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A thermodynamically consistent two surface/bubble thermo-mechanical model considering thermal and mechanical cyclic behaviour of fine-grained soils

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ABSTRACT

The formulation of a two surface/bubble thermo-mechanical constitutive model consistent with the principles of thermodynamics is presented. This allows plastic deformations inside the outer yield surface, resulting in a smooth stress–strain prediction and progressive cyclic deformations. This is achieved by the translation of the inner yield surface (also known as the bubble surface) with the stress state of the soil, inside the outer yield surface, by using a kinematic rule. The constitutive equations, including the hardening rules, are derived by specifying a Gibbs-type energy potential and a rate of dissipation potential function, ensuring thermodynamic consistency. The kinematic rule is divided into isothermal and non-isothermal parts. With the isothermal component, the model is capable of capturing the hysteresis behaviour of soils during cyclic mechanical loading. With the non-isothermal part, the model is able to predict the shakedown behaviour of soils observed when they are subjected to heating–cooling cycles. The performance of the model is compared with various experimental data for isothermal and non-isothermal conditions, and is shown to be in good agreement.

1. Introduction

Geo-structures such as energy-piles, thermal retaining walls, heat storage tanks, thermal quay walls, pipelines and radioactive waste disposal, impose thermal loads and temperature changes on the ground (Brandl, 2006; Thomas et al., 2014). It has been demonstrated that temperature variation affects the mechanical behaviour of fine-grained soils in terms of both the volumetric and shear behaviour. Therefore, it is important to account for the thermo-mechanical response of soils in the design process of these structures. With this aim, a number of thermo-mechanical constitutive models have been developed (Cheng et al., 2020; Di Donna and Laloui, 2015; Hamidi et al., 2017; Laloui and François, 2009; Maranha et al., 2018; Zhou et al., 2017; Zhou and Ng, 2015).

To account for plastic strains in over-consolidated states and to capture the cyclic mechanical behaviour of soils, several theories and models in plasticity of soils have been developed, of which multi-surface models and bounding surface models are examples. Multi-surface models (also known as nested surface models), which were proposed originally for metals, were developed independently by Mróz (1967) and Iwan (1967). In these models, it is assumed that several yield

surfaces exist, each with an associated hardening modulus, that can kinematically translate with the stress state of the material. In bounding surface models, which were first proposed by Dafalias and Popov (1975) and Krieg (1975) for metals, and later extended for soils by Dafalias (1986) and Dafalias and Herrmann (1986), mainly two surfaces were employed; a “bounding surface” which encloses all the admissible stress states and a “yield surface” which translates within the bounding surface. The stress state, when located on the yield surface, is mapped (with different approaches) onto the bounding surface (image stress) and the distance between these stresses indicates how far the stress state is from the bounding surface, which controls the magnitude of plastic strain increments through the plastic modulus formulation. The greater the distance, the smaller the plastic strain increment. These concepts have been employed by many researchers (for example, Corti et al., 2016; Dafalias and Manzari, 2004; Elia and Rouainia, 2016; Gajo and Muir Wood, 2001; Golchin and Lashkari, 2014; Kavvadas and Amorosi, 2000; Khalili et al., 2005; Lashkari and Golchin, 2014; Li and Meissner, 2002; Manzari and Dafalias, 1997; Mroz et al., 1978; Mróz et al., 1979; Puzrin and Kirschenboim, 2001; Rezaian and Dejaloud, 2021; Stallebrass and Taylor, 1997; Sun et al., 2020) and have been successful in capturing the main behaviour of soils.

The concept of bounding surface plasticity has been also extended to

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List of symbols*Roman*

1	Second order identity tensor
$\langle \rangle$	Macaulay bracket
A_i	Stress-like function for the inner yield surface
A_o	Stress-like function for the outer yield surface
B_i	Stress-like function for the inner yield surface
B_o	Stress-like function for the outer yield surface
b	Parameter controlling the accumulated thermo-plastic strains
C_i	Stress-like function for the inner yield surface
C_o	Stress-like function for the outer yield surface
C_χ	Parameter controlling the plastic strains
CS	Critical State
C^e	Elastic compliance matrix
c_1	Thermodynamical constraint function
c_2	Thermodynamical constraint function
$(c_i)_p$	Thermodynamical constraint function under isothermal conditions along the p -axis
$(c_H)_p$	Thermodynamical constraint function under heating phase along the p -axis
$(c_i)_q$	Thermodynamical constraint function under isothermal conditions along the q -axis
$(c_H)_q$	Thermodynamical constraint function under heating phase along the q -axis
D^e	Elastic stiffness matrix
d	Rate of dissipation potential function
d_i	Sub-rate of dissipation function related to inner yield surface
d_o	Sub-rate of dissipation function related to outer yield surface
e	Void ratio
e_0	Initial void ratio
e^p	Plastic deviatoric strain tensor
g	Gibbs free energy potential
g_1	Isothermal Gibbs free energy potential
$\bar{G}$	Maximum elastic shear modulus
H	Function
$\bar{K}$	Maximum elastic bulk modulus
M	Critical State stress ratio
M_0	Critical State stress ratio at ambient temperature
n	Power of p for shear modulus
OCR	Over consolidation ratio
p	Hydrostatic pressure
p_{cT}	Apparent pre-consolidation pressure
p_{c0}	Initial pre-consolidation pressure
$\dot{p}_{cT}$	Rate of the apparent pre-consolidation pressure
p_{iT}	Compression apex of the inner yield surface
$\dot{p}_{iT}$	Rate of change of the compression axis of the inner yield surface
$\tilde{p}_{iT}$	Size of the major-axis of the inner yield surface
$\tilde{p}_{i0}$	Initial size of the major-axis of the inner yield surface
$\dot{\tilde{p}}_{iT}$	Rate of change of the major-axis of the inner yield surface
$\bar{p}$	Hydrostatic coordinate of projection centre
$\bar{p}$	Hydrostatic coordinate of image stress
p_a	Atmospheric hydrostatic pressure
q	Deviatoric stress
$\hat{q}$	Deviatoric coordinate of projection centre
$\bar{q}$	Deviatoric coordinate of image stress
r_p^i	Plastic flow along the p -axis
r_q^i	Plastic flow along the q -axis
r^i	Plastic flow tensor
r^i	Plastic flow vector in triaxial space

R_{ini}	Ratio between the inner and outer yield surfaces
s	Deviatoric stress tensor
T	Absolute temperature
T_0	Initial absolute temperature
$\dot{T}$	Temperature increment
u_p	Pore water pressure
W^p	Total plastic work increment
y_i	Inner yield surface in true stress space
y_o	Outer yield surface in true stress space
y_i^d	Inner yield surface in dissipative stress space
y_o^d	Outer yield surface in dissipative stress space
$\dot{y}_i$	Rate of change of inner yield surface
<i>Greek</i>	
α	Parameter related to the shape of the yield surface
α^*	Coefficient of linear thermal expansion
γ	Parameter related to the shape of the yield surface
ϵ	Strain tensor
ϵ^p	Plastic strain tensor
ϵ^e	Elastic strain vector in triaxial space
ϵ^{Therm}	Thermo-elastic strain vector in triaxial space
ϵ^p	Plastic strain vector in triaxial space
ϵ_i^p	Plastic strain vector related to d_i
ϵ_o^p	Plastic strain vector related to d_o
$\dot{\epsilon}$	Rate of strain vector
$\dot{\epsilon}_i^p$	Rate of plastic strain vector related to inner yield surface
ϵ_v	Total volumetric strain
ϵ_s	Total deviatoric strain
ϵ_v^e	Elastic volumetric strain
ϵ_s^e	Elastic deviatoric strain
$\dot{\epsilon}_v^e$	Elastic volumetric strain increment
$\dot{\epsilon}_s^e$	Elastic deviatoric strain increment
ϵ_v^p	Plastic volumetric strain
ϵ_s^p	Plastic deviatoric strain
$\dot{\epsilon}_v^p$	Plastic volumetric strain increment
$\dot{\epsilon}_s^p$	Plastic deviatoric strain increment
$\epsilon_{v,i}^p$	Plastic volumetric strain related to the inner yield surface
$\epsilon_{v,o}^p$	Plastic volumetric strain related to the outer yield surface
$\epsilon_{s,i}^p$	Plastic deviatoric strain related to the inner yield surface
$\epsilon_{s,o}^p$	Plastic deviatoric strain related to the outer yield surface
$\dot{\epsilon}_{v,i}^p$	Plastic volumetric strain increment related to the inner yield surface
$\dot{\epsilon}_{v,o}^p$	Plastic volumetric strain increment related to the outer yield surface
$\dot{\epsilon}_{s,i}^p$	Plastic deviatoric strain increment related to the inner yield surface
$\dot{\epsilon}_{s,o}^p$	Plastic deviatoric strain increment related to the outer yield surface
κ	Elastic compression index
λ	Elasto-plastic compression index
Λ_1	Lagrangian multiplier
Λ_2	Lagrangian multiplier
$\dot{\Lambda}$	Plastic multiplier
μ	Coefficient of thermal softening of the inner yield surface
μ_0	Coefficient of thermal softening of the outer yield surface
v	Specific volume
π^*	Coefficient of Critical State stress ratio variation with temperature
ρ	Coordinates of the decompression apex of the inner yield surface
$\dot{\rho}$	Kinematic rule
$\dot{\rho}_I$	Kinematic rule under isothermal conditions
$\dot{\rho}_H$	Kinematic rule during heating phase

ρ_p	Hydrostatic coordinate of decompression apex of the inner yield surface	$\dot{\bar{\sigma}}$	Incremental changes of the projection centre
ρ_q	Deviatoric coordinate of decompression apex of the inner yield surface	$\bar{\sigma}$	Coordinates of the image stress in triaxial space
$\dot{\rho}_p$	Kinematic rule along the p -axis	χ_i	Dissipative stress related to d_i
$\left(\dot{\rho}_I\right)_p$	Kinematic rule along the p -axis under isothermal conditions	χ_o	Dissipative stress related to d_o
$\left(\dot{\rho}_H\right)_p$	Kinematic rule along the p -axis under heating phase	χ_ρ	Dissipative stress related to kinematic variable ρ
$\dot{\rho}_p$	Kinematic rule along the q -axis	χ_p^i	Hydrostatic dissipative stress related to d_i
$\left(\dot{\rho}_I\right)_q$	Kinematic rule along the q -axis under isothermal conditions	χ_p^o	Hydrostatic dissipative stress related to d_o
σ	Stress tensor	χ_q^i	Deviatoric dissipative stress related to d_i
σ	Stress vector	χ_q^o	Deviatoric dissipative stress related to d_o
$\dot{\sigma}$	Stress increment vector	$\bar{\chi}_i$	Generalised stress related to ϵ_i^p
$\bar{\sigma}$	Stress coordinates of the projection centre	$\bar{\chi}_o$	Generalised stress related to ϵ_o^p
		$\bar{\chi}_p^i$	Hydrostatic generalised stress related to ϵ_i^p
		$\bar{\chi}_p^o$	Hydrostatic generalised stress related to ϵ_o^p
		$\bar{\chi}_q^i$	Deviatoric generalised stress related to ϵ_i^p
		$\bar{\chi}_q^o$	Deviatoric generalised stress related to ϵ_o^p

account for the effects of temperature on the behaviour of soils and for capturing the thermal cyclic shakedown behaviour of soils (Cheng et al., 2020; Di Donna and Laloui, 2015; Zhou et al., 2017; Zhou and Ng, 2018). It should be noted that these types of constitutive model may not always satisfy the laws of thermodynamics; i.e., by employing a certain range of parameters or following a certain stress path these models may not conserve energy (first law of thermodynamics) or result in negative dissipation (second law of thermodynamics) for deformations associated with plastic behaviour and, therefore, they are not unconditionally thermodynamically consistent. For example, Cheng et al. (2020) showed that the response of their thermo-mechanical model is stress path independent when it is subjected to stress paths under isotropic conditions (where the deviatoric stress $q = 0$). However, for stress paths including non-isotropic stress states (where $q \neq 0$) their model fails to satisfy thermodynamics criteria.

Constitutive equations may also be obtained by using the principles of thermodynamics and several frameworks have been proposed (see Houlsby and Puzrin (2007); Rajagopal and Srinivasa (2019), for example). Hyperplasticity, firstly developed by Collins and Houlsby (1997), is one of those approaches, in which constitutive equations are derived by introducing an energy potential function and a rate of dissipation potential function (for the rate-independent behaviour of materials). One of the advantages of developing constitutive equations within these frameworks is that the predicted behaviour of the material is guaranteed to satisfy the laws of thermodynamics (conservation of energy and non-negative dissipation during irreversible/plastic deformations). Following the hyperplasticity approach, many constitutive models have been proposed for soils (Aung et al., 2019; Collins and Hilder, 2002; Collins and Houlsby, 1997; Collins and Kelly, 2002; Coombs, 2017; Coombs et al., 2013; Houlsby et al., 2017; Houlsby and Puzrin, 2007; Lai et al., 2014; Likitlersuang and Houlsby, 2006; Rollo and Amorosi, 2020; Zhang et al., 2018). Golchin et al. (2022) developed a single surface thermo-mechanical constitutive model within this framework which successfully captured the monotonic response under isothermal conditions, the temperature effects on volumetric behaviour and shear behaviour, and the cyclic response of normally and slightly over-consolidated fine-grained soils when subjected to a single heating-cooling cycle. These results were consistent with the majority of thermo-mechanical models, but had the advantage of being unconditionally thermodynamically consistent.

Similar also to the majority of the models found in literature (e.g., Cui et al., 2000; Graham et al., 2001; Hueckel and Baldi, 1990; Zhou and Ng, 2015), the model of Golchin et al. (2022) failed to capture a number of key features for advanced stress paths. This included not capturing a smooth stress-strain response for highly over-consolidated states, due to

the use of a single yield surface which resulted in an abrupt change in response when the mechanical behaviour changed from an elastic to an elasto-plastic response. Moreover, when simulating the mechanical cyclic behaviour of soils (loading-unloading) for over-consolidated states, since the state of the material is inside the yield surface, a portion of the response (depending on the OCR) was predicted to be purely elastic. This issue was also attributed to the use of a single yield surface in the model. In addition, the model could not predict the accumulated plastic strains when soils were subjected to several heating-cooling cycles. After the first heating-cooling cycle (for normally and slightly over-consolidated soils), the model predicted the subsequent behaviour as a thermo-elastic response. Therefore, to some extent, the model was incapable of predicting the thermo-plastic behaviour of slightly over-consolidated soils during heating-cooling cycles. One approach to resolve the aforementioned shortcomings is to utilise an inner yield surface, similar to bounding surface plasticity models, which can translate (move) in the stress space within the outer (bounding) surface. Thus, a certain amount of plasticity within the outer yield surface is developed.

In this paper, the single surface thermo-mechanical constitutive model of Golchin et al. (2022) is upgraded to a two surface (bubble-type) constitutive model. The model is developed using the framework of hyperplasticity with a newly defined rate of dissipation potential function (resulting in the yield surface formulation proposed by Golchin et al. (2021)) and a newly proposed temperature-dependent kinematic hardening rule. Consequently, the model is capable of capturing the mechanical cyclic behaviour and thermal (heating-cooling) cyclic shakedown behaviour of fine-grained soils, as well as their monotonic thermo-mechanical behaviour, and the formulation is ensured to be consistent with the principles of thermodynamics. Note that the model is considered to be suitable for fine grained soils subjected to temperatures between 0 °C and 100 °C.

The formulations presented here are in accordance with geotechnical conventions, where compressive stresses and contractive strains are considered to be positive and all stresses are effective stresses.

2. Thermo-mechanical behaviour of fine-grained soils

At elevated temperatures, fine-grained soils exhibit a reduction in pre-consolidation pressure in comparison with the pre-consolidation pressure at ambient temperature (Abuel-Naga et al., 2007a; Cekerevac and Laloui, 2004; Tanaka et al., 1997; Uchaipichat and Khalili, 2009). The observed variation of normalised pre-consolidation pressure (p_{cT}/p_{cT0} , where p_{cT0} is the pre-consolidation pressure at ambient temperature, $T = T_0$) with temperature (T) for several soils is shown in Fig. 1(a).

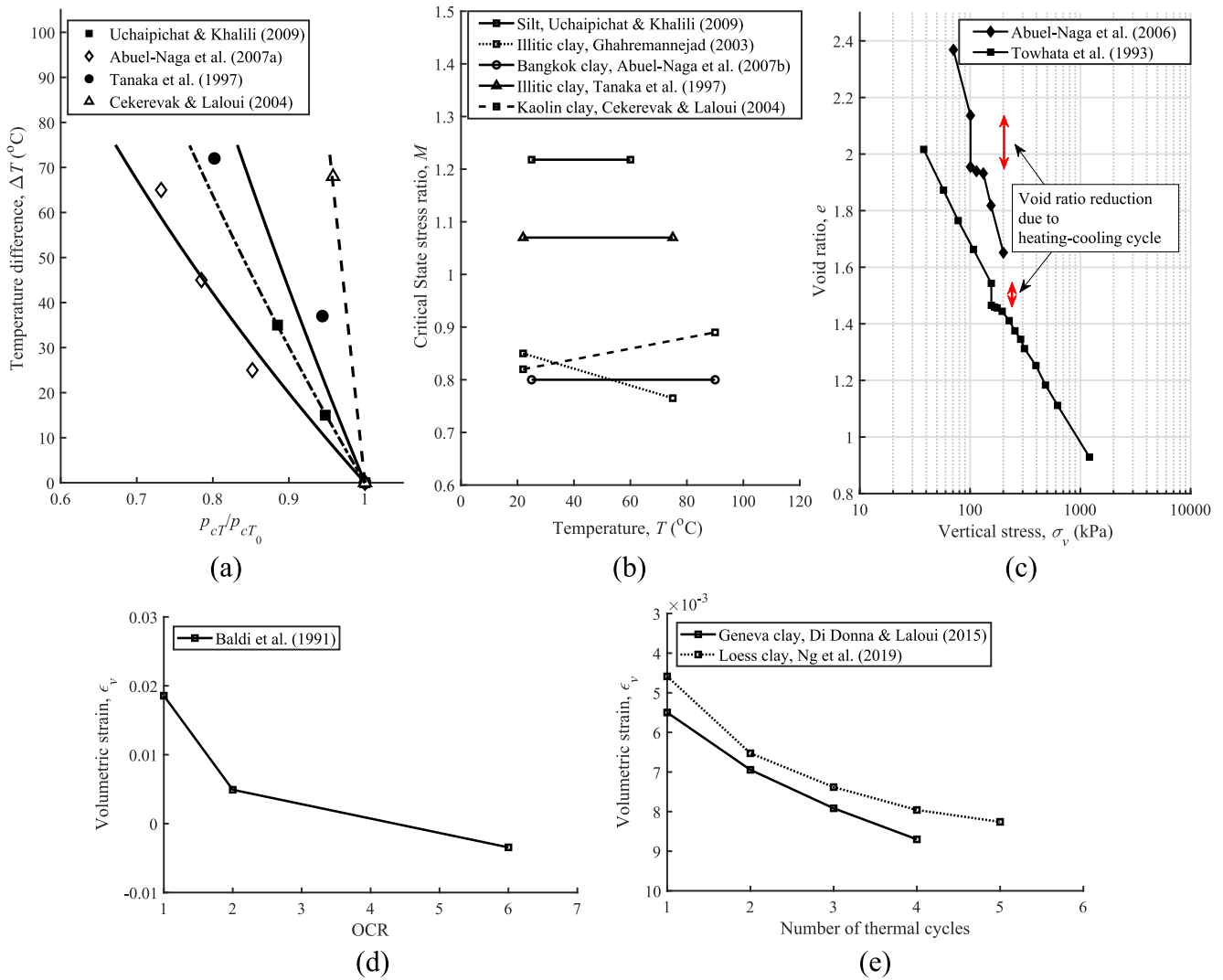


Fig. 1. Thermo-mechanical behaviour of fine-grained soils: (a) normalised pre-consolidation pressure variation with temperature; (b) variation of Critical State stress ratio with temperature; (c) oedometer test results of void ratio reduction due to heating-cooling cycle; (d) thermo-plastic behaviour of soils subjected to heating at various OCRs; (e) shakedown behaviour of soils when subjected to several heating-cooling cycles.

