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Modelling and Optimal Control of MIMO System - France Macroeconomic Model Case*

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Abstract— **In this paper, we focus on the French Macroeconomic model. We use real economic data, available as time series, starting from 1980s and openly provided by the INSEE. Variables such as Gross Domestic Production, Exportation, Importation, Household Consumption, Gross Fixed Capital Formation and Public expenditure are included in the analysis. Our objective is to maintain a constant economic growth rate according to the available resources. We implement an optimal control policy via LQR to achieve that. Since we aim to maintain a constant growth rate, the control system is modified for this purpose. We prove the efficiency with three experiments based on real data, and we test the method robustness with respect to: (1) variation of LQR parameters, (2) realistic constraints on inputs, and (3) perturbations on outputs. Results show that our designed control system can guide the output to the desired growth rate.**

MIMO model, LQR, Optimal control, Macroeconomic data.

I. Introduction

Control theory is commonly applied to mechanical and physical systems. But it is also a strong tool to solve optimization problem for various other systems [6], [21] with noisy data [20]. Applying control theory for economic problems has been studied since 1970s [19]. [8] summarizes the development of stochastic control theory in macroeconomic policy analysis in three periods: pre-1970 when the major ideas of policy analysis and of optimization were formed [10], early and middle 1970s when formal stochastic control theory was rapidly developed for and applied to the study of macroeconomic policy [12] and late 1970s with the introduction of the idea of rational expectations in economic analysis [4].

Recent works focus more on applications. [15] proposes a general class of PID-based monetary policy rules, the feedback rules let the model use a control signal (e.g. central bank's policy interest rate) responds to movements in a small number of macroeconomic factors, such as the current amount of labor market slack and the deviation of the rate of inflation from its target. Under an optimal control of monetary policy [5], the current and expected future path of the policy is instead typically calculated with a procedure that minimizes a cost function subject to certain constraints. For fiscal policy [18] and resource allocation [9], [16] problems, they follow the same ideas of optimal control, differences are the cost function and the constraints. To estimate the asset holdings of a portfolio, [13] uses algorithms applied to nonlinear dynamic systems to estimate the state with a discrete-time observer.

In this paper, we will apply control techniques to a French macroeconomic model using real economic data (available as time series) starting from 1980s. Our objective would be to design a meaningful control policy that would allow a constant growth rate of the Gross Domestic Product. After choosing the appropriate variables, the orders and parameters of MIMO model are estimated and validated. To satisfy our objective, an optimal LQR solutionis designed and implemented. Simulations are developed with and without constraints on input signals. Results show that our system can quickly recover from the disturbance, and constraints on input signals delay the recovery.

II. System Identification

In this section, the process of estimating the economic model is given. After the data preparation, a MIMO model is computed. The parameters of the model are then estimated.

A. Data Preparation

All the data used throughout this paper are obtained from the INSEE (Institut National de la Statistique et des Études Économiques) which regroups all the official economic French Data[1]. A plethora of data is available, which for a non-expert makes it impossible to decide which makes sense in our particular case. After an in depth analysis on the economical meaning of each variable and long discussions with experts, we decide to study 6 time series namely: Gross Domestic Production (GDP), Exportation (EXP), Importation (IMP), Household Consumption (HC), Gross Fixed Capital Formation (GFCF) and Public Expenditure (PE). Moreover, these variables have a causal relationship between each other and a meaningful influence on the GDP. In the INSEE database, all the data are quarterly ranged from the first quarter of 1980 to the fourth quarter of 2018, 1980T1 to 2018T4[2], to have a meaningful analysis we decide to take all the data. Following a similar analysis as the one performed in [22] we decide the inputs and outputs of the system.

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[1] https://www.insee.fr/fr/accueil

[2] The reader must note that especially due to the arrival of Covid-19 pandemics, data from 2019 were not taken into account in this study.

The inputs of the model would be variables on which the appropriate governmental structures can act and incite their modification, namely HC, GFCF and PE. On the other side, the outputs of the model are variables, which on a regular basis, are only measured, but the government can not directly act on: GDP, EXP, IMP.

The reader should note that the original data from INSEE are presented on the values of current price and make no adjustment for inflation. This is problematic as the current price measure measures for example GDP, inflation or asset prices using the actual prices we notice in the economy not the real, deflated, one. Before anything else, we therefore need to deflate it by using France GDP deflator (base year: 2014) obtained from the World Bank. Deflated time series are showed in Fig. 1.

