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On random tridiagonal matrices and the beta log-gas

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"On random tridiagonal matrices and the beta log-gas"

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Summary

In this thesis the beta log-gas probability density function is discussed. It is shown that there is a strong link between this density function and Jacobi matrices. A change of variables exercise shows that the distribution of eigenvalues is exactly like the quadratic beta log-gas. The change of variables gives the normalization constant for the quadratic beta log-gas. Finally, it is made likely that the Jacobi matrix adheres to Wigners semicircle law, and that the beta log-gas is limited by the semicircle distribution.

1 The beta log-gas

In this section, we will discuss the beta log-gas, a probablity density function, and some of its physical properties. In sections 3 and 4 we will look at random Jacobi matrices and show that the distribution of eigenvalues is linked to the beta log-gas. In section 5 we will link these notions by finding the normalization constant, and making it likely that the beta log-gas is limited by the semicircle distribution for sufficiently large n.

1.1 Definition and physical interpretation

Definition 1.1. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let $\beta \geq 0$, let $V(x)$ be a function on $\mathbb{R}$, then the joint density function

$$p_{n,\beta}^V(x) = \frac{1}{Z_{\beta,n}^V} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^n e^{-\beta n V(x_k)} \tag{1}$$

is called the *beta log-gas*, where $Z_{\beta,n}^V$ is the normalisation constant.

We can also write this as

$$p_{n,\beta}^V(x) = \frac{1}{Z_{\beta,n}^V} \exp\{-\beta H_n^V(x)\},$$

where

$$H_n^V(x) = n \sum_{k=1}^n V(x_k) - \sum_{i<j} \log|x_i - x_j|.$$

This second expression gives a physical interpretation of the beta log-gas. In statistical mechanics any system (in this case $\mathbb{R}^n$, the space of configurations of n particles) is defined by an energy function H (in this case H_n^V). The probability or probability density of a configuration is proportional to $\exp\{-\beta H(x)\}$, where β is a tunable parameter. At $\beta = 0$ all configurations are equally likely, while as $\beta \to +\infty$, the probability concentrates on configurations with the lowest possible energy. Its reciprocal $1/\beta$ is what is called temperature in physics.

In this specific case, we can first look at the simpler probability density $\exp\{-\sum_{k=1}^n V(x_k)\}$, which corresponds to n independent variables with density $e^{V(x)}$. Physically it describes n non-interacting charges in an electric potential well given by V (that is, a particle at x has energy $V(x)$) so that the total energy is just $V(x_1) + \cdots + V(x_n)$.
Similarly, the beta log-gas has the physical interpretation of n unit charges in an electric potential V and with interaction energy $\log(1/|x-y|)$. The

total energy is then given by H_n^V.

We expect the interaction energy $log(1/|x-y|)$ to blow up to $+\infty$ when x and y get close. Configurations where x and y are close are thus highly unlikely, which indicates that under this probability distribution tend to stay away from each other. This *repulsion* is an important feature of the beta log-gas.

Mathematically speaking, we are very interested in what this probability density represents. What does its density function look like? Within what interval, if any, can we expect the values of x to lie? How can we determine the normalization constant? The dependent variables of the beta log-gas make it hard to answer these questions.
In this report we will link the beta log-gas to the eigenvalues of a random Jacobi matrix to find a formula for the normalization constant. We will also make it likely that the beta log-gas is limited by the semicircle distribution as n grows large.

2 Tridiagonal Matrices

In this section, we will introduce Jacobi matrices. We will show that it's eigenvalues are distinct and strictly interlace with the eigenvalues of its principal submatrix. This all follows from a very strong recursion for the characteristic polynomial. We will introduce auxiliary variables to complete a change of variables from the entries of a Jacobi matrix to its eigenvalues (and the auxiliary variables). The key point is finding the Jacobian determinant of this transformation.

2.1 Definition

Definition 2.1. Let n be a positive integer. Let $a_1, \ldots, a_n$ be real numbers and let $b_1, \ldots, b_{n-1}$ be strictly positive real numbers. Then

$$T = T_n(a,b) = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & b_2 & \ddots & 0 & 0 \\ 0 & b_2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & b_{n-2} & 0 \\ 0 & 0 & \ddots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & b_{n-1} & a_n \end{bmatrix} \tag{2}$$

is the *Jacobi matrix* associated with $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_{n-1})$.

Note that we could assume that $b_1, \ldots, b_{n-1}$ are all real numbers, but that this is synonymous with the assumption made in the definition.
In fact, we can assume that $b_n \neq 0$ for all n, because otherwise the matrix would break into a direct sum of two matrices, in which case we could regard the eigenvalues of the separate matrices.
Now imagine that $b_1, \ldots, b_{n-1}$ are real numbers with $b_k \neq 0$ for $k = 1, \ldots, n-1$. Regard diagonal matrix $D = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, with $\epsilon_i = \pm 1$. Then $DT_n(a,b)D^{-1} = T_n(a,c)$, where $c_k = \epsilon_k \epsilon_{k+1} b_k$. It is clear how we can pick values for ϵ_i so that all c_k are positive.
As eigenvalues don't change under conjugation it follows that we might as well start by assuming that all b_k are positive.

The question we will answer in this section is as follows: If a_ks and b_ks are random variables with some joint distribution, what will be the distribution of eigenvalues of $T_n(a,b)$?

2.2 The characteristic polynomial

Let $T_n = T_n(a,b)$ be a Jacobi matrix. Let T_k denote the top-left $k \times k$ principal submatrix of T_n. Let φ_k denote the characteristic polynomial of

T_k, i.e., $\varphi_k(z) = \det(zI - T_k)$. Let $\lambda_j^{(k)}$, $1 \leq j \leq k$, denote the zeroes of φ_k, or in other words, the eigenvalues of T_k.
If we expand along the last row, we get the recursion:

$$\varphi_k(z) = (z - a_k)\varphi_{k-1}(z) - b_{k-1}^2 \varphi_{k-2}(z), \tag{3}$$

which is also valid for $k = 1$ and $k = 0$ provided we set $\varphi_0 = 1$ and $\varphi_{-1} = 0$ (and $b_0 = 0$).

Proposition 2.1. *Let $T_n(a, b)$ be a Jacobi matrix. Then the eigenvalues of T_n are distinct.*

Proof. The eigenvalue equations $T_n \underline{v} = \lambda \underline{v}$ are as follows:

$$\begin{aligned} (a_1 - \lambda)\underline{v}_1 + b_1 \underline{v}_2 &= 0, \\ b_{i-1}\underline{v}_{i-1} + (a_i - \lambda)\underline{v}_i + b_i \underline{v}_{i+1} &= 0,\ 2 \leq i \leq n-1, \\ b_{n-1}\underline{v}_{n-1} + (a_n - \lambda)\underline{v}_n &= 0. \end{aligned}$$

Notice that if $\underline{v}_1 = 0$, then every $\underline{v}_k = 0$ by the equations above. So we know that $\underline{v}_1 \neq 0$. Now let $\underline{v}_1 \neq 0$, then $\underline{v}_2$ is determined by the first equation, and every $\underline{v}_k$, $k = 3, \ldots, n$, is determined by the middle set of equations above (key here is that no b_k can be zero). Let λ be an eigenvalue of T_n. We have found the nullspace of $T_n - \lambda I$ is one-dimensional at most. We know that every eigenvalue has at least one eigenvector, so the nullspace of $T_n - \lambda I$ must be at least one-dimensional. In other words, there is exactly one vector $\underline{v}$ that satisfies the eigenvalue equation $(T_n - \lambda I)\underline{v} = 0$. Thus every eigenvalue of T_n has exactly one eigenvector, so it follows that the eigenvalues of T_n are distinct. ∎

Proposition 2.2. *Let $T_n(a, b)$ be a Jacobi matrix. Define $\psi_k(z) = \frac{1}{\prod_{i=1}^k b_i} \varphi_k(z)$ for $0 \leq k \leq n-1$. Let λ be an eigenvalue of T_n. Then*

$$\underline{v} = (\psi_0(\lambda), \psi_1(\lambda), \ldots, \psi_{n-1}(\lambda))^T$$

is the eigenvector associated with λ.

