in this range, so that there can be no minimum on the part of Λ_B defined by $\epsilon_1 = k/\omega - \mu_1$ or $\epsilon_1 = \mu_1 - k/\omega$. Similar arguments apply for $\epsilon_2 = 1 - \mu_2$ and $\epsilon_2 = \mu_2 - 1, \ldots., \epsilon_9 = \mu_9$, $\epsilon_9 = -\mu_9$. Thus no minima exists in $v_1(\epsilon_1, \ldots, \epsilon_9)$ for $\epsilon_1, \ldots, \epsilon_9$ in Λ_B. It may be concluded that $\text{lub}v_1$ is the value of v_1 at one of the "corner" values $(v_1)_c$ and these are readily evaluated. Thus

$$\min (v_1)_c \Big]_{\Lambda_B} = \text{lub}v_1 \tag{7.10}$$

8. Application of the Existence Proof when c_3 is Small

It is intended to apply the existence proof of Section 2 in the case when c_3 is small.

Some guidance to the choice of $\mu_1, \ldots, \mu_9$ may be obtained from the linearized second approximations obtained in Section 3. It can be seen from these that provided $c_3 x_s^3$ is chosen to be sufficiently small, then $\epsilon_1, \ldots, \epsilon_9$ will always be small. Designate the values given by (3.95) by

$$|\epsilon_{i1}| = E_{i1} |c_3|, \tag{8.1}$$

where these quantities define a nine-cell, Λ_ϵ. A satisfactory choice for Λ is then a cell slightly larger than Λ_ϵ and such that Λ_ϵ is contained in the interior of Λ. For this purpose the values of $\mu_1, \ldots, \mu_9$ defining Λ may be chosen to be

$$\left.\begin{aligned} \mu_1 &= |k/\omega| + (1+\zeta)E_{11}|c_3| \\ \mu_2 &= 1 + (1+\zeta)E_{21}|c_3| \\ \mu_3 &= (1+\zeta)E_{31}|c_3| \\ &\cdots\cdots\cdots\cdots \\ \mu_9 &= (1+\zeta)E_{91}|c_3| \end{aligned}\right\}, \quad \zeta > 0 \tag{8.2}$$

It then follows from (5.8) that

$$\tau_1 = (1 + |k/\omega|) + (1+\zeta)|c_3| \, \Sigma \, E_{i1} = T_o + T_1|c_3| \tag{8.3}$$

and

$$\tau_2 = R_o + R_1|c_3|. \tag{8.4}$$

From (5.18), ϕ_1 may be expressed in the form

$$\phi_1 = \phi_{11}|c_3| + \phi_{12}|c_3|^2, \qquad \phi_{11}, \phi_{12} > 0 \tag{8.5}$$

and from (5.22)

$$\phi_2 = \phi_{21}|c_3| + \phi_{22}|c_3|^2 + \phi_{23}|c_3|^3, \quad \phi_{21}, \phi_{22}, \phi_{23} > 0 \tag{8.6}$$

The inequality (5.29) may, alternatively, be written as the equation

$$N = B\delta \tag{8.7}$$

with $0 \leqslant B \leqslant 1$. Similarly the contraction condition may be written

$$(\omega^2 + k^2) + g|h| + |c_3|\{6|x_s|d + 3d^2\} = Cg \tag{8.8}$$

where $0 < C < 1$. Now (8.7) may be re-arranged to give

$$|c_3|g^{-1}(1+g)^{-3}\delta^3 + 3|c_3|g^{-1}(1+g)^{-2}(\tau_1 + |x_s|)\delta^2$$

$$+ (A - B)\delta + |c_3|g^{-1}(3|x_s|\phi_1 + \phi_2) = 0, \tag{8.9}$$

where

$$A = (\omega^2 + k^2)g^{-1}(1+g)^{-1} + |h|(1+g)^{-1} + 3|c_3|g^{-1}\tau_1(\tau_1 + 2|x_s|) \tag{8.10}$$