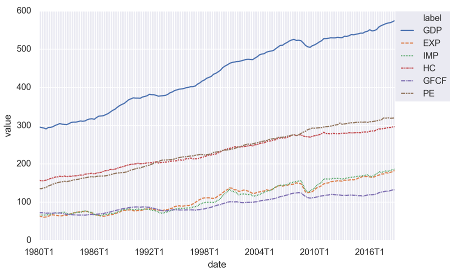


Fig. 1: Original data (Unit: Billion Euro)

As we deal with time series having an economic meaning, we prefer to use natural logarithm to better linearize them. Moreover, as we will use these time series to do linear regression, we must ensure that all these series are stationary. Augmented Dickey–Fuller (ADF) test [7] is a commonly used statistical unit root test to examine whether a given time series is stationary or not. In our case, we would test the stationary for the natural logarithm of our original time series as well as the first difference of natural logarithm. Results are shown in Tab. I where LGDP, LEXP, LIMP, LHC, LGFCF and LPE denote the natural logarithm of our 6 time series introduced before. From Tab. I we can therefore observe that the test, in the case of the natural logarithm of the original data, cannot reject the null hypothesis that the variable contains a unit root, which implies that the original time series are therefore not stationary. Nevertheless, we can notice that the results for the first difference of natural logarithm of all the series rejects the null hypothesis at 99% (i.e., note the $***$ following the values as indicator). We can therefore infer that these time series using the first difference of natural logarithm are stationary. Moreover, in later analysis we decide to use the first difference of natural logarithm of DLGDP (- for 1^{st} difference logarithm of GDP), DLEXP, DLIMP, DLHC, DLGFCF and DLPE.

B. Selection of MIMO model order

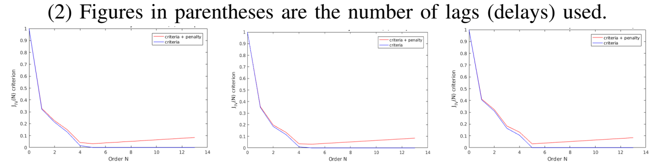
In order to introduce our economic model, we first define the input and output vectors as follows:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \text{DLGDP} \\ \text{DLEXP} \\ \text{DLIMP} \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \text{DLHC} \\ \text{DLGFCF} \\ \text{DLPE} \end{bmatrix} \quad (1)$$

TABLE I: Augmented Dickey–Fuller test for unit root. Null hypotheses: Variable contains an unit root

Variables	**ADF Level**	**ADF First Difference**
LGDP	-1.10548(1)	-7.66072(0)***
LEXP	-3.28665(2)	-6.29252(3)***
LIMP	-2.87331(4)	-6.20372(3)***
LHC	-1.45805(3)	-5.59492(2)***
LGFCF	-3.60257(2)	-4.01801(1)***
LPE	-1.92627(1)	-5.69958(1)***

Notes: (1) ***, ** and * denote significance at 1%, 5%, and 10% levels. (2) Figures in parentheses are the number of lags (delays) used.



(a) y_1 - DLGDP (b) y_2 - DLEXP (c) y_3 - DLIMP

Fig. 2: Order Selection

where y is the output (endogenous variables) of our model, u is the input (exogenous variables) of our model. Note that the selections of y and u are done according to their economic interpretation and attributes. Input (exogenous) variables are therefore the factors we could manipulate in an economic system (e.g., increasing public expenditure for example) while output (endogenous) variables are the consequences that we could only observe in our case but not directly interfere with (e.g., the variation of importations for example).

Consider an "m-input-p-output" system represented by a canonical input-output representation [14], for $i = 1, 2, ...p$:

$$\begin{aligned} y_i(k) = & \sum_{j=1}^{p} \sum_{q=1}^{n_{ij}} a_{ijq} y_j(k + q - n_i - 1) \\ & + \sum_{j=1}^{m} \sum_{q=1}^{n_i} b_{ijq} u_j(k + q - n_j - 1) + e_i(k) \end{aligned} \quad (2)$$

where p and m is the numbers of outputs and inputs, $y_i(k)$ denotes the value of output y_i at time k, a_{ijq} and b_{ijq} are the coefficients of y_i and u_i, $e_i(k)$ is the white noise. Moreover, the observability indices n_i are given by:

$$n_{ij} = \min\{n_i, n_j\}, \text{ if } i \leq j \quad (3)$$

and

$$n_{ij} = \min\{n_i + 1, n_j\}, \text{ if } i > j \quad (4)$$

We apply this method for each output using the technique of instrumental variables, and implementing a criterion which penalizes the model complexity as we want to estimate models of reduced order (see for example [11], [17] among others). Fig. 2 shows the order selection process while $Criteria$ variable calculates the average of estimation errors. The *Criteria*, (namely $J(\hat{n})$) is defined as:

$$J(\hat{n}) = \min_{\hat{\theta}} \frac{1}{N} \|Y(t) - R(\hat{n})\hat{\theta}\|^2 \quad (5)$$

where $\hat{n}$ is the estimated system order, N is the number of data, $\hat{\theta}$ is the estimation of parameters. $Y(t)$ is the real value of the data and $R(\hat{n})\hat{\theta}$ is the estimated one. S($\hat{n}$, N) is the part served as *penalty* on the order of the model and defined as follows:

$$S(\hat{n}, N) = \frac{2\hat{n}log(N)}{N} \tag{6}$$

We see that $J(\hat{n})$ goes towards 0 as the estimated order approaches the true one. Therefore, as order increases, we can see from Fig. 2 that $criteria$ decreases to 0. But $S(\hat{n}, N)$ increases with the increase of the chosen order.

From Fig. 2, the estimated order of the model between the inputs and each of the three outputs is: 5 for y_1, 4 for y_2 and 5 for y_3. One should notice that these estimated orders are not definitive, as we still need to pass the validation process.

C. Estimation and Validation of parameters

After finding the model order, Least Squares method is used to estimate the parameters of the 3 equations for $y1$, $y2$ and $y3$. After each estimation, a whiteness test (autocorrelation test) will be applied to make sure the residuals from the estimated equation are white noise, which means the estimated model extracted all the knowledge from training data [2]. The algorithms of calculation differ, but the goal remains the same: testing if values are mutually uncorrelated. In this paper, we choose to use the implementation of autocorrelation test function from [3].

Consider T as the total number of data points in the dataset, we can therefore conclude that all the autocorrelation values should be in the range $(0 \pm \frac{1.96}{T^{0.5}})$. In our case, from 1980T1 to 2018T4 we have 156 data points, so the limit is ± 0.157 here. Fig. 3 shows the final autocorrelation test for estimation residuals of y_1, y_2 and y_3 where the limit of ± 0.157 is manifested by the horizontal blue lines.

Please recall that our objective is to find a model with the lowest order (minimum number of variables), but still valid. Therefore, once we find a valid candidate model, we would still try to reduce the order by eliminating the variables whose coefficient is much smaller than the others by comparing their absolute value. Nevertheless, this can pose problems concerning the validity of the mode (for example the residual not being a white noise anymore). Therefore, every time we delete one variable we need to re-estimate all the parameters, re-do the whiteness test on residuals to check the validity of the new model. We continue to remove variables until none can be removed without compromising the validity of the model. The final orders of the reduced model that we found are: 5 for y_1, 4 for y_2 and 4 for y_3. If we define our state vector X as:

$$\begin{aligned}\text{X(k)} = [&y_1(k-1)\; y_1(k-2)\; y_1(k-3)\; y_1(k-4)\; y_1(k-5)\\ &y_2(k-1)\; y_2(k-2)\; y_2(k-3)\; y_2(k-4)\\ &y_3(k-1)\; y_3(k-2)\; y_3(k-3)\; y_3(k-4)]^T\end{aligned} \tag{7}$$

Our system can be written in discrete-time as:

$$\begin{aligned}\text{X(k+1)} &= A \cdot \text{X(k)} + B \cdot \text{u(k)}\\ \text{Y(k)} &= C \cdot \text{X(k)} + D \cdot \text{u(k)}\end{aligned} \tag{8}$$

where $k \in \mathbb{Z}^+$, output y and control u vectors are given by Eq. (1), $A \in \mathbb{R}^{13\times13}$; $B \in \mathbb{R}^{13\times3}$; $C \in \mathbb{R}^{1\times13}$; $D \in \mathbb{R}^{1\times3}$. The values of A, B, C, D are showed in Eq. (9) below, where $O_{i\times j}$ is the zero matrix of size $i \times j$.