Proof. Let $\underline{v} = (\psi_0(\lambda), \psi_1(\lambda), \ldots, \psi_{n-1}(\lambda))^T$. We get the following eigenvalue equations $T_n \underline{v} = \lambda \underline{v}$.

$$\begin{aligned} \lambda\psi_0(\lambda) &= a_1\psi_0(\lambda) + b_1\psi_1(\lambda), \\ \lambda\psi_i(\lambda) &= b_i\psi_{i-1}(\lambda) + a_{i+1}\psi_i(\lambda) + b_{i+1}\psi_{i+1}(\lambda),\ i = 1, \ldots, n-2, \\ \lambda\psi_{n-1}(\lambda) &= b_{n-1}\psi_{n-2}(\lambda) + a_n\psi_{n-1}(\lambda) \end{aligned}$$

We want to show that ϕ_k, $k = 0, \ldots, n-1$, indeed satisfy these equations. For the first equation we find

$$\begin{aligned}
\lambda\varphi_0(\lambda) &= a_1\varphi_0(\lambda) + \frac{b_1}{b_1}\varphi_1(\lambda), \\
\lambda &= a_1 + \lambda - a_1, \\
\lambda &= \lambda.
\end{aligned}$$

For the middle equations we find

$$\begin{aligned}
\frac{\lambda}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) &= \frac{b_i}{\prod_{j=1}^{i-1} b_j}\varphi_{i-1}(\lambda) + \frac{a_{i+1}}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) + \frac{b_{i+1}}{\prod_{j=1}^{i+1} b_j}\varphi_{i+1}(\lambda), \\
&= \frac{b_i}{\prod_{j=1}^{i-1} b_j}\varphi_{i-1}(\lambda) + \frac{a_{i+1}}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) + \frac{1}{\prod_{j=1}^{i} b_j}\varphi_{i+1}(\lambda).
\end{aligned}$$

Now we use the fact that

$$\begin{aligned}
\frac{1}{\prod_{j=1}^{i} b_j}\varphi_{i+1}(\lambda) &= \frac{\lambda - a_{i+1}}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) - \frac{b_i^2}{\prod_{j=1}^{i} b_j}\varphi_{i-1}(\lambda), \\
&= \frac{\lambda - a_{i+1}}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) - \frac{b_i}{\prod_{j=1}^{i-1} b_j}\varphi_{i-1}(\lambda)
\end{aligned}$$

Thus the middle equations become

$$\frac{\lambda}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda) = \frac{\lambda}{\prod_{j=1}^{i} b_j}\varphi_i(\lambda).$$

Then for the final equation, we get

$$\begin{aligned}
\frac{\lambda}{\prod_{i=1}^{n-1} b_i}\varphi_{n-1}(\lambda) &= \frac{b_{n-1}}{\prod_{i=1}^{n-2} b_i}\varphi_{n-2}(\lambda) + \frac{a_n}{\prod_{i=1}^{n-1} b_i}\varphi_{n-1}(\lambda), \\
\frac{\lambda}{\prod_{i=1}^{n-1} b_i}\varphi_{n-1}(\lambda) &= \frac{b_{n-1}^2}{\prod_{i=1}^{n-1} b_i}\varphi_{n-2}(\lambda) + \frac{a_n}{\prod_{i=1}^{n-1} b_i}\varphi_{n-1}(\lambda), \\
\lambda\varphi_{n-1}(\lambda) &= b_{n-1}^2\varphi_{n-2}(\lambda) + a_n\varphi_{n-1}(\lambda).
\end{aligned}$$

This is just another way to write the recurrence for the characteristic polynomial, so this must be true! We find that $\underline{v}$ satisfies the eigenvalue equations. This completes the proof. ■

Proposition 2.3. *Let $T_k(a,b)$, $k \geq 2$, be a Jacobi matrix. Let T_{k-1} denote its principal submatrix. Then the eigenvalues of T_k and T_{k-1} strictly interlace. That is*

$$\lambda_1^{(k)} < \lambda_1^{(k-1)} < \lambda_2^{(k)} < \cdots < \lambda_{k-1}^{(k-1)} < \lambda_k^{(k)}.$$

Proof. Let $k \geq 2$. We will show that the eigenvalues of T_k and T_{k-1} strictly interlace. We will give a proof by induction on k. Note that for any k, the eigenvalues of T_k are the same as the zeroes of $\varphi_k(z)$.

First let $k = 2$. We have $\varphi_1(z) = z - a_1$. Thus $\varphi_1(z)$ has a zero at $z = a_1$. In other words, $\lambda_1^{(1)} = a_1$.
$\varphi_2(z) = (z-a_1)(z-a_2)-b_1^2$. Since $b_1^2 > 0$, it is clear that the zeroes of $\varphi_2(z)$ occur once for $z > \max\{a_1, a_2\}$ and once for $z < \min\{a_1, a_2\}$, for the term $(z - a_1)(z - a_2)$ will always be negative for $\min\{a_1, a_2\} < z < \max\{a_1, a_2\}$. Thus we find that $\lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)}$, so the eigenvalues of T_2 and T_1 strictly interlace.

Now fix $k > 2$. Assume that the eigenvalues of T_{k-1} and T_{k-2} strictly interlace. From the interlacing of the zeroes of $\varphi_{k-1}(z)$ and $\varphi_{k-2}(z)$ we know that $\varphi_{k-2}(z) \neq 0$ at the zeroes of $\varphi_{k-1}(z)$. By recurrence (3) we know that $\varphi_k(z)$ and $\varphi_{k-2}(z)$ have opposite sign at the zeroes of $\varphi_{k-1}(z)$, as $b_k^2 > 0$. Because the zeroes of $\varphi_{k-2}(z)$ strictly interlace with the zeroes of $\varphi_{k-1}(z)$, its sign changes in between every two consecutive zeroes of $\varphi_{k-1}(z)$, so the same must apply for $\varphi_k(z)$. This means we have found $n-2$ zeroes of $\varphi_k(z)$ that are strictly in between the zeroes of $\varphi_{k-1}(z)$. It remains to find a zero that is larger than $\lambda_{k-1}^{(k-1)}$ and a zero that is smaller than $\lambda_1^{(k-1)}$.

We know that $\varphi_{k-2}(z) = \prod_{j=1}^{k-2}(\lambda_j^{(k-2)} - z) = (-1)^{k-2}z^{k-2} + [\dots]$, where $[\dots]$ contains terms with lower powers of z. Similarly, we can see that $\varphi_k(z) = \prod_{j=1}^{k}(\lambda_j^{(k)} - z) = (-1)^k z^k + [\dots]$, where $[\dots]$ contains terms with lower powers of z. Note that $\varphi_{k-2}(z)$ does not change its sign for $z \leq \lambda_1^{(k-1)}$ or for $z \geq \lambda_{k-1}^{(k-1)}$. At $z = \lambda_1^{(k-1)}$ and at $z = \lambda_{k-1}^{(k-1)}$ $\varphi_k(z)$ has the opposite sign. However, as $z \to \pm\infty$, it becomes clear that $\varphi_{k-2}(z)$ and $\varphi_k(z)$ must have the same sign, so it follows that we find the two zeroes we were looking for. Thus the eigenvalues of T_k and T_{k-1} strictly interlace.
This concludes the proof. ■

2.3 Auxiliary variables

We want to do a change of variables to find the eigenvalues of T_n. Note that T_n has $2n - 1$ variables, but there are only n eigenvalues. We shall have to find another $n - 1$ auxiliary variables to complete the change of variables exercise.