Depending on the mineralogy and constituents of the soil, the influence of temperature on the shear behaviour of soils does not show a unique pattern. Fig. 1(b) shows the variation of Critical State stress ratio (M) against temperature reported for a wide range of soils. It is observed that M may increase, decrease or remain unchanged at elevated temperatures (Abuel-Naga et al., 2007b; Cekerevak and Laloui, 2004; Ghahremannejad, 2003; Tanaka et al., 1997; Uchaipichat and Khalili, 2009).

When a fine-grained soil is subjected to heating at normally and slightly over-consolidated states, the soil undergoes permanent (plastic) volumetric contraction (Di Donna and Laloui, 2015; Towhata et al., 1993; Uchaipichat and Khalili, 2009). As the soil attains higher over-consolidated states (higher OCRs), the severity of permanent volumetric strains due to heating reduces (Baldi et al., 1991) and the thermo-elastic behaviour, caused by the volumetric expansion of soil particles, dominates the deformation of the soil. This is demonstrated in Fig. 1(d), where the soil samples were subjected to a single heating-cooling cycle between 20 °C and 95 °C at different OCRs.

Fine-grained soils also exhibit hardening behaviour when they are subjected to heating-cooling cycles; i.e., after being subjected to thermal cycles, the soil attains a denser state due to the reduction of its volume. The oedometer results for the void ratio reduction of two different clays

(Abuel-Naga et al., 2006; Towhata et al., 1993) subjected to one heating-cooling cycle are presented in Fig. 1(c). Moreover, permanent volumetric strains are accumulated when the soil is subjected to several thermal cycles (Di Donna and Laloui, 2015; Ng et al., 2019). However, the amount of permanent strains reduces as the number of thermal cycles increase, and this continues until the behaviour becomes thermo-elastic (Fig. 1(e)). This is known as cyclic thermal shakedown behaviour and is shown in Fig. 1(e) for Geneva clay subjected to thermal cycles between 5 °C and 60 °C (Di Donna and Laloui, 2015) and Loess clay subjected to heating-cooling cycles between 70 °C and 15 °C (Ng et al., 2019).

3. Conceptual framework of the two-surface/bubble constitutive model

The proposed constitutive model utilises the formulation of an adapted Modified Cam-Clay type surface which is able to have non-elliptical shapes, proposed by Golchin et al. (2021), for both the outer yield surface and the inner yield surface. The outer yield surface is extended into the temperature domain, in a manner consistent with Cui et al. (2000), so that the observed reduction in pre-consolidation pressure is captured. However, this constitutive model uses the existing

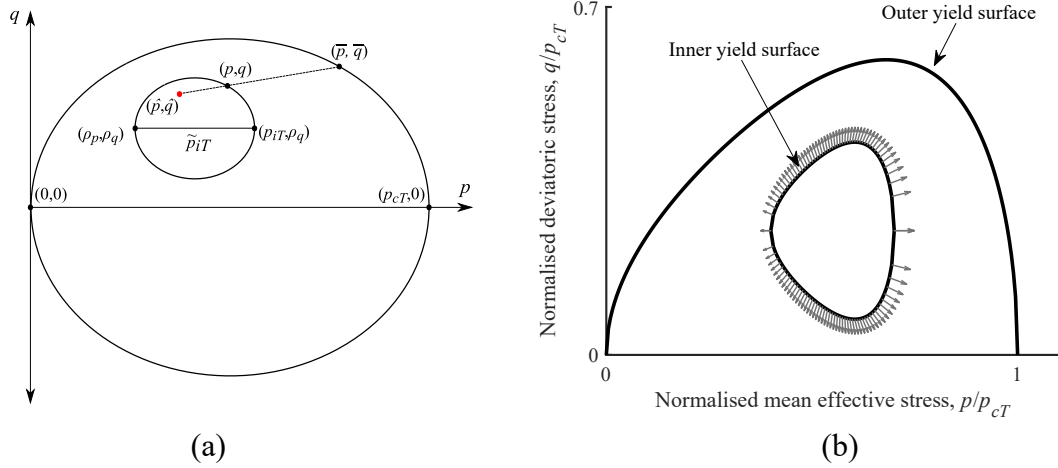


Fig. 2. (a) The inner and outer yield surfaces ($\alpha = \gamma = 0$) in meridian plane; (b) non-associated flow rule of the inner yield surface ($\alpha = \gamma = 0.5$).

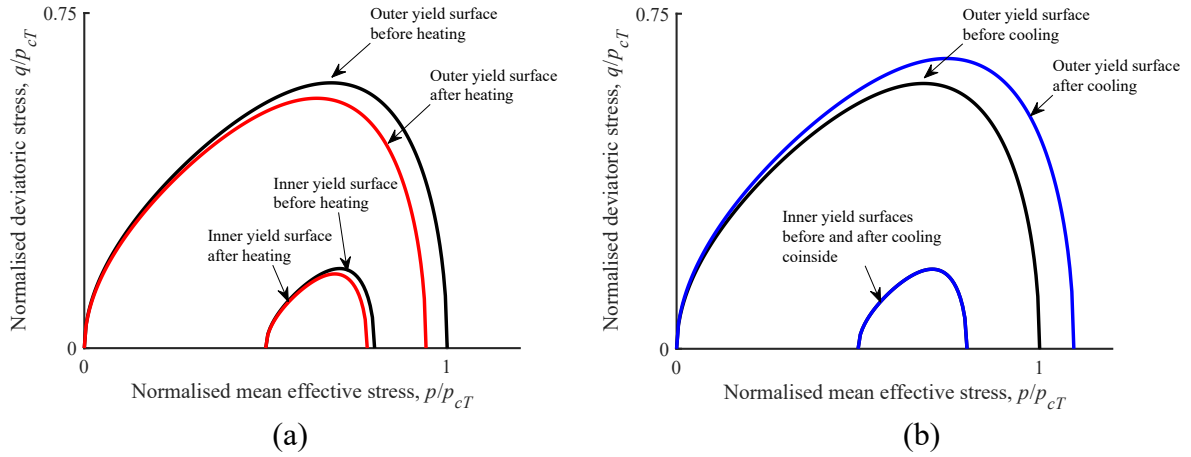


Fig. 3. (a) The inner and outer yield surfaces under heating phase in meridian $(p - q)$ space ($\alpha = \gamma = 0.5$); (b) the inner and outer yield surfaces under cooling phase.

formulation of Golchin et al. (2022) as a starting point, as this approach is thermodynamically consistent. In order to capture a smooth stress–strain response and the observed thermo-mechanical cyclic behaviour, a second surface (the inner yield or bubble surface) is included, as is shown schematically in Fig. 2(a).

The two surfaces are geometrically similar in shape, with the possibility of having elliptical (Fig. 2(a)) and non-elliptical shapes (Fig. 2(b)) in the meridian $(p-q)$ plane. The outer yield surface is pinned to the origin of the stress space $(p,q)=(0,0)$ and is bounded at the apparent pre-consolidation pressure at the current temperature $(p,q)=(p_{cT},0)$ (Fig. 2(a)). This surface is regarded as the surface encapsulating the loading history that the soil has experienced (i.e., the bounding surface) and can expand/shrink with temperature without hardening/softening (temperature dependent yield surface) (Fig. 3) and with hardening/softening for plastic (volumetric) strains, consistent with the observed behaviour of fine-grained soils. The inner yield surface indicates the elastic domain and is bounded on the p -axis between ρ_p , as the decompression apex, and p_{iT} , as the compression apex, with a major-axis size (hydrostatic extent) of $\tilde{p}_T$ ($\tilde{p}_T = p_{iT} - \rho_p$) (Fig. 2(a)). When the state of the soil is inside the inner yield surface, the mechanical behaviour is (thermo) elastic, and stress states lying on the inner yield surface indicate an elasto-plastic response. The magnitude of the plastic strains is controlled by the proximity of the inner yield surface to the outer yield surface (i.e., controlled by the distance between the stress

state (p,q) lying on the inner yield surface, and the geometrically similar position on the outer yield surface ($\bar{p}, \bar{q}$) (also called the image stress)). The inner surface can move (translate) with the state of the soil inside the outer yield surface; i.e., the domain in which the inner yield surface can move is limited by the enclosed area of the outer yield surface in temperature and stress space.

The translation of the inner yield surface is triggered when plastic strain increments are developed and is governed by a kinematic rule, which is controlled by the distance between the stress (on the inner yield surface) and the image stress (on the outer yield surface) and the growth of the inner and outer yield surfaces.

The inner yield surface, similar to the outer yield surface, shrinks in size in response to an increase in temperature (heating phase) (Fig. 3(a)). To capture the progress of plasticity during consecutive heating–cooling cycles, the hydrostatic extent of this surface reacts neutrally to a decrease in temperature, i.e., the size of the major-axis of the surface remains unchanged during cooling (Fig. 3(b)). This feature enables the state of the material to encounter the inner yield surface during the heating phase and is in contrast to the outer yield surface which expands as temperature decreases. With these features, the model is able to predict a smooth stress–strain response at any state, simulate the mechanical cyclic response, and predict the shakedown behaviour and accumulated plastic strains observed when soils are subjected to several heating–cooling cycles.

The elasticity formulation used for the model is non-linear and the

plastic flow is non-associated, i.e., not necessarily perpendicular to the inner yield surface (Fig. 2(b)), without the necessity of defining a separate plastic potential as is the case for conventional plasticity models. This feature is obtained by following the thermodynamically consistent hyperplastic approach (Collins and Houlsby, 1997; Houlsby and Puzrin, 2007). The temperature effects on the shear behaviour are accounted for by defining a temperature-dependent Critical State stress ratio (M). The effect of temperature on volumetric behaviour is encapsulated in the temperature-dependency of the outer yield surface, and in the temperature-dependent isotropic hardening rule of the inner yield surface (Fig. 3). In addition, the thermal-dependency of the inner yield surface is proposed to be dependent on the state of the soil, to capture the thermo-mechanical behaviour of the soil at highly over-consolidated states, as well as slightly over and normally consolidated states. Note that all the aforementioned mechanisms are embedded in the formulation of a newly proposed rate of dissipation potential.

4. Formulation

The constitutive equations presented here are defined in triaxial stress space and are developed within the framework of hyperplasticity theory (Collins and Houlsby, 1997; Houlsby and Puzrin, 2007), where the formulations are derived by specifying an energy potential and a rate of dissipation function (for rate independent materials). A detailed description of the thermodynamics equations relating to the thermo-mechanical aspects can be found in Golchin et al. (2022). A Gibbs-type energy potential, defined by the independent variables, is selected for the energy potential. The independent variables are the true stress invariants in triaxial stress space (hydrostatic pressure or mean effective stress, $p = \text{tr}(\boldsymbol{\sigma})/3$ (kPa) and deviatoric stress, $q = (3/2 \mathbf{s}:\mathbf{s})^{1/2}$ (kPa), where $\boldsymbol{\sigma}$ and $\mathbf{s} = \boldsymbol{\sigma} - \text{tr}(\boldsymbol{\sigma})/3 \mathbf{1}$ are, respectively, the stress and deviatoric stress tensors), internal variables (which here are the plastic strain invariants, i.e., the plastic volumetric strain, $\epsilon_v^p = \text{tr}(\boldsymbol{\epsilon}^p)$, and the plastic deviatoric strain, $\epsilon_s^p = (2/3 \boldsymbol{\epsilon}^p:\boldsymbol{\epsilon}^p)^{1/2}$, $\boldsymbol{\epsilon}^p = \boldsymbol{\epsilon}^p - \text{tr}(\boldsymbol{\epsilon}^p)/3 \mathbf{1}$, where $\boldsymbol{\epsilon}^p$, $\boldsymbol{\epsilon}^p$ and $\mathbf{1}$ are the plastic strain, plastic deviatoric strain, and identity tensors, respectively), and the absolute temperature T (K). The energy potential is defined as

$$g = g_1(p, q) - (p \epsilon_{v,o}^p + q \epsilon_{s,o}^p) - (p \epsilon_{v,i}^p + q \epsilon_{s,i}^p) - 3\alpha^* p(T - T_0) \quad (1)$$

where “i” and “o”, respectively, correspond to the variables of the inner and outer yield surfaces, α^* is the linear thermal expansion coefficient of the soil skeleton and $g_1(p, q)$ represents the Gibbs energy potential at isothermal conditions, where the elastic component of the model is derived from Appendix A. The form of the energy potential considered here (Eq. (1)) results in a “decoupled” material behaviour (Collins and Houlsby, 1997) through the decoupling terms $p \epsilon_v^p$ and $q \epsilon_s^p$, in which the elastic and plastic behaviours became independent of each other. The choice for a decoupled formulation is made in order to obtain a less complex formulation and ease of implementing the constitutive equations into boundary-value solvers such as FEM and MPM. Energy potentials incorporating elastic–plastic coupling can be found in the works of Collins (2002), Golchin and Lashkari (2014) and Lashkari and Golchin (2014). Note that some researchers may argue that plastic strains should not be considered as internal variables (see for example Maugin, 2015; Rubin, 2001) and assume that by nature the behaviour of the material is the sum of elastic and plastic strains. For such an assumption, the decoupling terms drop out of the energy potential function. In addition, when dealing with large deformations an appropriate framework is required to incorporate the elastic and plastic constitutive equations of the material. For such a problem the (strain energy/Helmholtz) energy potential function could be defined by elastic strains (and not a combination of the total strains and plastic strains) and incorporated into the general large deformation theory (Dafalias, 1998).

To derive the rate-independent plasticity formulation, the definition of an appropriate rate of dissipation potential function (d) is necessary

and this is required to be a first order homogenous function of plastic strain rates (Collins and Houlsby, 1997; Golchin et al., 2022). Following the approach of Golchin et al. (2022), and unlike Collins and Hilder (2002) and Coombs et al. (2013), the rate of dissipation function (Eq. (2)) is defined in such a way that the use of a shift stress to transfer stresses from dissipative stress space to true stress space is eliminated (Houlsby and Puzrin, 2007; Rollo and Amorosi, 2020). As a result, the derived formulations become simpler. The total rate of dissipation potential consists of two sub-rate of dissipation potentials (d_o and d_i) relating to the rate of dissipation of the outer yield surface and the inner yield (bubble) surface. The constitutive model is enriched by a kinematic rule assigned to the internal variable ρ . The approach proposed by Rollo (2018) and Rollo and Amorosi (2020) is followed here to ensure that the proposed kinematic rule is formulated in a consistent manner with thermodynamics. This operation is performed via the addition of thermodynamic complementary constraint functions (c_1 and c_2) using the standard method of Lagrangian multipliers. The rate of dissipation function is defined as

$$d = d_o + d_i + \Lambda_1 c_1 + \Lambda_2 c_2 \quad (2a)$$

$$d_o = C_o \dot{\epsilon}_{v,o}^p + \sqrt{A_o^2 \left(\dot{\epsilon}_{v,o}^p \right)^2 + B_o^2 \left(\dot{\epsilon}_{s,o}^p \right)^2} \quad (2b)$$

$$d_i = (C_i + \rho_p) \dot{\epsilon}_{v,i}^p + \rho_q \dot{\epsilon}_{s,i}^p + \sqrt{A_i^2 \left(\dot{\epsilon}_{v,i}^p \right)^2 + B_i^2 \left(\dot{\epsilon}_{s,i}^p \right)^2} \quad (2c)$$

$$c_1 \left(\dot{\epsilon}_{v,i}^p, \dot{\rho}_p, p, \rho_p, p_{cT}, \tilde{p}_{iT} \right) = 0 \quad (2d)$$

$$c_2 \left(\dot{\epsilon}_{s,i}^p, \dot{\rho}_q, q, \rho_q, p_{cT}, \tilde{p}_{iT} \right) = 0 \quad (2e)$$

where d_o and d_i are, respectively, the rate of dissipation potentials that the outer and inner yield surfaces are derived from; $\dot{\epsilon}_{v,k}^p$ and $\dot{\epsilon}_{s,k}^p$ ($k = i, o$) are plastic volumetric and deviatoric strain increments corresponding to the inner and outer yield surfaces; A_k , B_k and C_k are stress-like functions of the inner and outer yield surfaces; ρ_p and ρ_q are, respectively, the hydrostatic and deviatoric components of the kinematic variable ρ which represent the coordinates (location) of the de-compression apex of the inner yield surface in meridian stress space; and, $\dot{\rho}_p$ and $\dot{\rho}_q$ are the incremental translations of the inner yield surface along the p -axis and q -axis, respectively. Note that the sub-rate of dissipation potentials considered in Eq. (2a) have the general formulation of the rate of dissipation function in the existing formulation of Golchin et al. (2022) without considering the rotation/shearing of the surface with respect to the p -axis.

The thermodynamic constraint functions, c_1 and c_2 , are augmented by Lagrangian multipliers (Λ_1 and Λ_2) in the rate of dissipation potential d (Eq. (2a)) (Rollo, 2018; Rollo and Amorosi, 2020). Note that c_1 and c_2 are numerically zero (Eq. (2d) and Eq. (2e)), which implies that

$$d = d_o + d_i \quad (2f)$$

This means that the terms $\Lambda_1 c_1$ and $\Lambda_2 c_2$ in Eq. (2a) do not produce any work and they represent constraints to optimise the maximum rate of dissipation, d (Eq. (2a)) (Karrech et al., 2011); i.e., the addition of constraints restricts the solutions for maximising d to a subset where the conditions $c_1 = 0$ and $c_2 = 0$ are valid. The specific formulations of c_1 and c_2 are considered later in this section.