Eigenvalues of the state matrix A are checked and all are within the unit circle which means the open-loop model is stable as we expected. Controllability and observability of the system are also tested, the rank of the $observability$ and $controllability$ matrices are equal to the number of states, which suggests that our system is controllable and observable.

III. Optimal Control Policy

The theory of optimal control is concerned with operating a dynamic system at minimum cost. As the system dynamics are described by a set of linear differential equations and the cost by a quadratic function, we have an LQ problem [1].

In this section, we introduce Linear–Quadratic Regulator (LQR) and then, according to the physical nature of our input and output, we develop several modifications of the method. for a discrete-time linear system given by:

$$\text{X(k+1)} = A \cdot \text{X(k)} + B \cdot \text{u(k)}$$

the cost function of discrete time LQR in finite horizon is presented as follows:

$$\begin{aligned}J = X(N)^T Q X(N) + \sum_{k=0}^{N-1} (X(k)^T Q X(k)\\ + u(k)^T R u(k) + 2x(k)^T N u(k))\end{aligned} \tag{10}$$

where Q, R are the weighting matrices for state and input while the cross term matrix N is set to 0 in our case as the states and input vectors do not have the same economic meaning in order to be multiplied together. The control law that minimizes the cost function J is the usual:

$$u(k) = -K \cdot X(k) \tag{11}$$

More details about the practical implementation of LQR in our case are given in Sec. IV-A below.

A. Reference Input

As we will focus on controlling GDP, then we need to implement our designed reference to let output reach the desirable value. Recall that y_1 from Eq. 1 is the first difference of natural logarithm of GDP, illustrated below:

$$y_1(k+1) = ln(\text{GDP(k)}) - ln(\text{GDP(k-1)}) = ln(\frac{\text{GDP(k)}}{\text{GDP(k-1)}}) \tag{12}$$

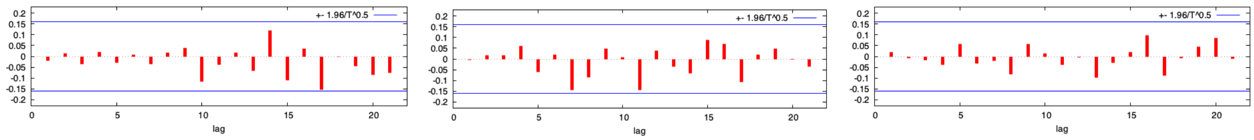
If we want our GDP to have constant p percent (%) increasing, we have:

$$\frac{\text{GDP(k)}}{\text{GDP(k-1)}} = 1 + \frac{p}{100}$$

and then the quarterly GDP increasing ratio p is:

$$p = (e^{y_1(k)} - 1) \times 100 \tag{13}$$

where e is the base of the natural logarithm. One can remark that as we want a constant increasing ratio, i.e. p constant, the y_1 needs also to be constant.



(a) y_1 Residuals (b) y_2 Residuals (c) y_3 Residuals

Fig. 3: Autocorrelation test for the estimation of residuals

$$A = \begin{bmatrix} 0.397 & 0.06 & -0.126 & 0.193 & 0.066 & 0.037 & 0.05 & 0 & 0 & 0 & 0 & -0.29 & -0.038 \\ \dots & I_4 & \dots & \mathcal{O}_{4\times 9} & \dots & & & & & & & & \\ 0.264 & 0.072 & -1.055 & 0.958 & 0 & 0.225 & 0.331 & 0.008 & -0.223 & 0.208 & -0.185 & 0 & 0 \\ \dots & \mathcal{O}_{3\times 6} & \dots & I_3 & \dots & \mathcal{O}_{3\times 4} & \dots & & & & & & \\ -1.0538 & 0.97 & -0.6 & 0 & 0 & 0.356 & 0.399 & 0 & 0 & 0.05 & -0.315 & 0.035 & -0.218 \\ \dots & \mathcal{O}_{3\times 9} & \dots & I_3 & \dots & \mathcal{O}_{3\times 1} & \dots & & & & & & \end{bmatrix}$$

$$B = \begin{bmatrix} 0.177 & 0 & 0.118 \\ \dots & \mathcal{O}_{4\times 3} & \dots \\ 0.342 & 0 & 0.364 \\ \dots & \mathcal{O}_{3\times 3} & \dots \\ 0 & 0.869 & 0.765 \\ \dots & \mathcal{O}_{3\times 3} & \dots \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad (9)$$