We can write $T_n = \lambda_1 \underline{v}_1 \underline{v}_1^T + \lambda_2 \underline{v}_2 \underline{v}_2^T + \cdots + \lambda_n \underline{v}_n \underline{v}_n^T = QDQ^T$, the spectral decomposition of T_n, where $\underline{v}_1, \dots, \underline{v}_n$ is some choice of orthonormal eigenvectors of T_n. D is the diagonal matrix with the eigenvalues on the diagonal, and Q is the orthogonal matrix containing the corresponding

eigenvectors.
Now for any integer $m \geq 1$ we can see that $T_n^m = QD^mQ^T = \lambda_1^m \underline{v}_1\underline{v}_1^T + \lambda_2^m \underline{v}_2\underline{v}_2^T + \cdots + \lambda_n^m \underline{v}_n\underline{v}_n^T$. We find the following equality:

$$\langle T_n^m \underline{e}_1, \underline{e}_1 \rangle = \sum_{k=1}^{n} \lambda_k^m p_k, \text{ where } p_k = |\langle \underline{v}_k, \underline{e}_1 \rangle|^2. \tag{4}$$

This states that the first element of the m-th power of T_n depends only on the eigenvalues and the squares of exactly the first element of each eigenvector (note that the p_ks, $k = 1, \ldots, n$, are just the squares of the first-row entries of Q. Since we have orthonormal eigenvectors, it is evident that $p_1 + p_2 + \cdots + p_n = 1$. Therefore, we may take $p = (p_1, p_2, \ldots, p_{n-1})$ as the auxiliary variables.

We will now highlight a particular case of equality (4), as we will need it in section 4. Let λ_k, $k = 1, \ldots, n$, denote the eigenvalues of T_n. Then the eigenvalues of $zI_n - T_n$ are $z - \lambda_k$. Note that the eigenvectors do not change, as $zI_n - T_n = QzI_nQ^T - QDQ^T = Q(zI_n - D)Q^T$. We can take the inverse of this matrix. We know from linear algebra that the eigenvalues of $(zI_n - T_n)^{-1}$ are $\frac{1}{z-\lambda_k}$. Note that again the eigenvectors remain unchanged, as $(zI_n - T_n)^{-1} = Q(zI_n - D)^{-1}Q^T$. We find the identity

$$\left((zI_n - T_n)^{-1}\right)_{1,1} = \sum_{k=1}^{n} \frac{p_k}{z - \lambda_k}.$$

The only caveat is that z can not be equal to any eigenvalue of T_n, as a matrix is invertible if and only if its determinant is not equal to zero, which is the case for exactly those z which are not eigenvalues of T_n.

2.4 The Jacobian determinant

To complete the change of variables, we need to find the Jacobian determinant of the transformation that maps (a, b) to (λ, p). This is the key point. We will first look more precisely at the spaces involved.

The set of all $n \times n$ Jacobi matrices is naturally identified, via the parameters (a, b), by $\mathcal{I}_n := \mathbb{R}^n \times \mathbb{R}_+^{n-1}$, where $\mathbb{R}_+ = (0, \infty)$.

Next, define

$$\Delta_n = \{p \in \mathbb{R}^{n-1} : p_1 + p_2 + \cdots + p_{n-1} < 1, p_i > 0\}, \text{ and}$$
$$\mathbb{R}_\uparrow^n = \{x \in \mathbb{R}^n : x_1 < x_2 < \cdots < x_n\}.$$

The set of all probability measure on $\mathbb{R}$ whose support has exactly n point is naturally identified with $\mathcal{M}_n := \mathbb{R}_\uparrow^n \times \Delta_n$ by identifying $\sum_{k=1}^n p_k \delta_{\lambda_k}$

with (λ, p).

It should be noted that $\mathcal{I}_n$ is not strictly a matrix space and $\mathcal{M}_n$ is not strictly a measure space, however, for any element of $\mathcal{I}_n$ there is clearly exactly one associated Jacobi matrix, and for any element of $\mathcal{M}_n$ there is exactly one associated measure of the form $\sum_{k=1}^{n} p_k \delta_{\lambda_k}$.

Note: from now on, when we write p_n, it is to be regarded this as the short form of $1 - (p_1 + p_2 + \cdots + p_{n-1})$. Now we have all the neccesary ingredients to pose the following theorem.

Theorem 2.4. *Fix $n \geq 1$, and let $G : \mathcal{I}_n \mapsto \mathcal{M}_n$ be defined as $G(T) = \boldsymbol{\nu}_T$. Then*

1. *G is a bijection from $\mathcal{I}_n$ onto $\mathcal{M}_n$.*
2. *If $T = T_n(a, b)$ and $\boldsymbol{\nu}_T = \sum_{k=1}^{n} p_k \delta_{\lambda_k}$, then*

$$\prod_{k=1}^{n-1} b_k^{2(n-k)} = \prod_{k=1}^{n} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^2 \tag{5}$$

3. *The Jacobian determinant of G^{-1} is equal to (up to a sign that depends on the ordering of variables)*

$$J_{G^{-1}}(\lambda, p) = \frac{\prod_{k=1}^{n} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^4}{2^{n-1} \prod_{k=1}^{n-1} b_k^{4(n-k)-1}} = \frac{\prod_{k=1}^{n-1} b_k}{2^{n-1} \prod_{k=1}^{n} p_k} \tag{6}$$

The proof of this theorem is far from trivial and is central to this project. We will prove it in section 4, but first we will discuss some of its consequences.

3 A class of random Jacobi matrices

In this section we will use the Jacobian determinant found in section 3 to determine the distribution of eigenvalues of a Jacobi matrix with a specific joint distribution for (a, b). We will also find the associated normalization constant.

3.1 The distribution of eigenvalues

Let $N(\mu, \sigma^2)$ be the normal distribution with mean μ and variance σ^2.
Let χ_m^2, $m > 0$, be the sum of squares of m independent standard normal variables. It is the same as the $Gamma(\frac{m}{2}, 2)$ distribution with shape $\frac{m}{2}$ and scale 2. The density is

$$\frac{1}{\Gamma(\frac{m}{2})2^m} e^{-\frac{x}{2}} x^{m-1} \text{ for } x > 0. \tag{7}$$

Theorem 3.1. *Let a_ks be independent $N(0,2)$ random variables and let b_k^2s be independent $\chi^2_{\beta(n-k)}$ variables also independent of the a_ks. Then the eigenvalues of the Jacobi matrix $T_n(a, b)$ have density*

$$\frac{1}{Z''_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^{n} \lambda_k^2\right\} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \tag{8}$$

with respect to the Lebesgue measure on $\mathbb{R}^n_\uparrow$. $Z''_{\beta,n}$ is the normalization constant.

The density (8) is called the beta Hermite ensemble.

Proof. If (a, b) has joint density $f(a, b)$ with respect to the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^{n-1}_+$, then by the change of variable formula, the density of (λ, p) with respect to the Lebesgue measure on $\mathbb{R}^n_\uparrow \times \Delta_n$ is given by

$$g(\lambda, p) = f(a, b) J_{G^{-1}}(\lambda, p) = f(a, b) \frac{\prod_{k=1}^{n-1} b_k}{2^{n-1} \prod_{k=1}^{n} p_k}.$$

On the right, (a, b) is written as a short form for the image of (λ, p) under the bijection from $\mathcal{M}_n$ to $\mathcal{I}_n$.

Let $a_k \sim N(0, 2)$ and $b_k^2 \sim \chi^2_{\beta(n-k)}$ i.i.d., then we find the following joint distribution for a and b:

$$f(a, b) = \frac{1}{Z_{\beta,n}} \prod_{k=1}^{n} e^{-\frac{1}{4}a_k^2} \prod_{k=1}^{n-1} e^{-\frac{1}{2}b_k^2} b_k^{\beta(n-k)-1}.$$

Here $\beta > 0$ and $Z_{\beta,n}$ is the normalizing constant. We can rewrite this to get

$$f(a,b) = \frac{1}{Z_{\beta,n}} \exp\left\{-\frac{1}{4}\left[\sum_{k=1}^{n} a_k^2 + 2\sum_{k-1}^{n-1} b_k^2\right]\right\} \prod_{k=1}^{n-1} b_k^{\beta(n-k)-1}.$$

Now it becomes clear why we chose these specific distributions for a and b. The factor $\prod_{k=1}^{n-1} b_k^{\beta(n-k)-1}$ partly cancels the similar factor in the Jacobian determinant, and the remaining expression can be written in terms of (λ, p) by (5). The exponent can be written nicely in terms of the eigenvalues:

$$\sum_{k=1}^{n} a_k^2 + 2\sum_{k-1}^{n-1} b_k^2 = \operatorname{tr}(T^2) = \sum_{k=1}^{n} \lambda_k^2.$$

Thus, we arrive at

$$g(\lambda, p) = \frac{1}{Z'_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^{n} \lambda_k^2\right\} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{k=1}^{n} p_k^{\frac{\beta}{2}-1}.$$

Here, $Z'_{\beta,n}$ is the normalization constant so that g is a probability density.