According to the principles of thermodynamics, it is assumed that dissipative materials (e.g. geomaterials) prefer to follow a state path which produces the maximum rate of dissipation (to ensure the second law of thermodynamics is satisfied). When the rate of dissipation is maximised, i.e., when Ziegler's postulation is valid, it follows that $\Lambda_1 = \Lambda_2 = 0$ (see Appendix A). This means that constitutive equations may be

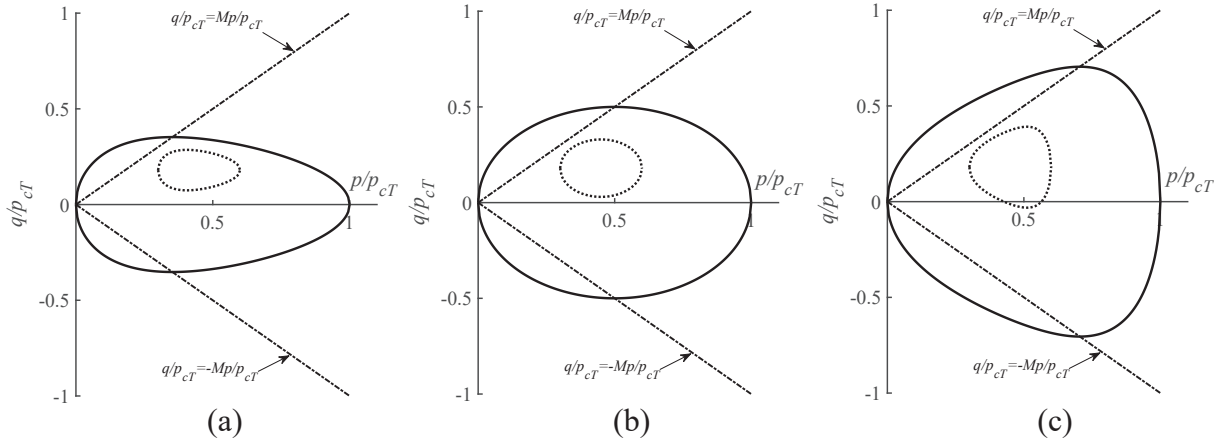


Fig. 4. The inner and outer yield surfaces in meridian ($p-q$) space: (a) $(\alpha, \gamma) = (0, -1)$; (b) $(\alpha, \gamma) = (0, 0)$; (c) $(\alpha, \gamma) = (0, 1.5)$.

obtained from the Gibbs energy potential (Eq. (1)) along with the rate of dissipation potential function Eq. (2f), with extra conditions that satisfy the constraint functions (c_1 and c_2). These additional conditions form the kinematic rule. In short, the addition of the constraint functions ensures that only the possible state paths (i.e., when the kinematic rule is valid) are considered. This approach has been used by other researchers to add several constraints to constitutive equations (Houlsby et al., 2017; Karrech et al., 2011; Rollo, 2018; Rollo and Amorosi, 2022, 2020).

The outer yield surface and the inner yield surface are derived from the sub-rate of dissipation potentials and are defined by dissipative stresses which are obtained by differentiating the rate of dissipation potential with respect to the internal variables. The flow rule is derived by considering associated conditions to the inner yield surface in dissipative stresses (i.e., the flow rule is perpendicular to the inner yield surface defined in dissipative stress space). The derivation of the outer yield surface, the inner yield surface, and the plastic flow in true (triaxial) stress space (p, q) are provided in Appendix A and are summarised, respectively, as

Outer yield surface:

$$y_o = B_o^2(p - C_o)^2 + A_o^2 q^2 - A_o^2 B_o^2 = 0 \quad (3)$$

Inner yield surface:

$$y_i = B_i^2(\tilde{p} - C_i)^2 + A_i^2 \tilde{q}^2 - A_i^2 B_i^2 = 0 \quad (4)$$

Plastic flow:

$$\{\mathbf{r}^i\} = \left\{ \begin{matrix} r_p^i \\ r_q^i \end{matrix} \right\} = 2 \left\{ \begin{matrix} B_i^2(\tilde{p} - C_i) \\ A_i^2 \tilde{q} \end{matrix} \right\} \quad (5)$$

where $\tilde{p} = p - \rho_p$ and $\tilde{q} = q - \rho_q$; r_p^i and r_q^i are, respectively, the plastic flow vector (in triaxial stress space), the plastic flow along the p -axis and the plastic flow along the q -axis. Note that, although the plastic flow in dissipative stress space (Eq. (A-15)) is associated with respect to the inner yield surface in dissipative stress space (Eq. (A-14)), the plastic flow in true stress space (Eq. (5)) may become non-associated with respect to the inner yield surface defined in true stress space (Eq. (4)) (see Fig. 2(b)). This observation is due to the stress dependency of the rate of dissipation potential function via stress-like functions (see Golchin et al. (2022) for more details).

The stress-like functions (A_k, B_k and C_k ; $k = o, i$) define the shape and the size of the surfaces in true stress space and are generally expressed by terms consisting of the true stresses, isotropic hardening variables and shape parameters. Golchin et al. (2021) proposed a new set of these functions, which provides flexibility for the yield surface to attain various non-elliptical shapes and improves the robustness when using

implicit stress integration schemes for numerically implementing in boundary-value solvers. This was achieved by eliminating the potential for false elastic nuclei and divergent zones. The stress-like functions for the outer yield surface are

$$A_o(p, p_{cT}) = \frac{p_{cT}}{2\pi} \left(2\arctan\left(\gamma \left(\frac{1}{2} - \frac{p}{p_{cT}}\right)\right) + \pi \right) \quad (6a)$$

$$B_o(p, p_{cT}) = MC_o \exp\left(\frac{\alpha(p - C_o)}{p_{cT}}\right) \quad (6b)$$

$$C_o(p_{cT}) = \frac{p_{cT}}{2\pi} \left(2\arctan\left(\frac{\gamma}{2}\right) + \pi \right) \quad (6c)$$

where α and γ are shape parameters that affect the shape of the surface to form a wide range of non-elliptical shapes (e.g. tear and bullet shapes) observed for fine-grained soils and have values in the range $-2 \leq \alpha, \gamma \leq 2$ (see Appendix F). The effect of these parameters on the shape of the outer and the inner yield surfaces in meridian stress space (p, q) for $(\alpha, \gamma) = (0, -1), (0, 0)$ and $(0, 1.5)$ is shown in Fig. 4. Note that for $(\alpha, \gamma) = (0, 0)$ the elliptical shape of the MCC model is obtained.

p_{cT} is the apparent pre-consolidation pressure defined as

$$p_{cT} = p_{c0} \exp\left(\frac{1 + e_0}{\lambda - \kappa} \epsilon_v^p\right) \exp(-\mu_0(T - T_0)) \quad (7)$$

where p_{c0} is the initial pre-consolidation pressure, λ and κ are the bi-logarithmic elasto-plastic and elastic compressibility indices, respectively, e_0 is the initial void ratio, and μ_0 is the coefficient of thermal shrinkage of the outer yield surface, first presented by Cui et al. (2000), to capture the observed pre-consolidation pressure reduction due to temperature increase (Fig. 1(a)). M is the stress ratio at Critical State conditions and is defined as

$$M = M_0 + \pi^*(T - T_0) \quad (8)$$

where π^* is the gradient of the variation of M with temperature T , and M_0 is the Critical State stress ratio at ambient temperature $T = T_0$, which is compatible with the observed behaviour in Fig. 1(b). The default for π^* can be considered to be 0, following the limited changes in observed behaviour seen in Fig. 1(b).

The corresponding stress-like functions for the inner yield surface are similar to those for the outer yield surface, where p and p_{cT} are respectively replaced by $\tilde{p}$ and $\tilde{p}_{iT}$:

$$A_i(\tilde{p}, \tilde{p}_{iT}) = \frac{\tilde{p}_{iT}}{2\pi} \left(2\arctan\left(\gamma \left(\frac{1}{2} - \frac{\tilde{p}}{\tilde{p}_{iT}}\right)\right) + \pi \right) \quad (9a)$$

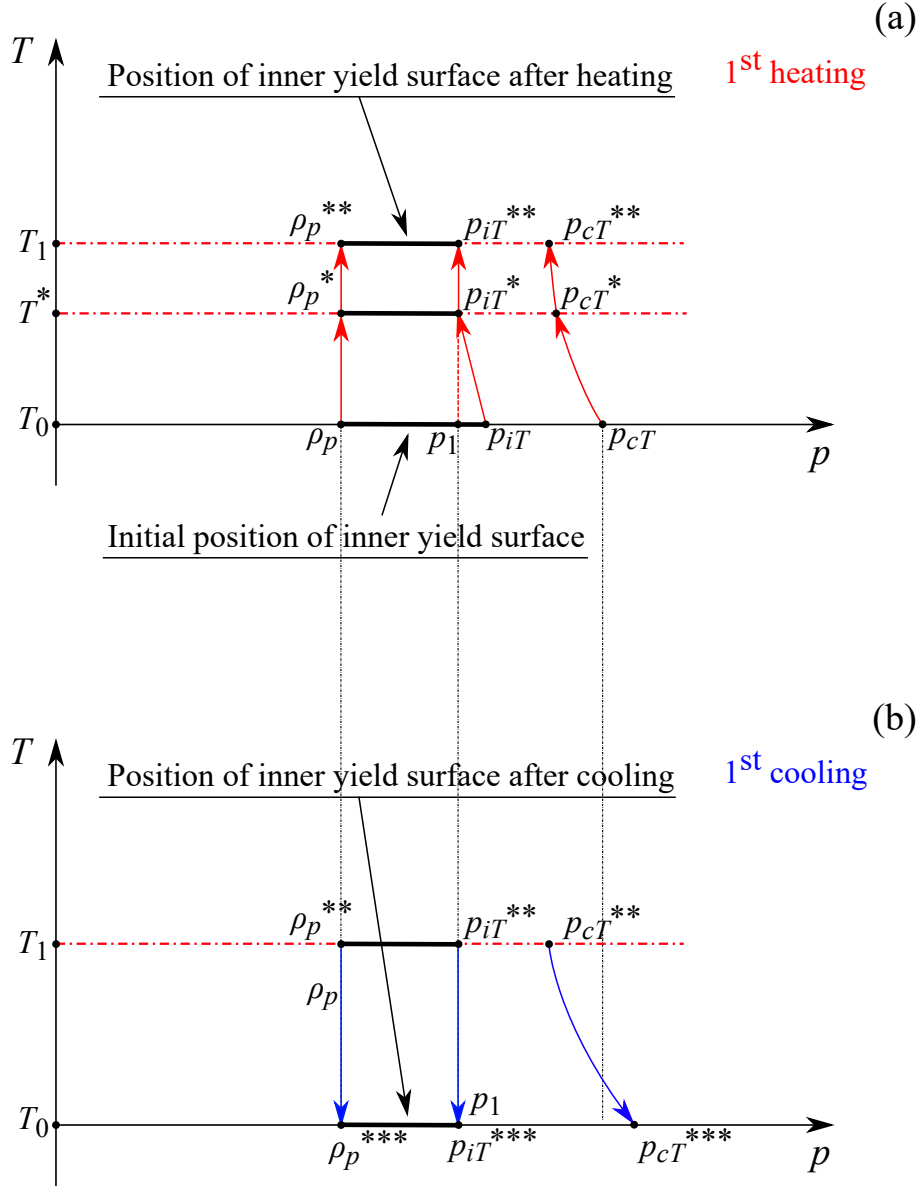


Fig. 5. Performance of the model: (a) during heating phase; (b) during cooling.

$$B_i(\tilde{p}, \tilde{p}_{iT}) = MC_i \exp\left(\frac{\alpha(\tilde{p} - C_i)}{\tilde{p}_{iT}}\right) \quad (9b)$$

$$C_i(\tilde{p}_{iT}) = \frac{\tilde{p}_{iT}}{2\pi} \left(2\arctan\left(\frac{\gamma}{2}\right) + \pi\right) \quad (9c)$$

where $\tilde{p}_{iT}$ is the isotropic hardening variable representing the size of the major-axis of the inner yield surface, and is defined as

$$\tilde{p}_{iT} = \tilde{p}_{i0} \exp\left(\frac{1 + e_0}{\lambda - \kappa} \varepsilon_v^p\right) \exp(-\mu \langle T - T_0 \rangle) \quad (10)$$

where $\tilde{p}_{i0}$ is the initial value of $\tilde{p}_{iT}$, $\langle \cdot \rangle$ is the Macaulay bracket which operates as $\langle x \rangle = (x + |x|)/2$, and

$$\mu = \mu_0 \left(\frac{p_{iT}}{p_{cT}}\right)^b \quad (11)$$

where p_{iT} ($=\rho_p = \tilde{p}_{iT}$) indicates the coordinates of the compression apex of the inner yield surface (Fig. 2(a)) along the p -axis. The coefficient of

thermal shrinkage of the inner yield surface, μ , varies with the ratio of the hydrostatic coordinate of the compression apex of the inner yield surface with respect to the hydrostatic coordinate of the compression apex of the outer yield surface, p_{iT}/p_{cT} , in order to capture the thermo-mechanical behaviour of soils at various over-consolidated stress states. For normally-consolidated states, $p_{iT}/p_{cT} = 1$, and thus $\mu = \mu_0$. For over-consolidated states, however, $p_{iT}/p_{cT} < 1$ which results in $\mu < \mu_0$. As a result, the coefficient of thermal shrinkage (and consequently, thermally induced plastic strain increments) reduces as the state of the soil becomes over-consolidated, which is compatible with experimental observations (Fig. 1(d)). The parameter b in Eq. (11) controls the amount of accumulated plastic strains when the soil is subjected to heating-cooling cycles.

Plastic strain increments in triaxial space are calculated as

$$\{\dot{\epsilon}^p\} = \dot{\lambda} \{r^j\} = \dot{\lambda} \begin{Bmatrix} r_p^j \\ r_q^j \end{Bmatrix} \quad (12)$$

where $\dot{\lambda}$ is the plastic multiplier representing the magnitude of the

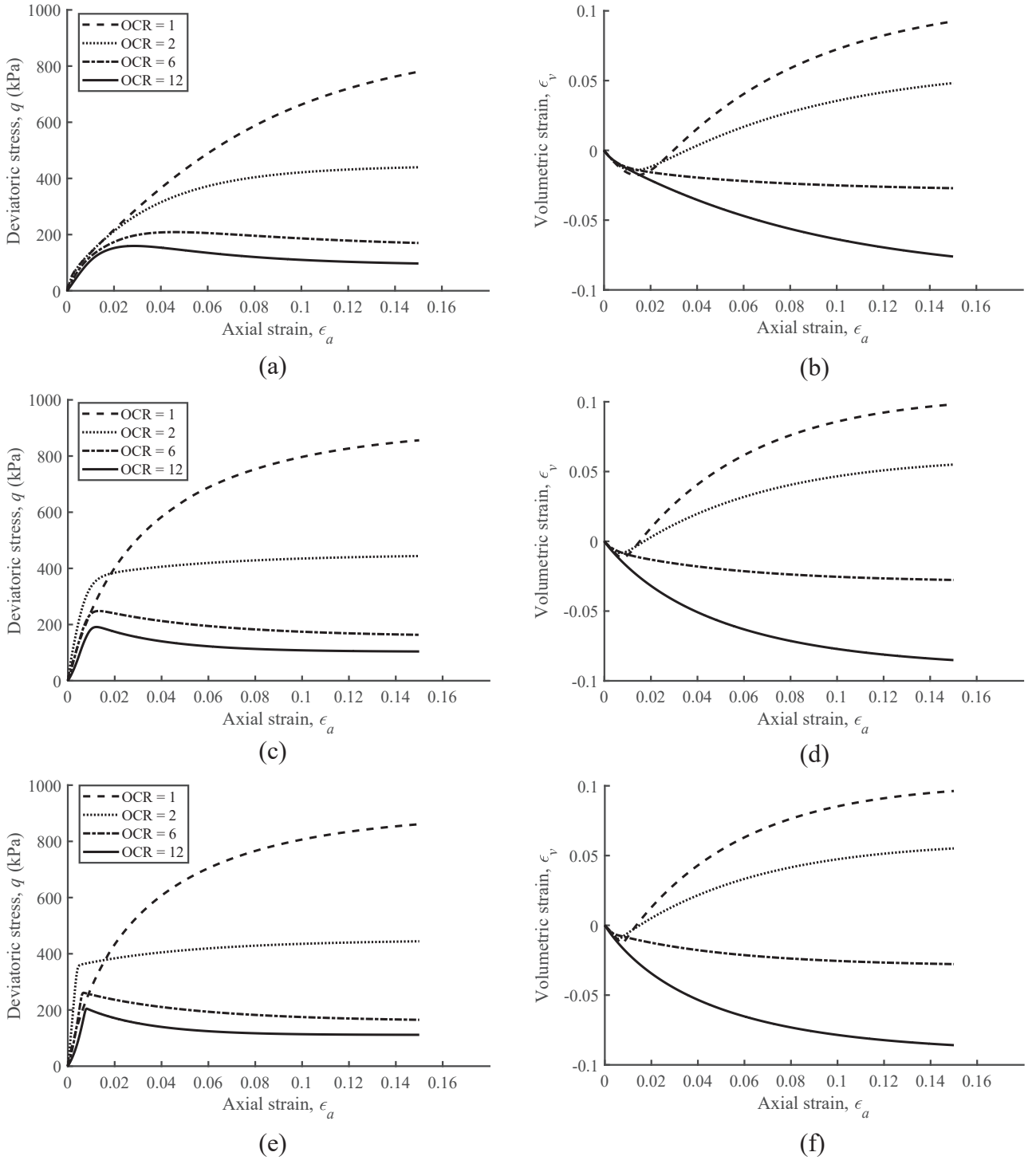


Fig. 6. Effect of parameter C_χ on drained behaviour response: (a) q - ϵ_a response with $C_\chi = 100$; (b) ϵ_v - ϵ_a response with $C_\chi = 100$; (c) q - ϵ_a response with $C_\chi = 1000$; (d) ϵ_v - ϵ_a response with $C_\chi = 1000$; (e) q - ϵ_a response with $C_\chi = 10000$; (f) ϵ_v - ϵ_a response with $C_\chi = 10000$.

plastic strain increments and is defined, for strain controlled conditions (see Appendix C), as

$$\dot{\lambda} = \frac{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T [D^e] \{ \dot{\epsilon} - \dot{\epsilon}^{Therm} \} + \frac{\partial y_i}{\partial p_{IT}} U_{p_{IT}} + \frac{\partial y_i}{\partial M} \dot{M}}{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T [D^e] \{ r^i \} + \left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T \{ M_I + M_H \} - \frac{\partial y_i}{\partial p_{IT}} M_{p_{IT}}} \quad (13)$$

where D^e is the elastic stiffness matrix; $\dot{\epsilon}$ and $\dot{\epsilon}^{Therm}$ are, respectively, the vectors of total strain and thermal strain increments in triaxial space; $\partial y_i / \partial \sigma$, $\partial y_i / \partial p_{IT}$ and $\partial y_i / \partial M$ are the derivatives of the inner yield surface

with respect to $\tilde{\sigma}$, $\tilde{p}_{IT}$ and M (see Appendix D); and $U_{p_{IT}}$, M_I , M_H and $M_{p_{IT}}$ are defined in Appendix C.