Since, by definition, δ must be real and positive, then only the real and positive roots of the cubic (8.9) are relevant. For the present purpose it is convenient to restrict the choice of c_3, g, x_s and B, and hence τ_1, ϕ_1 and ϕ_2, to ranges of values which cause (8.9) to have only one positive real root and such that this root is small compared with unity. That the choice of such a root is possible may be seen in the following way: Write (8.9) in the form

$$K_1|c_3|\delta^3 + K_2|c_3|\delta^2 + (A - B)\delta + |c_3|g^{-1}(3|x_s|\phi_1 + \phi_2) = 0$$

with $K_1, K_2 > 0$. Then if the desired root δ exists it must follow that

$\delta > \delta^2 > \delta^3$ and

$$K_1|c_3|\delta + K_2|c_3|\delta + (A - B)\delta + |c_3|g^{-1}(3|x_s|\phi_1 + \phi_2) > 0$$

or

$$\{B - A - |c_3|(K_1 + K_2)\}\delta - |c_3|g^{-1}(3|x_s|\phi_1 + \phi_2) < 0$$

or

$$\delta < \{B - A - c_3 K\}^{-1}|c_3|g^{-1}(3|x_s|\phi_1 + \phi_2), \qquad 8.11)$$

where

$$K = K_1 + K_2.$$

Thus $\delta > 0$ provided c_3 is sufficiently small and $B > A$.

Values of ω, k, x_s and c_3 may always be chosen to satisfy the contraction condition (8.8) and B chosen to be $A < B < 1$. Writing (8.8) in the form

$$(\omega^2 + k^2) + g|h| = Cg - |c_3| f_1(x_s, d)$$

and (8.10) in the form

$$(\omega^2 + k^2) + g|h| = Ag(1 + g) - |c_3| f_2(g, \tau_1, x_s)$$

then

$$A = g^{-1}(1 + g)^{-1}\{Cg + |c_3|(f_2 - f_1)\} \tag{8.12}$$

and

$$A \longrightarrow C/(1 + g) \quad \text{as } c_3 \longrightarrow 0.$$

It follows that with $0 < C < 1$ and c_3 sufficiently small then $A < 1$ in (8.12). B may now be chosen so that $A < B < 1$. Thus if ω, k, x_s and c_3 are chosen to satisfy the contraction condition and A and B calculated on the basis of (8.12), then (8.7) will be satisfied.

From (8.12), (8.5) and (8.6) it follows that δ may be written

$$\delta < (B - A - |c_3|K)^{-1} g^{-1}\{J_2|c_3|^2 + J_3|c_3|^3 + J_4|c_3|^4\}. \tag{8.13}$$

Hence from (6.2)

$$\text{glb}|(U, V) - (u, v)| < M_1(B - A - |c_3|K)^{-1} g^{-1}\{J_2|c_3|^2 + \ldots + J_4|c_3|^4\} \tag{8.14}$$

This quantity may be made as small as desired by taking $|c_3|$ sufficiently small.

From (2.19) and (3.77) it will be observed that

$$(v_1)_c = \pm 3c_3 \frac{k}{\omega} x_s^3 (\mu_1 - k/\omega) \pm 3c_3 x_s^3 (\mu_2 - 1) \pm x_s(\omega^2 + k^2)\mu_3$$
$$+ \frac{3}{2} c_3 x_s^3 \left[1 + (k/\omega)^2\right] + (G_1)_c.$$

Now $(G_1)_c$ must, from (3.86) and (8.1), have the form

$$(G_1)_c = G_{13}|c_3|^3$$

and, therefore, from (8.2)

$$(v_1)_c = \pm 3c_3 \frac{k}{\omega} x_s^3 (1 + \zeta)E_{11}|c_3| \pm 3c_3 x_s^3(1 + \zeta)E_{21}|c_3|$$

$$\pm x_s(\omega^2 + k^2)(1 + \zeta)E_{31}|c_3| + 3c_3 x_s^3 \left[1 + (k/\omega)^2\right] \pm G_{13}|c_3|^3.$$