It is clear that under $g(\lambda, p)$, the vector λ is independent of the vector p. The density of λ is proportional to

$$\exp\left\{-\frac{1}{4}\sum_{k=1}^{n} \lambda_k^2\right\} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta}.$$

This concludes the proof. ■

It should be noted that the density of p is proportional to $\prod_{k=1}^{n} p_k^{\frac{\beta}{2}-1}$ for $p \in \Delta_n$. The latter is the well known Dirichlet distribution with parameters n and $(\frac{\beta}{2}, \ldots, \frac{\beta}{2})$.

3.2 The normalization constant

Proposition 3.2. *Let a_ks be independent $N(0,2)$ random variables and let b_k^2s be independent $\chi^2_{\beta(n-k)}$ variables also independent of the a_ks.*
Then the normalization constant for the distribution of eigenvalues of Jacobi matrix $T_n(a,b)$ is

$$Z''_{\beta,n} = \pi^{\frac{n}{2}} 2^{n+\frac{1}{4}\beta n(n-1)} \frac{\prod_{k=1}^{n} \Gamma(\frac{1}{2}\beta k)}{\Gamma(\frac{\beta}{2})^n}. \tag{9}$$

Proof. Let $a_k \sim N(0,2)$ and $b_k^2 \sim \chi^2_{\beta(n-k)}$ i.i.d., then we find the following joint distribution for a and b:

$$\begin{aligned}
f(a,b) &= \prod_{k=1}^{n} \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a_k^2} \prod_{k=1}^{n-1} 2b_k \frac{1}{2^{\frac{1}{2}\beta(n-k)}\Gamma(\frac{1}{2}\beta(n-k))} b_k^{\beta(n-k)-2} e^{-\frac{1}{2}b_k^2}, \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \frac{1}{\prod_{k=1}^{n-1} 2^{\frac{1}{2}\beta(n-k)-1}\Gamma(\frac{1}{2}\beta(n-k))} h(a,b), \\
&= \frac{1}{2^n \pi^{\frac{n}{2}} 2^{\frac{1}{4}\beta n(n-1)-(n-1)} \prod_{k=1}^{n-1}\Gamma(\frac{1}{2}\beta(n-k))} h(a,b), \\
&= \frac{1}{\pi^{\frac{n}{2}} 2^{1+\frac{1}{4}\beta n(n-1)} \prod_{k=1}^{n-1}\Gamma(\frac{1}{2}\beta k)} h(a,b).
\end{aligned}$$

Here $h(a,b) = \prod\limits_{k=1}^{n} e^{-\frac{1}{4}a_k^2} \prod\limits_{k=1}^{n-1} b_k^{\beta(n-k)-1} e^{-\frac{1}{2}b_k^2}$.
Now we can do the transformation to $g(\lambda, p)$.

$$\begin{aligned}
g(\lambda,p) &= f(a,b) \frac{\prod\limits_{k=1}^{n-1} b_k}{2^{n-1}\prod\limits_{k=1}^{n} p_k}, \\
&= \frac{1}{\pi^{\frac{n}{2}} 2^{n+\frac{1}{4}\beta n(n-1)} \prod_{k=1}^{n-1}\Gamma(\frac{1}{2}\beta k)} h'(\lambda,p).
\end{aligned}$$

Where $h'(\lambda,p) = \exp\left\{-\frac{1}{4}\sum\limits_{k=1}^{n}\lambda_k^2\right\} \prod\limits_{i<j} |\lambda_i - \lambda_j|^\beta \cdot \prod\limits_{k=1}^{n} p_k^{\frac{\beta}{2}-1}$.

Now we have found the normalization constant for the joint distribution of λ and p. We know that p is proportional to the Dirichlet distribution with parameters n and $(\frac{\beta}{2}, \dots, \frac{\beta}{2})$, for which we know the normalization constant! It is

$$\frac{\prod_{k=1}^{n}\Gamma(\frac{\beta}{2})}{\Gamma(\sum_{k=1}^{n}\frac{\beta}{2})} = \frac{\Gamma(\frac{\beta}{2})^n}{\Gamma(\frac{\beta}{2}n)}$$

We can divide this out of the normalization constant for the joint distribution to find the following distribution (including explicit nomalization constant) for λ:

$$\frac{1}{\pi^{\frac{n}{2}} 2^{n+\frac{1}{4}\beta n(n-1)} \frac{\prod_{k=1}^{n}\Gamma(\frac{1}{2}\beta k)}{\Gamma(\frac{\beta}{2})^n}} h'(\lambda,p).$$

This is the desired result. ∎

4 Proving the theorem

We will now prove Theorem 2.4, then we will have all the results stated in the previous two sections. Embedded in the proof is a deep connection between Jacobi matrices, probability measures and orthogonal polynomials.

Proof of Theorem 2.4. Let $n \geq 1$ and let $G : \mathcal{I}_n \to \mathcal{M}_n$ be defined as $G(T) = \boldsymbol{\nu}_T$. We will now prove all three statements in the proof separately.

Proof of 2. Let $T_n = T_n(a,b)$ be a Jacobi matrix and let $\boldsymbol{\nu}_T = \sum_{k=1}^n p_k \delta_{\lambda_k}$. We will show that

$$\prod_{k=1}^{n-1} b_k^{2(n-k)} = \prod_{k=1}^{n} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^2.$$

Remember the recursion (3):

$$\varphi_k(z) = (z - a_k)\varphi_{k-1}(z) - b_{k-1}^2 \varphi_{k-2}(z)$$

which is also valid for $k = 1$ and $k = 0$ provided we set $\varphi_0 = 1$ and $\varphi_{-1} = 0$. Here $\varphi_k(z)$ denotes the characteristic polynomial of the $k \times k$ principal submatrix of T_n. Respectively, $\tilde{\varphi}_k(z)$ denotes the characteristic polynomial of the bottom-right $k \times k$ principal submatrix of T_n.

By Proposition 2.3, we know that the eigenvalues of T_k are distinct and strictly interlace with those of T_{k-1}.
Now put $z = \lambda_j^{(k-1)}$ in (3) and multiply over $j \leq k-1$ to get

$$\prod_{j=1}^{k-1} \varphi_k(\lambda_j^{(k-1)}) = (-1)^{k-1} b_{k-1}^{2(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda_j^{(k-1)}).$$

Now, for any two monic polynomials $P(z) = \prod_{j=1}^{p}(z - \alpha_j)$ and $Q(z) = \prod_{j=1}^{q}(z - \beta_j)$, we know that

$$\prod_{j=1}^{q} P(\beta_j) = \pm \prod_{j=1}^{p} Q(\alpha_j),$$

since both are equal (up to sign) to $\prod_i \prod_j (\alpha_i - \beta_j)$. Use this for φ_k and φ_{k-1}, to get

$$\prod_{j=1}^{k} \varphi_{k-1}(\lambda_j^{(k)}) = \pm b_{k-1}^{2(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda_j^{(k-1)}).$$

Take a product over k and telescope to get (we write λ_j for $\lambda_j^{(n)}$)

$$\prod_{j=1}^{n} \varphi_{n-1}(\lambda_j) = \pm \prod_{j=1}^{n-1} b_j^{2j}.$$

Clearly this can be done in reverse for the $\tilde{\varphi}_k$s to get

$$\prod_{j=1}^{n} \tilde{\varphi}_{n-1}(\lambda_j) = \pm \prod_{j=1}^{n-1} b_j^{2(n-j)}. \tag{10}$$

The spectral measure is related to $\tilde{\varphi}_{n-1}$ as follows.

$$\sum_{k=1}^{n} \frac{p_k}{z-\lambda_k} = \big((zI_n - T_n)^{-1}\big)_{1,1} = \frac{\tilde{\varphi}_{n-1}(z)}{\varphi_n(z)}.$$

The equality on the left follows from a note we made in section 2, remember that this is just a specific case of identity (4). $\varphi_n(z)$ is the determinant of the matrix $zI_n - T_n$. Note that $\tilde{\varphi}_{n-1}(z)$ is the cofactor corresponding to the top-left element of T_n, so the equality on the right follow from the inverse matrix formula $A^{-1} = \frac{1}{det(A)} adj(A)$, which we know from linear algebra.