4.1. Isotropic hardening rules

The model has two isotropic hardening rules assigned to the isotropic hardening variables of the outer and inner yield surfaces (p_{cT} and $\tilde{p}_{IT}$, respectively). Two mechanisms control the size of the outer yield surface, in which one, similar to the MCC model, is due to the development

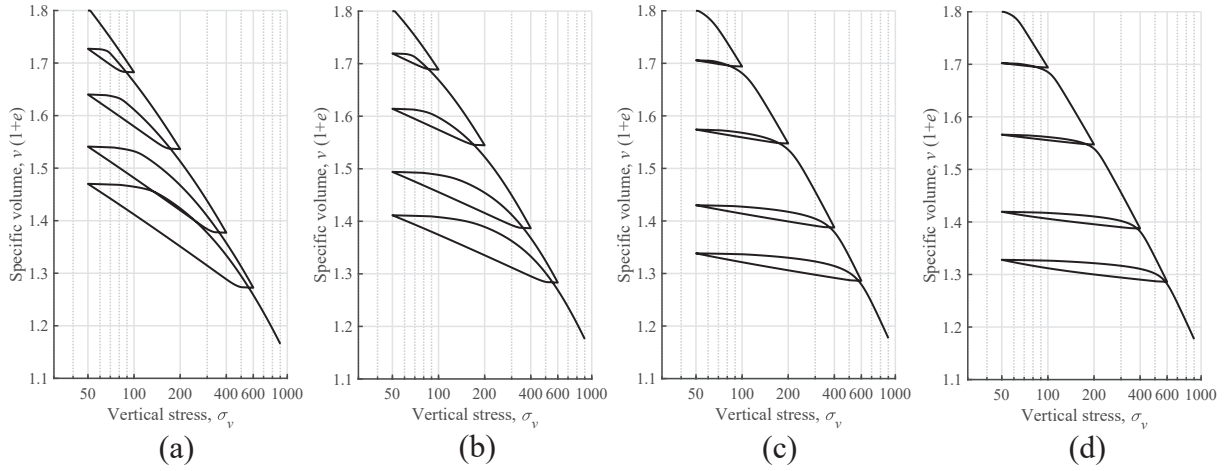


Fig. 7. Effect of C_x on hysteresis behaviour of isotropic compression loading–unloading–reloading with $R_{ini} = 0.2$: (a) $C_x = 50$; (b) $C_x = 100$; (c) $C_x = 500$; (d) $C_x = 1000$.

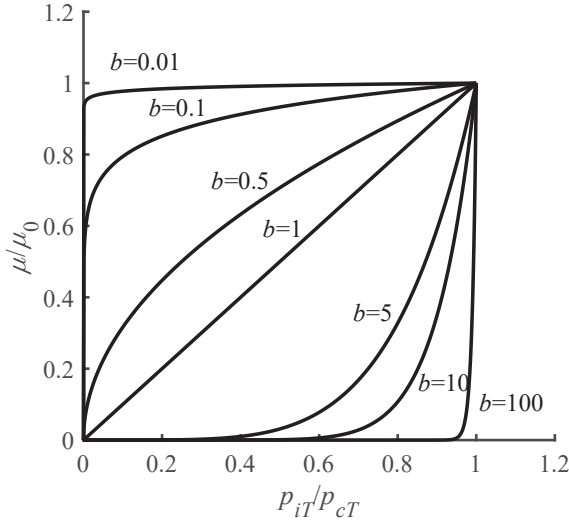


Fig. 8. Effect of parameter b on the variation of normalised μ/μ_0 with normalised p_{iT}/p_{cT} .

of plastic strains and the other is influenced by temperature changes. The isotropic hardening rule of the outer yield surface is determined from the rate of Eq. (7):

$$\dot{p}_{cT} = p_{cT} \left(\frac{1 + e_0}{\lambda - \kappa} \dot{\epsilon}_v^p - \mu_0 \dot{T} \right) \quad (14)$$

The rate of Eq. (10) defines the isotropic hardening rule for the inner yield surface:

$$\dot{\tilde{p}}_{iT} = \tilde{p}_{iT} \left(\frac{1 + e_0}{\lambda - \kappa} \dot{\epsilon}_v^p - \left(\mu \langle \dot{T} \rangle + \dot{\mu} \langle T - T_0 \rangle \right) \right) \quad (15)$$

The outer yield surface and the inner yield surface harden (grow) incrementally with plastic volumetric strain increments, via the respective terms $p_{cT}((1 + e_0)/(\lambda - \kappa))\dot{\epsilon}_v^p$ and $\tilde{p}_{iT}((1 + e_0)/(\lambda - \kappa))\dot{\epsilon}_v^p$.

The hydrostatic extent of the outer yield surface changes for any variation of temperature (heating and cooling). The surface shrinks or expands due to an increase or decrease of temperature, respectively, via the term $\dot{p}_{cT} = -\mu_0 p_{cT} \dot{T}$. However, due to the presence of Macaulay brackets in Eq. (15), the hydrostatic extent of the inner yield surface only shrinks as the temperature increases (heating phase) and remains unchanged when the temperature decreases (cooling phase) (see Fig. 3).

This behaviour is controlled via the term $\dot{\tilde{p}}_{iT} = -\tilde{p}_{iT}(\mu \langle \dot{T} \rangle + \dot{\mu} \langle T - T_0 \rangle)$.

Note that the constitutive assumption of independent behaviour of the inner yield surface in the cooling phase is consistent with experimental observations. For example, Ng et al. (2019) observed that when Loess clay samples were subjected to thermal cycles (and the vertical stress was kept constant), the gradients of the strain-temperature curves during cooling were almost always the same. In addition, these gradients were the same as the recoverable gradient after many thermal cycles. Therefore, it can be inferred that the response of the soil during cooling is recoverable (thermo-elastic) and that the inner surface is independent on cooling.

4.2. Kinematic rule

The purpose of introducing the inner yield surface is to capture the plastic strains that have been observed for stress states inside the outer yield surface. This requires the inner yield surface to move (translate), in accordance with the state of the soil, within the domain enclosed by the outer yield surface. Therefore, a so-called kinematic rule is needed. The kinematic rule can be assigned to any specific point of the inner yield surface. For example, in a number of models the kinematic rule is assigned to the centre of the inner yield surface (Coombs et al., 2013; Maranhã et al., 2016; Mroz et al., 1978; Mróz et al., 1979; Puzrin and Kirschenboim, 2001; Stallebrass and Taylor, 1997). For the model proposed here, the kinematic rule is assigned to the internal variable $\rho = (\rho_p, \rho_q)$, which represents the coordinates of the decompression apex of the inner surface. This reduces the geometrical complexity of the kinematic rule, specifically when the two surfaces have non-elliptical shapes.

The thermodynamic constraint functions in Eq. (2) are defined as

$$c_1 = \dot{\rho}_p - (c_I)_p - (c_H)_p = 0 \quad (16a)$$

$$c_2 = \dot{\rho}_q - (c_I)_q - (c_H)_q = 0 \quad (16b)$$

where $(c_I)_p$ and $(c_H)_p$ are the hydrostatic kinematic constraint functions under isothermal and heating conditions respectively, and $(c_I)_q$ and $(c_H)_q$ are the respective deviatoric kinematic constraint functions under isothermal and heating conditions, which, all together, form the kinematic rule of the inner yield surface. Note that these constraint functions should be first order homogenous functions with respect to their internal variables (a thermodynamic requirement), similar to the definition of

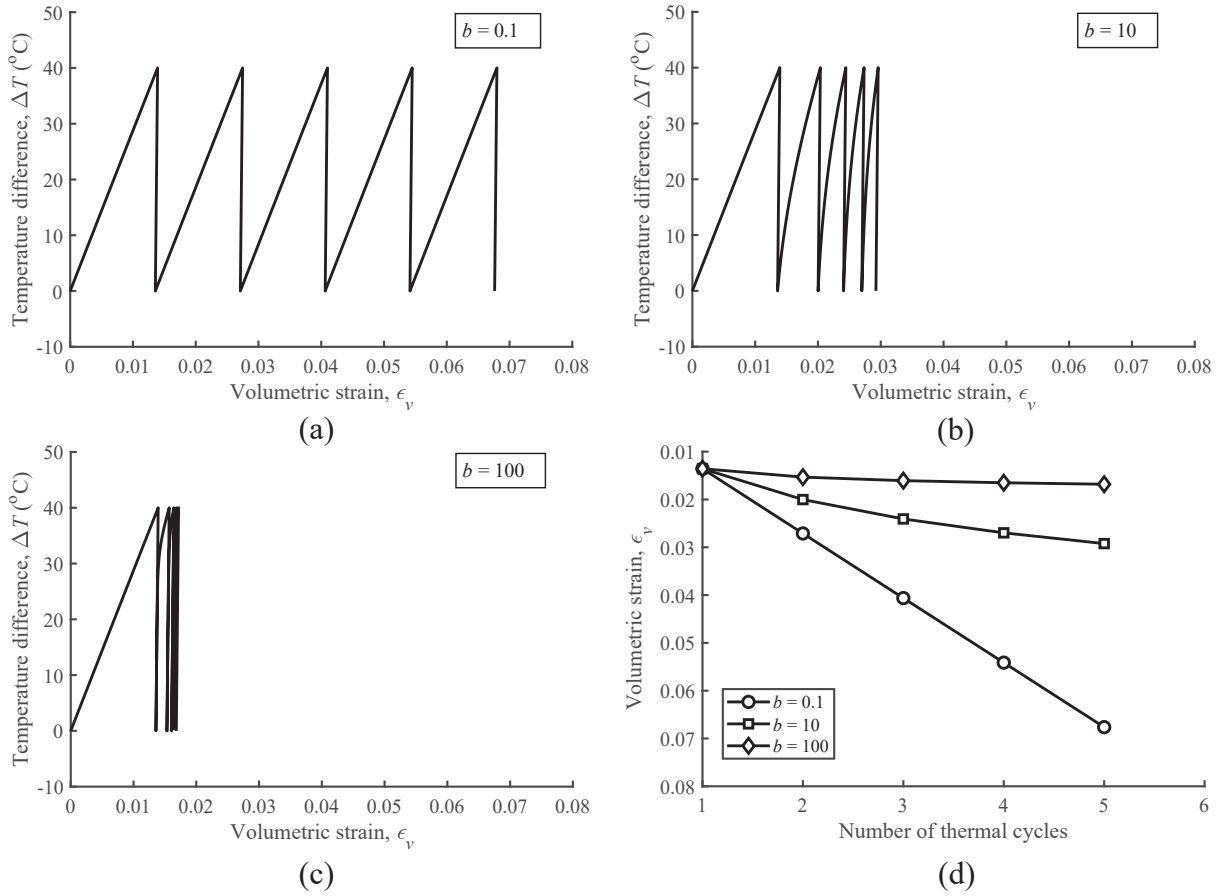


Fig. 9. Effect of parameter b on the volumetric strain vs. temperature difference response for: (a) $b = 0.1$; (b) $b = 10$; (c) $b = 100$; (d) effect of parameter b on the accumulated volumetric strain with thermal cycles.

Table 1
Summary of model parameters.

Parameters	Ng et al. (2019)	Di Donna and Laloui (2015)	Baldi et al. (1991)	Abuel-Naga et al. (2006)	Uchaipichat and Khalili (2009)	Ghahremannejad, (2003)	Suddepong et al. (2015)	Nakai and Hinokio (2004)
α (–)	0.0	0.0	0.0	0.0	0.2	–0.4	–0.05	0.5
γ (–)	0.0	0.0	0.0	–2.0	0.3	–0.2	0	0.5
λ (–)	0.095	0.062	0.180	0.450	0.090	0.150	0.155	0.098
κ (–)	0.060	0.017	0.020	0.090	0.006	0.030	0.012	0.005
M_0 (–)	–	–	–	0.80	1.20	0.85	1.00	1.04
$\bar{K}$ (–)	$1/\kappa$	$1/\kappa$	$1/\kappa$	18.5	45.0	$1/\kappa$	$1/\kappa$	700.0
$\bar{G}$ (–)	–	–	–	15	120	270	200	550
n (–)	1.00	1.00	1.00	1.00	0.70	1.00	1.00	1.00
α^* (1/K)	-3.1×10^{-6}	-3.50×10^{-6}	-1.75×10^{-5}	-3.10×10^{-6}	-3.40×10^{-5}	-3.00×10^{-5}	–	–
μ_0 (1/K)	2.01×10^{-3}	5.15×10^{-3}	3.42×10^{-3}	7.70×10^{-3}	3.40×10^{-3}	9.09×10^{-4}	–	–
π^* (1/K)	–	–	–	0.0	0.0	-1.6×10^{-3}	–	–
b (–)	30.70	21.50	0.75	–	–	–	–	–
C_x (–)	–	–	–	2000	2000	5000	3300	11,500
R_{ini} (–)	0.25	0.25	0.25	0.60	0.20	0.20	0.05	0.01

the rate of dissipation function d . As explained before, when the rate of dissipation is maximised, Eq. (2d) and Eq. (2e) are valid ($c_1 = c_2 = 0$) (see Appendix A). Therefore, from c_1 and c_2 (Eq. (16)), the kinematic rule along the hydrostatic and deviatoric axes are determined:

$$\{\dot{\rho}\} = \left\{ \begin{matrix} \dot{\rho}_p \\ \dot{\rho}_q \end{matrix} \right\} = \left\{ \begin{matrix} (c_I)_p \\ (c_I)_q \end{matrix} \right\} + \left\{ \begin{matrix} (c_H)_p \\ (c_H)_q \end{matrix} \right\} = \{\dot{\rho}_I\} + \{\dot{\rho}_H\} \quad (17)$$

The kinematic rule has been decomposed into components due to isothermal $\dot{\rho}_I$ and heating $\dot{\rho}_H$, which are explained in the following subsections.

4.3. Interaction between the inner yield surface and the outer yield surface under isothermal conditions

The derivation of the kinematic rule under isothermal conditions, by

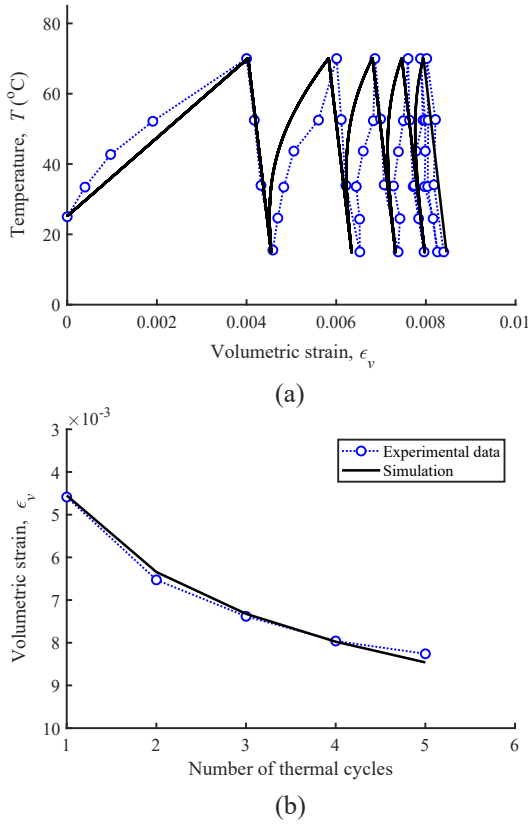


Fig. 10. Comparison of model predictions with experimental data of Ng et al. (2019): (a) temperature vs. volumetric strain; (b) accumulated volumetric strain vs. the number of thermal cycles.

modifying the approach of Coombs et al. (2013), is explained in detail in Appendix B. With this kinematic rule, the inner yield surface and the outer yield surface interact with each other, such that a stress point on the inner yield surface (with an elasto-plastic response) moves toward the image stress projected on the outer yield surface. The kinematic rule under isothermal conditions is defined as

$$\{\dot{\rho}_I\} = \left\{ \begin{pmatrix} c_I \end{pmatrix}_p \right\} = \left(\frac{1}{W_r} \left(\frac{\dot{p}_c}{p_{cT}} - \frac{\dot{p}_i}{p_{cT}} \right) - C_\chi \|\dot{\epsilon}^p\| \right) \{\rho\} + C_\chi \|\dot{\epsilon}^p\| W_r \{\sigma\} \quad (18)$$

where $\|\dot{\epsilon}^p\|$ is the magnitude of the plastic strain increment; C_χ is a parameter which controls the pace of the inner yield surface approaching the outer yield surface; and W_r , p_c , $\dot{p}_c$ and $\dot{p}_i$ are defined in Appendix B. This kinematic rule is activated when plastic strain increments are produced, i.e., when the response is elastic it is zero. In addition, it can be shown that the proposed formulation is a first order homogenous function of its rate variables (a thermodynamic requirement).

4.4. Kinematic rule under non-isothermal conditions

The kinematic rule under isothermal conditions (Eq. (18)) is extended in order to capture the accumulated plastic strains and shakedown behaviour observed when soils are subjected to consecutive heating-cooling cycles. This kinematic rule allows the simulation of progressive strains due to thermal cycles. The kinematic rule under non-isothermal conditions is defined as

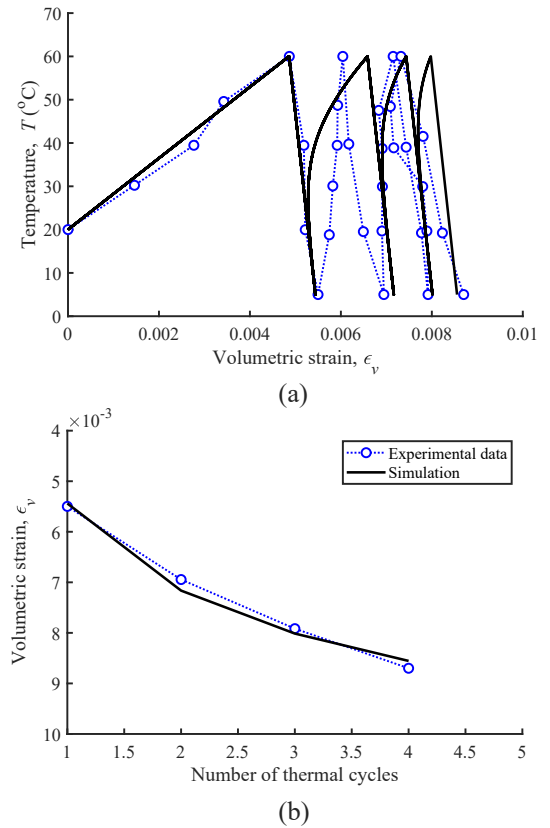


Fig. 11. Comparison of model predictions with experimental data of Di Donna and Laloui (2015): (a) temperature vs. volumetric strain; (b) accumulated volumetric strain vs. the number of thermal cycles.