The minimum corner value of v_1 on Λ_B has the form

$$\left|\min(v_1)_c\right|_{\Lambda_B} = \left\{L_1|c_3| + L_2|c_3|^2 + L_3|c_3|^3\right\}, \quad L_1, L_2, L_3 > 0 \tag{8. 15}$$

or

$$\text{lub}|(u, v) - 0| \geqslant \text{lub}|v_1| = \left|\min(v_1)_c\right|_{\Lambda_B} = \left\{L_1|c_3| + L_2|c_3|^2 + L_3|c_3|^3\right\}. \tag{8. 16}$$

It follows from (8. 14) and (8. 16) that, provided $|c_3|$ is chosen to be sufficiently small,

$$\text{glb}|(U, V) - (u, v)| < \text{lub}|(u, v) - 0|$$

and from (2. 21) the desired result follows.

The contraction condition may severely restrict the range of values of ω and k, or h, for which the result is valid. In order to widen the range of validity the value of s, and thereby g, may be made larger. This has the desired effect provided $|h| < C$. When s is increased the number of equations in the set corresponding to (3. 68), ... , (3. 76) increases and requires much more extensive manipulation. However, the basic result (3. 95) is left unchanged, and the equation defining v_1, although possessing more terms, will yield a form for $\left|\min(v_1)_c\right|_{\Lambda_B}$ which always has a leading term $L_1|c_3|$.

It follows that the existence theorem is valid for a wider range of ω and k than predicted by taking s = 3, but that the manipulation required to determine the ranges of ω and k is too lengthy to attempt by analytic means. Having shown that solutions of the form (2. 2) exist for $|c_3|$ small for some rangesof values of ω and k close to zero, the obvious next step is to develop a numerical technique for evaluating the coefficients ${}_j a_m$, ${}_j b_{mn}$, ${}_j c_{mn}$ on a digital computer, and this is being pursued.

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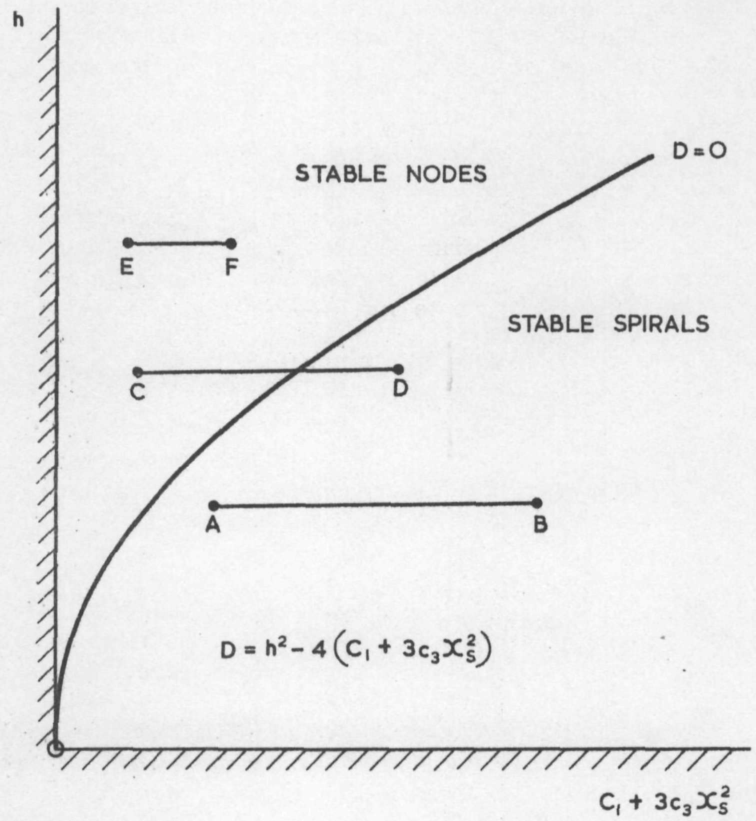

FIG.I. CLASSIFICATION OF POINTS OF STATIC EQUILIBRIUM.