Now we multipy by $(z-\lambda_j)$ to get

$$(z-\lambda_j) \sum_{k=1}^{n} \frac{p_k}{z-\lambda_k} = \frac{(z-\lambda_j)\tilde{\varphi}_{n-1}(z)}{\varphi_n(z)},$$

so that

$$p_j + \sum_{k \neq j} \frac{p_k(z-\lambda_j)}{z-\lambda_k} = \frac{(z-\lambda_j)\tilde{\varphi}_{n-1}(z)}{\varphi_n(z)}.$$

Note that $\varphi_n^{`}(\lambda_j) = \lim_{z\to\lambda_j} \frac{\varphi_n(\lambda_j+(z-\lambda_j))-\varphi_n(\lambda_j)}{z-\lambda_j} = \lim_{z\to\lambda_j} \frac{\varphi_n(z)}{z-\lambda_j}$. Now let $z \to \lambda_j$ and we find

$$p_j = \frac{\tilde{\varphi}_{n-1}(\lambda_j)}{\varphi_n^{`}(\lambda_j)}.$$

Take the product over j and use (10) to get

$$\prod_{j=1}^{n} \tilde{\varphi}_{n-1}(\lambda_j) = \prod_{j=1}^{n} p_j \prod_{j=1}^{n} \varphi_n^{`}(\lambda_j),$$

$$\prod_{j=1}^{n} b_j^{2(n-j)} = \pm \prod_{j=1}^{n} p_j \prod_{j=1}^{n} \varphi_n^{`}(\lambda_j) = \prod_{j=1}^{n} p_j \prod_{i<j} |\lambda_i - \lambda_j|^2$$

since $\varphi_n^{`} = \prod_{i\neq j}(\lambda_j - \lambda_i)$. In the end, both sides are positive, so we did not have to follow the sign. This proves the second part of the theorem. ∎

Proof of 1. We will show that G is a bijection from $\mathcal{I}_n$ onto $\mathcal{M}_n$.

Note that on the way of proving *2.* we have proven a part of *1.* already. Let $T \in \mathcal{I}_n$, then we have noted the distinctness of eigenvalues. Further, we know that $p_j = \frac{\tilde{\varphi}_{n-1}(\lambda_j)}{\varphi_n^{'}(\lambda_j)}$, which cannot be zero because of the strict interlacing of eigenvalues of T_n and $\tilde{T}_{n-1}$. Thus, $\boldsymbol{\nu}_T$ belongs to $\mathcal{M}_n$. This shows that G maps $\mathcal{I}_n$ into $\mathcal{M}_n$.

Proposition 4.1. *Let $\boldsymbol{\nu} \in \mathcal{M}_n$. Let $H(\boldsymbol{\nu}) = T_n(a,b)$. Then $G \circ H$ is the identity map from $\mathcal{M}_n$ into itself.*

Proof. Let $\boldsymbol{\nu} = \sum_{j=1}^{n} p_j \delta_{\lambda_j} \in \mathcal{M}_n$. Observe that $L^2(\boldsymbol{\nu})$ has dimension exactly equal to n and that $1, x, \dots, x^{n-1}$ are linearly independent in $L^2(\boldsymbol{\nu})$ (Remember, $L^2(\boldsymbol{\nu})$ is the space of all functions f for which $\left(\int_{\mathcal{M}_n} |f|^2 d\boldsymbol{\nu}\right)^{1/2} < \infty$. This has dimension n because there are exactly n points in $\mathcal{M}_n$).

Therefore, we may apply Gram-Schmidt procedure to get $\psi_0, \dots, \psi_{n-1}$, where ψ_j is a polynomial of degree j. We fix some k and expand $x\psi_k(x)$ in this orthonormal basis to write (note that there is no ψ_n)

$$\begin{aligned} x\psi_k(x) &= c_{k,k+1}\psi_{k+1}(x) + \dots + c_{k,0}\psi_0(x) \text{ for } k \le n-2, \\ x\psi_{n-1}(x) &= c_{n,n-1}\psi_{n-1}(x) + \dots + c_{n,0}\psi_0(x). \end{aligned}$$

Note that with Gramm-Schmidt we can pick ψ_j such that they all have strictly positive leading coefficients. This means that both $x\psi_k(x)$ and $\psi_{k+1}(x)$ have strictly positive leading coefficients, so it follows that $c_{k,k+1}$ must be strictly positive for $k \le n-2$. Further, observe that $\langle x\psi_k(x), \psi_j(x)\rangle$ $= \langle \psi_k(x), x\psi_j(x)\rangle$ which is zero if $j < k-1$ as ψ_k is orthogonal to all polynomials of degree lower that k. That leaves

$$c_{k,k+1} = \int x\psi_k(x)\psi_{k+1}(x)d\boldsymbol{\nu}(x), c_{k.k} = \int x\psi_k^2(x)d\boldsymbol{\nu}(x).$$

From this it is clear that $c_{k,k+1} = c_{k+1,k}$ for $k \le n-1$. Now set $a_k = c_{k-1,k-1}, 1 \le k \le n$ and $b_k = c_{k-1,k}, 1 \le k \le n-1$. We have already shown that $b_k > 0$ for all $k \le n-1$. Remember, $H(\boldsymbol{\nu}) = T_n(a,b)$, so H maps $\mathcal{M}_n$ into $\mathcal{I}_n$.

With all this, the recursions can now be written as

$$\begin{aligned} x\psi_0(x) &= a_1\psi_0(x) + b_1\psi_1(x), \\ x\psi_k(x) &= b_k\psi_{k-1}(x) + a_{k+1}\psi_k(x) + b_{k+1}\psi_{k+1}(x), \text{ for } 2 \le k \le n-2, \\ x\psi_{n-1}(x) &= b_{n-1}\psi_{n-2}(x) + a_n\psi_{n-1}(x). \end{aligned}$$

These equalities are in $L^2(\boldsymbol{\nu})$, meaning that it holds for $x \in \{\lambda_1, \dots, \lambda_n\}$. In short, the above equation are saying that T_n has eigenvalues λ_j with eigenvector

$$v_j = \sqrt{p_j}(\psi_0(\lambda_j), \dots, \psi_{n-1}(\lambda_j))^T.$$

We have introduced the factor $\sqrt{p_j}$ because then the rows of the matrix $[v_1\ v_2\ \dots\ v_n]$ become orthonormal. As $\psi_0 = 1$, we get $|v_j(1)|^2 = p_j$ and hence the spectral measure at e_1 is $\sum_{j=1}^n p_j \delta_{\lambda_j} = \boldsymbol{\nu}$. Thus, $G \circ H$ is the identity map from $\mathcal{M}_n$ into itself. In particular, G maps $\mathcal{I}_n$ onto $\mathcal{M}_n$. ■

The proof will be complete if we show that G is bijective. Because $G \circ H$ is the identity map from $\mathcal{M}_n$ into itself, it is bijective, so it follows that G is surjective. There are many ways to show that G is injective. We will refer to the equations (11) in the next part of the proof, from which it is clear that if we know (λ, p), then we can recover (a unique set of) $a_1, b_1, a_2, b_2, \dots$. This completes te proof that G is a bijection mapping $\mathcal{I}_n$ into $\mathcal{M}_n$. ■

Proof of part 3. We will show that

$$J_{G^{-1}}(\lambda, p) = \frac{\prod_{k=1}^{n} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^4}{2^{n-1} \prod_{k=1}^{n-1} b_k^{4(n-k)-1}} = \frac{\prod_{k=1}^{n-1} b_k}{2^{n-1} \prod_{k=1}^{n} p_k}.$$

Let $T_n = T_n(a, b) \in \mathcal{I}_n$ correspond to $\boldsymbol{\nu} = \sum_{j=1}^n p_j \delta_{\lambda_j} \in \mathcal{M}_n$.