B. Overall control system

At equilibrium, the corresponding input vector u_r and the desired output Y_r, are satisfying the following equations according to Eq. (8):

$$\begin{aligned} X_r(k) &= A \cdot X_r(k) + B \cdot u_r(k) \\ Y_r(k) &= C \cdot X_r(k) + D \cdot u_r(k) \end{aligned} \quad (14)$$

As our state vector contains only the previous values of the output we do not need an observer before implementing our control law. To drive the error between $X(k)$ and $X_r(k)$ to 0, we make a change on Eq. (8), namely:

$$\begin{aligned} X(k) - X_r(k) &= A \cdot X(k) + B \cdot u(k) - X_r(k) \\ &= A \cdot X(k) + B \cdot u(k) - A \cdot X_r(k) - B \cdot u_r(k) \\ &= A \cdot (X(k) - X_r(k)) + B \cdot (u(k) - u_r(k)) \end{aligned} \quad (15)$$

if we define $\Delta X(k) = X(k) - X_r(k)$ and $\Delta u(k) = u(k) - u_r(k)$, then we have a new linear system:

$$\Delta X(k) = A \cdot (\Delta X(k)) + B \cdot (\Delta u(k)) \quad (16)$$

and re-write the cost function of LQR (10) as:

$$\begin{aligned} J_r = \sum_{k=0}^{N-1} &((\Delta X(k))^T Q (\Delta X(k)) + (\Delta u(k))^T R (\Delta u(k)) \\ &+ 2(\Delta X(k))^T N (\Delta u(k))) + (\Delta X(N))^T Q (\Delta X(N)) \end{aligned} \quad (17)$$

the feedback control law that minimizes J_r can be written as:

$$\Delta u(k) = -K \cdot \Delta X(k) \quad (18)$$

where $K = (R + BP^T B)^T (B^T PA + N^T)$ is independent of state and input vectors. We can notice therefore that the K in (11) and (18) does not change.

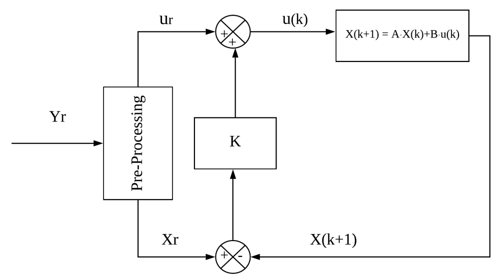


Fig. 4: Modified LQR Control problem

By grouping terms, the control law (18) can also be written:

$$u(k) = -K \cdot (X(k) - X_r(k)) + u_r(k) \quad (19)$$

The new control problem is graphically depicted in Fig. 4. According to Eq. (19) we can not directly apply designed output into the feedback, we will need a pre-processing function to transfer the desired output Y_r to the desired state vector X_r and input vector u_r when the system reaches the state-space. One thing to notice is that there will not be only one pair of X_r and u_r to satisfy the pre-processing function condition, it will be a range for both value, we will let experts to choose the values which make more sense in real world.

Recall the cost function of LQR (10) where N is set to 0. In this case, we set the weight matrix $Q = C^T Q' C$, since $Y(k) = C \cdot X(k)$, and the auxiliary matrix Q' weights the plant output. We find therefore the usual conclusions: when $R >> C^T Q' C$, the cost function is dominated by the control effort u, and so the controller minimizes the control action itself, this control strategy is used when the control signal is constrained; when $R << C^T Q' C$, the cost function is dominated by the output Y, and there is less penalty for using large u.

IV. System Evaluation

A. Setting-up the evaluation

According to the reality, a yearly GDP growth ratio of 3.2% is interesting to study[3]. To reach this level, the quarterly GDP increasing ratio p is around 0.8% (i.e. $(1.008)^4 \approx 1.032$).

Consider $Y_r = [y_{1r} \quad y_{2r} \quad y_{3r}]$ as the reference vector signal.

From (13) we know:

$$y_{1r} = ln(1 + \frac{p}{100}) = ln(1.008) = 0.007968 \approx 0.008$$

Following the same computation process as above, we can find $y_{2r} = 0.0175$ and $y_{3r} = 0.0086$.