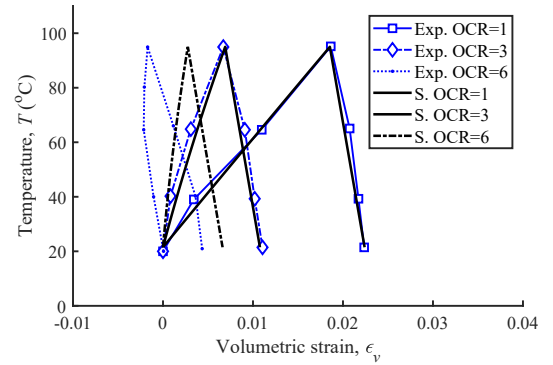


Fig. 12. Comparison of the model predictions with experimental data of Baldi et al. (1991): temperature vs. volumetric strain.

$$\{\dot{\rho}_H\} = \left\{ \begin{pmatrix} \dot{\rho}_H \end{pmatrix}_p \right\} = \left\{ \begin{pmatrix} c_H \end{pmatrix}_p \right\} = -H(\dot{T}) \left\{ \begin{pmatrix} \dot{\rho}_I \end{pmatrix}_p \right\} \quad (19)$$

where $H(\dot{T})$ is a function similar to the Heaviside function, defined as

$$H(\dot{T}) = \begin{cases} 0 & \text{when } \dot{T} \leq 0 \text{ (under isothermal and cooling phase)} \\ 1 & \text{when } \dot{T} > 0 \text{ (heating phase)} \end{cases} \quad (20)$$

Due to the presence of $H(\dot{T})$, the model activates the kinematic rule $\dot{\rho}_H$ only during the heating phase. For isothermal conditions and during cooling, $\dot{T} \leq 0$ which results in $H = 0$. On the other hand, during heating

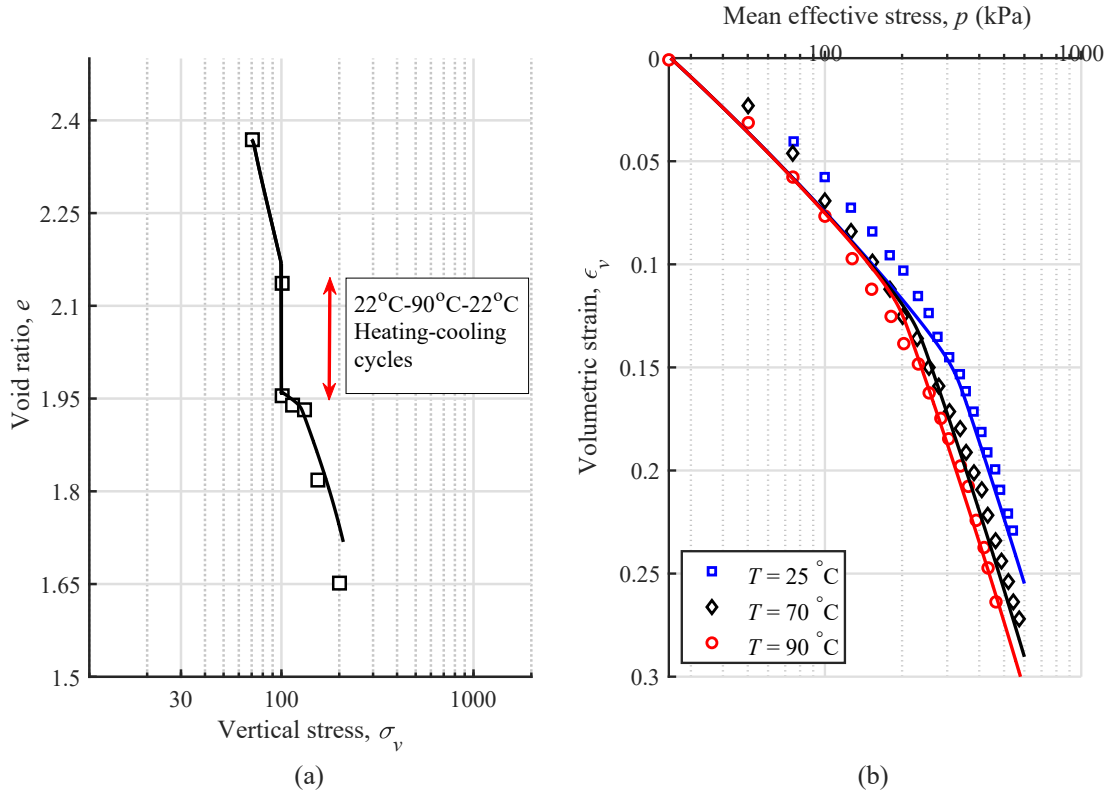


Fig. 13. Comparison of model predictions with experimental data of Abuel-Naga et al. (2006): (a) void ratio vs. vertical stress in oedometer test; (b) volumetric strain vs. mean effective stress during isotropic triaxial compression at different temperatures.

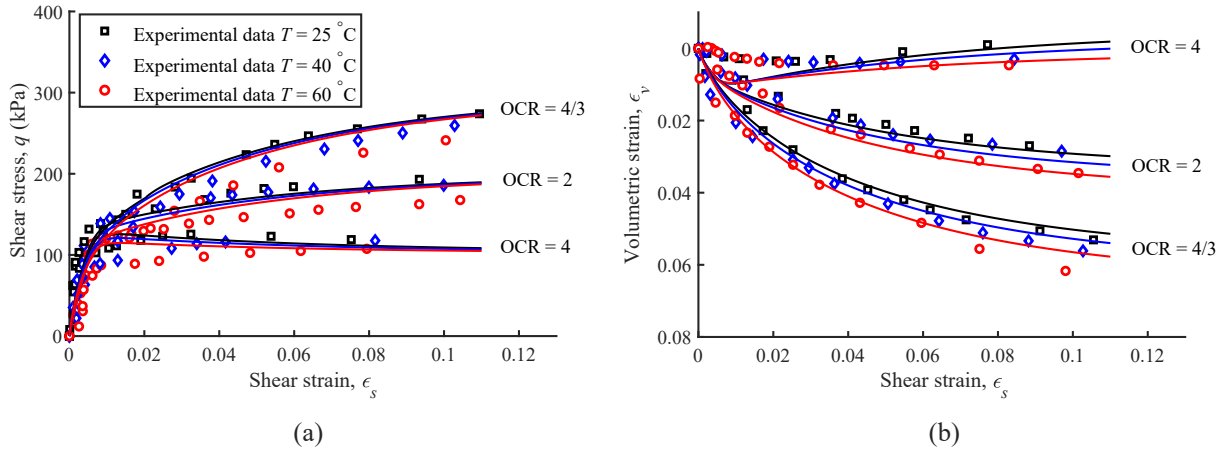


Fig. 14. Comparison of model predictions with experimental data of Uchaipichat and Khalili (2009): (a) deviatoric stress vs. deviatoric strain at different OCRs and temperatures; (b) volumetric strain vs. deviatoric strain at different OCRs and temperatures.

$\dot{T} > 0$ which results in $H = 1$ and activates the kinematic rule $\dot{p}_H$. An example of a soil element that is subjected to thermal cycles is provided to explain the role of the kinematic rule $\dot{p}_H$ in capturing the shakedown behaviour and the progressive development of plastic strains due to thermal cycles.

Assume a soil in an isotropic state ($q = 0$) and at an ambient temperature ($T = T_0$) has a hydrostatic pressure of p_1 and has experienced a maximum hydrostatic pressure of p_{cT} (pre-consolidation pressure) where $p_{cT} > p_1$. The state of the soil is $(p, q, T) = (p_1, 0, T_0)$ and is inside the inner yield surface. While the hydrostatic pressure is held constant ($\dot{p} = 0$), the soil is subjected to heating from $T = T_0$ to $T = T_1$ (as shown in Fig. 5 (a) in p - T space). Due to heating and considering that the state of the soil is inside the yield surface (thermo-elastic response), the isotropic

hardening rules for the outer and inner yield surfaces (Eqs. (14) and (15), respectively) reduce to $\dot{p}_{cT} = -\mu_0 p_{cT} \dot{T}$ and $\dot{p}_{iT} = -\tilde{p}_{iT} (\mu \langle \dot{T} \rangle + \dot{\mu} (T - T_0))$. Therefore, the outer and inner yield surfaces shrink in accordance with their respective isotropic rule. While the state of the material is inside the inner yield surface, the behaviour is thermo-elastic and the kinematic rule $\dot{p}_i$ is deactivated. The shrinkage of the inner surface continues (as the temperature increases) until the state of the soil encounters it at $T = T^*$ (position $p_{iT}^* = p_1$). Due to the consistency condition, which requires that the stress state remains on the yield surface, and, since p is kept constant, $\dot{p}_{iT} = 0$. Hence, plastic volumetric strain increments are then produced to compensate the thermal shrinkage and are determined from Eq. (15) as $\dot{\epsilon}_v^p =$

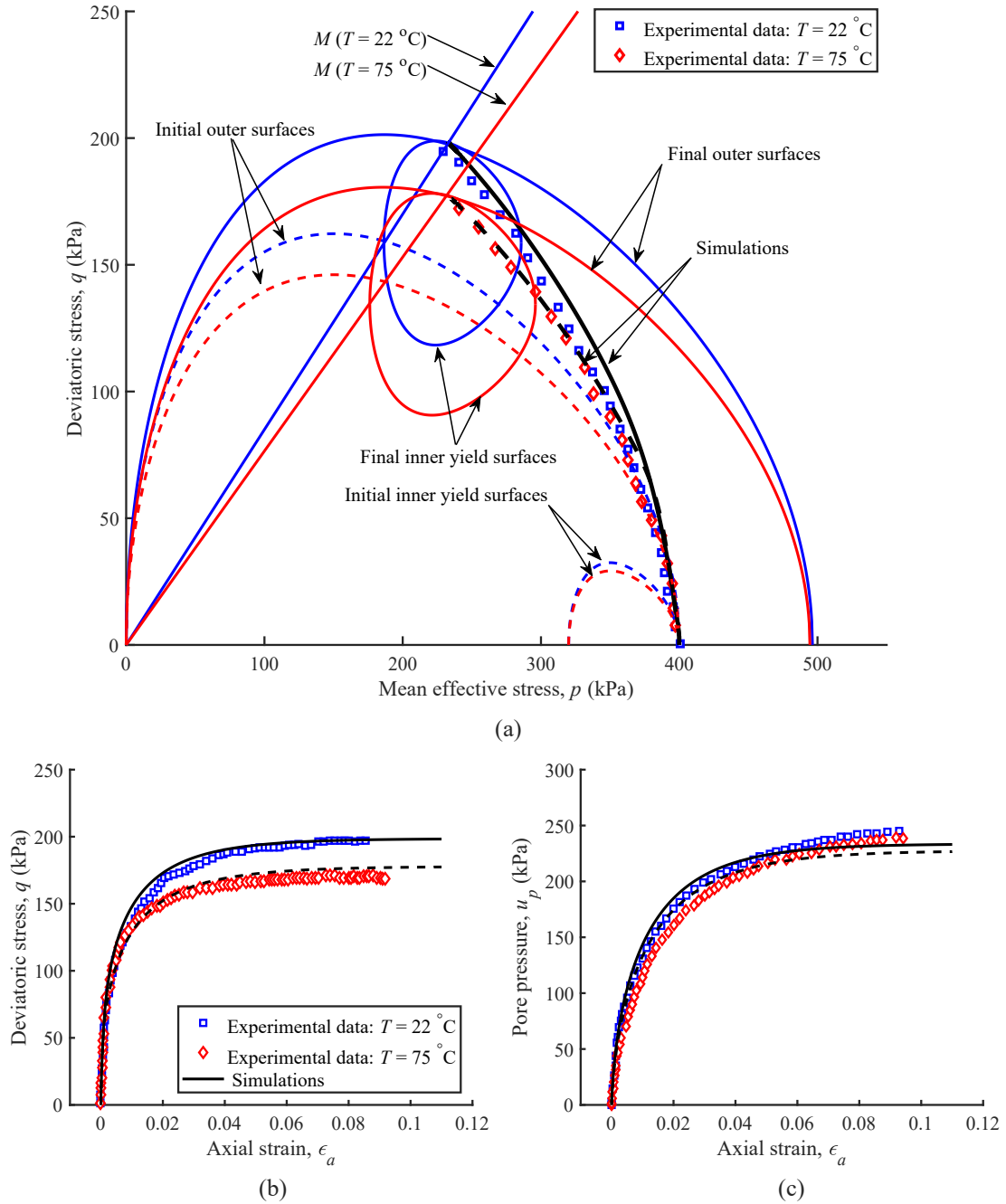


Fig. 15. Comparison of model predictions with undrained experimental data of [Ghahremannejad \(2003\)](#) at different temperatures: (a) stress path; (b) deviatoric stress vs. axial strain; (c) pore pressure vs. axial strain.

$\left(\frac{1+\epsilon_0}{\lambda-\kappa}\right)^{-1} \left(\mu \langle \dot{T} \rangle + \dot{\mu} (T - T_0) \right)$. As plastic volumetric strain increments are produced, the kinematic rule $\dot{\rho}_I$ activates and drags the inner yield surface to the right. By using the heating kinematic rule $\dot{\rho}_H$ (Eq. (19)), the inner yield surface is moved by the same magnitude of $\dot{\rho}_I$ in the opposite direction (position ρ_p^{**}). Therefore, during the heating phase, plastic volumetric strains are produced progressively while the state of the soil (p, q, T) remains on the compression apex of the inner yield surface (position p_{IT}^{**}) and the size of the inner surface and the coordinates of the decompression apex remain unchanged (Fig. 5(a)). This observation is mathematically expressed as $\dot{p}_{IT} \left(= \dot{p}_{IT} + \dot{\rho}_p \right) = \dot{p}_{IT} = \dot{\rho}_p = 0$.

For a cooling phase following the heating phase (from $T = T_1$ to $T =$

T_0 in Fig. 5(b)), due to the Macaulay bracket in Eq. (15) the isotropic rule for the inner yield surface is deactivated ($\dot{p}_{IT} = 0$). In addition, during cooling the state of the soil does not go outside the inner yield surface, and thus the response of the soil becomes thermo-elastic and no plastic strains are developed. This implies that the kinematic rule (Eq. (17)) is not activated ($\dot{\rho} = 0$). Thus, the size and position of the inner yield surface does not change (i.e. $\rho_p^{***} = \rho_p^{**}$ and $p_{IT}^{***} = p_{IT}^{**}$). Therefore, at the end of the cooling phase (at $T = T_0$) the state of the soil remains at the compression apex of the inner yield surface ($p_{IT}^{***} = p_1$). Upon subsequent heating, plastic strains are produced from the beginning of heating. Moreover, during the cooling phase the isotropic rule for the outer yield surface is activated (Eq. (14)) and is $\dot{p}_{cT} = -\mu_0 p_{cT} \dot{T}$ (where $\dot{T} < 0$). Consequently, at the end of cooling the size of the out surface is larger than before being subjected to heating-cooling ($p_{cT}^{***} > p_{cT}$).

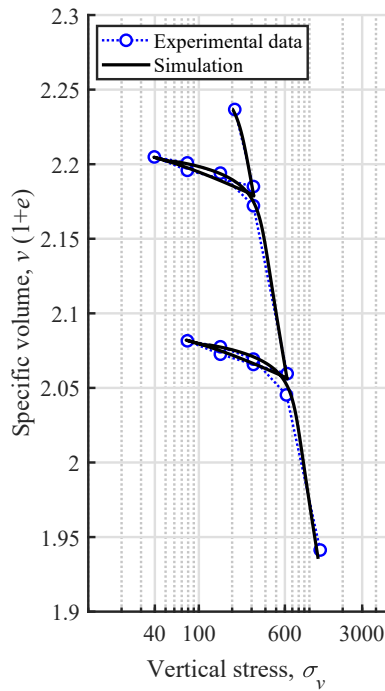


Fig. 16. Comparison of the model predictions with oedometer experimental data of [Suddepong et al. \(2015\)](#).

The aforementioned strategy allows the inner yield surface to capture the plastic strains in successive thermal cycles and the shakedown behaviour. The magnitude of plastic strains developed during the heating phase is related to the coefficient of thermal shrinkage μ (Eq. (11)). After each heating-cooling cycle, the size of the outer yield surface, p_{cT} , increases, while the coordinate of p_{iT} remains unchanged (e.g. $p_{cT}^{***} > p_{cT}$ and $p_{iT}^{***} = p_{iT}^{**} = p_1$ in Fig. 5(b)). According to Eq. (11), the coefficient of thermal shrinkage (μ) decreases, i.e., $\mu^{***} = \mu_0(p_{iT}^{***}/p_{cT}^{***})^b < \mu = \mu_0(p_{iT}/p_{cT})^b$. This implies that the magnitude of plastic strains produced during the subsequent heating phase decreases. Therefore, the generated plastic strains in subsequent thermal cycles decrease progressively, until at a sufficiently high number of thermal cycles the produced plastic strains become negligible, resulting in shakedown behaviour.

5. Calibration and the role of parameters

The model is defined by 13 parameters and, except for the new parameters of C_γ and b defined in this model, the others are the same as those defined in [Golchin et al.'s \(2022\)](#) model and are calibrated in accordance with [Golchin et al. \(2022\)](#). Therefore, the original parameters are briefly explained, and then the new parameters are investigated comprehensively.

λ , κ and M_0 are defined similarly to the ones in the MCC model. Hence, λ and κ are the bi-logarithmic elasto-plastic and elastic compressibility indices, respectively, and can be determined from oedometer and isotropic triaxial tests. M_0 is the Critical State stress ratio at ambient temperature ($T = T_0$) and is determined as the gradient between the stress ratio (q/p) at Critical State and the origin (p, q) = (0,0) in the stress space of drained and undrained triaxial tests.