Proposition 4.2. *Let $T_n(a, b)$ be a Jacobi matrix. Let (a, b) be ordered as $(a_1, b_1, a_2, \dots, b_{n-1}, a_n)$. Then the $(1, 1)$ entry of T_n^m contains exactly one term containing $(a, b)_m$.*
In fact, if $(a, b)_m = a_i$ for some $i \in \{1, \dots, n\}$, then that term is $a_i \prod_{j=1}^{i-1} b_j^2$.
If $(a, b)_m = b_i$ for some $i \in \{1, \dots, n-1\}$, then that term is $\prod_{j=1}^{i} b_i^2$.

Proof. Note that T_n is tridiagonal. Let A be any $n \times n$ matrix. A has exactly $2n - 1$ diagonals (from botom-left to top-right). When we multiply A by T_n (either on the left or the right), the elements of each diagonal will only appear in the same diagonal and the two adjacent diagonals in the resulting matrix. Key here are the many zeroes of T_n.

If we regard T_n, we see that all the individual a_k and b_k appear only on exactly one bottom-left to top-right diagonal. Now take any power T_n^m, $m = 1, \dots, 2n - 2$, then the individual a_k and b_k will appear exactly one diagonal closer to the top-left corner in T_n^{m+1} than in T_n^m (we are interested in the top-left entry of every power of T_n). As such, every power of T_n has exactly one a_k or b_k in its top-left entry that did not appear in the top-left entry of any lower power of T_n.

Now it remains for us to show that for each power of T_n, the term with the 'new' a_k or b_k in the top-left entry is exactly such as shown in the proposition.

Without loss of generality, we can assume that we do left-side multiplication when we calculate $T_n^{m+1} = T_n T_n^m$, $m = 1, \ldots, 2n-2$. This means that the top-left entry of T_n^{m+1} is b_1 times the $(2,1)$ entry of T_n^m. Similarly, the $(2,1)$ entry of T_n^m is b_1 times the $(2,2)$ entry of T_n^{m-1}.

We will now create a 'path' that all the a_k and b_k take to get to the top-left entry of some power of T_n.
Let C be any $n \times n$ matrix. Let $i \in \{2, \ldots, n\}$. $C_{i,i}$ denote the (i,i) entry of C. The $(i-1,i)$ entry of $T_n C$ contains the term $b_{i-1} C_{i,i}$. The $(i-1,i-1)$ entry of $T_n^2 C$ contains the term $b_{i-1}^2 C_{i,i}$. We can repeat this process to find that the $(1,1)$ entry of $T_n^{2i-2} C$ contains the term $b_{i-1}^2 b_{i-2}^2 \ldots b_1^2 C_{i,i}$. From our earlier work we know that this must be the only term in the $(1,1)$ entry to contain $C_{i,i}$.

Let $i \in \{1, \ldots, n\}$. a_i is the (i,i) entry of T_n. Thus the only term in the $(1,1)$ entry of T_n^{2i-1} that contains a_i is $a_i \prod_{j=1}^{i-1} b_j^2$.
Let $i \in \{1, \ldots, n-1\}$. b_i is the $(i+1,i)$ entry of T_n. Thus, the (i,i) entry of T_n^2 contains the term b_i^2. Thus the only term in the $(1,1)$ entry of T_n^{2i} that contains b_i is $\prod_{j=1}^{i} b_i^2$. ■

We write the identities $(T_n^m)_{1,1} = \sum_{j=1}^n p_j \lambda_j^m$ for $m = 1, 2, \ldots, 2n-1$.

$$\begin{aligned}
\sum_{j=1}^n p_j \lambda_j &= (T_n)_{1,1} = a_1 & \sum_{j=1}^n p_j \lambda_j^2 &= (T_n^2)_{1,1} = b_1^2 + [\ldots] \\
\sum_{j=1}^n p_j \lambda_j^3 &= (T_n^3)_{1,1} = a_2 b_1^2 & \sum_{j=1}^n p_j \lambda_j^4 &= (T_n^4)_{1,1} = b_2^2 b_1^2 + [\ldots] \\
\sum_{j=1}^n p_j \lambda_j^5 &= (T_n^5)_{1,1} = a_3 b_2^2 b_1^2 & \sum_{j=1}^n p_j \lambda_j^6 &= (T_n^6)_{1,1} = b_3^2 b_2^2 b_1^2 + [\ldots]
\end{aligned} \tag{11}$$

Here the[...] include many terms, but all the a_k, b_k that appear there have appeared in previous equations. For example, $(T^2)_{1,1} = a_1^2 + b_1^2$ and as a_1 appeared in the first equation, we have brushed it under [...] as they will not matter.

Now, let $u = (u_1, \ldots, u_{2n-1})$ where $u_j = (T_n^j)_{1,1}$. The right hand sides of the above equation express u as $F(a,b)$ while the left hand sides express

u as $H(\lambda, p)$. We find the Jacobian determinants of F and H as follows.

Jacobian determinant of F: We will now make use of Proposition 4.2 about F we found earlier. If we order (a, b) as $(a_1, b_1, a_2, b_2, \ldots, b_{n-1}, a_n)$, it is easy to see that the derivative matrix of u with respect to (a, b) becomes a triangular matrix. This follows from the fact that the a_k and b_k appear in the equations of F exactly in the same order as we ordered (a, b). Thus, the determinant of the derivative matrix is simply the product of its diagonal entries. We also know all the terms of the 'new' a_k and b_k that appear in each equation. So the determinant is as follows

$$J_F(a, b) = 2^{n-1} \prod_{k=1}^{n-1} b_k^{4(n-)-1}. \tag{12}$$

Jacobian determinant of H: We order (λ, p) as $(\lambda_1, \ldots, \lambda_n, p_1, \ldots, p_{n-1})$. This gives the derivative matrix of H to be

$$\begin{bmatrix} p_1 & \ldots & p_n & \lambda_1 - \lambda_n & \ldots & \lambda_{n-1} - \lambda_n \\ 2p_1\lambda_1 & \ldots & 2p_n\lambda_n & \lambda_1^2 - \lambda_n^2 & \ldots & \lambda_{n-1}^2 - \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (2n-1)p_1\lambda_1^{2n-2} & \ldots & (2n-1)p_n\lambda_n^{2n-2} & \lambda_1^{2n-1} - \lambda_n^{2n-1} & \ldots & \lambda_{n-1}^{2n-1} - \lambda_n^{2n-1} \end{bmatrix}$$

Proposition 4.3. *Let T_n be a Jacobi matrix. Let $H(\lambda, p)$ denote the left-side equations of (11). Let (λ, p) be ordered as $(\lambda_1, \ldots, \lambda_n, p_1, \ldots, p_{n-1}$. Then $J_H(\lambda, p)$, the derivative matrix of H, has determinant*

$$J_H(\lambda, p) = \pm|\Delta(\lambda)|^4 \prod_{i=1}^{n} p_i. \tag{13}$$

Proof. Let C_i denote the ith column of this matrix. Since we want to find the determinant of this matrix, we can factor out the p_i from the C_i (remember, we can do this in a determinant). The resulting determinant is clearly a polynomial in $\lambda_1, \ldots, \lambda_n$. It must also be symmetric in λ_ks, as we can choose to order of the eigenvalues any way we want. Note that we have already found the factor $\prod_{i=1}^{n} p_i$.