Recall the relations between Y_r, X_r and u_r in (14) and since the output is identical to the first entry of the state vector, the matrix D is 0. These equations can be re-written as:

$$\begin{aligned} X_r(k) &= (I - A)^{-1} \cdot B \cdot u_r(k) \\ Y_r(k) &= C \cdot X_r(k) \end{aligned} \tag{20}$$

where I is the identity matrix of suitable dimensions. As we explained in the end of Sec. III-B, there is not a single pair of X_r and u_r that satisfy (20). For example, one reasonable pair of X_r and u_r is:

$$\begin{aligned} X_r(k) = [&0.008, 0.008, 0.008, 0.008, 0.008, 0.0173, 0.0173, \\ &0.0173, 0.0173, 0.0086, 0.0086, 0.0086, 0.0086]^T \end{aligned}$$

$$u_r(k) = [0.024, 0.003, 0.003]^T$$

For testing, purposes, diagonal weights of Q and R are used.

$$Q = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_{n_q} \end{bmatrix}; R = \rho \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_{n_r} \end{bmatrix} \tag{21}$$

where $n_q = rank(A) = 13$, $n_r = rank(B) = 3$. For the sake of simplicity, we will let $q_i = 1$ for all $i \in [1, 13]$, and $r_j = 1$ for all $j \in [1, 3]$, we will use ρ to adjust the input/state balance. We choose $\rho \in \{1, 10, 100\}$ to conduct experiments comparing the converging speed and observing the input signal range.

B. Experimental Evaluation

For the evaluation of the proposed control strategy, we implement the system detailed in Fig. 4.

The initial state vector X_0 of the state-space model is set by a linear regression over the real data only from 1980T1 to 1981T1. The resulting values are given below:

$$\begin{aligned} X_0 = [&0.001, 0.012, -0.008, -0.004, -0.004, 0.036, \\ &0.032, 0.013, -0.03, 0.007, 0.017, 0.006, -0.002]^T \end{aligned}$$

[3]Actually, the recent 10 years (2010-2019) average GDP growth ratio of France is 1.38%, but if we look back 25 years ago, the highest GDP growth ratio are showing during 1998-2001, which the average ratio is around 3.2%. Therefore, we want to study what measurements should be implemented to sustain this growth ratio.

Moreover, due to the meaning of our variables we have the following constraints regarding the three inputs: $u_1 \in$ **[0, 0.3]**, $u_2 \leq$ **[0, 0.008]**, $u_3 \leq$ **[0, 0.008]**. Details about these limits are given below.

For the sake of better presenting our results, throughout all the evaluation procedure we will consider the same simulation time, namely 50 time points (i.e. quarters).

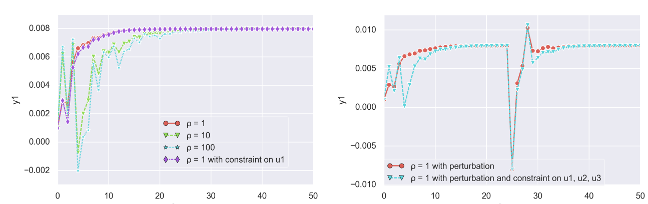
TABLE II: Output behavior. Please note that the state-space error is always 0 due to the type of the controller we consider. I.C. is for Input Constraint

ρ	**response time (quarter)**	**output variation range**
ρ =1	18	(0.001, 0.008)
ρ= 10	32	(-0.001, 0.008)
ρ= 100	37	(-0.002, 0.008)
ρ= 1 & I.C.	20	(0.001, 0.008)

1) *Variation of ρ without Constraints on Input Signals:*

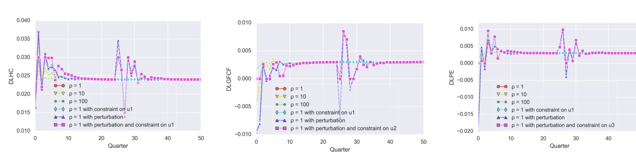
Let us first discard the limitations on the control signal in order to observe the behavior of the control system as well as it's limitations for different values of ρ. The output (i.e. y_1) is illustrated in Fig. 5a for $\rho \in \{1, 10, 100\}$. A more detailed result is showed in Tab. II where the settling time as well as the variation range of the output are given. From these results, we can clearly conclude that increasing the ρ value will also increase the convergence time to the state-space. Nevertheless, this comes with a significant impact on the inputs.

As we expected, the benefits of decreasing ρ come with cost, translated by the fact that the input exceeds the maximum allowed limits (see Fig.6). We can see that when ρ is small, the input signal fluctuates more.