$\bar{G}$ and $\bar{K}$ are the coefficients representing the maximum shear and bulk moduli (which are related to the initial small strains), respectively, and can be determined from resonant column tests, bender element tests or approximately from the tangential values of the q - ϵ_s and p - ϵ_v curves of the drained and undrained triaxial tests at low strain levels. The pressure-dependency (p) of the shear and bulk moduli are linked

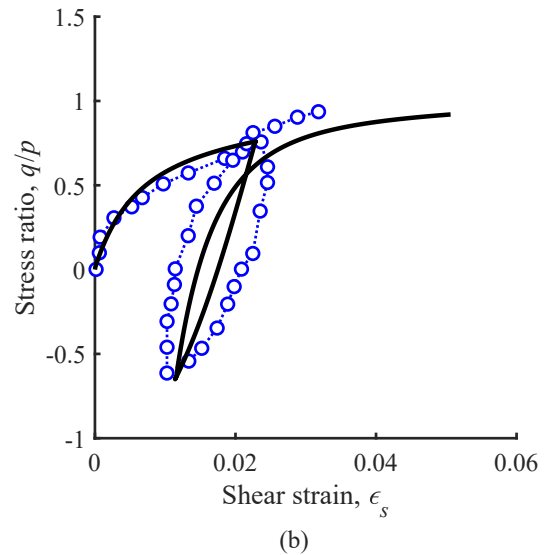
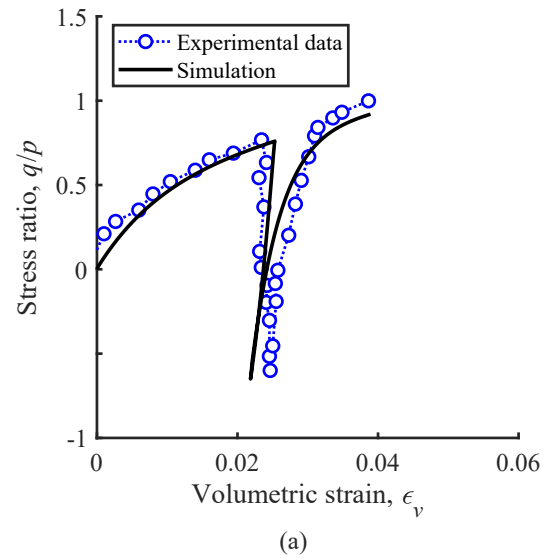


Fig. 17. Comparison of the model predictions with drained cyclic triaxial experimental data of [Nakai and Hinokio \(2004\)](#).

through the parameter n which usually has a value less than one.

The flexibility of the inner and outer yield surfaces, i.e., adapting them with experimentally determined loci of the yield stresses of geo-materials, is via parameters α and γ . Typically, at least three triaxial tests, consisting of one isotropic compression test at a normally consolidated state, and two triaxial shear tests at lightly over-consolidated and highly over-consolidated states, are needed to adjust the parameters α and γ via regression analysis to provide the best fit to the yield stresses.

$3\alpha^*$ is the volumetric thermal expansion coefficient. As shown in Fig. 1(e), when fine-grained soils at isotropic stress states ($q = 0$) are subjected to several heating-cooling cycles, thermal (plastic) volumetric strains are accumulated at each thermal cycle and, when the soil is subjected to a sufficiently high number of thermal cycles, the accumulated strains level-off. This means that, in subsequent thermal cycles, the response of the soil becomes thermo-elastic and the volumetric behaviour of the soil is mostly dominated by the volumetric thermal expansion of the grains. $3\alpha^*$ can then be determined as $3\alpha^* = \Delta\epsilon_v/\Delta T$, where $\Delta\epsilon_v$ is the volumetric change due to heating ΔT . In addition, the gradient of the ΔT - ϵ_v curve during cooling may also be used to determine $3\alpha^*$, because it is assumed that soils behave thermo-elastically during cooling.

π^* is the gradient of M with respect to temperature changes ΔT (Fig. 1(b)) and is determined as $\pi^* = (M_T - M_{T0})/(T - T_0)$ (Eq. (8)), where M_T and M_{T0} are, respectively, the Critical State stress ratio at the current temperature and the ambient temperature.

μ_0 represents the variation of the apparent pre-consolidation pressure, p_{cT} , as the temperature changes (Fig. 1(a)). μ_0 is calculated from Eq. (7) as $\mu_0 = -\ln(p_{cT}/p_{cT0})/(T - T_0)$, where p_{cT} and p_{cT0} are the pre-consolidation pressures at the current temperature and the ambient temperature, respectively. This equation indicates that at least two triaxial tests with the same pre-consolidation pressure at two different temperatures are needed to determine μ_0 .

C_χ controls the pace with which the inner yield surface approaches the outer yield surface, i.e., it controls the magnitude of the generated plastic strains when the inner yield surface is inside the outer yield surface. Fig. 6 shows the deviatoric stress versus axial strain (q - ϵ_a) and volumetric strain versus axial strain (ϵ_v - ϵ_a) curves of the soil response for C_χ values of 100, 1000 and 10,000 at different stress states (OCRs of 1, 2, 6 and 12), when subjected to drained triaxial shearing under isothermal conditions. By comparing the q - ϵ_a curves, it can be seen that, at a specific deviatoric stress (e.g. $q = 150$ kPa), lower (plastic) axial strains are produced at larger values of C_χ , which implies that a stiffer response is obtained when a higher C_χ is used. Moreover, although all responses are smooth, a sharper response is observed at higher C_χ when the inner yield surface approaches the outer yield surface. In addition, by comparing Fig. 6(b) with Fig. 6(f), it is seen that the amount of dilation of the soil specimens at high OCRs (e.g. 6 and 12) is smaller at higher C_χ .

The stiffening effect of C_χ is more pronounced on the hysteresis behaviour of the soil. Fig. 7 presents the response of a soil subjected to several isotropic loading–unloading–reloading cycles at ambient temperature, for various C_χ values. For lower C_χ (e.g. $C_\chi = 100$ in Fig. 7(a)), more plastic strains are produced upon reloading, which results in bigger hysteresis loops. At higher values of C_χ the hysteresis loops become flatter, which indicates a stiffer response of the soil where less plastic strains are produced during reloading. In the aforementioned analysis, C_χ can be determined by matching the best fit to the size of the hysteresis response (loop) in cyclic loading tests, including conventional isotropic loading–unloading–reloading compression tests. Note that R_{ini} in Fig. 7 is the initial ratio between the size of the inner yield surface and the outer yield surface ($R_{ini} = \tilde{p}_{IT}/p_{cT}$).

The coefficient of thermal shrinkage of the inner yield surface, Eq. (11), varies between 0 and μ_0 as a function of the coordinates of the compression apex of the inner yield surface with respect to the outer yield surface, through the value of the term p_{IT}/p_{cT} which is controlled by the parameter b . This mechanism is designed to capture the thermo-plastic behaviour observed at over-consolidated states and the accumulated thermo-plastic strains during cyclic thermal loadings. The variation of the normalised thermal shrinkage coefficient, μ/μ_0 , with p_{IT}/p_{cT} for various values of b are shown in Fig. 8. For low values of b (e.g. $b = 0.01$), μ is almost the same as μ_0 for most stress states inside the outer yield surface, while for higher values (e.g. $b = 100$) μ becomes nearly zero at stress states slightly less than the normally-consolidated states.

The effect of parameter b on the response of the soil is best observed when the soil is subjected to several heating–cooling cycles (Fig. 9). Fig. 9(a)–(c) show the volumetric strains developed with $b = 0.01$, 1 and 100, when the soil is subjected to five thermal cycles at a normally consolidated state ($p_{IT}/p_{cT} = 1$ and $q = 0$). In the first heating phase of all the simulations, $p_{IT}/p_{cT} = 1$ (the inner and outer yield surfaces coincide on p_{cT}); thus, $\mu = \mu_0$ and the magnitudes of thermally-induced volumetric strains are identical. Simultaneously, the outer yield surface becomes bigger (see Eq. (14)). Upon cooling, the response of the soil is thermo-elastic and the compression apex of the inner yield surface (p_{IT}) remains unchanged. Therefore, after the first heating–cooling cycle, $p/p_{IT} = 1$ and $p_{IT}/p_{cT} < 1$, which represents an over-consolidated state. For the second heating phase when using lower b values (e.g. $b = 0.01$ in Fig. 9(a)), $\mu \approx \mu_0$, and hence the amount of plastic volumetric strains produced during subsequent heating–cooling cycles remains almost the

same as for the first heating phase. As the parameter b increases (e.g. $b = 100$), μ becomes smaller during the subsequent heating phase, and so the amount of plastic strains in each thermal cycle becomes smaller. The accumulated volumetric strains after each thermal cycle for different values of b are plotted in Fig. 9(d). It can be seen that, for lower values of b (e.g. $b = 0.01$), the accumulated strains due to the heating–cooling cycles have not levelled-off after five thermal cycles, whereas when using higher values (e.g. $b = 100$) the accumulated volumetric strains remain almost steady after three cycles, thereby representing how the model can successfully capture the shakedown behaviour of fine-grained soils when they are subjected to several thermal cycles. From the knowledge of accumulated volumetric strains and number of cycles (e.g. Fig. 1(e)), the parameter b can be calibrated.

6. Performance of the model

The performance of the model has been validated by comparing simulations with a variety of experimental data, including oedometer and triaxial tests subjected to thermal cycles and tests under drained and undrained conditions on different soil types. The tests cover a wide range of stress paths and loading histories. The calibrated parameters used for the simulations are reported in Table 1.

6.1. Ng et al. (2019) test

Ng et al. (2019) investigated the thermo-mechanical behaviour of a reconstituted Loess clay, subjected to several heating–cooling cycles, by using a temperature-controlled oedometer apparatus. The soil specimen was first consolidated to a vertical stress of 50 kPa and then subjected to five heating–cooling cycles between 70 °C and 15 °C, starting from an ambient temperature of 25 °C. The performance of the model is compared with experimental data in T - ϵ_v space and the accumulated volumetric strain versus number of thermal cycles in Fig. 10(a) and Fig. 10(b), respectively. The experimental data show a shakedown response with respect to thermal cycles, in which, after five thermal cycles, the response of the soil becomes almost thermo-elastic. It is observed that the model can capture the amplitude of thermally-induced volumetric strains at each thermal cycle accurately, and the accumulated strains with respect to the number of cycles matches well with the experimental data.

6.2. Di Donna and Laloui (2015) test

The cyclic response of Geneva clay, subjected to several heating–cooling cycles, was investigated by Di Donna and Laloui (2015) by using a temperature-controlled oedometer apparatus. The soil specimen was compressed under a vertical stress of 250 kPa and then subjected to four thermal cycles between 60 °C and 5 °C, starting from an ambient temperature of 20 °C. The predictions of the model are compared with experimental data in Fig. 11. The soil, similar to Loess clay (Fig. 10), shows a shakedown behaviour which is well captured by the model.

6.3. Baldi et al. (1991) tests

Baldi et al. (1991) studied the thermo-mechanical behaviour of Boom clay by performing a single thermal cycle at different stress states. Soil specimens with an initial pre-consolidation pressure of 6 MPa were subjected to heating and cooling between 20 °C and 95 °C at hydrostatic pressures of 1 MPa, 2 MPa and 6 MPa, giving OCRs of 6, 3 and 1, respectively. The predictions of the model (with calibrated parameters) and experimental data are shown in Fig. 12 (variation of volumetric strain with temperature at different stress states). The model accurately predicts the volume changes of the soil at OCRs of 1 and 3 and slightly over-predicts the thermo-mechanical compressive strain behaviour of the highly over-consolidated specimen (OCR = 6).

6.4. Abuel-Naga et al. (2006) tests

The thermo-mechanical behaviour of Bangkok clay, using a temperature-controlled oedometer and triaxial apparatus, is shown in Fig. 13.

The void ratio (e) variation versus the vertical stress of a soil specimen in the temperature-controlled oedometer is shown in Fig. 13(a). At an ambient temperature of 22 °C, the soil specimen was compressed to 100 kPa. Next, with the vertical stress being held constant, the specimen was subjected to a single heating–cooling cycle between 95 °C and 22 °C. As a result, the soil experienced a reduction in void ratio, i.e., it attained a denser state. The test was ended by further compression with the vertical stress increasing to 200 kPa at room temperature. Due to the heating–cooling cycle, the soil exhibited an initial stiffer response upon the second compression phase, i.e., the soil behaved as an over-consolidated soil. This behaviour is attributed to the denser state that the soil had attained after the thermal cycle. This behaviour, as well as the void ratio reduction during the thermal cycle, is well captured by the model.

Fig. 13(b) presents the volumetric behaviour of the same soil in an isotropic compression test using a triaxial apparatus, at three different temperatures of 25, 70 and 90 °C. Samples were first compressed to 300 kPa at an ambient temperature ($T = 25$ °C) and then unloaded to 25 kPa to attain an OCR of 12. This was followed by a heating phase to reach the desired temperature. Finally, the specimens were isotropically loaded with the hydrostatic pressure increasing beyond 300 kPa. The predictions of the model are also depicted in Fig. 13(b) and show satisfactory results. It should be noted that the stress–strain curves predicted by the model are smooth, in contrast to the model of Golchin et al. (2022) which exhibited a bilinear response.

6.5. Uchaipichat and Khalili (2009) tests

The drained behaviours of a saturated silty soil at various stress states and three temperatures of 25, 40 and 60 °C were investigated by Uchaipichat and Khalili (2009) by using a temperature-controlled triaxial apparatus. At a room temperature of 25 °C, all the samples were first isotropically compressed under a hydrostatic pressure of 200 kPa, and then unloaded by the hydrostatic pressure decreasing to 50 kPa, 100 kPa and 150 kPa, resulting in respective stress states with OCR = 4, OCR = 2 and OCR = 4/3. The temperatures of the samples for each over-consolidated state were then elevated to the targeted temperatures and this was followed by shearing under drained conditions. Experimental data of deviatoric stress versus deviatoric strain (q – ϵ_s) and volumetric strain versus deviatoric strain (ϵ_v – ϵ_s), as well as the corresponding model predictions, are presented in Fig. 14. The predictions of the model, similar to those of Golchin et al. (2022), are in good agreement with experimental data, except for the early stage of ϵ_v – ϵ_s curves for over-consolidated states (OCR = 4). Note that Uchaipichat and Khalili (2009) reported that the Critical State stress ratio (M) of the studied soil was temperature independent. Therefore, the simulations were conducted using $\pi^* = 0$.

6.6. Ghahremannejad (2003) tests

In the study of Ghahremannejad (2003), illitic clay samples were first heated to reach the target temperatures of 22 °C (room temperature) and 75 °C. With the temperature held constant, specimens were subjected to drained isotropic compression with the hydrostatic pressure increasing to 400 kPa, resulting in a normally-consolidated stress state. Then, the specimens were sheared under undrained conditions. The stress path in p – q stress space, and the corresponding deviatoric stress and pore pressure (u_p) versus deviatoric strain experimental data, along with model predictions are shown, respectively, in Fig. 15(a)–(c). Comparison of the simulations with experimental data indicates that the current model

successfully captures the undrained behaviour of fine-grained soils at various temperatures.

For clarification on how the model performs, the initial and final configurations of the inner (bubble) and outer yield surfaces are also plotted in p – q stress space in Fig. 15(a). The initial states of the surfaces are shown by dashed lines and the final configurations of the surfaces are depicted by solid lines. The stress path, Critical State stress ratio (M) and yield surfaces in blue correspond to test results and model predictions at $T = 22$ °C, while those in red correspond to model predictions and tests results at $T = 75$ °C. The soil exhibits a temperature-dependent Critical State stress ratio (M) where its value is reduced at elevated temperatures. This observation is well captured by the model through the parameter π^* . As a result, the sizes of the initial and final inner and outer yield surfaces are smaller at $T = 75$ °C, compared to the corresponding surfaces at $T = 22$ °C. Therefore, the inner yield surface corresponding to $T = 75$ °C follows a shorter stress path to reach the outer yield surface during undrained shearing.

6.7. Other simulations

While the main focus of this section is to validate the proposed model with thermo-mechanical test results, the capability of the model in capturing the hysteresis behaviour of soils is investigated here.

Fig. 16 shows the specific volume ($v = 1 + e$) versus logarithm of vertical stress from an oedometer test on undisturbed Ariake clay (Suddepong et al., 2015), along with the model prediction. The specimen, under isothermal conditions, was subjected to two loading–unloading–reloading paths and hysteresis loops were thereby formed. The model well captures the hysteresis loops through the parameter C_v .

Nakai and Hinokio (2004) performed drained cyclic tests with different stress paths on Fujinomori clay by using a triaxial apparatus under isothermal conditions. Fig. 17 shows the results of a normally-consolidated clay which was subjected to one cyclic drained shearing at a hydrostatic pressure of 196 kPa. The model predictions (q/p – ϵ_v and q/p – ϵ_s curves) are compared with experimental data in Fig. 17(a) and Fig. 17(b), respectively. It can be seen that the model can successfully predict the narrow hysteresis loop of the experimental q/p – ϵ_v curve and the wider hysteresis loop in the experimental q/p – ϵ_s curve.