Let $h := \lambda_1 - \lambda_n \to 0$, then we can find at least four zeroes in the determinant by combining columns of the determinant. It is clear that C_{n+1} and $C_1 - C_n$ will both become columns of zeroes as $\lambda_1 \to \lambda_n$ (remember that $\lambda_1^i - \lambda_n^i = (\lambda_1 - \lambda_n)(\lambda_1^{i-1} + \lambda_1^{i-2}\lambda_n + \lambda_1^{i-3}\lambda_n^2 + \cdots + \lambda_1\lambda_n^{i-1} + \lambda_n^{i-1}))$. Note that the first element of $C_1 - C_n$ is zero.

We can also check that $C' = C_{n+1} - h(C_1 + C_n)/2$ has two zeroes as $\lambda_1 \to \lambda_n$. Note that the first two elements of C' are zero. For the i-th element ($i > 2$) of this column we can do the following calculation

$$C_i' = \lambda_1^i - \lambda_n^i - \frac{1}{2}(\lambda_1 - \lambda_n)(i\lambda_1^{i-1} + i\lambda_n^{i-1}),$$
$$= (\lambda_1 - \lambda_n)\left([\lambda_1^{i-1} + \lambda_1^{i-2}\lambda_n + \cdots + \lambda_1\lambda_n^{i-1} + \lambda_n^{i-1}] - \frac{i}{2}[\lambda_1^{i-1} + \lambda_n^{i-1}]\right).$$

Let $\lambda_1 \to \lambda_n$, then clearly the first term becomes zero. The second term becomes $i\lambda_n^{i-1} - \frac{i}{2}(2\lambda_n)$, which is also zero.

Now, in the determinant, put the first column to be $C_1 - C_n$, put the n-th column to be $C_{n+1} - h(C_1 + C_n)/2$ and put the $n+1$-th column to be C_{n+1}. Now if we expand the determinant along these three columns in any order, we will get a polynomial with at least a four fold zero when $\lambda_1 \to \lambda_n$. Thus for fixed $\lambda_k, k \geq 2$, the polynomial in λ_1 had (at least) a four fold zero at λ_n. By symmetry, the determinant has a factor $\Delta(\lambda)^4 = \prod_{i<j}(\lambda_i - \lambda_j)^4$.

Note that $\Delta(\lambda)^4$ is a polynomial of degree $4(n-1)$ in λ_1. Notice that in the matrix, λ_1 only appears in columns 1 and $n+1$. We are looking for the maximum possible degree of λ_1 in the determinant. If we expand the determinant along row $2n-1$, the maximum degree it can have is $(2n-2) + (2n-2) = 4(n-1)$ (the first element of row $2n-1$ multiplied by the n-th element of row $2n-2$ of the remaining subdeterminant), or otherwise $(2n-1) + (2n-3) = 4(n-1)$ (the $n+1$-th element of row $2n-1$ multiplied by the first element of row $2n-2$ of the remaining subdeterminant).

So the determinant and $\Delta(\lambda)^4$ have the same degree in λ_1. Further, the leading coefficient of λ_1^{4n-4} is the same in $\Delta(\lambda)^4$ and in the determinant above. We can see this fact as follows. In $\Delta(\lambda)^4$, the leading coefficient is $\prod_{i<j}(\lambda_i - \lambda_j)^4$, $2 \leq i \leq n$.

In the determinant, we can develop by the last row, and subsequently by the last row of the remaining determinant. As we are interested in the coefficitent of λ_1^{4n-4}, we only need to check two terms. These are the first element of the last row multiplied by the $n+1$-th element of the second to last row, and the opposite. The first is $(2n-1)\lambda_1^{4n-4}$, and the second is $-(2n-2)\lambda_1^{4n-4}$. Note that the signs of these terms are always the same, regardless of whether n is even or odd. Because these two terms have the exact same sub-determinant (that is, one where the last two rows and the first and $n+1$-th column are taken out) we can combine these terms to get that the coefficient of λ_1^{4n-4} is 1 times whatever polynomial follows from the sub-determinant.

Now notice that this sub-determinant looks exactly like the original, the only difference is that n is exactly one smaller than before, and λ_1 does not appear. Therefore, we can do the same thing as before for λ_2, which only appears in the first and n-th column of the sub-determinant. For fixed λ_i, $3 \leq i \leq n$ we can find a polynomial in λ_2 that has a four fold zero at every other λ_i, $3 \leq i \leq n$. We also find that the coefficient of $\lambda_2^{4(n-2)}$ is 1 times the polynomial from the sub-sub-determinant.

We can see that through induction, we can keep doing this until we have calculated the full determinant. In other words, the coefficient of $\lambda_1^{4(n-1)}$ in the determinant is exactly $\prod_{i<j}(\lambda_i - \lambda_j)^4$, $2 \leq i \leq n$. Therefore, the determinant must be the same as $\Delta(\lambda)^4$.

Therefore, we get the Jacobian determinant (with all p_i factored back in)

$$J_H(\lambda, p) = \pm|\Delta(\lambda)|^4 \prod_{i=1}^{n} p_i.$$

■

From (12) and (13) we deduce that

$$|J_{G^{-1}}(\lambda, p)| = \frac{J_H(\lambda, p)}{J_F(a, b)} = \frac{\prod_{i=1}^{n} p_i \prod_{i<j} |\lambda_i - \lambda_j|^4}{2^{n-1} \prod_{k=1}^{n-1} b_k^{4(n-k)-1}}.$$

This proves the first equality in (6), the second equality follows from (5). ■

That completes the proof of Theorem 2.4. ■

5 The beta log-gas and the distribution of eigenvalues

We have found the density function (8) for the eigenvalues of a specific Jacobi matrix. Remember the beta log-gas density function (1). If we take $V(x) = \frac{x^2}{4}$, the density is

$$p^V_{n,\beta}(x) = \frac{1}{Z^V_{\beta,n}} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{n} e^{-\beta n \frac{x_k^2}{4}}. \tag{14}$$

This density is refered to as the *quadratic beta log-gas*, which can also be written as

$$p^V_{n,\beta}(x) = \frac{1}{Z^V_{\beta,n}} \exp\left\{-\beta\left[\frac{n}{4}\sum_{k=1}^{n} x_k^2 - \sum_{i<j} log|x_i - x_j|\right]\right\}.$$

This looks very close to the density function of the eigenvalues we found earlier. In fact, by theorem 3.1, this is exactly the distribution of eigenvalues of the matrix $\frac{1}{\sqrt{\beta n}}T_n$, where T_n is the Jacobi matrix of size n with $a_k \sim N(0,2)$ and $b_k^2 \sim \chi^2_{\beta(n-k)}$.

5.1 The Metha integral

The normalization constant of the quadratic beta log-gas can be found through an identity conjectured by Metha in de 60s:

$$\int\limits_{\mathbb{R}^n} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} dx = (2\pi)^{\frac{n}{2}} \prod_{k=1}^{n} \frac{\Gamma(1+\frac{\beta k}{2})}{\Gamma(1+\frac{\beta}{2})}. \tag{15}$$

Proof of identity (15). It is far from trivial to prove this identity, as the integral on the left is very hard to work out. Luckily, the random matrix approach has given us a much easier proof of this identity.
In equation (8), put $\lambda_k = \sqrt{2}x_k$ to get

$$\frac{1}{Z''_{\beta,n}} 2^{\frac{n}{2}} 2^{\frac{1}{4}\beta n(n-1)} \exp\left\{-\frac{1}{2}\sum_{k=1}^{n} x_k^2\right\} \prod_{i<j} |x_i - x_j|^\beta.$$

Remember that we know the value of $Z''_{\beta,n}$, so from all this we find that

$$\int\limits_{\mathbb{R}^n_\uparrow} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} dx = (2\pi)^{\frac{n}{2}} \frac{\prod_{k=1}^{n} \Gamma(\frac{1}{2}\beta k)}{\Gamma(\frac{\beta}{2})^n}.$$

We can now use the fact that $n\Gamma(n) = \Gamma(1+n)$ to see that $\Gamma(n) = \frac{1}{n}\Gamma(1+n)$. We find

$$\int_{\mathbb{R}^n_\uparrow} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2} dx = (2\pi)^{\frac{n}{2}} \prod_{k=1}^{n} \frac{\frac{1}{\frac{1}{2}\beta k}\Gamma(1+\frac{1}{2}\beta k)}{\frac{1}{\frac{1}{2}\beta}\Gamma(1+\frac{\beta}{2})}$$

$$= \frac{1}{n!}(2\pi)^{\frac{n}{2}} \prod_{k=1}^{n} \frac{\Gamma(1+\frac{\beta k}{2})}{\Gamma(1+\frac{\beta}{2})}.$$

Remember that $\mathbb{R}^n_\uparrow$ is simply $\mathbb{R}^n$ with the constraint that its elements are ordered. If we then extend the integral to $\mathbb{R}^n$, the result will be $n!$ times the result on the right. This gives us exactly what we want, and proves identity (15). ■

5.2 Wigner's semicircle law

We would like to get an idea of the distribution of eigenvalues in the limit (that is, what if $n \to \infty$?). Sadly, we will not be able to prove anything about this, but I would like to show a result by Wigner, known as the semicircle law (Krishnapur (2003, p. 17)).