(a) Output $y1$ and constraints on u (b) Output $y1$ and perturbation

Fig. 5: Output $y1$ under different ρ values



(a) u_1 - DLHC (b) u_2 - DLGFCF (c) u_3 - DLPE

Fig. 6: Corresponding input u under different ρ values

2) *Constraints on Input Signals:*

In our first experiments, we do not impose constraints on input signals, but in reality, there are some levels that input signals cannot reach. Therefore, in this experiment, we set $\rho = 1$ as well as a maximum limit 0.03 on signal u_1: $u_1 \leq 0.03$ (which is strictly lower than the maximum value of u_1 in the non-constrained case). Input signals results of "$\rho = 1$ with constraint on u_1" are showed in Fig. 6a, 6b and 6c and Tab. III. Comparing to only $\rho = 1$ result, we can see for signal u_1, that the maximum value of the range becomes 0.03 (which is the limit). For u_2 and u_3, the maximum value increases s they are being used to compensate for the insufficiency of u_1. For the sake of readability, all the input behavior is detailed in Table III.

From Fig. 5a, 6a, 6b and 6c, we can observe that even u_2 and u_3 react to compensate limitation of u_1, at 1^{st} and 4^{th} quarter, new results are slightly lower than the result without constraint. As for the converging speed, from Tab. II, we can also notice that experiment with constraint converges slower than the experiment without constraint. But still better than the results of $\rho = 10$ or 100.

TABLE III: Summary of Input variation range. C. is short for constraint, P. is short for perturbation.

ρ	u_1	u_2	u_3
1	(0.016, 0.038)	(-0.010, 0.003)	(-0.020, 0.007)
10	(0.022, 0.027)	(-0.005, 0.003)	(-0.007, 0.005)
100	(0.024, 0.025)	(0.002, 0.003)	(0.001, 0.003)
1 with C.	(0.016, 0.030)	(-0.010, 0.003)	(-0.020, 0.008)
1 with P.	(0.015, 0.038)	(-0.010, 0.009)	(-0.020, 0.010)
1 with C.&P.	(0.015, 0.03)	(0.0, 0.009)	(0.0, 0.010)

3) Perturbation on Output Signals: In this experiment, we keep $\rho = 1$, and we add perturbation on output signals to simulate economic crisis. From the reference signals setting in Sec. IV-A, we know that when the system is stable, y_1, y_2, y_3 should equal to 0.008, 0.0173 and 0.0086. And u_1, u_2, u_3 should equal 0.024, 0.003 and 0.003.

From Fig. 5b, 6a, 6b and 6c, we can see that at 24th quarter, the system has converged to a stable state. **Then we add a negative perturbation pulse signal -0.16 on y_1, y_2 and y_3 at 25th quarter.** The *"$\rho = 1$ with perturbation and constraint on u_i"* curves are the scenario where we not only implement the perturbation, but also implement constraints on all the input signals. Besides enforcing all to be positive we also have $u_1 \in$ **[0, 0.3]**, $u_2 \leq$ **[0, 0.008]**, $u_3 \leq$ **[0, 0.008]**.

Fig. 5b shows that after 25th quarter, the two systems totally recover from the perturbation, apparently the curve without constraint recovers faster than the other. One interesting point in Fig. 6b and 6c reveal that if we do not impose constraint on input signals, inputs u_2 and u_3 can be negative, recall that u_2 represents the first difference of logarithm of GFCF (also called investment). A negative signal means instead of investing during the crisis, we should sell our assets. As u_3 represents the first difference of logarithm of PE, negative means we need to reduce government spending. Nevertheless, all these conclusions, although correct from the engineering point of view, need to be coordinated with expert's advice.

V. Conclusion

Applying control theory to economic problems has been successfully studied in many cases, resource allocation is one of the well-established problems in this area. It demands dynamically choosing available resources with constraints over time to maximize or minimize an objective function.

In this paper, to apply optimal control, French macroeconomic quarterly data from 1980T1 to 2018T4 are used, more precisely we use 6 valiables which are representative from an economic point of view: GDP, EXP, IMP, HC, GFCF and PE. After estimating the model, an optimal LQR control solution is designed for our problem, which is to maintain a constant GDP increase ratio. Simulations using different realistic constraints and perturbations are performed. The control structure designed in this paper has good applicability and extensibility for other economic systems as well. This work can be further extended by considering more variables or by imposing certain characteristic constraints on inputs/outputs.

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