7. Conclusion

A thermodynamically consistent two surface/bubble model has been developed which successfully simulates the hysteresis behaviour of soils during cyclic mechanical loadings and shakedown behaviour when they are subjected to heating–cooling cycles. These behaviours have been captured by assigning a kinematic rule for the heating phase along with an isothermal kinematic rule. As a result, the model predicts plastic strains at highly over-consolidated states and during the heating phase of cyclic thermal loads. The magnitude of thermally-induced plastic strains reduces as the state of the soil becomes over-consolidated and the model has a control on the accumulated plastic strains due to thermal loads.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Energy potential, yield surface and plastic flow

The Gibbs energy potential proposed by [Houlsby et al. \(2005\)](#) is used as the energy potential for isothermal conditions:

$$g_1(p, q) = -\frac{p_o^{2-n}}{\bar{K}(2-n)(1-n)p_a^{1-n}} - \frac{p}{\bar{K}(1-n)} \quad (\text{A-1a})$$

where

$$p_o^2 = p^2 + \frac{\bar{K}(1-n)}{3\bar{G}} q^2 \quad (\text{A-1b})$$

and p_a (kPa) is the atmospheric pressure (101 kPa); $\bar{K}$ and $\bar{G}$ are material constants, respectively related to the elastic bulk modulus and shear modulus; and n ($0 < n < 1$) is the constant representing the power dependence of the bulk and shear moduli on the hydrostatic pressure p . For the case when the elastic bulk and shear moduli are linearly proportional to p (for cases in which $n=1$), the free energy is natural-log dependent on p :

$$g_1(p, q) = -\frac{1}{\bar{K}} p \left(\ln\left(\frac{p}{p_a}\right) - 1 \right) - \frac{q^2}{6\bar{G}p} \quad (\text{A-1c})$$

The total strain and generalised stresses (in triaxial space) are derived, respectively, by differentiating the energy potential (Eq. (1)) with respect to the true triaxial stresses ($\sigma = \{p, q\}^T$) and the internal variables ($\epsilon_i^p = \{\epsilon_{v,i}^p, \epsilon_{s,i}^p\}^T$, $\epsilon_o^p = \{\epsilon_{v,o}^p, \epsilon_{s,o}^p\}^T$ and $\rho = \{\rho_p, \rho_q\}^T$):

$$\{\epsilon\} = -\left\{ \frac{\partial g}{\partial \sigma} \right\} = \{\epsilon^e\} + \{\epsilon_i^p\} + \{\epsilon_o^p\} + \{\epsilon^{Therm}\} = -\left\{ \frac{\partial g_1}{\partial p} \right\} + \left\{ \frac{\epsilon_{v,i}^p}{\epsilon_{s,i}^p} \right\} + \left\{ \frac{\epsilon_{v,o}^p}{\epsilon_{s,o}^p} \right\} + 3\alpha^* (T - T_0) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad (\text{A-2})$$

$$\{\bar{\chi}_i\} = -\left\{ \frac{\partial g}{\partial \epsilon_i^p} \right\} = \{\sigma\} = \begin{Bmatrix} p \\ q \end{Bmatrix} \quad (\text{A-3})$$

$$\{\bar{\chi}_o\} = -\left\{ \frac{\partial g}{\partial \epsilon_o^p} \right\} = \{\sigma\} = \begin{Bmatrix} p \\ q \end{Bmatrix} \quad (\text{A-4})$$

$$\{\bar{\chi}_\rho\} = -\left\{ \frac{\partial g}{\partial \rho} \right\} = -\begin{Bmatrix} \frac{\partial g}{\partial \rho_p} \\ \frac{\partial g}{\partial \rho_q} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A-5})$$

where ϵ is the total strain (in triaxial space), and $\bar{\chi}_i$, $\bar{\chi}_o$ and $\bar{\chi}_\rho$, respectively, are the generalised stresses related to the inner yield surface, outer yield surface and kinematic variable. It should be noted that ρ is an internal variable that is introduced in the definition of the rate of dissipation potential function (Eq. (2)) (and not in the definition of the energy potential, Eq. (1)).

Dissipative stresses are derived by differentiating the total rate of dissipation potential (Eq. (2a)) with respect to the rates of the internal variables ($\dot{\epsilon}_i^p = \{\dot{\epsilon}_{v,i}^p, \dot{\epsilon}_{s,i}^p\}^T$, $\dot{\epsilon}_o^p = \{\dot{\epsilon}_{v,o}^p, \dot{\epsilon}_{s,o}^p\}^T$ and $\dot{\rho} = \{\dot{\rho}_p, \dot{\rho}_q\}^T$):

$$\{\chi_i\} = \left\{ \frac{\partial d}{\partial \dot{\epsilon}_i^p} \right\} = \begin{Bmatrix} \frac{\partial d}{\partial \dot{\epsilon}_{v,i}^p} \\ \frac{\partial d}{\partial \dot{\epsilon}_{s,i}^p} \end{Bmatrix} = \begin{Bmatrix} (C_i + \rho_p) + \frac{A_i^2 \dot{\epsilon}_{v,i}^p}{\sqrt{A_i^2 (\dot{\epsilon}_{v,i}^p)^2 + B_i^2 (\dot{\epsilon}_{s,i}^p)^2}} \\ \rho_q + \frac{B_i^2 \dot{\epsilon}_{s,i}^p}{\sqrt{A_i^2 (\dot{\epsilon}_{v,i}^p)^2 + B_i^2 (\dot{\epsilon}_{s,i}^p)^2}} \end{Bmatrix} + \begin{Bmatrix} \Lambda_1 \frac{\partial c_1}{\partial \dot{\epsilon}_{v,i}^p} \\ \Lambda_2 \frac{\partial c_2}{\partial \dot{\epsilon}_{s,i}^p} \end{Bmatrix} \quad (\text{A-6})$$

$$\{\chi_o\} = \left\{ \frac{\partial d}{\partial \dot{\epsilon}_o^p} \right\} = \begin{Bmatrix} \frac{\partial d}{\partial \dot{\epsilon}_{v,o}^p} \\ \frac{\partial d}{\partial \dot{\epsilon}_{s,o}^p} \end{Bmatrix} = \begin{Bmatrix} C_o + \frac{A_o^2 \dot{\epsilon}_{v,o}^p}{\sqrt{A_o^2 (\dot{\epsilon}_{v,o}^p)^2 + B_o^2 (\dot{\epsilon}_{s,o}^p)^2}} \\ \frac{B_o^2 \dot{\epsilon}_{s,o}^p}{\sqrt{A_o^2 (\dot{\epsilon}_{v,o}^p)^2 + B_o^2 (\dot{\epsilon}_{s,o}^p)^2}} \end{Bmatrix} \quad (\text{A-7})$$

$$\{\chi_\rho\} = \left\{ \frac{\partial d}{\partial \dot{\rho}} \right\} = \begin{Bmatrix} \frac{\partial d}{\partial \dot{\rho}_p} \\ \frac{\partial d}{\partial \dot{\rho}_q} \end{Bmatrix} = \begin{Bmatrix} \Lambda_1 \frac{\partial c_1}{\partial \dot{\rho}_p} \\ \Lambda_2 \frac{\partial c_2}{\partial \dot{\rho}_p} \end{Bmatrix} \quad (\text{A-8})$$

where χ_o and χ_i are the dissipative stresses related to d_o and d_i , respectively, and χ_p is the dissipative stress related to the kinematic variable p . Note that $\partial c_1 / \partial \dot{\epsilon}_{v,i}^p$, $\partial c_2 / \partial \dot{\epsilon}_{s,i}^p$, $\partial c_1 / \partial \dot{\rho}_p$, and $\partial c_2 / \partial \dot{\rho}_q$ are non-zero terms.

By maximising the rate of dissipation, i.e., using Ziegler's postulate (Collins and Houlsby, 1997), where $\chi = \bar{\chi}$, $\chi_i = \bar{\chi}_i$ (equating Eq. (A-3) with Eq. (A-6)), $\chi_o = \bar{\chi}_o$ (equating Eq. (A-4) with Eq. (A-7)) and $\chi_p = \bar{\chi}_p$ (equating Eq. (A-5) with Eq. (A-8)), it is determined that:

$$\{\chi_i\} = \{\chi_o\} = \{\sigma\} = \begin{Bmatrix} p \\ q \end{Bmatrix} \quad (\text{A-9})$$

$$\Lambda = 0 \quad (\text{A-10})$$

By substituting Eq. (A-10) in Eqs. (A-6) and (A-8), the dissipative stresses related to the inner yield surface and kinematic variable are simplified to

$$\{\chi_i\} = \begin{Bmatrix} \frac{\partial d}{\partial \dot{\epsilon}_{v,i}^p} \\ \frac{\partial d}{\partial \dot{\epsilon}_{s,i}^p} \end{Bmatrix} = \begin{Bmatrix} (C_i + \rho_p) + \frac{A_i^2 \dot{\epsilon}_{v,i}^p}{\sqrt{A_i^2 (\dot{\epsilon}_{v,i}^p)^2 + B_i^2 (\dot{\epsilon}_{s,i}^p)^2}} \\ \rho_q + \frac{B_i^2 \dot{\epsilon}_{s,i}^p}{\sqrt{A_i^2 (\dot{\epsilon}_{v,i}^p)^2 + B_i^2 (\dot{\epsilon}_{s,i}^p)^2}} \end{Bmatrix} \quad (\text{A-11})$$

$$\{\chi_p\} = \begin{Bmatrix} \frac{\partial d}{\partial \dot{\rho}_p} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A-12})$$

The general approach of deriving the inner and outer yield surfaces for rate-independent materials, with the rate of dissipation potential being a first order homogenous function of plastic strain increments, is by invoking the Legendre-Fenchel transform on the rate of dissipation potential. This procedure can also be done in a simpler way by eliminating the sub-rate of dissipation functions d_o and d_i in Eq. (A-7) and Eq. (A-11), respectively. Eventually, the surfaces in the dissipative stress space (χ_p - χ_q) are determined as

$$\text{Outer yield surface: } y_o^d = B_o^2 (\chi_{p,o} - C_o)^2 + A_o^2 (\chi_{q,o})^2 - A_o^2 B_o^2 = 0 \quad (\text{A-13})$$

$$\text{Inner yield surface: } y_i^d = B_i^2 (\chi_{p,i} - \rho_p - C_i)^2 + A_i^2 (\chi_{q,i} - \rho_q)^2 - A_i^2 B_i^2 = 0 \quad (\text{A-14})$$

which form ellipses in χ_p - χ_q space (dissipative stress space).

The flow rule (direction of plastic strain increments) is always normal to the inner yield surface in dissipative stress space (y_i^d), i.e., it is along the derivatives of the inner yield surface with respect to dissipative stresses corresponding to the inner yield surface ($\chi_{p,i}$ and $\chi_{q,i}$):

$$\begin{Bmatrix} \dot{\rho}_p \\ \dot{\rho}_q \end{Bmatrix} = 2 \begin{Bmatrix} B_i^2 (\chi_{p,i} - \rho_p - C_i) \\ A_i^2 (\chi_{q,i} - \rho_q) \end{Bmatrix} \quad (\text{A-15})$$

Substituting Eq. (A-9) in Eqs. (A-13)-(A-15), the inner yield surface, the outer yield surface and the plastic flow in true stress space (p - q) are derived as those in Eqs. (3)-(5), respectively.

The incremental formulations for strain controlled conditions and stress controlled conditions (as the input) are presented in Appendix C.

Appendix B. Isothermal kinematic rule

The projection or similarity centre is the point about which the inner yield surface and the outer yield surface are geometrically similar, such that a ray extending from the projection centre passes through points on the inner yield surface and the outer yield surface with a common normal. The projection point can be determined by considering the intersection of two lines. Each line connects two points, one on the inner yield surface and the other on the outer yield surface, that have an identical normal on each surface. Here, these two lines are selected to connect the compression and decompression apexes of both surfaces (Fig. B-1(a)). The decompression apexes of the inner and the outer yield surfaces, respectively, have coordinates $(p,q)=(\rho_p,\rho_q)$ and $(p,q)=(0,0)$, and the compression apexes for corresponding surfaces have coordinates $(p,q)=(p_{iT},\rho_q)$ and $(p,q)=(p_{eT},0)$, where p_{iT} is geometrically defined as

$$p_{iT} = \rho_p + \tilde{p}_{iT} \quad (\text{B-1})$$

By determining these two lines, the projection centre is their point of intersection ($\hat{p}$, $\hat{q}$) and is determined as

$$\begin{Bmatrix} \hat{p} = \rho_p / W_r \\ \hat{q} = \rho_q / W_r \end{Bmatrix} \rightarrow \begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix} = \frac{1}{W_r} \begin{Bmatrix} \rho_p \\ \rho_q \end{Bmatrix} \quad (\text{B-2a})$$

or

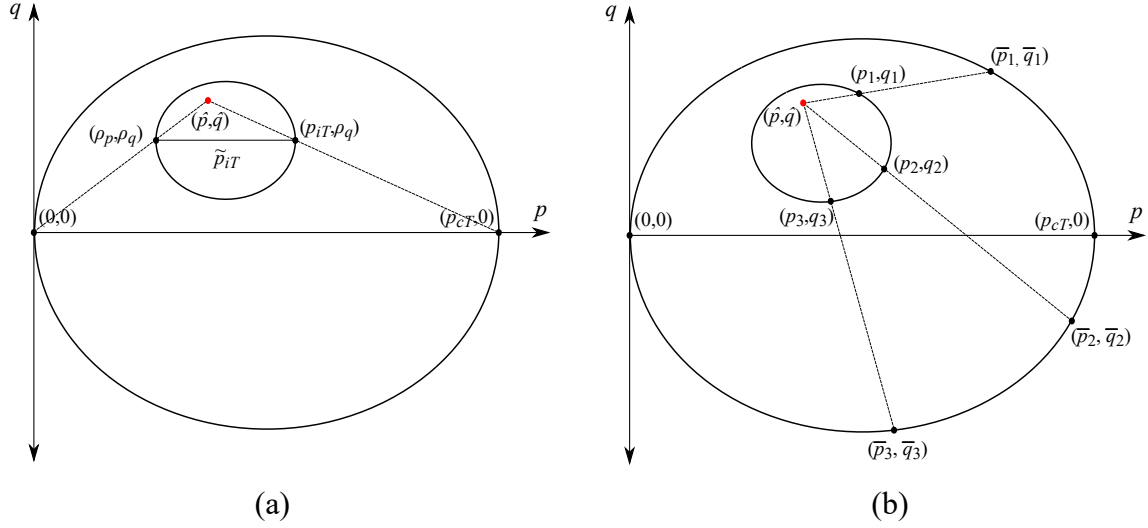


Fig. B-1. (a) Definition of projection centre; (b) definition of image stress.

$$\hat{\sigma} = \frac{\rho}{W_r} \quad (\text{B-2b})$$

where W_r is defined as

$$W_r = 1 - \frac{\tilde{p}_{iT}}{p_{cT}} \quad (\text{B-3})$$

where $\tilde{p}_{iT}$ and p_{cT} are the size of the major-axis of the inner and outer yield surfaces, respectively. The coordinates of the projection centre are controlled by W_r and the coordinates of the decompression apex of the inner yield surface, ρ (Eq. (B-2b)). Vice versa, the decompression apex of the inner yield surface is linked to the projection centre:

$$\rho = W_r \hat{\sigma} \quad (\text{B-4})$$

The corresponding image stress on the outer yield surface can be obtained from any stress on the inner yield surface (Fig. B-1(b)). The image stress, $(\bar{p}, \bar{q})$, on the outer yield surface is determined from the properties of similar triangles (Fig. B-1(a)):

$$\left. \begin{aligned} \bar{p} &= p + \left(\frac{W_r}{1 - W_r} \right) (p - \hat{p}) \\ \bar{q} &= q + \left(\frac{W_r}{1 - W_r} \right) (q - \hat{q}) \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \bar{p} \\ \bar{q} \end{aligned} \right\} = \left\{ \begin{aligned} p \\ q \end{aligned} \right\} + \left(\frac{W_r}{1 - W_r} \right) \left(\left\{ \begin{aligned} p \\ q \end{aligned} \right\} - \left\{ \begin{aligned} \hat{p} \\ \hat{q} \end{aligned} \right\} \right) \quad (\text{B-5a})$$

or, equivalently,

$$\{\bar{\sigma}\} = \{\sigma\} + \left(\frac{W_r}{1 - W_r} \right) (\{\sigma\} - \{\hat{\sigma}\}) \quad (\text{B-5b})$$

The kinematic rule is the rate of Eq. (B-4):

$$\{\dot{\rho}\} = \dot{W}_r \{\hat{\sigma}\} + W_r \{\dot{\hat{\sigma}}\} \quad (\text{B-6})$$

where

$$\dot{W}_r = (1 - W_r) \frac{\dot{p}_{cT}}{p_{cT}} - \frac{\dot{\tilde{p}}_{iT}}{p_{cT}} \quad (\text{B-7})$$

Under isothermal conditions, the temperature dependency terms of $\dot{\tilde{p}}_{iT}$ (Eq. (14)) and $\dot{p}_{cT}$ (Eq. (15)) are dropped; thereby, they are respectively defined by $\dot{\tilde{p}}_i$ and $\dot{p}_c$:

$$\dot{\tilde{p}}_i = \tilde{p}_{iT} \frac{1 + e_0}{\lambda - \kappa} \dot{\epsilon}_v^p \quad (\text{B-8})$$

$$\dot{p}_c = p_{cT} \frac{1 + e_0}{\lambda - \kappa} \dot{\epsilon}_v^p \quad (\text{B-9})$$

The isothermal kinematic rule, assigned to the decompression apex ($\hat{\rho}$), is complete once the evolution rule of the projection centre ($\hat{\sigma}$) is designed. Two mechanisms form the evolution rule of the projection centre, $\hat{\sigma} = (\hat{p}, \hat{q})$:

- When the inner yield surface translates with the stress state along the hydrostatic pressure axis (p -axis) under isothermal conditions (no temperature change), incremental plastic volumetric strains are produced and, simultaneously, the outer yield surface hardens isotropically with the plastic volumetric strain increments (see Eq. (B-9)). The hydrostatic translation (along the p -axis) of the inner yield surface is restricted by the isotropic growth of the outer yield surface, which implies that the projection centre translates proportionally to the hydrostatic incremental growth of the outer yield surface (isotropic hardening). Mathematically, this means that:

$$\{\dot{\hat{\sigma}}_1\} = \frac{\dot{p}_c}{p_{cT}} \{\hat{\sigma}\} \quad (\text{B-10})$$

- When plastic strains are developed, the inner yield surface translates toward the image stress, projected on the outer yield surface, which is mathematically expressed as

$$\{\dot{\hat{\sigma}}_2\} = C_\chi \|\dot{\epsilon}^p\| (\{\sigma\} - \{\hat{\sigma}\}) \quad (\text{B-11})$$

where $\|\dot{\epsilon}^p\|$ is the magnitude of the plastic strain increment and C_χ is a parameter which controls the pace of the inner yield surface approaching the outer yield surface.

The evolution rule of the projection centre is the sum of $\{\dot{\hat{\sigma}}_1\}$ and $\{\dot{\hat{\sigma}}_2\}$,

$$\{\dot{\hat{\sigma}}\} = C_\chi \|\dot{\epsilon}^p\| (\{\sigma\} - \{\hat{\sigma}\}) + \frac{\dot{p}_c}{p_{cT}} \{\hat{\sigma}\} \quad (\text{B-12})$$

and consequently, by using Eq. (B-6), the kinematic rule assigned to the decompression apex for isothermal conditions is determined as

$$\{\dot{\rho}_I\} = \left(\frac{\dot{p}_c}{p_{cT}} - \frac{\dot{p}_i}{p_{cT}} \right) \{\hat{\sigma}\} + C_\chi \|\dot{\epsilon}^p\| W_r (\{\sigma\} - \{\hat{\sigma}\}) \quad (\text{B-13})$$

By further substituting Eq. (B-2b) in Eq. (B-13) and eliminating $\{\hat{\sigma}\}$, the kinematic rule (Eq. (18)) is derived.