Theorem 5.1 (Wigner's Semicircle Law)**.** *Let $A = (a)_{ij}$ be a Hermitian $N \times N$ matrix, such that*

- *The two sets $\{a_{ij} | 1 \leq i < j \leq N\} \in \mathbb{C}$ and $\{a_{ii} | 1 \leq i \leq N\} \in \mathbb{R}$ are sets of independent identically distributed random variables.*
- $\mathbb{E}(a_{ij}) = 0$ *for all i and j ($\mathbb{E}$ denotes the expectation).*
- $\mathbb{E}(a_{ij}^2) = \begin{cases} 1 & i \neq j \\ 2 & i = j. \end{cases}$

If N is large enough, then the distribution of eigenvalues of $A/\sqrt{N}$ approximates the semicircle distribution $\frac{1}{2\pi}\sqrt{4-x^2}$.

This is a more general result than what we discussed in this report. It is unclear if the exact assumptions of this theorem are met by the Jacobi matrix T_n, but if we assume they are, then the theorem implies that the eigenvalues will tend to the semicircle distribution as $n \to \infty$. Because of the relation between the quadratic beta log-gas and the eigenvalues of T_n, we would expect the quadratic beta log-gas to tend to the semicircle distribution as n grows large. We will try to make this likely through numeric processes.

The semicircle distribution as described in the theorem has values on the interval $[-2, 2]$, so we expect the eigenvalues of the scaled matrix to fall within this interval.

Below are the results of a script in python that creates a large matrix T_n, with $a_k \sim N(0, 2)$ and $b_k^2 \sim \chi^2_{\beta(n-k)}$, where the script 'picks' the values of each a_k and b_k according to their distribution. It then calculates the eigenvalues of this matrix and shows them in a (normalized) histogram, along with a red line showing the probability density function of the semicircle distribution. Figure 1 shows the results for $n = 3000$ and various values of β, however, since we scale the matrix with $\sqrt{\beta}$, we expect that this will not make a difference. Indeed, in the figures we can see that the histogram is very close to the semicircle distribution for all values of β, this makes it likely that the beta log-gas approximates the semicircle distribution as $n \to \infty$.

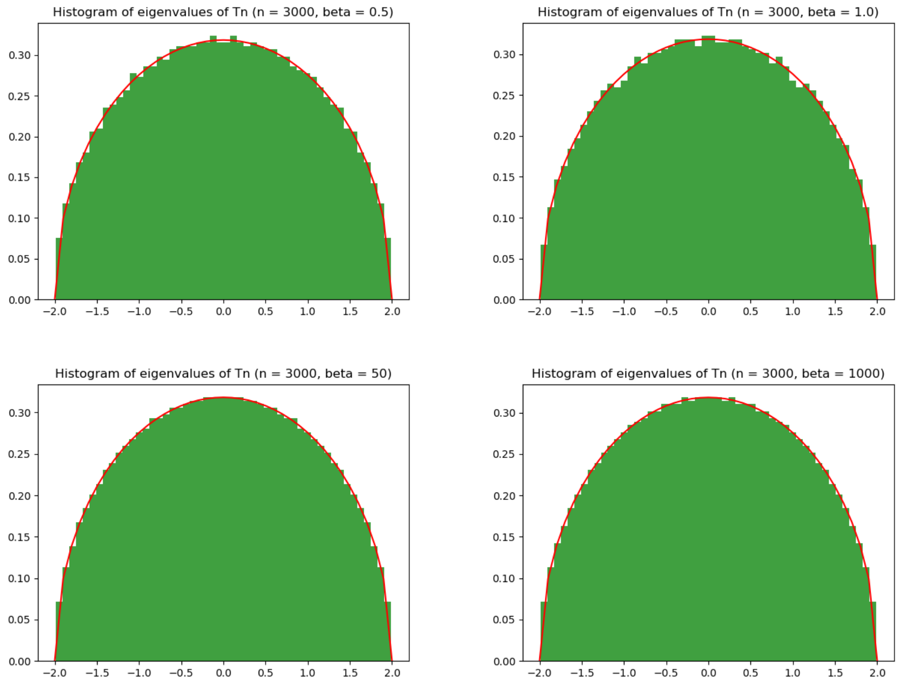

Figure 1: Histograms

6 Conclusion and Discussion

We saw that the dependent variables of the beta log-gas make it difficult to analyse. However, the Jacobi matrix approach allowed us to make some progress.
Finding the Jacobian determinant for the transformation of the elements of the Jacobi matrix to its eigenvalues was key in determining the distribution of eigenvalues for Jacobi matrices.
In the end we found that if we choose the right distribution for the elements of the Jacobi matrix, the distribution of eigenvalues is a specific case of the beta log-gas known as the quadratic beta log-gas. This helped us find the normalization constant (and prove a nice identity), and make it likely that the quadratic beta log-gas is limited by the semicircle distribution. Proving that this is the case would be a good direction for future research.

Other extensions for this research could include finding a limiting distribution for the beta log-gas in general, and providing a proof for Wigners semicircle law.

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A Python code

```
import matplotlib.pyplot as plt
import numpy as np

#define parameteres beta and n (here n=3000 and beta = 1)
n = 3000
beta = 1.

#define distribution of a and b
mu, sigma = 0, 2./(beta*n) #ak scaled by sqrt(beta*n), sigma^2 scaled by (beta*n)
a = np.array([])
for k in range(1,n+1):
    ak = np.random.normal(mu, sigma, 1)
    a = np.append(a,ak[0])

b = np.array([])
for k in range(1,n):
    shape = (beta/2.)*(n-k)
    scale = 2./(beta*n)
    bk = np.random.gamma(shape, scale, 1)
    b = np.append(b,np.sqrt(bk[0]))

#construct tridiagonal matrix Tn
mat = []
for i in range(0,n):
    mat.append([])

for i in range(0,n):
    if i==0:
        for j in range(0,n):
            if j==i:
                mat[i].append(a[i])
            elif j==i+1:
                mat[i].append(b[i])
            else:
                mat[i].append(0)
    elif i==n-1:
        for j in range(0,n):
            if j==i-1:
                mat[i].append(b[i-1])
            elif j==i:
                mat[i].append(a[i])
            else:
                mat[i].append(0)
    else:
        for j in range(0,n):
            if j==i-1:
                mat[i].append(b[i-1])
            elif j==i:
                mat[i].append(a[i])
            elif j==i+1:
                mat[i].append(b[i])
            else:
                mat[i].append(0)

#find eigenvalues and eigenvectors of matrix Tn
w, v = np.linalg.eig(mat)

#plot the eigenvalues in a histogram
count, bins, patches = plt.hist(w, 50, density=True, stacked=True, facecolor='green', alpha=0.75)

#plot the semicircle distribution
def semicircle(x):
    return (1/(2*np.pi))*np.sqrt(4-x**2)
t = np.arange(-2.0, 2.0, 0.1)
t = np.append(t,2.0)
plt.plot(t,semicircle(t),'r')
plt.title('Histogram of eigenvalues of Tn (n = '+str(n)+', beta = '+str(beta)+')')
plt.show()
```