Appendix C. Rate-form formulation

Strain-controlled incremental formulation

In order to determine the strain-controlled rate-form formulation, all the incremental formulations, including the isotropic and kinematic rules, need to be re-written. Here, the rate equations are split into plastic multiplier ($\dot{\lambda}$) dependent and independent terms, which are respectively denoted by M_i and U_i , where subscript i represents the variable.

The isotropic hardening rule for the outer yield surface (Eq. (14)) is rewritten as

$$\dot{p}_{cT} = \dot{\lambda} M_{p_{cT}} + U_{p_{cT}} \quad (\text{C-1a})$$

where

$$M_{p_{cT}} = p_{cT} \frac{1 + e_0}{\lambda - \kappa} r_p^j \quad (\text{C-1b})$$

$$U_{p_{cT}} = -\mu_0 p_{cT} \dot{T} \quad (\text{C-1c})$$

where r_p^j is the flow rule along the p -axis and $\dot{\epsilon}_v^p = \dot{\lambda} r_p^j$. The isotropic hardening of the inner yield surface (Eq. (15)) is rewritten as

$$\dot{\tilde{p}}_{iT} = \dot{\lambda} \tilde{M}_{\tilde{p}_{iT}} + U_{\tilde{p}_{iT}} \quad (\text{C-2a})$$

where

$$\tilde{M}_{\tilde{p}_{iT}} = \tilde{p}_{iT} \left(\frac{1 + e_0}{\lambda - \kappa} r_p^j - M_\mu \langle T - T_0 \rangle \right) \quad (\text{C-2b})$$

$$U_{\tilde{p}_{iT}} = - \left(\mu \langle \dot{T} \rangle + U_\mu \langle T - T_0 \rangle \right) \tilde{p}_{iT} \quad (\text{C-2c})$$

The kinematic rule under isothermal conditions (Eq. (18)) is rewritten as

$$\{\dot{\rho}_I\} = \dot{\lambda} \{M_I\} \quad (\text{C-3a})$$

where

$$\{M_I\} = \left\{ \begin{pmatrix} M_I \end{pmatrix}_p \right\} = \left(\frac{1}{W_r} \left(\frac{M_{p_c}}{p_{cT}} - \frac{M_{p_i}}{p_{cT}} \right) - C_\chi \|\dot{\epsilon}^p\| \right) \left\{ \begin{pmatrix} \rho_p \end{pmatrix} \right\} + C_\chi \|\dot{\epsilon}^p\| W_r \left\{ \begin{pmatrix} \rho_q \end{pmatrix} \right\} \quad (\text{C-3b})$$

and

$$\bar{M}_{p_i} = \tilde{p}_{iT} \frac{1 + e_0}{\lambda - \kappa} r_p^i \quad (C-3c)$$

$$\bar{M}_{p_c} = p_{cT} \frac{1 + e_0}{\lambda - \kappa} r_p^i \quad (C-3d)$$

The kinematic rule under non-isothermal conditions (Eq. (18)) is rewritten as

$$\{\dot{\rho}_H\} = \dot{\lambda} \{M_H\} \quad (C-4a)$$

where

$$\{M_H\} = \left\{ \begin{pmatrix} M_H \end{pmatrix}_p \right\} = -H(\dot{T}) \left\{ \begin{pmatrix} M_I \end{pmatrix}_p \right\} \quad (C-4b)$$

The kinematic rule is therefore, the sum of (C-3a) and (C-4a):

$$\{\dot{\rho}\} = \{\dot{\rho}_I\} + \{\dot{\rho}_H\} \quad (C-5)$$

$\dot{\mu}$ is the rate of μ (Eq. (11)):

$$\dot{\mu} = \mu b \left(\frac{\dot{p}_{iT}}{p_{iT}} - \frac{\dot{p}_{cT}}{p_{cT}} \right) \quad (C-6)$$

which can be rewritten as

$$\dot{\mu} = \dot{\lambda} M_\mu + U_\mu \quad (C-7a)$$

where

$$M_\mu = \left(\frac{1}{\mu b} + \frac{\tilde{p}_{iT}}{p_{iT}} \langle T - T_0 \rangle \right)^{-1} \left(\frac{(M_I)_p + (M_H)_p}{p_{iT}} - \frac{M_{p_{cT}}}{p_{cT}} + \frac{1 + e_0}{\lambda - \kappa} \frac{\tilde{p}_{iT}}{p_{iT}} r_p^i \right) \quad (C-7b)$$

$$U_\mu = - \left(\frac{1}{\mu b} + \frac{\tilde{p}_{iT}}{p_{iT}} \langle T - T_0 \rangle \right)^{-1} \left(\mu \frac{\tilde{p}_{iT}}{p_{iT}} \langle \dot{T} \rangle + \frac{U_{p_{cT}}}{p_{cT}} \right) \quad (C-7c)$$

The total strain increment can be determined by differentiating Eq. (A-2):

$$\{\dot{\epsilon}\} = \{\dot{\epsilon}^e\} + \{\dot{\epsilon}_i^p\} + \{\dot{\epsilon}_o^p\} + \{\dot{\epsilon}^{Therm}\} \quad (C-8)$$

where the elastic strain increments are

$$\{\dot{\epsilon}^e\} = \left\{ \begin{pmatrix} \dot{\epsilon}_v^e \\ \dot{\epsilon}_s^e \end{pmatrix} \right\} = [C^e] \{\dot{\sigma}\} = - \begin{bmatrix} \frac{\partial^2 g_1}{\partial p^2} & \frac{\partial^2 g_1}{\partial p \partial q} \\ \frac{\partial^2 g_1}{\partial q \partial p} & \frac{\partial^2 g_1}{\partial q^2} \end{bmatrix} \left\{ \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} \right\} \quad (C-9)$$

where $[C^e]$ is the elastic compliance (or flexibility) matrix. With respect to Eq. (A-1a) or Eq. (A-1c), the components of the elastic compliance matrix can be derived, respectively, as

$$[C^e] = \frac{1}{3\bar{G}p_a^{1-n}p_o^n} \begin{bmatrix} \frac{3\bar{G}}{\bar{K}(1-n)} \left(1 - \frac{np^2}{p_o^2} \right) & -\frac{npq}{p_o^2} \\ -\frac{npq}{p_o^2} & 1 - \frac{n(1-n)\bar{K}}{3\bar{G}} \frac{q^2}{p_o^2} \end{bmatrix} \quad (C-10a)$$

$$[C^e] = \frac{1}{3\bar{G}p} \begin{bmatrix} \frac{3\bar{G}}{\bar{K}} \left(1 + \frac{\bar{K}}{3\bar{G}} \frac{q^2}{p^2} \right) - \frac{q}{p} & \\ -\frac{q}{p} & 1 \end{bmatrix} \quad (C-10b)$$

The thermo-elastic strain increments are

$$\{\dot{\epsilon}^{Therm}\} = \left\{ \begin{pmatrix} \dot{\epsilon}_v^{Therm} \\ \dot{\epsilon}_s^{Therm} \end{pmatrix} \right\} = 3\alpha^* \dot{T} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \quad (C-11)$$

By using Eqs. (12) and (C-8), the stress increments are determined:

$$\{\dot{\sigma}\} = [D^e] (\{\dot{\epsilon}\} - \{\dot{\epsilon}^{Therm}\} - \dot{\lambda} \{r^i\}) \quad (C-12)$$

where $[D^e]$ is the elastic stiffness matrix, which is the inverse of elastic flexibility matrix $[C^e]$.

The consistency condition of the inner yield surface ($\dot{y}_i = 0$) is

$$\dot{y}_i = \left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T \left\{ \dot{\sigma} \right\} + \frac{\partial y_i}{\partial \tilde{p}_{iT}} \dot{\tilde{p}}_{iT} + \frac{\partial y_i}{\partial M} \dot{M} \quad (C-13)$$

where $\left\{ \frac{\partial y_i}{\partial \sigma} \right\} = \left\{ \frac{\partial y_i / \partial \tilde{p}}{\partial y_i / \partial \tilde{q}} \right\}$, $\left\{ \dot{\sigma} \right\} = \left\{ \frac{\dot{\tilde{p}}}{\dot{\tilde{q}}} \right\}$ and $\dot{M} = \pi^* \dot{T}$.

By further substituting Eq. (C-2), Eq. (C-5) and Eq. (C-12) in Eq. (C-13), the plastic multiplier for strain-controlled loading conditions is derived:

$$\dot{\lambda} = \frac{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T [D^e] \{\dot{\epsilon} - \dot{\epsilon}^{Therm}\} + \frac{\partial y_i}{\partial \tilde{p}_{iT}} \dot{\tilde{p}}_{iT} + \frac{\partial y_i}{\partial M} \dot{M}}{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T [D^e] \{r^i\} + \left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T \{M_I + M_H\} - \frac{\partial y_i}{\partial \tilde{p}_{iT}} \dot{\tilde{p}}_{iT}} \quad (C-14)$$

The derivatives of the yield surface with respect to its variables $\left(\partial y_i / \partial \sigma, \partial y_i / \partial \tilde{p}_{iT}, \partial y_i / \partial M \right)$ are presented in Appendix D.

Stress-controlled incremental formulation

For stress-controlled conditions, the stress increment $\{\dot{\sigma}\}$ is the input and the corresponding strain increment is calculated. To derive the stress-controlled incremental formulations, the plastic multiplier (Eq. (C-14)) needs to be revised. By using Eq. (C-8) and substituting for the term $\{\dot{\epsilon} - \dot{\epsilon}^{Therm}\}$ in Eq. (C-14), and using Eq. (C-10), the plastic multiplier is revised as

$$\dot{\lambda} = \frac{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T \{\dot{\sigma}\} + \frac{\partial y_i}{\partial \tilde{p}_{iT}} \dot{\tilde{p}}_{iT} + \frac{\partial y_i}{\partial M} \dot{M}}{\left\{ \frac{\partial y_i}{\partial \sigma} \right\}^T \{M_I + M_H\} - \frac{\partial y_i}{\partial \tilde{p}_{iT}} \dot{\tilde{p}}_{iT}} \quad (C-15)$$

Appendix D. Derivative of the yield surface with respect to its variables

The derivatives of the yield surface with respect to its variables are presented here:

$$\left\{ \frac{\partial y_i}{\partial \sigma} \right\} = \left\{ \frac{\partial y_i}{\partial \tilde{p}} \right\} = \left\{ \begin{matrix} r_p^i + \frac{\partial y_i}{\partial A_i} \frac{\partial A_i}{\partial \tilde{p}} + \frac{\partial y_i}{\partial B_i} \frac{\partial B_i}{\partial \tilde{p}} \\ r_q^i \end{matrix} \right\} \quad (D-1)$$

$$\frac{\partial y_i}{\partial \tilde{p}_{iT}} = \frac{\partial y_i}{\partial A_i} \frac{\partial A_i}{\partial \tilde{p}_{iT}} + \frac{\partial y_i}{\partial B_i} \frac{\partial B_i}{\partial \tilde{p}_{iT}} + \frac{\partial y_i}{\partial C_i} \frac{\partial C_i}{\partial \tilde{p}_{iT}} \quad (D-2)$$

$$\frac{\partial y_i}{\partial M} = \frac{\partial y_i}{\partial B_i} \frac{\partial B_i}{\partial M} \quad (D-3)$$

where

$$\frac{\partial y_i}{\partial A_i} = 2A_i (\tilde{q}^2 - B_i^2) \quad (D-4)$$

$$\frac{\partial y_i}{\partial B_i} = 2B_i ((\tilde{p} - C_i)^2 - A_i^2) \quad (D-5)$$

$$\frac{\partial y_i}{\partial C_i} = -2B_i^2 (\tilde{p} - C_i) + \frac{\partial y_i}{\partial B_i} \frac{\partial B_i}{\partial C_i} \quad (D-6)$$

$$\frac{\partial B_i}{\partial C_i} = B_i \left(\frac{1}{C_i} - \frac{\alpha}{\tilde{p}_{iT}} \right) \quad (D-7)$$

$$\frac{\partial A_i}{\partial \tilde{p}} = -\frac{\gamma}{\pi} \left(1 + \left(\frac{\gamma}{2} - \frac{\gamma \tilde{p}}{\tilde{p}_{iT}} \right)^2 \right)^{-1} \quad (\text{D-8})$$

$$\frac{\partial A_i}{\partial \tilde{p}_{iT}} = \frac{A_i}{\tilde{p}_{iT}} + \left(\frac{\gamma \tilde{p}}{\pi \tilde{p}_{iT}} \right) \left(1 + \left(\frac{\gamma}{2} - \frac{\gamma \tilde{p}}{\tilde{p}_{iT}} \right)^2 \right)^{-1} \quad (\text{D-9})$$

$$\frac{\partial B_i}{\partial \tilde{p}} = \frac{\alpha B_i}{\tilde{p}_{iT}} \quad (\text{D-10})$$

$$\frac{\partial B_i}{\partial \tilde{p}_{iT}} = -\frac{\alpha(\tilde{p} - C_i) B_i}{\tilde{p}_{iT}^2} \quad (\text{D-11})$$

$$\frac{\partial B_i}{\partial M} = \frac{B_i}{M} \quad (\text{D-12})$$

$$\frac{\partial C_i}{\partial \tilde{p}_{iT}} = \frac{C_i}{\tilde{p}_{iT}} \quad (\text{D-13})$$

$$\dot{M} = \pi^* \dot{T} \quad (\text{D-14})$$

Appendix E. The magnitude of plastic strain increments in triaxial space

The magnitude of the plastic strain increment produced by activation of the inner yield surface, $\|\dot{\epsilon}^p\|$, is calculated here. The plastic volumetric strain increment and plastic deviatoric strain increment are defined respectively as

$$\dot{\epsilon}_v^p = \text{tr}(\dot{\epsilon}^p) \quad (\text{E-1})$$

$$\dot{\epsilon}_s^p = (2/3 \dot{\epsilon}^p : \dot{\epsilon}^p)^{1/2} \quad (\text{E-2})$$

where “:” is the double contract operator, and $\dot{\epsilon}^p$ and $\dot{\epsilon}^p$ are, respectively, the plastic strain increment tensor and the plastic deviatoric strain increment tensor defined as

$$\dot{\epsilon}^p = \dot{\epsilon}^p - \left(\dot{\epsilon}_v^p / 3 \right) : \mathbf{1} \quad (\text{E-3})$$

where $\mathbf{1}$ is the second order identity tensor.

The magnitude of the plastic strain increment is

$$\|\dot{\epsilon}^p\| = (\dot{\epsilon}^p : \dot{\epsilon}^p)^{1/2} \quad (\text{E-4})$$

Substituting Eq. (E-3) in Eq. (E-4) results in

$$\|\dot{\epsilon}^p\| = \left(\underbrace{\dot{\epsilon}^p : \dot{\epsilon}^p}_{=3/2 \left(\dot{\epsilon}_s^p \right)^2} + \underbrace{\left(2\dot{\epsilon}_v^p / 3 \right) \mathbf{1} : \dot{\epsilon}^p}_{=0} + \underbrace{\left(\dot{\epsilon}_v^p / 3 \right)^2 \mathbf{1} : \mathbf{1}}_{=3} \right)^{1/2} \quad (\text{E-5})$$

which simplifies to

$$\|\dot{\epsilon}^p\| = \left(\frac{3}{2} \left(\dot{\epsilon}_s^p \right)^2 + \frac{1}{3} \left(\dot{\epsilon}_v^p \right)^2 \right)^{1/2} \quad (\text{E-6})$$

It can also be rewritten in terms of the plastic multiplier and flow rule as

$$\|\dot{\epsilon}^p\| = \dot{\lambda} \|\mathbf{r}^i\| = \dot{\lambda} \left(\frac{3}{2} \left(r_q^i \right)^2 + \frac{1}{3} \left(r_p^i \right)^2 \right)^{1/2} \quad (\text{E-7})$$

Appendix F. Range of shape parameters α and γ

The range of shape parameters α and γ are investigated from two perspectives:

- Convexity and range of shape parameters α and γ

From constitutive modelling and numerical perspectives, yield surfaces should be convex to avoid numerical difficulties. Golchin et al. (2021), with the use of convexity analysis, demonstrated that the yield surface presented by Eq. (3) and stress-like functions defined in Eq. (6) are always convex when $-2 \leq \alpha, \gamma \leq 2$. The proposed yield surface was shown to successfully represent the yield stress points of a wide range of geomaterials.

• Thermodynamics and range of shape parameters α and γ

From a thermodynamics perspective, the rate of dissipation is required to be non-negative. Therefore, shape parameters defining the rate of dissipation and the yield surfaces may need to be limited within a range that satisfies this thermodynamical requirement.

The total plastic work increment $\dot{W}^p$ associated with a continuum element is the product of the effective stress σ and plastic strain increment $\dot{\epsilon}^p$ and is equal to the rate of dissipation d (since there is no shift stress) (Collins and Hilder, 2002):

$$\dot{W}^p = \sigma \dot{\epsilon}^p = d \quad (\text{F-1a})$$

where, in triaxial stress space, it is rewritten as:

$$\dot{W}^p = p \dot{\epsilon}_v^p + q \dot{\epsilon}_s^p = d; \text{ where } d_i \geq 0 \quad (\text{F-1b})$$

Using Eq. (2f) and considering that plastic deformations are associated with the inner yield surface, Eq. (F-1b) reduces to

$$\dot{W}^p = (\tilde{p} + \rho_p) \dot{\epsilon}_v^p + (\tilde{q} + \rho_q) \dot{\epsilon}_s^p = d_i \quad (\text{F-1c})$$

where $\tilde{p} = p - \rho_p$ and $\tilde{q} = q - \rho_q$. Substituting Eq. (2c) in Eq. (F-1c) results in

$$(\tilde{p} - C_i) \dot{\epsilon}_v^p + \tilde{q} \dot{\epsilon}_s^p = \sqrt{A_i^2 \left(\dot{\epsilon}_{v,i}^p \right)^2 + B_i^2 \left(\dot{\epsilon}_{s,i}^p \right)^2} \quad (\text{F-2})$$

The right-hand side of Eq. (F-2) is non-negative. Therefore, the ranges of the shape parameters α and γ are restricted to values resulting in non-negative values of the term on the left-hand side of Eq. (F-2). The range of values of the term $(\tilde{p} - C_i) \dot{\epsilon}_v^p + \tilde{q} \dot{\epsilon}_s^p$ (left-hand side of Eq. (F-2)) was investigated for $-2 \leq \alpha, \gamma \leq 2$ (obtained from convexity analysis) and $0 \leq \tilde{p}/p_{IT} \leq 1$ (the range of $\tilde{p}$ varying inside the inner yield surface) which were non-negative, similar to the right-hand side of Eq. (F-2). Therefore, it was concluded that the accepted range of the shape parameters α and γ is $-2 \leq \alpha, \gamma \leq 2$.

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